TTK4225 - Systems Theory, Autumn 2020

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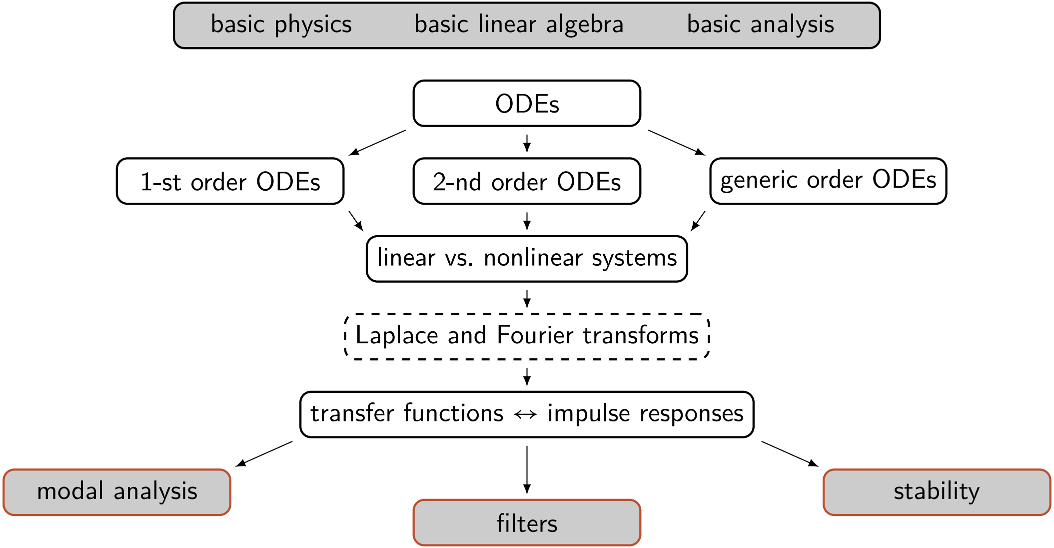
Transfer functions

Discussion: what did Laplace transforms enable us to do?

Summary / Roadmap of TTK4225



Roadmap

- definition of transfer function
- connections with impulse responses
- transfer functions of generic ARMA models
- examples

Laplace transforms: not only a way of solving dynamics

initial value problem

algebraic problem

$$\begin{tikzcd} \boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\ \boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \arrow[l, "\mathcal{L}^{-1}"'] \end{tikzcd}$$

solution in the time domain

solution in the frequency domain

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solution in the time domain solution in the frequency domain

This unit = transfer functions

i.e., complete description of the behavior of a LTI system in terms of a Laplace-object

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solution in the time domain — solution in the frequency domain

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i.e., complete description of the behavior of a LTI system in terms of a Laplace-object

Discussion: which other object was a complete description of a LTI system? *impulse responses and transfer functions need to be connected somehow*
(and now we will see how)

Towards the formal definition of transfer functions

$$\ddot{x} = a_1\dot{x} + a_0x = u(t), \quad x(0) = x_0, \quad \dot{x}(0) = x_1$$

$$\downarrow$$

$$s^2X - sx(0) - \dot{x}(0) = a_1\left(sX - x(0)\right) + a_0X + U$$

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$$X(s) = \underbrace{\frac{1}{s^2 - a_1 s - a_0}}_{H(s)} \left(\underbrace{(s - a_1)x(0) + \dot{x}(0)}_{M(s)} + U(s) \right)$$

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$$X(s) = H(s)M(s) + H(s)U(s)$$

Towards the formal definition of transfer functions

What are the roles of the objects here?

$$X(s) = H(s)M(s) + H(s)U(s)$$

- $H(s)$ = response of the system, the actual *transfer function*
- $M(s)$ = effect of the initial conditions, its combination with $H(s)$ gives the free evolution
- $U(s)$ = input, its combination with $H(s)$ gives the forced response

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this is another viewpoint that confirms the superposition of the effects

Connecting transfer functions with impulse responses

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$$X(s) = H(s)M(s) + H(s)U(s)$$

Discussion: what about $M(s) = 0$, $U(s) = 1$? Answer: $X(s) = H(s)$

but $U(s) = 1$ implies u a Dirac delta, so $X(s) = H(s)$ must be the Laplace transform of the impulse response $h(t)$

$$X(s) = H(s)M(s) + H(s)U(s)$$

$$x(t) = x_0 h(t) + h * u(t)$$

remember: "multiplication in frequency = convolution in time"

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implication: knowing $H(s)$ means knowing everything about the system

Are there alternative ways to compute $H(s)$?

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$$\Longrightarrow \quad H(s) = \frac{X(s)}{U(s)}$$

and, if we choose $u(t) = \delta(t)$ so that $U(s) = 1$,

$$\Longrightarrow \quad H(s) = X(s)$$

Computing $H(s)$ in the generic ARMA case

$$x^{(n)} = a_{n-1}x^{(n-1)} + \ldots + a_0 x + b_m u^{(m)} + \ldots + b_0 u$$

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Thus, since $H(s) = X(s)$,

$$H = \frac{b_m s^m + \ldots + b_0}{s^n - a_{n-1}s^{n-1} - \ldots - a_0}$$

How is knowing $H(s)$ simplify our life w.r.t. knowing $h(t)$?

$$H = \frac{b_m s^m + \ldots + b_1 s + b_0}{s^n - a_{n-1} s^{n-1} - \ldots a_1 s - a_0}$$

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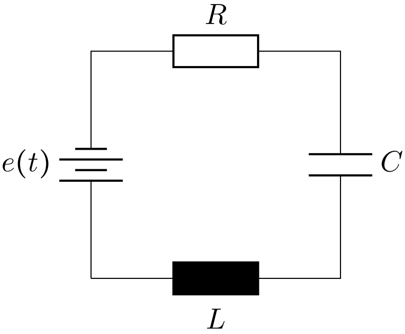
$$H = \frac{b_m s^m + \ldots + b_1 s + b_0}{s^n - a_{n-1} s^{n-1} - \ldots a_1 s - a_0}$$

- simpler analysis of the stability properties
- simpler analysis of the natural modes of the system
  (and thus its oscillatory behaviors)
- simpler analysis of the time constants of the system

Example: RCL circuits



$$L\ddot{x} + R\dot{x} + \frac{1}{C}x = \dot{e}(t), \quad e(t) = \delta(t) \implies H(s) = X(s),\ E(s) = 1 \tag{1}$$

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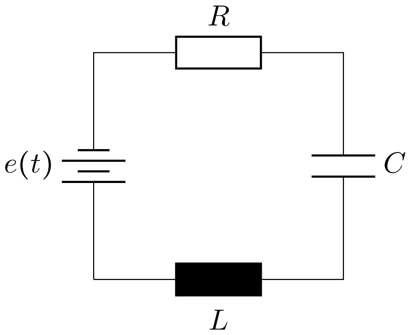

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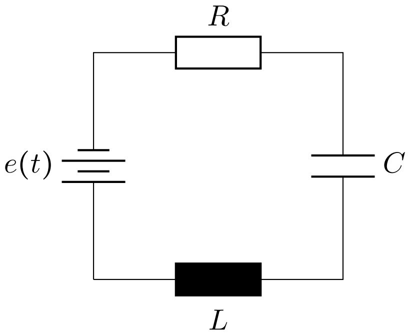

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$$\left(Ls^2 + Rs + 1/C\right) X(s) = s \quad \mapsto \quad H(s) = \frac{s}{20s^2 + 160s + 500} \tag{2}$$

Connecting transfer functions and impulse responses, take 2

$$H(s) = \frac{s}{20s^2 + 160s + 500} = \frac{1}{20}\frac{s}{s^2 + 8s + 25}$$

Discussion: how do we find the impulse response $h(t)$?

Connecting transfer functions and impulse responses, take 2

$$H(s) = \frac{s}{20s^2 + 160s + 500} = \frac{1}{20}\frac{s}{s^2 + 8s + 25}$$

Discussion: how do we find the impulse response $h(t)$? Solution: inverse-transform $H(s)$ using a partial fraction expansion $\implies$ we need to understand how the denominator looks like!

$$\begin{aligned} s^2 + 8s + 25 = 0 \quad &\implies s^2 + 8s = -25 \\ &s^2 + 2 \cdot 4s = -25 \\ &s^2 + 2 \cdot 4s + 4^2 = -25 + 4^2 \\ &\mapsto (s+4)^2 + 9 = 0 \end{aligned} \tag{3}$$

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Discussion: may the impulse response be an oscillatory one?

Laplace-transforms of oscillatory behaviors = **the second most useful formulas that you may remember**

$$\mathcal{L}\left\{e^{at}\sin\omega t\right\} = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\left\{e^{at}\cos\omega t\right\} = \frac{s-a}{(s-a)^2+\omega^2}$$

with limit case $a = 0$, so that

$$\mathcal{L}\left\{\sin\omega t\right\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\left\{\cos\omega t\right\} = \frac{s}{s^2+\omega^2}$$

Continuing the example above

$$H(s) = \frac{1}{20}\frac{s}{s^2+8s+25} = \frac{1}{20}\frac{(s+4)-4}{(s+4)^2+9}$$

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$$h(t) = \frac{1}{20}e^{-4t}\cos 3t - \frac{4}{20}e^{-4t}\sin 3t$$

?

Representing rational transfer functions through gain, zeros, and poles

Roadmap

- checking that not all the transfer functions are rational
- decomposition of rational transfer functions into gain, zeros, and poles

Computing transfer functions from differential equations

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$$\downarrow$$

$$\mathcal{L}\left\{\ddot{y}\right\} - a_1\mathcal{L}\left\{\dot{y}\right\} - a_2\mathcal{L}\left\{y\right\} = b_0\mathcal{L}\left\{\ddot{u}\right\} + b_1\mathcal{L}\left\{\dot{u}\right\} + b_2\mathcal{L}\left\{u\right\}$$

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$$\mathcal{L}\{\ddot{y}\} - a_1\mathcal{L}\{\dot{y}\} - a_2\mathcal{L}\{y\} = b_0\mathcal{L}\{\ddot{u}\} + b_1\mathcal{L}\{\dot{u}\} + b_2\mathcal{L}\{u\}$$
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$$s^2Y(s) - a_1 sY(s) - a_2 Y(s) = b_0 s^2 U(s) + b_1 sU(s) + b_2 U(s)$$

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$$s^2Y(s) - a_1 sY(s) - a_2Y(s) = b_0 s^2U(s) + b_1 sU(s) + b_2U(s)$$

$$\downarrow$$

$$Y(s) = \frac{b_0 s^2 + b_1 s + b_2}{s^2 - a_1 s - a_2}U(s)$$

Computing transfer functions from differential equations

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$$s^2Y(s) - a_1 sY(s) - a_2 Y(s) = b_0 s^2 U(s) + b_1 sU(s) + b_2 U(s)$$
$$\downarrow$$
$$Y(s) = \frac{b_0 s^2 + b_1 s + b_2}{s^2 - a_1 s - a_2}U(s)$$

problem: rational means finite polynomials; but not all the TFs are rational!

Delays

Remember: time shifting in Laplace transforms:

$$\mathcal{L}\left\{y(t-\tau)H\left(t-\tau\right)\right\} = e^{-\tau s}Y\left(s\right)$$

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example: $\mathcal{L}\left\{\delta t-\tau\right\}=?$ Solution: $e^{-\tau s}$

$$
\begin{aligned}
h(t) &\quad\mapsto\quad h(t-\tau) \\
H(s) &\quad\mapsto\quad e^{-\tau s} H(s) = \frac{1}{e^{\tau s}} H(s)
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Problem: $e^{\tau s} = \sum_{n=0}^{+\infty} \frac{(\tau s)^n}{n!}$:

$$
\frac{b_0 s^2 + b_1 s + b_2}{e^{\tau s}\left(s^2 - a_1 s - a_2\right)}
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$$\frac{b_0 s^2 + b_1 s + b_2}{e^{\tau s}\left(s^2 - a_1 s - a_2\right)}$$

Discussion: if $H(s)$ is rational, is $e^{-\tau s}H(s)$ rational too?

Partially solving the issue: Padé approximations

definition of Padé approximant: the "best" approximation of a function by a rational function of given order *(thus a concept applicable to any function)*

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explicit formulas a bit boring:

`https://mathoverflow.net/questions/41226/pade-approximant-to-exponential-function`

ZPK decompositions

Assumption:

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n - a_1 s^{n-1} - \ldots - a_n}$$

ZPK decompositions

Assumption:

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n - a_1 s^{n-1} - \ldots - a_n}$$

then

$$H(s) = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{j=1}^{n} (s - p_j)}$$

with

- $z_1, \ldots, z_m$ =: zeros of $H(s)$
- $p_1, \ldots, p_n$ =: poles of $H(s)$
- K =: gain of $H(s)$

How do we find ZPK representations?

I.e., how do we go from $\dfrac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n - a_1 s^{n-1} - \ldots - a_n}$ to $K \dfrac{\prod_{i=1}^{m} (s - z_i)}{\prod_{j=1}^{n} (s - p_j)}$?

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[1] Every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots.

Solution = rely on numerical aids

Solving the "polynomial roots" problem:

Matlab: `r = roots(p)`

Python: `r = numpy.roots(p)`

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Finding ZPK representations:

Matlab: `zpksys = zpk(sys)`

Python: `zkpsys = tf2zpk(b, a)` *(requires the "python-control" package)*

Ok, and so what?

$$H(s) = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{j=1}^{n} (s - p_j)}$$

Spoiler alert:

- p_j's $=$ poles $=$ natural modes of the system
- z_i's $=$ zeros $=$ complex exponentials that are killed by $H(s)$, but also factors modulating the amplitudes of the modes
- K $=$ gain $=$ how the system asypmtotically responds to a step-input

Ok, and so what?

$$H(s) = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{j=1}^{n} (s - p_j)}$$

Spoiler alert:

- p_j's $=$ poles $=$ natural modes of the system
- z_i's $=$ zeros $=$ complex exponentials that are killed by $H(s)$, but also factors modulating the amplitudes of the modes
- K $=$ gain $=$ how the system asypmtotically responds to a step-input

in other words, ZPK representations are interpretable

?

What are the poles of a system?

Roadmap

- partial fraction expansions
- differences between single and multiple poles
- examples

Physical meaning of a pole *(partial fraction expansion)*

$$
\begin{aligned}
H(s) = K\frac{\prod(s-z_i)}{\prod(s-p_j)} = \quad & \frac{\kappa_{1,1}}{s-p_1} + \frac{\kappa_{1,2}}{(s-p_1)^2} + \frac{\kappa_{1,3}}{(s-p_1)^3} + \dots \\
& + \frac{\kappa_{2,1}}{s-p_2} + \frac{\kappa_{2,2}}{(s-p_2)^2} + \dots \\
& + \dots
\end{aligned}
$$

Physical meaning of a pole *(partial fraction expansion)*

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& + \frac{\kappa_{2,1}}{s - p_2} + \frac{\kappa_{2,2}}{(s - p_2)^2} + \ldots \\
& + \ldots
\end{aligned}
$$

Thanks to the linearity of $\mathcal{L}$ and $\mathcal{L}^{-1}$,

$$
\begin{gathered}
H(s) = H_{1,1}(s) + H_{1,2}(s) + \ldots \\
\Updownarrow \\
h(t) = h_{1,1}(t) + h_{1,2}(t) + \ldots
\end{gathered}
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In words, $h(t)$ = sum of *all* the impulse responses relative to the various poles in the partial fraction expansion

The cover-up method

if $H(s) = K \frac{\prod_i (s - z_i)}{\prod_j (s - p_j)}$ does not have multiple poles then

$$H(s) = \sum_j \frac{\kappa_j}{(s - p_j)}$$

with

$$\kappa_j = (s - p_j) H(s)\Big|_{s=p_j}$$

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Example:

$$H(s) = \frac{3}{(s+1)(s+2)} \quad \Longrightarrow \quad h(t) = \ldots?$$

Physical meaning of a pole - What is $h_{j,\beta}(t)$?

$$H(s) = H_{1,1}(s) + H_{1,2}(s) + \dots$$

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$$H_{j,1}(s) = \frac{\kappa_{j,1}}{s - p_j} \qquad \Longrightarrow \qquad h_{j,1}(t) \propto e^{p_j t}$$

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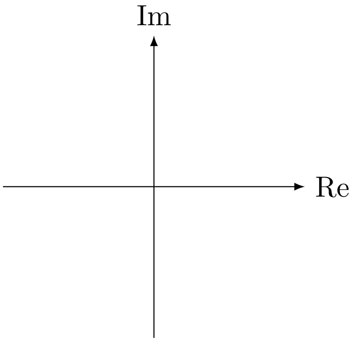
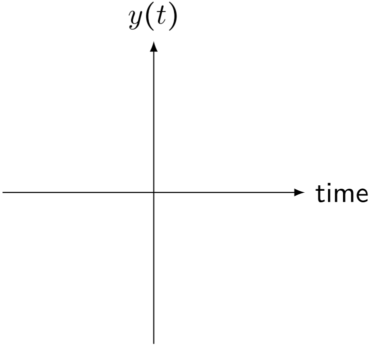
multiple root:

$$H_{j,\beta}(s) = \frac{\kappa_{j,\beta}}{(s - p_j)^\beta} \qquad \Longrightarrow \qquad h_{j,\beta}(t) \propto t^{\beta-1} e^{p_j t}$$

What is $h_{j,1}(t)$? (i.e., when the roots are simple)

if $\quad H(s) = K \dfrac{\prod(s - z_i)}{\prod(s - p_j)} = \sum_j \dfrac{\kappa_{j,1}}{s - p_j} \quad$ then $\quad \dfrac{\kappa_{j,1}}{s - p_j} \mapsto h_{j,1}(t) \propto e^{p_j t}$





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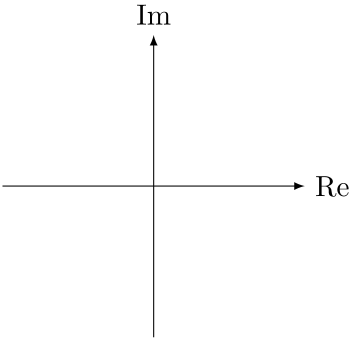
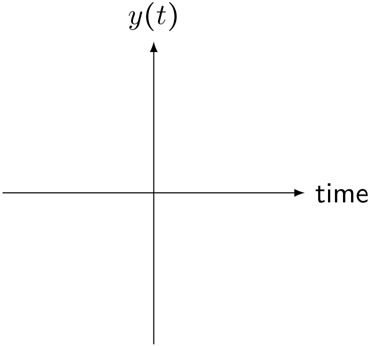
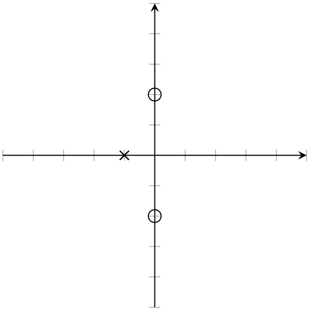




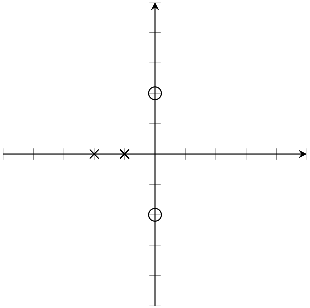
Physical meaning of a pole - Example

$$H(s) = K\frac{s^2+4}{s^3+3s^2+3s+1} = K\frac{(s+2j)(s-2j)}{(s+1)^3} \implies h(t) = \kappa_1 e^{-t} + \kappa_2 t e^{-t} + \kappa_3 t^2 e^{-t}$$



Physical meaning of a pole - Example

$$H(s) = K\frac{(s+2j)(s-2j)}{(s+1)^3(s+2)} \quad \Longrightarrow \quad h(t) = \ldots ??$$



?

What are the zeros of a system?

Roadmap

- generic definition
- definition for the LTI systems case
- examples of the effects of choosing different zeros

Definition (Zero of a function)

$$z_i = \textit{ zero of } H(\cdot) \textit{ if } H(z_i) = 0$$

Physical meaning of a zero of a transfer function

$$H \text{ LTI:} \qquad u(t) = e^{\overline{s}t} \qquad \Longrightarrow \qquad y(t) = H(\overline{s})e^{\overline{s}t}$$

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$$\text{thus} \quad y(0) = 0, \quad H(s) = K\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)}, \quad \text{and} \quad u(t) \propto e^{z_i t} \qquad \Longrightarrow \qquad y(t) = 0$$

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What is $e^{z_i t}$? Euler's formulae ($t \in \mathbb{R}$, $z_i \in \mathbb{C}$):

- $e^{z_i t} = e^{(\sigma + j\omega)t} = e^{\sigma t}\Big(\cos(\omega t) + j \sin(\omega t) \Big)$
- $\cos(\omega t) = \frac{1}{2}\Big(e^{j\omega t} + e^{-j\omega t}\Big)$

What is $e^{z_i t}$?

- $e^{z_i t} = e^{(\sigma + j\omega)t} = e^{\sigma t}\Big(\cos(\omega t) + j\sin(\omega t)\Big)$
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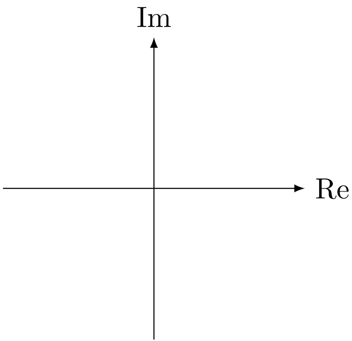
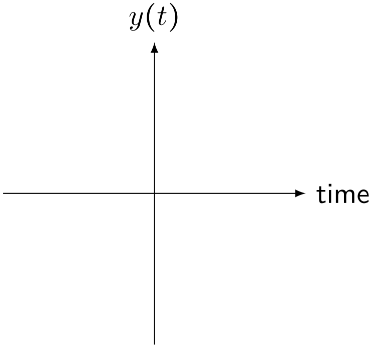
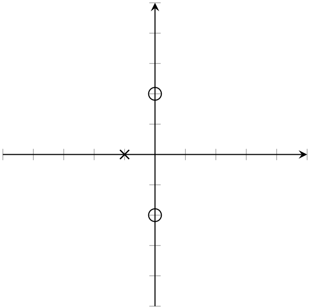




Physical meaning of a zero - Example

$$H(s) = K\frac{s^2+4}{s^3+3s^2+3s+1} = K\frac{(s+2j)(s-2j)}{(s+1)^3}$$



Thus $\quad u(t) = 2\cos(2t) = \left(e^{j2t} + e^{-j2t}\right) \quad \implies \quad y(t) = H(2j)e^{2jt} + H(-2j)e^{-2jt} = 0$

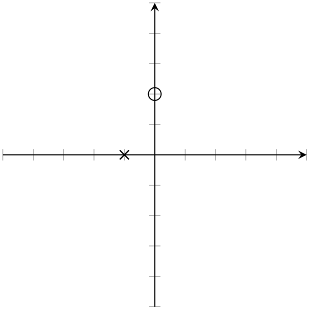
Discussion

Is it that

$$H(s) = K\frac{(s-2j)}{(s+1)^3}$$

implies

$$u(t) = 2\cos(2t) = \left(e^{j2t} + e^{-j2t}\right) \quad \implies \quad y(t) = 0 \quad ?$$



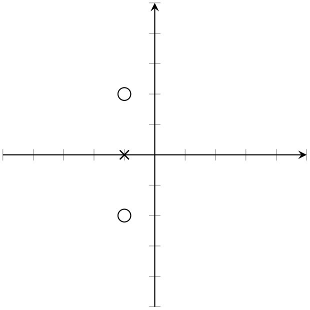
Discussion

Is it that

$$H(s) = K\frac{(s+1-2j)(s+1+2j)}{(s+1)^3}$$

implies

$$u(t) = 2\cos(2t) = \left(e^{j2t} + e^{-j2t}\right) \implies y(t) = 0 \quad ?$$



Example

$$H(s) = \frac{(s+\alpha)^2 + 1}{(s+1)\left((s+0.1)^2 + 1\right)}$$

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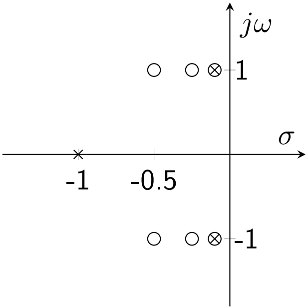

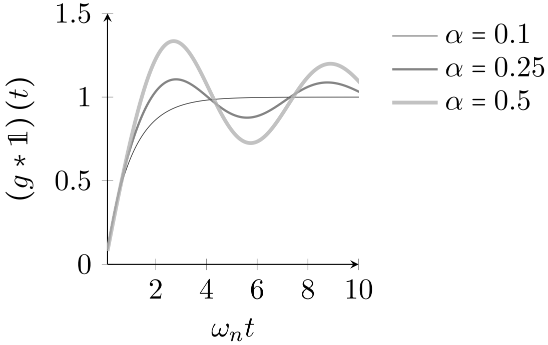

?

What is the gain of a system?

Roadmap

- definition from an intuitive perspective
- final value theorem
- examples

Gain, in generic terms

Concept, intuitively:

- I use an unitary step as the input
- I check what is the output at the end of the times

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important implication of linearity: for a LTI system, the ratio $\Delta u / \Delta y$ when the transient has passed is independent of the original u!

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- I use an unitary step as the input
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important implication of linearity: for a LTI system, the ratio $\Delta u / \Delta y$ when the transient has passed is independent of the original u!

Discussion: does this hold also for nonlinear systems?

How to compute the gain associated to a transfer function

Tool to be used: *final value theorem* *(but remember the caveats on its validity[2])*.

[2] I.e., the limit must exist and be finite

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$$\lim_{t \to +\infty} y(t) = \lim_{s \to 0} sY(s)$$

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[2] I.e., the limit must exist and be finite

How to compute the gain associated to a transfer function

Tool to be used: *final value theorem* *(but remember the caveats on its validity[2])*. For continuous time systems:

$$\lim_{t \to +\infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG(s)U(s) = \lim_{s \to 0} sG(s)\frac{1}{s} = \lim_{s \to 0} G(s)$$

Discussion: what is the gain associated to $\dfrac{10}{s+20}$?

[2] I.e., the limit must exist and be finite

For the sake of completeness ...

Final value theorem:

$$\lim_{t \to +\infty} y(t) = \lim_{s \to 0} sY(s)$$

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Final value theorem:

$$\lim_{t \to +\infty} y(t) = \lim_{s \to 0} sY(s)$$

Initial value theorem:

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} sY(s)$$

Discussion: why $s \to \infty$ and not $s \to +\infty$?

Final value theorem – proof

$$\lim_{s \to 0} sF(s) = \lim_{s \to 0} \left(\mathcal{L}\left\{ \dot{f}(t) \right\} + f(0) \right)$$

Final value theorem – proof

$$\begin{aligned}\lim_{s\to 0} sF(s) &= \lim_{s\to 0}\left(\mathcal{L}\left\{\dot{f}(t)\right\} + f(0)\right)\\ &= \lim_{s\to 0}\left(\int_0^{+\infty} \dot{f}(t)e^{-st}dt\right) + f(0)\end{aligned}$$

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&= \int_0^{+\infty} \lim_{s\to 0}\left(\dot{f}(t)e^{-st}dt\right) + f(0) \\
&= \int_0^{+\infty} \dot{f}(t)dt + f(0) \\
&= f(+\infty) - f(0) + f(0)
\end{aligned}$$

Final value theorem - example of how to use it correctly

$$X(s) = \frac{1}{s} \cdot \frac{6}{s+2} \qquad \lim_{t \to +\infty} x(t) = ?$$

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true?

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true? *no!* $\lim_{t \to +\infty} x(t)$ must exist and be finite!

no stability, no party!

?

Connections with step responses

Roadmap

- definition of step response
- properties for first order systems
- other important examples

Definition: step response

$$Y(s) = H(s)U(s), \qquad U(s) = \frac{1}{s}$$

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Discussion: may we easily implement step responses in real systems? What are the limitations?

Step responses for first order systems

Example: $\dot{x}(t) = ax(t) + bu(t)$, with $u(t) = H(t)$, implies

$$sX(s) - x_0 = aX(s) + bU(s) \qquad \Longrightarrow \qquad X(s) = \frac{1}{s-a}x_0 + \frac{b}{s(s-a)}$$

and thus $x(t) = e^{at}x_0 + \dfrac{b}{a}\left(e^{at} - H(t)\right)$.

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$\kappa\left(H(t) - e^{at}\right)$ = generic step response of first order systems

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- does the system converge somewhere?

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Discussion:

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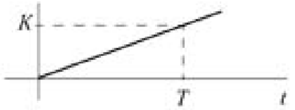
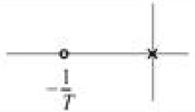
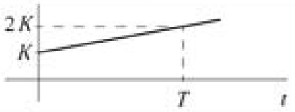
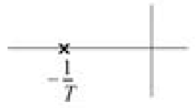
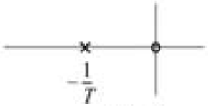
Discussion:

- does the system converge somewhere?
- may the system response pass this value?
- may we compute the time constant of the system in this case?

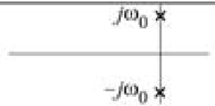
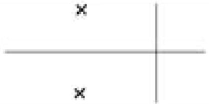
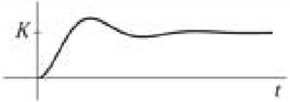
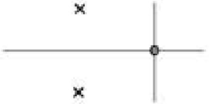
Examples for zeroth order systems

K		

Examples for first order systems

$K\frac{1}{Ts}$		
$K\frac{1+Ts}{Ts}$		
$K\frac{1}{1+Ts}$		
$K\frac{Ts}{1+Ts}$		

Examples for second order systems

$\dfrac{K}{1+\left(\dfrac{s}{\omega_0}\right)^2}$		
$\dfrac{K}{1+2\zeta\dfrac{s}{\omega_0}+\left(\dfrac{s}{\omega_0}\right)^2}$		
$\dfrac{Ks}{1+2\zeta\dfrac{s}{\omega_0}+\left(\dfrac{s}{\omega_0}\right)^2}$		

Important message

if the input is $\dot{u}$ then the step response gives the impulse response that we would get if the input were u, instead

Discussion

What about $\dfrac{K(s+a)}{1+2\xi\frac{s}{\omega}+\left(\frac{s}{\omega}\right)^2}$?

?

Block diagrams

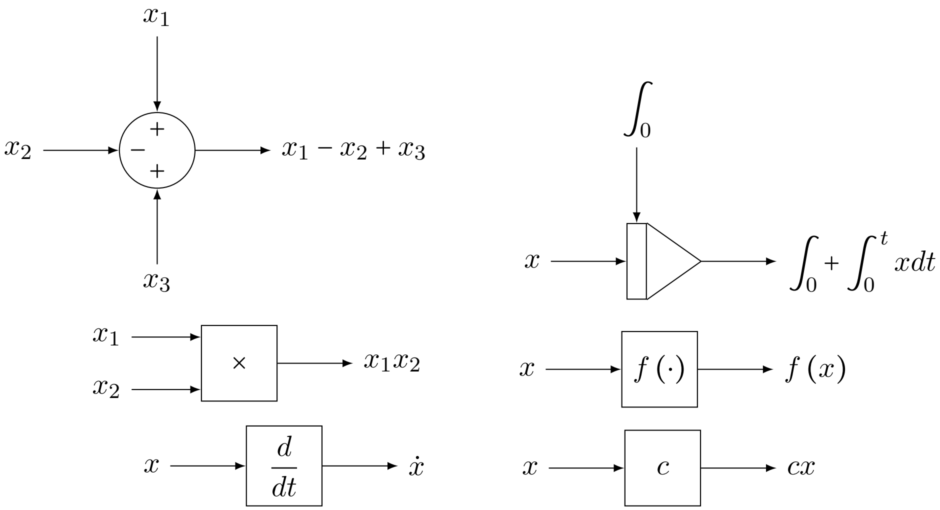
Roadmap

- recap of the diagrams in the time domain
- recap of the diagrams in the frequency domain
- rules for how to transform the diagrams
- examples

Block diagrams - why?

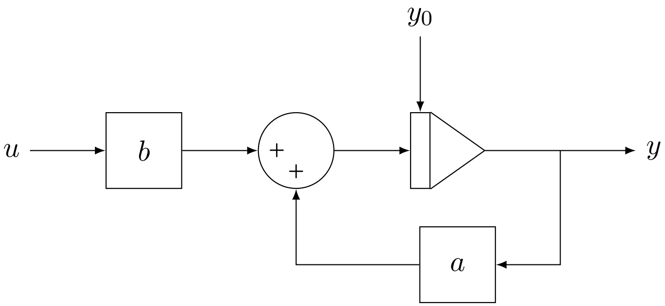
- used very often in companies
- aid visualization *(until a certain complexity is reached. . .)*
- enable “drag & drop” way of programming
- here primarily used for interpretations

Most common block diagrams in the time domain



Representing a first order DE with a block scheme

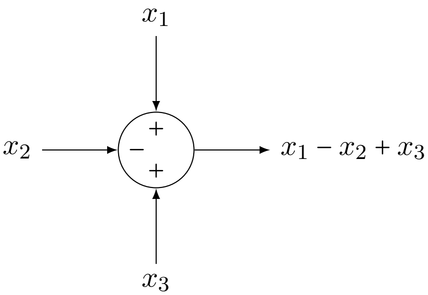
$$\dot{y} = ay + bu$$

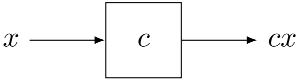


Discussion: how do we represent $\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}u$?

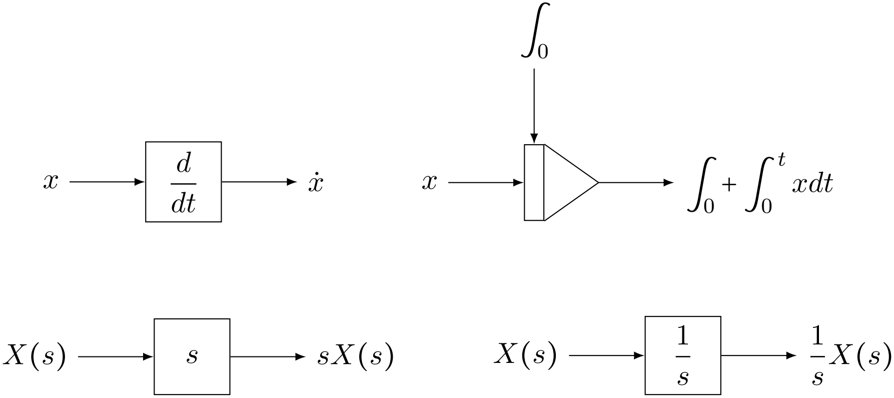
?

Block diagrams that are equal in both time and frequency domains

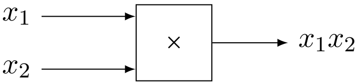


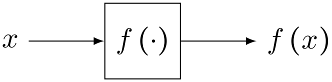


Block diagrams that are logically the same in both time and frequency domains



Block diagrams that do not exist in the frequency domain





Discussion: why?

rules for manipulating block diagrams in the frequency domain

Series of transfer functions

$$\begin{tikzcd} U(s) \arrow[r] & \boxed{H_a(s)} \arrow[r] & \boxed{H_b(s)} \arrow[r] & Y(s) \end{tikzcd}$$

is equivalent to

$$\begin{tikzcd} U(s) \arrow[r] & \boxed{H_a(s)H_b(s)} \arrow[r] & Y(s) \end{tikzcd}$$

Series of transfer functions

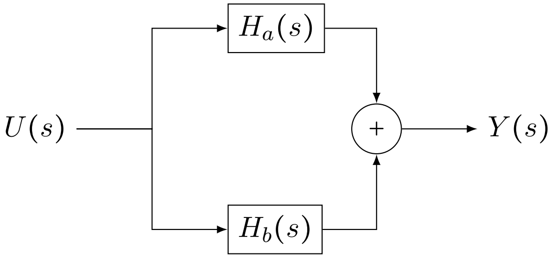
$$U(s) \longrightarrow \boxed{H_a(s)} \longrightarrow \boxed{H_b(s)} \longrightarrow Y(s)$$

is equivalent to

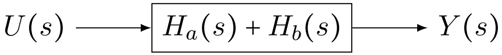
$$U(s) \longrightarrow \boxed{H_a(s)H_b(s)} \longrightarrow Y(s)$$

Discussion: why?

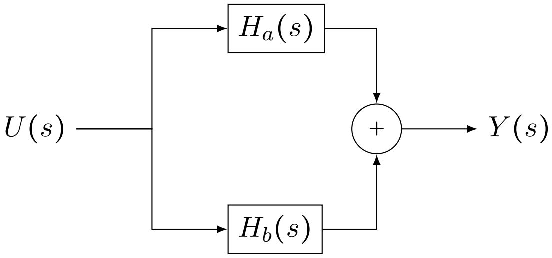
Parallel of transfer functions



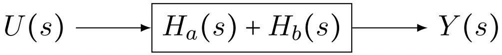
is equivalent to



Parallel of transfer functions

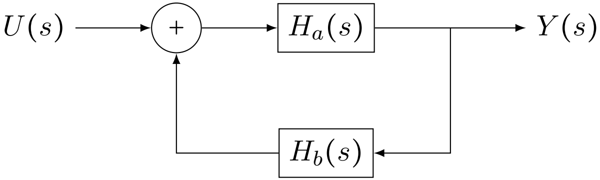


is equivalent to

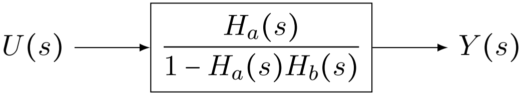


Discussion: why?

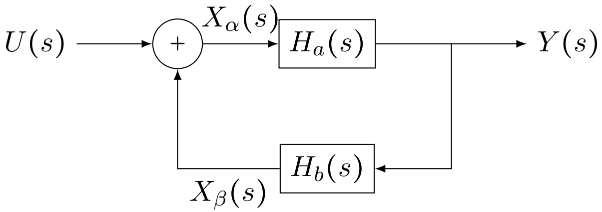
Elimination of feedback loops



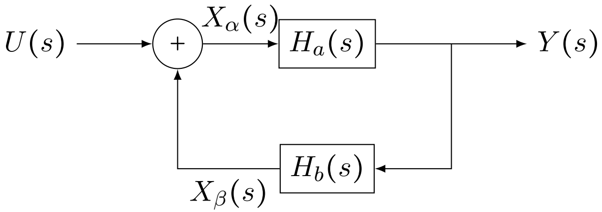
is equivalent to



Elimination of feedback loops: how to remember the formula

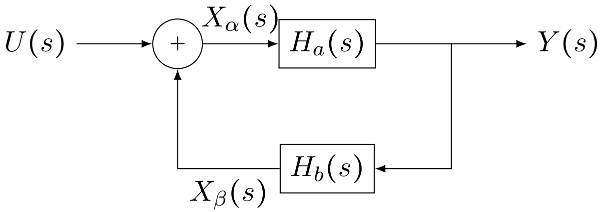


Elimination of feedback loops: how to remember the formula



- $Y = H_a X_\alpha$
- $X_\alpha = U + X_\beta$
- $X_\beta = H_b Y$

Elimination of feedback loops: how to remember the formula



- $Y = H_a X_\alpha$
- $X_\alpha = U + X_\beta$
- $X_\beta = H_b Y$
- $\implies Y = H_a \left(U + H_b Y\right)$