ECH4905 ChemE Optimization HW 4

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1 Problem 1

Consider the following integer programming problem:

$$\begin{aligned} \text{maximize} \quad & 1.2y_1 + y_2 \\ \text{subject to} \quad & y_1 + y_2 \leq 1 \\ & 1.2y_1 + 0.5y_2 \leq 1 \\ & y_1, y_2 \in \{0, 1\} \end{aligned}$$

1.1 Part a

Solve the first relaxed LP subproblem by hand using the simplex method and derive Gomory cuts based on the LP relaxation.

Solution: To tackle this problem we first relax the problem and then turn the problem above into the standard form so we can create a simplex tableau from it.

$$\begin{aligned} \text{maximize} \quad & 1.2y_1 + y_2 \\ \text{subject to} \quad & y_1 + y_2 + s_1 = 1 \\ & 1.2y_1 + 0.5y_2 + s_2 = 1 \\ & y_1 + s_3 = 1 \\ & y_2 + s_4 = 1 \\ & y_1, y_2, s_1, s_2, s_3, s_4 \geq 0 \end{aligned}$$

In matrix notation,

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \succeq 0 \end{aligned}$$

where

$$\mathbf{c} = \begin{bmatrix} -1.2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1.2 & 0.5 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} y_1 \\ y_2 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$$

(with a flipped objective component)
The initial simplex tableau for the problem is as follows:

Basic Var	y_1	y_2	s_1	s_2	s_3	s_4	RHS	α
s_1	1	1	1	0	0	0	1	
s_2	1.2	0.5	0	1	0	0	1	
s_3	1	0	0	0	1	0	1	
s_4	0	1	0	0	0	1	1	
obj	-1.2	-1	0	0	0	0	$-$	$-$

we can define the slack variables equal to the right hand side, and this is in turn a basic feasible solution, so we can jump into phase 2.
We select the y_1 as the entering variable and calculate the alpha value for each basic variable

Basic Var	y_1	y_2	s_1	s_2	s_3	s_4	RHS	α
s_1	1	1	1	0	0	0	1	$\frac{1}{1}$
s_2	1.2	0.5	0	1	0	0	1	$\frac{1}{1.2}$
s_3	1	0	0	0	1	0	1	$\frac{1}{1}$
s_4	0	1	0	0	0	1	1	$\frac{1}{0}$
obj	-1.2	-1	0	0	0	0	$-$	$-$

We pivot this on the 1st column (y_1) and the 2nd row (s_2)

Basic Var	y_1	y_2	s_1	s_2	s_3	s_4	RHS	α
s_1	0	$\frac{7}{12}$	1	$-\frac{5}{6}$	0	0	$\frac{1}{6}$	
y_1	1	$\frac{5}{12}$	0	$\frac{5}{6}$	0	0	$\frac{5}{6}$	
s_3	0	$-\frac{5}{12}$	0	$-\frac{5}{6}$	1	0	$\frac{1}{6}$	
s_4	0	1	0	0	0	1	1	
obj	0	-0.5	$-$	$-$	$-$	$-$	$-$	$-$

With blands rule, we pick y_2 and calculate the alpha value for each basic variable.

Basic Var	y_1	y_2	s_1	s_2	s_3	s_4	RHS	α
s_1	0	$\frac{7}{12}$	1	$-\frac{5}{6}$	0	0	$\frac{1}{6}$	$\frac{2}{7}$
y_1	1	$\frac{5}{12}$	0	$\frac{5}{6}$	0	0	$\frac{5}{6}$	$\frac{2}{1}$
s_3	0	$-\frac{5}{12}$	0	$-\frac{5}{6}$	1	0	$\frac{1}{6}$	$-\frac{1}{5}$
s_4	0	1	0	0	0	1	1	$\frac{1}{1}$
obj	0	-0.5	$-$	$-$	$-$	$-$	$-$	$-$

We pivot on the 2nd column (y_2) and the 1st row (s_1).

Basic Var	y_1	y_2	s_1	s_2	s_3	s_4	RHS	α
y_2	0	1	$\frac{12}{7}$	$-\frac{10}{7}$	0	0	$\frac{2}{7}$	
y_1	1	0	$-\frac{5}{7}$	$\frac{60}{42}$	0	0	$\frac{30}{42}$	
s_3	0	0	$\frac{5}{7}$	$-\frac{60}{42}$	1	0	$\frac{12}{42}$	
s_4	0	0	$-\frac{12}{7}$	$\frac{60}{42}$	0	1	$\frac{30}{42}$	
obj	0	0	0.857	0.286	0	0	0	-1.143

This is the optimal solution to the LP relaxed problem. Now we will derive Gomory cuts from this LP relaxed problem. Since each constraint has a non-integer solution, we can generate a Gomory cut on each constraint.

$$
\begin{aligned}
y_2 + \text{floor}(\frac{12}{7})s_1 + \text{floor}(\frac{-10}{7})s_2 &\leq \text{floor}(\frac{2}{7}) \\
y_1 + \text{floor}(\frac{-5}{7})s_1 + \text{floor}(\frac{10}{7})s_2 &\leq \text{floor}(\frac{5}{7}) \\
s_3 + \text{floor}(\frac{5}{7})s_1 + \text{floor}(\frac{-10}{7})s_2 &\leq \text{floor}(\frac{2}{7}) \\
s_4 + \text{floor}(\frac{-12}{7})s_1 + \text{floor}(\frac{10}{7})s_2 &\leq \text{floor}(\frac{5}{7})
\end{aligned}
$$

These turn into the cuts

$$
\begin{aligned}
y_2 + s_1 - 2s_2 &\leq 0 \\
y_1 - s_1 + s_2 &\leq 0 \\
s_3 - 2s_2 &\leq 0 \\
s_4 - 2s_1 + s_2 &\leq 0
\end{aligned}
$$

1.2 Part b

Solve the above problem with the branch and bound method by enumerating nodes in the tree and solving the LP subproblems using GAMS. **Solution:**