# ECH4905 ChemE Optimization HW 4

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## 1 Problem 1

Consider the following integer programming problem:

$$\begin{aligned} \text{maximize} \quad & 1.2y_1 + y_2 \\ \text{subject to} \quad & y_1 + y_2 \leq 1 \\ & 1.2y_1 + 0.5y_2 \leq 1 \\ & y_1, y_2 \in \{0, 1\} \end{aligned}$$

### 1.1 Part a

Solve the first relaxed LP subproblem by hand using the simplex method and derive Gomory cuts based on the LP relaxation.

**Solution:** To tackle this problem we first relax the problem and then turn the problem above into the standard form so we can create a simplex tableau from it.

$$\begin{aligned} \text{maximize} \quad & 1.2y_1 + y_2 \\ \text{subject to} \quad & y_1 + y_2 + s_1 = 1 \\ & 1.2y_1 + 0.5y_2 + s_2 = 1 \\ & y_1 + s_3 = 1 \\ & y_2 + s_4 = 1 \\ & y_1, y_2, s_1, s_2, s_3, s_4 \geq 0 \end{aligned}$$

In matrix notation,

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \succeq 0 \end{aligned}$$

where

$$\mathbf{c} = \begin{bmatrix} -1.2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1.2 & 0.5 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} y_1 \\ y_2 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$$

(with a flipped objective component)
The initial simplex tableau for the problem is as follows:

| Basic Var | $y_1$ | $y_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | RHS | $\alpha$ |
|---|---|---|---|---|---|---|---|---|
| $s_1$ | 1 | 1 | 1 | 0 | 0 | 0 | 1 | |
| $s_2$ | 1.2 | 0.5 | 0 | 1 | 0 | 0 | 1 | |
| $s_3$ | 1 | 0 | 0 | 0 | 1 | 0 | 1 | |
| $s_4$ | 0 | 1 | 0 | 0 | 0 | 1 | 1 | |
| obj | $-1.2$ | $-1$ | 0 | 0 | 0 | 0 | $-$ | $-$ |

we can define the slack variables equal to the right hand side, and this is in turn a basic feasible solution, so we can jump into phase 2.
We select the $y_1$ as the entering variable and calculate the alpha value for each basic variable

| Basic Var | $y_1$ | $y_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | RHS | $\alpha$ |
|---|---|---|---|---|---|---|---|---|
| $s_1$ | 1 | 1 | 1 | 0 | 0 | 0 | 1 | $\frac{1}{1}$ |
| $s_2$ | 1.2 | 0.5 | 0 | 1 | 0 | 0 | 1 | $\frac{1}{1.2}$ |
| $s_3$ | 1 | 0 | 0 | 0 | 1 | 0 | 1 | $\frac{1}{1}$ |
| $s_4$ | 0 | 1 | 0 | 0 | 0 | 1 | 1 | $\frac{1}{0}$ |
| obj | $-1.2$ | $-1$ | 0 | 0 | 0 | 0 | $-$ | $-$ |

We pivot this on the 1st column ($y_1$) and the 2nd row ($s_2$)

| Basic Var | $y_1$ | $y_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | RHS | $\alpha$ |
|---|---|---|---|---|---|---|---|---|
| $s_1$ | 0 | $\frac{7}{12}$ | 1 | $-\frac{5}{6}$ | 0 | 0 | $\frac{1}{6}$ | |
| $y_1$ | 1 | $\frac{5}{12}$ | 0 | $\frac{5}{6}$ | 0 | 0 | $\frac{5}{6}$ | |
| $s_3$ | 0 | $-\frac{5}{12}$ | 0 | $-\frac{5}{6}$ | 1 | 0 | $\frac{1}{6}$ | |
| $s_4$ | 0 | 1 | 0 | 0 | 0 | 1 | 1 | |
| obj | 0 | $-0.5$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |

With blands rule, we pick $y_2$ and calculate the alpha value for each basic variable.

| Basic Var | $y_1$ | $y_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | RHS | $\alpha$ |
|---|---|---|---|---|---|---|---|---|
| $s_1$ | 0 | $\frac{7}{12}$ | 1 | $-\frac{5}{6}$ | 0 | 0 | $\frac{1}{6}$ | $\frac{2}{7}$ |
| $y_1$ | 1 | $\frac{5}{12}$ | 0 | $\frac{5}{6}$ | 0 | 0 | $\frac{5}{6}$ | $\frac{2}{1}$ |
| $s_3$ | 0 | $-\frac{5}{12}$ | 0 | $-\frac{5}{6}$ | 1 | 0 | $\frac{1}{6}$ | $-\frac{1}{5}$ |
| $s_4$ | 0 | 1 | 0 | 0 | 0 | 1 | 1 | $\frac{1}{1}$ |
| obj | 0 | $-0.5$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |

We pivot on the 2nd column ($y_2$) and the 1st row ($s_1$).

| Basic Var | $y_1$ | $y_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | RHS | $\alpha$ |
|---|---|---|---|---|---|---|---|---|
| $y_2$ | 0 | 1 | $\frac{12}{7}$ | $-\frac{10}{7}$ | 0 | 0 | $\frac{2}{7}$ | |
| $y_1$ | 1 | 0 | $-\frac{5}{7}$ | $\frac{60}{42}$ | 0 | 0 | $\frac{30}{42}$ | |
| $s_3$ | 0 | 0 | $\frac{5}{7}$ | $-\frac{60}{42}$ | 1 | 0 | $\frac{12}{42}$ | |
| $s_4$ | 0 | 0 | $-\frac{12}{7}$ | $\frac{60}{42}$ | 0 | 1 | $\frac{30}{42}$ | |
| obj | 0 | 0 | 0.857 | 0.286 | 0 | 0 | 0 | $-1.143$ |

This is the optimal solution to the LP relaxed problem. Now we will derive Gomory cuts from this LP relaxed problem. Since each constraint has a non-integer solution, we can generate a Gomory cut on each constraint.

$$
\begin{aligned}
y_2 + \text{floor}(\frac{12}{7})s_1 + \text{floor}(\frac{-10}{7})s_2 &\leq \text{floor}(\frac{2}{7}) \\
y_1 + \text{floor}(\frac{-5}{7})s_1 + \text{floor}(\frac{10}{7})s_2 &\leq \text{floor}(\frac{5}{7}) \\
s_3 + \text{floor}(\frac{5}{7})s_1 + \text{floor}(\frac{-10}{7})s_2 &\leq \text{floor}(\frac{2}{7}) \\
s_4 + \text{floor}(\frac{-12}{7})s_1 + \text{floor}(\frac{10}{7})s_2 &\leq \text{floor}(\frac{5}{7})
\end{aligned}
$$

These turn into the cuts

$$
\begin{aligned}
y_2 + s_1 - 2s_2 &\leq 0 \\
y_1 - s_1 + s_2 &\leq 0 \\
s_3 - 2s_2 &\leq 0 \\
s_4 - 2s_1 + s_2 &\leq 0
\end{aligned}
$$

## 1.2 Part b

Solve the above problem with the branch and bound method by enumerating nodes in the tree and solving the LP subproblems using GAMS.

**Solution:** The initial LP relaxed problem is solved in 1.1, so we can start with the parent node. An important note for this question, I will be solving the LPs in my custom `gatorpy` LP solver so that I can use them as verification tests.

$$
\boxed{
\begin{aligned}
&[\frac{5}{7}, \frac{2}{7}, 0, 0, \frac{2}{7}, \frac{5}{7}], z = \frac{8}{7} \\
&\text{maximize} \quad 1.2y_1 + y_2 \\
&\text{subject to} \quad y_1 + y_2 + s_1 = 1, \\
&\quad 1.2y_1 + 0.5y_2 + s_2 = 1, \\
&\quad y_1 + s_3 = 1, \\
&\quad y_2 + s_4 = 1, \\
&\quad y_1, y_2, s_1, s_2, s_3, s_4 \geq 0. \\
&UB = \frac{8}{7}, LB = -inf
\end{aligned}
}
$$

Since all variables are fractional, we can pick the first one $y_1$ to branch on

$[\frac{5}{7}, \frac{2}{7}, 0, 0, \frac{2}{7}, \frac{5}{7}], z = \frac{8}{7}$

maximize $\quad 1.2y_1 + y_2$

subject to $\quad y_1 + y_2 + s_1 = 1,$

$\quad 1.2y_1 + 0.5y_2 + s_2 = 1,$

$\quad y_1 + s_3 = 1,$

$\quad y_2 + s_4 = 1,$

$\quad y_1, y_2, s_1, s_2, s_3, s_4 \geq 0.$

$UB = \frac{8}{7}, LB = -inf$

$y_1 \quad = \quad 0$

$[0, \frac{2}{3}, 0, \frac{1}{3}, \frac{1}{3}, 1], z = \frac{2}{3}$

maximize $\quad 1.2y_1 + y_2$

subject to $\quad y_1 \leq 0,$

$\quad y_1 + y_2 + s_1 = 1,$

$\quad 1.2y_1 + 0.5y_2 + s_2 = 1,$

$\quad y_1 + s_3 = 1,$

$\quad y_2 + s_4 = 1,$

$\quad y_1, y_2, s_1, s_2, s_3, s_4 \geq 0.$

$UB = \frac{2}{3}, LB = -inf$

$y_1 \quad = \quad 1$

$[1, 0, 0, 0, 0, 1], z = 1.2$

maximize $\quad 1.2y_1 + y_2$

subject to $\quad y_1 \geq 1,$

$\quad y_1 + y_2 + s_1 = 1,$

$\quad 1.2y_1 + 0.5y_2 + s_2 = 1,$

$\quad y_1 + s_3 = 1,$

$\quad y_2 + s_4 = 1,$

$\quad y_1, y_2, s_1, s_2, s_3, s_4 \geq 0.$

$UB = 1.2, LB = -inf$