Particle dynamics



- Particle overview
- Particle system
- Forces
- Constraints
- Second order motion analysis

Particle system

- Particles are objects that have mass, position, and velocity, but without spatial extent
- Particles are the easiest objects to simulate but they can be made to exhibit a wide range of objects

Particle animation

- Each particle has a position, mass, and velocity
  - maybe color, age, temperature
- Seeded randomly at start
  - maybe some created each frame
- Move each frame according to physics
- Eventually die when some condition met

Sparks from a campfire

- Add 2-3 particles at each frame
  - initialize position and temperature randomly
- Move in specified turbulent smoke flow and decrease temperature as evolving
- Render as a glowing dot
- Kill when too cold to glow visibly

Rendering

- Simplest rendering: color dots
- Animated sprites
- Deformable blobs
- Transparent spheres
- Shadows

A Newtonian particle

- First order motion is sufficient, if
  - a particle state only contains position
  - no inertia
  - particles are extremely light
- Most likely particles have inertia and are affected by gravity and other forces
- This puts us in the realm of second order motion

Second-order ODE

What is the differential equation that describes the behavior of a mass point?

$$\mathbf{f} = m\mathbf{a}$$

What does **f** depend on?

$$\ddot{\mathbf{x}}(t) = \frac{\mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t))}{m}$$

Second-order ODE

$$\ddot{\mathbf{x}}(t) = \frac{\mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t))}{m} = f(\mathbf{x}, \dot{\mathbf{x}})$$

This is not a first oder ODE because it has second derivatives

Add a new variable, $\mathbf{v}(t)$, to get a pair of coupled first order equations

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{v} \\ \dot{\mathbf{v}} = \mathbf{f}/m \end{cases}$$

Phase space

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Concatenate position and velocity to form a 6-vector: *position* in phase space

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \frac{\mathbf{f}}{m} \end{bmatrix}$$

First order differential equation: *velocity* in the phase space

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Particle structure

Particle

$\mathbf{x}$ position

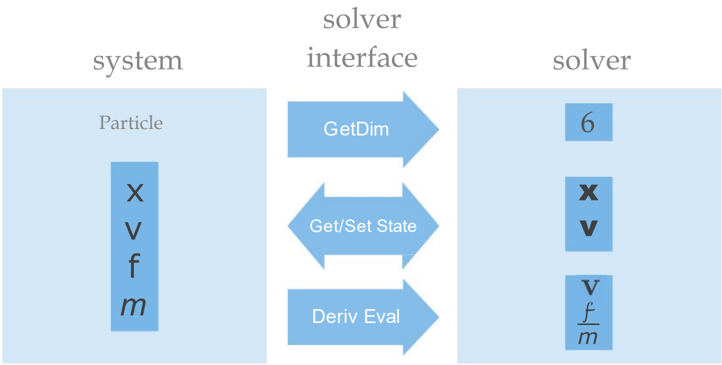
$\mathbf{v}$ velocity

→ a point in the phase space

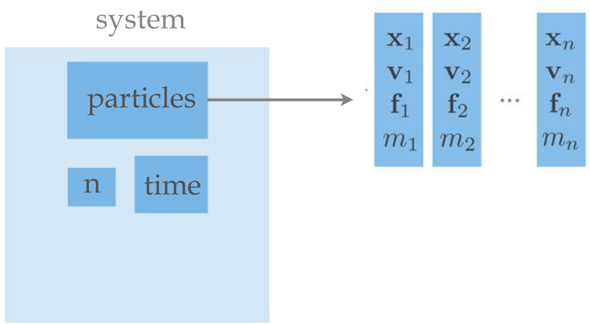
$\mathbf{f}$ force accumulator

m mass

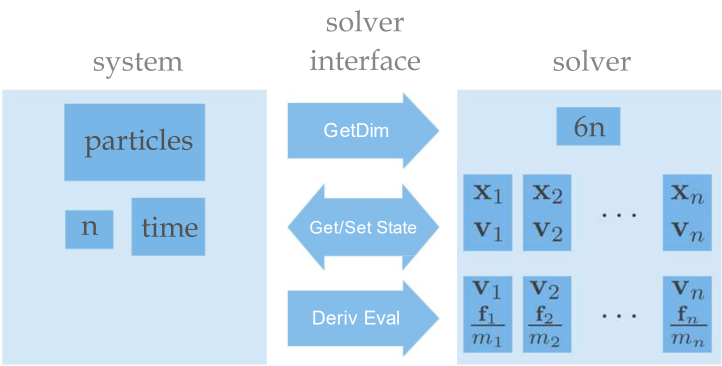
Solver interface



Particle system structure



Particle system structure



Deriv Eval

Clear forces: loop over particles, zero force accumulator

Calculate forces: sum all forces into accumulator

Gather: loop over particles, copy **v** and **f**/m into destination array

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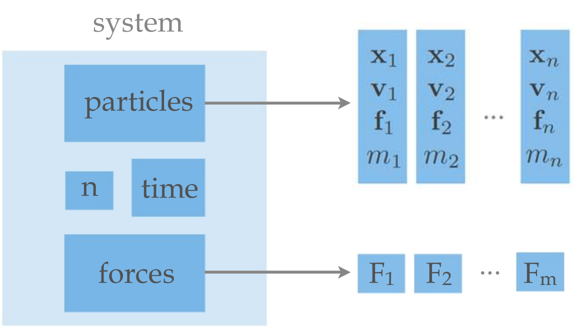
Forces

- Constant
  - gravity
- Position/time dependent
  - force fields, springs
- Velocity dependent
  - drag

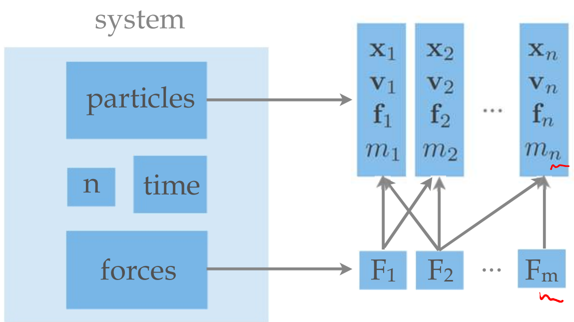
Particle systems with forces



Force structure

- Unlike particles, forces are heterogeneous (type-dependent)
- Each force object "knows"
  - which particles it influences
  - how much contribution it adds to the force accumulator

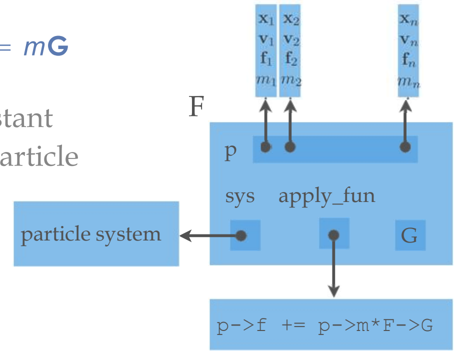
Particle systems with forces



Gravity

Unary force: $\mathbf{f} = m\mathbf{G}$

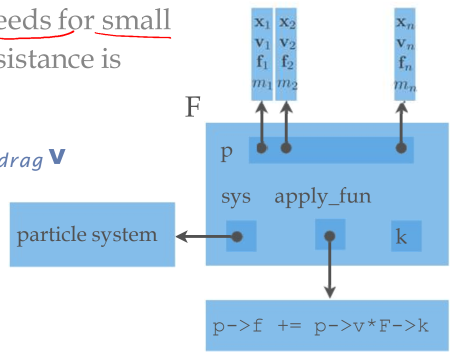
Exerting a constant force on each particle



Viscous drag

At very low speeds for small particles, air resistance is approximately:

$$\mathbf{f}_{drag} = -k_{drag}\mathbf{v}$$



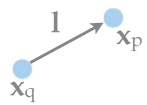
Attraction

Act on any or all **pairs of particles**, depending on their positions

$$\mathbf{f}_p = -k\frac{m_p m_q}{\|\mathbf{l}\|^2}\frac{\mathbf{l}}{\|\mathbf{l}\|}$$

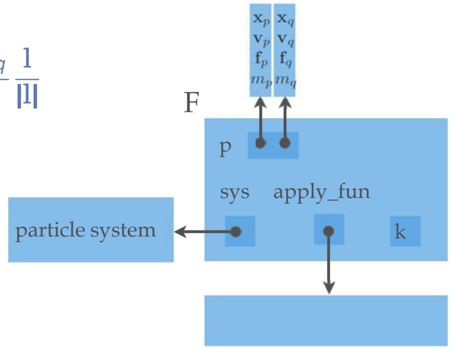
$$\mathbf{f}_q = -\mathbf{f}_p$$

$$\mathbf{l} = \mathbf{x}_p - \mathbf{x}_q$$



Attraction

$$\mathbf{f}_p = -k\,\frac{m_p m_q}{|\mathbf{l}|^2}\,\frac{\mathbf{l}}{|\mathbf{l}|}$$

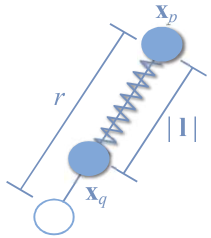


Damped spring

$$\mathbf{f}_p = -\left[k_s(|\mathbf{l}| - r) + k_d\frac{\dot{\mathbf{l}}\cdot\mathbf{l}}{|\mathbf{l}|}\right]\frac{\mathbf{l}}{|\mathbf{l}|}$$

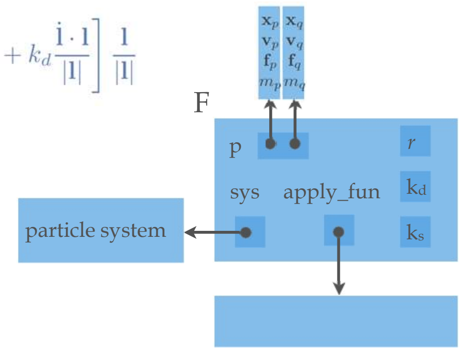
$$\mathbf{f}_q = -\mathbf{f}_p$$

$$\mathbf{l} = \mathbf{x}_p - \mathbf{x}_q$$



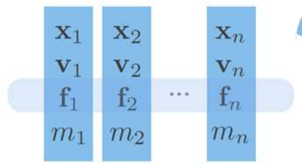
Damped spring

$$\mathbf{f}_p = -\left[k_s(|\mathbf{l}| - r) + k_d\frac{\dot{\mathbf{l}}\cdot\mathbf{l}}{|\mathbf{l}|}\right]\frac{\mathbf{l}}{|\mathbf{l}|}$$



Deriv Eval

1. Clear force accumulators



2. Invoke `apply_force` functions

$\mathbf{F}_1$ $\mathbf{F}_2$ ··· $\mathbf{F}_m$

3. Return derivatives to solver

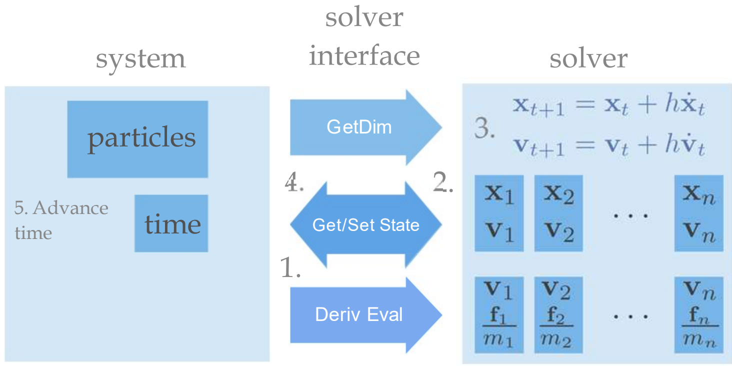
$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{v}}\end{bmatrix} = \begin{bmatrix}\mathbf{v}\\ \frac{\mathbf{f}}{m}\end{bmatrix}$$

ODE solver

Euler's method: $\mathbf{x}(t_0 + h) = \mathbf{x}(t_0) + hf(\mathbf{x}, t)$

$$\mathbf{x}_{t+1} = \mathbf{x}_t + h\dot{\mathbf{x}}_t$$
$$\mathbf{v}_{t+1} = \mathbf{v}_t + h\dot{\mathbf{v}}_t$$

Euler step



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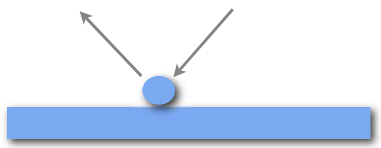
Particle Interaction

- We will revisit collision when we talk about rigid body simulation
- For now, just simple point-plane collisions

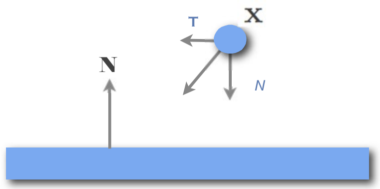


Collision detection

Normal and tangential components

$\mathbf{v}_N = (\mathbf{N} \cdot \mathbf{v})\mathbf{N}$

$\mathbf{v}_T = \mathbf{v} - \mathbf{v}_N$



Collision detection

Particle is on the legal side if

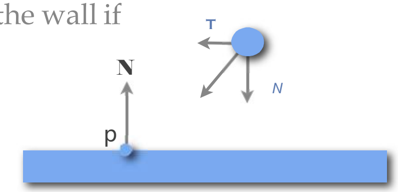
$(\mathbf{x} - \mathbf{p}) \cdot \mathbf{N} \geq 0$

Particle is within ϵ of the wall if

$(\mathbf{x} - \mathbf{p}) \cdot \mathbf{N} < \epsilon$

Particle is heading in if

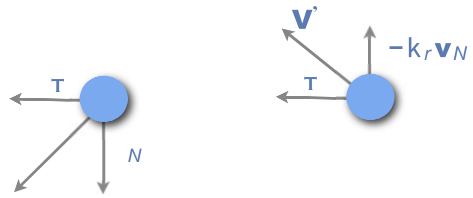
$\mathbf{v} \cdot \mathbf{N} < 0$



Collision response

Before collision

After collision



$\mathbf{V}' = \mathbf{v}_T - k_r \mathbf{v}_N$

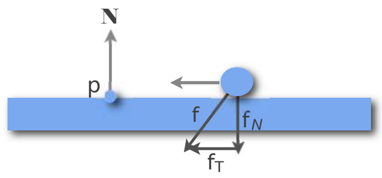
coefficient of restitution: $0 \leq k_r < 1$

Contact

Conditions for resting contact:

1. particle is on the collision surface
2. zero normal velocity

If a particle is pushed into the contact plane a contact force $\mathbf{f}_c$ is exerted to cancel the normal component of $\mathbf{f}$



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Linear analysis

- Linearly approximate acceleration

$$\mathbf{a}(\mathbf{x}, \mathbf{v}) \approx \mathbf{a}_0 - \mathbf{K}\mathbf{x} - \mathbf{D}\mathbf{v}$$

- Split up analysis into different cases
  - constant acceleration
  - linear acceleration

Constant acceleration

- Solution is

$$v(t) = v_0 + a_0 t$$

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2$$

- $\mathbf{v}(t)$ only needs 1st order accuracy, but $\mathbf{x}(t)$ demands 2nd order accuracy

Linear acceleration

- When K (or D) dominates ODE, what type of motion does it correspond to?

$$\mathbf{a}(\mathbf{x}, \mathbf{v}) = -\mathbf{K}\mathbf{x} - \mathbf{D}\mathbf{v}$$

$$\frac{d}{dt}\begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix}$$

- Need to compute the eigenvalues of A

Linear acceleration

Assume α is an eigenvalue of $\mathbf{A}$, $\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$ is the corresponding eigenvector

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \alpha \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

The eigenvector of $\mathbf{A}$ has the form $\begin{bmatrix} \mathbf{u} \\ \alpha\mathbf{u} \end{bmatrix}$

Often, D is linear combination of K and I (Rayleigh damping)

That means K and D have the same eigenvectors

Linear acceleration

For any $\mathbf{u}$, if $\begin{bmatrix} \mathbf{u} \\ \alpha\mathbf{u} \end{bmatrix}$ is an eigenvector of A, following must be true

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \alpha\mathbf{u} \end{bmatrix} = \alpha \begin{bmatrix} \mathbf{u} \\ \alpha\mathbf{u} \end{bmatrix}$$

Now assume $\mathbf{u}$ is an eigenvector for both $\mathbf{K}$ and $\mathbf{D}$

$$-\lambda_k \mathbf{u} - \alpha\lambda_d \mathbf{u} = \alpha^2 \mathbf{u}$$

$$\alpha = -\frac{1}{2}\lambda_d \pm \sqrt{(\frac{1}{2}\lambda_d)^2 - \lambda_k}$$

Eigenvalue approximation

- If $\mathbf{D}$ dominates

  $\alpha \approx -\lambda_d, 0$

  - exponential decay
- If $\mathbf{K}$ dominates

  $$\alpha \approx \pm\sqrt{-1}\sqrt{\lambda_k}$$

  - oscillation

Analysis

- Constant acceleration (e.g. gravity)
  - demands 2nd order accuracy for position
- Position dependence (e.g. spring force)
  - demands stability but low or zero damping
  - looks at imaginary axis
- Velocity dependence (e.g. damping)
  - demands stability, exponential decay
  - looks at negative real axis

Explicit methods

- First-order explicit Euler method
  - constant acceleration: bad (1st order)
  - position dependence: very bad (unstable)
  - velocity dependence: ok (conditionally stable)
- RK3 and RK4
  - constant acceleration: great (high order)
  - position dependence: ok (conditionally stable)
  - velocity dependence: ok (conditionally stable)

Implicit methods

- Implicit Euler method
  - constant acceleration: bad (1st order)
  - position dependence: ok (stable but damped)
  - velocity dependence: great (monotone)
- Trapezoidal rule
  - constant acceleration: great (2nd order)
  - position dependence: great (stable and no damp)
  - velocity dependence: good (stable, not monotone)