TTK4225 - Systems Theory, Autumn 2020

Damiano Varagnolo

NTNU

Complex numbers - introduction

Roadmap

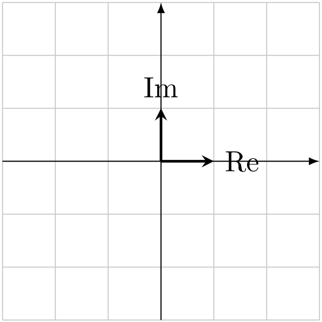
- definition
- sum, subtraction, multiplication, division
- why is this important?

What is a complex number, and why did we introduce them?

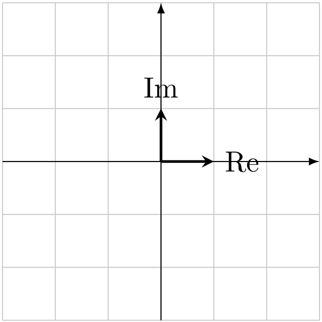
What is a complex number, and why did we introduce them?

In essence:

1. a point in the Cartesian plane
2. to be sure to find all the roots of polynomials *(i.e., be able to write polynomials in convenient forms)*

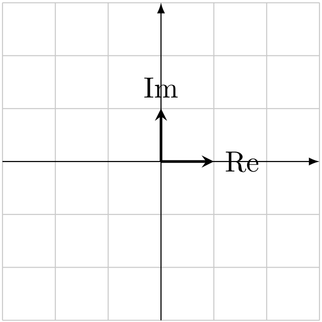


The “imaginary unit”



$$i \quad : \quad i^2 = -1$$

Simple operations with complex numbers: sums

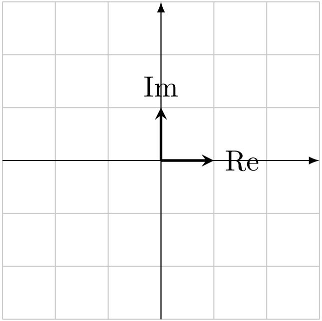


$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

implies

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i\,(b_1 + b_2)$$

Simple operations with complex numbers: subtractions

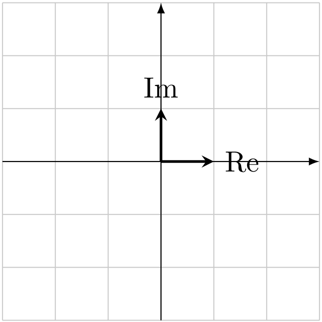


$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

implies

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i\,(b_1 - b_2)$$

Simple operations with complex numbers: multiplication

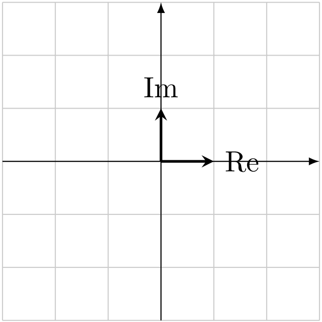


$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

implies

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Simple operations with complex numbers: inversion

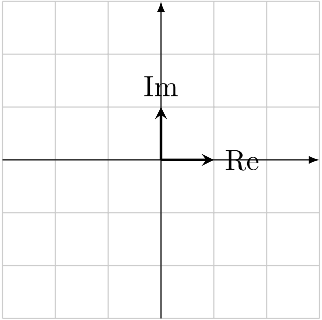


$$z_1 = a_1 + ib_1$$

implies

$$z_1^{-1} = \frac{a_1}{a_1^2 + b_1^2} - i\frac{b_1}{a_1^2 + b_1^2}$$

Simple operations with complex numbers: division

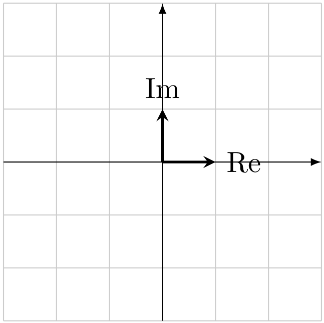


$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

implies

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}$$

Simple operations with complex numbers: conjugation



$$z_1 = a_1 + ib_1$$

implies

$$\overline{z_1} = a_1 - ib_1$$

Conjugacy: a good way of simplifying the previous operations

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- subtraction: $z - \overline{z} = a + ib - a + ib = 2ib$, thus $\mathrm{Im}\,(z) = \dfrac{1}{2i}\,(z + \overline{z})$

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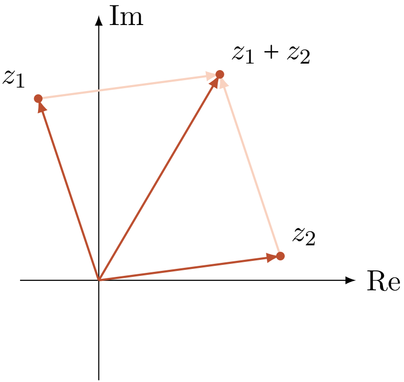
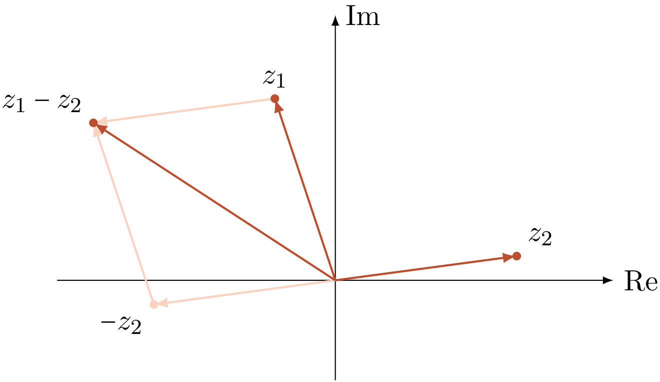
- subtraction: $z - \overline{z} = a + ib - a + ib = 2ib$, thus $\mathrm{Im}\left(z\right) = \dfrac{1}{2i}\left(z + \overline{z}\right)$
- multiplication: $z\overline{z} = \left(a + ib\right)\left(a - ib\right) = a^2 + b^2$, thus $\left|z\right|^2 = z\overline{z}$

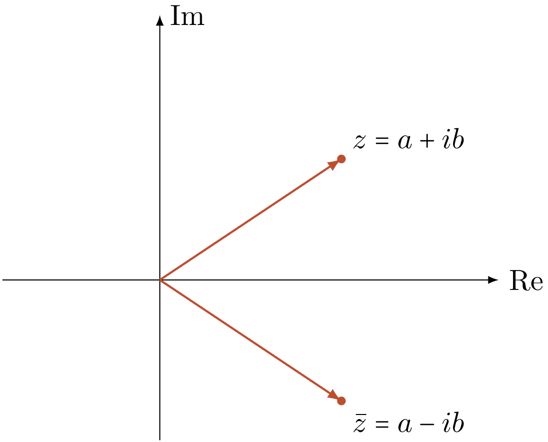
Once again, graphically: addition



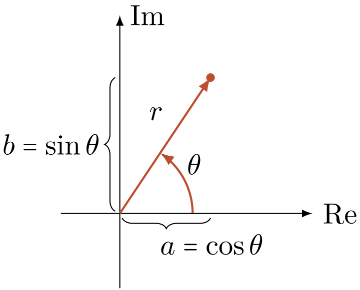
Once again, graphically: subtraction



Once again, graphically: conjugation



Polar coordinates



$$z = a + ib$$

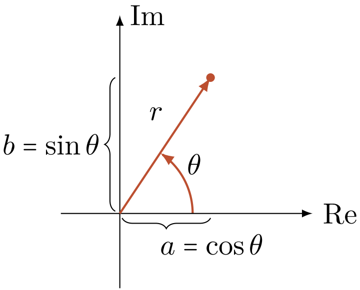
can be rewritten through r and θ so that

$$a = r\cos\theta \text{ and } b = r\sin\theta$$

so that

$$z = r\left(\cos\theta + i\sin\theta\right)$$

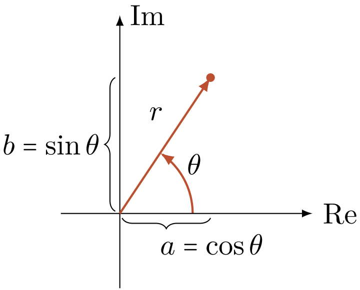
Polar coordinates



Equations:

$$
\begin{aligned}
r &= |z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}} \\
\theta &= \arg z = \operatorname{atan}(b, a) = \tan^{-1}\left(\frac{b}{a}\right)
\end{aligned}
$$

Polar coordinates



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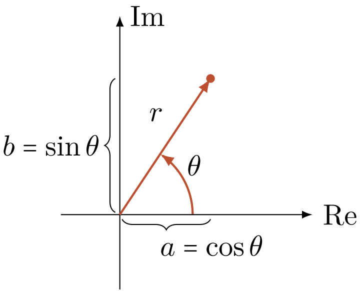
Notation:

- r = absolute value or modulus of z
- θ = argument, angle, or phase of z

Problem: different θ's lead to the same z



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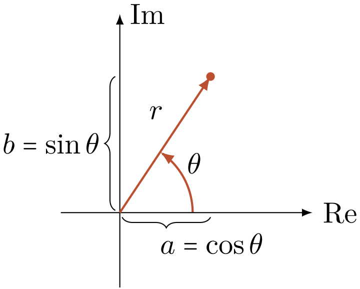
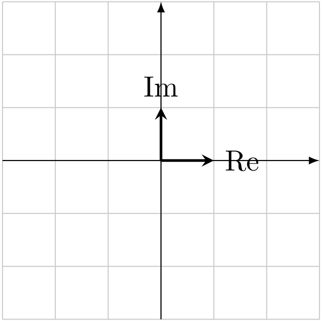


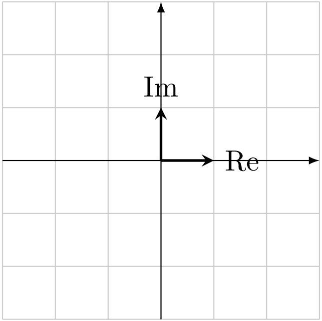
Definition: principal value of z = that value of θ that is in $[-\pi, \pi]$

Usefulness of polar forms: the multiplication is immediate



$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i \sin\left(\theta_1 + \theta_2\right)\right]$$

Implication: the division is immediate



$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$$

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Taylor expansions: a tool to do not underestimate

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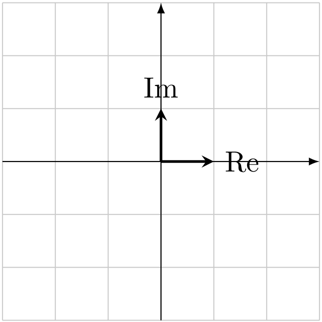
E.g., thus

$$e^z = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \ldots$$

?

The absolute value of a complex number



Meaning: Euclidean length of the vector. Very important for control, since very often we compute the absolute value of a transfer function at a specific $s = i\omega$ *(and very very often the transfer function is rational)*

Thus, chain of implications highlighting the importance of complex numbers for control perspectives

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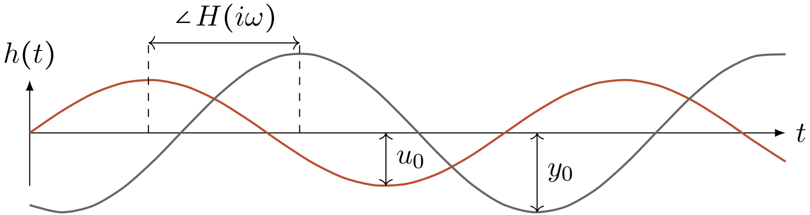
1: LTI systems have sinusoidal fidelity

$$u(t) = u_0 \sin(\omega t) \quad \Longrightarrow \quad y(t) = u_0 |H(i\omega)| \sin(\omega t + \angle H(i\omega))$$



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$$\Longrightarrow \qquad K \frac{\prod (s - z_i)}{\prod (s - p_i)}$$

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$$\Longrightarrow \quad K \frac{\prod (s - z_i)}{\prod (s - p_i)}$$

3: we very often consider rational H's

this means that to compute $|H(i\omega)|$ we need to do multiplications and divisions among the complex numbers $s - \star_i$

Example: $H(s) = \dfrac{1+2s}{1+2s+s^2}$. What is $|H(i\omega)|$?

$$
\begin{aligned}
|H(i\omega)| &= \sqrt{H(i\omega)\overline{H(i\omega)}} \\
&= \sqrt{\frac{1+2i\omega}{1+2i\omega-\omega^2} \cdot \frac{1-2i\omega}{1-2i\omega-\omega^2}} \\
&= \sqrt{\frac{1+4\omega^2}{(1-\omega^2)^2+4\omega^2}} \\
&= \sqrt{\frac{1+4\omega^2}{\omega^4+2\omega^2+1}}
\end{aligned}
$$

?

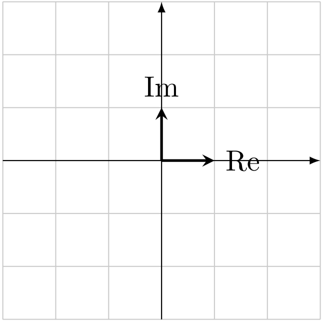
Complex functions

Roadmap

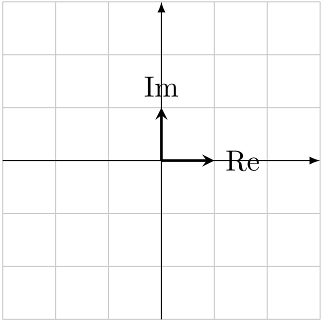
- definition
- why are they important?

Complex function: definition

$$f : \mathbb{C} \mapsto \mathbb{C}$$



$\mapsto$



Complex function: caveats

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thus, also in polar representations, $(r,\theta) \mapsto \left(r', \theta'\right)$ with in general both r' and θ' functions of both r and θ

Example: if $f(z) = z^2 + 3z$ then what is $f(1+3j)$?

$$
\begin{aligned}
f(z) &= (x+iy)(x+iy) + 3x + 3iy \\
&= x^2 + 2ixy - y^2 + 3x + 3iy \\
&= x^2 - y^2 + 3x + i(2xy + 3y)
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thus

$$\begin{aligned} u(x,y) &= x^2 - y^2 + 3x \\ v(x,y) &= 2xy + 3y \end{aligned}$$

thus

$$\begin{aligned} f(1+3j) &= u(1,3) + iv(1,3) \\ &= 1^3 - 3^2 + 3 + i(2 \cdot 1 \cdot 3 + 3 \cdot 3) \\ &= -5 + 15i \end{aligned}$$

Complex function: why are they important?

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Recall: the forced evolution is given by

$$Y(s) = H(s)U(s)$$

with $H(s)$ very often rational, i.e., ratio of polynomials.

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Recall: the forced evolution is given by

$$Y(s) = H(s)U(s)$$

with $H(s)$ very often rational, i.e., ratio of polynomials. *Essential tool for automatic control people: roots of complex polynomials*

Roots of complex functions

Roots of complex functions

Primary definition: root of a complex number

if $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then the n complex roots of z are the n complex numbers $z_0, \ldots, z_{n-1}$ for which $z_k^n = z$, i.e.,

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \quad \text{for} \quad k = 0, 1, \ldots, n-1$$

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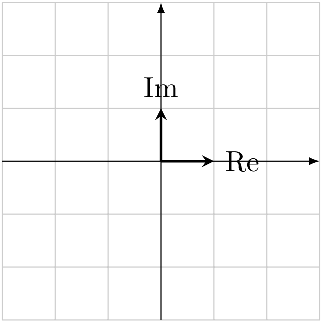
The intuition on how to get them follows from:

$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right]$$

Roots of complex functions

Geometrically:



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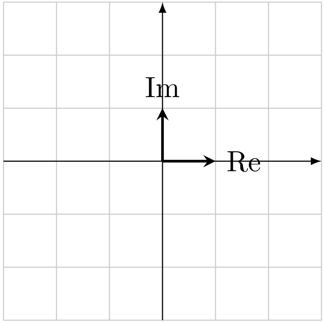
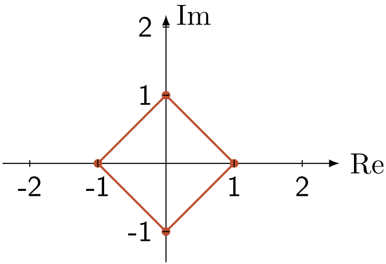


these n roots always exist

Roots of complex functions, example: quartic roots of 1

$$\sqrt[4]{1} = \{1, i, -1, -i\}$$

(note that only two of them are in $\mathbb{R}$)



IMPORTANT: ONE SHOULD CONSIDER THE PRINCIPAL VALUE

...otherwise one may artificially add $2\pi k$ to the phase of $w = \sqrt[n]{z}$ and have an infinite number of roots ...

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Because we often have to do with objects of the type $z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$, thus we need to know what we are dealing with!

Why are we using so much time on this?

Because we often have to do with objects of the type $z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$, thus we need to know what we are dealing with! *Essential results:*

- n-order polynomials have always from 0 to n real roots
- n-order polynomials have always n complex roots

Example

$$z^4 - 6iz^2 + 16 = 0$$

implies

$$z_1 = 2 + 2i \quad z_2 = -2 - 2i \quad z_3 = -1 + i \quad z_4 = 1 - i$$

(to get the solution let $y = z^2$, and then do a bit of massaging)

?

Complex exponentials

Roadmap

- intuitions
- definition
- Euler's identities
- complex logarithms

In the previous episodes . . .

- complex sums and multiplications
- complex roots
- complex polynomials

In the previous episodes ...

- complex sums and multiplications
- complex roots
- complex polynomials

$\rightarrow$ generalizing everything, even the functions

Discussion

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Because they are the essence of the modes of LTI systems with rational transfer functions, and LTI systems are often good approximations of nonlinear systems around their equilibria

First usefulness of complex exponentials: simplify notation even further

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$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots$$

with $\sin$ and $\cos$?

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with $\sin$ and $\cos$? Of course, Euler's formula!

Why does Euler's formula work? (so that one may remember it more. . .)

Starting point:

$$e^z = e^{x+iy} = e^x e^{iy}$$

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$$\begin{aligned} e^{iy} &= 1 + iy + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \ldots + \frac{1}{k!}(iy)^k + \ldots \\ &= \underbrace{\left(1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \frac{1}{6!}y^6 + \ldots\right)}_{=\cos(y)} + i \underbrace{\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \frac{1}{7!}y^7 + \ldots\right)}_{=\sin(y)} \end{aligned}$$

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thus

$$e^z = e^x\left(\cos y + i\sin y\right)$$

The new representation given by Euler's formula and polar representations

$$
\begin{aligned}
z &= x + iy = r\left(\cos\theta + i\sin\theta\right) \\
z &= re^{i\theta}
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with

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r = \sqrt{x^2 + y^2} \qquad \theta = \operatorname{atan}\frac{y}{x}
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Examples

$$ze^{i\alpha} = re^{i\theta}e^{i\alpha} = re^{i(\theta+\alpha)}$$

The new representation given by Euler's formula and polar representations

$$\begin{aligned} z &= x + iy = r\left(\cos\theta + i\sin\theta\right) \\ z &= re^{i\theta} \end{aligned}$$

with

$$r = \sqrt{x^2 + y^2} \qquad \theta = \operatorname{atan}\frac{y}{x}$$

This confirms the intuition that multiplying z in the complex plane by $e^{i\theta}$ means rotating z of θ radiants *anti-clockwise* in $\mathbb{C}$

Examples

$$\begin{aligned} ze^{i\alpha} &= re^{i\theta}e^{i\alpha} = re^{i(\theta+\alpha)} \\ zi &= re^{i\theta}e^{i\frac{\pi}{2}} = re^{i\left(\theta+\frac{\pi}{2}\right)} \end{aligned}$$

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$$

that, by the way, implies $(x + iy)\,i = -y + ix$, i.e., a 90-degrees rotation

How to remember the trigonometric identities

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Starting point:

$$e^{iy} = \underbrace{\left(1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \frac{1}{6!}y^6 + \dots\right)}_{=\cos(y)} + i\underbrace{\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \frac{1}{7!}y^7 + \dots\right)}_{=\sin(y)}$$

(must be in this way, because “cos” is even, “sin” is odd).

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thus

$$\sin y = \frac{1}{2i}\left(e^{iy} - e^{-iy}\right) \qquad \cos y = \frac{1}{2}\left(e^{iy} + e^{-iy}\right)$$

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- $e^{i\pi} = -1$, $e^{\pi i/2} = i$, $e^{-\pi i/2} = -i$, $e^{-\pi i} = -1$
- exponentials are never equal to 0, i.e., $e^z \neq 0$ independently of z
- exponentials are periodic, i.e., $e^{z+2\pi i} = e^z$

Notation: "fundamental region of the exponential"

$-\pi < \mathrm{Im}\,(z) \leq \pi$

Multiplications and divisions through the complex functions

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- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

Roots through the complex functions

$w = z^n$ is s.t. $w = re^{i\theta+2\pi k}$ and is equal to

$$z_k = r^{1/n} e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)}$$

(note that besides $k = 0, 1, \ldots, n-1$*, for other* k*'s we get the same roots as before)*

Complex logarithms

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Starting point: real logarithms

- if $x \in \mathbb{R}$ then $\ln(x)$ is s.t. $e^{\ln(x)} = x$
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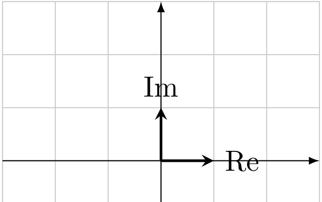
Graphically:



Very important difference

as soon as $z = re^{i\theta}$ is s.t. $r > 0$ then

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exists, thus the complex logarithm is defined for every $z \neq 0$!

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Example

$$\ln(-10) = 2.30259 + i\pi.$$

LTI filters - motivations

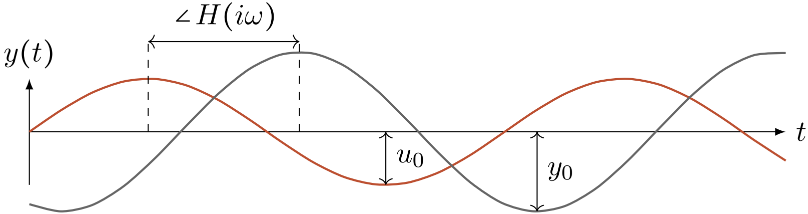
Roadmap

- sinusoidal fidelity
- how Fourier transforms help to analyse LTI systems
- introduction to Bode plots

Recall: why are Fourier transforms important for control people?

Among others: $H(i\omega)$ says how to apply the sinusoidal fidelity property, i.e., the fact that

$$u(t) = u_0 \sin(\omega t) \quad \Longrightarrow \quad y(t) = u_0 |H(i\omega)| \sin(\omega t + \angle H(i\omega))$$



A motivating example: sinusoidal fidelity

$$\dot{y} = ay + u \quad u(t) = \sin \omega t \qquad \Longrightarrow \qquad y(t) = y_0 e^{at} + e^{at} \int_0^t e^{-a\tau} \sin(\omega\tau)\, d\tau$$

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$$\int e^{-a\tau} \sin(\omega t)\, d\tau = \frac{e^{-a\tau}}{a^2 + \omega^2} (-a \sin \omega\tau - \omega \cos \omega\tau) + C$$

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Sinusoidal fidelity of LTIs, in details

$$\dot{y} = ay + u \quad u(t) = \sin \omega t$$

implies

$$y(t) = y_0 e^{at} + \frac{\omega}{a^2 + \omega^2} e^{at} + \frac{1}{a^2 + \omega^2} \left(-a \sin \omega t - \omega \cos \omega t \right)$$

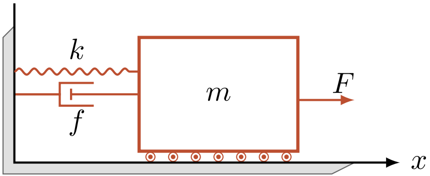
Notation:

- first term = free evolution
- second term = transient response *(part of the forced response)*
- third term = stationary response *(part of the forced response)*

Why do we have free evolution, plus transient & stationary response?

Example: spring-mass system:

$$y(t) = y_0 e^{at} + \frac{\omega}{a^2 + \omega^2} e^{at} + \frac{1}{a^2 + \omega^2} \left(-a \sin \omega t - \omega \cos \omega t \right)$$



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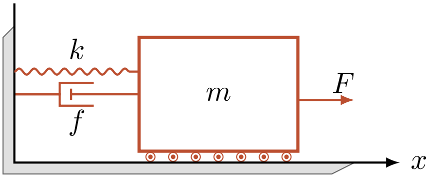
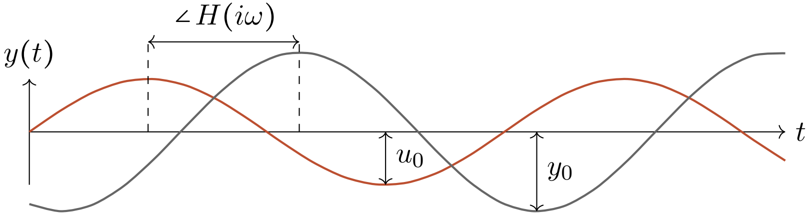


Answer: even if we have $y_0 = 0$, the machine needs to "warm up"

LTI means sinusoidal fidelity, and sinusoidal fidelity for every sine means LTI

Idea: we can estimate H by repeating experiments with different sinusoidal u's!

$$y(t) \approx u_0 \left|H(i\omega)\right| \sin\left(\omega t + \angle H(i\omega)\right)$$



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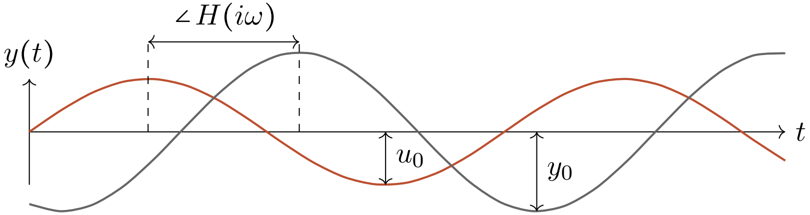
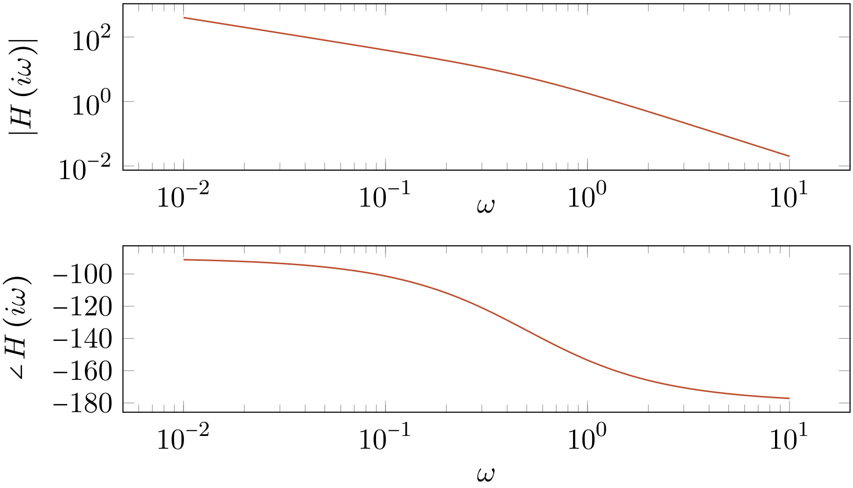


Good idea: check the "ETFE" system identification approach if you want to know more

Idea: we can estimate h by first estimating H, and then inverse-Laplacing!

. . . actually not. There are better ways of doing this → will be seen in courses that deal with system identification

Our goal: arrive at Bode plots



(will see them better what they are in the next modules)

?

Frequency response of LTI filters

Roadmap

- decomposing the output of a LTI system
- what is called how

Recap of how to decompose the output of a LTI system

$$y(t) = y_{\text{free}}(t) + y_{\text{forced,transient}}(t) + y_{\text{forced,stationary}}(t)$$

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Recap of how to decompose the output of a LTI system

$$y(t) = y_{\text{free}}(t) + y_{\text{tran}}(t) + y_{\text{stat}}(t)$$

with:

- free evolution: starts from y_0 and vanishes as $t \to +\infty$ if the system is asymptotically stable
- transient response: part of forced response that behaves in a similar way than the free evolution
- stationary response: part of forced response such that if $u(t) = \sin \omega t$ then

$$y_{\text{stat}}(t) = A(\omega) \sin\left(\omega t + \varphi(\omega)\right)$$

A deeper look on the stationary response

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Objects that define the stationary response:

- frequency ω
- amplitude $A(\omega)$
- phase $\varphi(\omega)$

I.e., $A(\cdot)$ and $\varphi(\cdot)$ are two functions of ω and each system has its own ones

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- frequency ω
- amplitude $A(\omega)$
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I.e., $A(\cdot)$ and $\varphi(\cdot)$ are two functions of ω and each system has its own ones

> $A(\cdot)$ and $\varphi(\cdot)$ define entirely how the sys-
> tem behaves <u>after</u> the initial transient vanished

very common misconception: A and φ **do not** define the behavior of the whole system. They define only the behavior after the transient has passed. I.e., they **do not** say what happens during the transient!

$$u(t) = \sin \omega t \qquad \Longrightarrow \qquad y_{\text{stat}}(t) = A(\omega) \sin \left(\omega t + \varphi(\omega)\right)$$

Frequency response of a LTI system: formal definition

:= the stationary part of $y(t)$ that results from a sinusoidal $u(t)$

Frequency response of a LTI system "in time"

Question: how can we get $y(t)$ from the frequency domain:

$$Y(s) = H(s)\frac{u_0\omega}{s^2+\omega^2}?$$

Answer = partial fraction decomposition, that leads to

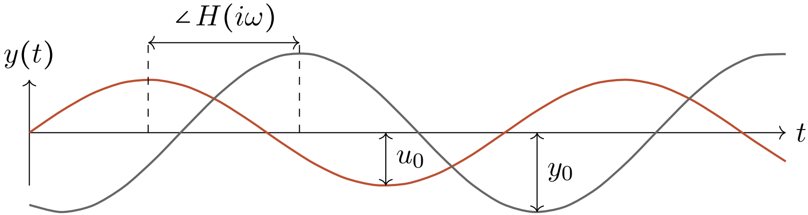
$$y(t) = \underbrace{u_0 H(i\omega)\frac{e^{i\omega t}}{2i} - \frac{e^{-i\omega t}}{2i}}_{\text{stationary response}} + \underbrace{\sum_{i=1}^{n} \left.\frac{(s-s_i)H(s)u_0\omega e^{ts}}{s^2+\omega^2}\right|_{s=s_i}}_{\text{transient response}}$$

with the sum being over the poles of $H(s)$ *(note that in the transient response the numerators are being computed using the residues theorem)*

Back to the stationary response

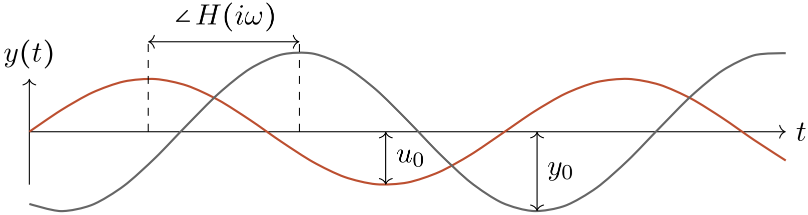
thus the stationary response to $u(t) = u_0 \sin \omega t$ is

$$y_{\text{stat}}(t) = u_0 H(i\omega) \frac{e^{i\omega t}}{2i} - \frac{e^{-i\omega t}}{2i}$$



Back to the stationary response – using Euler formulas

$$y_{\text{stat}}(t) = u_0 \, |H(i\omega)| \sin\left(\omega t + \angle H(i\omega)\right)$$



Notation

$$y_{\text{stat}}(t) = u_0 \left|H(i\omega)\right| \sin\left(\omega t + \angle H(i\omega)\right)$$

- $H(i\omega)$ = "frequency response"
- $y_0/u_0 = \left|H(i\omega)\right|$ = "amplitude response"
- $\angle H(i\omega)$ = "phase response"

?

Important to realize

- $H(s)$, i.e., the transfer function, tells us everything, since from it we can get the impulse response $h(t)$ and from $h(t)$ we can compute everything
- $H(i\omega)$, i.e., the frequency response, tells us only what happens to the stationary response. Indeed $H(i\omega)$ is only *part of* $H(s)$

?

Bode plots

Roadmap

- why and how to use Bode plots
- their main property: a sort of "additivity"
- examples
- remarks

Summarizing

- using $H(s)$ we can compute the whole $y(t)$
- using $H(i\omega)$ we can compute only $y_{\mathrm{stat}}(t)$

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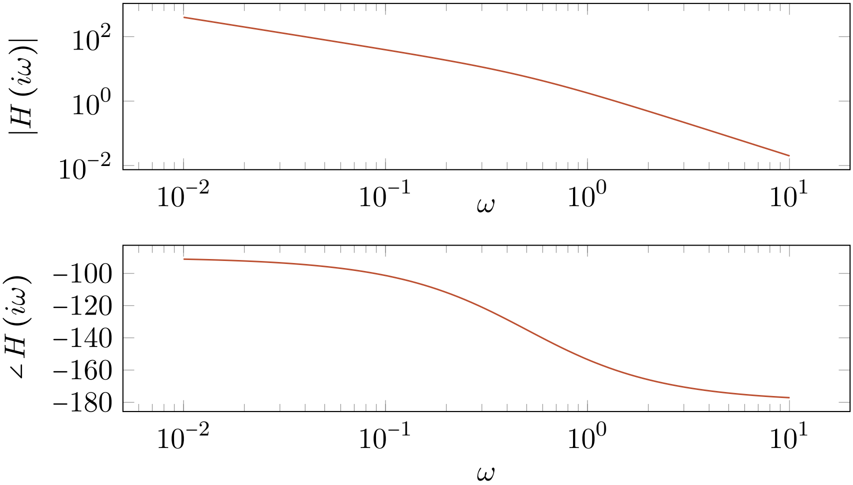
- using $H(s)$ we can compute the whole $y(t)$
- using $H(i\omega)$ we can compute only $y_{\text{stat}}(t)$

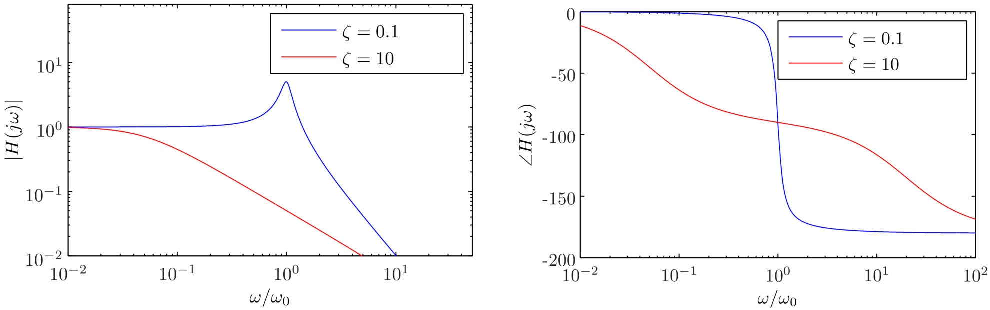
Good thing of $H(i\omega)$ is that we can visualize it! *(and visualizations help understanding and communicating with people)*

Bode diagrams, i.e., visualizing $H(i\omega)$ (and thus somehow also $y_{\text{stat}}(t)$)



Note: typically uses Decibels in the vertical axis, i.e., $|H(i\omega)|[\mathrm{dB}] = 20\log|H(i\omega)|$, and a logarithmic horizontal axis

Example: resonance



Example: resonance, in more details, through a spring-mass system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{f}{m} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}$$

with

$$H(s) = \frac{Y(s)}{U(s)} = \frac{m/k}{\dfrac{m}{k}s^2 + \dfrac{f}{k}s + 1}$$

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with

$$H(s) = \frac{Y(s)}{U(s)} = \frac{m/k}{\dfrac{m}{k}s^2 + \dfrac{f}{k}s + 1}$$

letting: $\omega_0 = \sqrt{\dfrac{k}{m}}$ $\quad \zeta = \dfrac{1}{2}\dfrac{f}{\sqrt{mk}}$ $\quad K = \dfrac{m}{k}$ then

$$H(s) = \frac{K}{1 + 2\zeta\dfrac{s}{\omega_0} + \left(\dfrac{s}{\omega_0}\right)^2} \qquad \Longrightarrow \qquad H(i\omega) = \frac{K}{1 + 2i\zeta\dfrac{\omega}{\omega_0} - \left(\dfrac{\omega}{\omega_0}\right)^2}$$

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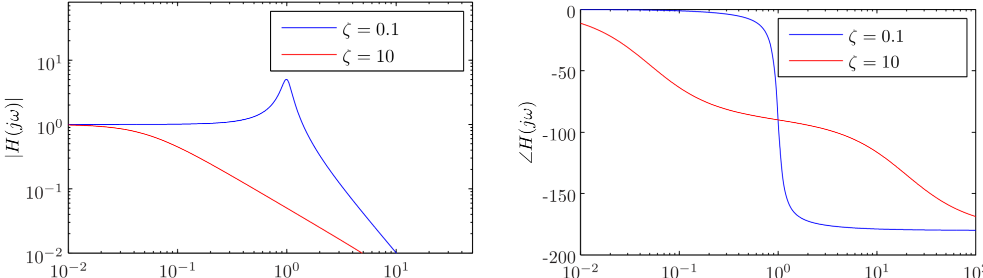
$$H(i\omega) = \frac{K}{1 + 2i\zeta\frac{\omega}{\omega_0} - \left(\frac{\omega}{\omega_0}\right)^2}$$

Discussion:

- what happens to $|H(i\omega)|$ when $\omega \to 0$?
- what happens to $|H(i\omega)|$ when $\omega \to +\infty$?
- what happens to $|H(i\omega)|$ when $\zeta \to 0$ and $\omega \to \omega_0$?



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What is the inner structure of Bode plots?

Assumption: $H(s)$ rational, so that

$$H(s) = \frac{G_1(s) \cdots G_m(s)}{F_1(s) \cdots F_n(s)}$$

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Rearranging amplitudes with amplitudes and phases with phases:

$$\angle H(j\omega) = \angle G_1(j\omega) + \angle G_m(j\omega) - \angle F_1(j\omega) - \angle F_n(j\omega)$$

$$\log |H(j\omega)| = \log |G_1(j\omega)| + \ldots + \log |G_m(j\omega)| - \log |F_1(j\omega)| - \ldots - \log |F_n(j\omega)|$$

What does it mean?

$$\angle H(j\omega) = \angle G_1(j\omega) + \angle G_m(j\omega) - \angle F_1(j\omega) - \angle F_n(j\omega)$$

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$\Longrightarrow$ every pole and zero contributes by its own

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$$\angle H(j\omega) = \angle G_1(j\omega) + \angle G_m(j\omega) - \angle F_1(j\omega) - \angle F_n(j\omega)$$

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$\Longrightarrow$ every pole and zero contributes by its own $\Longrightarrow$ we can decompose the Bode plot in "atomic" contributions!

What does it mean?

$$\angle H(j\omega) = \angle G_1(j\omega) + \angle G_m(j\omega) - \angle F_1(j\omega) - \angle F_n(j\omega)$$

$$\log|H(j\omega)| = \log|G_1(j\omega)| + \ldots + \log|G_m(j\omega)| - \log|F_1(j\omega)| - \ldots - \log|F_n(j\omega)|$$

$\implies$ every pole and zero contributes by its own $\implies$ we can decompose the Bode plot in “atomic” contributions!

Example:

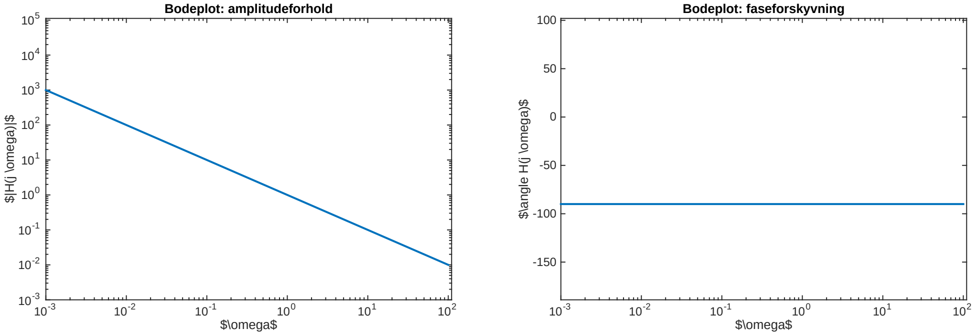
Bode plot of $\dfrac{s+3}{s(s-2)(s-3)}$ = “sum” of the Bode plots of

$$s+3 \qquad \frac{1}{s} \qquad \frac{1}{s-2} \qquad \frac{1}{s-3}$$

(with the meaning of “sum” explained better below)

What is the contribution of $\dfrac{1}{s}$, i.e., an integrator?

- contribution in amplitude: $\left|\dfrac{1}{i\omega}\right| [\mathrm{dB}] = -20 \log \omega$
- contribution in phase: $\angle \dfrac{1}{i\omega} = -\dfrac{\pi}{2}$



What is the contribution of $\dfrac{1}{Ts+1}$, i.e., a real pole?

Contribution in amplitude, letting $\omega = \dfrac{1}{T}$:

$$
\begin{aligned}
\left|\frac{1}{1+iT\omega}\right|[dB] &= \frac{1}{\sqrt{1+(T\omega)^2}}[dB] \\
&= -20\log\sqrt{1+(T\omega)^2} \\
&= \begin{cases} \approx 0[dB] & \text{for } \omega \ll 1/T \\ -20\log\sqrt{2} \approx -3[dB] & \text{for } \omega = 1/T \\ \approx -20\log\omega - 20\log T & \text{for } \omega \gg 1/T \end{cases}
\end{aligned}
$$

What is the contribution of $\dfrac{1}{Ts+1}$, i.e., a real pole?

Contribution in phase, letting $\omega = \dfrac{1}{T}$:

$$\angle \frac{1}{1+iT\omega} = \begin{cases} \approx 0 & \text{for } \omega \ll 1/T \\ = -\dfrac{\pi}{4} = -45^\circ & \text{for } \omega = 1/T \\ \approx -\dfrac{\pi}{2} = -90^\circ & \text{for } \omega \gg 1/T \end{cases}$$

What is the contribution of $\dfrac{1}{\frac{s^2}{\omega_0^2} + 2\frac{\zeta}{\omega_0}s + 1}$, i.e., a pair of complex conjugate poles?

Contribution in amplitude:

$$
\begin{aligned}
\left|\frac{1}{1-\left(\frac{\omega}{\omega_0}\right)^2 + i2\zeta\frac{\omega}{\omega_0}}\right|[dB] &= \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}}[dB] \\
&= -20\log\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2} \\
&= \begin{cases} \approx 0[dB] & \text{for } \omega \ll \omega_0 \\ = -20\log 2\zeta & \text{for } \omega = \omega_0 \\ \approx -40\log\omega + 40\log\omega_0 & \text{for } \omega \gg \omega_0 \end{cases}
\end{aligned}
$$

What is the contribution of $\dfrac{1}{\frac{s^2}{\omega_0^2} + 2\frac{\zeta}{\omega_0}s + 1}$, i.e., a pair of complex conjugate poles?

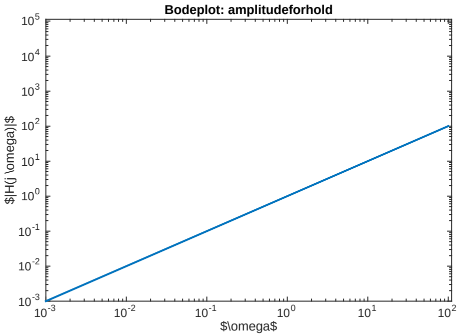
Contribution in phase:

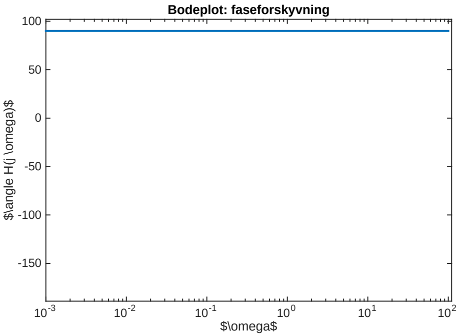
$$\angle \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + i2\zeta\frac{\omega}{\omega_0}} = -\text{atan}\frac{2\zeta\frac{\omega}{\omega_0}}{1 - \left(\frac{\omega}{\omega_0}\right)^2}$$

$$= \begin{cases} \approx 0 & \text{for } \omega \ll \omega_0 \\ = -\dfrac{\pi}{2} = -90^\circ & \text{for } \omega = \omega_0 \\ \approx -\pi = -180^\circ & \text{for } \omega \gg \omega_0 \end{cases}$$

What is the contribution of s, i.e., a derivator?

- contribution in amplitude: $|i\omega|\,[\mathrm{dB}] = 20 \log \omega$
- contribution in phase: $\angle i\omega = \dfrac{\pi}{2}$





What is the contribution of $Ts + 1$, i.e., a zero?

Contribution in amplitude, letting $\omega = \dfrac{1}{T}$:

$$|1 + iT\omega|\,[dB] = 20\log\sqrt{1 + (\omega T)^2} = \begin{cases} \approx 0 & \text{for } \omega \ll \omega_0 \\ 20\log\sqrt{2} & \text{for } \omega = \omega_0 \\ \approx 20\log\omega + 20\log T & \text{for } \omega \gg \omega_0 \end{cases}$$

What is the contribution of $Ts + 1$, i.e., a zero?

Contribution in phase, letting $\omega = \dfrac{1}{T}$:

$$\angle 1 + iT\omega = \begin{cases} \approx 0 & \text{for } \omega \ll 1/T \\ -\dfrac{\pi}{4} = 45^\circ & \text{for } \omega = 1/T \\ \approx \dfrac{\pi}{2} = 90^\circ & \text{for } \omega \gg 1/T \end{cases}$$

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Discussion: what happens if a zero is in the right-hand plane? $\mapsto$ non-minimum phase systems, will see them soon!

Summary of the most important contributions

constant

$$H(s) = K \quad \mapsto \quad |H(j\omega)|[dB] = 20\log K dB \quad \angle H(j\omega) = 0^\circ \ \forall\omega$$

single pole

$$H(s) = \frac{1}{1+Ts} \qquad \begin{cases} |H(j\omega)|[dB] = \ 0dB & \angle H(j\omega) = \ 0^\circ & \text{for } \omega \ll \frac{1}{T} \\ |H(j\omega)|[dB] = \ -20dB & \angle H(j\omega) = \ -90^\circ & \text{for } \omega \gg \frac{1}{T} \end{cases}$$

single zero

$$H(s) = 1 + Ts \qquad \begin{cases} |H(j\omega)|[dB] = \ 0dB & \angle H(j\omega) = \ 0^\circ & \text{for } \omega \ll \frac{1}{T} \\ |H(j\omega)|[dB] = \ 20dB & \angle H(j\omega) = \ 90^\circ & \text{for } \omega \gg \frac{1}{T} \end{cases}$$

Summary of the most important contributions

And the others?

integrators and derivators = limit cases for $T \to +\infty$

Note: nobody nows draw stuff by hand. You should nonetheless remember the general behaviors, so to do not need to actually draw Bode plots if you just need to check something (and in any case you should know the concepts)

?

Non-minimum phase systems

Roadmap

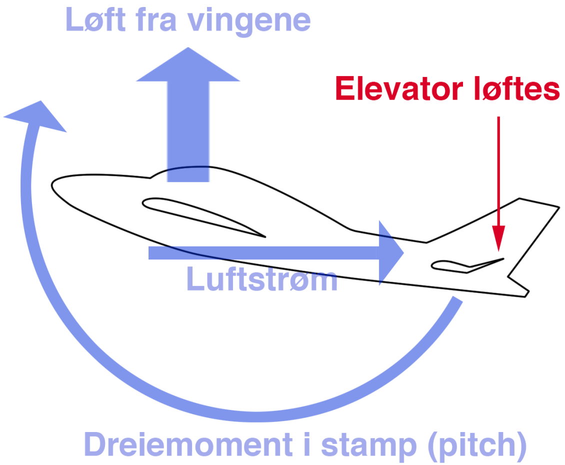
- what is a minumum phase system?
- how can I see if a LTI system is non-minimum phase from its Bode plot?
- which properties do minimum phase systems have?

Example 1: putting some wood in a fireplace

- does the amount of heat produced by the fireplace increase immediately?

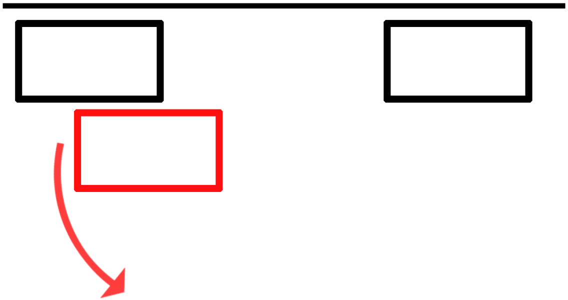
Example 2: changing the altitude of an airplane

- does the altitude of the plane increase immediately?



Example 3: parking a car backwards

- does the distance from the curb decrease immediately?



Non-minimum-phase systems: systems that have a step response that start “in the wrong direction”

Non-minimum-phase systems: systems that have a step response that start “in the wrong direction”

How do we see this from Bode plots? → this module

Considerations about Bode plots

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- typically Bode plots have phase plots that are entirely negative

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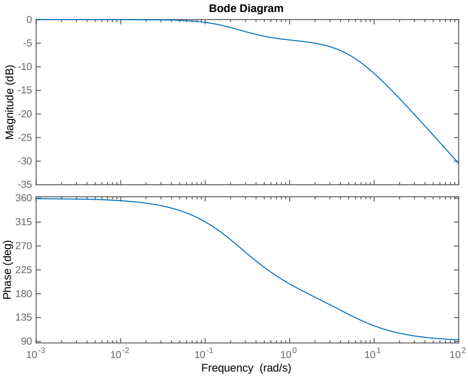
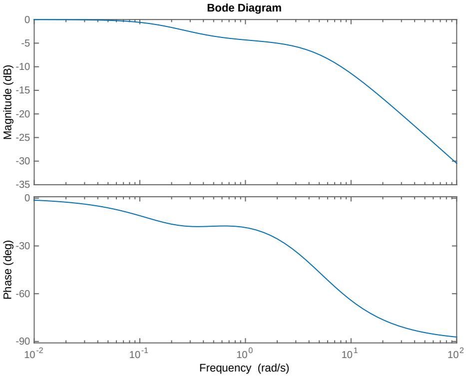
- typically Bode plots have phase plots that are entirely negative
- there may be different $H(s)$ that have the same $|H(i\omega)|$ but different $\angle H(i\omega)$

Example about "there may be different $H(s)$ that have the same $|H(i\omega)|$ but different $\angle H(i\omega)$"

$$H_1(s) = \frac{K(1 - T_3 s)}{(1 + T_1 s)(1 + T_2 s)} \qquad H_2(s) = \frac{K(1 + T_3 s)}{(1 + T_1 s)(1 + T_2 s)}$$





But if “there may be different $H(s)$ that have the same $|H(i\omega)|$ but different $\angle H(i\omega)$”, then which $H(s)$ is that one that has the “most negative” phase?

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Conditions:

- all the poles of $H(s)$ are in the left plane
- all the zeros of $H(s)$ are in the left plane
- $H(s)$ does not contain delays (i.e., e^{-as}). In other words, $H(s)$ is rational

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Conditions:

- all the poles of $H(s)$ are in the left plane
- all the zeros of $H(s)$ are in the left plane
- $H(s)$ does not contain delays (i.e., e^{-as}). In other words, $H(s)$ is rational

one can show that flipping the zeros on the right plane does not change the amplitude response, but changes the phase one!

Sometimes systems are always minimum phase

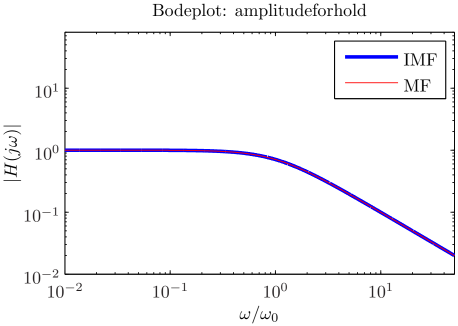
Example: spring-mass systems for which $m, f, k > 0$:

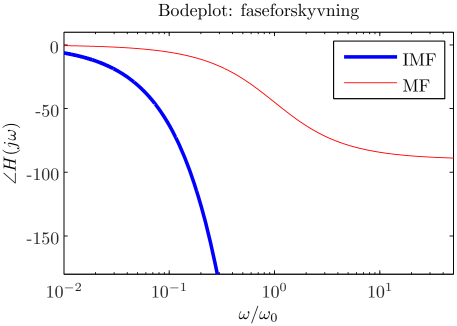
$$H(s) = \frac{m/k}{\frac{m}{k}s^2 + \frac{f}{k}s + 1}$$

Sometimes systems are always non-minimum phase

Example: system with time-delay:

$$H(s) = \frac{e^{-s/T_2}}{1 + T_1 s}$$



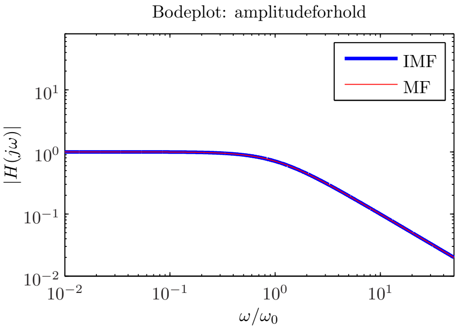


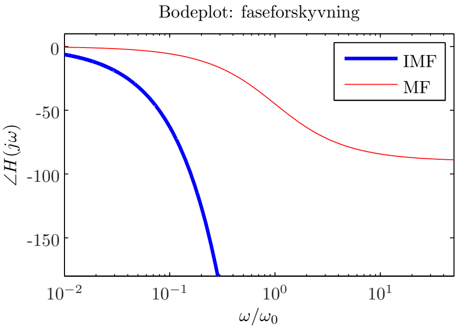
Sometimes systems are always non-minimum phase

Example: system with time-delay:

$$H(s) = \frac{e^{-s/T_2}}{1 + T_1 s}$$

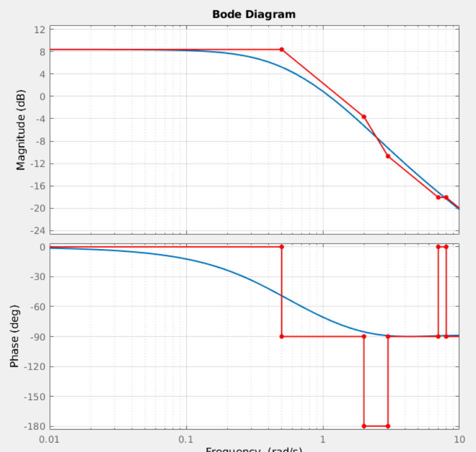
Note how $H'(s) = \dfrac{1}{1 + T_1 s}$ has the same amplitude response, but higher phase response:

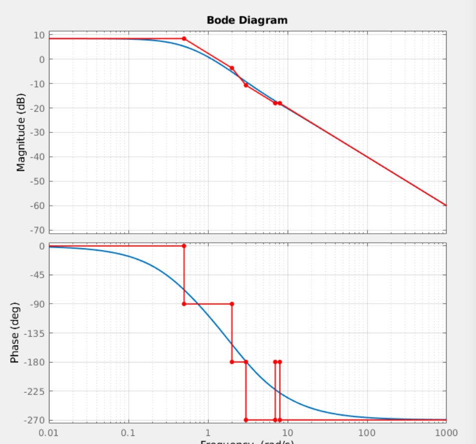




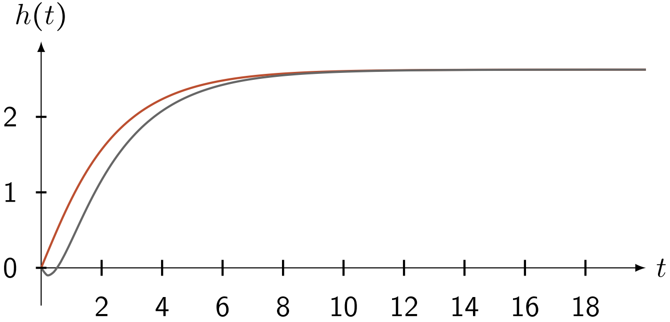
Another example

$$H_1(s) = \frac{(s+3)(s+7)}{(s+0.5)(s+2)(s+8)} \qquad H_2(s) = -\frac{(s-3)(s+7)}{(s+0.5)(s+2)(s+8)}$$





The most important concept in this topic: the smallest the phase, the 'earliest' the impulse response happens



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Filters

Roadmap

- motivations
- the simplest filters
- hints on how to make them more complicated

Why and where do we use filters?

Example 1: managing time signals, e.g., to

- eliminate some undesired frequencies
- separate different signals with different components
- create models that can then be used to do data-compression
- removing trends and biases

Why and where do we use filters?

Example 2: make the actuators only follow meaningful signals, e.g., by

- remove dithering
- remove offsets and low-frequency biases

Why and where do we use filters?

Example 3: help the estimation of the state of the system, e.g., by

- filtering noise

(this is connected with the topics of Luenberger observers and Kalman filters)

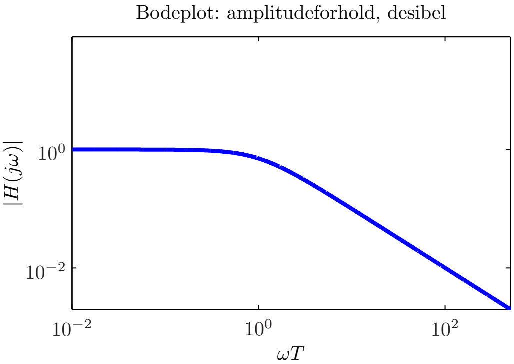
The 4 fundamental types of filters

- low-pass
- high-pass
- band-pass
- notch / band-stop

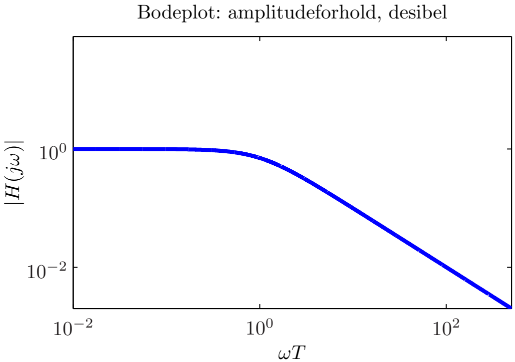
Low-pass filters, conceptually

- make the frequencies that are sufficiently low pass undisturbed
- stop the higher frequencies

Low-pass filters, Bode plots



Low-pass filters, Bode plots



Simplest low pass filter:

$$H(s) = \frac{1}{1+Ts} \quad \Longrightarrow \quad \begin{cases} |H(i\omega)| \approx 1 \text{ for } \omega \ll 1/T \\ |H(i\omega)| \approx 0 \text{ for } \omega \gg 1/T \end{cases}$$

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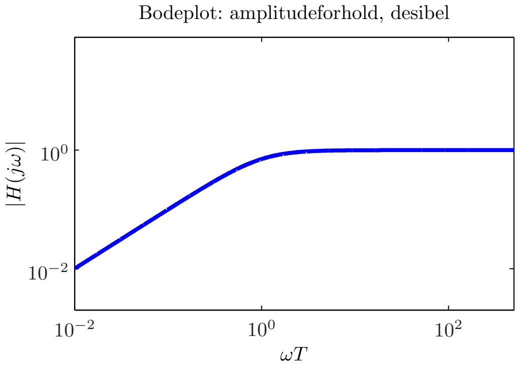
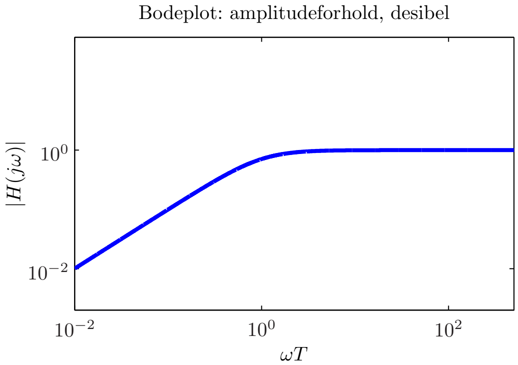
High-pass filters, conceptually

- stop frequencies that are sufficiently low
- make the higher frequencies pass undisturbed

High-pass filters, Bode plots



High-pass filters, Bode plots



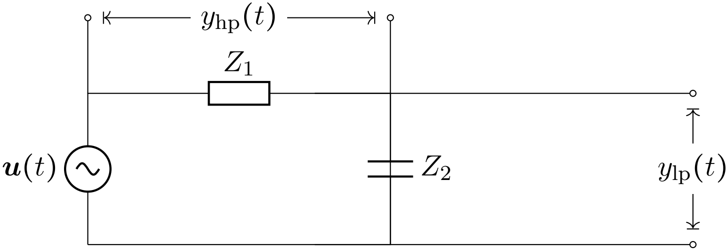
Simplest high pass filter:

$$H(s) = \frac{Ts}{1+Ts} \quad \Longrightarrow \quad \begin{cases} |H(i\omega)| \approx 0 \text{ for } \omega \ll 1/T \\ |H(i\omega)| \approx 1 \text{ for } \omega \gg 1/T \end{cases}$$

Do you like electronics? Then you may build your own RC-filter!

Concepts:

- condensers work as low-pass filters
- resistances can be used to work as high-pass filters



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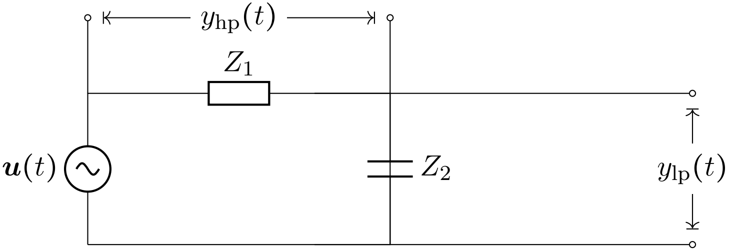


this is how filters were all implemented once, and this is still a valid strategy in cheap embedded systems

?

Band-pass filters, conceptually

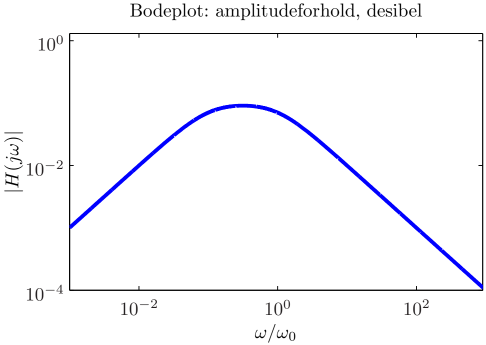
- make frequencies within a defined band pass undisturbed
- stop frequencies that are either too low or too high

Band-pass filters, conceptually

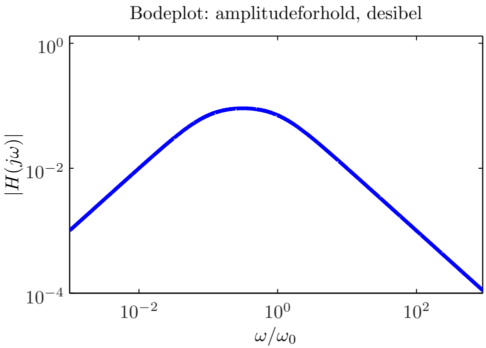
- make frequencies within a defined band pass undisturbed
- stop frequencies that are either too low or too high

i.e., combine a low-pass and a high-pass filter

Band-pass filters, Bode plots



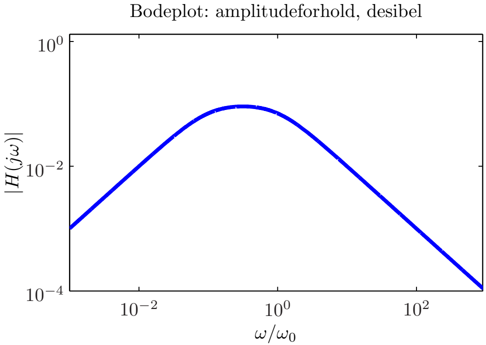
Band-pass filters, Bode plots



Simplest band-pass filter: series (i.e., kind of a logical "and") of low- and high-pass:

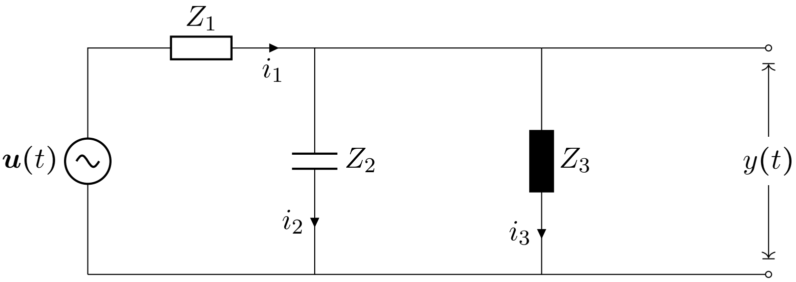
$$H_{\mathrm{lp}}(s) = \frac{1}{1+T_1 s} \text{ and } H_{\mathrm{hp}}(s) = \frac{s}{1+T_2 s} \quad \text{thus} \quad H(s) = \frac{s}{1+(T_1+T_2)\,s+T_1T_2s^2}$$

Band-pass filters, Bode plots



- all the frequencies $\gg 1/T_1$ will be stopped by the low pass part
- all the frequencies $\ll 1/T_2$ will be stopped by the high pass part

Band-pass via RLC-circuits



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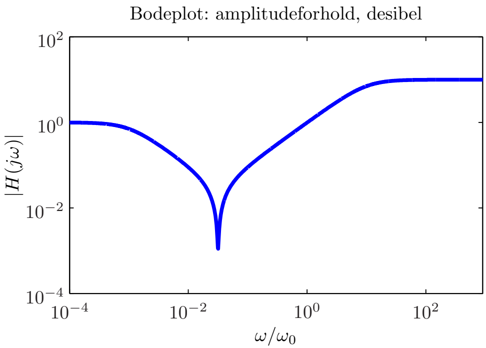
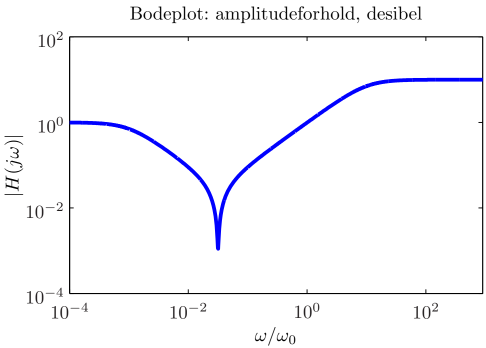
Notch / band-stop filters, conceptually

- stop a certain band of frequencies
- let the others pass

Notch / band-stop filters, Bode plots



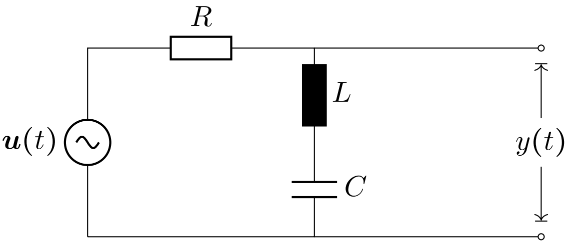
Notch / band-stop filters, Bode plots



Simplest band-stop filter: parallel (i.e., kind of a logical "or") of high- and low-pass filter

$$H_{\mathrm{lp}}(s) = \frac{1}{1+T_1 s} \text{ or } H_{\mathrm{hp}}(s) = \frac{sT_2}{1+T_2 s} \quad \text{thus} \quad H(s) = \frac{T_2 s}{T_1 T_2 s^2 + (T_1 + T_2)\, s + 1}$$

Notch / band-stop via RLC-circuits



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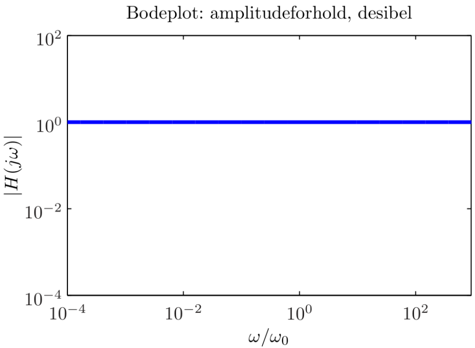
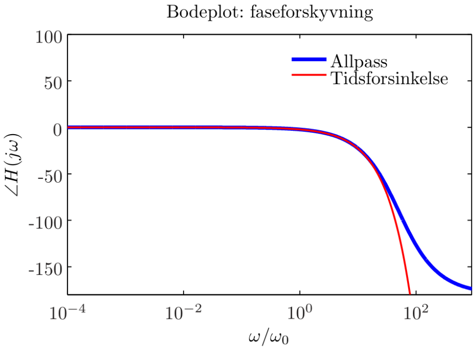
An interesting filter: all-pass, but adding some phase

Motivation: model or compensate for time-delays:





An interesting filter: all-pass, but adding some phase

requirement: $|H(i\omega)| = 1$ for all the ω's but $\angle H(i\omega) \neq 0$. Simplest one:

$$H(s) = \frac{s-p}{s+p} \quad p > 0$$

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$$H(s) = \frac{s-p}{s+p} \quad p > 0$$

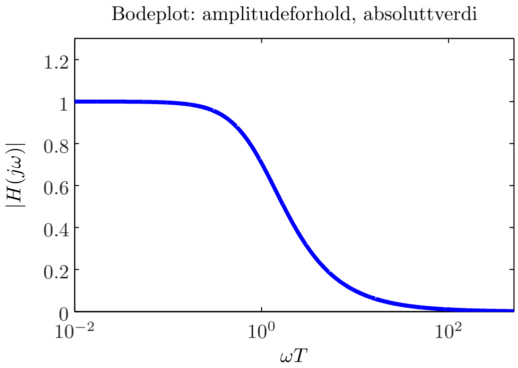
Notes:

- this is the 1-st order Pade-approximation of $e^{-\frac{2}{p}s}$
- all the frequencies have no distortions in the amplitudes, but the higher frequencies are delayed of half a period

?

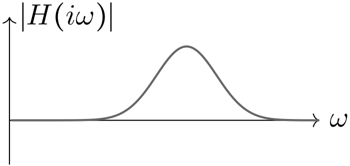
Problem: the filters we saw up to now are not too “sharp”

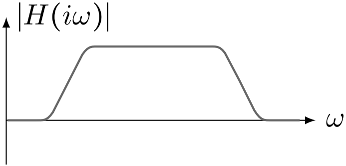
... i.e., there are a too many frequencies that are in between “low” and “high”. We would like the transitions “low” to “high” to be sharper!



Butterworth filters

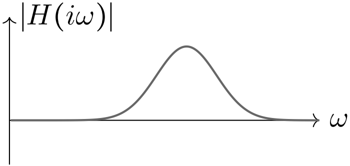
Motivation: design "sharper" filters

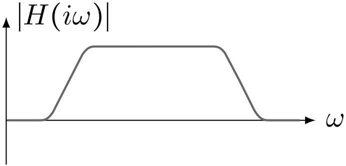




Butterworth filters

Motivation: design "sharper" filters





Strategy: add poles and zeros opportunely. They come in all the forms: low-pass, high-pass, band-pass, notch / band-stop

Butterworth filters

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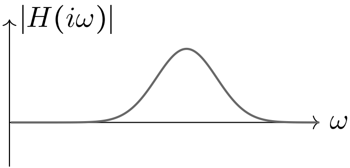

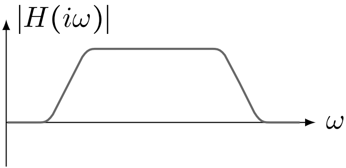

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interested in more info? check wikipedia!

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