TTK4225 - Systems Theory, Autumn 2020

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Linear algebra - why, if we are doing control?

Roadmap

- why?
- spoilers

Motivations, in very brief

$$\ddot{x} + a_1\dot{x} + a_0 x = bu(t) \tag{1}$$

is equivalent to

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t)$$

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$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu,$$

and every analysis problem on the system becomes a linear algebra one

(e.g., computing the equilibria)

How does this connect with the first part of the course?

Via the scalar form:

$$\ddot{y} + a_1 \dot{y} + a_0 y = bu \qquad \Longrightarrow \qquad Y(s) = \frac{bU(s)}{s^2 + a_1 s + a_0}$$

with the modes defined by the solutions of $s^2 + a_1 s + a_0 = 0$

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Via the matrix form:

$$sX(s) - AX(s) = \quad BU(s)$$

$$\begin{cases} \dot{\boldsymbol{x}} & = A\boldsymbol{x} + Bu \\ y & = C \end{cases} \qquad \Longrightarrow$$

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$$\begin{aligned} s\boldsymbol{X}(s) - A\boldsymbol{X}(s) &= BU(s) \\ (sI - A)\boldsymbol{X}(s) &= BU(s) \end{aligned}$$

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Spoilers

- the poles of $\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$ will be the eigenvalues of A
- the structure of A will determine the multiplicity of the poles and much more *(for the brave ones, check the "Rosenbrock's theorem", but only* <u>*after*</u> *the course has ended)*

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the concepts of eigenvalues, eigenvectors, eigenspaces plus their generalized counterparts are as fundamental as the concepts of modes

List of the knowledge we used in the previous slides

- matrix inverses, i.e., M and M^{-1}
- adjugate of a matrix, i.e., $\operatorname{adj}(M)$
- determinant of a matrix, i.e., $\det(M)$
- eigenvalues / eigenvectors / eigenspaces of a matrix

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- fundamentals: addition and multiplication of vectors and matrices

Suggested additional resources:

- 3blue1brown: Essence of linear algebra
- Khan Academy: 44 videos on linear algebra
- Khan Academy: Introduction to vectors
- Gilbert Strang: Linear algebra

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Basic operations

Roadmap

- inner products
- matrix vector products
- matrix matrix products

Notation

$$\text{matrices: } A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

$$\text{column vectors: } \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n}$$

$$\text{row vectors: } \boldsymbol{x} = \begin{bmatrix} x_1 & \dots & x_m \end{bmatrix} \in \mathbb{R}^{m}$$

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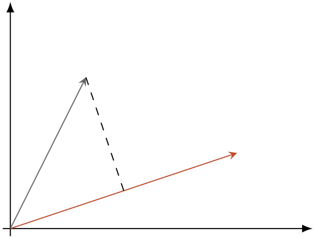
Important: saying "vector" means column vector; to indicate row vectors say "row vectors"!

Transposition

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{x}^\top = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$$

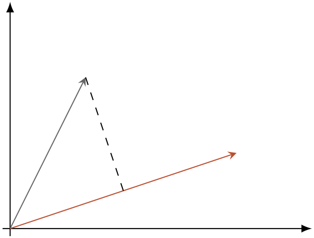
$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \quad \Longrightarrow \quad A^\top = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1m} & \cdots & a_{nm} \end{bmatrix}$$

Inner product



$$\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \qquad \Longrightarrow \qquad \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top \boldsymbol{y} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n$$

Geometrical meaning of inner product, some notes



note: $\boldsymbol{x}$ and $\boldsymbol{y}$ must live in the same space, thus they must have the same length

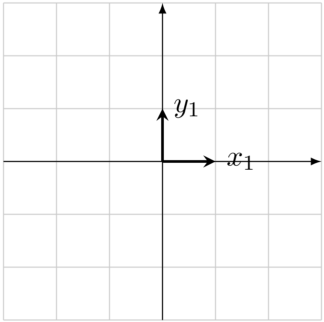
suggested material: 3 blue 1 brown, Dot products and duality, Essence of linear algebra, chapter 9

Matrix-vector product, mathematical definition

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} \star_1 \\ \vdots \\ \star_n \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} \star_1 \ + \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix} \star_2 \ + \begin{bmatrix} a_{13} \\ \vdots \\ a_{n3} \end{bmatrix} \star_3 \ + \ \ldots$$

Matrix-vector product, geometrically

Starting point: canonical basis: $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad y_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



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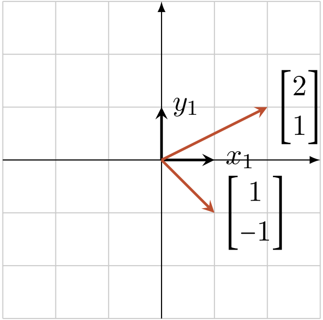
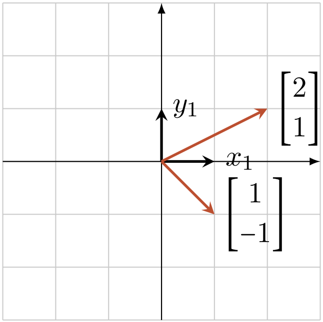
what are then $Ax_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $Ax_2 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?



VERY IMPORTANT INTERPRETATION

the columns of A are where the elements of the canonical basis are mapped by A

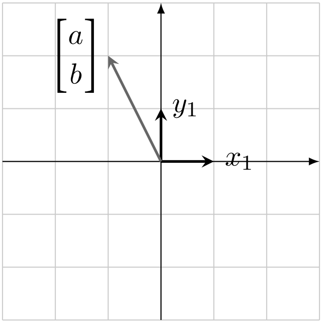


Remember: not all the A's are square

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

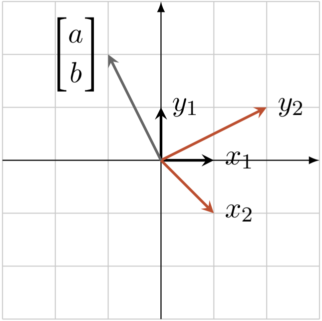
Matrix-vector product, geometrically

$$c = \begin{bmatrix} a \\ b \end{bmatrix} = ax_1 + by_1$$



Matrix-vector product, geometrically

$$c = \begin{bmatrix} a \\ b \end{bmatrix} = ax_1 + by_1 \qquad A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \qquad \Longrightarrow \qquad Ac = ?$$



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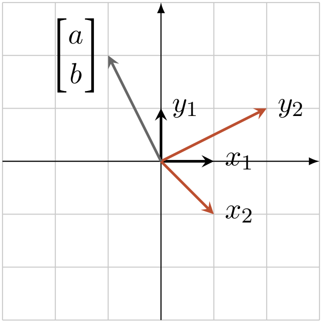
$$x_2 = Ax_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad y_2 = Ay_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



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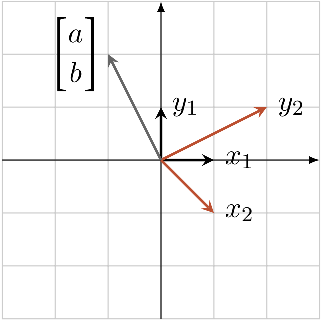

Matrix-vector product, geometrically

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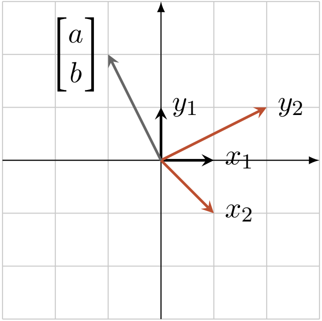

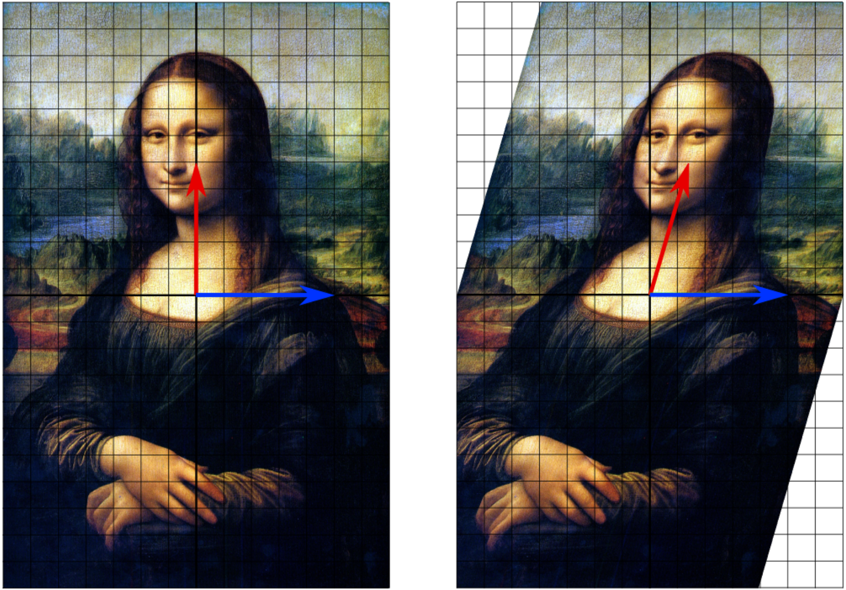

(By TreyGreer62 - Image:Mona Lisa-restored.jpg, CC0, https://commons.wikimedia.org/w/index.php?curid=12768508)

?

How do we go now from

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to

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} \star_1 & \triangle_1 \\ \vdots & \vdots \\ \star_n & \triangle_n \end{bmatrix} =?$$

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to

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} \star_1 & \triangle_1 \\ \vdots & \vdots \\ \star_n & \triangle_n \end{bmatrix} = ?$$

$$AB = A \begin{bmatrix} \boldsymbol{b}_1 & \dots & \boldsymbol{b}_p \end{bmatrix} = \begin{bmatrix} A\boldsymbol{b}_1 & \dots & A\boldsymbol{b}_p \end{bmatrix}$$

Matrix multiplication

$$C = AB$$

Discussion: how must the dimensions of A and B be?

- $A \in \mathbb{R}^{r_A \times c_A}$
- $B \in \mathbb{R}^{r_B \times c_B}$

Matrix multiplication

$$C = AB$$

Discussion: how must the dimensions of A and B be?

- $A \in \mathbb{R}^{r_A \times c_A}$
- $B \in \mathbb{R}^{r_B \times c_B}$
- $c_A = r_B$
- $\implies C \in \mathbb{R}^{r_A \times c_B}$

Examples

$$\begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ 19 & 2 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ 1 & 3 & 0 \\ 3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 11 & 4 \\ 3 & 15 & 4 \end{bmatrix}$$

Examples

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$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & 11 \end{bmatrix}$$

Do you see why this does *not* work?

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

In general, $AB \neq BA$

(even if it may actually happen, depending on the eigendecompositions of A and B ...)

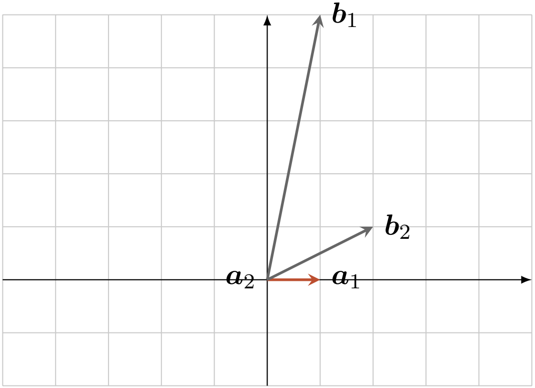
Numerical example:

$$\begin{bmatrix}1 & 2\\5 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\5 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\5 & 1\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$$

Ok that in general $AB \neq BA$, but why?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix}$$



Important points

- matrix multiplications are not communitative: $AB \neq BA$
- if $AB = BA$ then we say that A and B commute

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Alternative way of expressing matrix - column multiplications

$$\begin{bmatrix} — & \boldsymbol{a}_1 & — \\ — & \boldsymbol{a}_2 & — \\ & \vdots & \\ — & \boldsymbol{a}_n & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & \dots & \boldsymbol{b}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 & \boldsymbol{a}_1\boldsymbol{b}_2 & \dots & \boldsymbol{a}_1\boldsymbol{b}_n \\ \boldsymbol{a}_2\boldsymbol{b}_1 & \boldsymbol{a}_2\boldsymbol{b}_2 & \dots & \boldsymbol{a}_2\boldsymbol{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a}_n\boldsymbol{b}_1 & \boldsymbol{a}_n\boldsymbol{b}_2 & \dots & \boldsymbol{a}_n\boldsymbol{b}_n \end{bmatrix}$$

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$$\begin{bmatrix} — & \boldsymbol{a}_1 & — \\ — & \boldsymbol{a}_2 & — \\ & \vdots & \\ — & \boldsymbol{a}_n & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & \dots & \boldsymbol{b}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 & \boldsymbol{a}_1\boldsymbol{b}_2 & \dots & \boldsymbol{a}_1\boldsymbol{b}_n \\ \boldsymbol{a}_2\boldsymbol{b}_1 & \boldsymbol{a}_2\boldsymbol{b}_2 & \dots & \boldsymbol{a}_2\boldsymbol{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a}_n\boldsymbol{b}_1 & \boldsymbol{a}_n\boldsymbol{b}_2 & \dots & \boldsymbol{a}_n\boldsymbol{b}_n \end{bmatrix}$$

different interpretations; typically (but not always):

- "columns of the product $=$ linear combinations of the columns of A" more useful when doing control
- "elements of the product $=$ angles between the rows of A and columns of B" more useful when doing data science

?

How to change between bases, and why

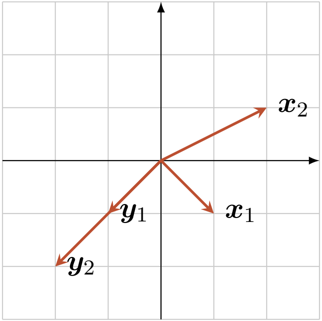
Roadmap

- what is a basis?
- what happens when there are two bases?
- how do I change between the two bases?

Linear independency

$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n$ are said to be *linearly independent* if and only if

$$\sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i = \mathbf{0} \quad \Leftrightarrow \quad \lambda_1 = \ldots = \lambda_m = 0$$



Additional basic definitions

$$\operatorname{span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle = \text{set of all the linear combinations of } \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$$

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dimension of a space: max. number of linearly independent vectors in that space

Basis of a vector space

Definition (basis)

$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^n$ *form a basis for* $\mathbb{R}^n$ *if they are linearly independent vectors*

Basis of a vector space

Definition (basis)

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Definition (basis of a subspace $\mathcal{B}$)

$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathcal{B} \subset \mathbb{R}^n, m < n$ *are a basis for* $\mathcal{B}$ *if they are linearly independent*

Basis of a vector space

Definition (basis)

$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^n$ *form a basis for* $\mathbb{R}^n$ *if they are linearly independent vectors*

Definition (basis of a subspace $\mathcal{B}$)

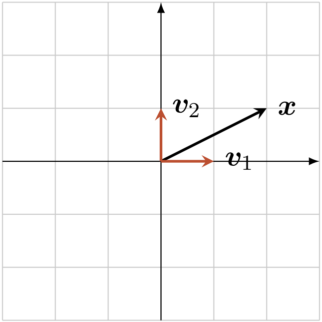
$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathcal{B} \subset \mathbb{R}^n, m < n$ *are a basis for* $\mathcal{B}$ *if they are linearly independent*

important point: they must be as many as there are dimensions in the vectors space we are looking for a basis

How to use a basis

if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ basis of $\mathbb{R}^n$ then

$$\forall \boldsymbol{x} \in \mathbb{R}^n \quad \exists \lambda_1, \ldots, \lambda_n \in \mathbb{R} \quad \text{s.t.} \quad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$



How to use a basis

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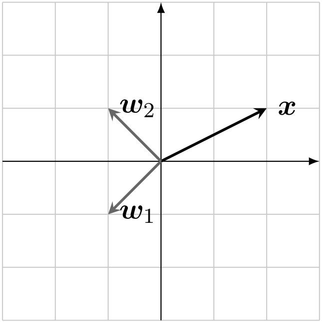
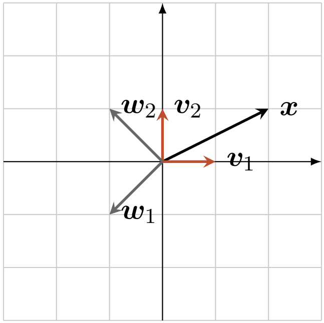
$$\forall \boldsymbol{x} \in \mathbb{R}^n \quad \exists \lambda_1, \ldots, \lambda_n \in \mathbb{R} \quad \text{s.t.} \quad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

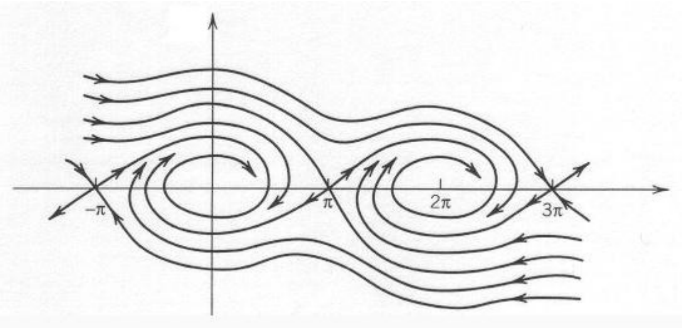


Important message: $\boldsymbol{x}$ is the same object, independently of the basis.

Thus we must be able to "change" between the coordinate systems!



Changing between bases - physical intuitions



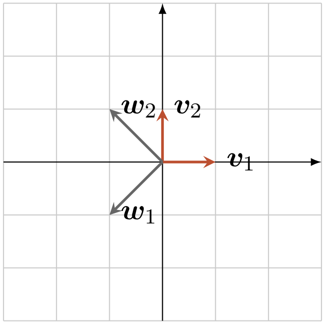
the system is the same system, even if I decide to measure things in a different way

Changing between bases

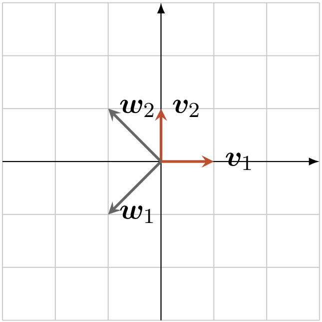
if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ are two separate bases of $\mathbb{R}^n$ then

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 \ \boldsymbol{v}_2 \cdots \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{w}_1 \ \boldsymbol{w}_2 \cdots \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}$$

Changing between bases

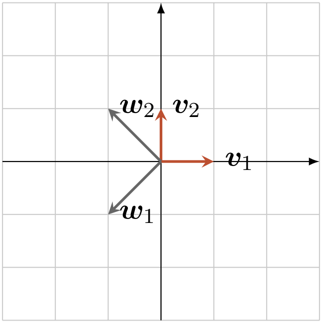


Changing between bases



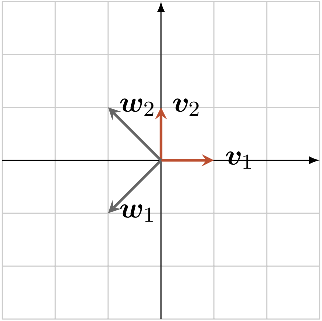
$$\boldsymbol{v}_1 = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Changing between bases



$$\boldsymbol{v}_1 = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_{1\to 1} \\ \gamma_{1\to 2} \\ \vdots \\ \gamma_{1\to n} \end{bmatrix}$$

Changing between bases



$$\boldsymbol{v}_1 = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_{1\to 1} \\ \gamma_{1\to 2} \\ \vdots \\ \gamma_{1\to n} \end{bmatrix} \implies \boldsymbol{v}_m = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_{m\to 1} \\ \gamma_{m\to 2} \\ \vdots \\ \gamma_{m\to n} \end{bmatrix}$$

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$$\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n\end{bmatrix}\begin{bmatrix}\gamma_{1\to1} & \gamma_{2\to1} & \cdots & \gamma_{n\to1}\\\gamma_{1\to2} & \gamma_{2\to2} & \cdots & \gamma_{n\to2}\\\vdots & \vdots & & \vdots\\\gamma_{1\to n} & \gamma_{2\to n} & \cdots & \gamma_{n\to n}\end{bmatrix}$$

or, compactly, $V = W\Gamma_{v\to w}$

Change of basis

$$V = W\Gamma_{v\to w}$$

Change of basis

$$V = W\Gamma_{v\to w} \quad W = V\Gamma_{w\to v}$$

Change of basis

$$V = W\Gamma_{v\to w} \quad W = V\Gamma_{w\to v} \qquad \Longrightarrow \qquad V = V\Gamma_{w\to v}\Gamma_{v\to w}$$

Change of basis

$$V = W\Gamma_{v\to w} \quad W = V\Gamma_{w\to v} \quad \implies \quad V = V\Gamma_{w\to v}\Gamma_{v\to w} \quad \implies \quad \Gamma_{w\to v} = \Gamma_{v\to w}^{-1}$$

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$$V = W\Gamma_{v\to w} \quad W = V\Gamma_{w\to v} \quad \Longrightarrow \quad V = V\Gamma_{w\to v}\Gamma_{v\to w} \quad \Longrightarrow \quad \Gamma_{w\to v} = \Gamma_{v\to w}^{-1}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

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$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = V\boldsymbol{\lambda}$$

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$$V = W\Gamma_{v\to w} \quad W = V\Gamma_{w\to v} \quad \Longrightarrow \quad V = V\Gamma_{w\to v}\Gamma_{v\to w} \quad \Longrightarrow \quad \Gamma_{w\to v} = \Gamma_{v\to w}^{-1}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 \; \boldsymbol{v}_2 \; \cdots \; \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = V\boldsymbol{\lambda} = W\Gamma_{v\to w}\boldsymbol{\lambda}$$

Change of basis

$$V = W\Gamma_{v\to w} \quad W = V\Gamma_{w\to v} \quad \Longrightarrow \quad V = V\Gamma_{w\to v}\Gamma_{v\to w} \quad \Longrightarrow \quad \Gamma_{w\to v} = \Gamma_{v\to w}^{-1}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \cdots \ \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = V\boldsymbol{\lambda} = W\Gamma_{v\to w}\boldsymbol{\lambda} = W\boldsymbol{\lambda}'$$

Exercise: change the basis of $\boldsymbol{x}$ from V to W

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{w}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \boldsymbol{w}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \boldsymbol{x} = V \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

hint: remember that $\boldsymbol{v}_m = \begin{bmatrix} \boldsymbol{w}_1 \; \boldsymbol{w}_2 \cdots \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_{m\to 1} \\ \gamma_{m\to 2} \\ \vdots \\ \gamma_{m\to n} \end{bmatrix}$ and try to form $\begin{bmatrix} \boldsymbol{v}_1 \; \boldsymbol{v}_2 \cdots \boldsymbol{v}_n \end{bmatrix}$

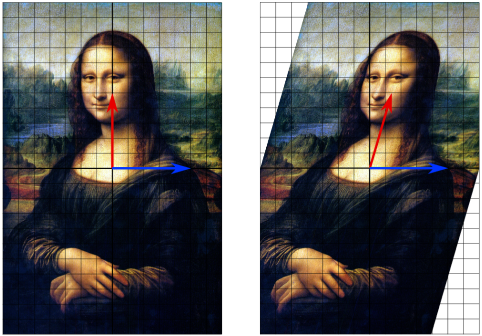
?

Linear transformations

Roadmap

- linear transformations as matrices
- the difference between “linear transformation” and “matrix”
- the effect of changing bases

Linear transformations and matrices



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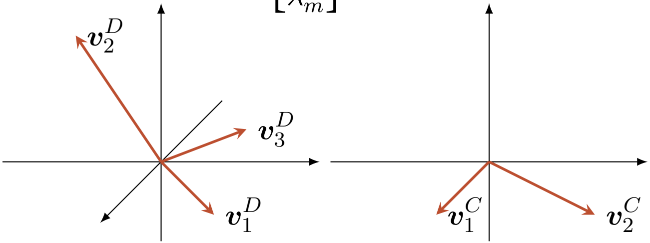
linear transformation $\mathcal{A} \neq$ matrix A

How can I express a linear transformation as a matrix?

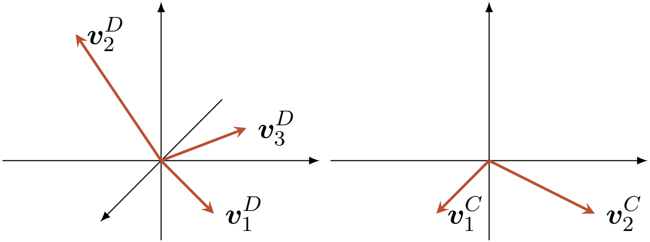
i.e., knowing $\mathcal{A} : D \mapsto C$ $\quad D = \mathbb{R}^m = \langle \boldsymbol{v}_1^D, \ldots, \boldsymbol{v}_m^D \rangle$ $\quad C = \mathbb{R}^n = \langle \boldsymbol{v}_1^C, \ldots, \boldsymbol{v}_n^C \rangle$

how to go from $\boldsymbol{x} = \left[\boldsymbol{v}_1^D \; \boldsymbol{v}_2^D \cdots \boldsymbol{v}_m^D\right] \begin{bmatrix} \lambda_1^D \\ \lambda_2^D \\ \vdots \\ \lambda_m^D \end{bmatrix}$ to $\boldsymbol{y} = \mathcal{A}\boldsymbol{x} = \left[\boldsymbol{v}_1^C \; \boldsymbol{v}_2^C \cdots \boldsymbol{v}_n^C\right] \begin{bmatrix} \lambda_1^C \\ \lambda_2^C \\ \vdots \\ \lambda_n^C \end{bmatrix}$?



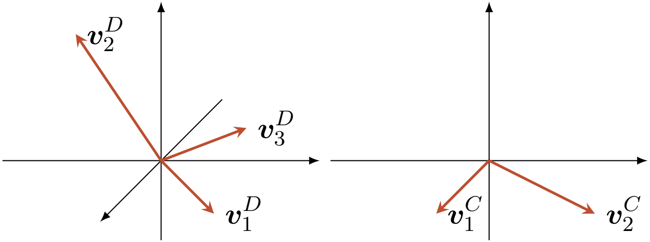
How can I express a linear transformation as a matrix?

$$\mathcal{A}\boldsymbol{v}_1^D = \left[\boldsymbol{v}_1^C \ \boldsymbol{v}_2^C \ \cdots \ \boldsymbol{v}_n^C\right] \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix}$$



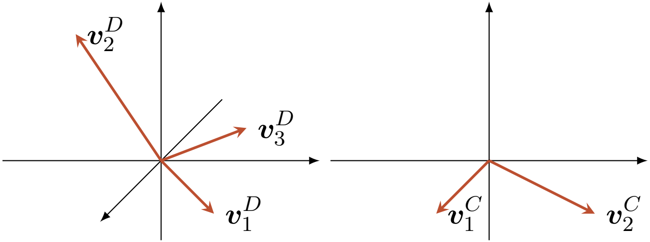
How can I express a linear transformation as a matrix?

$$\mathcal{A}\boldsymbol{v}_2^D = \begin{bmatrix}\boldsymbol{v}_1^C & \boldsymbol{v}_2^C & \cdots & \boldsymbol{v}_n^C\end{bmatrix}\begin{bmatrix}a_{21}\\ a_{22}\\ \vdots \\ a_{2n}\end{bmatrix}$$



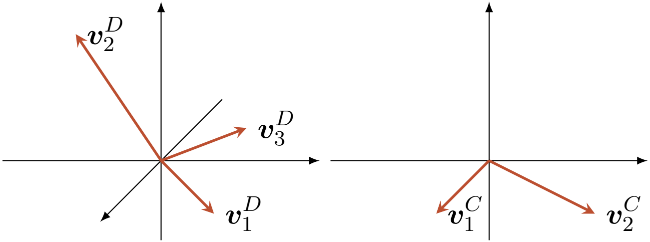
How can I express a linear transformation as a matrix?

$$\mathcal{A}\boldsymbol{v}_3^D = \begin{bmatrix} \boldsymbol{v}_1^C & \boldsymbol{v}_2^C & \cdots & \boldsymbol{v}_n^C \end{bmatrix} \begin{bmatrix} a_{31} \\ a_{32} \\ \vdots \\ a_{3n} \end{bmatrix}$$



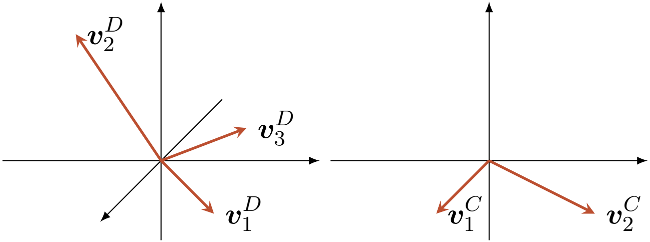
How can I express a linear transformation as a matrix?

$$\left[\mathcal{A}\boldsymbol{v}_1^D \; \ldots \; \mathcal{A}\boldsymbol{v}_m^D\right] = \left[\boldsymbol{v}_1^C \; \boldsymbol{v}_2^C \; \cdots \; \boldsymbol{v}_n^C\right] \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ a_{12} & \cdots & a_{m2} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}$$



How can I express a linear transformation as a matrix?

$$\mathcal{A}\boldsymbol{x} = \begin{bmatrix} \mathcal{A}\boldsymbol{v}_1^D & \mathcal{A}\boldsymbol{v}_2^D & \cdots & \mathcal{A}\boldsymbol{v}_m^D \end{bmatrix} \begin{bmatrix} \lambda_1^D \\ \lambda_2^D \\ \vdots \\ \lambda_m^D \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1^C & \boldsymbol{v}_2^C & \cdots & \boldsymbol{v}_n^C \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ a_{12} & \cdots & a_{m2} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \lambda_1^D \\ \lambda_2^D \\ \vdots \\ \lambda_m^D \end{bmatrix}$$

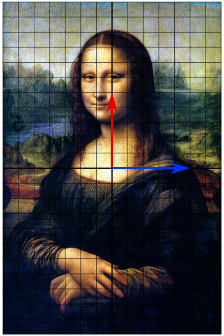


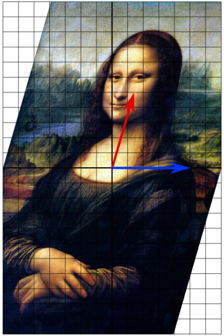
Summary

$$\mathcal{A}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1^C & \boldsymbol{v}_2^C & \cdots & \boldsymbol{v}_n^C \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ a_{12} & \cdots & a_{m2} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \lambda_1^D \\ \lambda_2^D \\ \vdots \\ \lambda_m^D \end{bmatrix} \qquad \Longrightarrow \qquad \text{"}A\boldsymbol{x} \mapsto \boldsymbol{y}\text{"}$$

i.e., to go from $\boldsymbol{x}$ to $\boldsymbol{y}$ start from the coordinates of $\boldsymbol{x}$ in the basis of the domain, transform the coordinates through the matrix A transforming the basis in the domain into the basis of the codomain, and consider the new coordinates $\boldsymbol{y}$ as expressed in the basis of the codomain

Linear transformations and matrices





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the transformation is defined by $\mathcal{A}$, not by A

And what about *square* matrices?

if $\mathcal{A} : \mathbb{R}^n \mapsto \mathbb{R}^n \quad \Longrightarrow \quad C = D$ we can choose the same basis, i.e.,

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solution: $$\mathcal{A}\boldsymbol{x} = \left[\boldsymbol{v}_1 \ \boldsymbol{v}_2 \cdots \boldsymbol{v}_n \right] \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ a_{12} & \cdots & a_{n2} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \quad \Longrightarrow \quad \text{"} A\boldsymbol{x} \mapsto \boldsymbol{y} \text{"}$$

same concepts as before, just that both $\boldsymbol{x}$ and $\boldsymbol{y}$ are expressed in the the same basis, so that A expresses how the elements *of the given basis* are transformed

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how do A *and* A' *relate?*

Changes of bases (summary)

$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ bases of $\mathbb{R}^n$ $\Longrightarrow$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}$$

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Effects of changing bases on the representations of $\mathcal{A}$

$$\begin{bmatrix}\mathcal{A}\boldsymbol{v}_1 & \ldots & \mathcal{A}\boldsymbol{v}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{v}_1 \cdots \boldsymbol{v}_n\end{bmatrix}\begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn}\end{bmatrix} = VA$$

$$\begin{bmatrix}\mathcal{A}\boldsymbol{w}_1 & \ldots & \mathcal{A}\boldsymbol{w}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{w}_1 \cdots \boldsymbol{w}_n\end{bmatrix}\begin{bmatrix} a'_{11} & \cdots & a'_{n1} \\ \vdots & & \vdots \\ a'_{1n} & \cdots & a'_{nn}\end{bmatrix} = WA'$$

Effects of changing bases on the representations of $\mathcal{A}$

$$V = W\Gamma_{v\to w} \qquad W = V\Gamma_{w\to v}$$

$$\Downarrow$$

$$A' = \Gamma_{v\to w} A \Gamma_{w\to v}$$

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Convenient notation:

$$\Gamma_{v\to w} = T \qquad A' = TAT^{-1}$$

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Convenient notation:

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> operations of the type $A' = TAT^{-1}$ with T full-rank mean changing the basis, i.e., "looking at the linear transformation from a different perspective" (more precisely, the perspective defined by the columns of T)

?

The spaces associated to a matrix

Roadmap

- rank and range
- determinants
- kernel
- connections among the various concepts

Recall:

$\mathrm{span}\,(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle$ = set of all the linear combinations of these vectors

$\mathrm{range}\,(A)$ = span of the columns of A

dimension of a space: max. number of linearly independent vectors

Just to make the importance of the concepts clear:

when does this system have a solution?

$$A\boldsymbol{x} = \begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \boldsymbol{y}$$

(Column) Rank of a matrix

$$\text{rank}\,(A) = \text{rank}\left(\begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix}\right) = \text{ number of linearly independent columns}$$

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Important result: column-rank = row-rank *(i.e., there are as many linearly independent rows as linearly independent columns)*

$$\operatorname{rank}(A) = \operatorname{rank}\left(A^\top\right) = \operatorname{rank}\left(A^\top A\right) = \operatorname{rank}\left(AA^\top\right)$$

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$$\text{rank}\,(A) = \text{rank}\left(A^\top\right) = \text{rank}\left(A^\top A\right) = \text{rank}\left(AA^\top\right)$$

Example: what is the maximal rank of $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$?

Reconnecting with automatic control

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

$\implies$ structure of A determines how the time derivative $\dot{\boldsymbol{x}}$ is, and how the time derivative is determines the stability and time-evolution properties of the system.

Reconnecting with automatic control

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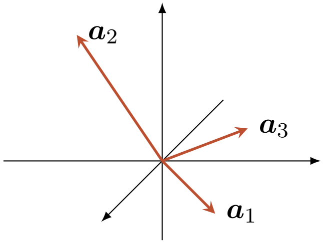
$\Longrightarrow$ structure of A determines how the time derivative $\dot{\boldsymbol{x}}$ is, and how the time derivative is determines the stability and time-evolution properties of the system. E.g.,

$$\mathrm{span}\,(A) = \begin{bmatrix} +1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad \text{if } x_1 \text{ grows then } x_2 \text{ diminishes, and viceversa}$$

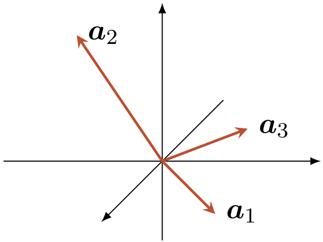
Determinant of a square matrix

$$\det(A) = \det\left(\begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix}\right) = $$ (signed) volume of the parallelepiped defined by $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$



Determinant of a square matrix

Remember: $\begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix}$ represent where the elements of the basis are mapped into



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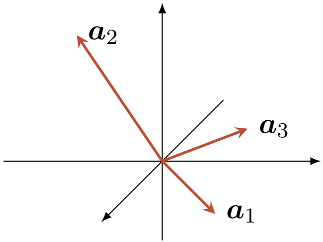

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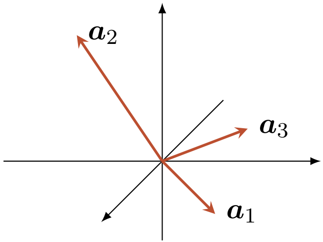
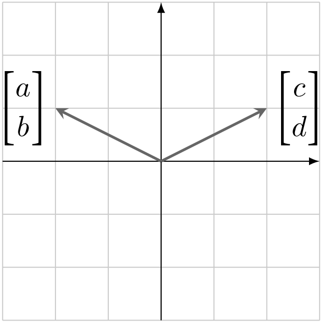


the determinant is a property of the linear transformation $\mathcal{A}$*,*
thus if T *is a change of basis then* $det(A) = det\left(TAT^{-1}\right)$*,*
since changing the basis does not change the underlying transformation

Likely the unique (other) case you should remember on how to compute determinants

$$\det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = ad - bc$$



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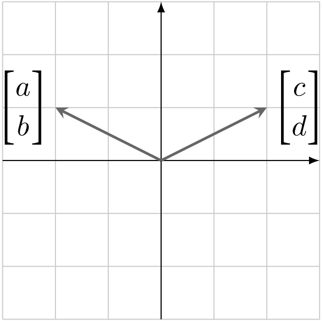

Which other case do you know? (you know for sure already one more)

Determinants and invertibility of linear maps

Immediate implications:

$$\det(A) \neq 0 \quad \Leftrightarrow \quad \mathcal{A} \text{ invertible}$$

$$\det(A) = 0 \quad \Leftrightarrow \quad \mathcal{A} \text{ not-invertible}$$

Why is invertibility important?

because if you want to solve $A\boldsymbol{x} = b$ for generic b then you need A^{-1}

Connections between the determinant and the rank of a square matrix

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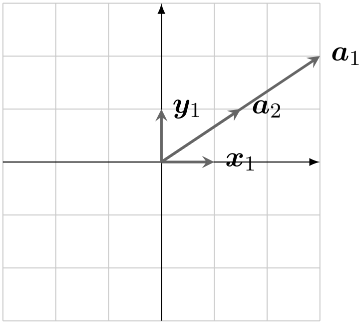
What does it mean that the columns are linearly dependent?



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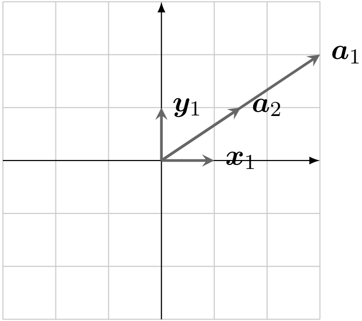
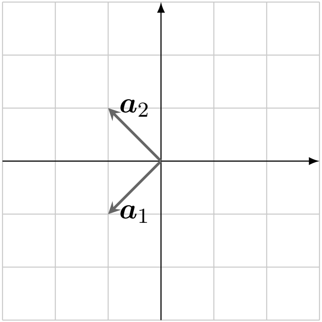


indeed, in this case we cannot "un-map"...

Summary until now and examples: $A \in \mathbb{R}^{2\times 2}$



determinant $=$ *area* spanned by the columns of A

- if $\operatorname{rank}(A) = 2$ then the column vectors span an area
- if $\operatorname{rank}(A) = 1$ then the column vectors span a line
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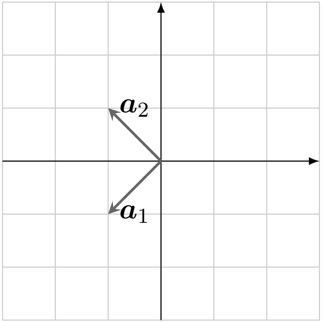

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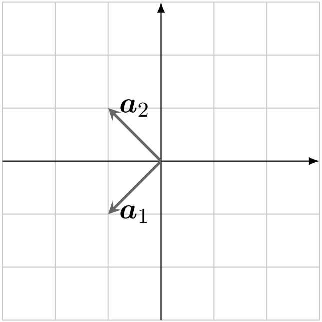

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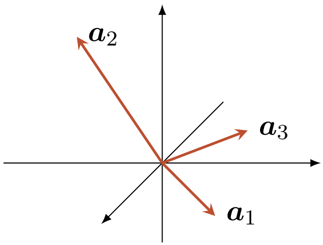
Summary until now and examples: $A \in \mathbb{R}^{3\times 3}$



determinant $=$ *volume* spanned by the columns of A

- if $\mathrm{rank}\,(A) = 3$ then the column vectors span a volume
- if $\mathrm{rank}\,(A) = 2$ then the column vectors span an area
- if $\mathrm{rank}\,(A) = 1$ then the column vectors span a line
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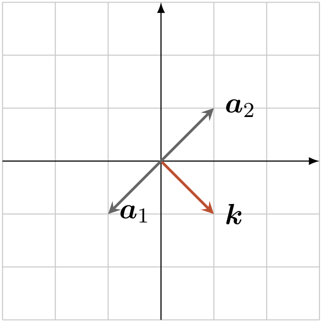
?

Kernel (or null-space) of a matrix $A \in \mathbb{R}^{n \times m}$

$$\ker(A) = \{\boldsymbol{x} \in \mathbb{R}^m \text{ s.t. } A\boldsymbol{x} = \boldsymbol{0}\}$$

Example 1: $\ker(A) = \operatorname{span}(\boldsymbol{k})$ with $\boldsymbol{k} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



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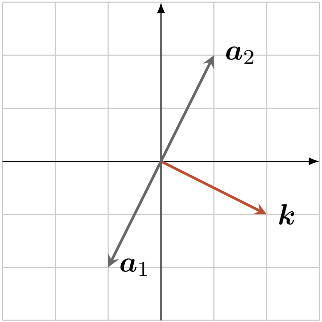
Example 2: $\ker(A) = \operatorname{span}(\boldsymbol{k})$ with $\boldsymbol{k} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$



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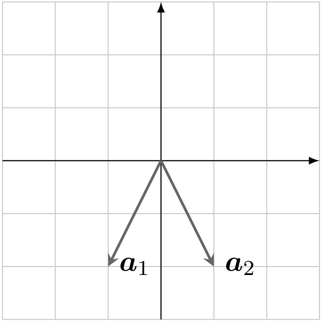
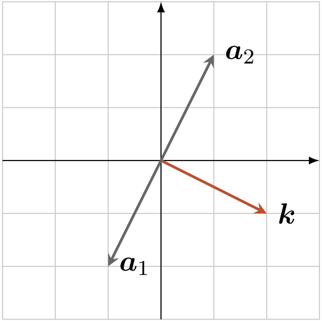
Example 3: $\ker(A) = \{\mathbf{0}\}$



Extremely important result: $\ker(A) \perp \operatorname{range}(A)$



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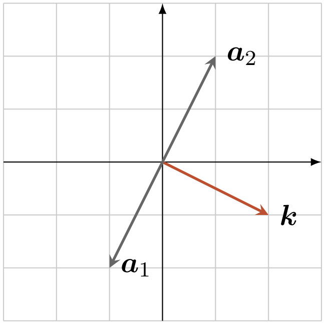


$\implies \operatorname{rank}(A) + \dim(\ker(A))$ = number of columns of A

Alternative viewpoint on the kernel of A

$$A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_m \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{a}_1 \boldsymbol{x} \\ \vdots \\ \boldsymbol{a}_m \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{if and only if} \quad \boldsymbol{a}_i \perp \boldsymbol{x} \ \forall i$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

- $\ker(A) = ?$
- $\text{range}(A) = ?$

bigger matrices, and needing to compute ranges, determinants, or kernels?

$\rightarrow$ use Matlab, python, Wolfram Alpha, whatever

Some useful general rules

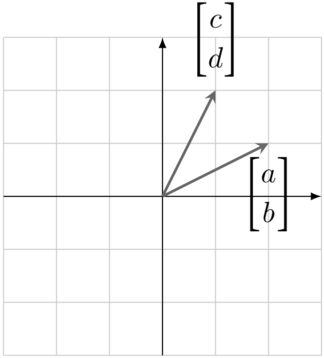
$$
\begin{aligned}
\left(A^\top\right)^\top &= A \\
(A+B)^\top &= A^\top + B^\top \\
(cA)^\top &= cA^\top \\
(AB)^\top &= B^\top A^\top \\
\det\left(A^\top\right) &= \det(A) \\
\left(A^{-1}\right)^\top &= \left(A^\top\right)^{-1}
\end{aligned}
$$

Eigenvectors, eigenspaces, and eigenvalues of a square matrix

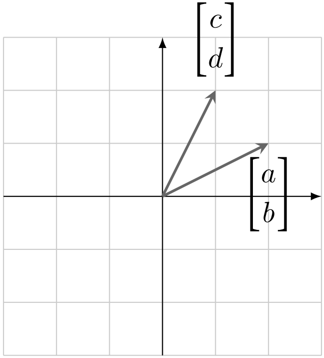
Roadmap

- eigenvectors
- eigenspaces
- eigenvalues
- connections with ranks and determinants

Eigenvectors



Eigenvectors



are there some directions that get only stretched, i.e., that do not rotate?

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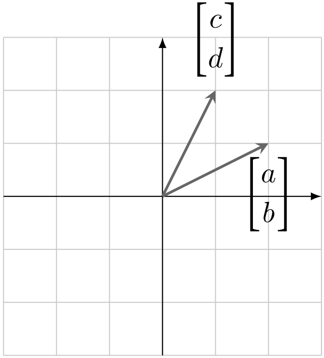
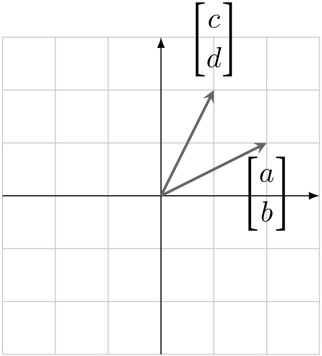


are there some directions that get only stretched, i.e., that do not rotate?

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Eigenvectors



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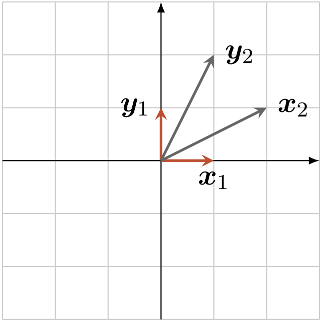
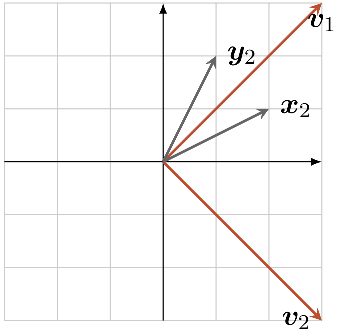
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mapsto \quad \boldsymbol{v}_{\lambda=1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvectors: sometimes you may seem them from the transformation of the hypercube



Why do we like eigenvectors?



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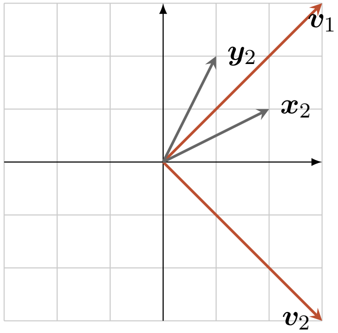

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they are connected to (and actually generalize) the modes of a LTI system

(more information in the next units)

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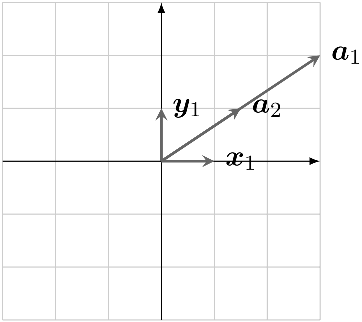
$$\lambda \text{ eigenvalue iff } \det(A - \lambda I) = 0$$

Remember: what does it mean that the determinant of A is zero?

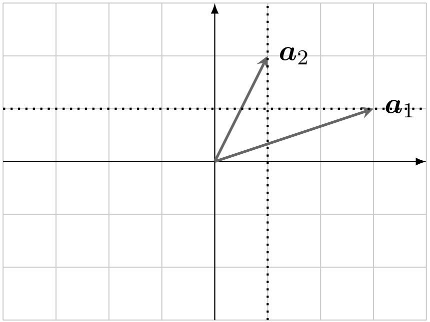
It means that the columns of A are linearly dependent, and this means



What does this mean, geometrically?

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \quad \Longrightarrow \quad (A - \lambda I) = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

question $=$ is $\det(A - \lambda I)$ zero, i.e., searching for stretchings that make *at least* 2 columns of $A - \lambda I$ align



How does one compute eigenvectors, more in general?

$$\lambda \text{ eigenvalue iff } \det(A - \lambda I) = 0$$

1. consider s as a complex variable
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but why is it a polynomial?

In brief, $\det(sI - A)$ is a polynomial because of how determinants are computed

$$\text{thus } A \in \mathbb{R}^{n \times n} \quad \Longrightarrow \quad \det(sI - A) = \prod_{i=1}^{n} (s - \lambda_i) \text{ for opportune } \exists \lambda_1, \ldots, \lambda_n \in \mathbb{C}$$

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note: this is reminiscent of the RTFs in ZPK representation $H(s) = K \dfrac{\prod_j (s - z_j)}{\prod_i (s - p_i)}$

How to find the eigenvalues: numerical example

Definition:

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \quad \Longrightarrow \quad A\boldsymbol{x} - \lambda\boldsymbol{x} = \mathbf{0} \quad \Longrightarrow \quad (A - \lambda I)\,\boldsymbol{x} = \mathbf{0}.$$

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$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \quad \Longrightarrow \quad (A - \lambda I) = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$$

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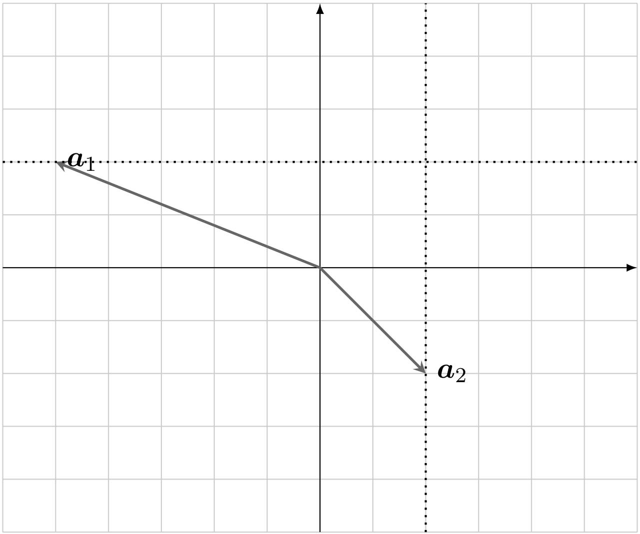
Eigenvalues $= \{-1, -6\}$

The same example, graphically



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How do we find the associated eigenvectors?

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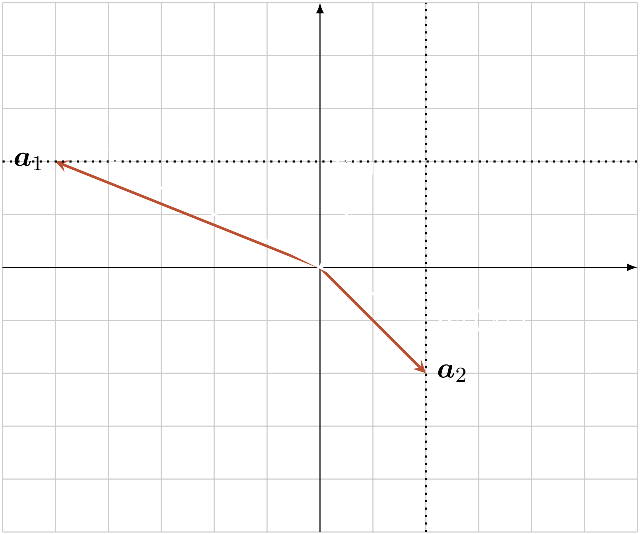
$$\implies x_2 = 2x_1$$

Again, graphically



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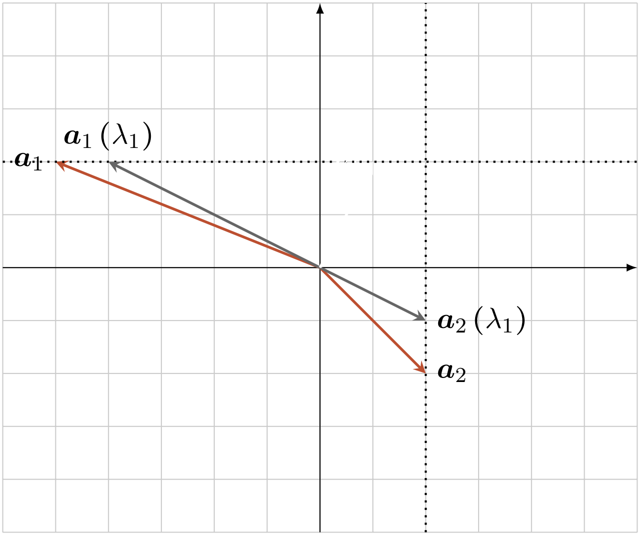

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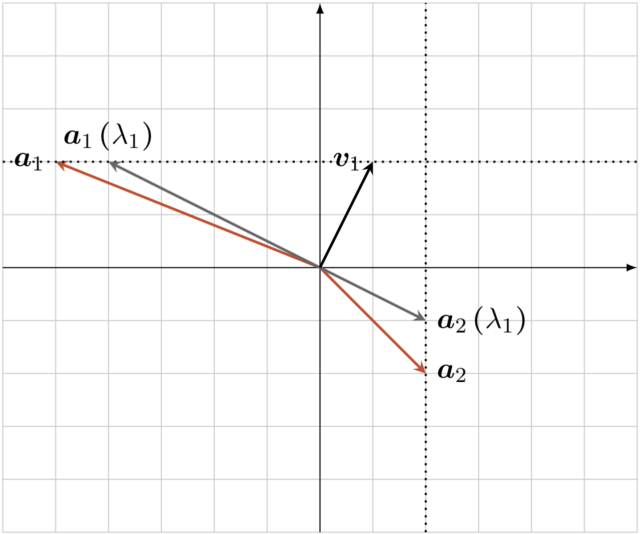

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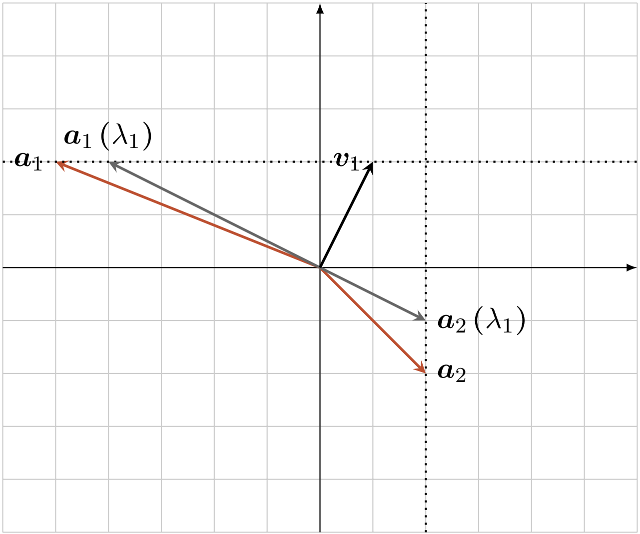

can you see what λ_2 is, and what the corresponding $\boldsymbol{v}_2$ will be?

Summarizing

eigenvectors: directions along which A does not introduce rotations

eigenspaces: set of all the vectors in these directions

eigenvalues: amplification that A causes along the eigenspaces
(remember: along each direction there may be a different amplification!)

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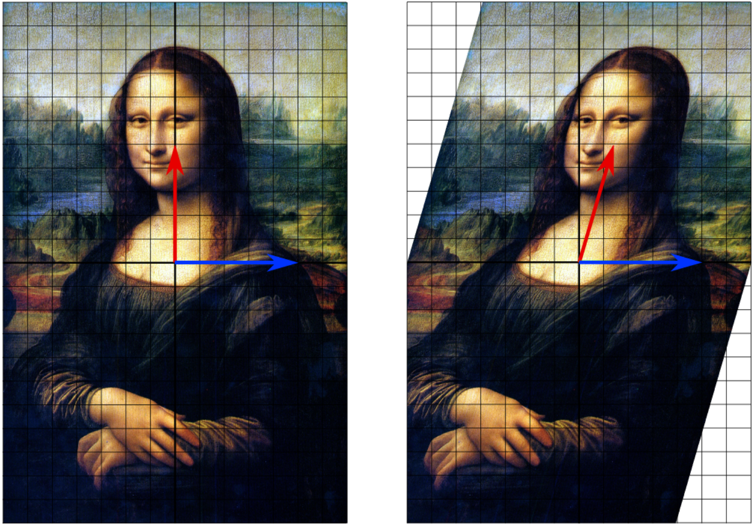
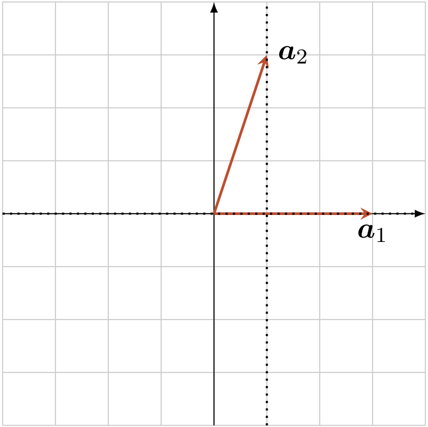
characteristic polynomial: $\det(sI - A)$

algebraic multiplicity of the eigenvalues: the power $\mu(\lambda_i')$ associated to each λ_i in the characteristic polynomial

Discussion: what are the eigenspaces in this case?



Generalizing to the case of a Jordan miniblock (something that will be an <u>extremely</u> important case)



how many "stretchings" can we find so to make the stretched columns align?

Important result

there may be fewer 1-dimensional eigenspaces than columns of A

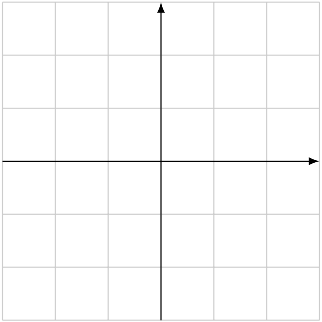
$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

?

Connections between determinant and rank

$$\det(A) = 0 \quad \Leftrightarrow \quad A \text{ rank-deficient}$$

since $\det(A) = 0$ means that the unitary hypercube gets mapped into a degenerate parallelepiped, and this is equivalent to say that A has linearly dependent columns



Connections between determinant and eigenvalues

$$\det(A) = \prod_i \lambda_i$$

i.e., the volume of the mapped parallelepiped is equal to the product of the expansions along the eigenspaces[1]

this is immedate and obvious when considering the Jordan form of A

[1] Kind of imprecise; correct in case the algebraic multiplicity of the eigenvalues is 1. Otherwise they need to be considered with their multiplicities

Connections between determinant and rank

From the previous episodes:

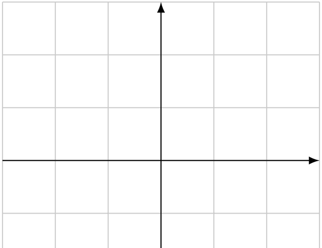
$$\det(A) = 0 \quad \Leftrightarrow \quad A \text{ rank-deficient}$$

and

$$\det(A) = \prod_i \lambda_i$$

thus

$$\exists \lambda_i = 0 \quad \Leftrightarrow \quad A \text{ rank-deficient}$$

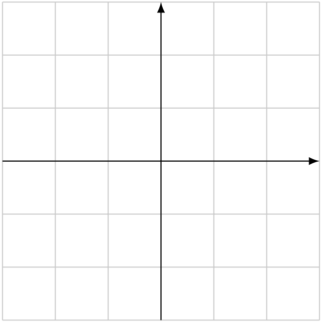


Connections between eigenvalues and invertibility

$$\exists \lambda_i = 0 \quad \Leftrightarrow \quad A \text{ rank-deficient}$$

but also

$$A \text{ rank-deficient} \quad \Leftrightarrow \quad A \text{ not invertible}$$



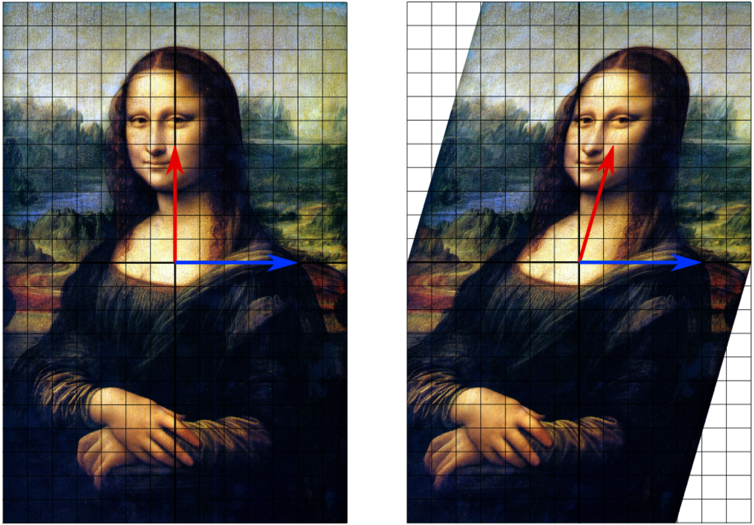
Definitions

singular matrix $=$ non-invertible matrix

non-singular matrix $=$ invertible matrix

?

Do eigenvalues change if we do a change of basis?



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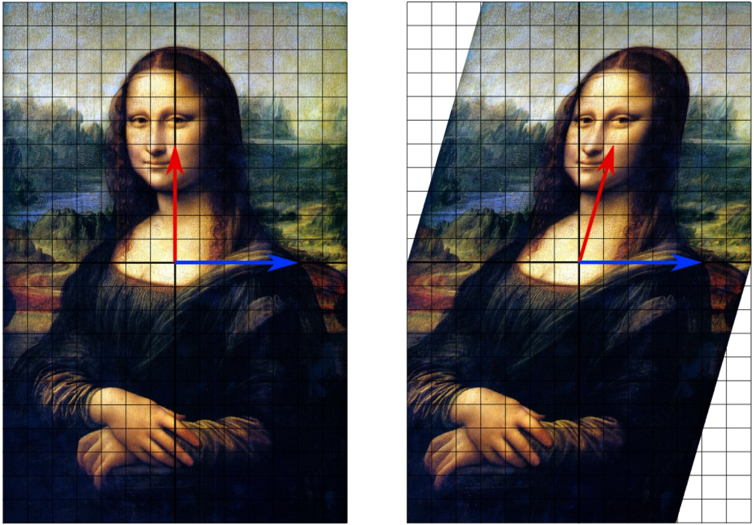

obviously not!

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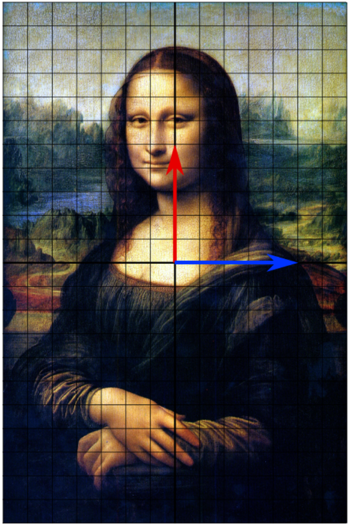

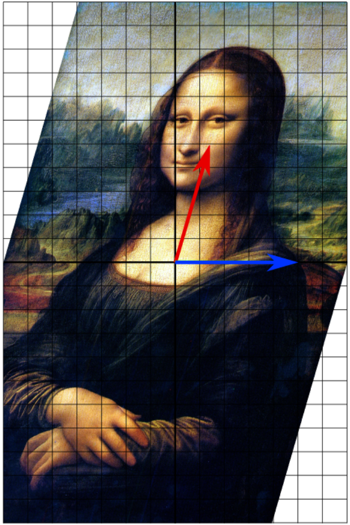

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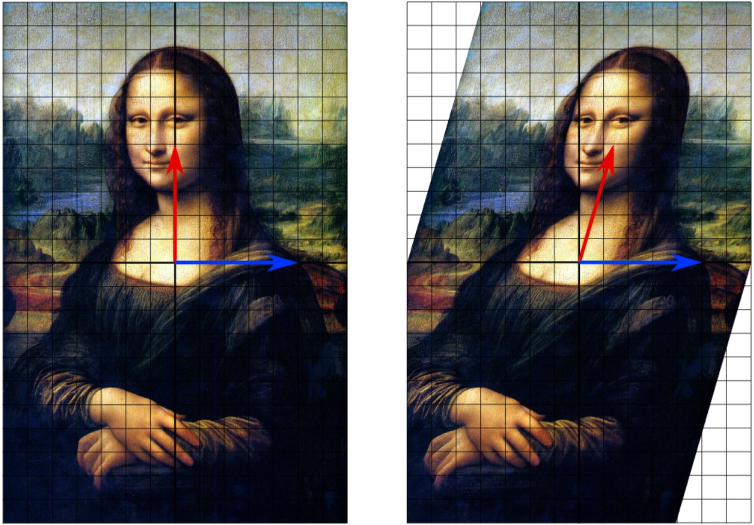

they change "name", but from a physical perspective they are the same object!

Extremely important facts!!!

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   $\ker(\lambda_i I - A) = \ker\left(\lambda_i I - TAT^{-1}\right)$

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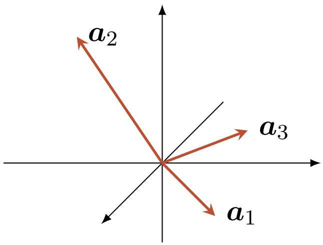
Diagonalization

Roadmap

- what happens if the eigenvectors of A form a basis of $\mathbb{R}^n$?
- what diagonalization means algebraically
- what diagonalization means geometrically
- what diagonalization means in practice

An interesting case: what if the eigenvectors of A form a basis of $\mathbb{R}^n$?



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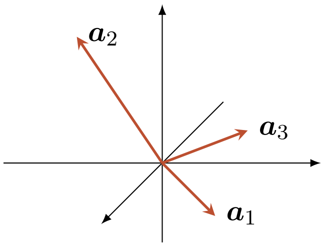

what happens if in this case I choose a new basis formed by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$*?*

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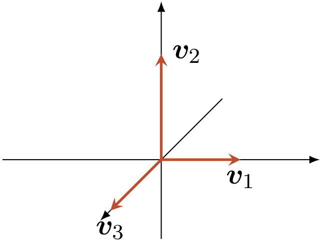

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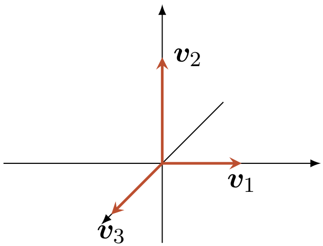

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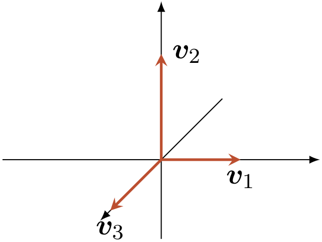


How does $\mathcal{A}$ look like, with respect to this basis?

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

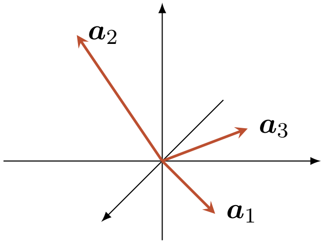
Note that the λ_i's may also be the same! Example:

$$A = \begin{bmatrix} 2.3 & & & \\ & 2.3 & & \\ & & \ddots & \\ & & & 2.3 \end{bmatrix}$$

Diagonalizing a square matrix

hypothesis: A is s.t. there exist $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ linearly independent eigenvectors

thesis: $T = \left[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\right]$ is s.t. $\Lambda = T^{-1}AT = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_n\right)$



Diagonalizing a square matrix: proof that $AT = T\Lambda$

$$AT \overset{(1)}{=} A\,[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n] \overset{(2)}{=} [A\boldsymbol{v}_1, \ldots, A\boldsymbol{v}_n] \overset{(3)}{=} [\lambda_1 \boldsymbol{v}_1, \ldots, \lambda_n \boldsymbol{v}_n] \overset{(4)}{=} T\Lambda$$

(1) recall that the columns of T are the eigenvectors

(2) this follows by the geometrical interpretation of matrix-columns multiplications

(3) this is because $\boldsymbol{v}_i$ is an eigenvector

(4) we can rewrite things as a product with a diagonal matrix

What about matrices with multiple eigenvalues?

Example:

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \quad \Longrightarrow \quad \det\left(A - sI\right) = -s^3 - s^2 + 21s + 45 = \left(s - 5\right)\left(s + 3\right)^2$$

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Eigenspaces-eigenvectors couples:

$$\left\{\lambda_1, \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right)\right\} \qquad \left\{\lambda_2 = \lambda_3, \mathrm{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right)\right\}$$

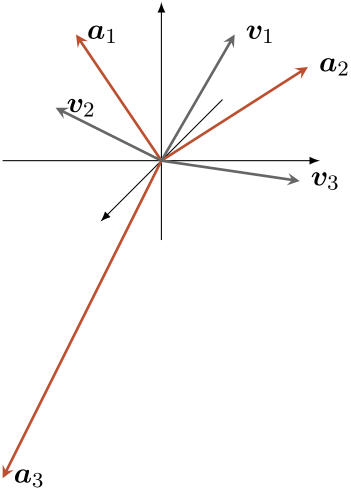
important point: to diagonalize we need n different and linearly independent eigenvectors, <u>not</u> n different eigenvalues

Graphically

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\Lambda = T^{-1}AT = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$



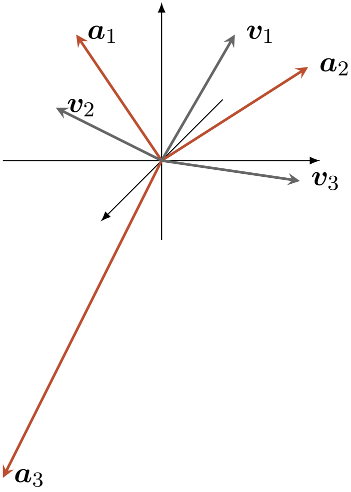
Diagonalization, in numbers

$$A = T\Lambda T^{-1}$$

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0.125 & 0.25 & -0.375 \\ -0.250 & 0.50 & 0.750 \\ 0.125 & 0.25 & 0.625 \end{bmatrix}$$

What does diagonalization mean, graphically?

I look at the world considering as the new axes the eigenspaces



What does diagonalization mean, physically?

Original system:

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The same system, but after the change of basis T:

$$\begin{bmatrix} \dot{\widetilde{x}}_1 \\ \dot{\widetilde{x}}_2 \\ \dot{\widetilde{x}}_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \end{bmatrix}$$

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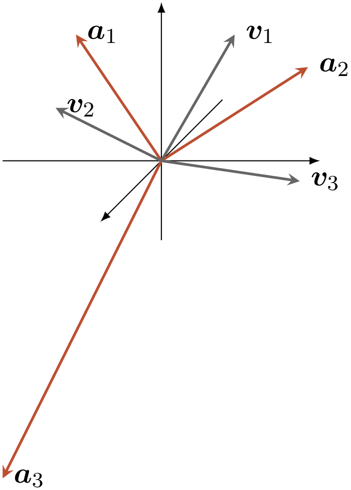
this means that the original system is actually the juxtaposition of 3 independent systems that evolve "ignoring" what is happening in the other ones

Thus diagonalizing = decomposing the dynamics in a set of independent 1-dimensional dynamics



the eigenspaces are where these 1-dimensional dynamics live

Messages of this unit:

- to be able to diagonalize means to be able to split up a system in independent pieces

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- to be able to diagonalize means to be able to split up a system in independent pieces
- however we can do this diagonalization only if the eigenvectors of A form a basis for $\mathbb{R}^n$, and this is not guaranteed in general

Generalization

Consider

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & & & \\ & \tilde{A}_2 & & \\ & & \ddots & \\ & & & \tilde{A}_k \end{bmatrix};$$

also this means "dividing the system in independent sub-systems"! However "diagonalizing" means finding independent subsystems of dimension 1, while in this general case the dimensions are potentially bigger than 1

?

Towards stranger things: recall that state space representations are ways of expressing LTI systems

$$\ddot{y} + a_1 \dot{y} + a_0 y = b u(t)$$

is equivalent to

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t) \\ y(t) & = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

and thus to

$$\begin{cases} \dot{\boldsymbol{x}} & = A \boldsymbol{x} + B u \\ y & = C \boldsymbol{x} \end{cases}$$

Towards stranger things: how was this connecting with the first part of the course?

$$\ddot{y} + a_1 \dot{y} + a_0 y = bu \quad \Longrightarrow \quad Y(s) = \frac{bU(s)}{s^2 + a_1 s + a_0} \quad \Longrightarrow \quad \text{modes} = \text{solutions of } s^2 + a_1 s + a_0$$

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$$s\boldsymbol{X}(s) - A\boldsymbol{X}(s) = BU(s)$$

$$\begin{cases} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + Bu \\ y &= C \end{cases} \quad \Longrightarrow$$

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$$sX(s) - AX(s) = BU(s)$$
$$(sI - A)X(s) = BU(s)$$

$$\begin{cases} \dot{\boldsymbol{x}} & = A\boldsymbol{x} + Bu \\ y & = C \end{cases} \quad \Longrightarrow$$

Towards stranger things: how was this connecting with the first part of the course?

$$\ddot{y} + a_1 \dot{y} + a_0 y = bu \quad \Longrightarrow \quad Y(s) = \frac{bU(s)}{s^2 + a_1 s + a_0} \quad \Longrightarrow \quad \text{modes} = \text{solutions of } s^2 + a_1 s + a_0$$

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with the denominator of the TF equal to $\det(sI - A)$

Towards stranger things: remember this basic fact

$$Y(s) = C\frac{\operatorname{adj}(sI - A)}{\det(sI - A)}BU(s)$$

- changing the basis does not change the characteristic polynomial, thus

$$\det(sI - A) = \det\left(sI - T^{-1}AT\right)$$

(in other words, changing the basis for the state space does not change the poles of the TF, and thus the modes of the LTI system – as it should obviously be)

Stranger things

...but if $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ implies $\det(A - sI) = (s-5)(s+3)^2$ then there is a double pole in -3, corresponding to a mode of the type te^{-3t};

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Solution (and we will see this in the next unit): the presence or not of the mode te^{-3t} depends on the structure of the eigenspaces of A → we need to study Jordan forms *doing systems theory for LTI systems means studying the inner structure of* $\dot{\boldsymbol{x}} = A\boldsymbol{x}$

?

Jordan forms

Roadmap

- non-diagonalizable matrices
- Jordan forms
- connections with dynamical systems
- summary of the differences between diagonalizable and non-diagonalizable matrices

A small trick, to make things faster

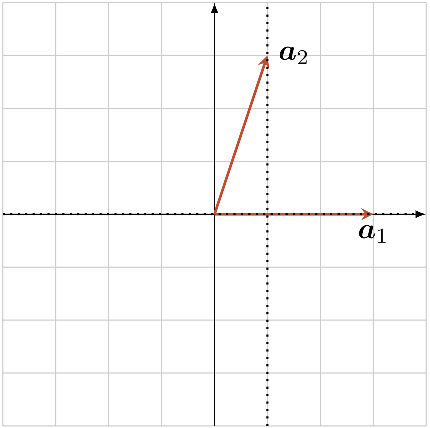
if A is upper triangular or lower triangular then its characteristic polynomial is given by $\prod (s - d_i)$ with the d_i's the elements on the diagonal, i.e.,

$$A = \begin{bmatrix} d_1 & \star & \star & \cdots \\ 0 & d_2 & \star & \cdots \\ 0 & 0 & \ddots & \cdots \\ 0 & 0 & 0 & d_n \end{bmatrix} \implies \det(sI - A) = \prod_i (s - d_i)$$

The case of Jordan miniblocks

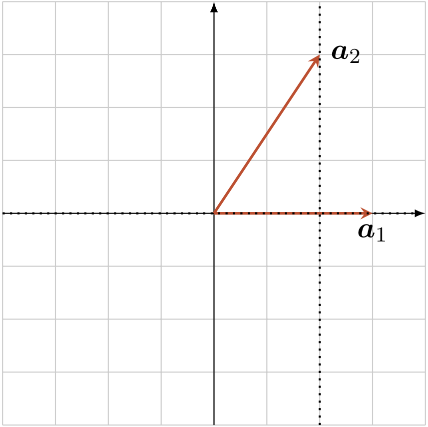
$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \Longrightarrow \quad \text{characteristic polynomial} = (s-\lambda)^2$$

How many 1-dimensional eigenspaces do Jordan miniblocks have?



in this case there is only one “stretching” for which the stretched columns align

Note that this can be generalized to Jordan miniblocks with α instead of 1



(we though like more to write Jordan miniblocks as $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$*)*

"The eigenspaces of Jordan miniblocks have dimension 1": algebraic proof

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad \Longrightarrow \quad \det(sI - A) = (s - \lambda)^3$$

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remember: $\boldsymbol{x} \neq \mathbf{0}$ is an eigenvector if $(\lambda I - A)\,\boldsymbol{x} = \mathbf{0}$

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$$\text{here } (\lambda I - A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

"The eigenspaces of Jordan miniblocks have dimension 1": algebraic proof

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad \Longrightarrow \quad \det\left(sI - A\right) = \left(s - \lambda\right)^3$$

remember: $\boldsymbol{x} \neq \mathbf{0}$ is an eigenvector if $\left(\lambda I - A\right)\boldsymbol{x} = \mathbf{0}$

$$\text{here } \left(\lambda I - A\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{and thus } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \star \\ 0 \\ 0 \end{bmatrix}$$

Summarizing

$$A = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda \end{bmatrix}$$

- the eigenspace is 1-dimensional and it is equal to $\ker(\lambda I - A) = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$

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- thus we cannot find a basis of $\mathbb{R}^n$ composed by eigenvectors

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- thus we cannot diagonalize, i.e., we cannot write $A = T \Lambda T^{-1}$

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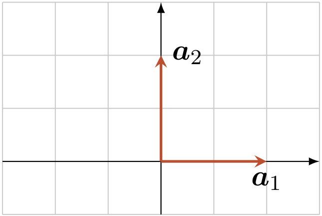
- thus we cannot find a basis of $\mathbb{R}^n$ composed by eigenvectors
- thus we cannot diagonalize, i.e., we cannot write $A = T \Lambda T^{-1}$
- thus the system $\dot{\boldsymbol{y}} = A \boldsymbol{y}$ cannot be divided into a series of independent 1-dimensional dynamics

?

An example, to make things in practice. System "N":

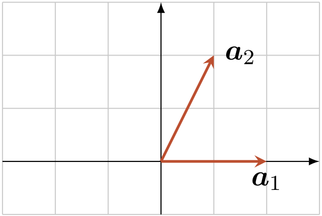
$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \\ y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases} \implies \dim\Big(\ker(2I - A) \Big) = 2$$



$\implies$ two independent 1-dimensional systems, each with a mode e^{2t}

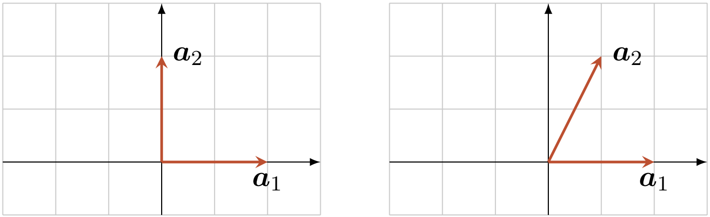
An example, to make things in practice. System "J":

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \\ y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases} \implies \dim\Big(\ker\left(2I - A\right) \Big) = 1$$



$\implies$ a truly 2-dimensional system, with modes e^{2t} and te^{2t}

Comparing “N” against “J”:



“J” contains an intrinsic shearing that “N” does not contain

(but remember that for the case “N” we are looking at the space through the directions defined by its eigenvectors)

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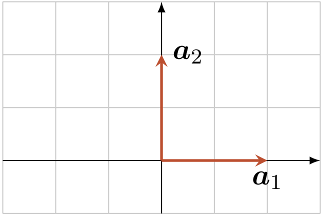

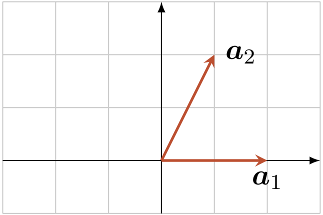

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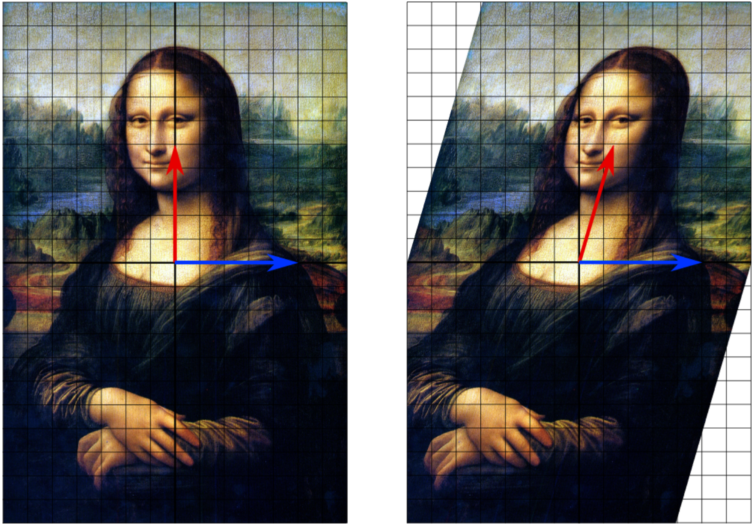
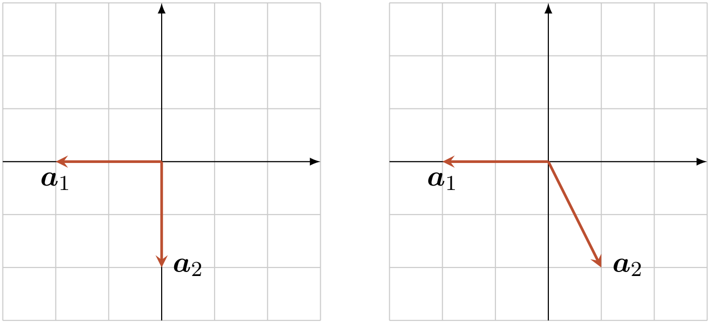
(but remember that for the case “N” we are looking at the space through the directions defined by its eigenvectors)

the same applies to $J = \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$ or the higher-dimensions cases

Discussion: is this due to a Jordan map?



Watch out that to have asymptotic stability the eigenvalues must have real part strictly negative!



?

Summarizing

$$\det(sI - A) = \prod_{i=1}^{d} (s - \lambda_i)^{\mu(\lambda_i)}$$

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$$\det(sI - A) = \prod_{i=1}^{d} (s - \lambda_i)^{\mu(\lambda_i)}$$

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- $\det(sI - A)$:= characteristic polynomial
- $\mu(\lambda_i)$:= algebraic multiplicity of λ_i
- $\ker(\lambda_i I - A)$:= eigenspace associated to λ_i
- $\dim\Big(\ker(\lambda_i I - A)\Big)$:= *geometric multiplicity* of λ_i

$$\boldsymbol{v} \neq \mathbf{0}, \qquad A\boldsymbol{v} = \lambda \boldsymbol{v}$$

$$\dim\Big(\ker\left(\lambda_i I - A\right) \Big)$$

$$\det\left(sI - A\right) = \prod_{i=1}^{d} \left(\lambda - \lambda_i'\right)^{\mu\left(\lambda_i'\right)}$$

$$\mu\left(\lambda_i'\right)$$

our aim: understand how these components relate

$\Longrightarrow$ need to go back to the geometric interpretations

(but, before, we need a couple of theoretical results)

Definition (diagonalizable matrix)

A is diagonalizable if $\exists T$ s.t. $T^{-1}AT = \Lambda$ with Λ diagonal

Theorem

A is diagonalizable if and only if A has n linearly independent eigenvectors

Theorem

not all the A's are diagonalizable; e.g., Jordan matrices are not

Theorem (Jordan canonical form)

all the matrices that can not be diagonalized can always be transformed, by using an opportune change of coordinates, to a block diagonal matrix

$$A = \begin{bmatrix} A_1 & & 0 \\ & \dots & \\ 0 & & A_{n'} \end{bmatrix}$$

with $n' < n$ *and at least one block* A_i *of the form*

$$A_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

Example

$$A \quad \mapsto \quad \widetilde{A} = \begin{bmatrix} 2 &&&&&&&&& \\ & 2 & 1 &&&&&&& \\ && 2 &&&&&&& \\ &&& 2 & 1 &&&&& \\ &&&& 2 & 1 &&&& \\ &&&&& 2 &&&& \\ &&&&&& 3 &&& \\ &&&&&&& 3 & 1 & \\ &&&&&&&& 3 & 1 \\ &&&&&&&&& 3 \end{bmatrix}$$

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- algebraic multiplicity $=$ dimension of the Jordan block *(since each element on the diagonal adds a term "$(s-\lambda)$" in the characteristic polynomial)*

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- algebraic multiplicity $=$ dimension of the Jordan block *(since each element on the diagonal adds a term "$(s-\lambda)$" in the characteristic polynomial)*
- geometric multiplicity $=$ number of Jordan miniblocks *(since each miniblock adds its own $dim\Big(\ker(2I - A)\Big) = 1$)*

Extremely important facts to remember!!!

Assume T to be a generic change of basis. Then:

1. the eigenvectors and eigenvalues depend only on $\mathcal{A}$, and not on the used basis: λ_i eigenvalue of $A \Leftrightarrow \lambda_i$ eigenvalue of $A' = TAT^{-1}$

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4. the eigenspaces depend only on $\mathcal{A}$, and not on the used basis: $\ker(\lambda_i I - A) = \ker\left(\lambda_i I - TAT^{-1}\right)$
5. the geometric multiplicities depend only on $\mathcal{A}$, and not on the used basis

Recap of the connections

If A is diagonalizable then:

- there exist a basis for $\mathbb{R}^n$ that is composed of eigenvectors of A
- the sum of the geometric multiplicities of the various eigenspaces of A is n
- the various eigenspaces of A span the whole $\mathbb{R}^n$
- the associated system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is actually a series of independent 1-dimensional systems
- the modes of the associated system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ are of the form $e^{\lambda t}$

Recap of the connections

The case "A is not diagonalizable"

- in any case there exists a change of basis that maps A into its Jordan form
- there must be at least one Jordan minibloc, and the effect of this miniblock is to introduce some sort of shearing in some directions
- the eigenvectors of A do not span the entire $\mathbb{R}^n$, but only a part of it
- the sum of the geometric multiplicities of the various eigenspaces of A is smaller than n; actually it is equal to the number of Jordan miniblocks
- the associated system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is actually a series of independent systems, each one corresponding to one of the Jordan miniblocks
- the modes of the associated system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ are not only of the form $e^{\lambda t}$, but there must be some $te^{\lambda t}$ or even higher powers of t

How do we find Jordan forms?

$$\text{i.e., how can we go from } A = \begin{bmatrix} 3 & 4 & 8 \\ 1 & -5 & 2 \\ -5 & 9 & 1 \end{bmatrix} \text{ to } J = TAT^{-1}?$$

→ needs the concepts of <u>generalized eigenvectors</u>, but this is a bit too much for this course . . . In any case just use numerical tools!

?