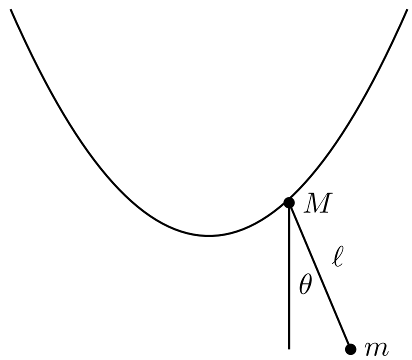
Tutorial Exercises: Incorporating constraints

1. A simple pendulum of length ℓ and mass m is suspended from a pivot of mass M that is free to slide on a frictionless wire frame in the shape of a parabola $y = ax^2$. The pendulum moves in the plane of the frame.



   (a) Write down the cartesian coordinates of both masses in terms of x and θ.

   (b) Calculate the time derivatives of the cartesian coordinates.

   (c) Write down the kinetic and potential energies using x and θ as generalised coordinates.

   (d) Write down the Lagrangian using the approximation that x, θ and their derivatives are small, and solve the corresponding linear Lagrange equations.

1. **Solution:**

   (a) The cartesian coordinates of mass M are

$$\begin{aligned} x_1 &= x \\ y_1 &= ax^2 \end{aligned}$$

   The cartesian coordinates of mass m are

$$\begin{aligned} x_2 &= x + \ell \sin\theta \\ y_2 &= ax^2 - \ell\cos\theta \end{aligned}$$

   (b) The time derivatives of these coordinates are

$$\begin{aligned} \dot{x}_1 &= \dot{x} \\ \dot{y}_1 &= 2ax\dot{x} \end{aligned}$$

   and

$$\begin{aligned} \dot{x}_2 &= \dot{x} + \ell\cos\theta\dot{\theta} \\ \dot{y}_2 &= 2ax\dot{x} + \ell\sin\theta\dot{\theta} \end{aligned}$$

   (c) The kinetic energy is

$$T = \frac{1}{2}M(\dot{x}_1^2+\dot{y}_1^2)+\frac{1}{2}m(\dot{x}_2^2+\dot{y}_2^2) = \frac{1}{2}(M+m)(1+4a^2x^2)\dot{x}^2+\frac{1}{2}m\ell^2\dot{\theta}^2+m\ell\dot{x}\dot{\theta}(\cos\theta+2ax\sin\theta)$$

   the potential energy is

$$V = Mgy_1 + mgy_2 = (M+m)gax^2 - mg\ell\cos\theta$$

   and the Lagrangian is then $L = T - V$.

(d) The Lagrangian including only the quadratic terms is

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + m\ell\dot{x}\dot{\theta} - (M+m)gax^2 - \frac{1}{2}mg\ell\theta^2$$

The Euler-Lagrange equations are

$$(M+m)\ddot{x} + m\ell\ddot{\theta} + 2(M+m)gax = 0$$

and

$$m\ell^2\ddot{\theta} + m\ell\ddot{x} + mg\ell\theta = 0$$

which can be simplified to

$$\ddot{x} + \frac{m}{M+m}\ell\ddot{\theta} + 2gax = 0$$

and

$$\ddot{x} + \ell\ddot{\theta} + g\theta = 0$$

These are linear coupled ODEs, the solutions will be trigonometric so we try solutions of the form

$$x = A\exp(i\omega t)$$

and

$$y = B\exp(i\omega t)$$

Substituting gives

$$(-\omega^2 + 2ag)A - \frac{m}{M+m}\ell\omega^2 B = 0$$

and

$$-\omega^2 A + (g - \ell\omega^2)B = 0$$

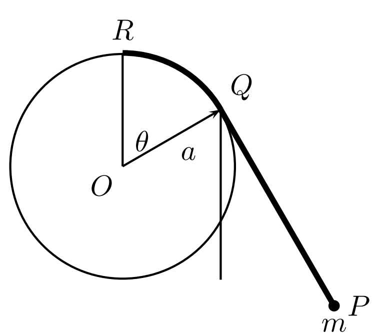
The determinant of this system gives

$$(2ag - \omega^2)(g - \ell\omega^2) - \frac{m}{M+m}\ell\omega^4 = \frac{M}{M+m}\ell\omega^4 - g(1+2a\ell)\omega^2 + 2ag^2 = 0$$

Thus,

$$\omega^2 = \frac{g(1+2a\ell) \pm \sqrt{g^2(1+2a\ell)^2 - 8ag^2\ell\frac{M}{M+m}}}{2\frac{M}{M+m}\ell}$$

2. A pendulum of length ℓ with mass m at the end wraps around a circular obstacle of radius a as it swings (see diagram). The point R is fixed, the point Q is where the string first loses contact with the circle, and the point P is location of the mass. Both Q and P change with time.



You can assume that the string remains taut so that the line PQ is always tangent to the circle. In case you have forgotten your basic geometry, the angle between the tangent PQ and the radius OQ is 90 degrees.

(a) Start by using cartesian coordinates for both Q and P. What is the length of the part of the string between R and Q? Thus, what are the coordinates of Q?

(b) Then determine the length of the part of the string between Q and P, and thus, the coordinates of P.

(c) Write down the kinetic and potential energies in terms of the cartesian coordinates of P.

(d) Rewrite everything in terms of θ and $\dot{\theta}$ and thus write down the Lagrangian.

(e) Find approximate solutions of the dynamics of the pendulum by expanding the Lagrangian up to quadratic order near the equilibrium point.

(f) If the string starts out horizontally, how far round the circle does the point Q move? Express your answer in terms of ℓ and a and assume that ℓ is sufficiently large.

2. **Solution:**

(a) The length of the part of the string between R and Q is $a\theta$. The coordinates of Q are $(a\sin\theta, a\cos\theta)$.

(b) The length of the part of the string between Q and P is $\ell - a\theta$. The coordinates of P are

$$x = a\sin\theta + (\ell - a\theta)\cos\theta$$

and

$$y = a\cos\theta - (\ell - a\theta)\sin\theta$$

The time derivatives are

$$\dot{x} = a\cos\theta\dot{\theta} - a\cos\theta\dot{\theta} - (\ell - a\theta)\sin\theta\dot{\theta} = -(\ell - a\theta)\sin\theta\dot{\theta}$$

and

$$\dot{y} = -a\sin\theta\dot{\theta} + a\sin\theta\dot{\theta} - (\ell - a\theta)\cos\theta\dot{\theta} = -(\ell - a\theta)\cos\theta\dot{\theta}$$

(c) Thus

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

and

$$V = mgy$$

(d) Thus

$$T = \frac{1}{2}m(\ell - a\theta)^2\dot{\theta}^2$$

and

$$V = mga\cos\theta - mg(\ell - a\theta)\sin\theta$$

and

$$L = \frac{1}{2}m(\ell - a\theta)^2\dot{\theta}^2 - mga\cos\theta + mg(\ell - a\theta)\sin\theta$$

(e) The general solution for this mechanical system is rather difficult to find. Particular solutions like equilibria are always easier to find. There is an equilibrium with the string hanging vertically down when $\theta = \pi/2$. Linearising near an equilibrium give approximate solutions. Approximating near the equilibrium point just found gives

$$L = \frac{1}{2}m(\ell - a\pi/2)^2\dot{\theta}^2 + \frac{1}{2}mg(\ell - a\pi/2)(\theta - \pi/2)^2$$

Which is obviously just the equation for a linearised pendulum of length $\ell - a\pi/2$.
In general this system is hard to solve. Writing down the energy which is a first integral we can separate the variables and are led to a complicated integral.

(f) If the system starts from rest with $\theta = 0$ then the initial $V = mga$. Thus the mass will swing around until $V = mga$ again. This occurs at

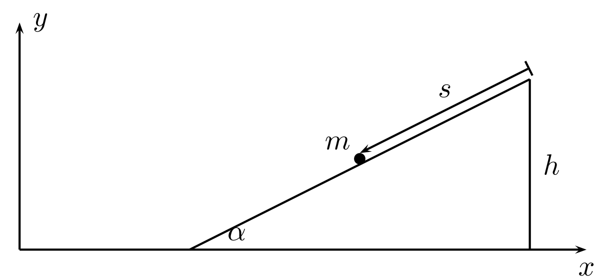
$$mga = mga\cos\theta - mg(\ell - a\theta)\sin\theta$$

thus cannot be solved in closed form but it simplifies to

$$\frac{\ell}{a} = \frac{\cos\theta + \theta\sin\theta - 1}{\sin\theta}$$

showing that the answer only depends on the ratio ℓ/a.
The shortest length for which this is possible is $\ell/a = 1 + 3\pi/2 \approx 5.7$, and then $\theta = 3\pi/2$. For larger $\ell/a > 1 + 3\pi/2$ one has to find θ as the root of the equation in the interval $[\pi, 3\pi/2]$ and for $\ell \to \infty$ the corresponding $\theta \to \pi$.

3. A particle of mass m slides freely from rest down a smooth plane inclined at an angle α to the horizontal. The plane is a wedge of mass M which can slide freely in the horizontal direction.



(a) Construct the Lagrangian in terms of the distance s moved down the slope and the distance x moved by the wedge. [Treat the wedge as a single particle of mass M.]

(b) Derive and solve the corresponding Lagrange equations. Hint: Rearrange your final set of DE's to give expressions for $\ddot{s}$ and $\ddot{x}$ first.

3. **Solution:**

(a) Using the coordinate system shown in the figure the coordinates of the particle are

$$x_1 = x - s\cos\alpha, \qquad y_1 = h - s\sin\alpha.$$

We have had to introduce another quantity h, the height of the initial position of the particle above the horizontal plane. Since it is not specified in the question, you may guess that it has no real significance.
The coordinates of the corner of the wedge are

$$x_2 = x, \qquad y_2 = 0.$$

Thus the velocities are

$$\begin{aligned} \dot{x}_1 &= \dot{x} - \dot{s}\cos\alpha, & \dot{y}_1 &= -\dot{s}\sin\alpha \\ \dot{x}_2 &= \dot{x}, & \dot{y}_2 &= 0\,. \end{aligned}$$

Note that we treat the wedge as a particle here.

Then

$$\begin{aligned}
T &= \frac{1}{2}m(\dot{x}_1^2+\dot{y}_1^2)+\frac{1}{2}M(\dot{x}_2^2+\dot{y}_2^2)\\
&= \frac{1}{2}(m+M)\dot{x}^2+\frac{1}{2}m\dot{s}^2-m\dot{s}\dot{x}\cos\alpha\\
V &= mgy_1+Mgy_2 = mg(h-s\sin\alpha)\\
L &= \frac{1}{2}(m+M)\dot{x}^2+\frac{1}{2}m\dot{s}^2-m\dot{s}\dot{x}\cos\alpha+mgs\sin\alpha\,.
\end{aligned}$$

Note firstly that the expression for the K.E. involves the cross term $\dot{s}\dot{x}$ because the s-axis and the x-axis are not orthogonal, and secondly that the constant term in the P.E. has been omitted from the Lagrangian, where it has no significance.

(b) There are two Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right)-\frac{\partial L}{\partial s}=0 \qquad \text{or} \qquad \frac{d}{dt}(m\dot{s}-m\dot{x}\cos\alpha)-mg\sin\alpha=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0 \qquad \text{or} \qquad \frac{d}{dt}\left((m+M)\dot{x}-m\dot{s}\cos\alpha\right)=0\,.$$

However, x is an ignorable co-ordinate so instead of the second Lagrange equations we can use a first integral instead

$$(m+M)\dot{x}-m\dot{s}\cos\alpha=C_1$$

In either case, since we require the accelerations, the Lagrange equations should be expanded, (or the first integral and can differentiated once to give)

$$\begin{aligned}
\ddot{s}-\ddot{x}\cos\alpha &= g\sin\alpha\\
(m+M)\ddot{x}-m\ddot{s}\cos\alpha &= 0\,.
\end{aligned}$$

The two accelerations are found by solving these simultaneously. This gives

$$\ddot{s}=\frac{(m+M)g\sin\alpha}{m\sin^2\alpha+M}\,, \qquad \ddot{x}=\frac{mg\sin\alpha\cos\alpha}{m\sin^2\alpha+M}\,.$$

These are now trivial to solve and give

$$s=\frac{(m+M)g\sin\alpha}{2(m\sin^2\alpha+M)}t^2+C_2t+C_3$$

and

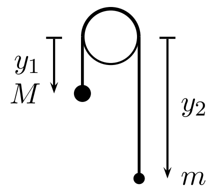
$$x=\frac{mg\sin\alpha\cos\alpha}{2(m\sin^2\alpha+M)}t^2+C_3t+C_4$$

If everything starts from rest when $x=0$ and $s=0$ then all the constants are zero.

The value of the first integral is $C_1=(m+M)C_3-m\cos\alpha C_2$.

4. An inextensible rope hangs over a pulley which has a moment of inertia I and radius a. The pulley can rotate freely about its centre and is rough so that the rope does not slip on the circumference. Masses M and m are attached at either end of the rope. If the system is released from rest, show that the acceleration of the masses is given by

$$\frac{(M-m)ga^2}{I+(M+m)a^2}\,.$$



4. **Solution:** Take coordinates from the centre line of the pulley with positive y downwards.

   Since the length of the string remains constant, there is a constraint

$$y_1 + \pi a + y_2 = \ell \,,$$

where ℓ is a constant. Thus

$$\begin{aligned} y_2 &= \ell - \pi a - y_1 \\ \dot{y}_2 &= -\dot{y}_1 \,. \end{aligned}$$

There is another constraint provided by the no-slip condition which requires the speed of the circumference of the pulley to equal that of the rope, i.e.

$$a\dot{\theta} = \dot{y}_1 = -\dot{y}_2 \,,$$

where $\dot{\theta}$ is the angular speed of the pulley as indicated.

Now

$$\begin{aligned} T &= \frac{1}{2}M\dot{y}_1^2 + \frac{1}{2}m\dot{y}_2^2 + \frac{1}{2}I\dot{\theta}^2 \\ &= \frac{1}{2}M\dot{y}_1^2 + \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}I\frac{\dot{y}_1^2}{a^2} \\ &= \frac{1}{2}\left(M + m + \frac{I}{a^2}\right)\dot{y}_1^2 \\ V &= -Mgy_1 - mgy_2 = -Mgy_1 - mg(\ell - \pi a - y_1) \,. \end{aligned}$$

Hence

$$L = \frac{1}{2}\left(M + m + \frac{I}{a^2}\right)\dot{y}_1^2 + (M - m)gy_1 \,,$$

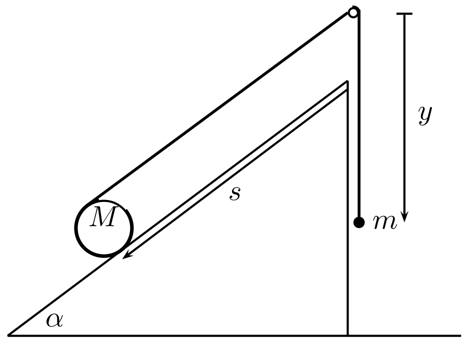
ignoring constant terms. There is now a single Lagrange equation

$$\frac{d}{dt}\left(\left(M + m + \frac{I}{a^2}\right)\dot{y}_1\right) - (M - m)g = 0 \,,$$

which gives

$$\ddot{y}_1 = \frac{(M - m)g}{M + m + I/a^2} \,, \quad \ddot{y}_2 = -\frac{(M - m)g}{M + m + I/a^2} \,.$$

5. A uniform solid cylinder, mass M and radius a, rolls without slipping on a rough plane inclined at angle α to the horizontal. Analyse the problem in two dimensions. As it rolls the cylinder rolls up a light string which passes over a fixed light pulley and supports a freely hanging mass m. Choose appropriate coordinates and no-slip condition. Write down the Lagrangian and solve to find the accelerations of the cylinder and the mass.



5. **Solution:** Let P be the point of attachment of the string on the cylinder. Let θ be the angle through the cylinder has rolled at time t, when the centre of mass has moved a distance s down the plane.

   The constant length of the string provides one constraint,

$$a\theta + a\pi + s + y + \text{constant} = \ell \,,$$

   where the constant is a function of the geometry of the pulley. Thus

$$a\dot{\theta} + \dot{s} + \dot{y} = 0 \,.$$

   The other constraint is the no-slip condition, $s = a\theta$ or

$$\dot{s} = a\dot{\theta} \,.$$

   Now we can form

$$\begin{aligned} T &= \frac{1}{2}M\dot{s}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\dot{y}^2 \\ &= \frac{1}{2}M\dot{s}^2 + \frac{1}{2}\left(\frac{1}{2}Ma^2\right)\left(\frac{\dot{s}}{a}\right)^2 + \frac{1}{2}(-2\dot{s})^2 \\ &= \frac{1}{2}\left(\frac{3}{2}M + 4m\right)\dot{s}^2 \\ V &= -Mgs\sin\alpha - mgy \\ &= -Mgs\sin\alpha + 2mgs + \text{constant} \\ L &= \frac{1}{2}\left(\frac{3}{2}M + 4m\right)\dot{s}^2 + (M\sin\alpha - 2m)gs \,. \end{aligned}$$

   The single Lagrange equation is then

$$\frac{d}{dt}\left(\left(\frac{3}{2}M + 4m\right)\dot{s}\right) - (M\sin\alpha - 2m)g = 0 \,.$$

   Hence

$$\ddot{s} = \frac{(M\sin\alpha - 2m)g}{\frac{3}{2}M + 4m} \,,$$

   and

$$\ddot{y} = -2\ddot{s} = -\frac{2(M\sin\alpha - 2m)g}{\frac{3}{2}M + 4m} \,.$$

6. **Physical Pendulum** The so called mathematical pendulum (as treated in the lecture notes) is a point mass suspended on a massless rod. In the physical pendulum the rod has mass, or more generally, the body of the pendulum has a suspension point, a centre of mass, and some moment of inertia I_G about the centre of mass. Let the distance from the suspension point and the centre of mass of the pendulum body be l. Denote by θ the angle between the vertical and the line connecting the suspension point and the centre of mass.

   (a) The *parallel axis theorem* states that if a body of mass M with moment of inertia I_G (about the centre of mass) rotates about an axis that has distance d from the centre of mass then it has moment of inertia $I = I_G + Md^2$. Prove this theorem by considering a body as composed of point masses.

   (b) Show that the kinetic energy of the pendulum can be written as

$$T = \frac{1}{2}Ml^2\dot{\theta}^2 + \frac{1}{2}I_G\dot{\theta}^2 \,.$$

6. **Solution: Physical Pendulum**

   (a) About the centre of mass we have $I_G = \sum m_i|\mathbf{r}_i|^2$, where $\mathbf{r}_i$ is the vector from the origin, which coincides with the centre of mass. If the body is now rotating about a point $\mathbf{s}$ instead the moment of inertia with respect to that point is

$$I_s = \sum m_i|\mathbf{r}_i + \mathbf{s}|^2 = \sum m_i(|\mathbf{r}_i|^2 + 2\mathbf{r}_i \cdot \mathbf{s} + |\mathbf{s}|^2) = \sum m_i|\mathbf{r}_i|^2 + M|\mathbf{s}|^2 = I_G + Md^2$$

   since $\sum m_i\mathbf{r}_i = 0$.

   (b) Since can be seen as a special case of the decomposition into motion of the centre of mass and the rotation about the centre of mass. Alternatively, using the parallel axis theorem this can also be read as the rotational energy $\frac{1}{2}I_s\dot{\theta}^2$ for rotation about the suspension point, which has distance l from the centre of mass.

7. **Double Pendulum** Consider a double pendulum made of two point masses. The first mass m_1 is suspended on a massless rod a distance c_1 from the point about which the rod is rotating. The first rod is rotating about the origin $(x, y) = (0, 0)$. Denote the counterclockwise angle between the rod and the downward vertical by θ_1. The second mass m_2 is suspended on a second massless rod a distance c_2 from the point about which the rod is rotating. The second rod that is rotating about the "end" for the first rod which is a distance l away from its suspension point. Denote the counterclockwise angle between the second rod and the downward vertical by θ_2. Gravity is on.

   (a) Write down the Lagrangian of this double pendulum by writing down the position (x_1, y_1) of mass m_1 and (x_2, y_2) of mass m_2 as a function of the generalised coordinates θ_1 and θ_2.

   (b) Write down the Lagrangian if instead of point masses we would have extended bodies of masses m_1 and m_2 with moments of inertia I_1^G and I_2^G, respectively. Here the G indicates that these moments of inertia are defined with respect to the centre of mass (or centre of gravity).
   From now on work with this general Lagrangian

   (c) Can you choose parameters c_1, c_2, l such that the potential energy vanishes? What does this mean? Give two constants of motion linear in angular velocities in this case.

   (d) After setting the potential to zero (e.g. by arranging the double pendulum so that it rotates in a horizontal plane), find a linear change of coordinates that introduces an ignorable coordinate. Choose the transformation matrix so that its determinant is 1.

(e) In the most general case the origin, the centre of mass of the first pendulum, and the suspension point of the second pendulum are not collinear. Give the additional terms in the Lagrangian in this case.

7. **Solution: Double Pendulum**

(a) From basic geometry $(x_1, y_1) = c_1(\sin\theta_1, -\cos\theta_1)$ and $(x_2, y_2) = l(\sin\theta_1, -\cos\theta_1) + c_2(\sin\theta_2, -\cos\theta_2)$. The kinetic energy is

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

and the potential energy is

$$V = m_1 g y_1 + m_2 g y_2 \,.$$

Inserting the expression for (x_i, y_i) and collecting terms gives

$$L = \frac{m_1}{2}c_1^2\dot{\theta}_1^2 + \frac{m_2}{2}\left(l^2\dot{\theta}_1^2 + 2ls_2\cos(\theta_2 - \theta_1)\dot{\theta}_1\dot{\theta}_2 + c_2^2\dot{\theta}_2^2\right) + g(m_1c_1 + m_2 l)\cos\theta_1 + gm_2c_2\cos\theta_2 \,.$$

(b) Use the general observation that kinetic energy can be decomposed into contributions from translation of the centre of mass and rotation about the centre of mass. This means to add the two terms

$$\frac{1}{2}I_1^G\dot{\theta}_1^2 + \frac{1}{2}I_2^G\dot{\theta}_2^2$$

to the kinetic energy, since we already have the contribution from the motion of the centres of mass. Notice that even though the angles are defined with respect to the massless rods, they do give the rotational angle of the centres of mass as well.

(c) Yes, $c_2 = 0$ and $m_1c_1 + m_2 l = 0$. It means that the second pendulum is suspended in its centre of mass, and the first pendulum is balanced so that the centre of mass and the suspension point of the second pendulum are on opposite sides of the origin (the suspension point of the first pendulum). The Lagrangian is $L = \frac{1}{2}I_2^G\dot{\theta}_2^2 + \frac{1}{2}(I_1^G + c_1^2m_1 + l^2m_2)\dot{\theta}_1^2$, which describes two uncoupled free rotors. The corresponding angular momenta are $I_2^G\dot{\theta}_2 = const$ and $(I_1^G + c_1^2m_1 + l^2m_2)\theta_1 = const$. Of course we would eliminate either l or c_1.

(d) The kinetic energy depends on the difference of the angles $\theta_1 - \theta_2$ only, so we should choose this combination as a new coordinate. A possible transformation whose matrix has determinant one is $u = \theta_1$, $v = \theta_2 - \theta_1$. The velocities are $\dot{\theta}_1 = \dot{u}$ and $\dot{\theta}_2 = \dot{u} + \dot{v}$. Then u is ignorable, and

$$\frac{\partial L}{\partial \dot{u}} = const.$$

(e) This requires going back to the beginning and replacing the term $l(\sin\theta_1, -\cos\theta_1)$ in (x_2, y_2) by the more general $l_x(\cos\theta_1, \sin\theta_1) + l_y(\sin\theta_1, -\cos\theta_1)$, where $l = l_y$. In the kinetic energy this replaces l^2 by $l_x^2 + l_y^2$ and any other l by l_y. The additional cross term in the kinetic energy is $-2l_xc_2\sin(\theta_2 - \theta_1)$, and in the potential the additional term is $m_2gl_x\sin\theta_1$.