Exam1MTH451

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1 -

π **Four digit rounding:** 3.142
Absolute Error: $|\pi - 3.142| \approx .000407$
Relative: $\frac{|\pi-3.142|}{\pi} \approx .000130$
Significant Digits: $\frac{|\pi-3.142|}{\pi} < 5 * 10^{-4}$.
$= 4$

$10^{200}\pi$ **Four digit rounding:** $3.142 * 10^{200}$
Absolute Error: $|10^{200}\pi - 3.142 * 10^{200}| \approx 4.07 * 10^{196}$
Relative: $\frac{|\pi 10^{200} - 3.142 * 10^{200}|}{\pi * 10^{200}} \approx .000130$
Significant Digits: $\frac{|\pi 10^{200} - 3.142 * 10^{200}|}{\pi * 10^{200}} < 5 * 10^{-4}$.
$= 4$

2 -

Bisection $f(x) = x^3 + x - 4$:
Iteration 1: upper = 4.00000 , lower = 1.00000
Mid-point 2.50000 and f(mid) 14.12500.
Iteration 2: upper = 2.50000 , lower = 1.00000
Mid-point 1.75000 and f(mid) 3.10938.
Iteration 3: upper = 1.75000 , lower = 1.00000
Mid-point 1.37500 and f(mid) -0.02539.
Iteration 4: upper = 1.75000 , lower = 1.37500
Mid-point 1.56250 and f(mid) 1.37720.
Iteration 5: upper = 1.56250 , lower = 1.37500
Mid-point 1.46875 and f(mid) 0.63718.
Iteration 6: upper = 1.46875 , lower = 1.37500
Mid-point 1.42188 and f(mid) 0.29652.
Iteration 7: upper = 1.42188 , lower = 1.37500
Mid-point 1.39844 and f(mid) 0.13326.
Iteration 8: upper = 1.39844 , lower = 1.37500
Mid-point 1.38672 and f(mid) 0.05336.
Iteration 9: upper = 1.38672 , lower = 1.37500
Mid-point 1.38086 and f(mid) 0.01384.
Iteration 10: upper = 1.38086 , lower = 1.37500
Mid-point 1.37793 and f(mid) -0.00581.
Iteration 11: upper = 1.38086 , lower = 1.37793
Mid-point 1.37939 and f(mid) 0.00401.
Iteration 12: upper = 1.37939 , lower = 1.37793
Mid-point 1.37866 and f(mid) -0.00092.
Zero: 1.37866

$$1.37866^3 + 1.37866 - 4 \approx -0.00092$$

To estimate the number of iterations we use the **Theorem 2.1** from the textbook. We have:

$$.001 \leq \frac{3}{2^n}$$

Which yields 12 necessary iterations.

3 -

We let $f(x) = .5(\cos(x) + \sin(x))$ be used for the fixed point approximation. Consider the interval [-1,1]. We note that f is continuous and bounded in [-1,1] and $|f(x)| < 1$ since the peaks of the sin and cos functions occur at different points. The derivative $= .5(\cos(x) - \sin(x))$. On the boundaries, $f'(-1) \approx .69, f'(1) \approx -.15$ and the function obtains a max at $cos(x) = -sin(x)$ which is $-\pi/4$. This has a value of $\sqrt{2}/2$. Thus from theorem 2.4, the fixed point theorem, for any starting point in [-1,1], the series $p_{n+1} = .5(\cos(p_n) + \sin(p_n))$ will converge to the fixed point.
We use $p_0 = 0$ to approximate.
Iteration 1: $p_0 = 0.000000$ p = 0.500000
Iteration 2: $p_1 = 0.500000$ p = 0.678504
Iteration 3: $p_2 = 0.678504$ p = 0.703071
Iteration 4: $p_3 = 0.703071$ p = 0.704712
Iteration 5: $p_4 = 0.704712$ p = 0.704806
Iteration 6: $p_5 = 0.704806$ p = 0.704812
Fixed Point:

$$.5(\cos(0.704812) + \sin(0.704812))) \approx 0.704812$$

or

$$2(0.704812) - (\cos(0.704812) + \sin(0.704812)) \approx 0$$

Estimated iterations:
We note that from the corollary 2.5, there are two estimates for the number of iterations:

$$|p_n - p| \leq (\sqrt{2}/2)^n$$

and

$$|p_n - p| \leq .5(\frac{(\sqrt{2}/2)^n}{1 - \sqrt{2}/2})$$

The first estimate yields 34 iterations.
The second estimation yields 35 iterations.
As expected these are guarantees and thus greatly over-estimate the necessary computations.

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We begin by noting that $g(x) = x(2 - \pi x)$ has two fixed points, one at 0 and another at $1/\pi \approx .31831$ ($x = x(2 - \pi x)$ has these solutions trivially).
With $p_0 = .5$, $p_1 = 1 - \pi/4 \approx .21$. Consider the interval [.2,.4]. $g(x) \in [.2, .4]$ for $x \in [.2, .4]$. The function and its derivative are continuous on this interval, and $g' = 2 - 2\pi x$ has a maximum value of $g'(.2) \approx .74$ and a minimum value $g'(.4) \approx -.51$. Thus we can apply the fixed point theorem (2.4) and conclude that since the iteration reaches $.21 \in$ [.2,.4], further iterations from this point will converge to $1/\pi$.
Convergence: $\lim_{n\to\infty} p_n = 1/\pi$.
Now we consider the rate of convergence. We note that on the interval [.2,.4] $|g'| < .75$. Applying Corollary 2.5 we conclude that the rate of convergence is O($(\frac{3}{4})^n$) since $|p_n - p| \le k^n max(p - .2, .4 - p)$ and with $k = .75$.
Rate of Convergence: $p_n = 1/\pi +$ O($(\frac{3}{4})^n$)

5 -

Newton's Method for $f(x) = e^x - 3x^2$, $f'(x) = e^x - 6x$
Iteration 1, p_0 = 0.500000, f(p_0) = 0.898721
Iteration 2, p_1 = 1.165089, f(p_1) = -0.866091
Iteration 3, p_2 = 0.936227, f(p_2) = -0.079222
Iteration 4, p_3 = 0.910397, f(p_3) = -0.001158
Iteration 5, p_4 = 0.910008, f(p_4) = -0.000000

$$e^{.910008} - 3(.910008)^2 \approx .000000$$

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Secant Method for $f(x) = e^x - 3x^2$:
Iteration 1, p_0 = 3.000, p_1 = 5.000 , $f(p_0)$ = -6.914 , $f(p_1)$ = 73.413
p:3.172157
Iteration 2, p_1 = 5.000, p_2 = 3.172 , $f(p_1)$ = 73.413 , $f(p_2)$ = -6.329
p:3.317226
Iteration 3, p_2 = 3.172, p_3 = 3.317 , $f(p_2)$ = -6.329 , $f(p_3)$ = -5.428
p:4.191600
Iteration 4, p_3 = 3.317, p_4 = 4.192 , $f(p_3)$ = -5.428 , $f(p_4)$ = 13.420
p:3.569044
Iteration 5, p_4 = 4.192, p_5 = 3.569 , $f(p_4)$ = 13.420 , $f(p_5)$ = -2.732
p:3.674331
Iteration 6, p_5 = 3.569, p_6 = 3.674 , $f(p_5)$ = -2.732 , $f(p_6)$ = -1.080
p:3.743165
Iteration 7, p_6 = 3.674, p_7 = 3.743 , $f(p_6)$ = -1.080 , $f(p_7)$ = 0.198
p:3.732518
Iteration 8, p_7 = 3.743, p_8 = 3.733 , $f(p_7)$ = 0.198 , $f(p_8)$ = -0.011
p:3.733074
Iteration 9, p_8 = 3.733, p_9 = 3.733 , $f(p_8)$ = -0.011 , $f(p_9)$ = -0.000
p:3.733079

$$e^{3.733079} - 3(3.733079)^2 \approx -.00000056$$