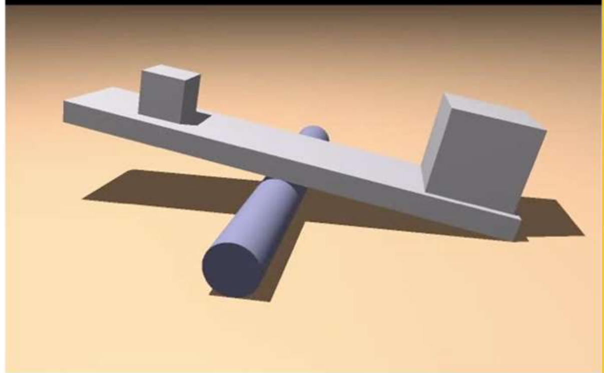
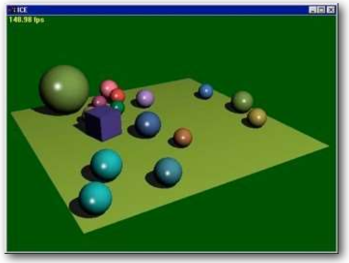
# Rigid body dynamics

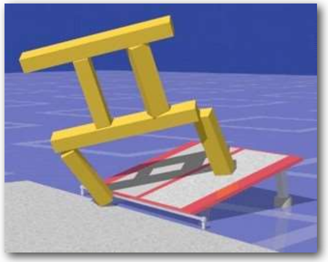


## Rigid body simulation



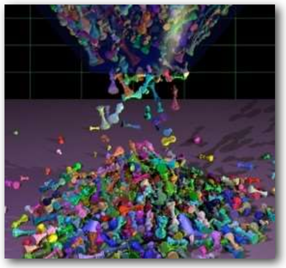
Once we consider an object with spatial extent, particle system simulation is no longer sufficient

## Rigid body simulation



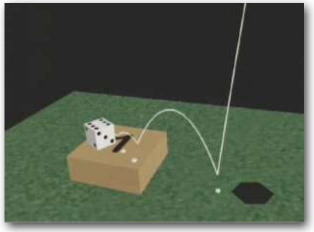
- Unconstrained system
  - no contact
- Constrained system
  - collision and contact

## Problems



Performance is important!

# Problems



Control is difficult!

# Particle simulation

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{v}(t) \end{bmatrix}$$

Position in phase space

$$\dot{\mathbf{Y}}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{f}(t)/m \end{bmatrix}$$

Velocity in phase space

# Rigid body concepts

| Translation | Rotation |
|---|---|
| Position | Orientation |
| Linear velocity | Angular velocity |
| Mass | Inertia tensor |
| Linear momentum | Angular momentum |
| Force | Torque |

- Position and orientation
- Linear and angular velocity
- Mass and Inertia
- Force and torques
- Simulation

# Position and orientation

Translation of the body

$$\mathbf{x}(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

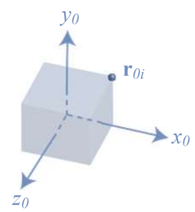
Rotation of the body

$$\mathbf{R}(t) = \begin{bmatrix} r_{xx} & r_{yx} & r_{zx} \\ r_{xy} & r_{yy} & r_{zy} \\ r_{xz} & r_{yz} & r_{zz} \end{bmatrix}$$

***x(t)*** and ***R(t)*** are called *spatial variables* of a rigid body

# Body space

Body space

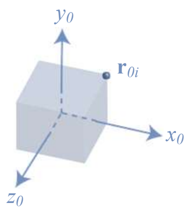



A fixed and unchanged space where the shape of a rigid body is defined

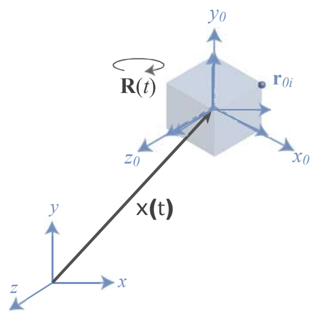
The geometric center of the rigid body lies at the origin of the body space

# Position and orientation

Body space




World space



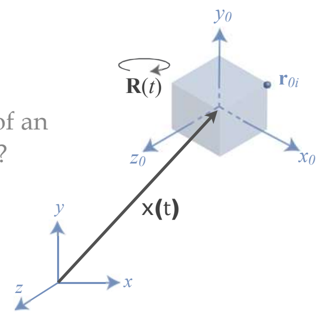

# Position and orientation

Use $\mathbf{x}(t)$ and $\mathbf{R}(t)$ to transform the body space into world space

What's the world coordinate of an arbitrary point $\mathbf{r}_{0i}$ on the body?

$$r_i(t) = x(t) + R(t)r_{0i}$$

World space



# Position and orientation

- Assume the rigid body has uniform density, what is the physical meaning of $\mathbf{x}(t)$?
  - center of mass over time
- What is the physical meaning of $\mathbf{R}(t)$?
  - it's a bit tricky

# Position and orientation

Consider the x-axis in body space, (1, 0, 0), what is the direction of this vector in world space at time $t$?

$$\mathbf{R}(t)\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}r_{xx}\\r_{xy}\\r_{xz}\end{bmatrix}$$

which is the first column of $\mathbf{R}(t)$

$\mathbf{R}(t)$ represents the directions of x, y, and z axes of the body space in world space at time $t$

# Position and orientation

- So $\mathbf{x}(t)$ and $\mathbf{R}(t)$ define the position and the orientation of the body at time $t$
- Next we need to define how the position and orientation change over time

- Position and orientation
- Linear and angular velocity
- Mass and Inertia
- Force and torques
- Simulation

## Linear velocity

Since $x(t)$ is the position of the center of mass in world space, $\dot{x}(t)$ is the velocity of the center of mass in world space

$$v(t) = \dot{x}(t)$$

## Angular velocity

- If we freeze the position of the COM in space
  - then any movement is due to the body spinning about some axis that passes through the COM
  - Otherwise, the COM would itself be moving

## Angular velocity

We describe that spin as a vector $\omega(t)$

Direction of $\omega(t)$?

Magnitude of $|\omega(t)|$?

Linear position and velocity are related by $v(t) = \frac{d}{dt}x(t)$

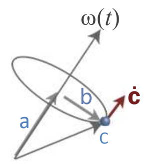
How are angular position (orientation) and velocity related?

## Angular velocity

How are $R(t)$ and $\omega(t)$ related?

Hint:

Consider a vector $c(t)$ at time $t$ specified in world space, how do we represent $\dot{\mathbf{c}}(t)$ in terms of $\omega(t)$




$$|\dot{\mathbf{c}}(t)| = |\mathbf{b}||\omega(t)| = |\omega(t) \times \mathbf{b}|$$

$$\dot{\mathbf{c}}(t) = \omega(t) \times \mathbf{b} = \omega(t) \times \mathbf{b} + \omega(t) \times \mathbf{a}$$

$$\dot{\mathbf{c}}(t) = \omega(t) \times \mathbf{c}(t)$$

# Angular velocity

Given the physical meaning of $\mathbf{R}(t)$, what does each column of $\dot{\mathbf{R}}(t)$ mean?

At time $t$, the direction of x-axis of the rigid body in world space is the first column of $\mathbf{R}(t)$

$$\begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix}$$

At time $t$, what is the derivative of the first column of $\mathbf{R}(t)$?

$$\begin{bmatrix} \dot{r}_{xx} \\ \dot{r}_{xy} \\ \dot{r}_{xz} \end{bmatrix} = \omega(t) \times \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix}$$

# Angular velocity

$$\dot{\mathbf{R}}(t) = \left[ \omega(t) \times \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \quad \omega(t) \times \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \quad \omega(t) \times \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \right]$$

This is the relation between angular velocity and the orientation, but it is too cumbersome

We can use a trick to simplify this expression

# Angular velocity

Consider two 3 by 1 vectors: **a** and **b**, the cross product of them is

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y - b_x a_y \end{bmatrix}$$

Given **a**, let's define **a*** to be a skew symmetric matrix

$$\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$A$ skew-symmetric $\iff A^T = -A$

$A$ skew-symmetric $\iff a_{ji} = -a_{ij}$

then

$$\mathbf{a}^*\mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \mathbf{a} \times \mathbf{b}$$

# Angular velocity

$$\dot{\mathbf{R}}(t) = \left[ \omega(t)^* \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \quad \omega(t)^* \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \quad \omega(t)^* \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \right]$$

$$= \omega(t)^* \left[ \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \right]$$

$$= \omega(t)^* \mathbf{R}(t)$$

Vector relation: $\dot{\mathbf{c}}(t) = \omega(t) \times \mathbf{c}(t)$

Matrix relation: $\dot{\mathbf{R}}(t) = \omega(t)^* \mathbf{R}(t)$

# Perspective of particles

- Imagine a rigid body is composed of a large number of small particles
  - the particles are indexed from 1 to N
  - each particle has a constant location $\mathbf{r}_{0i}$ in body space
  - the location of $i$-th particle in world space at time $t$ is $r_i(t) = x(t) + R(t)r_{0i}$

# Velocity of a particle

$$\dot{\mathbf{r}}(t) = \frac{d}{dt}\mathbf{r}(t) = \omega^*\mathbf{R}(t)\mathbf{r}_{0i} + \mathbf{v}(t)$$

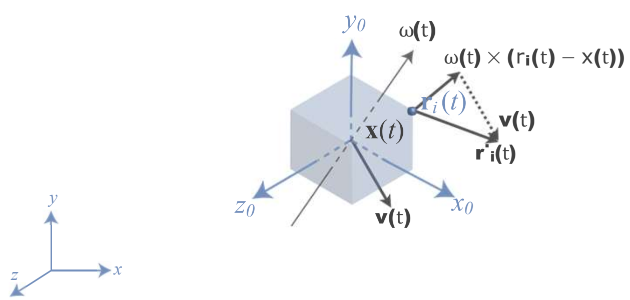
$$= \omega^*(\mathbf{R}(t)\mathbf{r}_{0i} + \mathbf{x}(t) - \mathbf{x}(t)) + \mathbf{v}(t)$$

$$= \omega^*(\mathbf{r}_i(t) - \mathbf{x}(t)) + \mathbf{v}(t)$$

$$\dot{\mathbf{r}}_i(t) = \underbrace{\omega \times (\mathbf{r}_i(t) - \mathbf{x}(t))}_{\text{angular component}} + \underbrace{\mathbf{v}(t)}_{\text{linear component}}$$

# Velocity of a particle

$$\dot{\mathbf{r}}_i(t) = \omega \times (\mathbf{r}_i(t) - \mathbf{x}(t)) + \mathbf{v}(t)$$




- Position and orientation
- Linear and angular velocity
- Mass and Inertia
- Force and torques
- Simulation

# Mass

The mass of the $i$-th particle is $m_i$

Mass $M = \sum_{i=1}^{N} m_i$

Center of mass in world space $\frac{\sum m_i r_i(t)}{M}$

What about center of mass in body space? (0, 0, 0)

# Center of mass

Proof that the center of mass at time $t$ in world space is $\mathbf{x}(t)$

$$\frac{\sum m_i \mathbf{r}_i(t)}{M} =$$

$= x(t)$

Additionally,

$$\sum m_i(r_i(t) - x(t)) =$$

$= \mathbf{0}$

# Inertia tensor

Inertia tensor describes how the mass of a rigid body is distributed relative to the center of mass

$$\mathbf{I} = \sum_i \begin{bmatrix} m_i(r'^2_{iy} + r'^2_{iz}) & -m_i r'_{ix} r'_{iy} & -m_i r'_{ix} r'_{iz} \\ -m_i r'_{iy} r'_{ix} & m_i(r'^2_{ix} + r'^2_{iz}) & -m_i r'_{iy} r'_{iz} \\ -m_i r'_{iz} r'_{ix} & -m_i r'_{iz} r'_{iy} & m_i(r'^2_{ix} + r'^2_{iy}) \end{bmatrix}$$

$$\mathbf{r}'_i = \mathbf{r}_i(t) - \mathbf{x}(t)$$

$\mathbf{I}(t)$ depends on the orientation of a body, but not the translation

For an actual implementation, we replace the finite sum with the integrals over a body's volume in world space

# Inertia tensor

- Inertia tensors vary in world space over time
- But are **constant** in the body space
- Pre-compute the integral part in the body space to save time

# Inertia tensor

Pre-compute $\mathbf{I}_{body}$ that does not vary over time

$$\mathbf{I}(t) = \sum m_i \mathbf{r}_i'^T \mathbf{r}_i' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} m_i \mathbf{r}_{ix}'^2 & m_i \mathbf{r}_{ix}' \mathbf{r}_{iy}' & m_i \mathbf{r}_{ix}' \mathbf{r}_{iz}' \\ m_i \mathbf{r}_{iy}' \mathbf{r}_{ix}' & m_i \mathbf{r}_{iy}'^2 & m_i \mathbf{r}_{iy}' \mathbf{r}_{iz}' \\ m_i \mathbf{r}_{iz}' \mathbf{r}_{ix}' & m_i \mathbf{r}_{iz}' \mathbf{r}_{iy}' & m_i \mathbf{r}_{iz}'^2 \end{bmatrix}$$

$$\mathbf{I} = \sum_i \begin{bmatrix} m_i(r_{iy}'^2 + r_{iz}'^2) & -m_i r_{ix}' r_{iy}' & -m_i r_{ix}' r_{iz}' \\ -m_i r_{iy}' r_{ix}' & m_i(r_{ix}'^2 + r_{iz}'^2) & -m_i r_{iy}' r_{iz}' \\ -m_i r_{iz}' r_{ix}' & -m_i r_{iz}' r_{iy}' & m_i(r_{ix}'^2 + r_{iy}'^2) \end{bmatrix}$$

$$\mathbf{r}_i' = \mathbf{r}_i(t) - \mathbf{x}(t)$$

$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{body}\mathbf{R}(t)^T \qquad \mathbf{I}_{body} = \sum_i m_i((\mathbf{r}_{0i}^T \mathbf{r}_{0i})\mathbf{1} - \mathbf{r}_{0i}\mathbf{r}_{0i}^T)$$

# Inertia tensor

Pre-compute $\mathbf{I}_{body}$ that does not vary over time

$$\mathbf{I}(t) = \sum m_i \mathbf{r}_i'^T \mathbf{r}_i' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} m_i \mathbf{r}_{ix}'^2 & m_i \mathbf{r}_{ix}' \mathbf{r}_{iy}' & m_i \mathbf{r}_{ix}' \mathbf{r}_{iz}' \\ m_i \mathbf{r}_{iy}' \mathbf{r}_{ix}' & m_i \mathbf{r}_{iy}'^2 & m_i \mathbf{r}_{iy}' \mathbf{r}_{iz}' \\ m_i \mathbf{r}_{iz}' \mathbf{r}_{ix}' & m_i \mathbf{r}_{iz}' \mathbf{r}_{iy}' & m_i \mathbf{r}_{iz}'^2 \end{bmatrix}$$

$$\begin{aligned} I(t) &= \sum m_i((r_i'^T r_i')\mathbf{1} - r_i' r_i'^T) \\ &= \sum m_i((R(t)r_{0i})^T (R(t)r_{0i})\mathbf{1} - (R(t)r_{0i})(R(t)r_{0i})^T) \\ &= \sum m_i(r_{0i}^T R(t)^T R(t) r_{0i}\mathbf{1} - R(t) r_{0i} r_{0i}^T R(t)^T) \\ &= \sum m_i((r_{0i}^T r_{0i})\mathbf{1} - R(t) r_{0i} r_{0i}^T R(t)^T). \\ &= \sum m_i(R(t)(r_{0i}^T r_{0i}) R(t)^T \mathbf{1} - R(t) r_{0i} r_{0i}^T R(t)^T) \\ &= R(t)\left(\sum m_i((r_{0i}^T r_{0i})\mathbf{1} - r_{0i} r_{0i}^T)\right) R(t)^T. \end{aligned}$$

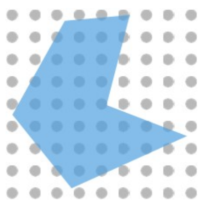
$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{body}\mathbf{R}(t)^T \qquad \mathbf{I}_{body} = \sum m_i((\mathbf{r}_{0i}^T \mathbf{r}_{0i})\mathbf{1} - \mathbf{r}_{0i}\mathbf{r}_{0i}^T)$$

# Approximate inertia tensor

- Bounding boxes
  - Pros: simple
  - Cons: inaccurate

# Approximate inertia tensor

- Point sampling
  - Pros: simple, fairly accurate, no B-rep needed.
  - Cons: expensive, requires volume test

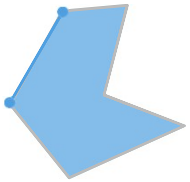


# Approximate inertia tensor

- Green's theorem

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \int\int_D (\nabla \times \mathbf{F}) \cdot d\mathbf{a}$$

  - Pros: simple, exact, no volumes needed
  - Cons: require boundary representation



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- Position and orientation
- Linear and angular velocity
- Mass and Inertia
- Force and torques
- Simulation

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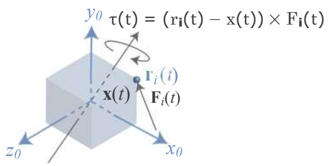
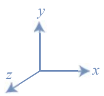
# Force and torque

$\mathbf{F}_i(t)$ denotes the total force from external forces acting on the $i$-th particle at time $t$

$$\mathbf{F}(t) = \sum_i \mathbf{F}_i(t)$$

$$\tau(t) = \sum_i (\mathbf{r}_i(t) - \mathbf{x}(t)) \times \mathbf{F}_i(t)$$

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# Force and torque

- $\mathbf{F}(t)$ conveys no information about where the various forces acted on the body
- $\boldsymbol{\tau}(t)$ contains the information about the distribution of the forces over the body
- Which one depends on the location of the particle relative to the center of mass?

## Linear momentum

$$\begin{aligned}\mathbf{P}(t) &= \sum_i m_i \dot{r}_i(t) \\ &= \sum_i m_i \mathbf{v}(t) + \omega(t) \times \sum_i m_i(\mathbf{r}_i(t) - \mathbf{x}(t)) \\ &= M\mathbf{v}(t)\end{aligned}$$

Total linear moment of the rigid body is the same as if the body was simply a particle with mass $M$ and velocity $\mathbf{v}(t)$

## Angular momentum

Similar to linear momentum, angular momentum is defined as

$$L(t) = I(t)\omega(t)$$

Does $\mathbf{L}(t)$ depend on the translational effect $\mathbf{x}(t)$?

Does $\mathbf{L}(t)$ depend on the rotational effect $\mathbf{R}(t)$?

What about $\mathbf{P}(t)$?

## Derivative of momentum

Change in linear momentum is equivalent to the total forces acting on the rigid body

$$\dot{P}(t) = M\,\dot{v}(t) = F(t)$$

The relation between angular momentum and the total torque is analogous to the linear case

$$\dot{L}(t) = \tau(t)$$

## Derivative of momentum

Proof $\dot{\mathbf{L}}(t) = \tau(t) = \sum \mathbf{r}'_i \times \mathbf{F}_i$

$$m_i \ddot{r}_i - \mathbf{F}_i = m_i(\dot{\mathbf{v}} - \dot{\mathbf{r}}'^*_i \omega - \mathbf{r}'^*_i \dot{\omega}) - \mathbf{F}_i = \mathbf{0}$$

$$\sum \mathbf{r}'^*_i m_i(\dot{\mathbf{v}} - \dot{\mathbf{r}}'^*_i \omega - \mathbf{r}'^*_i \dot{\omega}) - \sum \mathbf{r}'^*_i \mathbf{F}_i = \mathbf{0}$$

$$-\left(\sum m_i \mathbf{r}'^*_i \dot{\mathbf{r}}'^*_i\right)\omega - \left(\sum m_i \mathbf{r}'^*_i \mathbf{r}'^*_i\right)\dot{\omega} = \tau$$

$$\sum -m_i \mathbf{r}'^*_i \mathbf{r}'^*_i = \sum m_i((\mathbf{r}'^T_i \mathbf{r}'_i)\mathbf{1} - \mathbf{r}'_i \mathbf{r}'^T_i) = \mathbf{I}(t)$$

$$-\left(\sum m_i \mathbf{r}'^*_i \dot{\mathbf{r}}'^*_i\right)\omega + \mathbf{I}(t)\dot{\omega} = \tau$$

$$\dot{\mathbf{I}}(t) = \frac{d}{dt}\sum -m_i \mathbf{r}'^*_i \mathbf{r}'^*_i = \sum -m_i \mathbf{r}'^*_i \dot{\mathbf{r}}'^*_i - m_i \dot{\mathbf{r}}'^*_i \mathbf{r}'^*_i$$

$$\dot{\mathbf{I}}(t)\omega + \mathbf{I}(t)\dot{\omega} = \frac{d}{dt}(\mathbf{I}(t)\omega) = \dot{\mathbf{L}}(t) = \tau$$

- Position and orientation
- Linear and angular velocity
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# Equation of motion

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{R}(t) \\ \mathbf{P}(t) \\ \mathbf{L}(t) \end{bmatrix} \begin{matrix} \text{position} \\ \text{orientation} \\ \text{linear momentum} \\ \text{angular momentum} \end{matrix} \qquad \frac{d}{dt}\mathbf{Y}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \omega(t)^{*}\mathbf{R}(t) \\ \mathbf{F}(t) \\ \tau(t) \end{bmatrix}$$

Constants: $M$ and $\mathbf{I}_{body}$

$$\mathbf{v}(t) = \frac{\mathbf{P}(t)}{M}$$

$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{body}\mathbf{R}(t)^T$$

$$\omega(t) = \mathbf{I}(t)^{-1}\mathbf{L}(t)$$

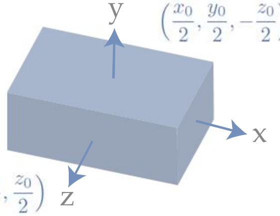
# Momentum vs. velocity

- Why do we use momentum in the phase space instead of velocity?
  - Because the relation of angular momentum and torque is simple
  - Because the angular momentum is constant when there is no torques acting on the object
  - Use linear momentum $\mathbf{P}(t)$ to be consistent with angular velocity and acceleration

# Example:

1. compute the $\mathbf{I}_{body}$ in body space

$$\left(\frac{x_0}{2}, \frac{y_0}{2}, -\frac{z_0}{2}\right)$$

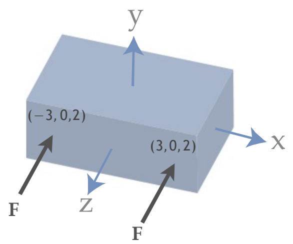



$$\left(-\frac{x_0}{2}, -\frac{y_0}{2}, \frac{z_0}{2}\right)$$

$$\mathbf{I}_{body} = \frac{M}{12}\begin{pmatrix} y_0^2 + z_0^2 & 0 & 0 \\ 0 & x_0^2 + z_0^2 & 0 \\ 0 & 0 & x_0^2 + y_0^2 \end{pmatrix}$$

# Example:

1. compute the $\mathbf{I}_{body}$ in body space
2. rotation free movement




### 5.3 Rotation Free Movement of a Body

Now, let us consider some forces acting on the block of figure 8. Suppose that an external force $F = (0, 0, f)$ acts on the body at points $x(t) + (-3, 0, -2)$ and $x(t) + (3, 0, -2)$. We would expect that this would cause the body to accelerate linearly, without accelerating angularly. The net force acting on the body is $(0, 0, 2f)$, so the acceleration of the center of mass is

$$\frac{2f}{M}$$

along the $z$ axis. The torque due to the force acting at $x(t) + (-3, 0, -2)$ is

$$((x(t) + \begin{pmatrix} -3 \\ 0 \\ -2 \end{pmatrix}) - x(t)) \times F = \begin{pmatrix} -3 \\ 0 \\ -2 \end{pmatrix} \times F$$

while the torque due to the force acting at $x(t) + (3, 0, -2)$ is

$$((x(t) + \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}) - x(t)) \times F = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \times F.$$

The total torque $\tau$ is therefore

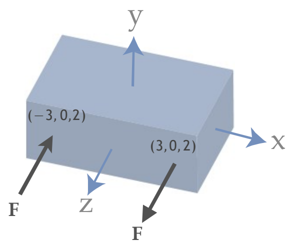
$$\tau = \begin{pmatrix} -3 \\ 0 \\ -2 \end{pmatrix} \times F + \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \times F = (\begin{pmatrix} -3 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}) \times F = \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix} \times F.$$

But this gives

$$\tau = \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} = \mathbf{0}.$$

# Example:

1. compute the $\mathbf{I}_{body}$ in body space
2. rotation free movement
3. translation free movement



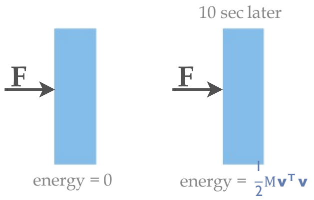

### 5.4 Translation Free Movement of a Body

Suppose now that an external force $F_1 = (0, 0, f)$ acts on the body at point $x(t) + (-3, 0, -2)$ and an external force $F_2 = (0, 0, -f)$ acts on the body at point $x(t) + (3, 0, 2)$ (figure 9). Since $F_1 = -F_2$, the net force acting on the block is $F_1 + F_2 = \mathbf{0}$, so there is no acceleration of the center of mass. On the other hand, the net torque is

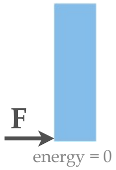
$$\begin{aligned} &((x(t) + \begin{pmatrix} -3 \\ 0 \\ -2 \end{pmatrix}) - x(t)) \times F_1 + \\ &((x(t) + \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}) - x(t)) \times F_2 = \begin{pmatrix} -3 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 3f \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3f \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6f \\ 0 \end{pmatrix}. \end{aligned} \quad (5\text{–}6)$$

Thus, the net torque is $(0, 6f, 0)$, which is parallel to the $y$ axis. The final result is that the forces acting on the block cause it to angularly accelerate about the $y$ axis.

# Force vs. torque puzzle

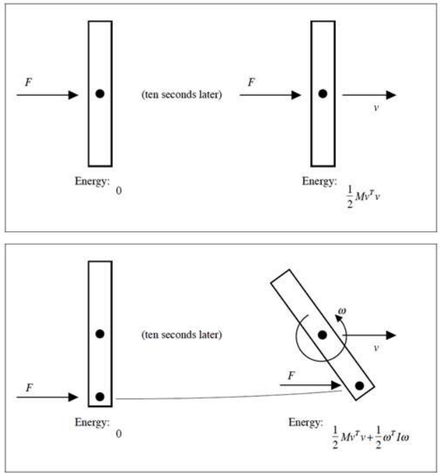



Suppose a force F acts on the block at the center of mass for 10 seconds. Since there is no torque acting on the block, the body will only acquire linear velocity **v** after 10 seconds. The kinetic energy will be $\frac{1}{2}M\mathbf{v}^T\mathbf{v}$




Now, consider the same force acting off-center to the body for 10 seconds. Since it is the same force, the velocity of the center of mass after 10 seconds is the same **v**. However, the block will also pick up some angular velocity ω. The kinetic energy will be $\frac{1}{2}M\,\mathbf{v}^T\mathbf{v} + \frac{1}{2}\omega^T I\omega$

If identical forces push the block in both cases, how can the energy of the block be different?




# Notes on implementation

- Using quaternion instead of transformation matrix
  - more compact representation
  - less numerical drift

# Quaternion

$$\mathbf{q}(t) = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \qquad \dot{\mathbf{q}}(t) = \frac{1}{2}\begin{bmatrix} 0 \\ \omega(t) \end{bmatrix}\mathbf{q}(t)$$

quaternion multiplication

# Equation of motion

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{q}(t) \\ \mathbf{P}(t) \\ \mathbf{L}(t) \end{bmatrix} \begin{matrix} \text{position} \\ \text{orientation} \\ \text{linear momentum} \\ \text{angular momentum} \end{matrix} \qquad \frac{d}{dt}\mathbf{Y}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \frac{1}{2}\begin{bmatrix} 0 \\ \omega(t) \end{bmatrix}\mathbf{q}(t) \\ \mathbf{F}(t) \\ \tau(t) \end{bmatrix}$$

Constants: $M$ and $\mathbf{I}_{body}$

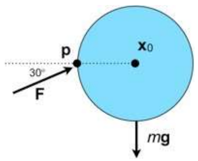
$$\mathbf{v}(t) = \frac{\mathbf{P}(t)}{M}$$

$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{body}\mathbf{R}(t)^T$$

$$\omega(t) = \mathbf{I}(t)^{-1}\mathbf{L}(t)$$

# Exercise

Consider a 3D sphere with radius 1m, mass 1kg, and inertia $\mathbf{I}_{body}$. The initial linear and angular velocity are both zero. The initial position and the initial orientation are $\mathbf{x}_0$ and $\mathbf{R}_0$. The forces applied on the sphere include gravity ($g$) and an initial push $\mathbf{F}$ applied at point $\mathbf{p}$. Note that $\mathbf{F}$ is only applied for one time step at $t_0$. If we use Explicit Euler method with time step $h$ to integrate , what are the position and the orientation of the sphere at $t_2$? Use the actual numbers defined as below to compute your solution (except for $g$ and $h$).




$\mathbf{x}_0 = (0, 0, 0)$

$\mathbf{R}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\mathbf{p} = (-1, 0, 0)$

$\mathbf{F} = (4\cos(30^\circ), 4\sin(30^\circ), 0)$

$m = 1$

$\mathbf{I}_{body} = \begin{pmatrix} 2/5 & 0 & 0 \\ 0 & 2/5 & 0 \\ 0 & 0 & 2/5 \end{pmatrix}$