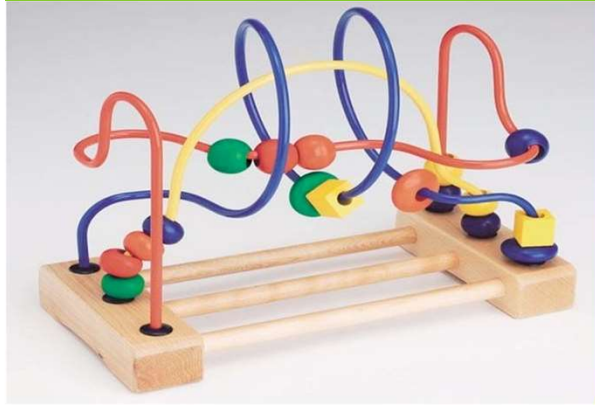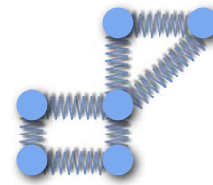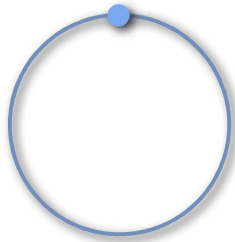# Constrained dynamics

Simple particle system

- In principle, you can make just about anything out of spring systems

- In practice, you can make just about anything as long as it's jello

Hard constraints

- Constraint force

- Single implicit constraint

- Multiple implicit constraint

- Parametric constraint

- Implementation
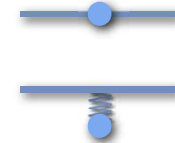
A simple example

A bead on a wire

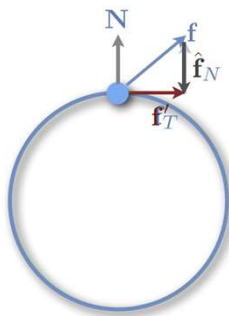The bead can slide freely along the wire, but cannot come off it no matter how hard you pull it.

How do we simulate the motion of the bead when arbitrary forces applied to it?

Penalty constraints

- Why not use a spring to hold the bead on the wire?

- Problems:
  - weak springs won't do the job

  - strong springs give you stiff systems

First order world

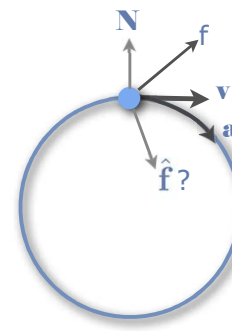In this world, $f = m\dot{v}$

What is the legal velocity?

What is the legal force?

Add constraint force $\hat{\mathbf{f}}$ to cancel the illegal part of $\mathbf{f}$

$$\mathbf{f}' = \mathbf{f} + \hat{\mathbf{f}}$$

$$\hat{\mathbf{f}} = -\frac{\mathbf{N} \cdot \mathbf{f}}{\mathbf{N} \cdot \mathbf{N}} \mathbf{N}$$

The real world

In the real world, $\mathbf{f} = m\mathbf{a}$

What is the legal acceleration?

depends on both $\mathbf{N}$ and $\mathbf{v}$

the faster you're going, the faster you have to turn

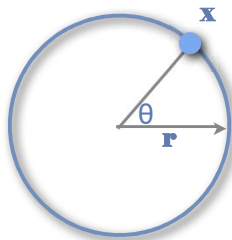Compute $\hat{\mathbf{f}}$ such that $\mathbf{f}'$ only generates legal acceleration

$$\mathbf{f}' = \mathbf{f} + \hat{\mathbf{f}}$$

Constraint force

- Need to compute constraint forces that cancel the illegal applied forces
- Which means we need to know what legal acceleration is

- Constraint force
- Single implicit constraint
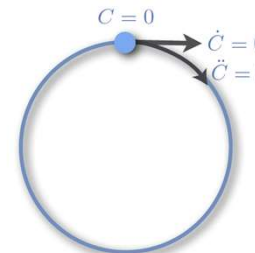- Multiple implicit constraint
- Parametric constraint
- Implementation

Constraints

Implicit:
$$C(\mathbf{x}) = |\mathbf{x}| - \mathbf{r} = 0$$

Parametric:
$$\mathbf{x} = r \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Legal acceleration

$C = 0$
$\dot{C} = 0$
$\ddot{C} = 0$

What is the legal position?
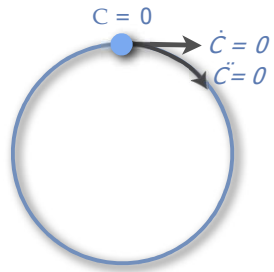$$C(\mathbf{x}) = \frac{1}{2}\mathbf{x} \cdot \mathbf{x} - \frac{1}{2} = 0$$

What is the legal velocity?
$$\dot{C}(\mathbf{x}) = \mathbf{x} \cdot \dot{\mathbf{x}} = 0$$

What is the legal acceleration?
$$\ddot{C}(\mathbf{x}) = \ddot{\mathbf{x}} \cdot \mathbf{x} + \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} = 0$$

Legal conditions

C = 0

$\dot{C} = 0$
$\ddot{C} = 0$

If we start with legal position and velocity

$$C(x) = 0$$
$$\dot{C}(x) = 0$$

We need only ensure the legal acceleration

$$\ddot{C}(\mathbf{x}) = 0$$

Constraint force

Use the legal condition to compute the constraint force

Rewrite the legal condition in a general form

$$\ddot{C}(\mathbf{x}) = \frac{\partial \dot{C}}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial C}{\partial \mathbf{x}} \cdot \ddot{\mathbf{x}} = 0 \qquad \frac{\partial C}{\partial \mathbf{x}} : \text{constraint gradient}$$

Substitute $\ddot{\mathbf{x}}$ with $\ddot{\mathbf{x}} = \frac{\mathbf{f} + \hat{\mathbf{f}}}{m}$

$$\ddot{C} = \frac{\partial \dot{C}}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial C}{\partial \mathbf{x}} \cdot \frac{\mathbf{f} + \hat{\mathbf{f}}}{m} = 0$$

$$\frac{\partial C}{\partial x} \cdot \hat{f} = -\frac{\partial C}{\partial x} \cdot f - m \frac{\partial \dot{C}}{\partial x} \cdot \dot{x}$$

Constraint force

$$\frac{\partial C}{\partial x} \cdot \hat{f} = -\frac{\partial C}{\partial x} \cdot f - m \frac{\partial \dot{C}}{\partial x} \cdot \dot{x}$$

How many variables do we have?

Need one more condition to solve the constraint force

Virtual work

Constraint force is passive - no energy gain or loss

Kinetic energy of the system: $T = \frac{1}{2} m \dot{\mathbf{x}} \cdot \dot{\mathbf{x}}$

Virtual work done by $\mathbf{f}$ and $\hat{\mathbf{f}}$: $\dot{T} = \dot{\mathbf{x}} \cdot m\ddot{\mathbf{x}} = \dot{\mathbf{x}} \cdot \mathbf{f} + \dot{\mathbf{x}} \cdot \hat{\mathbf{f}}$

Make sure $\hat{\mathbf{f}}$ does no work for every legal velocity:

$$\dot{\mathbf{x}} \cdot \hat{\mathbf{f}} = 0, \forall \dot{\mathbf{x}} \mid \frac{\partial C}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} = 0 \qquad \dot{C}(\mathbf{x}) = \frac{\partial C}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} = 0$$

Constraint force

$$\dot{\mathbf{x}} \cdot \hat{\mathbf{f}} = 0, \forall \dot{\mathbf{x}} \mid \frac{\partial C}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} = 0$$

$\hat{\mathbf{f}}$ must point in the direction of $\dfrac{\partial C}{\partial \mathbf{x}}$

$$\hat{\mathbf{f}} = \lambda \frac{\partial C}{\partial \mathbf{x}}$$

Substituting for $\hat{\mathbf{f}}$ in $\quad \dfrac{\partial C}{\partial x} \cdot \hat{f} = -\dfrac{\partial C}{\partial x} \cdot f - m\dfrac{\partial \dot{C}}{\partial x} \cdot \dot{x}$

$$\lambda = \frac{-\frac{\partial C}{\partial \mathbf{x}} \cdot \mathbf{f} - m\frac{\partial \dot{C}}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}}}{\frac{\partial C}{\partial \mathbf{x}} \cdot \frac{\partial C}{\partial \mathbf{x}}}$$
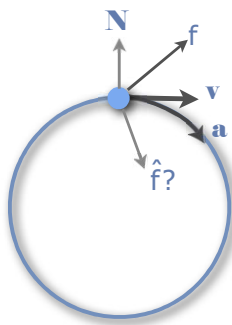
Two conditions

- Legal acceleration

$$\frac{\partial C}{\partial x} \cdot \hat{f} = -\frac{\partial C}{\partial x} \cdot f - m\frac{\partial \dot{C}}{\partial x} \cdot \dot{x}$$

- Principle of virtual work

$$\hat{\mathbf{f}} = \lambda \frac{\partial C}{\partial \mathbf{x}}$$

The Bead Example

$$C(\mathbf{x}) = \frac{1}{2}\mathbf{x} \cdot \mathbf{x} - \frac{1}{2} = 0$$

$$\hat{\mathbf{f}} = \lambda \frac{\partial C}{\partial \mathbf{x}}$$

$$\lambda = \frac{-\frac{\partial C}{\partial \mathbf{x}} \cdot \mathbf{f} - m\frac{\partial \dot{C}}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}}}{\frac{\partial C}{\partial \mathbf{x}} \cdot \frac{\partial C}{\partial \mathbf{x}}}$$

Feedback

- In principle, ensuring legal acceleration can keep the particle exactly on the circle
- In practice, two problems cause the particle to drift
  - numerical errors can accumulate when the ODE is not solved exactly
  - constraints might not be met initially

Feedback

- A feedback term handles both problems:

$$\ddot{C} = -k_s C - k_d \dot{C} \text{ instead of } \ddot{C} = 0$$

- Constraint force

- Single implicit constraint

- Multiple implicit constraint

- Parametric constraint

- Implementation

Tinkertoys

- Now we know how to simulate a bead on a wire
- Apply the simple idea, we can create a constrained particle system

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Constrained particles

- Particles: each particle represents a point in the phase space

- Forces: each force affects the acceleration of certain particles
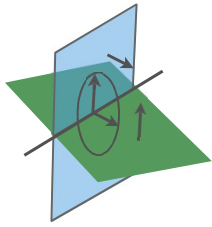
- Constraints:

  Each is a function $C_i(x_1, x_2, \ldots)$

  Legal state: $C_i(x_1, x_2, \ldots) = 0, \forall i$

  Constraint force: linear combination of **constraint gradients** $\frac{\partial C_i}{\partial x}, \forall i$

Constraint gradients

Normal of C_1: $\dfrac{\partial \mathbf{C}_1}{\partial \mathbf{x}}$

Normal of C_2: $\dfrac{\partial \mathbf{C}_2}{\partial \mathbf{x}}$

Legal states: the intersection of two planes

Normal of the legal states:

$$\lambda_1 \frac{\partial C_1}{\partial \mathbf{x}} + \lambda_2 \frac{\partial C_2}{\partial \mathbf{x}}$$

Implicit constraint

- Each constraint is represented by an implicit function
- What does the normal of a hypersurface mean? The direction where the particle is not allowed to move
- What does the intersection of the hypersurface represent? **Legal state**
- The constraint force lies in the space spanned by the constraint normals

Particle system notations

General 3D case

$\mathbf{q}$: $3n$ long position vector

$\mathbf{Q}$: $3n$ long force vector

$\mathbf{W}$: $3n \times 3n$ inverse mass matrix

Constraint notations

Additional notations

$\mathbf{C}$: m long constraint vector
λ: m Lagrangian multipliers
$\mathbf{J}$: $m \times 3n$ Jacobian matrix
$\dot{\mathbf{J}}$: time derivative of Jacobian matrix

Constraint equations

$$\ddot{C} = \frac{\partial \dot{C}}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial C}{\partial \mathbf{x}} \cdot \ddot{\mathbf{x}}$$

$$\ddot{C} = \frac{\partial \dot{C}}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial C}{\partial \mathbf{x}} \cdot \frac{\mathbf{f} + \hat{\mathbf{f}}}{m} = 0$$

$$\frac{\partial C}{\partial \mathbf{x}} \cdot \hat{\mathbf{f}} = -\frac{\partial C}{\partial \mathbf{x}} \cdot \mathbf{f} - m\frac{\dot{C}}{\partial \mathbf{x}} \dot{\mathbf{x}}$$

$$\frac{\partial C}{\partial \mathbf{x}} \cdot \lambda \frac{\partial C}{\partial \mathbf{x}} = -\frac{\partial C}{\partial \mathbf{x}} \cdot \mathbf{f} - m\frac{\dot{C}}{\partial \mathbf{x}} \dot{\mathbf{x}}$$

$$\ddot{\mathbf{C}} = \dot{\mathbf{J}}\dot{\mathbf{q}} + \mathbf{J}\ddot{\mathbf{q}}$$

$$\ddot{\mathbf{C}} = \dot{\mathbf{J}}\dot{\mathbf{q}} + \mathbf{J}\mathbf{W}(\mathbf{Q} + \hat{\mathbf{Q}}) = 0$$

$$\mathbf{J}\mathbf{W}\hat{\mathbf{Q}} = -\dot{\mathbf{J}}\dot{\mathbf{q}} - \mathbf{J}\mathbf{W}\mathbf{Q}$$

$$\mathbf{J}\mathbf{W}\mathbf{J}^T \lambda = -\dot{\mathbf{J}}\dot{\mathbf{q}} - \mathbf{J}\mathbf{W}\mathbf{Q}$$

Multiple force constraints

To solve for force constraints, we need to solve for this linear system

$$\mathbf{J}\mathbf{W}\mathbf{J}^T \lambda = -\dot{\mathbf{J}}\dot{\mathbf{q}} - \mathbf{J}\mathbf{W}\mathbf{Q}$$
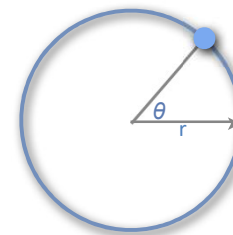
One nice property of $\mathbf{J}\mathbf{W}\mathbf{J}^T$ is that it is **symmetric positive definite matrix**

Can you prove that?

Once the linear system has been solved, the vector λ is multiplied by $\mathbf{J}^T$ to produce the **global constraint force vector** $\hat{\mathbf{Q}}$

- Constraint force
- Single implicit constraint
- Multiple implicit constraint
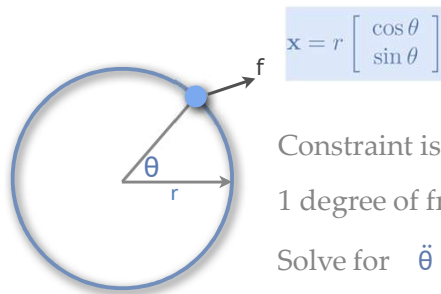- Parametric constraint
- Implementation

Constraints

Implicit:
$$C(x) = |x| - r = 0$$

Parametric:
$$\mathbf{x} = r \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Parametric constraints

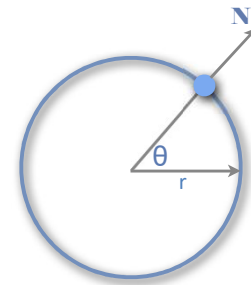$$\mathbf{x} = r \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Constraint is always met exactly.

1 degree of freedom:

Solve for $\ddot{\theta}$

The real world

In the real world, $\mathbf{f} = m\mathbf{a}$

$$\dot{\mathbf{x}} = \mathbf{T}\dot{\theta} \qquad \mathbf{T} = \frac{\partial \mathbf{x}}{\partial \theta}$$

$$\ddot{\mathbf{x}} = \dot{\mathbf{T}}\dot{\theta} + \mathbf{T}\ddot{\theta} = \frac{\mathbf{f} + \hat{\mathbf{f}}}{m}$$

$$\mathbf{T} \cdot \dot{\mathbf{T}}\dot{\theta} + \mathbf{T} \cdot \mathbf{T}\ddot{\theta} = \mathbf{T} \cdot \frac{\mathbf{f}}{m} + \cancel{\mathbf{T} \cdot \frac{\hat{\mathbf{f}}}{m}}$$

$$\ddot{\theta} = \frac{\mathbf{T} \cdot \frac{\mathbf{f}}{m} - \mathbf{T} \cdot \dot{\mathbf{T}}\dot{\theta}}{\mathbf{T} \cdot \mathbf{T}}$$

Particle system notations

General 2D case

$$\mathbf{q} \quad \mathbf{u} \quad \mathbf{Q} \quad \mathbf{M}$$

$\mathbf{q}$: $2n$ long position vector

$\mathbf{u}$: n long parameter vector

$\mathbf{Q}$: $2n$ long force vector

$\mathbf{M}$: $2n$ by $2n$ mass matrix

Constraint notations

Additional notations

$$\mathbf{J} \qquad \dot{\mathbf{J}}$$

$\mathbf{J}$: $2n$ by n Jacobian matrix

$\dot{\mathbf{J}}$ time derivative of $\mathbf{J}$

Constraint equations

$$\ddot{x} = \dot{T}\dot{\theta} + T\ddot{\theta} = \frac{f + \hat{f}}{m}$$

$$T \cdot \dot{T}\dot{\theta} + T \cdot T\ddot{\theta} = T \cdot \frac{f}{m}$$

$$\ddot{\theta} = \frac{T \cdot \frac{f}{m} - T \cdot \dot{T}\dot{\theta}}{T \cdot T}$$

where $\quad T = \frac{\partial x}{\partial \theta}$

$$M\ddot{q} = M(\dot{J}\dot{u} + J\ddot{u}) = Q + \hat{Q}$$

$$J^T M(\dot{J}\dot{u} + J\ddot{u}) = J^T Q$$

$$J^T M J\ddot{u} = -J^T M\dot{J}\dot{u} + J^T Q$$

where $\quad J = \frac{\partial q}{\partial u}$

Parametric vs. implicit

Parametric	Implicit
$J^T M J\ddot{u} = -J^T M\dot{J}\dot{u} + J^T Q$	$J W J^T \lambda = -\dot{J}\dot{q} - J W Q$
where $\quad J = \frac{\partial q}{\partial u}$	where $\quad J = \frac{\partial C}{\partial q}$

Lagrangian dynamics

Parametric constraints

- Advantages:
  - Fewer degrees of freedom
  - Constraints are always met
- Disadvantages:
  - Hard to formulate constraints
  - Hard to combine constraints

Impress your friends

- The requirement that constraints not add or remove energy is called the *Principle of Virtual Work*
- The λ's are called *Lagrangain Multipliers*
- The derivative matrix **J** is called the *Jacobian Matrix*

- Constraint force
- Single implicit constraint
- Multiple implicit constraint
- Parametric constraint
- Implementation

How do we implement all this?

- We have a global matrix equation
- We want to build a model on the fly
- Each constraint function knows how to evaluate the function itself and its various derivatives
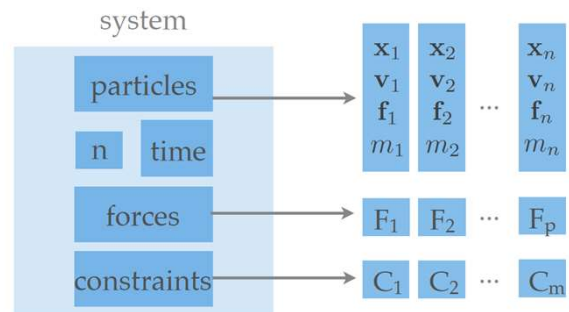
How do we hook this up?

- We have a basic particle system which main job is to perform derivative evaluations
- To add constraints to the basic system, we need to modify
  - the data structure of the basic system
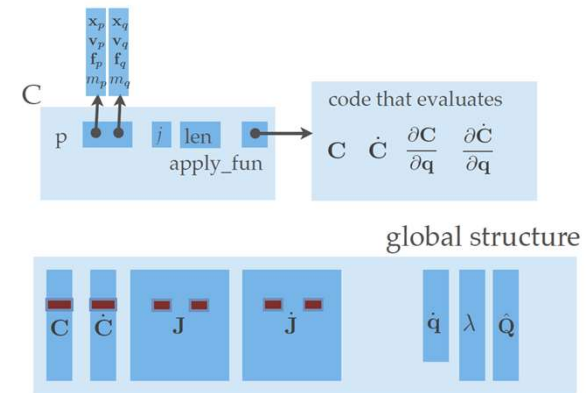 - the "deriv eval" loop
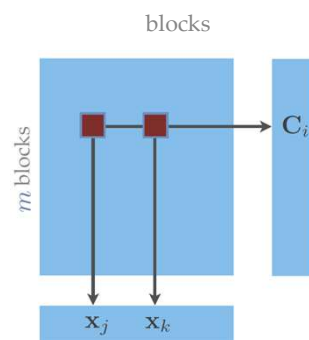
Basic particle system
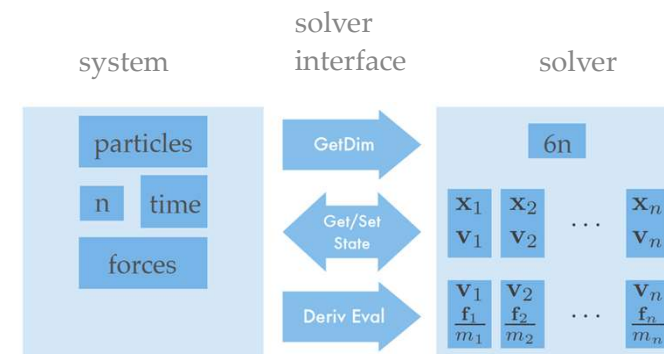
Constrained system

A Constraint

Jacobian matrix

Each constraint contributes one or more blocks to the Jacobian matrix
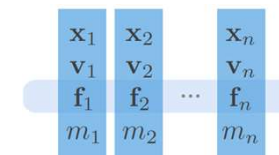
Sparsity: many empty blocks

Modularity: let each constraint compute its own blocks

Constraint and particle indices determine block location

Deriv Eval

1. Clear force accumulators
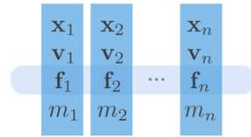

2. Invoke `apply_force` functions

3. Return derivatives to solver

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \frac{\mathbf{f}}{m} \end{bmatrix}$$

Modified "Deriv Eval" loop

1. Clear force accumulators

$$\begin{array}{c} \mathbf{x}_1 \\ \mathbf{v}_1 \\ \mathbf{f}_1 \\ m_1 \end{array} \quad \begin{array}{c} \mathbf{x}_2 \\ \mathbf{v}_2 \\ \mathbf{f}_2 \\ m_2 \end{array} \quad \cdots \quad \begin{array}{c} \mathbf{x}_n \\ \mathbf{v}_n \\ \mathbf{f}_n \\ m_n \end{array}$$

2. Invoke `apply_force` functions

$$\mathbf{F}_1 \quad \mathbf{F}_2 \quad \cdots \quad \mathbf{F}_p$$

4. Return derivatives to solver

$$\left[\begin{array}{c} \dot{\mathbf{x}} \\ \dot{\mathbf{v}} \end{array} \right] = \left[\begin{array}{c} \mathbf{v} \\ \frac{\mathbf{f}}{m} \end{array} \right]$$

3. Compute and apply constraint forces

$$\mathbf{C}_1 \quad \mathbf{C}_2 \quad \cdots \quad \mathbf{C}_m$$

Constraint force eval

- After computing ordinary forces:
  - loop over constraints, assemble global structure
  - call matrix solver to solve for $\boldsymbol{\lambda}$, multiply by $\mathbf{J}^T$ to get constraint force
  - add constraint force to the force accumulator in the corresponding particle

- Beyond mass points: equations of motion for real objects

- Read: Constrained dynamics by Witkin and Baraff