TTK4225 - Systems Theory, Autumn 2020

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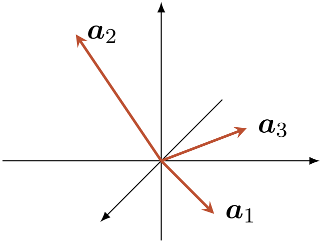
Diagonalization

Roadmap

- what happens if the eigenvectors of A form a basis of $\mathbb{R}^n$?
- what diagonalization means algebraically
- what diagonalization means geometrically
- what diagonalization means in practice

An interesting case: what if the eigenvectors of A form a basis of $\mathbb{R}^n$?



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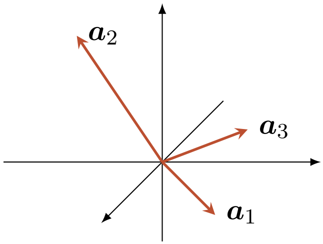

what happens if in this case I choose a new basis formed by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$*?*

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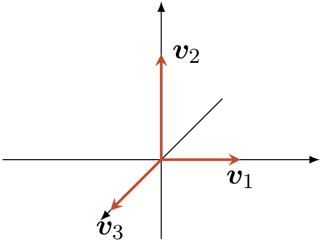

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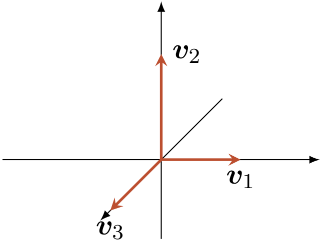

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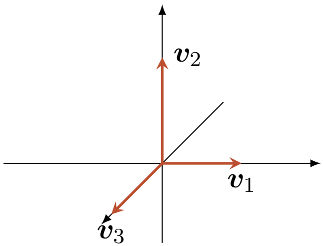


How does $\mathcal{A}$ look like, with respect to this basis?

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

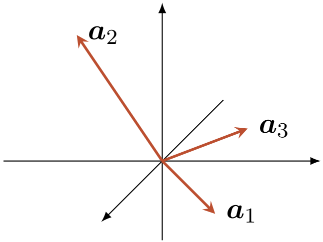
Note that the λ_i's may also be the same! Example:

$$A = \begin{bmatrix} 2.3 & & & \\ & 2.3 & & \\ & & \ddots & \\ & & & 2.3 \end{bmatrix}$$

Diagonalizing a square matrix

hypothesis: A is s.t. there exist $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ linearly independent eigenvectors

thesis: $T = \left[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\right]$ is s.t. $\Lambda = T^{-1} A T = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_n\right)$



Diagonalizing a square matrix: proof that $AT = T\Lambda$

$$AT \overset{(1)}{=} A\,[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n] \overset{(2)}{=} [A\boldsymbol{v}_1, \ldots, A\boldsymbol{v}_n] \overset{(3)}{=} [\lambda_1\boldsymbol{v}_1, \ldots, \lambda_n\boldsymbol{v}_n] \overset{(4)}{=} T\Lambda$$

(1) recall that the columns of T are the eigenvectors

(2) this follows by the geometrical interpretation of matrix-columns multiplications

(3) this is because $\boldsymbol{v}_i$ is an eigenvector

(4) we can rewrite things as a product with a diagonal matrix

What about matrices with multiple eigenvalues?

Example:

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \quad \Longrightarrow \quad \det\left(A - sI\right) = -s^3 - s^2 + 21s + 45 = \left(s-5\right)\left(s+3\right)^2$$

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Eigenspaces-eigenvectors couples:

$$\left\{\lambda_1, \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right)\right\} \qquad \left\{\lambda_2 = \lambda_3, \mathrm{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right)\right\}$$

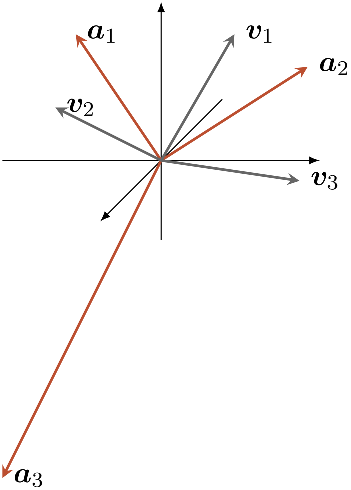
important point: to diagonalize we need n different and linearly independent eigenvectors, not n different eigenvalues

Graphically

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\Lambda = T^{-1}AT = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$



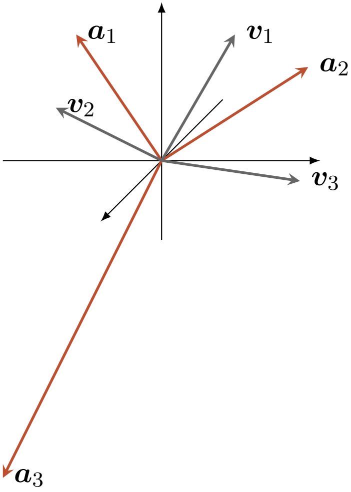
Diagonalization, in numbers

$$A = T\Lambda T^{-1}$$

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0.125 & 0.25 & -0.375 \\ -0.250 & 0.50 & 0.750 \\ 0.125 & 0.25 & 0.625 \end{bmatrix}$$

What does diagonalization mean, graphically?

I look at the world considering as the new axes the eigenspaces



What does diagonalization mean, physically?

Original system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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The same system, but after the change of basis T:

$$\begin{bmatrix} \dot{\widetilde{x}}_1 \\ \dot{\widetilde{x}}_2 \\ \dot{\widetilde{x}}_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \end{bmatrix}$$

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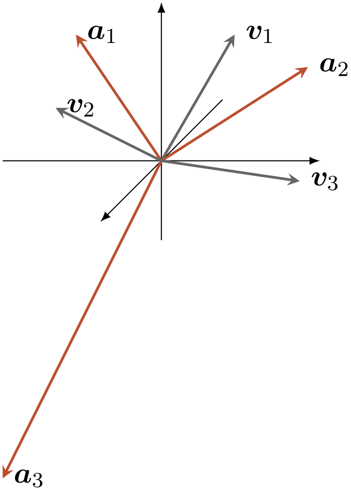
this means that the original system is actually the juxtaposition of 3 independent systems that evolve "ignoring" what is happening in the other ones

Thus diagonalizing = decomposing the dynamics in a set of independent 1-dimensional dynamics

the eigenspaces are where these 1-dimensional dynamics live



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- to be able to diagonalize means to be able to split up a system in independent pieces
- however we can do this diagonalization only if the eigenvectors of A form a basis for $\mathbb{R}^n$, and this is not guaranteed in general

Generalization

Consider

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & & & \\ & \tilde{A}_2 & & \\ & & \ddots & \\ & & & \tilde{A}_k \end{bmatrix};$$

also this means "dividing the system in independent sub-systems"! However "diagonalizing" means finding independent subsystems of dimension 1, while in this general case the dimensions are potentially bigger than 1

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Towards stranger things: recall that state space representations are ways of expressing LTI systems

$$\ddot{y} + a_1 \dot{y} + a_0 y = bu(t)$$

is equivalent to

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

and thus to

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \\ y = C\boldsymbol{x} \end{cases}$$

Towards stranger things: how was this connecting with the first part of the course?

$$\ddot{y} + a_1 \dot{y} + a_0 y = bu \implies Y(s) = \frac{bU(s)}{s^2 + a_1 s + a_0} \implies \text{modes} = \text{solutions of } s^2 + a_1 s + a_0 = 0$$

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with the denominator of the TF equal to $\det(sI - A)$

Towards stranger things: remember this basic fact

$$Y(s) = C \frac{\operatorname{adj}(sI - A)}{\det(sI - A)} B U(s)$$

- changing the basis does not change the characteristic polynomial, thus

$$\det(sI - A) = \det\left(sI - T^{-1} A T\right)$$

(in other words, changing the basis for the state space does not change the poles of the TF, and thus the modes of the LTI system – as it should obviously be)

Stranger things

…but if $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ implies $\det(A - sI) = (s-5)(s+3)^2$ then there is a double pole in -3, corresponding to a mode of the type te^{-3t};

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Solution (and we will see this in the next unit): the presence or not of the mode te^{-3t} depends on the structure of the eigenspaces of A → we need to study Jordan forms *doing systems theory for LTI systems means studying the inner structure of* $\dot{\boldsymbol{x}} = A\boldsymbol{x}$

?

Jordan forms

Roadmap

- non-diagonalizable matrices
- Jordan forms
- connections with dynamical systems
- summary of the differences between diagonalizable and non-diagonalizable matrices

A small trick, to make things faster

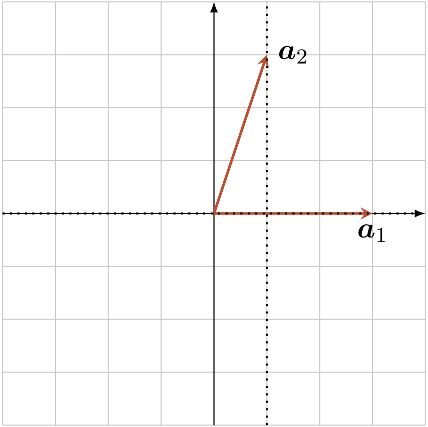
if A is upper triangular or lower triangular then its characteristic polynomial is given by $\prod(s - d_i)$ with the d_i's the elements on the diagonal, i.e.,

$$A = \begin{bmatrix} d_1 & \star & \star & \cdots \\ 0 & d_2 & \star & \cdots \\ 0 & 0 & \ddots & \cdots \\ 0 & 0 & 0 & d_n \end{bmatrix} \implies \det(sI - A) = \prod_i (s - d_i)$$

The case of Jordan miniblocks

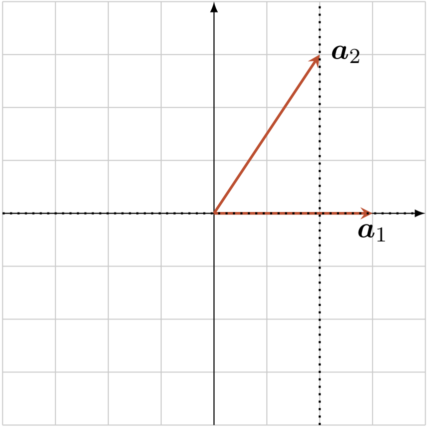
$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \Longrightarrow \quad \text{characteristic polynomial} = (s - \lambda)^2$$

How many 1-dimensional eigenspaces do Jordan miniblocks have?



in this case there is only one “stretching” for which the stretched columns align

Note that this can be generalized to Jordan miniblocks with α instead of 1



(we though like more to write Jordan miniblocks as $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$*)*

"The eigenspaces of Jordan miniblocks have dimension 1": algebraic proof

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad \Longrightarrow \quad \det\left(sI - A\right) = \left(s - \lambda\right)^3$$

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$$\text{and thus } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \star \\ 0 \\ 0 \end{bmatrix}$$

Summarizing

$$A = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda \end{bmatrix}$$

- the eigenspace is 1-dimensional and it is equal to $\ker(\lambda I - A) = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$

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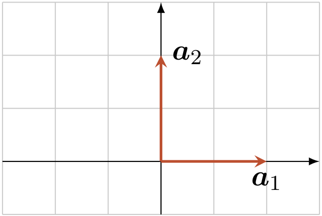
- thus we cannot find a basis of $\mathbb{R}^n$ composed by eigenvectors
- thus we cannot diagonalize, i.e., we cannot write $A = T \Lambda T^{-1}$
- thus the system $\dot{\boldsymbol{y}} = A \boldsymbol{y}$ cannot be divided into a series of independent 1-dimensional dynamics

?

An example, to make things in practice. System "N":

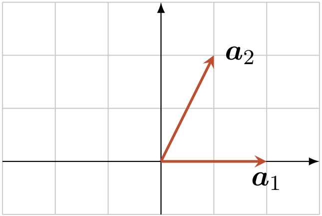
$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \\ y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases} \implies \dim\Big(\ker\left(2I - A\right) \Big) = 2$$



$\implies$ two independent 1-dimensional systems, each with a mode e^{2t}

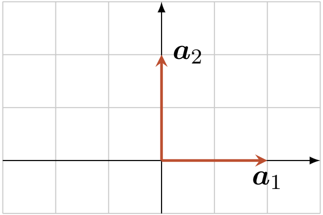
An example, to make things in practice. System "J":

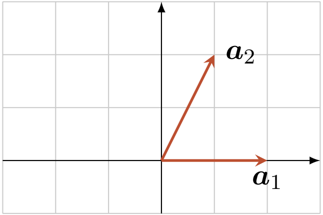
$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \\ y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases} \implies \dim\Big(\ker\left(2I - A\right) \Big) = 1$$



$\implies$ a truly 2-dimensional system, with modes e^{2t} and te^{2t}

Comparing "N" against "J":





"J" contains an intrinsic shearing that "N" does not contain

(but remember that for the case "N" we are looking at the space through the directions defined by its eigenvectors)

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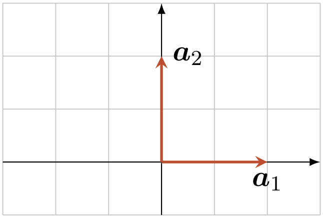
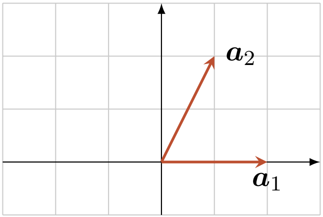


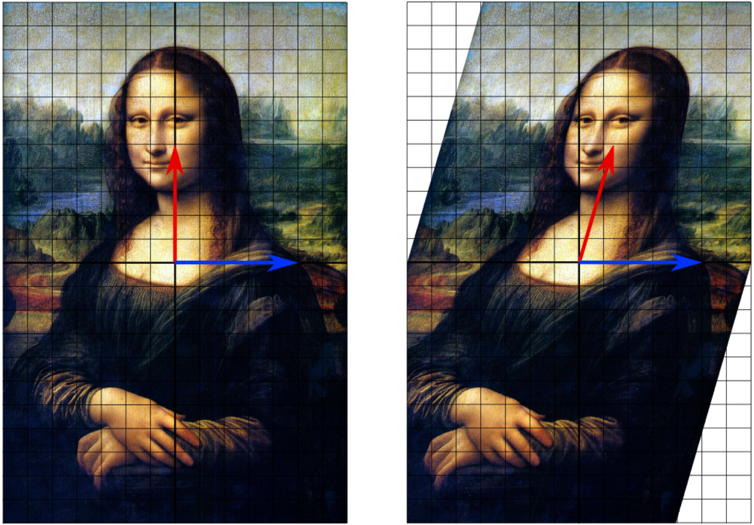


"J" contains an intrinsic shearing that "N" does not contain

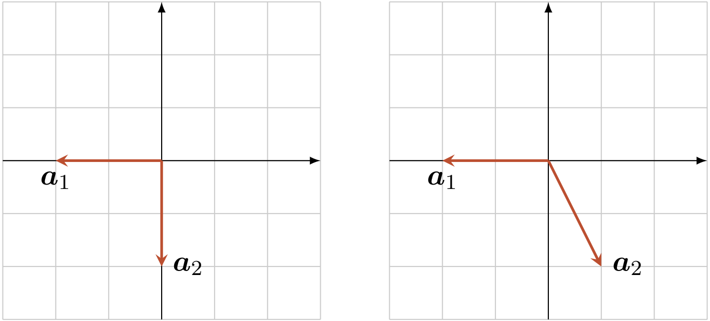
(but remember that for the case "N" we are looking at the space through the directions defined by its eigenvectors)

the same applies to $J = \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$ or the higher-dimensions cases

Discussion: is this due to a Jordan map?



Watch out that to have asymptotic stability the eigenvalues must have real part strictly negative!



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Summarizing

$$\det(sI - A) = \prod_{i=1}^{d} (s - \lambda_i)^{\mu(\lambda_i)}$$

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- $\mu(\lambda_i)$:= algebraic multiplicity of λ_i
- $\ker(\lambda_i I - A)$:= eigenspace associated to λ_i
- $\dim\Big(\ker(\lambda_i I - A)\Big)$:= *geometric multiplicity* of λ_i

$$\boldsymbol{v} \neq \mathbf{0}, \quad A\boldsymbol{v} = \lambda \boldsymbol{v}$$

$$\dim\Big(\ker\left(\lambda_i I - A\right) \Big)$$

$$\det\left(sI - A\right) = \prod_{i=1}^{d} \left(\lambda - \lambda_i'\right)^{\mu\left(\lambda_i'\right)}$$

$$\mu\left(\lambda_i'\right)$$

our aim: understand how these components relate

$\Longrightarrow$ need to go back to the geometric interpretations

(but, before, we need a couple of theoretical results)

Definition (diagonalizable matrix)

A is diagonalizable if $\exists T$ s.t. $T^{-1}AT = \Lambda$ with Λ diagonal

Theorem

A is diagonalizable if and only if A has n linearly independent eigenvectors

Theorem

not all the A's are diagonalizable; e.g., Jordan matrices are not

Theorem (Jordan canonical form)

all the matrices that can not be diagonalized can always be transformed, by using an opportune change of coordinates, to a block diagonal matrix

$$A = \begin{bmatrix} A_1 & & 0 \\ & \dots & \\ 0 & & A_{n'} \end{bmatrix}$$

with $n' < n$ *and at least one block* A_i *of the form*

$$A_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

Example

$$A \quad \mapsto \quad \widetilde{A} = \begin{bmatrix} 2 &&&&&&&&& \\ & 2 & 1 &&&&&&& \\ && 2 &&&&&&& \\ &&& 2 & 1 &&&&& \\ &&&& 2 & 1 &&&& \\ &&&&& 2 &&&& \\ &&&&&& 3 &&& \\ &&&&&&& 3 & 1 & \\ &&&&&&&& 3 & 1 \\ &&&&&&&&& 3 \end{bmatrix}$$

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- algebraic multiplicity $=$ dimension of the Jordan block *(since each element on the diagonal adds a term "$(s-\lambda)$" in the characteristic polynomial)*

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- algebraic multiplicity $=$ dimension of the Jordan block *(since each element on the diagonal adds a term "$(s - \lambda)$" in the characteristic polynomial)*
- geometric multiplicity $=$ number of Jordan miniblocks *(since each miniblock adds its own $dim\Big(\ker(2I - A)\Big) = 1$)*

Extremely important facts to remember!!!

Assume T to be a generic change of basis. Then:

1. the eigenvectors and eigenvalues depend only on $\mathcal{A}$, and not on the used basis: λ_i eigenvalue of $A \Leftrightarrow \lambda_i$ eigenvalue of $A' = TAT^{-1}$

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5. the geometric multiplicities depend only on $\mathcal{A}$, and not on the used basis

Recap of the connections

If A is diagonalizable then:

- there exist a basis for $\mathbb{R}^n$ that is composed of eigenvectors of A
- the sum of the geometric multiplicities of the various eigenspaces of A is n
- the various eigenspaces of A span the whole $\mathbb{R}^n$
- the associated system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is actually a series of independent 1-dimensional systems
- the modes of the associated system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ are of the form $e^{\lambda t}$

Recap of the connections

The case "A is not diagonalizable"

- in any case there exists a change of basis that maps A into its Jordan form
- there must be at least one Jordan minibloc, and the effect of this miniblock is to introduce some sort of shearing in some directions
- the eigenvectors of A do not span the entire $\mathbb{R}^n$, but only a part of it
- the sum of the geometric multiplicities of the various eigenspaces of A is smaller than n; actually it is equal to the number of Jordan miniblocks
- the associated system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is actually a series of independent systems, each one corresponding to one of the Jordan miniblocks
- the modes of the associated system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ are not only of the form $e^{\lambda t}$, but there must be some $te^{\lambda t}$ or even higher powers of t

How do we find Jordan forms?

i.e., how can we go from $A = \begin{bmatrix} 3 & 4 & 8 \\ 1 & -5 & 2 \\ -5 & 9 & 1 \end{bmatrix}$ to $J = TAT^{-1}$?

→ needs the concepts of generalized eigenvectors, but this is a bit too much for this course . . . In any case just use numerical tools!

?

matrix exponentials and ODEs

Roadmap

- matrix exponentials

Looking back: general solution of 1-st order differential equations

$$\dot{x} = ax + bu \qquad \Longrightarrow \qquad x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

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What about multi-dimensional systems? May it be

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \quad \Longrightarrow \quad \boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)} B\boldsymbol{u}(\tau) d\tau \quad ?$$

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What is e^{At}?

WRONG way of doing it:

$$A = \begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix} \implies e^{At} := \begin{bmatrix} e^{a_{11}t} & e^{a_{12}t} \\ e^{a_{21}t} & e^{a_{22}t} \end{bmatrix}$$

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This is wrong because exponentials must be s.t. $e^{as}e^{at} = e^{a(s+t)}$, and the definition above does not hold:

$$\begin{aligned} e^{As}e^{At} &= \begin{bmatrix} e^{a_{11}s} & e^{a_{12}s} \\ e^{a_{21}s} & e^{a_{22}s} \end{bmatrix}\begin{bmatrix} e^{a_{11}t} & e^{a_{12}t} \\ e^{a_{21}t} & e^{a_{22}t} \end{bmatrix} \\ &= \begin{bmatrix} e^{a_{11}s+a_{11}t} + e^{a_{12}s+a_{21}t} & e^{a_{11}s+a_{12}t} + e^{a_{12}s+a_{22}t} \\ e^{a_{21}s+a_{11}t} + e^{a_{22}s+a_{21}t} & e^{a_{21}s+a_{12}t} + e^{a_{22}s+a_{22}t} \end{bmatrix} \neq e^{A(s+t)} \end{aligned}$$

What is e^{At}?

Good definition = through Taylor expansions:

$$e^{at} = \sum_{n=0}^{+\infty} \frac{(at)^n}{n!} = 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \dots$$

implies

$$e^{At} = \sum_{n=0}^{+\infty} \frac{(At)^n}{n!} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots$$

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Implications

- $\frac{d}{dt}e^{At} = Ae^{At}$ *(i.e., as expected and desired)*
- $e^{As}e^{At} = e^{A(s+t)}$ *(i.e., as expected and desired)*

Additional results

derivation	$\frac{d}{dt}e^{At} = Ae^{At}$
product of same exponentials	$e^{As}e^{At} = e^{A(s+t)}$
preservation of commutativity	$AB = BA \quad \Longleftrightarrow \quad e^{A}e^{B} = e^{B}e^{A}$
exponential of zero	$e^{\mathbf{0}} = I$
non-null determinant	$\det(e^{A}) \neq \mathbf{0}$
inversion	$(e^{At})^{-1} = e^{-At}$
decomposition	$e^{\boldsymbol{P}B\boldsymbol{P}^{-1}} = \boldsymbol{P}e^{B}\boldsymbol{P}^{-1}$

?

Going back to the original problem: solving first-order ODEs

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \quad \Longrightarrow \quad \boldsymbol{x}(t) = \int_{t_0}^{t} e^{A(t-\tau)} B\boldsymbol{u}(\tau) d\tau + e^{A(t-t_0)}\boldsymbol{x}_0$$

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3. inserting this in the original ODE implies

$$e^{At}\dot{\boldsymbol{y}}(t) = B\boldsymbol{u} \qquad \Longleftrightarrow \qquad \dot{\boldsymbol{y}}(t) = e^{-At}B\boldsymbol{u} \qquad \Longleftrightarrow \qquad \boldsymbol{y}(t) = \int_{t_0}^{t} e^{-A\tau} B\boldsymbol{u}(\tau) d\tau + \boldsymbol{k}$$

with $\boldsymbol{k}$ an opportune constant

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4. plugging in this result back gives then

$$\boldsymbol{x}(t) = e^{At}\left(\int_{t_0}^{t} e^{-A\tau} B\boldsymbol{u}(\tau) d\tau + \boldsymbol{k}\right) = \int_{t_0}^{t} e^{A(t-\tau)} B\boldsymbol{u}(\tau) d\tau + e^{At}\boldsymbol{k} \quad \Longrightarrow \quad \boldsymbol{k} = \boldsymbol{x}_0 e^{-At_0}$$

e^{At} := transition matrix

Notation

e^{At} := transition matrix $= \Phi(t)$

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> computers will do the computations. You though need to know what are the concepts

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Finding the transition matrix $\Phi(t) = e^{\boldsymbol{At}}$ using Taylor expansions

$$e^{\boldsymbol{At}} \approx I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k$$

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$$e^{\boldsymbol{At}} \approx I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k$$

Note that this approximation may actually be exact if there exists m s.t. $A^m = \mathbf{0}$, since in this case

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{(m-1)!}A^{(m-1)}t^{(m-1)}$$

Example

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\implies \quad e^{At} = I + At + \frac{1}{2!}A^2t^2 = \begin{bmatrix} 1 & t & t + \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$