The double pendulum: Lagrangian formulation

Posted by Diego Assencio on 2014.02.28 under Physics (Mechanics)

Consider the double pendulum shown on figure 1. A double pendulum is formed by attaching a pendulum directly to another one. Each pendulum consists of a bob connected to a massless rigid rod which is only allowed to move along a vertical plane. The pivot of the first pendulum is fixed to a point O. All motion is frictionless.

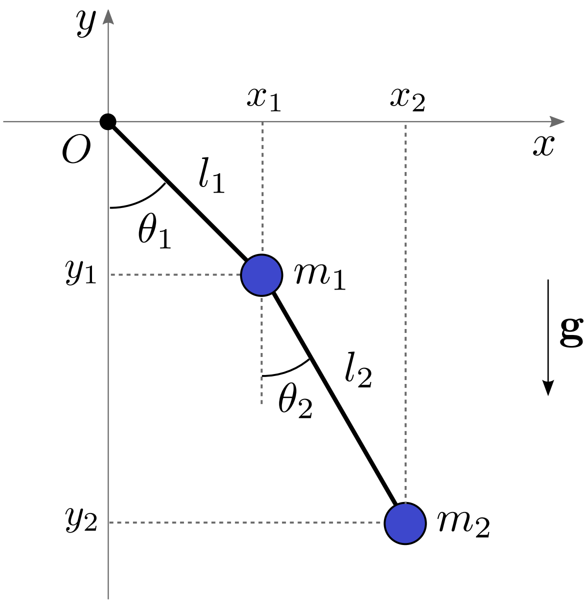

Fig. 1: A double pendulum.

In our discussion, the fixed point O will be taken as the origin of the Cartesian coordinate system with the x axis pointing along the horizontal direction and the y axis pointing vertically upwards. Let θ_1 and θ_2 be the angles which the first and second rods make with the vertical direction respectively. As can be seen on figure 1, the positions of the bobs are given by:

$$x_1 = l_1 \sin\theta_1 \qquad y_1 = -l_1\cos\theta_1 \tag{1}$$

$$x_2 = l_1 \sin\theta_1 + l_2\sin\theta_2 \qquad y_2 = -l_1\cos\theta_1 - l_2\cos\theta_2 \tag{2}$$

Differentiating the quantities above with respect to time, we obtain the velocities of the bobs:

$$\dot{x}_1 = l_1\dot{\theta}_1\cos\theta_1 \qquad \dot{y}_1 = l_1\dot{\theta}_1\sin\theta_1 \tag{3}$$

$$\dot{x}_2 = l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 \qquad \dot{y}_2 = l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2 \tag{4}$$

The double pendulum is a very interesting system as it is very simple but can show chaotic behavior for certain initial conditions. In this regime, slightly changing the initial values of the angles (θ_1, θ_2) and angular velocities $(\dot{\theta}_1, \dot{\theta}_2)$ makes the trajectories of the bobs become very different from the original ones.

The Lagrangian for the double pendulum is given by $L = T - V$, where T and V are the kinetic and potential energies of the system respectively. The kinetic energy T is given by:

$$\begin{aligned} T &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \\ &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right] \end{aligned} \tag{5}$$

where above we used the fact that $\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \cos(\theta_1 - \theta_2)$. The potential energy V is given by:

$$\begin{aligned} V &= m_1 g y_1 + m_2 g y_2 \\ &= -m_1 g l_1 \cos\theta_1 - m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2) \\ &= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \end{aligned} \tag{6}$$

The Lagrangian of the system is then:

$$\begin{aligned} L = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2 \end{aligned} \tag{7}$$

The canonical momenta associated with the coordinates θ_1 and θ_2 can be obtained directly from L:

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \tag{8}$$

$$p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \tag{9}$$

The equations of motion of the system are the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0 \Longrightarrow \frac{dp_{\theta_i}}{dt} - \frac{\partial L}{\partial \theta_i} = 0 \quad \text{for} \quad i = 1, 2 \tag{10}$$

Since:

$$\begin{aligned} \frac{dp_{\theta_1}}{dt} = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) \\ - m_2 l_1 l_2 \dot{\theta}_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \end{aligned} \tag{11}$$

$$\begin{aligned} \frac{dp_{\theta_2}}{dt} = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) \\ - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \end{aligned} \tag{12}$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin\theta_1 \tag{13}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2 \tag{14}$$

then equation (10) yields (after dividing by l_1 when $i = 1$ and by $m_2 l_2$ when $i = 2$):

$$\begin{aligned}(m_1 + m_2) l_1 \ddot\theta_1 + m_2 l_2 \ddot\theta_2 \cos(\theta_1 - \theta_2)& \\ + m_2 l_2 \dot\theta_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin\theta_1 &= 0\end{aligned} \tag{15}$$

$$l_2 \ddot\theta_2 + l_1 \ddot\theta_1 \cos(\theta_1 - \theta_2) - l_1 \dot\theta_1^2 \sin(\theta_1 - \theta_2) + g \sin\theta_2 = 0 \tag{16}$$

Equations (15) and (16) form a system of coupled second-order nonlinear differential equations. Dividing equation (15) by $(m_1 + m_2) l_1$ and equation (16) by l_2 and also moving all terms which do not involve $\ddot\theta_1$ and $\ddot\theta_2$ to the right-hand side, we obtain:

$$\ddot\theta_1 + \alpha_1(\theta_1, \theta_2) \ddot\theta_2 = f_1(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2) \tag{17}$$

$$\ddot\theta_2 + \alpha_2(\theta_1, \theta_2) \ddot\theta_1 = f_2(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2) \tag{18}$$

where:

$$\alpha_1(\theta_1, \theta_2) := \frac{l_2}{l_1} \left(\frac{m_2}{m_1 + m_2} \right) \cos(\theta_1 - \theta_2) \tag{19}$$

$$\alpha_2(\theta_1, \theta_2) := \frac{l_1}{l_2} \cos(\theta_1 - \theta_2) \tag{20}$$

and:

$$f_1(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2) := -\frac{l_2}{l_1} \left(\frac{m_2}{m_1 + m_2} \right) \dot\theta_2^2 \sin(\theta_1 - \theta_2) - \frac{g}{l_1} \sin\theta_1 \tag{21}$$

$$f_2(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2) := \frac{l_1}{l_2} \dot\theta_1^2 \sin(\theta_1 - \theta_2) - \frac{g}{l_2} \sin\theta_2 \tag{22}$$

Interestingly, f_1 does not depent on $\dot\theta_1$ and f_2 does not depend on $\dot\theta_2$. Equations (17) and (18) can be combined into a single equation:

$$A \begin{pmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{pmatrix} = \begin{pmatrix} 1 & \alpha_1 \\ \alpha_2 & 1 \end{pmatrix} \begin{pmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{23}$$

where the matrix A depends on θ_1 and θ_2 since α_1 and α_2 depend on these variables. Being a 2×2 matrix, A can be inverted directly:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 1 & -\alpha_1 \\ -\alpha_2 & 1 \end{pmatrix} = \frac{1}{1 - \alpha_1 \alpha_2} \begin{pmatrix} 1 & -\alpha_1 \\ -\alpha_2 & 1 \end{pmatrix} \tag{24}$$

Before we continue, notice that A is always invertible since:

$$\det(A) = 1 - \alpha_1\alpha_2 = 1 - \left(\frac{m_2}{m_1 + m_2}\right)\cos^2(\theta_1 - \theta_2) > 0 \tag{25}$$

because $m_2/(m_1+m_2) < 1$ and $\cos^2(x) \leq 1$ for all real values of x. From equations (23) and (24) we obtain:

$$\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix} = A^{-1}\begin{pmatrix}f_1\\ f_2\end{pmatrix} = \frac{1}{1-\alpha_1\alpha_2}\begin{pmatrix}f_1 - \alpha_1 f_2\\ -\alpha_2 f_1 + f_2\end{pmatrix} \tag{26}$$

Finally, letting $\omega_1 := \dot\theta_1$ and $\omega_2 := \dot\theta_2$, we can write the equations of motion of the double pendulum as a system of coupled first order differential equations on the variables θ_1, θ_2, ω_1, ω_2:

$$\frac{d}{dt}\begin{pmatrix}\theta_1\\ \theta_2\\ \omega_1\\ \omega_1\end{pmatrix} = \begin{pmatrix}\omega_1\\ \omega_2\\ g_1(\theta_1,\theta_2,\omega_1,\omega_2)\\ g_2(\theta_1,\theta_2,\omega_1,\omega_2)\end{pmatrix} \tag{27}$$

$$g_1 := \frac{f_1 - \alpha_1 f_2}{1-\alpha_1\alpha_2} \qquad g_2 := \frac{-\alpha_2 f_1 + f_2}{1-\alpha_1\alpha_2} \tag{28}$$

where $\alpha_i = \alpha_i(\theta_1,\theta_2)$ and $f_i = f_i(\theta_1,\theta_2,\omega_1,\omega_2)$ for $i = 1, 2$ are given on equations (19)-(22).

Equation (27) can be solved numerically using a Runge-Kutta (RK) method. A simulator based on the fourth-order RK method can be found here.

References

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