ECH4905 ChemE Optimization HW 2

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1 Problem 1

Write the following problem in standard form

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 = 2 \end{aligned}$$

Solution: To solve this problem we want to turn the above problem into the canonical form

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Flipping the maximize to minimize

$$\begin{aligned} \text{minimize} \quad & -3x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 = 2 \end{aligned}$$

Expanding the equality

$$\begin{aligned} \text{minimize} \quad & -3x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 2 \\ & x_1 + 3x_2 \leq 2 \end{aligned}$$

Flipping the $\geq$ inequalities

$$\begin{aligned} \text{minimize} \quad & -3x_1 - 2x_2 \\ \text{subject to} \quad & 2x_1 - x_2 \leq -1 \\ & -x_1 + -3x_2 \leq -2 \\ & x_1 + 3x_2 \leq 2 \end{aligned}$$

Bounding the variables to be ≥ 0

$$\begin{aligned}
\text{minimize} \quad & -3(x_1^+ - x_1^-) - 2(x_2^+ - x_2^-) \\
\text{subject to} \quad & 2(x_1^+ - x_1^-) - (x_2^+ - x_2^-) \leq -1 \\
& -(x_1^+ - x_1^-) + -3(x_2^+ - x_2^-) \leq -2 \\
& (x_1^+ - x_1^-) + 3(x_2^+ - x_2^-) \leq 2 \\
& x_1^+, x_1^-, x_2^+, x_2^- \geq 0
\end{aligned}$$

To solve this problem in standard form, we turn it into

$$\begin{aligned}
\text{minimize} \quad & c^\top x \\
\text{subject to} \quad & Ax = b \\
& x \geq 0
\end{aligned}$$

We can alter the canonical form easily by adding slack variables

$$\begin{aligned}
\text{minimize} \quad & -3(x_1^+ - x_1^-) - 2(x_2^+ - x_2^-) \\
\text{subject to} \quad & 2(x_1^+ - x_1^-) - (x_2^+ - x_2^-) + s_1 = -1 \\
& (x_1^+ - x_1^-) + 3(x_2^+ - x_2^-) = 2 \\
& x_1^+, x_1^-, x_2^+, x_2^-, s_1 \geq 0
\end{aligned}$$

2 Problem 2

An engineering factory makes seven products on the following machines: four grinders, two vertical drills, three horizontal drills, one borer and one planer. Each product yields a certain contribution to profit (defined in dollars per unit). These quantities together with the unit production times required on each process are given below.

	P1	P2	P3	P4	P5	P6	P7
Contribution to profit	10	6	8	4	11	9	3
Grinding (hours)	0.5	0.7	-	-	0.3	0.2	0.5
Vertical drilling (hours)	0.1	0.2	-	0.3	-	0.6	-
Horizontal drilling (hours)	0.2	-	0.8	-	-	-	0.6
Boring (hours)	0.05	0.03	-	0.07	0.1	-	0.08
Planning (hours)	-	-	0.01	-	0.05	-	0.05
Demand (units)	500	1000	300	300	800	200	100

Table 1: Production data for the engineering factory

There are some marketing limitations to the production (demand), these are given as the bottom row in table 1. We can assume that the factory works 24 days, and that each day each machine works for 16 hours. Formulate an LP to find the optimal product distribution.

Solution: Our objective function is pretty simple and can be expressed as $10p_1 + 6p_2 + 8p_3 + 4p_4 + 11p_5 + 9p_6 + 3p_7$. To handle the constraints, we have three types: demand, machine, and non-negativity constraints. Since the factory works 24 days and each machine works for 16 hours, each machine is constrained to work a maximum of $24 \times 16 = 384$ hours. The optimization problem can be expressed as

$$
\begin{aligned}
\text{maximize} \quad & 10p_1 + 6p_2 + 8p_3 + 4p_4 + 11p_5 + 9p_6 + 3p_7 \\
\text{subject to} \quad & 0.5p_1 + 0.7p_2 + 0.3p_5 + 0.2p_6 + 0.5p_7 \leq 384 \\
& 0.1p_1 + 0.2p_2 + 0.3p_4 + 0.6p_6 \leq 384 \\
& 0.2p_1 + 0.8p_3 + 0.6p_7 \leq 384 \\
& 0.05p_1 + 0.03p_2 + 0.07p_4 + 0.1p_5 + 0.08p_7 \leq 384 \\
& 0.01p_3 + 0.05p_5 + 0.05p_7 \leq 384 \\
& p_1 \leq 500 \\
& p_2 \leq 1000 \\
& p_3 \leq 300 \\
& p_4 \leq 300 \\
& p_5 \leq 800 \\
& p_6 \leq 200 \\
& p_7 \leq 100 \\
& p_1, p_2, p_3, p_4, p_5, p_6, p_7 \geq 0
\end{aligned}
$$

This can be equivalently expressed with matrix notation as

$$
\begin{aligned}
\text{maximize} \quad & \mathbf{c}^\top \mathbf{p} \\
\text{subject to} \quad & \mathbf{A}\mathbf{p} \preceq 384 \times \mathbf{1} \\
& \mathbf{p} \preceq \mathbf{d} \\
& \mathbf{p} \succeq 0
\end{aligned}
$$

With the following parameters

$$
\mathbf{c} = \begin{bmatrix} 10 \\ 6 \\ 8 \\ 4 \\ 11 \\ 9 \\ 3 \end{bmatrix}
\mathbf{A} = \begin{bmatrix} 0.5 & 0.7 & 0 & 0 & 0.3 & 0.2 & 0.5 \\ 0.1 & 0.2 & 0 & 0.3 & 0 & 0.6 & 0 \\ 0.2 & 0 & 0.8 & 0 & 0 & 0 & 0.6 \\ 0.05 & 0.03 & 0 & 0.07 & 0.1 & 0 & 0.08 \\ 0 & 0 & 0.01 & 0 & 0.05 & 0 & 0.05 \end{bmatrix}
\mathbf{d} = \begin{bmatrix} 500 \\ 1000 \\ 300 \\ 300 \\ 800 \\ 200 \\ 100 \end{bmatrix}
$$

and the decision variables

$$
\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{bmatrix}
$$

3 Problem 3

Solve problem 2 but consider that the demand of each product is a function of the month, i.e., that the demand for product i in month m is given by parameter $d_{i,m}$. In this case, assume that you can store a maximum amount of 100 units of product each month at a cost of 0.5/unit-month. At the beginning of the planning period there is no inventory, but at the end we want to have 50 units of each product in storage.

When and what should the factory produce to maximize profit?

Solution: In order to solve this problem with the introduction of periods, we can abstract the number of months as M and products P. We define three decision variables $\mathbf{P}, \mathbf{S}, \mathbf{X}, \mathbf{X}_{prev} \in \mathbb{R}^{P\times(M+1)}$.

- $\mathbf{P} \in \mathbb{R}^{P\times(M+1)}$ has entries p_{im} where each entry is the amount of product i produced during period m. This matrix is padded with zeros in the first column since we do not produce anything in period 0.

- $\mathbf{S} \in \mathbb{R}^{P\times(M+1)}$ has entries s_{im} where each entry is the amount of product i sold during period m. This matrix is padded with zeros in the first column since we do not produce anything in period 0.

- $\mathbf{X} \in \mathbb{R}^{P\times(M+1)}$ has entries x_{im} where each entry is the amount of product i left at the end of period m. The first column is a zero vector since we end period 0 with no inventory. The last column is a P length vector of 50s since we must end with 50 units of each product.

- $\mathbf{X}_{prev} \in \mathbb{R}^{P\times(M+1)}$ is a shift of $\mathbf{X}$ and represents the amount of product i left at the beginning of period m. This isn't really a decision variable, as each entry is shared with $\mathbf{X}$. The first two columns are zero vectors since we do not start period 0 or 1 with any inventory.

The parameters to this problem would be

- $\mathbf{c} \in \mathbb{R}^P$ is the contribution to profit of each product i.

- $\mathbf{A} \in \mathbb{R}^{K\times P}$ is the constraint matrix for the machines. This problem has five machines so $K = 5$

- $\mathbf{D} \in \mathbb{R}^{P\times(M+1)}$ has entries d_{im} with the amount of demand of product i in period m. The first column is a zero vector since there is no demand for period 0.

- $v \in \mathbb{R}$ is the maximum amount of product i at the end of each month. In this problem, it is equivalent to 100

- $l \in \mathbb{R}$ is the cost of storing a unit of product i in month m. In this problem, it is 0.5

- $u \in \mathbb{R}$ is the maximum amount of hours each machine can work in a month. It is equal to 384 in this problem.

The optimization problem can then be expressed as

$$
\begin{aligned}
\text{maximize} \quad & \mathbf{c}^\top \mathbf{S}\mathbf{1} - l(\mathbf{1}^\top \mathbf{X}\mathbf{1}) \\
\text{subject to} \quad & \mathbf{A}\mathbf{P} \preceq u \times \mathbf{1} \quad \mathbf{1} = \mathbf{1}^{K\times(M+1)} \\
& \mathbf{S} \preceq \mathbf{D} \\
& \mathbf{X} \preceq v \times \mathbf{1} \quad \mathbf{1} = \mathbf{1}^{P\times(M+1)} \\
& \mathbf{X} = \mathbf{P} - \mathbf{S} + \mathbf{X}_{prev} \\
& \mathbf{X}, \mathbf{P}, \mathbf{S}, \mathbf{X}_{prev} \succeq 0
\end{aligned}
$$

In the above problem, the inequalities $\preceq, \succeq$ are defined as element-wise inequality for both vectors and matrices.
To solve for the optimal solution, `cvxpy` was used with the following parameters:

```
import cvxpy as cp
import numpy as np

### Parameters
n_P = 7 # number of products
n_M = 4 # number of months
n_K = 5 # number of machines


c = np.array([10, 6, 8, 4, 11, 9, 3]) # selling price of product i
A = np.array([
    [0.5, 0.7, 0, 0, 0.3, 0.2, 0.5],
    [0.1, 0.2, 0, 0.3, 0, 0.6, 0],
    [0.2, 0, 0.8, 0, 0, 0, 0.6],
    [0.05, 0.03, 0, 0.07, 0.1, 0, 0.08],
    [0, 0, 0.01, 0, 0.05, 0, 0.05]
]) # machine constraint matrix

D = np.array(
    [500, 1000, 300, 300, 800, 200, 100] * n_M
).reshape(n_P, n_M)
D = np.c_[np.zeros(n_P), D] # demand constraint matrix

v = 100 # max amount of product i inventory in a month m (is constant)
l = 0.5 # cost of holding a unit to the next month
u = 384 # max amount of hours for a machine

### Variables
P = cp.Variable((n_P, n_M+1),name="P")
S = cp.Variable((n_P, n_M+1),name="S")
X = cp.Variable((n_P, n_M+1),name="X")
X_prev = cp.hstack([np.zeros((n_P, 2)), X[:, 1:-1]])

### Objective Function
obj = cp.Maximize(cp.sum(c.T @ S) - cp.sum(l* X))

### Constraints
constraints = [
    A @ P <= u * np.ones((n_K,n_M+1)),
    S <= D,
    X <= v * np.ones((n_P, n_M+1)),
    X == P - S + X_prev,
```

```
    X >= 0,
    P >= 0,
    S >= 0,
    X_prev >= 0,
    X[:,0] == 0,
    X[:,-1] == 50,
    P[:,0] == 0,
    S[:,0] == 0
]

### Problem
prob = cp.Problem(obj, constraints)

prob.solve()
>>> 58885.52385535276
```

The results of the variables were queried and resulted in:

$$\mathbf{S} = \begin{bmatrix} 0 & 448 & 448 & 141.43 & 300 \\ 0 & 0 & 0 & 0 & 5.71 \\ 0 & 368 & 300 & 300 & 405 \\ 0 & 200 & 100 & 500 & 476.19 \\ 0 & 300 & 300 & 800 & 200 \\ 0 & 100 & 500 & 466.43 & 300 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 0 & 0 & 0 & 0 & 50 \\ 0 & 0 & 0 & 0 & 50 \\ 0 & 0 & 0 & 100 & 50 \\ 0 & 69.33 & 100 & 100 & 50 \\ 0 & 100 & 0 & 0 & 50 \\ 0 & 100 & 100 & 0 & 50 \\ 0 & 0 & 0 & 0 & 50 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 0 & 448 & 448 & 141.43 & 350 \\ 0 & 0 & 0 & 0 & 55.71 \\ 0 & 368 & 300 & 400 & 355 \\ 0 & 269.33 & 130.67 & 500 & 426.19 \\ 0 & 400 & 200 & 800 & 250 \\ 0 & 200 & 500 & 366.43 & 350 \\ 0 & 0 & 0 & 0 & 50 \end{bmatrix}$$

4 Problem 4

Formulate a LP model for the following reaction network (image 1), assuming that you want to maximize the production rate of 3-methyltetrahydrofuran, and that you have an incoming flux of itaconic acid equal to 1. For the sake of simplicity you can ignore the water and hydrogen that would be required to balance the equations.

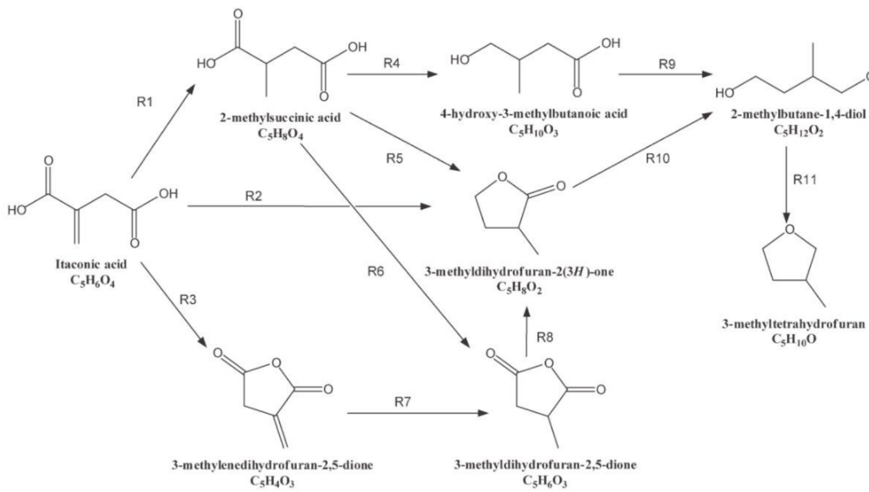

Figure 1: Problem 4 Reaction Network

Solution: This is a flow problem and can be modeled using the flow variables r_i.

$$\begin{aligned} \text{maximize} \quad & r_{11} \\ \text{subject to} \quad & 1 = r_1 + r_2 + r_3 \\ & r_1 = r_4 + r_5 + r_6 \\ & r_3 = r_7 \\ & r_6 + r_7 = r_8 \\ & r_2 + r_5 + r_8 = r_{10} \\ & r_4 = r_9 \\ & r_9 + r_{10} = r_{11} \end{aligned}$$

This can be expressed in matrix notation as

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\top \mathbf{r} \\ \text{subject to} \quad & \mathbf{N}\mathbf{r} = \mathbf{b} \\ & \mathbf{r} \succeq 0 \end{aligned}$$

with the following parameters

$$\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{N} = \text{flow conservation matrix } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the decision variables

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \\ r_9 \\ r_{10} \\ r_{11} \end{bmatrix}$$