# TTK4225 - Systems Theory, Autumn 2020

Damiano Varagnolo

NTNU

# Course Overview

welcome to TTK4225!

# Instructors

- main teacher: Damiano Varagnolo, `damiano.varagnolo@ntnu.no`, room D233
- theaching assistant: Iver Andreas Ugelvik, `iverau@stud.ntnu.no`

# Where is the information?

- Blackboard
- study guide
- GitHub repository
- mediasite.ntnu.no

# Reference group – VERY IMPORTANT

- at least 3 students
- will do 4 meetings (1 after the exam)
- shall represent the whole class $\Longrightarrow$ you will have meetings among yourselves too
- shall lead to a <u>referansegrupperapport</u> containing suggestions for improvements

# Expectations, teacher side

- be proactive
- be respectful
- be collaborative
- eventually, learn stuff

# Prerequisites

- basic physics
- basic linear algebra
- basic analysis

- better if you can Matlab and Python

this course teaches the foundations of automatic control. Not mastering this course $\Longrightarrow$ failing to understand all the next courses

# Expectations, learners side

?

# Implementation leitmotif

*maximize the interactions among us*

$\Longrightarrow$ things that you can do by yourself you do by yourself

# Recommendations

- read the study guide
- do the assignments in the suggested order & times
- signal bugs / imprecisions in the material
- charge the phone + bring pen and paper + bring your laptop

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# Structure of each lesson

1. frontal guidance
2. peer instructions
3. Jupyter coding

# Peer instructions

- purpose = be active
- algorithm = for each question:
  1. first time, answer individually
  2. then form groups, discuss, try to convince each other, but eventually answer individually
  3. then I show the solution
  4. after that you pose questions

# Peer instructions - test

`https://play.kahoot.it/v2/lobby?quizId=bef23121-b1a6-4cce-8565-b9631387bbbf`

# Jupyter notebooks

`http://localhost:8888/notebooks/GitHub/TTK4225/trunk/Jupyter/odeint-example.ipynb`

# Jupyter notebooks - instructions for the next lesson

1. download the notebooks in `https://github.com/damianovar/TTK4225-2020`
2. install *Jupyter notebooks* or *Anaconda*
3. launch and test `TTK4225/trunk/Jupyter/odeint-example.ipynb`

?

# “At-home” self-assessment activities

1. contents mapping
2. Blackboard questionnaires

# Contents mapping

why: enable you visually assess if you got the logical structure of the course contents

what: gamified mapping of the content units in the course

how: python script downloadable from Blackboard TODO

when: whenever you prefer – this is a self assessment activity!

*More info in the study guide!*

# Blackboard questionnaires

why: enable the continuous quantitative collection of how much the class is learning / adapt the course pace

what: private Blackboard questionnaires

how: set of questions indexed in terms of "what is asked" and "how difficult the question is"

when: whenever you prefer – this is a self assessment activity!

*More info in the study guide!*

leitmotif: we want to help you, but we need you to help us

?

# What about the CoVid-19?

Cornerstones:

- we may need to switch to fully digital teaching as in this past spring
- independently of this, who must be at home in quarantine is entitled to good teaching

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videotaping, etc $=$ all elements to account for this

tips are most welcome

?

# Course overview

# Aim of the course

learn a language for describing phenomena
in an abstract and formal way,
that will be used later on
to design feedback and feedforward control schemes

# What is control?

- adjusting the temperature in the shower
- riding a bike

# What is automatic control?

applying mechanisms so to operate processes automatically,
so they don't need continuous direct human intervention

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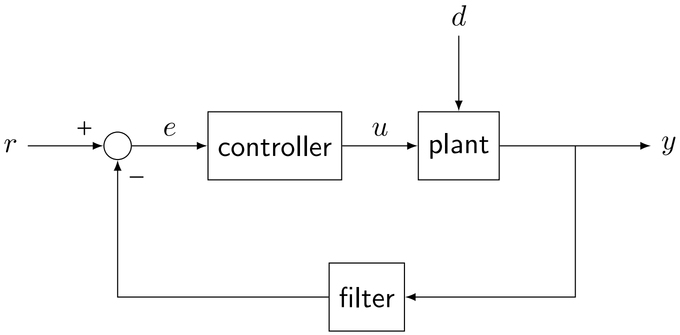
"remove the man from the loop"

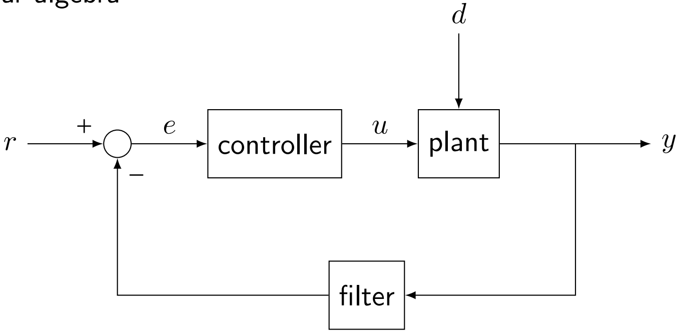
# What is (feedback) control?




*PS: this is not the only type of automatic control!*

# Intended Learning Outcomes, in brief:

- use differential equations to model continuous-time plants
- analyse the models in time and frequency domains
- learn to characterise important structural properties of these models
- learn a little bit about filters
- refresh linear algebra



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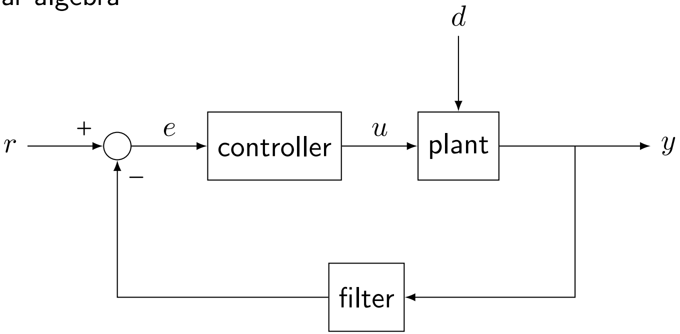
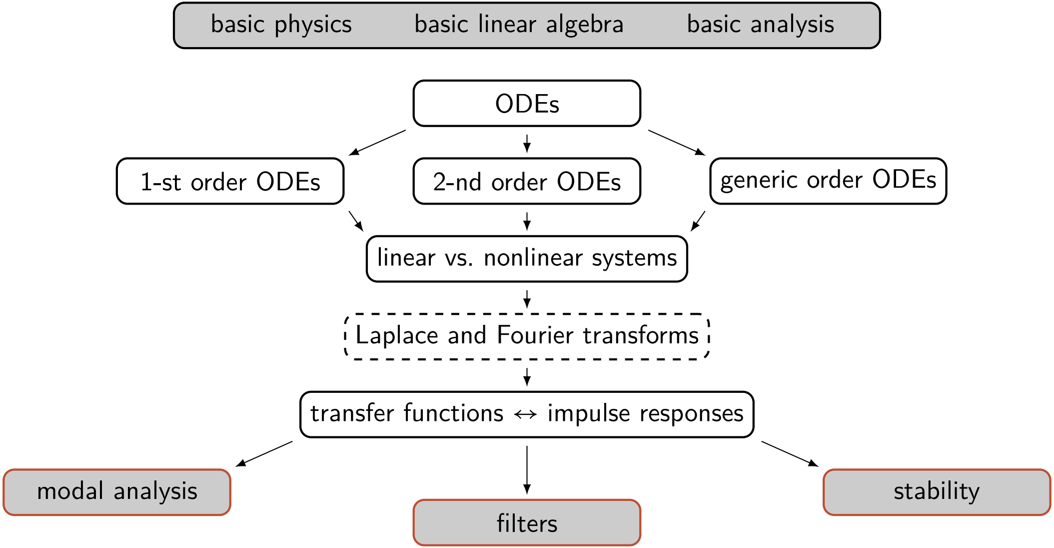



*why?* need all these ingredients to do advanced control!

# Summary / Roadmap of TTK4225



# Tentative time line

GitHub repository

$\downarrow$

`ContentsMap` folder

$\downarrow$

`plan-and-content-units.xlsx` file

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# Introduction to modelling and dynamics

# Roadmap

- what do we mean with
  - “dynamics”?
  - “modelling a dynamical system”?
  - “control-oriented modelling”?

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### From mechanics

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Even better:

$$\dot{v} = \frac{F}{m} \tag{2}$$

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Example:

$$F = ma = m\frac{dv}{dt} = m\dot{v} \tag{1}$$

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$$\dot{v} = \frac{F}{m} \tag{2}$$

**this is an ODE**

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- $y_{\text{pred}}$ := predator

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- $y_{\text{pred}}$ := predator

$$\begin{cases} \dot{y}_{\text{prey}} &= \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} &= -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

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`GitHub/TTK4225/trunk/Jupyter/Lotka-Volterra-introduction.ipynb`

# What do we mean with “modelling a dynamical system”?

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Example of a system from before:

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More generic definition: modelling a dynamical system = defining

$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\theta}\right),$$

thus defining:

- the variables
  - $\boldsymbol{u}$ = inputs
  - $\boldsymbol{y}$ = outputs
- the structure of the function $\boldsymbol{f}$
- the value of the parameters $\boldsymbol{\theta}$

# What do we mean with “modelling a dynamical system”?

Even more generic definition: defining a *state-space system*:

$$\begin{cases} \dot{\boldsymbol{x}} & = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}\right) \\ \boldsymbol{y} & = \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}\right) \end{cases}$$

thus defining:

- the variables
  - $\boldsymbol{u}$ = inputs
  - $\boldsymbol{x}$ = state
  - $\boldsymbol{y}$ = measured outputs
- the structure of the functions $\boldsymbol{f}$ and $\boldsymbol{g}$
- the value of the parameters $\boldsymbol{\theta}$

# Discussion: did we model the Lotka-Volterra dynamical system here?

```
def myModel(y, t):
#
# parameters
alpha = 1.1
beta  = 0.4
gamma = 0.4
delta = 0.1
#
# get the individual variables - for readability
yPrey = y[0]
yPred = y[1]
#
# individual derivatives
dyPreydt  =   alpha * yPrey - beta  * yPrey * yPred
dyPreddt  = - gamma * yPred + delta * yPrey * yPred
#
return [ dyPreydt, dyPreddt ]
```

# Discussion: do we need something else to simulate this system?

$$\begin{cases} \dot{y}_{\mathrm{prey}} &= \ \ 1.2y_{\mathrm{prey}} - 0.1y_{\mathrm{prey}}y_{\mathrm{pred}} \\ \dot{y}_{\mathrm{pred}} &= -0.6y_{\mathrm{pred}} + 0.2y_{\mathrm{prey}}y_{\mathrm{pred}} \end{cases}$$

# Connecting with next courses

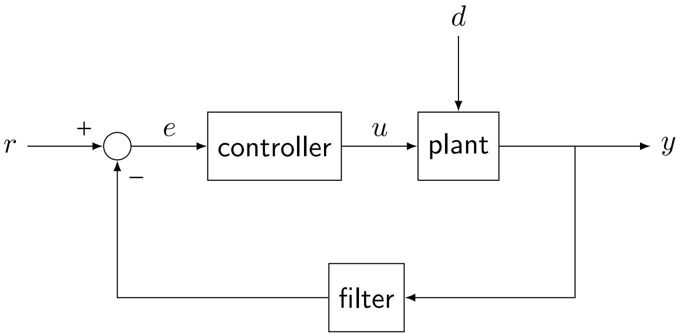
$$\begin{cases} \dot{\boldsymbol{x}} & = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}\right) \\ \boldsymbol{y} & = \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}\right) \end{cases}$$

- how to estimate the structure of $\boldsymbol{f}$, $\boldsymbol{g}$ and the values of $\boldsymbol{\theta}$ $=$ system identification
- how to estimate $\boldsymbol{x}(0)$ from $\boldsymbol{y}$ $=$ state observers

# Caveat: static $\neq$ dynamic

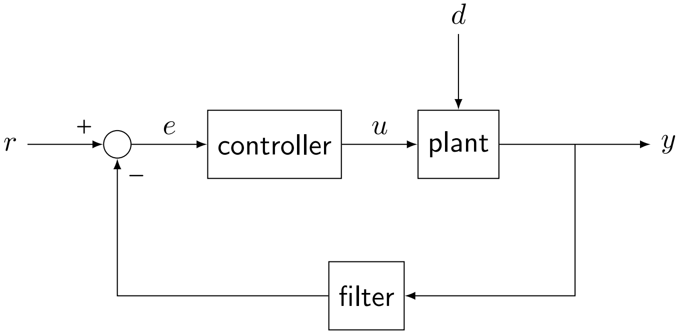
$$\boldsymbol{y} = \boldsymbol{f}\left(\boldsymbol{u}, \boldsymbol{\theta}\right) \qquad \neq \qquad \dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\theta}\right)$$

# The workflow to control a dynamical system




1. model the process
2. identify it *(sometimes not strictly necessary)*
3. design the controller

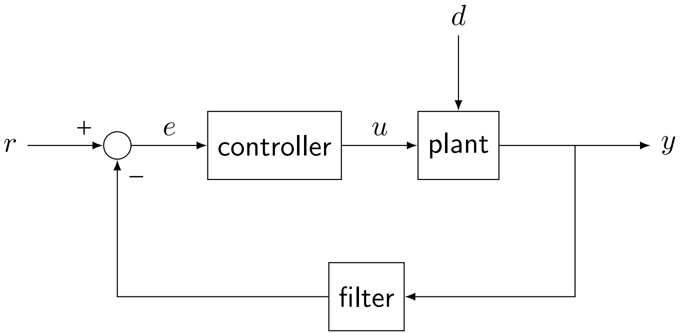
# What does it mean to design a controller?




*task:* complete the following sentence: "In this case, the controller is $u = h\,(\_\_\_\_)$, and it shall be designed so that ____ is as desired, independently of ____"

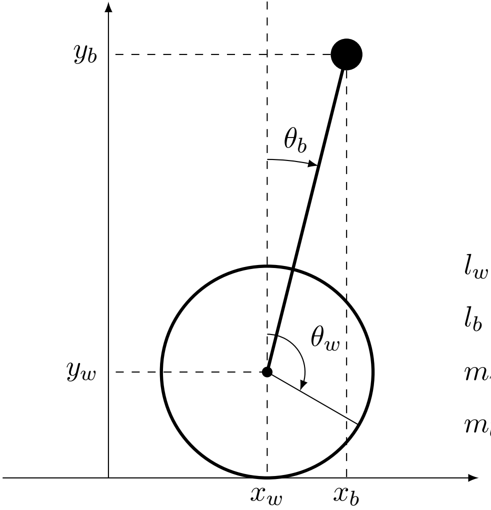
?

# What does it mean to control a dynamical system?




*= designing a controller = designing a function*

# Another example: balancing robot; what are $u$, $x$ and $y$ in this case?




$l_w$ = radius of the wheel

$l_b$ = body-wheel's center distance

$m_w$ = mass of the wheel

$m_b$ = mass of the body

# Modelling - simplest example: speed of a cart

- speed $v(t)$
- friction $F_f(t) = -kv(t)$
- force from the engine $F(t)$ *(this is our $u$!)*
- Newton's second law: $\sum$ forces $= ma(t)$

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$$ma(t) = F_t(t) + F(t)$$

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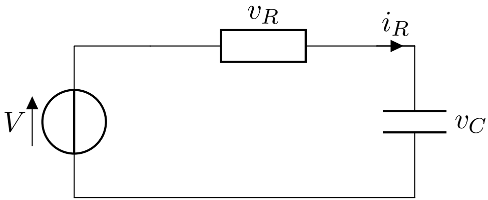
- Newton's second law: $\sum$ forces $= ma(t)$

$$\begin{aligned} ma(t) &= F_t(t) + F(t) \\ m\dot{v} &= -kv(t) + F(t) \\ \implies \dot{v} &= -\frac{k}{m}v(t) + \frac{1}{m}F(t) \end{aligned} \tag{3}$$

# Modelling - another example: RC-circuit

## Example 2.2 in Balchen's book



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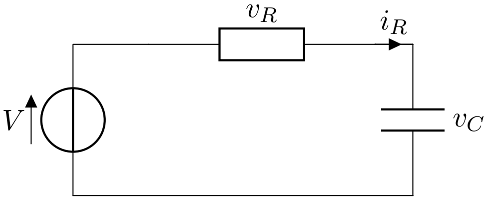


*Question:* what do we mean with dynamics in this case?

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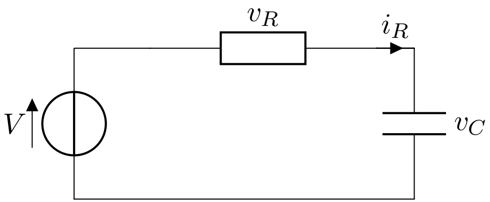


*Question:* what do we mean with dynamics in this case?

**Remember our acception**

dynamics = the study of the motion (i.e., temporal evolution) of the variables that characterize a system $\Longrightarrow$ modelling how tensions and currents evolve in time

# Modelling - another example: RC-circuit

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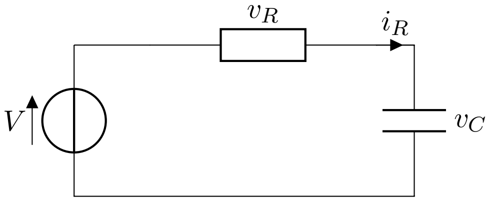


*Question:* what do we mean with modelling the dynamics in this case?

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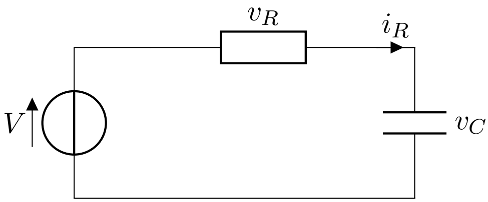


*Question:* what do we mean with modelling the dynamics in this case? We mean defining:

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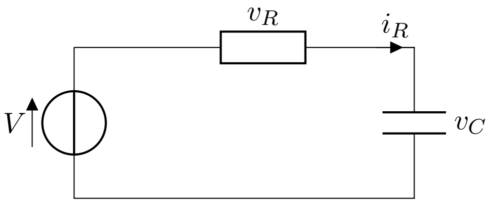


How to obtain a dynamical model = use Kirchhoff's laws (the sum of the voltages in a circuit is zero)

$$\frac{v_R}{R} = i_R = i_C = C\frac{dv_C}{dt} \tag{4}$$

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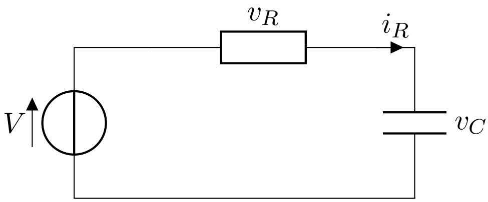


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$$V(t) = v_R + v_C \Rightarrow v_R = V(t) - v_C \quad \Longrightarrow \quad C\dot{v}_C = \frac{V(t) - v_C}{R} \tag{5}$$

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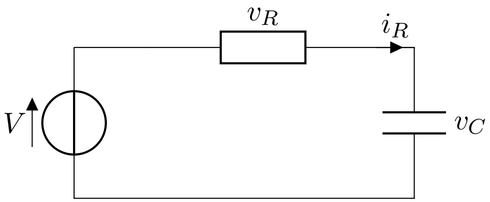


How to obtain a dynamical model = use Kirchhoff's laws (the sum of the voltages in a circuit is zero)

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$$V(t) = v_R + v_C \Rightarrow v_R = V(t) - v_C \quad \Longrightarrow \quad C\dot{v}_C = \frac{V(t) - v_C}{R} \tag{5}$$

$$\Longrightarrow \quad \dot{v}_C = -\frac{1}{RC}v_C + \frac{1}{RC}V(t) \tag{6}$$

?

# Standing examples

# Roadmap

Summary of the standing examples in this course:

- exponential growth
- RC-circuits
- RCL-circuits
- spring-mass system
- Lotka-Volterra
- Van-der-Pol oscillator
- balancing robot
- insuline concentration

# Exponential growth

## Generalization of "speed of a cart" and "RC circuit"

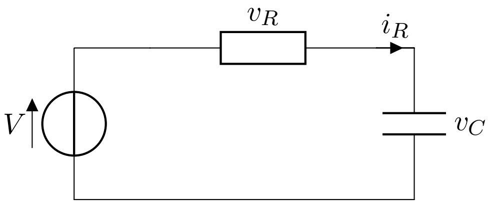
Scalar version:

$$\dot{y} = \alpha y + \beta u \tag{7}$$

Matricial version:

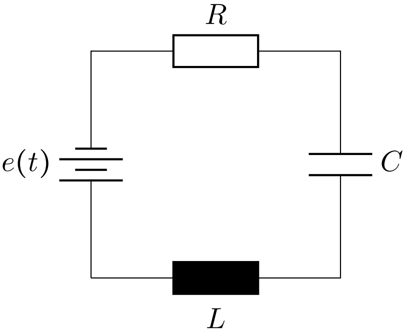
$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u} \tag{8}$$

# RC-circuit




$$\dot{v}_C = -\frac{1}{RC} v_C + \frac{1}{RC} V(t) \tag{9}$$

# RCL-circuit



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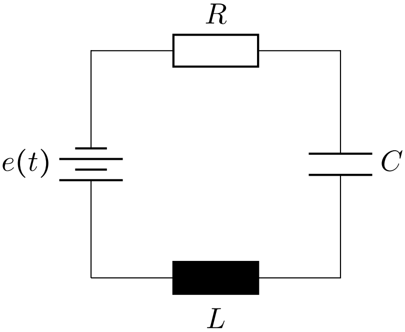
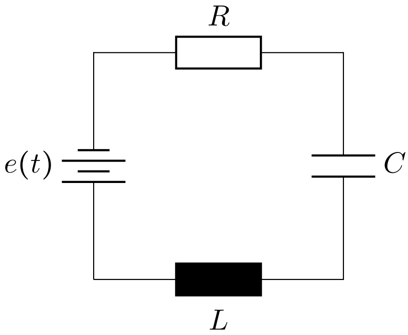



EOM: Kirkhoff laws

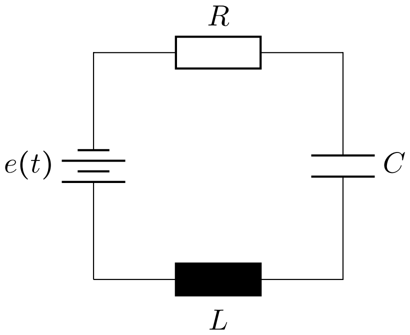
# RCL-circuit




EOM: Kirkhoff laws

$$v_L(t) = L\frac{di(t)}{dt} \qquad v_i(t) = Ri(t) \qquad v_C(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

# RCL-circuit

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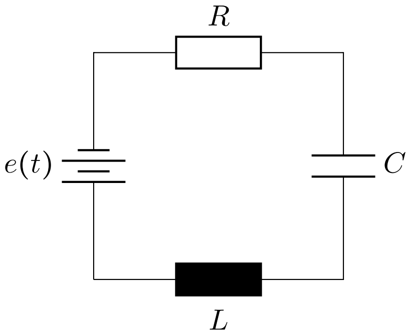
$$e(t) = v_L(t) + v_R(t) + v_C(t)$$

# RCL-circuit

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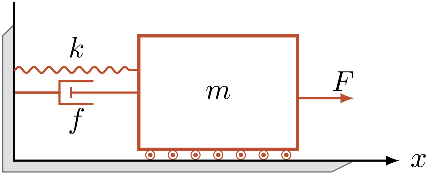
$$e(t) = v_L(t) + v_R(t) + v_C(t) \quad \mapsto \quad e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau \tag{10}$$

# Spring-mass systems

E.g., position of a cart fastened with a spring to a wall and subject to friction



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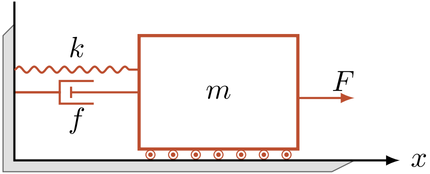


EOM:

- force from the spring: $F_x(t) = -kx(t)$
- friction: $F_f(t) = -f\dot{x}(t)$
- applied force: $F(t)$
- Newton's second law: $\sum F = m\ddot{x}(t)$

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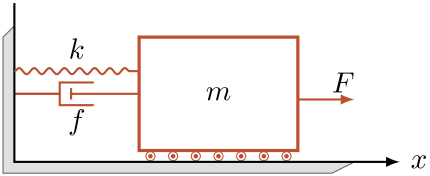


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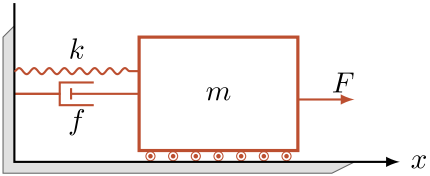
$$m\ddot{x}(t) = F_x(t) + F_f(t) + F(t)$$

# Spring-mass systems

E.g., position of a cart fastened with a spring to a wall and subject to friction




EOM:

- force from the spring: $F_x(t) = -kx(t)$
- friction: $F_f(t) = -f\dot{x}(t)$
- applied force: $F(t)$
- Newton's second law: $\sum F = m\ddot{x}(t)$

$$m\ddot{x}(t) = F_x(t) + F_f(t) + F(t) \qquad \mapsto \qquad m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + F(t)$$

# Lotka-Volterra

- $y_{\text{prey}}$ := prey
- $y_{\text{pred}}$ := predator

$$\begin{cases} \dot{y}_{\text{prey}} & = \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} & = -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

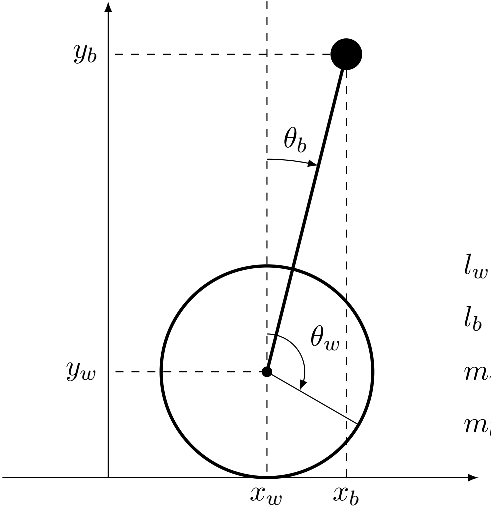
`GitHub/TTK4225/trunk/Jupyter/Lotka-Volterra-introduction.ipynb`

# Van-der-Pol oscillator

$$\begin{cases} \dot{y}_1 & = \mu \left( y_1 - \frac{y_1^3}{3} - y_2 \right) \\ \dot{y}_2 & = \frac{y_1}{\mu} \end{cases} \tag{11}$$

`GitHub/TTK4225/trunk/Jupyter/Van-der-Pol-introduction.ipynb`

# Balancing robot




$l_w$ = radius of the wheel

$l_b$ = body-wheel's center distance

$m_w$ = mass of the wheel

$m_b$ = mass of the body

# Balancing robot

$$
\begin{aligned}
\left(I_b + m_b l_b^2\right)\ddot{\theta}_b &= +m_b l_b g \sin\left(\theta_b\right) - m_b l_b \ddot{x}_w \cos\left(\theta_b\right) - \frac{K_t}{R_m} v_m + \left(\frac{K_e K_t}{R_m} + b_f\right)\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \\
\left(\frac{I_w}{l_w} + l_w m_b + l_w m_w\right)\ddot{x}_w &= -m_b l_b l_w \ddot{\theta}_b \cos\left(\theta_b\right) + m_b l_b l_w \dot{\theta}_b^2 \sin\left(\theta_b\right) + \frac{K_t}{R_m} v_m - \left(\frac{K_e K_t}{R_m} + b_f\right)\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right)
\end{aligned}
\tag{12}
$$

# Insulin concentration

- $x_1 :=$ sugar concentration
- $x_2 :=$ insulin concentration
- $u_1 :=$ food intake
- $u_2 :=$ insulin intake
- $c :=$ sugar concentration in fasting

$$\begin{cases} \dot{x}_2 = a_{21}\left(x_1 - c\right) - a_{22}x_2 + b_2u_2 & x_1 \geq c \\ \dot{x}_2 = -a_{22}x_2 + b_2u_2 & x_1 < c \end{cases}$$

$$\begin{cases} \dot{x}_1 = -a_{11}x_1x_2 - a_{12}\left(x_1 - c\right) + b_1u_1 & x_1 \geq c \\ \dot{x}_1 = -a_{11}x_1x_2 + b_1u_1 & x_1 < c \end{cases}$$

?

Seeing this course as a first step towards model predictive control

# Roadmap

- how can we do control?
- what does it mean to do model predictive control?
- why are we doing this course?

control = to take out humans from the loop

*(and do not waste time in doing things that machines could do)*

# Remember: control = design a function

$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\theta}\right)$$

where

- $\boldsymbol{f}$ and $\boldsymbol{\theta}$ = the model
- $\boldsymbol{y}$ the outputs
- $\boldsymbol{u}$ the inputs

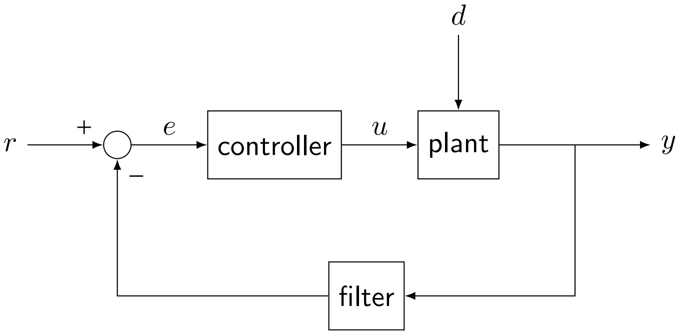
# Two important paradigms

$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\theta}\right)$$

model free control: $\boldsymbol{u}$ is built without knowing $\boldsymbol{f}$ and $\boldsymbol{\theta}$

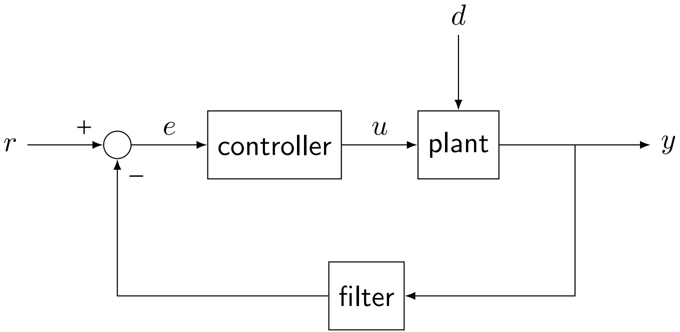
model based control: $\boldsymbol{u}$ is built through knowing $\boldsymbol{f}$ and/or $\boldsymbol{\theta}$

# The simplest controller: the $P$ control




$$u(t) = \kappa e(t)$$

# The simplest controller: the $P$ control




$$u(t) = \kappa e(t)$$

*Discussion:* we want to automate the control of the steering wheel while driving in a curve with a $P$ controller.

1. What is $u$? What is $r$? What is $e$?
2. What does it imply to make $\kappa$ bigger? And smaller?

# $P$ controlling a steering wheel

`GitHub/TTK4225/trunk/Jupyter/`

`first-order-system-introduction-with-P-controller.ipynb`

# $P$ controlling a steering wheel

`GitHub/TTK4225/trunk/Jupyter/`
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*Discussion: what do you expect may happen if there is a spring somewhere?*

important message: we can do 'model free' control

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*(and actually something like 90% of the controllers in the world are model-free...)*

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efficient, but kind of "dumb". We don't actually know what is happening!

# Discussion: can we do something better if we know the model?

# How can we do something better if we know the model?

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1. if I know $\boldsymbol{f}$ and my current initial condition, I can plug in a tentative $\widehat{\boldsymbol{u}}$, and see what would be the corresponding $\widehat{\boldsymbol{y}}$

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$$\boldsymbol{u}^{\star} = \arg \min_{\boldsymbol{u} \in \mathcal{U}, \boldsymbol{f}(\boldsymbol{u}) \in \mathcal{F}} \text{Cost}\left(\boldsymbol{f}\left(\boldsymbol{u}\right), \boldsymbol{u}\right)$$

# What does this require?

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- define "$\mathrm{Cost}$"

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- define "Cost"
- be able to compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

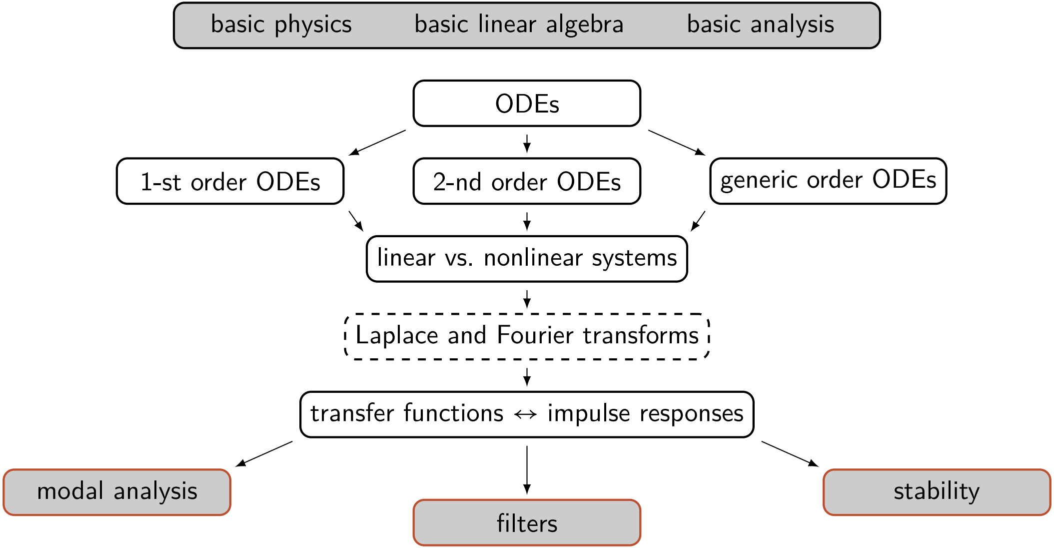
the most important point: to do model-based control we need to be able to simulate the most important point: to do model-based control we need to be able to simulate

*this course* = understanding the properties of the system,
& understanding how to simulate it with the purpose of doing advanced control

# The control courses at NTNU ITK

- how do I compute explicitly $\boldsymbol{u}^{\star}$ in special cases?
- how do I compute numerically $\boldsymbol{u}^{\star}$ in general?
- how do I estimate $\boldsymbol{f}$ from data?
- how do I handle uncertainties?
- which properties can I guarantee?
- ...

# Summary / Roadmap of TTK4225



*Caveat!* Better doing MPC than data-driven control:

Benjamin Recht (UC Berkeley, USA)

*Reflections on the Learning-to-Control Renaissance*

(and also `https://www.youtube.com/watch?v=nF2-39a29Pw`)

?

## Introduction to first order systems - the concept of free evolution

# Roadmap

- generalization of our examples of first order systems
- the concept of free evolution
- introduction to stability
- free evolution $\neq$ forced response

# First order dynamical models

- train velocity

$$\dot{v} = -\frac{k}{m} v(t) + \frac{k}{m} F(t) \tag{13}$$

- RC-circuit

$$\dot{v}_C = -\frac{1}{RC} v_C + \frac{1}{RC} V(t) \tag{14}$$

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In general:

$$\dot{y} = ay + bu \tag{15}$$

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# Overarching question towards model-based control: how does $y$ evolve?

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- what happens if $u = 0$, $y_0 > 0$, and $a > 0$?

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- what happens if $u = 0$, $y_0 \neq 0$, and $a < 0$?
- what happens if $u = 0$, and $a = 0$?

$\boxed{\text{stability} = \text{extremely important topic!}}$

will be analysed in more details later on in this course

and much more extensively in others

*(feat. Lyapunov, Krasovskii, La-Salle among others)*

# An important case: free evolution

$$\dot{y} = ay \cancel{+bu} \qquad\qquad y(0) = y_0 \neq 0 \tag{17}$$

# Towards the solution of the free evolution starting from a simplified case

$$\dot{y} = y \qquad\qquad y(0) = y_0 \neq 0 \tag{18}$$

*(in other words, which function is equal to its derivative?)*

# The solution of the free evolution in general

$$\dot{y} = ay \qquad\qquad y(0) = y_0 \neq 0 \tag{19}$$

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$$\dot{y} = ay \qquad\qquad y(0) = y_0 \neq 0 \tag{19}$$

Alternatively:

$$\dot{y} = \frac{dy}{dt} = ay(t) \tag{20}$$

$$\frac{dy}{y(t)} = a\ dt \tag{21}$$

$$\int_0^t \frac{dy}{y(\tau)} = \int_0^t a\ \mathsf{d}\tau \tag{22}$$

$$\Big[\ln y(\tau)\Big]_0^t = a\Big[\tau\Big]_0^t \tag{23}$$

$$\ln\big(v(t)\big) = \ln(y_0) + at \tag{24}$$

$$y(t) = e^{\ln(y_0)at} = e^{\ln(y_0)}e^{at} \tag{25}$$

$$y(t) = y_0 e^{at} \tag{26}$$

# And what about the vectorial generalization?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y}$$

$\Longrightarrow$ later in the course, and extensively!

# Analysis of first order systems in free evolution

$$\dot{y} = ay \qquad y(0) = y_0 \neq 0 \qquad \Longrightarrow \qquad y(t) = y_0 e^{at} \tag{27}$$

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*Discussion:* which case is this one?

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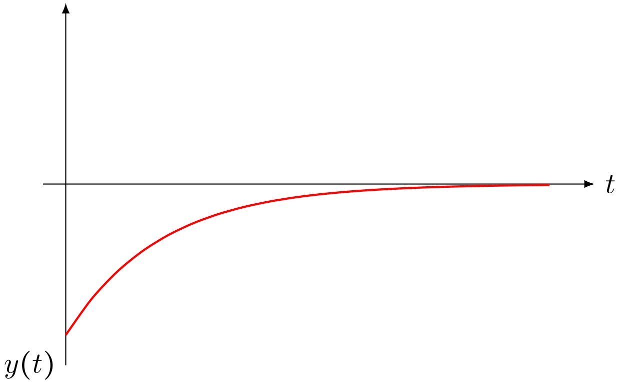

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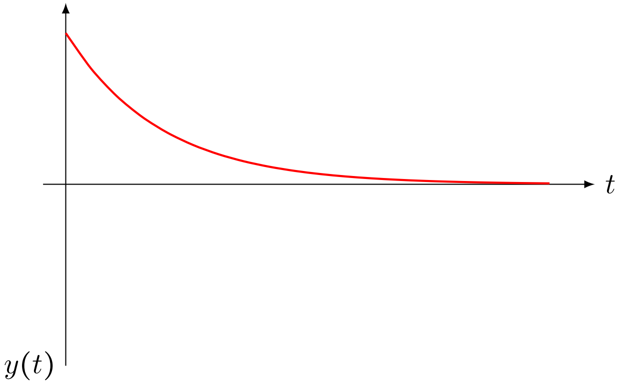

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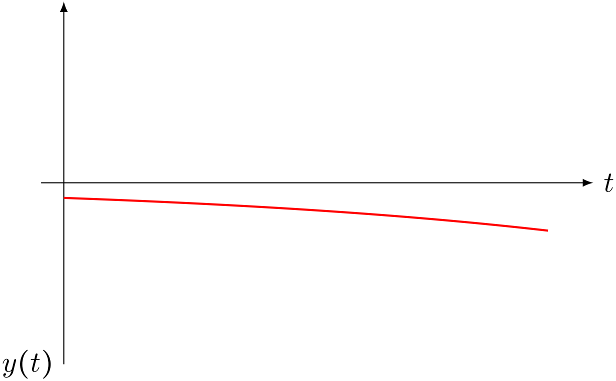

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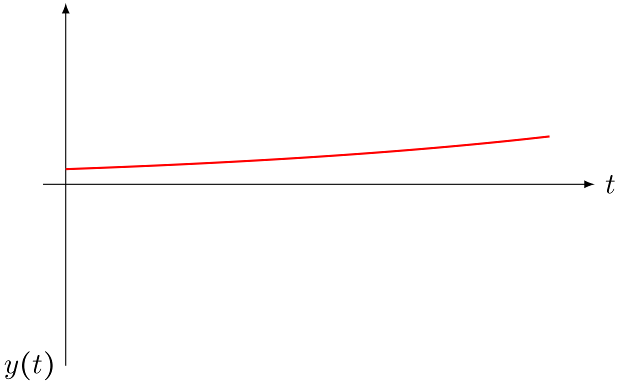

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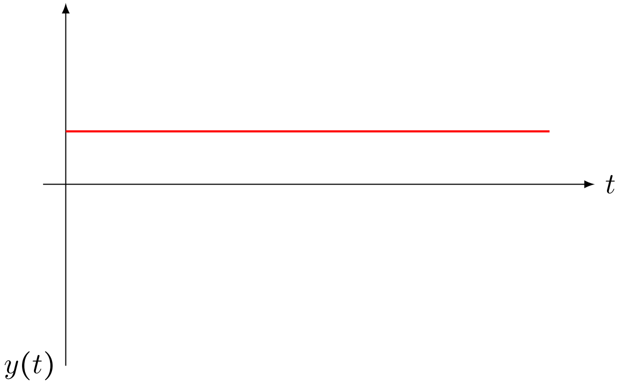

?

# Introduction to stability

stable system: $y(t) \to 0$ for $t \to +\infty$

unstable system: $|y(t)| \to +\infty$ for $t \to +\infty$

marginally stable system: $y(t) = y_0$ for every $t$

# Introduction to stability

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marginally stable system: $y(t) = y_0$ for every $t$

*much more on stability both later on & in later courses!*

# Are we done with studying the response of first order dynamical models?

$$\dot{y} = ay + bu \tag{28}$$

# Forced response

$$\dot{y} = ay + bu \qquad \color{darkred}{y(0) = y_0 = 0} \tag{29}$$

*(remember: free evolution =* "$u = 0, y_0 \neq 0$"*)*

# Forced response

$$\dot{y} = ay + bu \qquad\qquad y(0) = y_0 = 0 \tag{29}$$

*(remember: free evolution =* "$u = 0, y_0 \neq 0$"*)*

*Discussion:* does $y(0) = y_0 = 0$ imply that we should study $\dot{y} = bu$?

# Try it yourself at home!

`GitHub/TTK4225/trunk/Jupyter/`
`first-order-system-introduction-with-P-controller.ipynb`

?

# Introduction to first order systems - the concept of forced response

# Roadmap

- the concept of “forced response”
- the convolution operator
- some properties of convolution

# First order dynamical models

$$\dot{y} = ay + bu \tag{30}$$

`GitHub/TTK4225/trunk/Jupyter/`
`first-order-system-introduction-with-P-controller.ipynb`

# Overarching question towards model-based control: how does $y$ evolve?

$$\dot{y} = ay + bu \tag{31}$$

# Free evolution vs. forced response

$$\dot{y} = ay + bu \tag{32}$$

free evolution: $u = 0,\ y_0 \neq 0$

forced response: $u \neq 0,\ y_0 = 0$

# Solution to the forced response

(actual solution through equations (2.11)–(2.17) in Balchen's book)

$$\dot{y} = ay + bu, \quad y(0) = y_0 = 0 \qquad \Longrightarrow \qquad y(t) = \int_0^t e^{a(\tau)} bu(t-\tau) d\tau \tag{33}$$

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extremely important generalization:

$$h(t) = e^{at} \qquad \Longrightarrow \qquad y(t) = \int_0^{+\infty} h(\tau) u(t-\tau) d\tau$$

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extremely important definition:

$$\text{convolution} := h * u(t) := \int_{-\infty}^{+\infty} h(\tau) u(t-\tau) d\tau$$

# Visualizing $h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$

# Discussions

$$h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

1. is $h * u(t) = u * h(t)$?

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$$h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

1. is $h * u(t) = u * h(t)$?
2. is $(\alpha h_1 + \beta h_2) * u(t) = \alpha\left(h_1 * u(t)\right) + \beta\left(h_2 * u(t)\right)$?

# Discussions

$$h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

1. is $h * u(t) = u * h(t)$?
2. is $(\alpha h_1 + \beta h_2) * u(t) = \alpha\left(h_1 * u(t)\right) + \beta\left(h_2 * u(t)\right)$?
3. why is it that $\dot{y} = ay + bu, \quad y(0) = y_0 = 0$ implies

$$y(t) = \int_{0}^{+\infty} e^{a(\tau)} bu(t-\tau)d\tau$$

and *not*

$$y(t) = \int_{-\infty}^{+\infty} e^{a(\tau)} bu(t-\tau)d\tau \quad ?$$

# Example

$$y(t) = \int_0^{+\infty} h(\tau)u(t-\tau)d\tau,$$

$$u(t) = \begin{cases} 1 & \text{for } t \in [1,2] \\ 0 & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 2 & \text{for } t \in [0,1) \\ 1 & \text{for } t \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

# Discussion:

What is $y(t) = h * u(t)$ for the case

$$u(t) = \begin{cases} 1 & \text{for } t \in [1,2] \text{ and } t \in [2,3] \\ 0 & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 2 & \text{for } t \in [0,2) \\ 0 & \text{otherwise} \end{cases} \qquad ?$$

# TODO

`GitHub/TTK4225/trunk/Jupyter/TODO.ipynb`

# Summary

$$\dot{y} = ay + bu \tag{34}$$

free evolution: $u = 0,\ y_0 \neq 0$; implies $y(t) = y_{\text{free}}(t) = y_0 e^{at}$

forced response: $u \neq 0,\ y_0 = 0$; implies $y(t) = y_{\text{forced}}(t) = h * u(t)$ with $h(t) = e^{at}$

# Summary

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forced response: $u \neq 0$, $y_0 = 0$; implies $y(t) = y_{\text{forced}}(t) = h * u(t)$ with $h(t) = e^{at}$

$e^{at}$ = "impulse response"

*yet another pillar concept for automatic control*
→ *will be discussed better in a dedicated module*
*and re-discussed in* ***every*** *control-oriented course*

# General response

$$\dot{y} = ay + bu \qquad (35)$$

free evolution: $u = 0$, $y_0 \neq 0$; implies $y(t) = y_{\text{free}}(t) = y_0 e^{at}$

forced response: $u \neq 0$, $y_0 = 0$; implies $y(t) = y_{\text{forced}}(t) = h * u(t)$ with $h(t) = e^{at}$

# General response

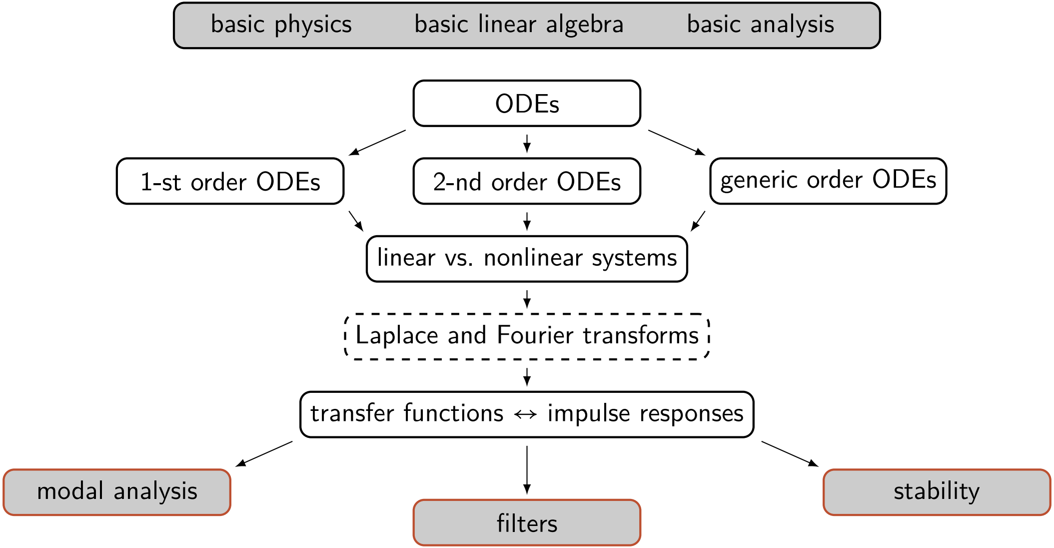
$$\dot{y} = ay + bu \tag{35}$$

free evolution: $u = 0$, $y_0 \neq 0$; implies $y(t) = y_{\text{free}}(t) = y_0 e^{at}$

forced response: $u \neq 0$, $y_0 = 0$; implies $y(t) = y_{\text{forced}}(t) = h * u(t)$ with $h(t) = e^{at}$

general response: $u \neq 0$, $y_0 \neq 0$; implies $y(t) = y_{\text{free}}(t) + y_{\text{forced}}(t)$ thanks to the fact that the model is *linear* *(will be seen in more details later on)*

# Summary / Roadmap of TTK4225



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# Introduction to first order systems - visualizing the system through a block scheme

# Roadmap

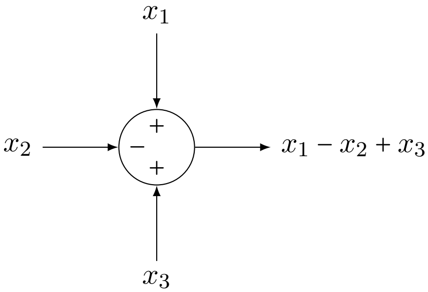
- the most common block schemes
- first order systems as block schemes

# Block diagrams - why?

- used very often in companies
- aid visualization *(until a certain complexity is reached. . . )*
- enable "drag & drop" way of programming

- in this course, primarily used for interpretations

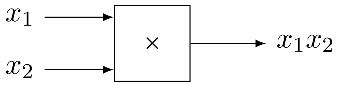
# Most common block diagrams - sum of $n$ signals



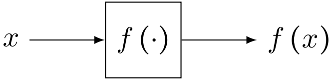
# Most common block diagrams - multiplication for a constant

$$x \longrightarrow \boxed{c} \longrightarrow cx$$

# Most common block diagrams - multiplication of two signals



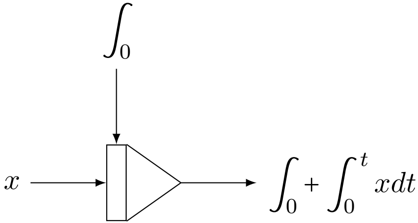
# Most common block diagrams - generic functions



# Most common block diagrams - derivatives

$$x \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \dot{x}$$

# Most common block diagrams - integrals

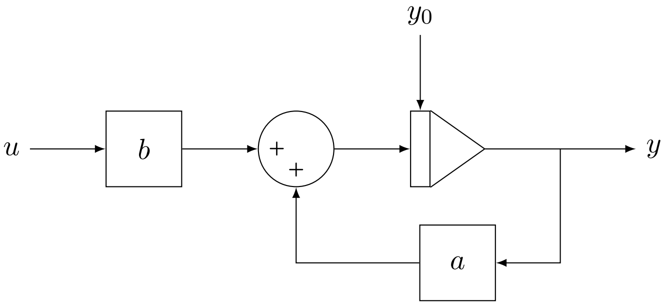


# Discussion: how do we represent a first order DE with a block scheme?

$$\dot{y} = ay + bu$$

# The solution

$$\dot{y} = ay + bu$$



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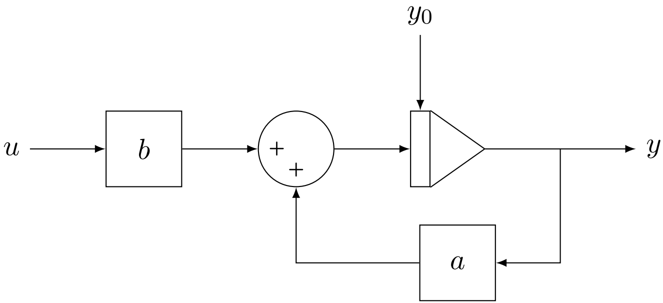



*Discussion:* do you note the presence of a feedback loop?

# Introduction to Simulink

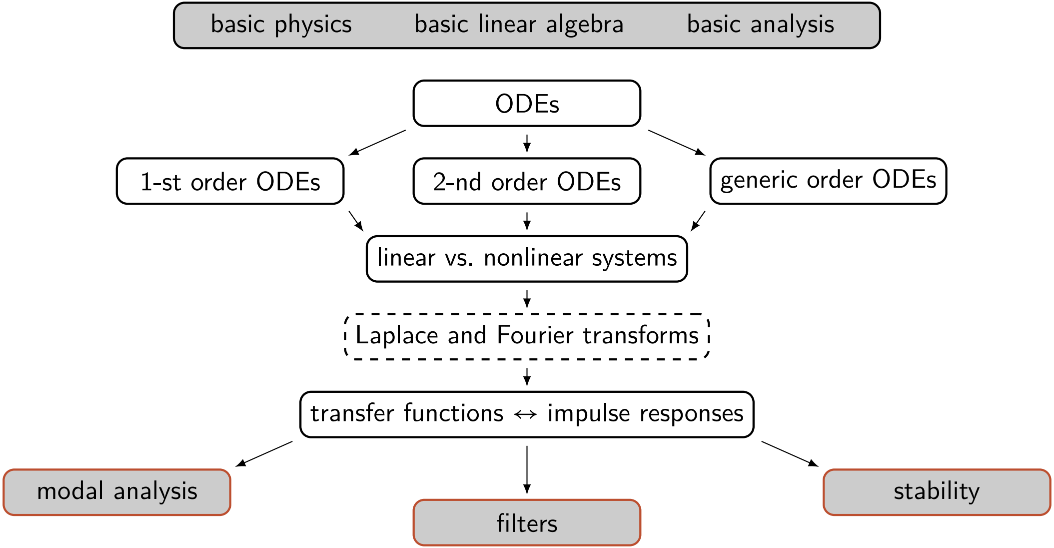
?

# Introduction to generic order systems

# Roadmap

- from first order to second order
- generic orders
- classification of the different types of ODEs

# Summary / Roadmap of TTK4225



# From first order to second order

*Discussion:* if we can write

$$\dot{y} = ay + bu\ ,$$

can we also write

$$\ddot{y} = a_1\dot{y} + a_0 y + bu\ ?$$

# From first order to second order, third order

*Discussion:* if we can write

$$\dot{y} = ay + bu \ ,$$

can we also write

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And at this point also

$$\dddot{y} = a_2\ddot{y} + a_1\dot{y} + a_0 y + bu \ ?$$

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And at this point also

$$\dddot{y} = a_2\ddot{y} + a_1\dot{y} + a_0 y + bu \ ?$$

And why not

$$\dddot{y} = a_2\ddot{y} + a_1\dot{y} + a_0 y + b_1\dot{u} + b_0 u \ ?$$

# ARMA models

$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u \,?$$

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$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u\ ?$$

*Discussion:* and why don't we have $a_n y^{(n)}$, instead of only $y^{(n)}$ on the LHS?

# Discussion: can we generalize even more?

$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u \,?$$

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time-varying: $\dot{y} = a(t)y + b(t)u$

non-homogeneous: $\dot{y} = a(t)y + b(t)u + c$

non-linear: $\dot{y} = a\sqrt{y} + bu^{2/3}$

# Discussion: can we make this linear again?

$$\dot{y} = ay + bu^{2/3}$$

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$$\dot{y} = ay + bu^{2/3}$$

*Another discussion:* can we apply the same "linearization trick" to $\dot{y} = a\sqrt{y} + bu$?

main focus in this course:

linear, time invariant, homogeneous (LTI)

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why?

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linear, time invariant, homogeneous (LTI)

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Remember: predictive control of the type

$$\boldsymbol{u}^{\star} = \arg \min_{\boldsymbol{u} \in \mathcal{U}, \boldsymbol{f}(\boldsymbol{u}) \in \mathcal{F}} \mathrm{Cost}\left(\boldsymbol{f}\left(\boldsymbol{u}\right), \boldsymbol{u}\right)$$

requires to be able to:

- define "$\mathrm{Cost}$"
- compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

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from the previous lessons:

1-st order LTI models $\mapsto$ impulse responses;

as will be clear later generic order LTI models $\mapsto$ also impulse responses

# The question we are trying to answer in this first part of the course

*"What will be the solution $y(t)$ to a generic-order LTI ODE starting from a generic initial condition $y(0) = y_0$ and generic input signal $u(t)$?"*

*(our hope: the solution will be structurally easy, and fast to compute)*

?

# What is what?

(i.e., a small quiz to see if we are aligned)

| | autonomous | linear | homogeneous | time constant |
|---|---|---|---|---|
| $\frac{\dot{x}}{x} = 1$ | | | | |
| $\pi = \dot{x}x$ | | | | |
| $2\dot{x} + \sin t = 0$ | | | | |
| $\sin(t\dot{x}) - x = 0$ | | | | |
| $t\dot{x} = x \sin t$ | | | | |
| $x - e^{-t} = \ddot{x} + \dot{x}$ | | | | |

# Introduction to generic order systems

# Roadmap

- state space representations
- from $n$-th order to vectorial first order

# State space representations - Definition

mathematical model (typically but not limited to of a physical system) as a finite set of inputs, outputs and state variables related by first-order differential equations satisfying the separation principle

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# State space representations - Definition

<u>mathematical model (typically but not limited to of a physical system) as a finite set of inputs, outputs and state variables related by first-order differential equations satisfying the separation principle</u>

Ingredients

- finite number of inputs, outputs and state variables
- first-order differential equations
- *satisfies the separation principle:* the current value of states contains all the information necessary to forecast the future evolution of the outputs and of the states given the inputs

# State space representations - Facts

- the future output depends only on the current state and the future input

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- the future output depends on the past input only through the current state
- the state summarizes the effect of past inputs on future output
  *(sort of a "memory" of the system)*

# Example

Rechargeable flashlight:

- state = level of charge of the battery & on / off button
- output = how much light the device is producing

# State space representations - Notation

$u_1, \ldots, u_m$ = inputs

$x_1, \ldots, x_n$ = states

$y_1, \ldots, y_p$ = outputs

# State space representations - Notation

$$
\begin{aligned}
\dot{x}_1 &= f_1\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
&\vdots \\
\dot{x}_n &= f_n\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
y_1 &= g_1\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
&\vdots \\
y_p &= g_n\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right)
\end{aligned}
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\dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\
\boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)
\end{aligned}
$$

- $\boldsymbol{f}$ = state transition map
- $\boldsymbol{g}$ = output map

# Discussion: is any $\boldsymbol{f}$ and $\boldsymbol{g}$ ok?

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u})\end{aligned}$$

# Discussion: is any $\boldsymbol{f}$ and $\boldsymbol{g}$ ok?

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned}$$

*No:* for every given $\boldsymbol{x}_0$ and $\boldsymbol{u}$ the solution must be:

- unique
- guaranteed to exist for $t \in [0, \bar{t})$ for some $\bar{t} > 0$

*(check "existence and uniqueness of solutions of ODEs")*

# State space representations - Example

$$\begin{cases} \dot{x} = -x^2 \\ y = x \end{cases} \quad \cup \quad x_0 = -1 \qquad \Longrightarrow x(t) = \frac{1}{t-1}$$

# State space representations - Example

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*Discussion:* *(a bit outside of the current flow of concepts on ODEs, but still relevant and very important)* what happens as $t \to 1$?

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*Discussion:* *(a bit outside of the current flow of concepts on ODEs, but still relevant and very important)* what happens as $t \to 1$?

*there is a finite escape time!*

# Discussion: can every physical system be described with a state space representation?

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Noticeable exceptions:

- non-lumped parameters model (i.e., PDEs)
- continuous-time systems with delays
  *(will be discussed better later on)*

# Example: dynamics of sugar and insulin concentrations in blood

- $x_1$ := sugar concentration
- $x_2$ := insulin concentration
- $u_1$ := food intake
- $u_2$ := insulin intake
- $c$ := sugar concentration in fasting

$$\begin{cases} \dot{x}_2 = a_{21}\left(x_1 - c\right) - a_{22}x_2 + b_2u_2 & x_1 \geq c \\ \dot{x}_2 = -a_{22}x_2 + b_2u_2 & x_1 < c \end{cases}$$

$$\begin{cases} \dot{x}_1 = -a_{11}x_1x_2 + a_{12}\left(c - x_1\right) + b_1u_1 & x_1 \geq c \\ \dot{x}_1 = -a_{11}x_1x_2 + b_1u_1 & x_1 < c \end{cases}$$

?

# How can I transform a generic-order ODE into a state space representation (that requires first order DEs)?

$$\dddot{y} = \phi\left(\ddot{y}, \dot{y}, y, \ddot{u}, \dot{u}, u\right);$$

# How can I transform a generic-order ODE into a state space representation (that requires first order DEs)?

$$\dddot{y} = \phi\left(\ddot{y}, \dot{y}, y, \ddot{u}, \dot{u}, u\right);$$

*change of variables:*

$$\boldsymbol{x} = \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix}, \ \boldsymbol{u} = \begin{bmatrix} \ddot{u} \\ \dot{u} \\ u \end{bmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{x}} = \begin{bmatrix} \dddot{y} \\ \ddot{y} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \phi\left(\boldsymbol{x}, \boldsymbol{u}\right) \\ [1\ 0\ 0]\, \boldsymbol{x} \\ [0\ 1\ 0]\, \boldsymbol{x} \end{bmatrix} = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)$$

# Thus, how do I transform a generic order LTI system into a first order LTI system?

$$\ddot{y} = 2\dot{y} + 3y + 4\ddot{u} + 5\dot{u} + 6u$$

# Transforming ARMA models

$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u \,?$$

# Discussion: but then any LTI system is a first order vectorial system?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$$

# Discussion: but then any LTI system is a first order vectorial system?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$$

YES, AND THIS IS WHY
LINEAR ALGEBRA IS ESSENTIAL
FOR CONTROL!

?

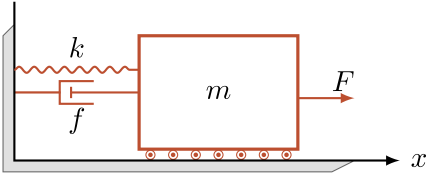
# Second order systems

# Roadmap

- interesting examples
- interesting responses
- differences between 1st order and 2nd order

# Example 1: spring-mass systems

Position of a cart fastened with a spring to a wall and subject to friction:



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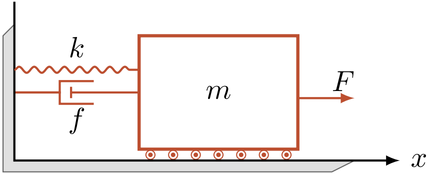


EOM:

- force from the spring: $F_x(t) = -kx(t)$
- friction: $F_f(t) = -f\dot{x}(t)$
- applied force: $F(t)$
- Newton's second law: $\sum F = m\ddot{x}(t)$

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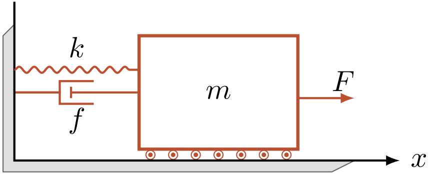


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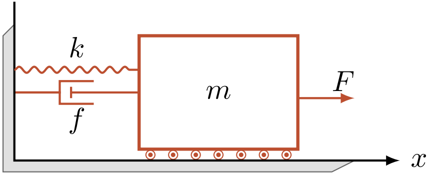


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- Newton's second law: $\sum F = m\ddot{x}(t)$

$$m\ddot{x}(t) = F_x(t) + F_f(t) + F(t) \qquad \mapsto \qquad m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + F(t)$$

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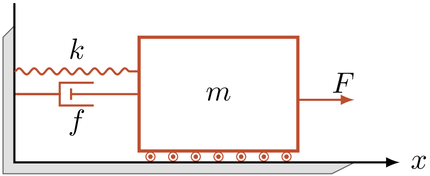


$$\ddot{x}(t) + \frac{f}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{1}{m}F(t) \tag{36}$$

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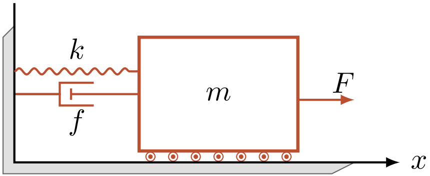


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Generalizing:

$$\ddot{x} = a_1\dot{x} + a_0 x + b_0 u \tag{37}$$

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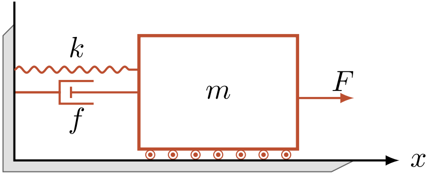


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Discussion: may we write this dynamics using a vectorial first order DE?

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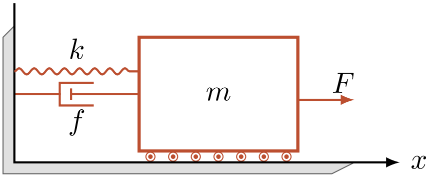


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Generalizing:

$$\ddot{x} = a_1\dot{x} + a_0 x + b_0 u \tag{37}$$

Discussion: may we write this dynamics using a vectorial first order DE?

Discussion: may we write this dynamics using a scalar first order DE?

# Example 1: spring-mass systems

`https://youtu.be/lZPtFDXYQRU?t=31`

*(here there is very little friction, though)*

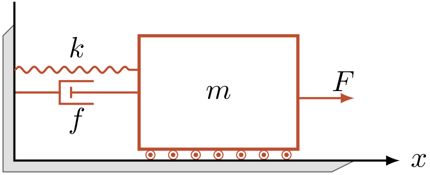
## Example 1: spring-mass systems - stability analysis from intuitive perspectives

*Discussion:* $\dot{x} = a_0 x + b_0 u$ unstable if . . . ?

# Example 1: spring-mass systems - stability analysis from intuitive perspectives

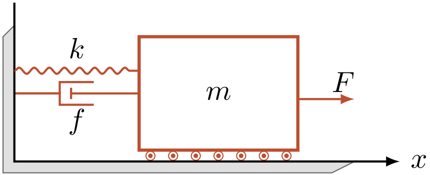
*Discussion:* $\dot{x} = a_0 x + b_0 u$ unstable if ...? And What about, from intuitive perspectives, when considering

$$\ddot{x} = a_1 \dot{x} + a_0 x + b_0 u \; ? \tag{38}$$



GitHub/TTK4225/trunk/Jupyter/spring-mass-systems-introduction.ipynb

# Intuition: asymptotic stability $\leftrightarrow$ dissipation of energy



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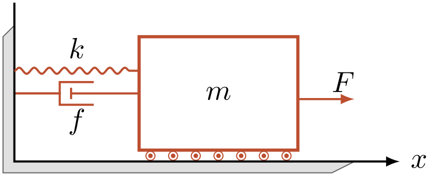
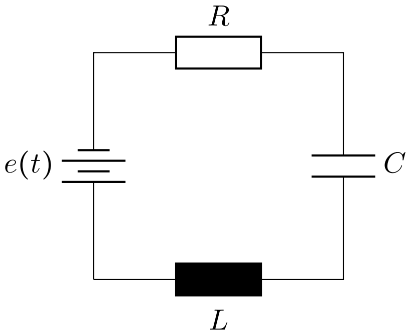



will see this MUCH better in later on courses
when discussing about Lyapunov stability

?

# Example 2: RCL-circuit



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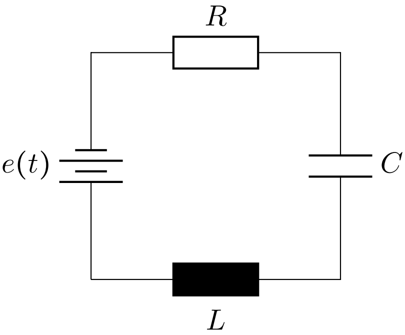


EOM: Kirkhoff laws

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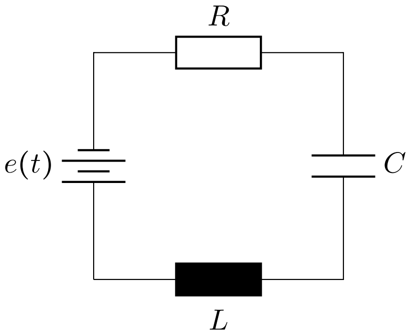
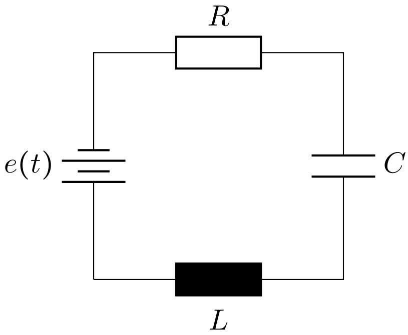



EOM: Kirkhoff laws

$$v_L(t) = L\frac{di(t)}{dt} \qquad v_i(t) = Ri(t) \qquad v_C(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

# Example 2: RCL-circuit




EOM: Kirkhoff laws

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$$e(t) = v_L(t) + v_R(t) + v_C(t)$$

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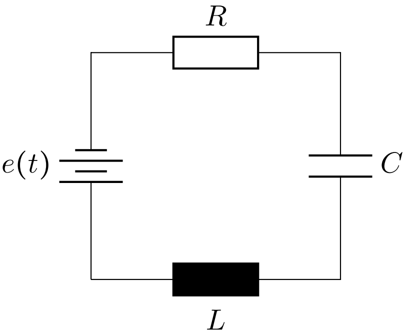


EOM: Kirkhoff laws

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$$e(t) = v_L(t) + v_R(t) + v_C(t) \quad \mapsto \quad e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau \tag{39}$$

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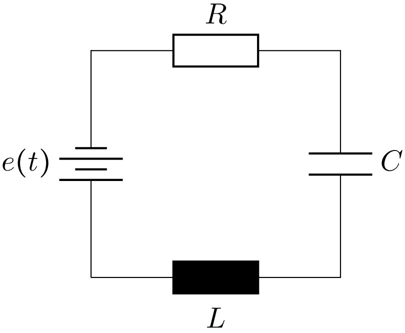


$$e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau$$

*Discussion:* how do we write it in terms of $x$, $\dot{x}$, $\ldots$?

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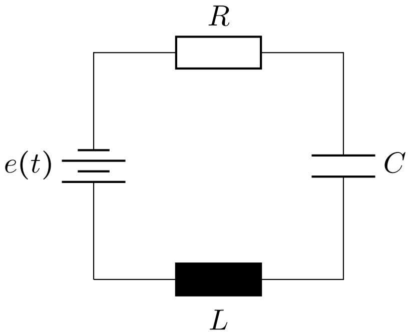


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$$x(t) = i(t), \quad u(t) = \dot{e}(t) \qquad \Longrightarrow \qquad \ddot{x} = -\frac{R}{L}\dot{x} - \frac{1}{LC}x + \frac{1}{L}u \tag{40}$$

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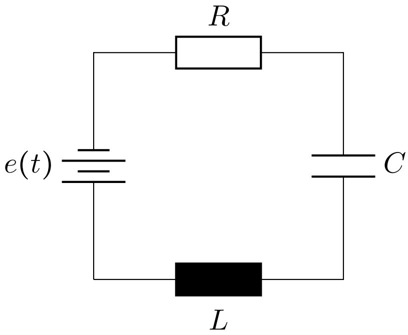


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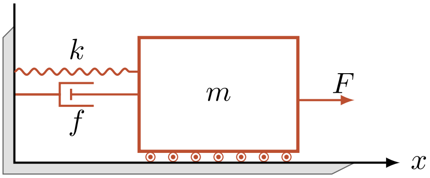
Generalizing:

?

# Discussion: what are the potential different ways the cart may move back to its equilibrium?



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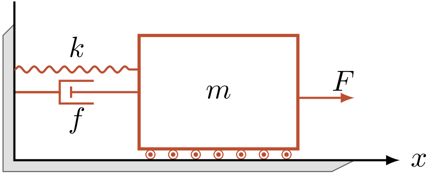
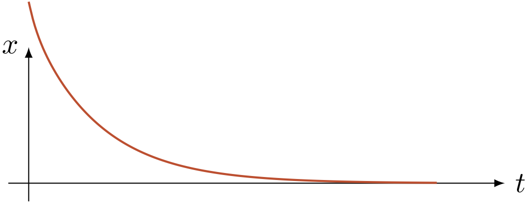



very big friction w.r.t. the spring stiffness $\implies$ overdamping:



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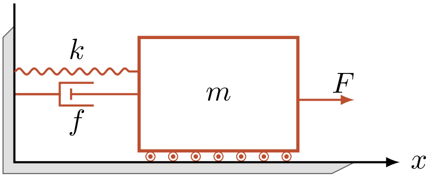


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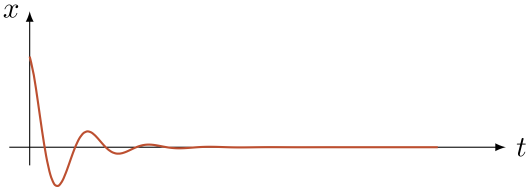
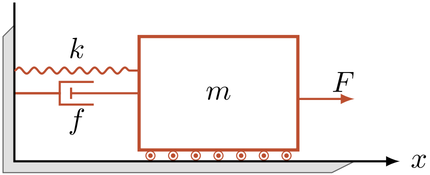
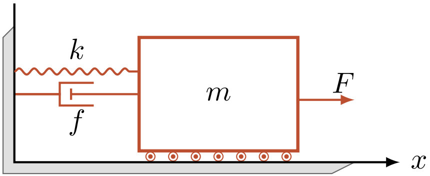


## Discussion: what are the potential different ways the cart may move back to its equilibrium?




just the right amount of friction w.r.t. the spring stiffness $\implies$ critical damping, i.e., *the combination that leads to the fastest decay to zero without oscillations*:

# Discussion: what are the potential different ways the cart may move back to its equilibrium?

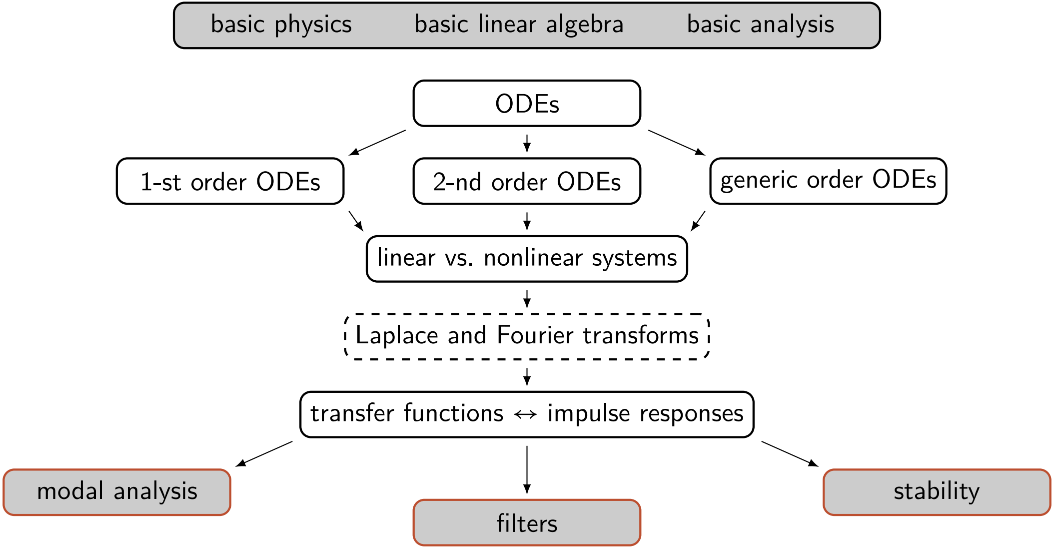



*Discussion:* can we get these behaviors with a first order system?

# How do the free evolutions and forced responses look like numerically?

*we will solve this problem with the aid of Laplace transforms*

# Summary / Roadmap of TTK4225



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