ECH4905 ChemE Optimization HW 1

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1 Problem 1

Consider the following matrix and perform the following calculations showing all your steps.

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

1.1 Part a

Determinant of A
Solution:

$$\begin{aligned} \det A &= 2(-1(1) - 0) - 2(1(1) + 0) + 3((1)(2) - (-1)(-1)) \\ &= -2 - 2 + 6 - 3 \\ \det A &= -1 \end{aligned}$$

1.2 Part b

Eigenvalues and eigenvectors of A
Solution:

$$\begin{gathered} (2-\lambda)((-1-\lambda)(1-\lambda) - 0) - 2(1(1-\lambda) + 0) + 3((1)(2) - (-1-\lambda)(-1)) = 0 \\ -(2-\lambda)(1+\lambda)(1-\lambda) - 2(1-\lambda) + 6 - 3(1+\lambda) = 0 \\ -(2-\lambda)(1+\lambda)(1-\lambda) + (1-\lambda) \\ (1-\lambda)((1+\lambda)(\lambda-2) + 1) \\ (1-\lambda)(\lambda^2 - \lambda - 2 + 1) \\ (1-\lambda)(\lambda^2 - \lambda - 1) \\ \lambda = 1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \end{gathered}$$

Now, we solve for $(A - \lambda I)v = 0$ to get the eigenvectors.

$$A - I = \begin{bmatrix} 2-1 & 2 & 3 \\ 1 & -1-1 & 0 \\ -1 & 2 & 1-1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 0 \\ -1 & 2 & 0 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & -4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$
$$v_1 = a \begin{bmatrix} -3/2 \\ -3/4 \\ 1 \end{bmatrix}$$

Using python, the other eigenvectors were found the same way and evaluated to be

$$v_2 = a \begin{bmatrix} 6 + 2\sqrt{10} \\ -1 \\ 1 \end{bmatrix}$$
$$v_3 = a \begin{bmatrix} 6 - 2\sqrt{10} \\ -1 \\ 1 \end{bmatrix}$$

2 Problem 2

Check if the set of all polynomials with real coefficients form a vector space.
Solution: The set of all polynomials with real coefficients looks like $\mathbb{P} = \{p(x) | a \in \mathbb{R}^n, x \in \mathbb{R}\}$ where

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0$$

This set is a vector space since each of the properties of a vector space hold.

- $p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3$
- $p_1 + p_2 = p_2 + p_1$
- $p + 0 = p \forall p \in \mathbb{P}$
- $p - p = 0 \forall p \in \mathbb{P}$
- $\alpha p \in \mathbb{P}$
- $(\alpha + \beta)p = \alpha p + \beta p$
- $\beta(p_1 + p_2) = \beta p_1 + \beta p_2$
- $(\alpha\beta)p = \alpha(\beta)p$
- $1p = p$

These statements are true for all p in the form above. Therefore, the set of all polynomials forms a vector space.

3 Problem 3

Consider the following function and perform the following calculations

$$f(x_1, x_2) = x_1^3 x_2 - x_1 x_2^3$$

3.1 Part a

Gradient of the function
Solution:

$$\nabla f(x_1, x_2) = \begin{bmatrix} 3x_1^2 x_2 - x_2^3 \\ x_1^3 - 3x_1 x_2^2 \end{bmatrix}$$

3.2 Part b

Hessian of the function
Solution:

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 6x_1 x_2 & 3x_1^2 - 3x_2^2 \\ 3x_1^2 - 3x_2^2 & 6x_1 x_2 \end{bmatrix}$$

3.3 Part c

Write the second order Taylor expansions around a point (x_1^*, x_2^*)
Solution: Defining a vector $\mathbf{x}^* = (x_1^*, x_2^*)$The second order taylor expansion has the form

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^\top (\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^\top \nabla^2 f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*)$$

We can evaluate the $f, \nabla f, \nabla^2 f$ at $\mathbf{x}^*$ and replace the symbols above with those values.

4 Problem 4

Check if the following function is convex

$$f(x_1, x_2) = (1 - x_1)^2 + 100(x_2 - x_1^2)^2$$

Solution: In order to check if f is convex, we calculate its hessian and evaluate if it is positive semi-definite.

$$\begin{aligned} \nabla f(x_1, x_2) &= \begin{bmatrix} -2(1 - x_1) - (2x_1)200(x_2 - x_1^2) \\ 200(x_2 - x_1^2) \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 - 2 - 400x_2 x_1 + 400x_1^3 \\ 200x_2 - 200x_1^2 \end{bmatrix} \end{aligned}$$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 2 - 400x_2 + 1200x_1^2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}$$

$$(H - \lambda I) = \begin{bmatrix} 2 - 400x_2 + 1200x_1^2 - \lambda & -400x_1 \\ -400x_1 & 200 - \lambda \end{bmatrix}$$
$$0 = (2 - 400x_2 + 1200x_1^2 - \lambda)(200 - \lambda) - (-400x_1)(-400x_1)$$
$$0 \leq 400 - 80,000x_2 + 240,000x_1^2 - 200\lambda - 2\lambda + 400\lambda x_2 - 1200x_1^2\lambda + \lambda^2 - 160,000x_1^2$$

The function f is convex when the above equation holds. Another way to evaluate if the function f is convex is by using rules that preserve convexity. We can split f into two functions $f = f_1 + f_2$.

$$f_1(x_1, x_2) = (1 - x_1)^2, \quad f_2(x_1, x_2) = 100(x_2 - x_1^2)^2$$

The non-negative weighted sum of two convex functions is convex so we break the problem down to verifying convexity of f_1 and f_2. f_1 is clearly convex since it is a quadratic function in one variable. To verify convexity of f_2, we can use composition rules.

$$f_2(x_1, x_2) = 100(\tilde{f}_2(x_1, x_2))^2, \quad \tilde{f}_2(x_1, x_2) = x_2 - x_1^2$$

The function f_2 is convex, non-increasing on $\tilde{f}_2(x_1, x_2) \leq 0$, and non-decreasing on $\tilde{f}_2(x_1, x_2) \geq 0$. The function $\tilde{f}_2$ is concave, therefore the overall function f is convex on the domain $\mathbf{dom} f = x_2 - x_1^2 \leq 0$.

5 Problem 5

Check if the following function is convex

$$g(x_1, x_2) = 5x_1^2 - 4x_1x_2$$

Solution: Like the previous problem, we calculate the hessian of this function and evaluate when it is positive semi-definite.

$$\nabla g(x_1, x_2) = \begin{bmatrix} 10x_1 - 4x_2 \\ -4x_1 \end{bmatrix}$$

$$\nabla^2 g(x_1, x_2) = \begin{bmatrix} 10 & -4 \\ -4 & 0 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} 10 - \lambda & -4 \\ -4 & -\lambda \end{bmatrix}$$
$$0 = (10 - \lambda)(-\lambda) - (-4)(-4)$$
$$0 = \lambda^2 - 10\lambda - 16$$
$$16 + 25 = \lambda^2 - 10\lambda + 25$$
$$(\lambda - 5)^2 = 41$$
$$\lambda - 5 = \pm\sqrt{41}$$
$$\lambda = 5 \pm \sqrt{41}$$
$$\lambda >= 0, \lambda <= 0.$$

Since we have one eigenvalue greater than 0 and one less than 0, the hessian is indefinite and g is not convex.

6 Problem 6

Consider a set of linear equalities $Ax = b$ as well as a set of convex nonlinear inequalities $g(x) \leq 0$. Consider the feasible region constrained by these linear and nonlinear inequalities. Assuming that this region is non-empty, show that this feasible region is convex.
Solution:

7 Problem 7

Consider the following optimization problem

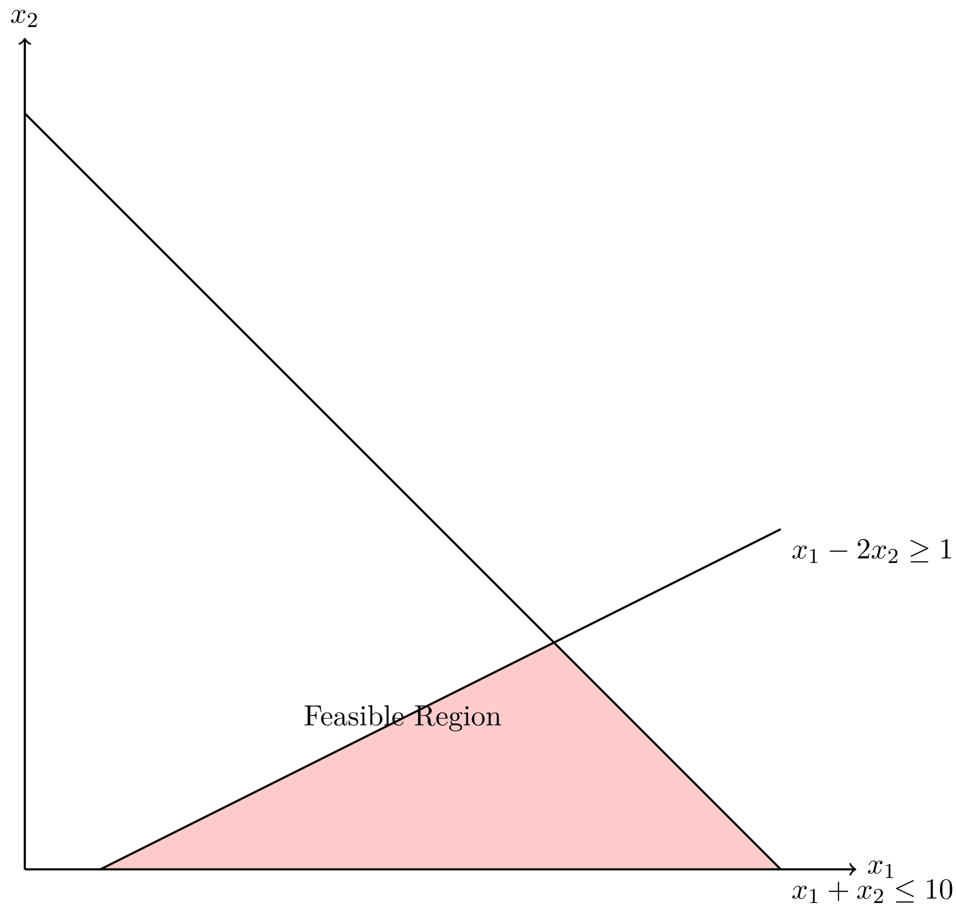
$$\begin{aligned}
\text{minimize} \quad & x_1 && (1)\\
\text{subject to} \quad & x_1 + x_2 \leq 10 && (2)\\
& x_1 - 2x_2 \geq 1 && (3)\\
& x_1, x_2 \geq 0 && (4)\\
& x_1, x_2 \in \mathbb{R} && (5)
\end{aligned}$$

7.1 Part a

What type of problem is this (MILP, MINLP,...)? Justify.
Solution: This problem is an LP. The constraints are all linear and the variables are continuous variables in $\mathbb{R}$.

7.2 Part b

Draw the feasible region of the problem
Solution:



7.3 Part c

Is the region convex or non-convex? Justify.
Solution: The problem is convex. The objective function is convex as it is linear. The feasible region is also convex because it is the intersection of half-spaces. The feasible region is a polyhedron.

8 Problem 8

Consider the following optimization problem

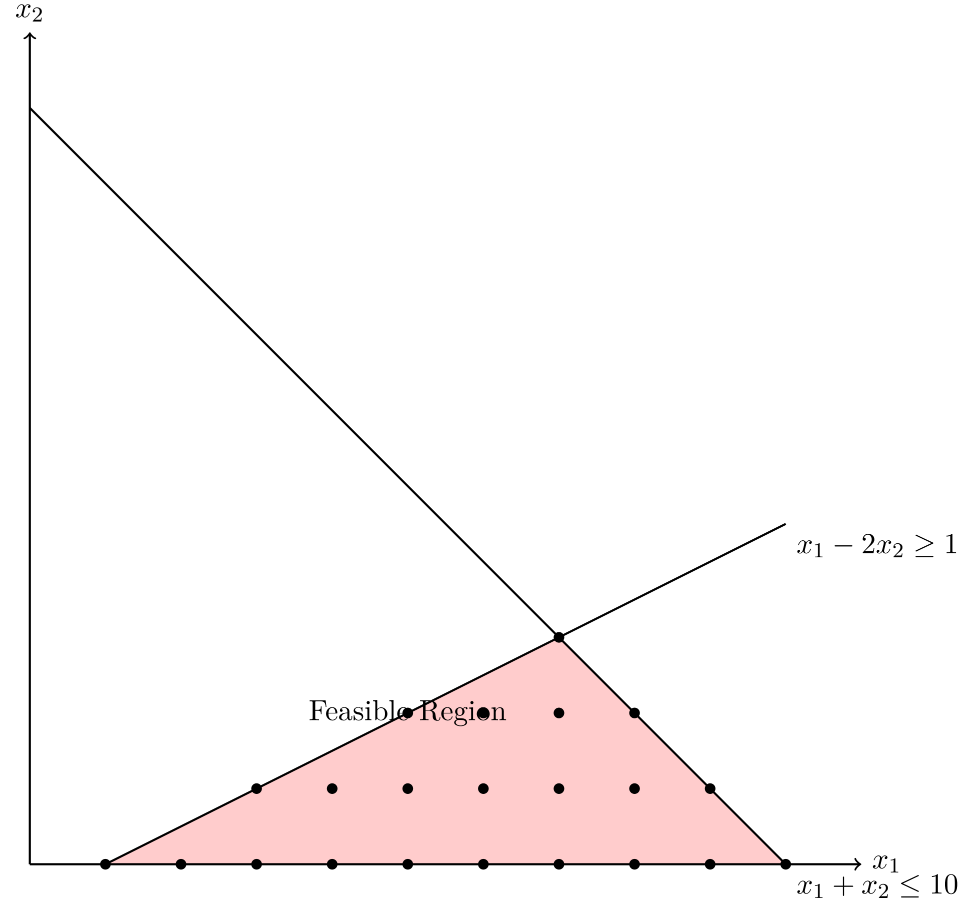
$$\begin{aligned} \text{minimize} \quad & x_1 && (6) \\ \text{subject to} \quad & x_1 + x_2 \leq 10 && (7) \\ & x_1 - 2x_2 \geq 1 && (8) \\ & x_1, x_2 \geq 0 && (9) \\ & x_1, x_2 \in \mathbb{R} && (10) \end{aligned}$$

8.1 Part a

What type of problem is this (MILP, MINLP, ...)? Justify.
Solution: This problem is a MILP. The constraints and objective function are all linear and the variables are binary integer variables. Therefore it is not a standard LP, but instead an MILP.

8.2 Part b

Draw the feasible region of the problem.
Solution:



8.3 Part c

Is the region convex or non-convex? Justify.
Solution:
The problem is non-convex because there are integer variables. The disjoint that integer variables introduce naturally make the problem non-convex.