ECH4905 ChemE Optimization HW 3

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1 Problem 1

Take the feasible set defined by the constraints

$$\begin{aligned} x_1 + x_2 &\leq 3 \\ x_1 + x_3 &\leq 7 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_3 &\geq 0 \end{aligned}$$

1.1 Part a

Express the constraints in standard form
Solution: To move this feasible set to standard form, we add slack variables to turn the inequalities into

$$\begin{aligned} x_1 + x_2 + s_1 &= 3 \\ x_1 + x_3 + s_2 &= 7 \\ x_1, x_2, x_3, s_1, s_2 &\geq 0 \end{aligned}$$

which can be expressed in matrix form as

$$\begin{aligned} A\mathbf{x} &= b \\ \mathbf{x} &\succeq 0 \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{bmatrix}$$

1.2 Part b

Identify the basic solutions, and among them, those that are feasible
Solution: To identify the basic solutions, we can group the columns into the following sets of two, select those columns as the basis of a matrix B, and solve for $\mathbf{x}$:

1. $J_1 = 1, 2$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \quad \mathbf{x}_B = B^{-1}b = \begin{bmatrix} 7 \\ -4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 7 \\ -4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2. $J_2 = 1, 3$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \mathbf{x}_B = B^{-1}b = \begin{bmatrix} 3 \\ 10 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 10 \\ 0 \\ 0 \end{bmatrix}$$

3. $J_3 = 1, 4$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \quad \mathbf{x}_B = B^{-1}b = \begin{bmatrix} 7 \\ -4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 7 \\ 0 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

4. $J_4 = 1, 5$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \mathbf{x}_B = B^{-1}b = \begin{bmatrix} 3 \\ 10 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

5. $J_5 = 2, 3$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{x}_B = B^{-1}b = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 7 \\ 0 \\ 0 \end{bmatrix}$$

6. $J_6 = 2, 4$ These columns are linearly dependent and therefore do not form a basis and are not a basic solution.

7. $J_7 = 2, 5$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{x}_B = B^{-1}b = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 7 \end{bmatrix}$$

8. $J_8 = 3, 4$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{x}_B = B^{-1}b = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 7 \\ 3 \\ 0 \end{bmatrix}$$

9. $J_9 = 3, 5$ This is not a basic solution since the columns are linearly dependent

10. $J_{10} = 4, 5$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{x}_B = B^{-1}b = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 7 \end{bmatrix}$$

The basic solutions are $J_i, \forall i \in [1, 2, 3, 4, 5, 7, 8, 10]$. The basic *feasible* solutions are $J_i, \forall i \in [2, 4, 5, 7, 8, 10]$.

1.3 Part c

For each point corresponding to a feasible basic solution identify the basic directions
Solution: Since there are 6 basic feasible solution points, and 3 non-basic variables for each, there will be 18 total basic directions. To be less confusing, I will index them with d_i^j, where i is the non-basic variable index and j is the index of the basic feasible solution that it corresponds to.

1. $j = 2 = [1, 3]$

$$d_2^2 = \mathbf{P} \begin{bmatrix} d_{B2} \\ d_{N2} \end{bmatrix} = \mathbf{P} \begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$d_4^2 = \mathbf{P} \begin{bmatrix} d_{B4} \\ d_{N4} \end{bmatrix} = \mathbf{P} \begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$d_5^2 = \mathbf{P} \begin{bmatrix} d_{B5} \\ d_{N5} \end{bmatrix} = \mathbf{P} \begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

2. $j = 4 = [1, 5]$

$$d_2^4 = \mathbf{P}\begin{bmatrix} d_{B2} \\ d_{N2} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$d_3^4 = \mathbf{P}\begin{bmatrix} d_{B3} \\ d_{N3} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$d_4^4 = \mathbf{P}\begin{bmatrix} d_{B4} \\ d_{N4} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

3. $j = 5 = [2, 3]$

$$d_1^5 = \mathbf{P}\begin{bmatrix} d_{B1} \\ d_{N1} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$d_4^5 = \mathbf{P}\begin{bmatrix} d_{B4} \\ d_{N4} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$d_5^5 = \mathbf{P}\begin{bmatrix} d_{B5} \\ d_{N5} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

4. $j = 7 = [2, 5]$

$$d_1^7 = \mathbf{P}\begin{bmatrix} d_{B1} \\ d_{N1} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$d_3^7 = \mathbf{P}\begin{bmatrix} d_{B3} \\ d_{N3} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$d_4^7 = \mathbf{P}\begin{bmatrix} d_{B4} \\ d_{N4} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

5. $j = 8 = [3, 4]$

$$d_1^8 = \mathbf{P}\begin{bmatrix} d_{B1} \\ d_{N1} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

$$d_2^8 = \mathbf{P}\begin{bmatrix} d_{B2} \\ d_{N2} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$d_5^8 = \mathbf{P}\begin{bmatrix} d_{B5} \\ d_{N5} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

6. $j = 10 = [4, 5]$

$$d_1^{10} = \mathbf{P}\begin{bmatrix} d_{B1} \\ d_{N1} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

$$d_2^{10} = \mathbf{P}\begin{bmatrix} d_{B2} \\ d_{N2} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$d_3^{10} = \mathbf{P}\begin{bmatrix} d_{B3} \\ d_{N3} \end{bmatrix} = \mathbf{P}\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

2 Problem 2

Solve the following optimization problem using the two-stage simplex method

$$\begin{aligned} \min\ & -9x_1 - 4x_2 \\ \text{s.t.}\ & 5x_1 + 2x_2 \le 31 \\ & -3x_1 + 2x_2 \le 5 \\ & -2x_1 - 3x_2 \le -1 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{aligned}$$

2.1 Part a

Use the two-stage simplex tableau

Solution:

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ -2 & -3 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 31 \\ 5 \\ -1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -9 \\ -4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

	x_1	x_2	s_1	s_2	s_3	RHS
s_1	5	2	1	0	0	31
s_2	−3	2	0	1	0	5
s_3	−2	−3	0	0	1	−1
obj	−9	−4	0	0	0	0

We can identify x_1 as the entering variable. To find the exiting variable we must first calculate the descent direction associated with x_1.

$$d_p = \mathbf{P}\begin{bmatrix} d_{Bp} \\ d_{Np} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -5 \\ 3 \\ 2 \end{bmatrix}$$

$$\alpha_3 = -\frac{31}{-5}, \alpha_4 = \infty, \alpha_5 = \infty$$

s_1 is the exiting variable.

	x_1	x_2	s_1	s_2	s_3	RHS
x_1	1	0.4	0.2	0	0	6.2
s_2	0	3.2	0.6	1	0	23.6
s_3	0	−2.6	0.4	0	1	5.8
obj	0	−0.4	1.8	0	0	55.8

We can identify x_2 as the entering variable. To find the exiting variable we must first calculate the descent direction associated with x_2.

$$d_p = \mathbf{P}\begin{bmatrix} d_{Bp} \\ d_{Np} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\begin{bmatrix} 0.4 \\ 3.2 \\ -2.6 \end{bmatrix} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 1 \\ 0 \\ -3.2 \\ 2.6 \end{bmatrix}$$

$$\alpha_1 = -\frac{6.2}{-0.4}, \alpha_4 = -\frac{23.6}{-3.2}, \alpha_5 = \infty$$

s_2 is the exiting variable.

	x_1	x_2	s_1	s_2	s_3	RHS
x_1	1	0.4	0.2	0	0	6.2
x_2	0	1	0.1875	0.3125	0	7.375
s_3	0	0	0.8875	0.8125	1	18.375
obj	0	0	1.925	1.25	0	58.75

The optimal solution is $x_1 = 3.25$, $x_2 = 7.375$, and the optimal value is -58.75.

2.2 Part b

Use GAMS to solve the optimization problem
Solution:

```
                LOWER          LEVEL          UPPER         MARGINAL

---- VAR x1                      .            3.2500         +INF             .
---- VAR x2                      .            7.3750         +INF             .
---- VAR obj                   -INF         -58.7500         +INF             .
```

3 Problem 3

An oil refinery purchases two crude oils (crude 1 and crude 2). These crude oils are put through four processes: distillation, reforming, cracking, and blending, to produce petrols and fuels that are sold.

Distillation: distillation separates each crude oil into fractions known as light naphtha, medium naphtha, heavy naphtha, light oil, heavy oil, and residuum according to their boiling points. Light, medium, and heavy naphthas have octane numbers of 90, 80, and 70, respectively. The fractions into which one barrel of each type of crude splits are given in the following table:

	Light naphtha	Medium naphtha	Heavy naphtha	Light oil	Heavy oil	Residuum
Crude 1	0.1	0.2	0.2	0.12	0.2	0.13
Crude 2	0.15	0.25	0.18	0.08	0.19	0.12

Note that there is a small wastage in the distillation operation (i.e., fractions do not add to 1).

Reforming: the naphthas can be used immediately for blending into different grades of petrol or can go through a process known as reforming. Reforming produces a product known as reformed gasoline with an octane number of 115. The yields of reformed gasoline from each barrel of the different naphthas are given as follows:

- 1 barrel of light naphtha yields 0.6 barrels of reformed gasoline
- 1 barrel of medium naphtha yields 0.52 barrels of reformed gasoline
- 1 barrel of heavy naphtha yields 0.45 barrels of reformed gasoline

Cracking: the oils (light and heavy) can either be used directly for blending into jet fuel or fuel oil or be put through a process known as catalytic cracking. The catalytic cracker produces cracked oil and cracked gasoline. Cracked gasoline has an octane number of 105.

- 1 barrel of light oil yields 0.68 barrels of cracked oil and 0.28 barrels of cracked gasoline
- 1 barrel of heavy oil yields 0.75 barrels of cracked oil and 0.2 barrels of cracked gasoline

Cracked oil is used for blending fuel oil and jet fuel; cracked gasoline is used for blending petrol. Residuum can be used for either producing lube oil or blending into jet fuel and fuel oil:

- 1 barrel of residuum yields 0.5 barrels of lube oil.

Blending:

- Petrols: there are two types of petrol, regular and premium, obtained by blending the naphtha, reformed gasoline, and cracked gasoline. The only stipulations concerning them are that regular must have an octane number of at least 84 and that premium must have an octane number of at least 94. It is assumed that octane numbers blend linearly by volume.
- Jet fuel: the stipulation concerning jet fuel is that its vapor pressure must not exceed 1 kg/cm^2. The vapor pressures for light, heavy, cracked oils, and residuum are 1.0, 0.6, 1.5, and 0.05 kg/cm^2. It may again be assumed that vapor pressures blend linearly by volume.
- Fuel oil: To produce fuel oil, we blend light oil, cracked oil, heavy oil, and residuum in a ratio of 10:4:3:1.

There are some availability and capacity limitations on the quantities and processes used as follows:

- The daily availability of crude 1 is 20000 barrels
- The daily availability of crude 2 is 30000 barrels
- At most 45000 barrels of crude can be distilled per day
- At most 10000 barrels of naphtha can be reformed per day
- At most 8000 barrels of oil can be cracked per day
- The daily production of lube oil must be between 500 and 1000 barrels
- Premium motor fuel production must be at least 40% of regular motor fuel production

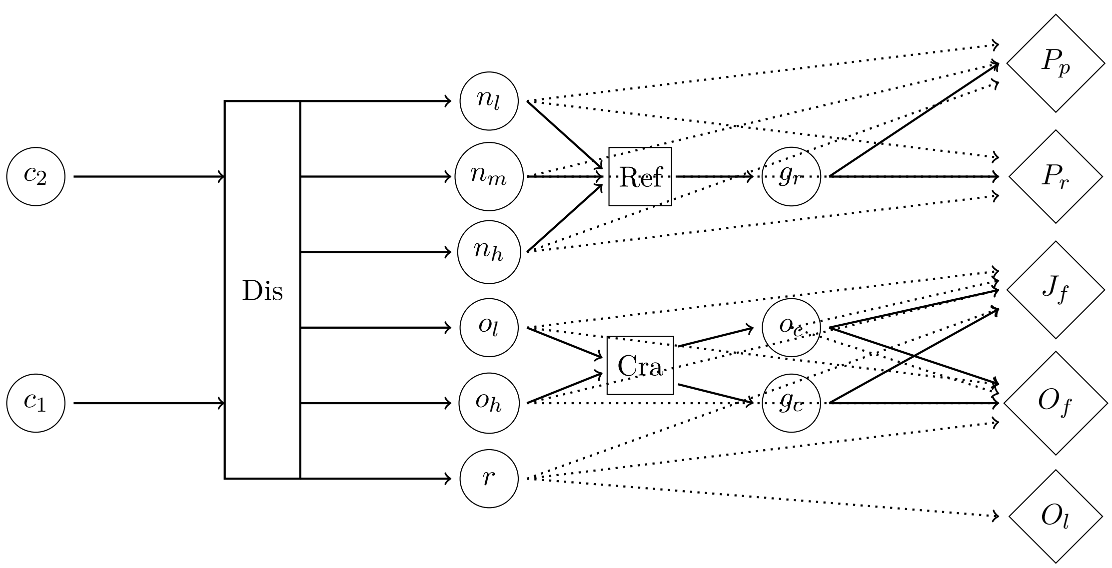
The profit contributions from the sale of the final products are (in dollars per barrel) as follows:

- Premium petrol: 700
- Regular petrol: 600
- Jet fuel: 400
- Fuel oil: 350
- Lube oil: 150

How should the operations of the refinery be planned in order to maximize the total profit?

3.1 Part a

Develop a model based on the provided information
Solution:



For this problem above, we can define the following parameters and variables:

Sets:
C : Set of crude oil inputs $o \in [1, 2]$
D : Set of distillations $o \in [3]$
R : Set of refineries $o \in [4]$
A : Set of cracking stations $o \in [5]$
B : Set of blenders $o \in [6, 7, 8, 9, 10, 11]$
I : Set of inputs and outputs $c_1, c_2, n_l, n_m, n_h, o_l, o_h, r, g_r, o_c, g_c, P_p, P_r, J_f, O_f, O_l$ $i \in [1, \ldots, 16]$
O : Set of outputs P_p, P_r, J_f, O_f, O_l
Parameters:
$y^o_{i,j}$: Yield of output j produced by input i and processed by op o.
q_j : Octane rating output j.
a_i : Daily availability of input i
v_i : Vapor pressure of output j
Profit contributions:
c_j : Profit per barrel of output j.
Variables:
$x^o_{i,j}$: Quantity of output j produced by input i and processed by op o.
If $j = i$, then it is the total amount produced by o. (Barrels)

The objective function can be formulated as

$$\text{maximize} \quad \sum_{j \in I} c_j \sum_{o \in O} x^o_{i,j}$$

The input and output yield constraints can be formulated as

$$\begin{aligned}
\text{Capacity and Availability Constraints} \quad & x^1_{1,1} \leq 20000 \\
& x^2_{2,2} \leq 30000 \\
& x^1_{1,1} + x^2_{2,2} \leq 45000 \\
& \sum_{i \in [3,4,5]} x^4_{i,9} \leq 10000 \\
& \sum_{j \in [10,11]} \sum_{i \in [6,7]} x^5_{i,j} \leq 8000 \\
& 500 \leq x^{11}_{16,16} \leq 1000 \\
& \sum_{o \in [4,6,7,8]} x^o_{12,12} \geq 0.40 \sum_{o \in [4,6,7,8]} x^o_{13,13}
\end{aligned}$$

$$\text{Yield Constraints} \quad x^o_{j,j} = \sum_{i \in I \backslash O} y^o_{i,j} x^o_{i,j}, \quad \forall j \in I \backslash C, \forall o \in D \cup R \cup A \cup B$$

$$\text{Pressure Constraints} \quad \sum_{o\in[5,9,10,11]} x^o_{15,15} \leq \sum_{o\in[5,9,10,11],j\in[10,6,7,8]} x^o_{j,j}$$

$$\text{Octane Constraints} \sum_{o\in O} \geq q_j \sum_{i\in[3,4,5,9],o\in O} q_i x^O_{i,j}, \quad \forall j \in [12,13]$$

3.2 Part b

Find the optimal value in GAMS
Solution: