TTK4225 - Systems Theory, Autumn 2020

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The most important Laplace transforms properties for control people

Roadmap

- make Laplace-transform more matematically ground
- derive some few important examples and properties

Laplace-properties we saw up to now:

(4.2.1 in Balchen's book)

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Linearity

$$\mathcal{L}\left\{af(t) + bg(t)\right\} = a\mathcal{L}\left\{f(t)\right\} + b\mathcal{L}\left\{g(t)\right\} \tag{1}$$

Laplace-properties we saw up to now:

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Derivatives

$$\mathcal{L}\{\dot{x}\} = sX(s) - x(0) \tag{2}$$

$$\mathcal{L}\{\ddot{x}\} = s^2X(s) - sx(0) - \dot{x}(0) \tag{3}$$

$$\vdots \tag{4}$$

Laplace-properties we saw up to now:

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Integration

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s) \tag{5}$$

Laplace-transforms we needed up to now:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
e^{at}	$\dfrac{1}{s-a}$
te^{at}	$\dfrac{1}{(s-a)^2}$
$e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos\omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$

Laplace transforms, take 2

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{+\infty} e^{-st} f(t) dt \tag{6}$$

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Important (but intuitive) teorem

If $f(t)$ is defined and piecewise continuous for $t > 0$,
and there exist two constants M and k so that $|f(t)| \leq Me^{kt}$ for every $t > 0$,
then $\mathcal{L}\{f(t)\}$ is well defined for every $s > k$

Important (but intuitive) property: linearity

Assuming that $f(t)$ and $g(t)$ are so that $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exists, then $\mathcal{L}\{af(t)+bg(t)\}$ exists and is s.t.

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

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Proof

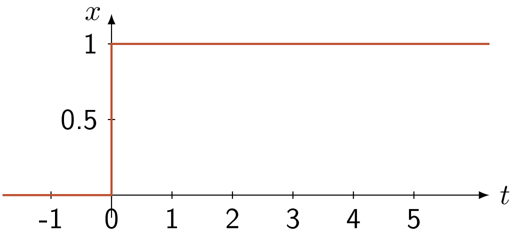
Integrals are linear, thus

$$\begin{aligned}\mathcal{L}\{af(t)+bg(t)\} &= \int_0^{+\infty} e^{-st}\left[af(t)+bg(t)\right]dt \\ &= a\int_0^{+\infty} e^{-st}f(t)dt + b\int_0^{+\infty} e^{-st}g(t)dt \\ &= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}\end{aligned}$$

?

Important example: Laplace-transforming the Heaviside-function

$$H(t) := \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$



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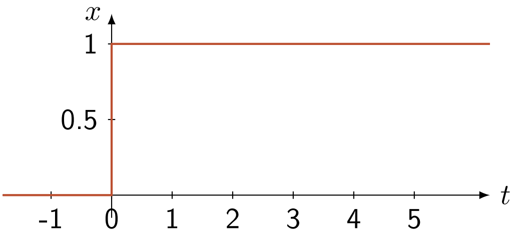


Discussion: is $H(0)$ defined?

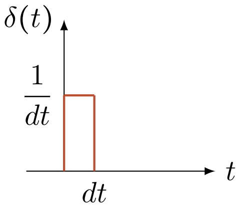
Deriving the Heaviside function from the Dirac's impulse

$$H(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

with, *very loosely,*

$$\delta(t) := \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{otherwise,} \end{cases}$$

or, better, as the limit of concentrating an unitary point mass at 0:



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$$\int_0^{+\infty} u\dot{v}dt = [uv]_0^{+\infty} - \int_0^{+\infty} \dot{u}vdt \tag{12}$$

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(this means that often one gets a ramp response and then one has to reconstruct back the impulse response)

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&= s^2X(s) - sx(0) - \dot{x}(0) && (31)\\
& && (32)
\end{aligned}$$

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$$\mathcal{L}\left\{x^{(n)}(t)\right\} = s^n\mathcal{L}\left\{x(t)\right\} - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - \ldots - x^{(n-1)}(0) \qquad (33)$$

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If $f(t)$ admits $F(s)$, then $f(t-a)\mu(t-a)$ transforms into $e^{-as}F(s)$

Discussion: for which s do these theorems hold?

Answer: only in the intersection of the ROCs of both "original" and "translated" transforms!

Translation theorems + Laplace-transforming polynomials = **the most useful formula that you may remember**

$$\begin{cases} \mathcal{L}\{t^n\} &= \dfrac{n!}{s^{n+1}} \\ \mathcal{L}\{e^{at}f(t)\} &= F(s-a) \end{cases} \implies \mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \tag{34}$$

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Discussion: what is the ROC?

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Discussion: what is the ROC? $\operatorname{Re}(s) > a$

Laplace-transforms of oscillatory behaviors = **the second most useful formulas that you may remember**

$$\mathcal{L}\left\{e^{at}\sin\omega t\right\} = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\left\{e^{at}\cos\omega t\right\} = \frac{s-a}{(s-a)^2+\omega^2}$$

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Limit cases: $a = 0$, so that

$$\mathcal{L}\left\{\sin\omega t\right\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\left\{\cos\omega t\right\} = \frac{s}{s^2+\omega^2}$$

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The potential points of view of this course:

simply a complex number, i.e., $s = \sigma + j\omega$, that gives rise to a *complex frequency* through the complex exponential $e^{-st} = e^{-\sigma t} e^{-j\omega t}$

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Discussion: which "unit of measurement" shall s have? e^{-st} is adimensional, thus st is adimensional, thus s is "time^{-1}"

Be careful on how to use s in this course!

Typically, in all the automatic control courses the interest is not too much on the value of $F(s)$ for different s, but rather on the fact that $F(s)$ enables performing algebraic operations to solve ODEs!

?

Using Laplace transforms with generic n order systems in free evolution

In the last episodes: the "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$, and to do something similar for second order systems in free evolution

$$\dot{y} = ay \qquad \rightarrow \qquad sY - y_0 = aY \qquad \rightarrow \qquad Y = y_0 \frac{1}{s - a}$$

Roadmap

- the same thing, but for generic systems

A small recap

- first oder homogeneous systems:

$$\dot{x} - ax = 0 \quad \mapsto \quad x(t) = x_0 e^{at} \tag{35}$$

- first oder non-homogeneous systems:

$$\dot{x} - ax = bu \quad \mapsto \quad x(t) = x(0)e^{at} + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \tag{36}$$

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The problem: the convolution integral gives us headaches, and we want to use Laplace to ease things up. To understand how, step-by-step path:

1. start from $\dot{x} - ax = 0$
2. move to $\ddot{x} - a_1\dot{x} - a_2 x = 0$
3. move to higher order
4. add the "u" part

A more detailed roadmap for this unit:

- generalize how to do partial expansions
- solve the problem: how to use Laplace transforms with generic n order systems in free evolution

?

Recalling some past results

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\Downarrow$$

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} \Downarrow$$

$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ (and the other cases)

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$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} \Downarrow$$
$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \text{ (and the other cases)}$$

Discussion: when is it that $\ddot{x} + \dfrac{f}{m}\dot{x} + \dfrac{k}{m}x = 0$ leads to $X(s) \neq 0$?

what about $\dddot{x} + \alpha_2 \ddot{x} + \alpha_1 \dot{x} + \alpha_0 x = 0$?

what about $\dddot{x} + \alpha_2 \ddot{x} + \alpha_1 \dot{x} + \alpha_0 x = 0$?

and what about higher orders?

Partial fraction decomposition

Partial fraction decomposition

case single poles: if $\dfrac{N(s)}{(s-a)(s-b)(s-c)\cdots}$ is s.t. $a \neq b \neq c \neq \cdots$ then there exist A, B, C, ... s.t.

$$\frac{N(s)}{(s-a)(s-b)(s-c)\cdots} = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots \tag{37}$$

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case repeated poles: if some poles are repeated, then there exist A_1, ..., A_{na}, B_1, ..., B_{nb}, s.t.

$$\frac{N(s)}{(s-a)^{na}(s-b)^{nb}\dots} = \frac{A_1}{s-a} + \dots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \dots + \frac{B_{nb}}{(s-b)^{nb}} + \dots \tag{38}$$

Partial fraction decomposition

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case repeated poles: if some poles are repeated, then there exist $A_1, \ldots, A_{na}, B_1, \ldots, B_{nb}$, s.t.

$$\frac{N(s)}{(s-a)^{na}(s-b)^{nb}\ldots} = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots \tag{38}$$

"But how do I compute A, B, etc.?" $\mapsto$
`en.wikipedia.org/wiki/Partial_fraction_decomposition`
(tip: start from `en.wikipedia.org/wiki/Heaviside_cover-up_method`)

?

Computing the free evolution for generic n order LTI systems

$$\text{start from } x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0 \tag{39}$$

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$$\text{for each } m \text{ apply } \mathcal{L}\left\{x^{(m)}(t)\right\} = s^m \mathcal{L}\left\{x(t)\right\} - s^{m-1} x(0) - s^{m-2} \dot{x}(0) - \ldots - x^{(m-1)}(0) \tag{40}$$

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$$\text{obtain } X(s) = \frac{N(s)}{(s-a)(s-b)(s-c)\cdots} \tag{41}$$

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$$\text{obtain } X(s) = \frac{N(s)}{(s-a)(s-b)(s-c)\cdots} \tag{41}$$

$$\text{do a partial fraction decomposition} \tag{42}$$

Computing the free evolution for generic n order LTI systems

Two main cases:

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$\text{either } X(s) = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots$$

$$\text{or } X(s) = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots$$

Computing the free evolution for generic n order LTI systems

Case single poles

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$X(s) = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots$$

Computing the free evolution for generic n order LTI systems

Case single poles

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$$X(s) = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots$$

Discussion: given $\mathcal{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = e^{\alpha t}$, what is $x(t)$?

Computing the free evolution for generic n order LTI systems

Case multiple poles

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$X(s) = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots$$

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Discussion: what was $\mathcal{L}^{-1}\left\{\frac{1}{(s-\alpha)^2}\right\}$?

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remember: $\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}$

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remember: $\mathcal{L}\left\{t^n e^{at}\right\} = \dfrac{n!}{(s-a)^{n+1}}$

Discussion: given the previous transform, what is $x(t)$? Solution: a sum of terms of the type $t^n e^{at}$

Extremely important result

a LTI in free evolution behaves as a combination of terms $e^{\alpha t}$, $te^{\alpha t}$, $t^2e^{\alpha t}$, etc. for a set of different αs and powers of t, called the *modes* of the system

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Discussion: assuming that we have two modes, $e^{-0.3t}$ and $e^{-1.6t}$, so that

$$x(t) = \gamma_1 e^{-0.3t} + \gamma_2 e^{-1.6t}.$$

What determines γ_1 and γ_2?

`comparison-of-exponentials.ipynb`

?

Using Laplace transforms with second order systems in free evolution

In the last episodes: the “Laplace way” to find how LTI systems freely evolve

$$\dot{y} = ay \qquad \rightarrow \qquad sY - y_0 = aY \qquad \rightarrow \qquad Y = y_0 \frac{1}{s-a}$$

Roadmap

- the same thing, but adding the forced response term

A small recap

- first oder homgeneous systems:

$$\dot{x} - ax = 0,\ x(0) = x_0 \quad \mapsto \quad x(t) = x_0 e^{at} \tag{43}$$

- first oder non-homgeneous systems:

$$\dot{x} - ax = bu,\ x(0) = x_0 \quad \mapsto \quad x(t) = x(0)e^{at} + \int_0^t e^{a\tau} bu(t-\tau)d\tau \tag{44}$$

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The problem, once again: the convolution integral gives us headaches - we want to use Laplace to ease things up

Laplace-ing everything

$$\dot{x} - ax = bu, \; x(0) = x_0 \tag{45}$$

$$\downarrow \tag{46}$$

$$sX(s) - x(0) - aX(s) = bU(s) \tag{47}$$

$$\downarrow \tag{48}$$

$$X(s) = \frac{x(0)}{s-a} + \frac{bU(s)}{s-a} \tag{49}$$

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$$x(t) = \mathcal{L}^{-1}\left\{\frac{x(0)}{s-a} + \frac{bU(s)}{s-a}\right\} \tag{51}$$

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if $u(t)$ has a rational Laplace transform, then

$$x(t) = \sum \mathcal{L}^{-1} \left\{ \frac{\alpha}{(s-a)^{\beta}} \right\}$$

Example 1

$$\dot{x} + x = 0, \quad x(0) = 2 \qquad \Longrightarrow \qquad x(t > 0) = ?$$

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$$\dot{x} + x = 0, \quad x(0) = 2 \qquad \Longrightarrow \qquad x(t > 0) = ?$$

$$sX(s) - x(0) + X(s) = 0$$

$$\downarrow$$

$$(s+1)X(s) = x(0)$$

$$\downarrow$$

$$X(s) = \frac{x(0)}{(s+1)} = \frac{2}{s-(-1)}$$

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Discussion: what is the time constant in this case?

Example 2

Discussion: $\mathcal{L}\left\{H\left(t\right)\right\} = ?$

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Discussion: $\mathcal{L}\{H(t)\} =?$ Solution: $\dfrac{1}{s}$

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$$sX(s) - x(0) + X(s) = \frac{1}{s}$$

$$\downarrow$$

$$(s+1)X(s) = \frac{1}{s} + 2$$

$$\downarrow$$

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

Example 2, continued

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

Example 2, continued

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1)+Bs}{s(s+1)} \tag{55}$$

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$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1)+Bs}{s(s+1)} \tag{55}$$

s: $As + Bs = 0 \implies A = -B$

1: $A = 1 \implies B = -1$

Example 2, continued

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Example 2, continued

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1)+Bs}{s(s+1)} \tag{55}$$

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$$x(t) = \mathcal{L}^{-1}\{X(s)\} = e^{-t} + \mu(t) \tag{57}$$

Example 3

$$\dot{x} + 4x = 3e^{-4t}, \quad x(0) = 1 \qquad \Longrightarrow \qquad x(t > 0) = ?$$

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$$sX(s) - 1 + 4X(s) = \frac{3}{s+4} \tag{58}$$

$$\downarrow \tag{59}$$

$$X(s) = \frac{1}{s+4} + \frac{3}{(s+4)^2} \tag{60}$$

$$\tag{61}$$

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$$sX(s) - 1 + 4X(s) = \frac{3}{s+4} \tag{58}$$

$$\downarrow \tag{59}$$

$$X(s) = \frac{1}{s+4} + \frac{3}{(s+4)^2} \tag{60}$$

$$\tag{61}$$

Remember: $\mathcal{L}\left\{t^n e^{at}\right\} = \dfrac{n!}{(s-a)^{n+1}}$, thus

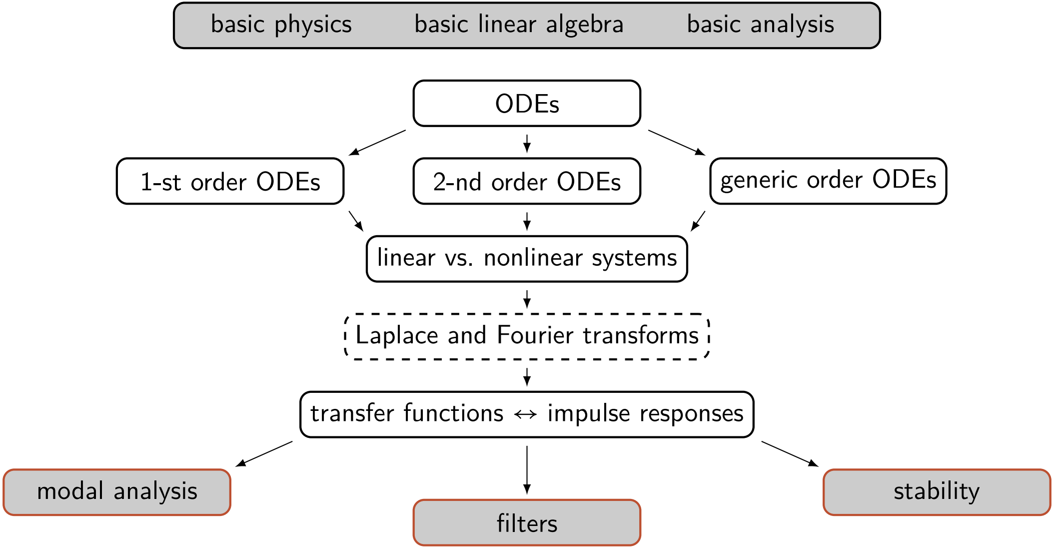
$$x(t) = e^{-4t} + 3te^{-4t} \tag{62}$$

?

Transfer functions

Discussion: what did Laplace transforms enable us to do?

Summary / Roadmap of TTK4225



Roadmap

- definition of transfer function
- connections with impulse responses
- transfer functions of generic ARMA models
- examples

Laplace transforms: not only a way of solving dynamics

$$
\begin{tikzcd}
\text{initial value problem} & \text{algebraic problem} \\
\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\
\boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \arrow[l, "\mathcal{L}^{-1}"'] \\
\text{solution in the time domain} & \text{solution in the frequency domain}
\end{tikzcd}
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Laplace transforms: not only a way of solving dynamics

initial value problem — algebraic problem

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solution in the time domain — solution in the frequency domain

This unit $=$ transfer functions

i.e., complete description of the behavior of a LTI system in terms of a Laplace-object

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This unit = transfer functions

i.e., complete description of the behavior of a LTI system in terms of a Laplace-object

Discussion: which other object was a complete description of a LTI system? *impulse responses and transfer functions need to be connected somehow (and now we will see how)*

Towards the formal definition of transfer functions

$$\ddot{x} = a_1\dot{x} + a_0 x = u(t), \quad x(0) = x_0, \quad \dot{x}(0) = x_1$$

$$\downarrow$$

$$s^2 X - s x(0) - \dot{x}(0) = a_1 \left(sX - x(0)\right) + a_0 X + U$$

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$$\left(s^2 - a_1 s - a_0\right) X = (s - a_1)x(0) + \dot{x}(0) + U$$

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$$X(s) = \underbrace{\frac{1}{s^2 - a_1 s - a_0}}_{H(s)} \left(\underbrace{(s - a_1) x(0) + \dot{x}(0)}_{M(s)} + U(s) \right)$$

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$$X(s) = H(s)M(s) + H(s)U(s)$$

Towards the formal definition of transfer functions

What are the roles of the objects here?

$$X(s) = H(s)M(s) + H(s)U(s)$$

- $H(s)$ = response of the system, the actual *transfer function*
- $M(s)$ = effect of the initial conditions, its combination with $H(s)$ gives the free evolution
- $U(s)$ = input, its combination with $H(s)$ gives the forced response

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this is another viewpoint that confirms the superposition of the effects

Connecting transfer functions with impulse responses

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Connecting transfer functions with impulse responses

$$X(s) = H(s)M(s) + H(s)U(s)$$

Discussion: what about $M(s) = 0$, $U(s) = 1$? Answer: $X(s) = H(s)$

but $U(s) = 1$ implies u a Dirac delta, so $X(s) = H(s)$ must be the Laplace transform of the impulse response $h(t)$

$$X(s) = H(s)M(s) + H(s)U(s)$$

$$x(t) = x_0 h(t) + h * u(t)$$

remember: "multiplication in frequency $=$ convolution in time"

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implication: knowing $H(s)$ means knowing everything about the system

Are there alternative ways to compute $H(s)$?

If $x(0) = 0, \dot{x}(0) = 0, \ddot{x}(0) = 0, \ldots$ then $M(s) = 0$.

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$$\implies \quad H(s) = \frac{X(s)}{U(s)}$$

and, if we choose $u(t) = \delta(t)$ so that $U(s) = 1$,

$$\implies \quad H(s) = X(s)$$

Computing $H(s)$ in the generic ARMA case

$$x^{(n)} = a_{n-1}x^{(n-1)} + \ldots + a_0 x + b_m u^{(m)} + \ldots + b_0 u$$

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Thus, since $H(s) = X(s)$,

$$H = \frac{b_m s^m + \ldots + b_0}{s^n - a_{n-1}s^{n-1} - \ldots - a_0}$$

How is knowing $H(s)$ simplify our life w.r.t. knowing $h(t)$?

$$H = \frac{b_m s^m + \ldots + b_1 s + b_0}{s^n - a_{n-1} s^{n-1} - \ldots a_1 s - a_0}$$

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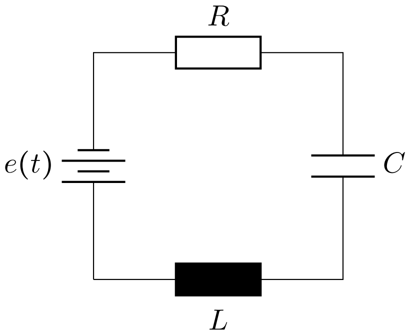
$$H = \frac{b_m s^m + \ldots + b_1 s + b_0}{s^n - a_{n-1} s^{n-1} - \ldots a_1 s - a_0}$$

- simpler analysis of the stability properties
- simpler analysis of the natural modes of the system
  (and thus its oscillatory behaviors)
- simpler analysis of the time constants of the system

Example: RCL circuits



$$L\ddot{x} + R\dot{x} + \frac{1}{C}x = \dot{e}(t), \quad e(t) = \delta(t) \implies H(s) = X(s),\ E(s) = 1 \tag{63}$$

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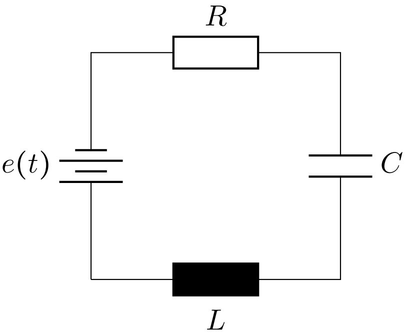

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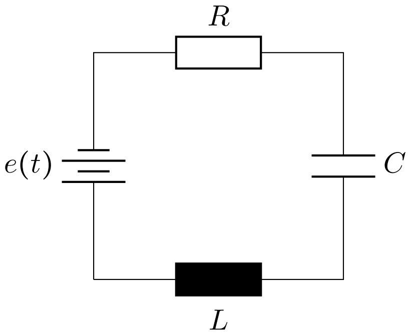

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$$\left(Ls^2 + Rs + 1/C\right)X(s) = s \quad \mapsto \quad H(s) = \frac{s}{20s^2 + 160s + 500} \tag{64}$$

Connecting transfer functions and impulse responses, take 2

$$H(s) = \frac{s}{20s^2 + 160s + 500} = \frac{1}{20}\frac{s}{s^2 + 8s + 25}$$

Discussion: how do we find the impulse response $h(t)$?

Connecting transfer functions and impulse responses, take 2

$$H(s) = \frac{s}{20s^2 + 160s + 500} = \frac{1}{20}\frac{s}{s^2 + 8s + 25}$$

Discussion: how do we find the impulse response $h(t)$? Solution: inverse-transform $H(s)$ using a partial fraction expansion $\implies$ we need to understand how the denominator looks like!

$$\begin{aligned}
s^2 + 8s + 25 = 0 \quad \implies \; s^2 + 8s &= -25 \\
s^2 + 2 \cdot 4s &= -25 \\
s^2 + 2 \cdot 4s + 4^2 &= -25 + 4^2 \\
\mapsto (s+4)^2 + 9 &= 0
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Discussion: may the impulse response be an oscillatory one?

Laplace-transforms of oscillatory behaviors = **the second most useful formulas that you may remember**

$$\mathcal{L}\left\{e^{at}\sin\omega t\right\} = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\left\{e^{at}\cos\omega t\right\} = \frac{s-a}{(s-a)^2+\omega^2}$$

with limit case $a = 0$, so that

$$\mathcal{L}\left\{\sin\omega t\right\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\left\{\cos\omega t\right\} = \frac{s}{s^2+\omega^2}$$

Continuing the example above

$$H(s) = \frac{1}{20}\frac{s}{s^2 + 8s + 25} = \frac{1}{20}\frac{(s+4) - 4}{(s+4)^2 + 9}$$

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$$H(s) = \frac{1}{20}\frac{s}{s^2+8s+25} = \frac{1}{20}\frac{(s+4)-4}{(s+4)^2+9}$$

$$h(t) = \frac{1}{20}e^{-4t}\cos 3t - \frac{4}{20}e^{-4t}\sin 3t$$

?

Representing rational transfer functions through gain, zeros, and poles

Roadmap

- checking that not all the transfer functions are rational
- decomposition of rational transfer functions into gain, zeros, and poles

Computing transfer functions from differential equations

$$\ddot{y} = a_1\dot{y} + a_2 y + b_0\ddot{u} + b_1\dot{u} + b_2 u \qquad \text{initial conditions } = 0$$

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$$\downarrow$$

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$$\downarrow$$

$$s^2 Y(s) - a_1 s Y(s) - a_2 Y(s) = b_0 s^2 U(s) + b_1 s U(s) + b_2 U(s)$$

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$$\downarrow$$
$$Y(s) = \frac{b_0 s^2 + b_1 s + b_2}{s^2 - a_1 s - a_2} U(s)$$

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$$\downarrow$$

$$s^2 Y(s) - a_1 s Y(s) - a_2 Y(s) = b_0 s^2 U(s) + b_1 s U(s) + b_2 U(s)$$

$$\downarrow$$

$$Y(s) = \frac{b_0 s^2 + b_1 s + b_2}{s^2 - a_1 s - a_2} U(s)$$

problem: rational means finite polynomials; but not all the TFs are rational!

Delays

Remember: time shifting in Laplace transforms:

$$\mathcal{L}\left\{y(t-\tau)H\left(t-\tau\right)\right\} = e^{-\tau s}Y\left(s\right)$$

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example: $\mathcal{L}\left\{\delta t-\tau\right\} =?$ Solution: $e^{-\tau s}$

$$
\begin{aligned}
h(t) \quad &\mapsto \quad h(t-\tau) \\
H(s) \quad &\mapsto \quad e^{-\tau s} H(s) = \frac{1}{e^{\tau s}} H(s)
\end{aligned}
$$

$$h(t) \quad \mapsto \quad h(t - \tau)$$

$$H(s) \quad \mapsto \quad e^{-\tau s} H(s) = \frac{1}{e^{\tau s}} H(s)$$

Problem: $e^{\tau s} = \sum_{n=0}^{+\infty} \frac{(\tau s)^n}{n!}$:

$$\frac{b_0 s^2 + b_1 s + b_2}{e^{\tau s} \left(s^2 - a_1 s - a_2\right)}$$

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Discussion: if $H(s)$ is rational, is $e^{-\tau s} H(s)$ rational too?

Partially solving the issue: Padé approximations

definition of Padé approximant: the "best" approximation of a function by a rational function of given order *(thus a concept applicable to any function)*

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explicit formulas a bit boring:

`https://mathoverflow.net/questions/41226/pade-approximant-to-exponential-function`

ZPK decompositions

Assumption:

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n - a_1 s^{n-1} - \ldots - a_n}$$

ZPK decompositions

Assumption:

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n - a_1 s^{n-1} - \ldots - a_n}$$

then

$$H(s) = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{j=1}^{n} (s - p_j)}$$

with

- $z_1, \ldots, z_m$ =: zeros of $H(s)$
- $p_1, \ldots, p_n$ =: poles of $H(s)$
- K =: gain of $H(s)$

How do we find ZPK representations?

I.e., how do we go from $\dfrac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n - a_1 s^{n-1} - \ldots - a_n}$ to $K \dfrac{\prod_{i=1}^{m} (s - z_i)}{\prod_{j=1}^{n} (s - p_j)}$?

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[1]Every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots.

Solution = rely on numerical aids

Solving the "polynomial roots" problem:

Matlab: `r = roots(p)`

Python: `r = numpy.roots(p)`

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Finding ZPK representations:

Matlab: `zpksys = zpk(sys)`

Python: `zkpsys = tf2zpk(b, a)` *(requires the "python-control" package)*

Ok, and so what?

$$H(s) = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{j=1}^{n} (s - p_j)}$$

Spoiler alert:

- p_j's $=$ poles $=$ natural modes of the system
- z_i's $=$ zeros $=$ complex exponentials that are killed by $H(s)$, but also factors modulating the amplitudes of the modes
- K $=$ gain $=$ how the system asypmtotically responds to a step-input

Ok, and so what?

$$H(s) = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{j=1}^{n} (s - p_j)}$$

Spoiler alert:

- p_j's $=$ poles $=$ natural modes of the system
- z_i's $=$ zeros $=$ complex exponentials that are killed by $H(s)$, but also factors modulating the amplitudes of the modes
- K $=$ gain $=$ how the system asypmtotically responds to a step-input

in other words, ZPK representations are interpretable

?

What are the poles of a system?

Roadmap

- partial fraction expansions
- differences between single and multiple poles
- examples

Physical meaning of a pole *(partial fraction expansion)*

$$
\begin{aligned}
H(s) = K\frac{\prod(s - z_i)}{\prod(s - p_j)} = \quad & \frac{\kappa_{1,1}}{s - p_1} + \frac{\kappa_{1,2}}{(s - p_1)^2} + \frac{\kappa_{1,3}}{(s - p_1)^3} + \dots \\
& + \frac{\kappa_{2,1}}{s - p_2} + \frac{\kappa_{2,2}}{(s - p_2)^2} + \dots \\
& + \dots
\end{aligned}
$$

Physical meaning of a pole *(partial fraction expansion)*

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$$+\frac{\kappa_{2,1}}{s-p_2} + \frac{\kappa_{2,2}}{(s-p_2)^2} + \dots$$
$$+\dots$$

Thanks to the linearity of $\mathcal{L}$ and $\mathcal{L}^{-1}$,

$$H(s) = H_{1,1}(s) + H_{1,2}(s) + \dots$$
$$\Updownarrow$$
$$h(t) = h_{1,1}(t) + h_{1,2}(t) + \dots$$

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In words, $h(t)$ = sum of *all* the impulse responses relative to the various poles in the partial fraction expansion

The cover-up method

if $H(s) = K \frac{\prod_i (s - z_i)}{\prod_j (s - p_j)}$ does not have multiple poles then

$$H(s) = \sum_j \frac{\kappa_j}{(s - p_j)}$$

with

$$\kappa_j = (s - p_j)\, H(s)\Big|_{s=p_j}$$

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Example:

$$H(s) = \frac{3}{(s+1)(s+2)} \quad \Longrightarrow \quad h(t) = \ldots ?$$

Physical meaning of a pole - What is $h_{j,\beta}(t)$?

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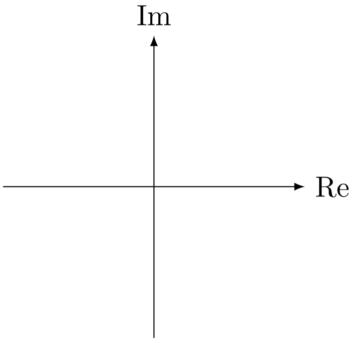
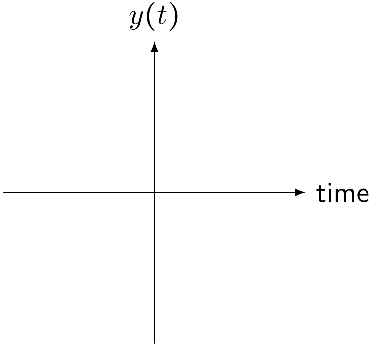
multiple root:

$$H_{j,\beta}(s) = \frac{\kappa_{j,\beta}}{(s - p_j)^\beta} \qquad \Longrightarrow \qquad h_{j,\beta}(t) \propto t^{\beta-1} e^{p_j t}$$

What is $h_{j,1}(t)$? (i.e., when the roots are simple)

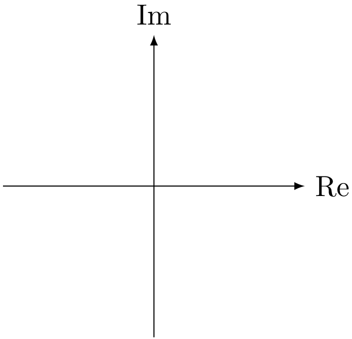
if $\quad H(s) = K\dfrac{\prod(s-z_i)}{\prod(s-p_j)} = \sum_j \dfrac{\kappa_{j,1}}{s-p_j}$ then $\quad \dfrac{\kappa_{j,1}}{s-p_j} \mapsto h_{j,1}(t) \propto e^{p_j t}$

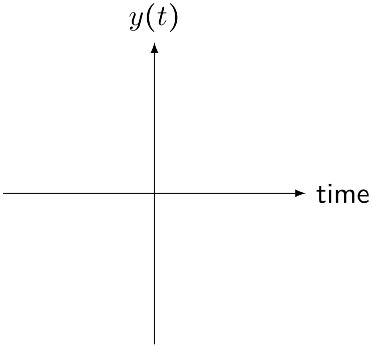




What is $h_{j,1}(t)$? (i.e., when the roots are multiple)

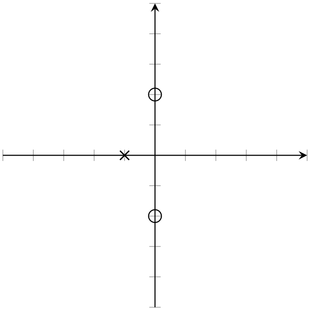
if $\quad H(s) = K \frac{\prod (s - z_i)}{\prod (s - p_j)} = \sum_j \sum_\beta \frac{\kappa_{j,\beta}}{(s - p_j)^\beta} \quad$ then $\quad \frac{\kappa_{j,\beta}}{(s - p_j)^\beta} \mapsto h_{j,\beta}(t) \propto t^{\beta - 1} e^{p_j t}$





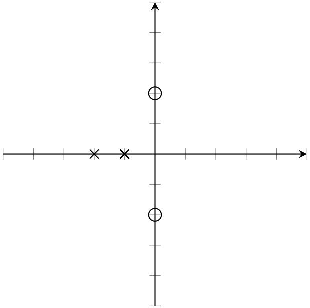
Physical meaning of a pole - Example

$$H(s) = K\frac{s^2+4}{s^3+3s^2+3s+1} = K\frac{(s+2j)(s-2j)}{(s+1)^3} \implies h(t) = \kappa_1 e^{-t} + \kappa_2 t e^{-t} + \kappa_3 t^2 e^{-t}$$



Physical meaning of a pole - Example

$$H(s) = K\frac{(s+2j)(s-2j)}{(s+1)^3(s+2)} \quad \Longrightarrow \quad h(t) = \ldots ??$$



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What are the zeros of a system?

Roadmap

- generic definition
- definition for the LTI systems case
- examples of the effects of choosing different zeros

Definition (Zero of a function)

$$z_i = \textit{zero of } H(\cdot) \;\; \textit{if } H(z_i) = 0$$

Physical meaning of a zero of a transfer function

$$H \text{ LTI:} \qquad u(t) = e^{\overline{s}t} \qquad \Longrightarrow \qquad y(t) = H(\overline{s})e^{\overline{s}t}$$

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$$\text{thus} \quad y(0) = 0, \quad H(s) = K\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)}, \quad \text{and} \quad u(t) \propto e^{z_i t} \qquad \Longrightarrow \qquad y(t) = 0$$

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What is $e^{z_i t}$? Euler's formulae ($t \in \mathbb{R}$, $z_i \in \mathbb{C}$):

- $e^{z_i t} = e^{(\sigma + j\omega)t} = e^{\sigma t}\Big(\cos(\omega t) + j\sin(\omega t)\Big)$
- $\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$

What is $e^{z_i t}$?

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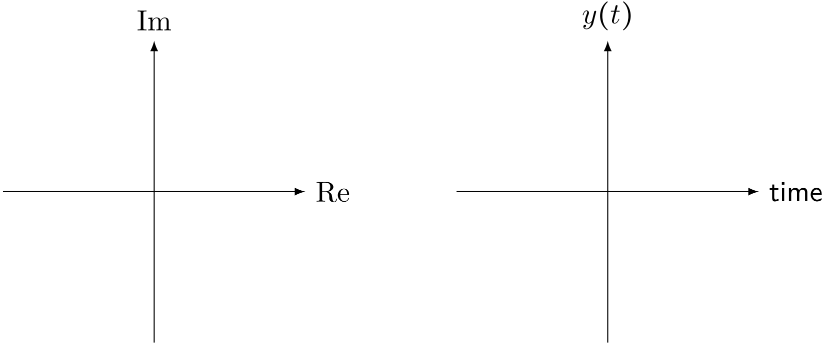
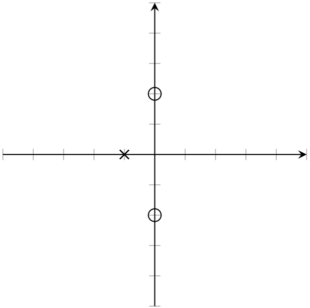


Physical meaning of a zero - Example

$$H(s) = K\frac{s^2+4}{s^3+3s^2+3s+1} = K\frac{(s+2j)(s-2j)}{(s+1)^3}$$



Thus $\quad u(t) = 2\cos(2t) = \left(e^{j2t} + e^{-j2t}\right) \quad \Longrightarrow \quad y(t) = H(2j)e^{2jt} + H(-2j)e^{-2jt} = 0$

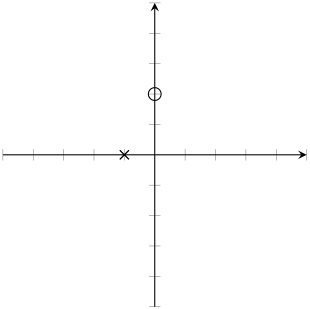
Discussion

Is it that

$$H(s) = K\frac{(s-2j)}{(s+1)^3}$$

implies

$$u(t) = 2\cos(2t) = \left(e^{j2t} + e^{-j2t}\right) \quad \Longrightarrow \quad y(t) = 0 \quad ?$$



Discussion

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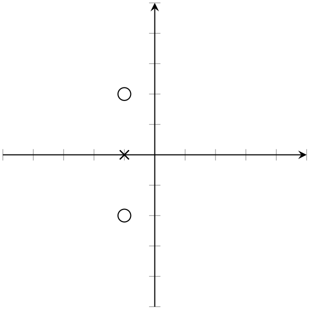
$$H(s) = K\frac{(s+1-2j)(s+1+2j)}{(s+1)^3}$$

implies

$$u(t) = 2\cos(2t) = \left(e^{j2t} + e^{-j2t}\right) \implies y(t) = 0 \quad ?$$



Example

$$H(s) = \frac{(s+\alpha)^2 + 1}{(s+1)\left((s+0.1)^2 + 1\right)}$$

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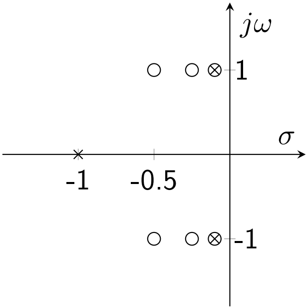

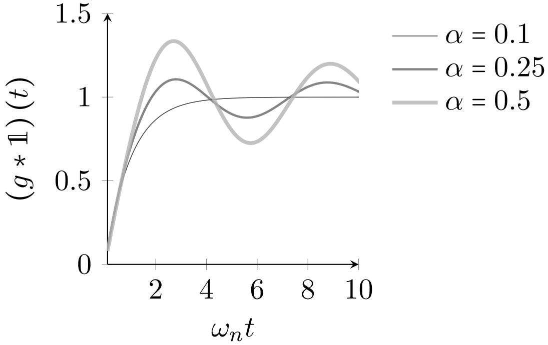

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What is the gain of a system?

Roadmap

- definition from an intuitive perspective
- final value theorem
- examples

Gain, in generic terms

Concept, intuitively:

- I use an unitary step as the input
- I check what is the output at the end of the times

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Discussion: does this hold also for nonlinear systems?

How to compute the gain associated to a transfer function

Tool to be used: *final value theorem* *(but remember the caveats on its validity*[2]*)*.

[2] I.e., the limit must exist and be finite

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Discussion: what is the gain associated to $\dfrac{10}{s+20}$?

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?