ECH4905 ChemE Optimization HW 2

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February 10, 2025

1 Problem 1

Write the following problem in standard form

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 = 2 \end{aligned}$$

Solution: To solve this problem we want to turn the above problem into the canonical form

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Flipping the maximize to minimize

$$\begin{aligned} \text{minimize} \quad & -3x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 = 2 \end{aligned}$$

Expanding the equality

$$\begin{aligned} \text{minimize} \quad & -3x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 2 \\ & x_1 + 3x_2 \leq 2 \end{aligned}$$

Flipping the $\geq$ inequalities

$$\begin{aligned} \text{minimize} \quad & -3x_1 - 2x_2 \\ \text{subject to} \quad & 2x_1 - x_2 \leq -1 \\ & -x_1 + -3x_2 \leq -2 \\ & x_1 + 3x_2 \leq 2 \end{aligned}$$

Bounding the variables to be ≥ 0

$$\begin{aligned}
\text{minimize} \quad & -3(x_1^+ - x_1^-) - 2(x_2^+ - x_2^-) \\
\text{subject to} \quad & 2(x_1^+ - x_1^-) - (x_2^+ - x_2^-) \leq -1 \\
& -(x_1^+ - x_1^-) + -3(x_2^+ - x_2^-) \leq -2 \\
& (x_1^+ - x_1^-) + 3(x_2^+ - x_2^-) \leq 2 \\
& x_1^+, x_1^-, x_2^+, x_2^- \geq 0
\end{aligned}$$

To solve this problem in standard form, we turn it into

$$\begin{aligned}
\text{minimize} \quad & c^\top x \\
\text{subject to} \quad & Ax = b \\
& x \geq 0
\end{aligned}$$

We can alter the canonical form easily by adding slack variables

$$\begin{aligned}
\text{minimize} \quad & -3(x_1^+ - x_1^-) - 2(x_2^+ - x_2^-) \\
\text{subject to} \quad & 2(x_1^+ - x_1^-) - (x_2^+ - x_2^-) + s_1 = -1 \\
& (x_1^+ - x_1^-) + 3(x_2^+ - x_2^-) = 2 \\
& x_1^+, x_1^-, x_2^+, x_2^-, s_1 \geq 0
\end{aligned}$$

2 Problem 2

An engineering factory makes seven products on the following machines: four grinders, two vertical drills, three horizontal drills, one borer and one planer. Each product yields a certain contribution to profit (defined in dollars per unit). These quantities together with the unit production times required on each process are given below.

	P1	P2	P3	P4	P5	P6	P7
Contribution to profit	10	6	8	4	11	9	3
Grinding (hours)	0.5	0.7	-	-	0.3	0.2	0.5
Vertical drilling (hours)	0.1	0.2	-	0.3	-	0.6	-
Horizontal drilling (hours)	0.2	-	0.8	-	-	-	0.6
Boring (hours)	0.05	0.03	-	0.07	0.1	-	0.08
Planning (hours)	-	-	0.01	-	0.05	-	0.05
Demand (units)	500	1000	300	300	800	200	100

Table 1: Production data for the engineering factory

There are some marketing limitations to the production (demand), these are given as the bottom row in table 1. We can assume that the factory works 24 days, and that each day each machine works for 16 hours. Formulate an LP to find the optimal product distribution.

Solution: Our objective function is pretty simple and can be expressed as $10p_1 + 6p_2 + 8p_3 + 4p_4 + 11p_5 + 9p_6 + 3p_7$. To handle the constraints, we have three types: demand, machine, and non-negativity constraints. Since the factory works 24 days and each machine works for 16 hours, each machine is constrained to work a maximum of $24 \times 16 = 384$ hours. The optimization problem can be expressed as

$$
\begin{aligned}
\text{maximize} \quad & 10p_1 + 6p_2 + 8p_3 + 4p_4 + 11p_5 + 9p_6 + 3p_7 \\
\text{subject to} \quad & 0.5p_1 + 0.7p_2 + 0.3p_5 + 0.2p_6 + 0.5p_7 \leq 384 \\
& 0.1p_1 + 0.2p_2 + 0.3p_4 + 0.6p_6 \leq 384 \\
& 0.2p_1 + 0.8p_3 + 0.6p_7 \leq 384 \\
& 0.05p_1 + 0.03p_2 + 0.07p_4 + 0.1p_5 + 0.08p_7 \leq 384 \\
& 0.01p_3 + 0.05p_5 + 0.05p_7 \leq 384 \\
& p_1 \leq 500 \\
& p_2 \leq 1000 \\
& p_3 \leq 300 \\
& p_4 \leq 300 \\
& p_5 \leq 800 \\
& p_6 \leq 200 \\
& p_7 \leq 100 \\
& p_1, p_2, p_3, p_4, p_5, p_6, p_7 \geq 0
\end{aligned}
$$

This can be equivalently expressed with matrix notation as

$$
\begin{aligned}
\text{minimize} \quad & \mathbf{c}^\top \mathbf{p} \\
\text{subject to} \quad & \mathbf{A}\mathbf{p} \preceq 384 \times \mathbf{1} \\
& \mathbf{p} \preceq \mathbf{d} \\
& \mathbf{p} \succeq 0
\end{aligned}
$$

With the following parameters

$$
\mathbf{c} = \begin{bmatrix} 10 \\ 6 \\ 8 \\ 4 \\ 11 \\ 9 \\ 3 \end{bmatrix}
\mathbf{A} = \begin{bmatrix} 0.5 & 0.7 & 0 & 0 & 0.3 & 0.2 & 0.5 \\ 0.1 & 0.2 & 0 & 0.3 & 0 & 0.6 & 0 \\ 0.2 & 0 & 0.8 & 0 & 0 & 0 & 0.6 \\ 0.05 & 0.03 & 0 & 0.07 & 0.1 & 0 & 0.08 \\ 0 & 0 & 0.01 & 0 & 0.05 & 0 & 0.05 \end{bmatrix}
\mathbf{d} = \begin{bmatrix} 500 \\ 1000 \\ 300 \\ 300 \\ 800 \\ 200 \\ 100 \end{bmatrix}
$$

and the decision variables

$$
\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{bmatrix}
$$

3 Problem 3

Solve problem 2 but consider that the demand of each product is a function of the month, i.e., that the demand for product i in month m is given by parameter $d_{i,m}$. In this case, assume that you

can store a maximum amount of 100 units of product each month at a cost of 0.5/unit-month. At the beginning of the planning period there is no inventory, but at the end we want to have 50 units of each product in storage.
When and what should the factory produce to maximize profit?
Solution:

4 Problem 4

Formulate a LP model for the following reaction network (image 1), assuming that you want to maximize the production rate of 3-methyltetrahydrofuran, and that you have an incoming flux of itaconic acid equal to 1. For the sake of simplicity you can ignore the water and hydrogen that would be required to balance the equations.

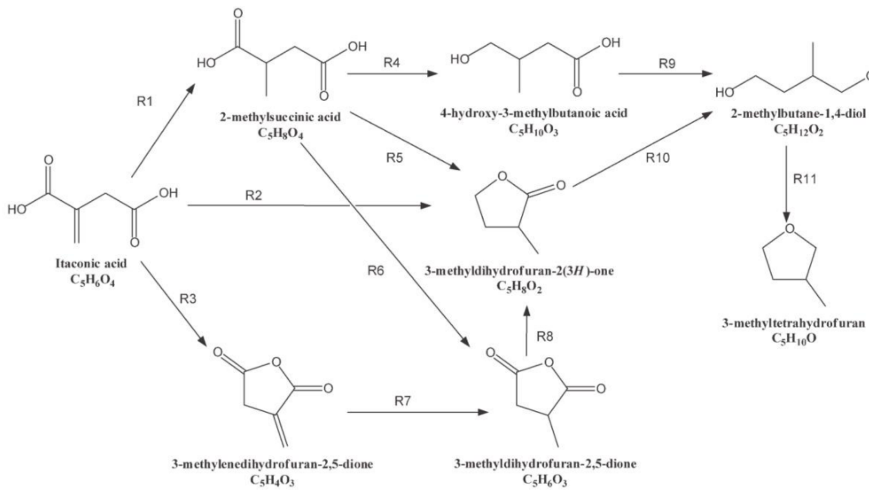

Figure 1: Problem 4 Reaction Network

Solution: