TTK4225 - Systems Theory, Autumn 2020

Damiano Varagnolo

NTNU

Course Overview

welcome to TTK4225!

Instructors

- main teacher: Damiano Varagnolo, `damiano.varagnolo@ntnu.no`, room D233
- theaching assistant: Iver Andreas Ugelvik, `iverau@stud.ntnu.no`

Where is the information?

- Blackboard
- study guide
- GitHub repository
- mediasite.ntnu.no

Reference group – VERY IMPORTANT

- at least 3 students
- will do 4 meetings (1 after the exam)
- shall represent the whole class $\implies$ you will have meetings among yourselves too
- shall lead to a <u>referansegrupperapport</u> containing suggestions for improvements

Expectations, teacher side

- be proactive
- be respectful
- be collaborative
- eventually, learn stuff

Prerequisites

- basic physics
- basic linear algebra
- basic analysis

- better if you can Matlab and Python

this course teaches the foundations of automatic control. Not mastering this course $\Longrightarrow$ failing to understand all the next courses

Expectations, learners side

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Implementation leitmotif

maximize the interactions among us

$\Longrightarrow$ things that you can do by yourself you do by yourself

Recommendations

- read the study guide
- do the assignments in the suggested order & times
- signal bugs / imprecisions in the material
- charge the phone + bring pen and paper + bring your laptop

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Structure of each lesson

1. frontal guidance
2. peer instructions
3. Jupyter coding

Peer instructions

- purpose = be active
- algorithm = for each question:
  1. first time, answer individually
  2. then form groups, discuss, try to convince each other, but eventually answer individually
  3. then I show the solution
  4. after that you pose questions

Peer instructions - test

Which of these effects:

a: the height of the sun in the sky during the day

b: the distance of the Earth from the sun

c: how many hours the sun is up each day

are reasons for the seasons?

1. a + b
2. a + c
3. b + c
4. a + b + c

Jupyter notebooks

`http://localhost:8888/notebooks/GitHub/TTK4225/trunk/Jupyter/odeint-example.ipynb`

Jupyter notebooks - instructions for the next lesson

1. download the notebooks in `https://github.com/damianovar/TTK4225-2020`
2. install *Jupyter notebooks* or *Anaconda*
3. launch and test `TTK4225/trunk/Jupyter/odeint-example.ipynb`

?

“At-home” self-assessment activities

1. contents mapping
2. Blackboard questionnaires

Contents mapping

why: enable you visually assess if you got the logical structure of the course contents

what: gamified mapping of the content units in the course

how: python script downloadable from Blackboard TODO

when: whenever you prefer – this is a self assessment activity!

More info in the study guide!

Blackboard questionnaires

why: enable the continuous quantitative collection of how much the class is learning / adapt the course pace

what: private Blackboard questionnaires

how: set of questions indexed in terms of “what is asked” and “how difficult the question is”

when: whenever you prefer – this is a self assessment activity!

More info in the study guide!

leitmotif: we want to help you, but we need you to help us

?

What about the CoVid-19?

Cornerstones:

- we may be in fully digital teaching as in this past spring for some time
- independently of this, who must be at home in quarantine is entitled to good teaching

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videotaping, etc = all elements to account for this

tips are most welcome

?

Course overview

Aim of the course

learn a language for describing phenomena
in an abstract and formal way,
that will be used later on
to design feedback and feedforward control schemes

What is control?

- adjusting the temperature in the shower
- riding a bike

What is automatic control?

applying mechanisms so to operate processes automatically,
so they don't need continuous direct human intervention

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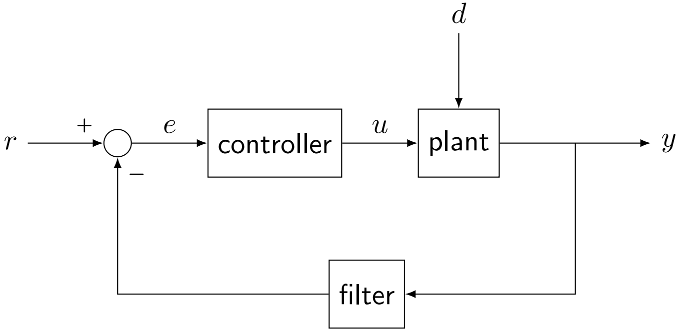
"remove the man from the loop"

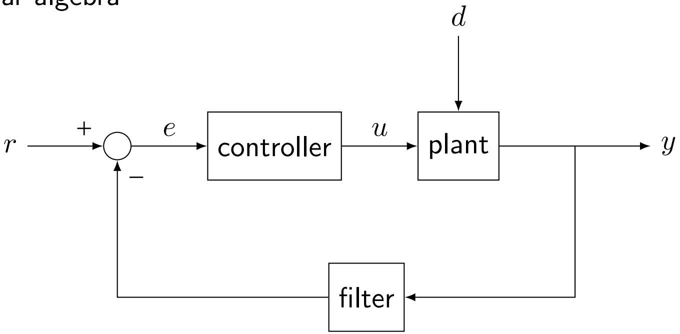
What is (feedback) control?



PS: this is not the only type of automatic control!

Intended Learning Outcomes, in brief:

- use differential equations to model continuous-time plants
- analyse the models in time and frequency domains
- learn to characterise important structural properties of these models
- learn a little bit about filters
- refresh linear algebra



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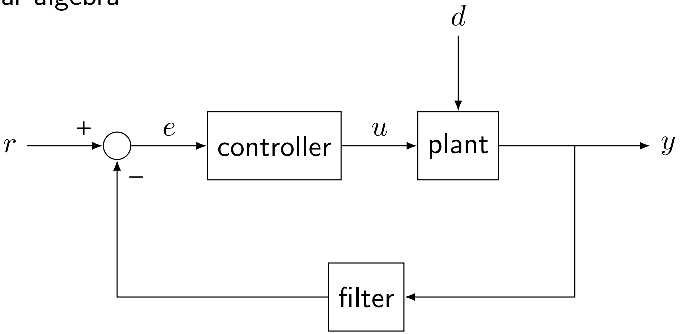
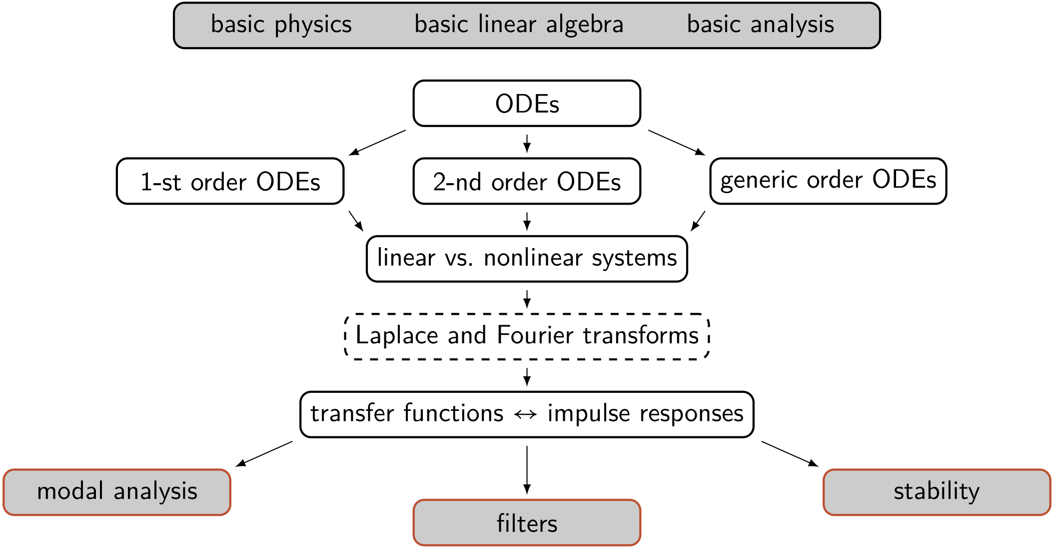


why? need all these ingredients to do advanced control!

Summary / Roadmap of TTK4225



Tentative time line

GitHub repository

↓

`ContentsMap` folder

↓

`plan-and-content-units.xlsx` file

assignments: will skip the first one on Matlab;

the first one will be the one on Laplace transforms

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Introduction to modelling and dynamics

Roadmap

- what do we mean with
  - “dynamics”?
  - “modelling a dynamical system”?
  - “control-oriented modelling”?

What do we mean with "dynamics"? Historical origins

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From mechanics

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Example:

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Even better:

$$\dot{v} = \frac{F}{m} \tag{2}$$

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$$F = ma = m\frac{dv}{dt} = m\dot{v} \tag{1}$$

Even better:

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this is an ODE

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Example: Lotka-Volterra:

- y_{prey} := prey
- y_{pred} := predator

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dynamics = the study of the motion (i.e., temporal evolution) of the variables that characterize a system

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$$\begin{cases} \dot{y}_{\text{prey}} &= \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} &= -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

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`GitHub/TTK4225/trunk/Jupyter/Lotka-Volterra-introduction.ipynb`

What do we mean with “modelling a dynamical system”?

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Example of a system from before:

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More generic definition: modelling a dynamical system = defining

$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\theta}\right),$$

thus defining:

- the variables
  - $\boldsymbol{u}$ = inputs
  - $\boldsymbol{y}$ = outputs
- the structure of the function $\boldsymbol{f}$
- the value of the parameters $\boldsymbol{\theta}$

What do we mean with “modelling a dynamical system”?

Even more generic definition: defining a *state-space system*:

$$\begin{cases} \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}\right) \\ \boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}\right) \end{cases}$$

thus defining:

- the variables
  - $\boldsymbol{u}$ = inputs
  - $\boldsymbol{x}$ = state
  - $\boldsymbol{y}$ = measured outputs
- the structure of the functions $\boldsymbol{f}$ and $\boldsymbol{g}$
- the value of the parameters $\boldsymbol{\theta}$

Discussion: did we model the Lotka-Volterra dynamical system here?

```
def myModel(y, t):
#
# parameters
alpha = 1.1
beta  = 0.4
gamma = 0.4
delta = 0.1
#
# get the individual variables - for readability
yPrey = y[0]
yPred = y[1]
#
# individual derivatives
dyPreydt  =   alpha * yPrey - beta  * yPrey * yPred
dyPreddt  = - gamma * yPred + delta * yPrey * yPred
#
return [ dyPreydt, dyPreddt ]
```

Discussion: do we need something else to simulate this system?

$$\begin{cases} \dot{y}_{\text{prey}} &= \ \ 1.2 y_{\text{prey}} - 0.1 y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} &= -0.6 y_{\text{pred}} + 0.2 y_{\text{prey}} y_{\text{pred}} \end{cases}$$

Connecting with next courses

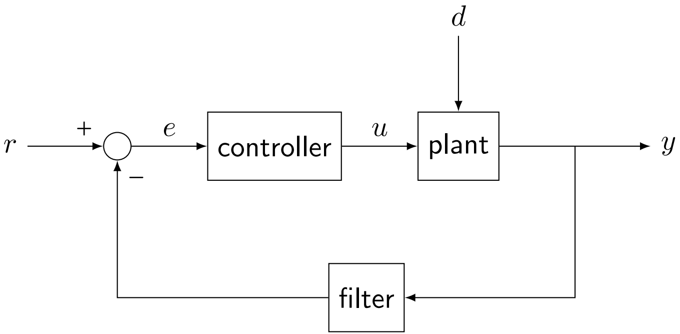
$$\begin{cases} \dot{\boldsymbol{x}} & = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}\right) \\ \boldsymbol{y} & = \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}\right) \end{cases}$$

- how to estimate the structure of $\boldsymbol{f}$, $\boldsymbol{g}$ and the values of $\boldsymbol{\theta}$ = system identification
- how to estimate $\boldsymbol{x}(0)$ from $\boldsymbol{y}$ = state observers

Caveat: static $\neq$ dynamic

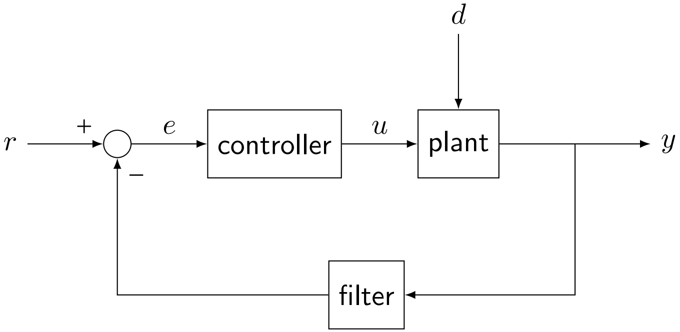
$$\boldsymbol{y} = \boldsymbol{f}\left(\boldsymbol{u}, \boldsymbol{\theta}\right) \qquad \neq \qquad \dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\theta}\right)$$

The workflow to control a dynamical system



1. model the process
2. identify it *(sometimes not strictly necessary)*
3. design the controller

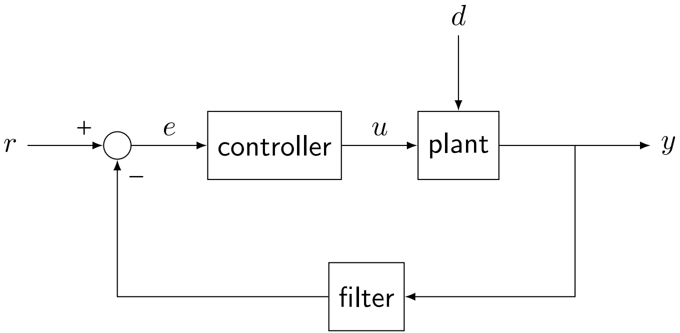
What does it mean to design a controller?



task: complete the following sentence: "In this case, the controller is $u = h\,(\_\_\_\_)$, and it shall be designed so that ____ is as desired, independently of ____"

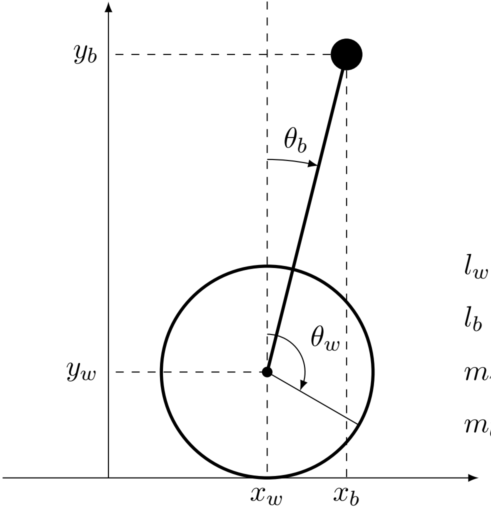
?

What does it mean to control a dynamical system?



= designing a controller = designing a function

Another example: balancing robot; what are u, x and y in this case?



l_w = radius of the wheel

l_b = body-wheel's center distance

m_w = mass of the wheel

m_b = mass of the body

Modelling - simplest example: speed of a cart

- speed $v(t)$
- friction $F_f(t) = -kv(t)$
- force from the engine $F(t)$ *(this is our u!)*
- Newton's second law: $\sum$ forces $= ma(t)$

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$$ma(t) = F_t(t) + F(t)$$

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$$ma(t) = F_t(t) + F(t)$$
$$m\dot{v} = -kv(t) + F(t)$$

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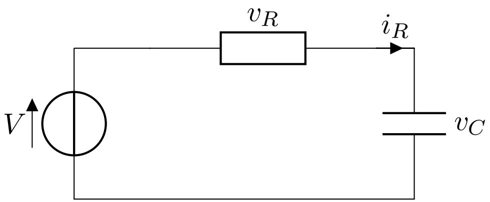
$$
\begin{aligned}
ma(t) &= F_t(t) + F(t) \\
m\dot{v} &= -kv(t) + F(t) \\
\Longrightarrow \quad \dot{v} &= -\frac{k}{m}v(t) + \frac{1}{m}F(t) \qquad (3)
\end{aligned}
$$

Modelling - another example: RC-circuit

Example 2.2 in Balchen's book



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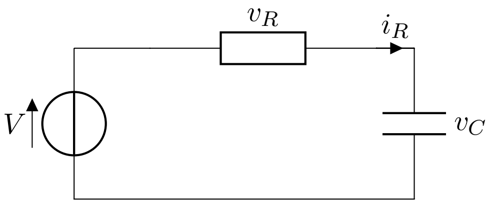

Question: what do we mean with dynamics in this case?

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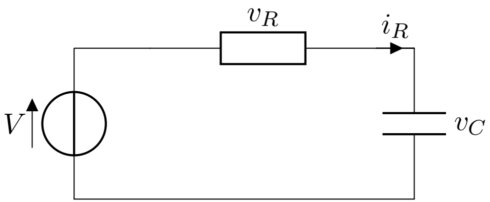


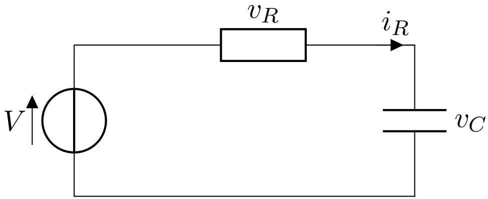
Question: what do we mean with dynamics in this case?

Remember our acception

dynamics = the study of the motion (i.e., temporal evolution) of the variables that characterize a system $\Longrightarrow$ modelling how tensions and currents evolve in time

Modelling - another example: RC-circuit

Example 2.2 in Balchen's book



Question: what do we mean with modelling the dynamics in this case?

Modelling - another example: RC-circuit

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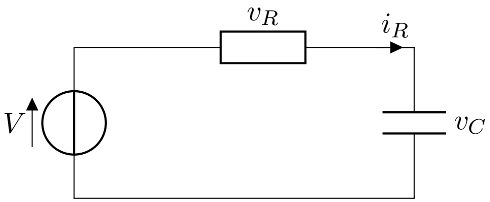

Question: what do we mean with modelling the dynamics in this case? We mean defining:

- the variables
  - $\boldsymbol{u}$ = inputs
  - $\boldsymbol{x}$ = state
  - $\boldsymbol{y}$ = measured outputs
- the structure of the functions $\boldsymbol{f}$ and $\boldsymbol{g}$
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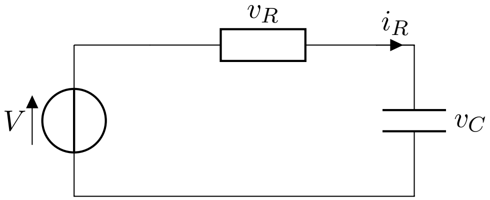

How to obtain a dynamical model = use Kirchhoff's laws (the sum of the voltages in a circuit is zero)

$$\frac{v_R}{R} = i_R = i_C = C\frac{dv_C}{dt} \tag{4}$$

Modelling - another example: RC-circuit

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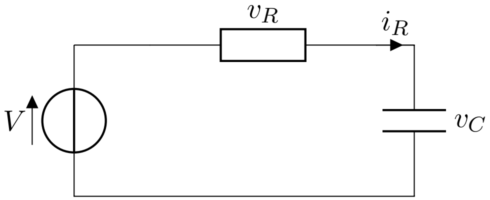

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$$V(t) = v_R + v_C \Rightarrow v_R = V(t) - v_C \quad \Longrightarrow \quad C\dot{v}_C = \frac{V(t) - v_C}{R} \tag{5}$$

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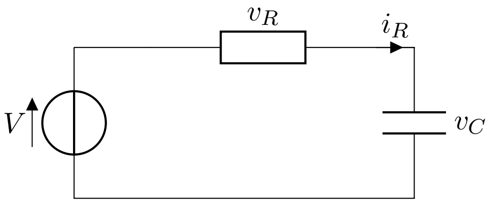

How to obtain a dynamical model = use Kirchhoff's laws (the sum of the voltages in a circuit is zero)

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$$V(t) = v_R + v_C \Rightarrow v_R = V(t) - v_C \quad \Longrightarrow \quad C\dot{v}_C = \frac{V(t) - v_C}{R} \tag{5}$$

$$\Longrightarrow \quad \dot{v}_C = -\frac{1}{RC}v_C + \frac{1}{RC}V(t) \tag{6}$$

?

Standing examples

Roadmap

Summary of the standing examples in this course:

- exponential growth
- RC-circuits
- RCL-circuits
- spring-mass system
- Lotka-Volterra
- Van-der-Pol oscillator
- balancing robot
- insuline concentration

Exponential growth

Generalization of "speed of a cart" and "RC circuit"

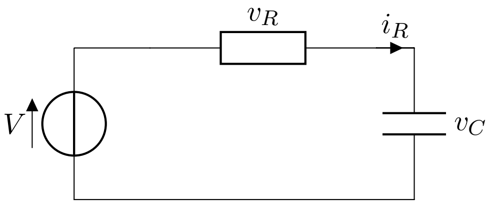
Scalar version:

$$\dot{y} = \alpha y + \beta u \tag{7}$$

Matricial version:

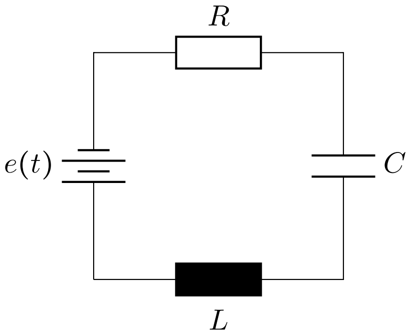
$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u} \tag{8}$$

RC-circuit



$$\dot{v}_C = -\frac{1}{RC} v_C + \frac{1}{RC} V(t) \tag{9}$$

RCL-circuit



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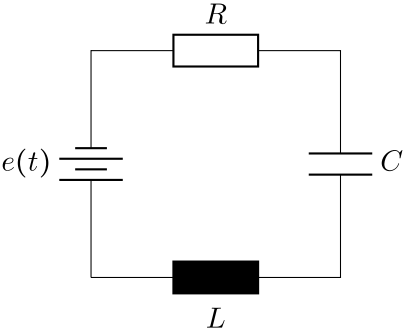
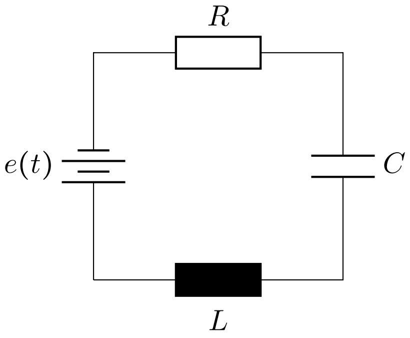


EOM: Kirkhoff laws

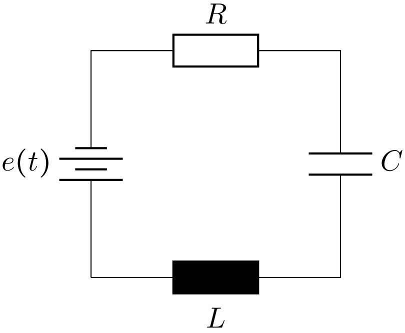
RCL-circuit



EOM: Kirkhoff laws

$$v_L(t) = L\frac{di(t)}{dt} \qquad v_i(t) = Ri(t) \qquad v_C(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

RCL-circuit



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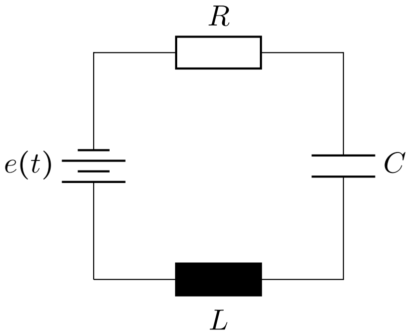
$$e(t) = v_L(t) + v_R(t) + v_C(t)$$

RCL-circuit



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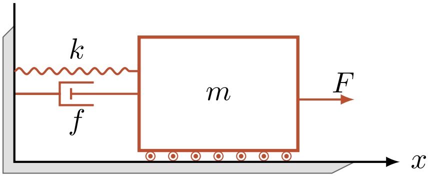
$$v_L(t) = L\frac{di(t)}{dt} \qquad v_i(t) = Ri(t) \qquad v_C(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

$$e(t) = v_L(t) + v_R(t) + v_C(t) \quad \mapsto \quad e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau \tag{10}$$

Spring-mass systems

E.g., position of a cart fastened with a spring to a wall and subject to friction



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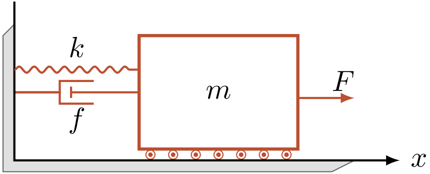

EOM:

- force from the spring: $F_x(t) = -kx(t)$
- friction: $F_f(t) = -f\dot{x}(t)$
- applied force: $F(t)$
- Newton's second law: $\sum F = m\ddot{x}(t)$

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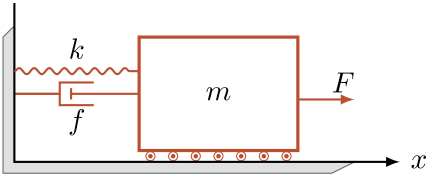

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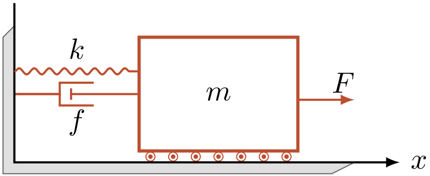
$$m\ddot{x}(t) = F_x(t) + F_f(t) + F(t)$$

Spring-mass systems

E.g., position of a cart fastened with a spring to a wall and subject to friction



EOM:

- force from the spring: $F_x(t) = -kx(t)$
- friction: $F_f(t) = -f\dot{x}(t)$
- applied force: $F(t)$
- Newton's second law: $\sum F = m\ddot{x}(t)$

$$m\ddot{x}(t) = F_x(t) + F_f(t) + F(t) \qquad \mapsto \qquad m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + F(t)$$

Lotka-Volterra

- y_{prey} := prey
- y_{pred} := predator

$$\begin{cases} \dot{y}_{\text{prey}} &= \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} &= -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

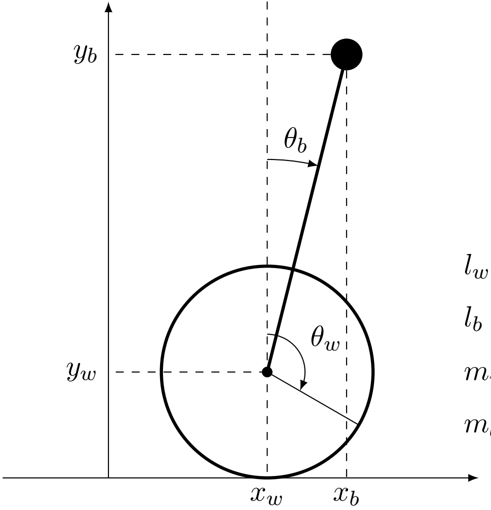
`GitHub/TTK4225/trunk/Jupyter/Lotka-Volterra-introduction.ipynb`

Van-der-Pol oscillator

$$\begin{cases} \dot{y}_1 &= \mu \left(y_1 - \dfrac{y_1^3}{3} - y_2 \right) \\ \dot{y}_2 &= \dfrac{y_1}{\mu} \end{cases} \tag{11}$$

`GitHub/TTK4225/trunk/Jupyter/Van-der-Pol-introduction.ipynb`

Balancing robot



l_w = radius of the wheel

l_b = body-wheel's center distance

m_w = mass of the wheel

m_b = mass of the body

Balancing robot

$$
\begin{aligned}
\left(I_b + m_b l_b^2\right)\ddot{\theta}_b &= +m_b l_b g \sin\left(\theta_b\right) - m_b l_b \ddot{x}_w \cos\left(\theta_b\right) - \frac{K_t}{R_m} v_m + \left(\frac{K_e K_t}{R_m} + b_f\right)\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \\
\left(\frac{I_w}{l_w} + l_w m_b + l_w m_w\right)\ddot{x}_w &= -m_b l_b l_w \ddot{\theta}_b \cos\left(\theta_b\right) + m_b l_b l_w \dot{\theta}_b^2 \sin\left(\theta_b\right) + \frac{K_t}{R_m} v_m - \left(\frac{K_e K_t}{R_m} + b_f\right)\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right)
\end{aligned}
\tag{12}
$$

Insulin concentration

- x_1 := sugar concentration
- x_2 := insulin concentration
- u_1 := food intake
- u_2 := insulin intake
- c := sugar concentration in fasting

$$\begin{cases} \dot{x}_2 = a_{21}\left(x_1 - c\right) - a_{22}x_2 + b_2u_2 & x_1 \geq c \\ \dot{x}_2 = -a_{22}x_2 + b_2u_2 & x_1 < c \end{cases}$$

$$\begin{cases} \dot{x}_1 = -a_{11}x_1x_2 - a_{12}\left(x_1 - c\right) + b_1u_1 & x_1 \geq c \\ \dot{x}_1 = -a_{11}x_1x_2 + b_1u_1 & x_1 < c \end{cases}$$

?

Seeing this course as a first step towards model predictive control

Roadmap

- how can we do control?
- what does it mean to do model predictive control?
- why are we doing this course?

control = to take out humans from the loop

(and do not waste time in doing things that machines could do)

Remember: control = design a function

$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\theta}\right)$$

where

- $\boldsymbol{f}$ and $\boldsymbol{\theta}$ = the model
- $\boldsymbol{y}$ the outputs
- $\boldsymbol{u}$ the inputs

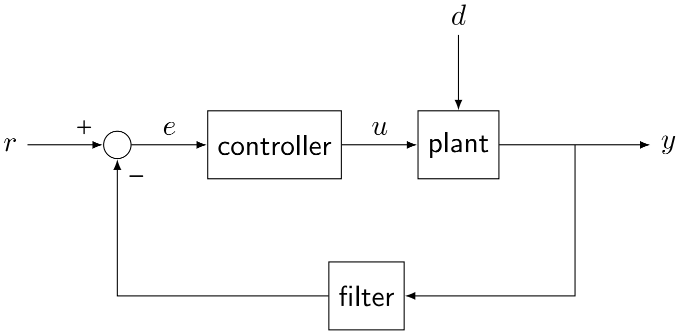
Two important paradigms

$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\theta}\right)$$

model free control: $\boldsymbol{u}$ is built without knowing $\boldsymbol{f}$ and $\boldsymbol{\theta}$

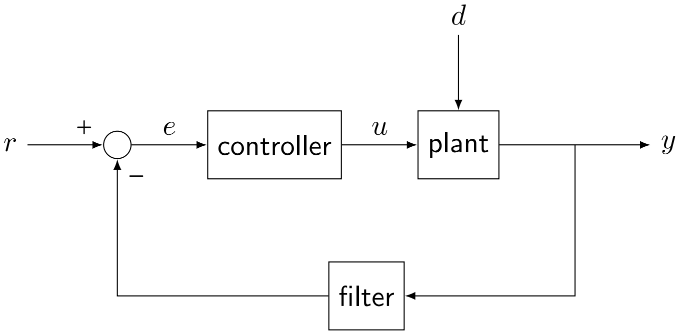
model based control: $\boldsymbol{u}$ is built through knowing $\boldsymbol{f}$ and/or $\boldsymbol{\theta}$

The simplest controller: the P control



$$u(t) = \kappa e(t)$$

The simplest controller: the P control



$$u(t) = \kappa e(t)$$

Discussion: we want to automate the control of the steering wheel while driving in a curve with a P controller.

1. What is u? What is r? What is e?
2. What does it imply to make κ bigger? And smaller?

P controlling a steering wheel

`GitHub/TTK4225/trunk/Jupyter/`

`first-order-system-introduction-with-P-controller.ipynb`

P controlling a steering wheel

`GitHub/TTK4225/trunk/Jupyter/`
`first-order-system-introduction-with-P-controller.ipynb`

Discussion: what do you expect may happen if there is a spring somewhere?

important message: we can do 'model free' control

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(and actually something like 90% of the controllers in the world are model-free...)

important message: we can do 'model free' control

(and actually something like 90% of the controllers in the world are model-free. . .)

efficient, but kind of "dumb". We don't actually know what is happening!

Discussion: can we do something better if we know the model?

How can we do something better if we know the model?

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1. if I know $\boldsymbol{f}$ and my current initial condition, I can plug in a tentative $\widehat{\boldsymbol{u}}$, and see what would be the corresponding $\widehat{\boldsymbol{y}}$

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$$\boldsymbol{u}^{\star} = \arg \min_{\boldsymbol{u} \in \mathcal{U}, \boldsymbol{f}(\boldsymbol{u}) \in \mathcal{F}} \text{Cost}\left(\boldsymbol{f}\left(\boldsymbol{u}\right), \boldsymbol{u}\right)$$

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- define “Cost”

What does this require?

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- define "Cost"
- be able to compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly

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- define "Cost"
- be able to compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

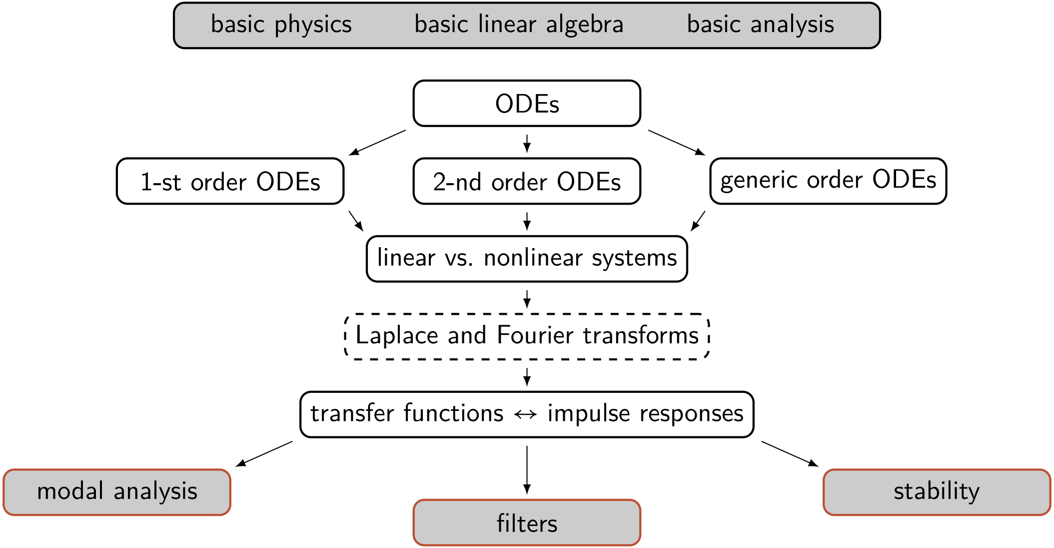
the most important point: to do model-based control we need to be able to simulate the most important point: to do model-based control we need to be able to simulate

this course = understanding the properties of the system,
& understanding how to simulate it with the purpose of doing advanced control

The control courses at NTNU ITK

- how do I compute explicitly $\boldsymbol{u}^\star$ in special cases?
- how do I compute numerically $\boldsymbol{u}^\star$ in general?
- how do I estimate $\boldsymbol{f}$ from data?
- how do I handle uncertainties?
- which properties can I guarantee?
- ...

Summary / Roadmap of TTK4225



Caveat! Better doing MPC than data-driven control:

Benjamin Recht (UC Berkeley, USA)

Reflections on the Learning-to-Control Renaissance

(and also `https://www.youtube.com/watch?v=nF2-39a29Pw`)

?

Introduction to first order systems - the concept of free evolution

Roadmap

- generalization of our examples of first order systems
- the concept of free evolution
- introduction to stability
- free evolution $\neq$ forced response

First order dynamical models

- train velocity

$$\dot{v} = -\frac{k}{m}v(t) + \frac{k}{m}F(t) \tag{13}$$

- RC-circuit

$$\dot{v}_C = -\frac{1}{RC}v_C + \frac{1}{RC}V(t) \tag{14}$$

First order dynamical models

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In general:

$$\dot{y} = ay + bu \tag{15}$$

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`GitHub/TTK4225/trunk/Jupyter/`
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Overarching question towards model-based control: how does y evolve?

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- what happens if $u = 0$, $y_0 > 0$, and $a > 0$?

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- what happens if $u = 0$, $y_0 \neq 0$, and $a < 0$?
- what happens if $u = 0$, and $a = 0$?

stability $=$ extremely important topic!

will be analysed in more details later on in this course

and much more extensively in others

(feat. Lyapunov, Krasovskii, La-Salle among others)

An important case: free evolution

$$\dot{y} = ay \cancel{+bu} \qquad\qquad y(0) = y_0 \neq 0 \tag{17}$$

Towards the solution of the free evolution starting from a simplified case

$$\dot{y} = y \qquad\qquad y(0) = y_0 \neq 0 \tag{18}$$

(in other words, which function is equal to its derivative?)

The solution of the free evolution in general

$$\dot{y} = ay \qquad y(0) = y_0 \neq 0 \tag{19}$$

The solution of the free evolution in general

$$\dot{y} = ay \qquad\qquad y(0) = y_0 \neq 0 \tag{19}$$

Alternatively:

$$\dot{y} = \frac{dy}{dt} = ay(t) \tag{20}$$

$$\frac{dy}{y(t)} = a\ dt \tag{21}$$

$$\int_0^t \frac{dy}{y(\tau)} = \int_0^t a\ \mathrm{d}\tau \tag{22}$$

$$\Big[\ln y(\tau)\Big]_0^t = a\Big[\tau\Big]_0^t \tag{23}$$

$$\ln\big(v(t)\big) = \ln(y_0) + at \tag{24}$$

$$y(t) = e^{\ln(y_0)at} = e^{\ln(y_0)}e^{at} \tag{25}$$

$$y(t) = y_0 e^{at} \tag{26}$$

And what about the vectorial generalization?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y}$$

$\Longrightarrow$ later in the course, and extensively!

Analysis of first order systems in free evolution

$$\dot{y} = ay \qquad y(0) = y_0 \neq 0 \qquad \Longrightarrow \qquad y(t) = y_0 e^{at} \tag{27}$$

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Discussion: which case is this one?

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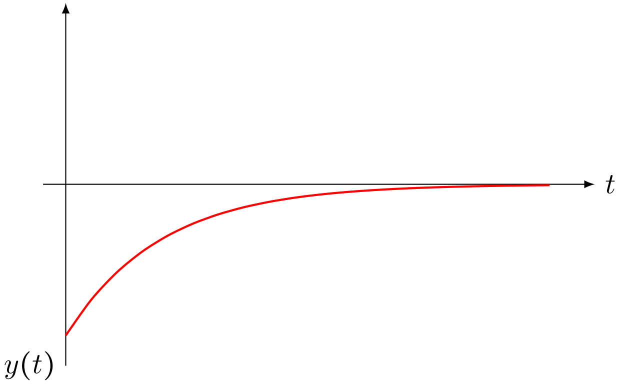

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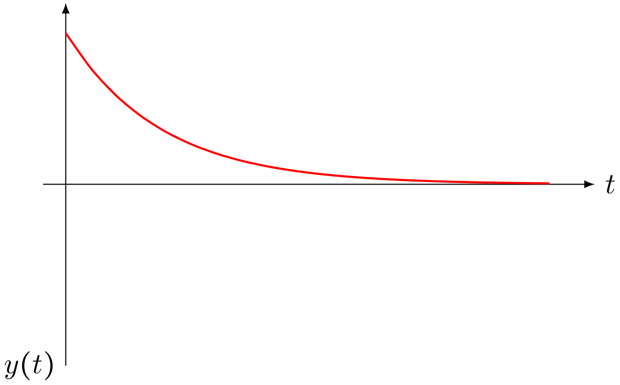

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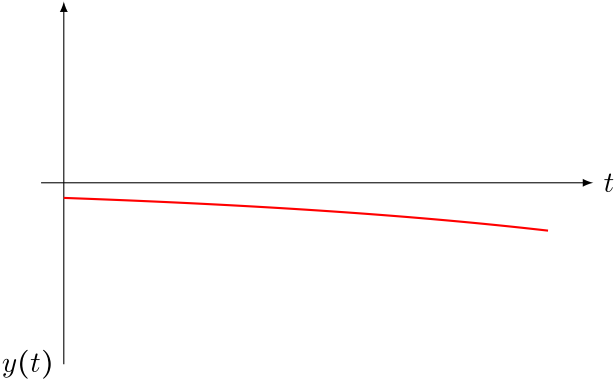

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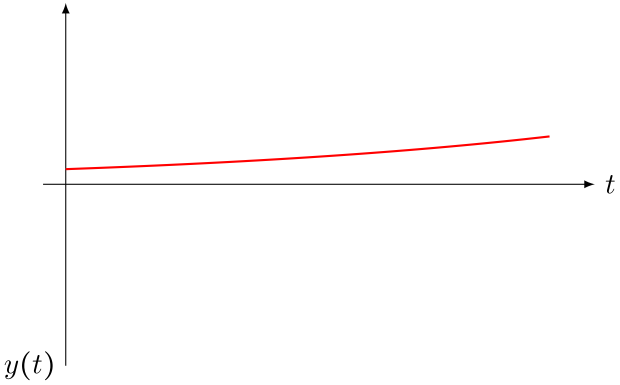

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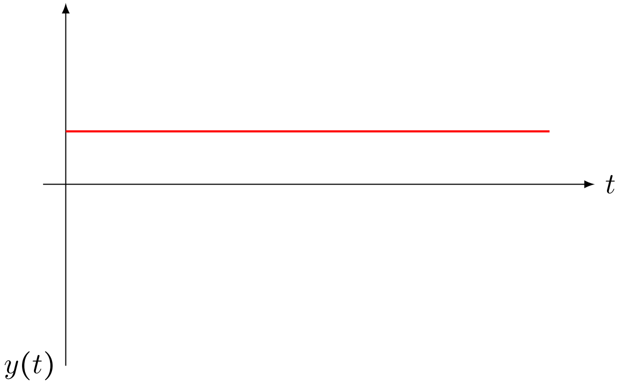

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Introduction to stability

stable system: $y(t) \to 0$ for $t \to +\infty$

unstable system: $|y(t)| \to +\infty$ for $t \to +\infty$

marginally stable system: $y(t) = y_0$ for every t

Introduction to stability

stable system: $y(t) \to 0$ for $t \to +\infty$

unstable system: $|y(t)| \to +\infty$ for $t \to +\infty$

marginally stable system: $y(t) = y_0$ for every t

much more on stability both later on & in later courses!

Are we done with studying the response of first order dynamical models?

$$\dot{y} = ay + bu \tag{28}$$

Forced response

$$\dot{y} = ay + bu \qquad\qquad y(0) = y_0 = 0 \tag{29}$$

(remember: free evolution = "$u = 0, y_0 \neq 0$")

Forced response

$$\dot{y} = ay + bu \qquad\qquad y(0) = y_0 = 0 \tag{29}$$

(remember: free evolution = "$u = 0, y_0 \neq 0$"*)*

Discussion: does $y(0) = y_0 = 0$ imply that we should study $\dot{y} = bu$?

Try it yourself at home!

`GitHub/TTK4225/trunk/Jupyter/`
`first-order-system-introduction-with-P-controller.ipynb`

?

Introduction to first order systems - the concept of forced response

Roadmap

- the concept of “forced response”
- the convolution operator
- some properties of convolution

First order dynamical models

$$\dot{y} = ay + bu \tag{30}$$

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`first-order-system-introduction-with-P-controller.ipynb`

Overarching question towards model-based control: how does y evolve?

$$\dot{y} = ay + bu \tag{31}$$

Free evolution vs. forced response

$$\dot{y} = ay + bu \tag{32}$$

free evolution: $u = 0,\ y_0 \neq 0$

forced response: $u \neq 0,\ y_0 = 0$

Solution to the forced response

(actual solution through equations (2.11)–(2.17) in Balchen's book)

$$\dot{y} = ay + bu, \quad y(0) = y_0 = 0 \qquad \Longrightarrow \qquad y(t) = \int_0^t e^{a(\tau)} bu(t - \tau) d\tau \tag{33}$$

Solution to the forced response

(actual solution through equations (2.11)–(2.17) in Balchen's book)

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extremely important generalization:

$$h(t) = e^{at} \qquad \Longrightarrow \qquad y(t) = \int_0^{+\infty} h(\tau) u(t - \tau) d\tau$$

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extremely important generalization:

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extremely important definition:

$$\text{convolution} := h * u(t) := \int_{-\infty}^{+\infty} h(\tau) u(t - \tau) d\tau$$

Visualizing $h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$

Discussions

$$h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

1. is $h * u(t) = u * h(t)$?

Discussions

$$h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

1. is $h * u(t) = u * h(t)$?
2. is $(\alpha h_1 + \beta h_2) * u(t) = \alpha\left(h_1 * u(t)\right) + \beta\left(h_2 * u(t)\right)$?

Discussions

$$h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

1. is $h * u(t) = u * h(t)$?
2. is $(\alpha h_1 + \beta h_2) * u(t) = \alpha\,(h_1 * u(t)) + \beta\,(h_2 * u(t))$?
3. why is it that $\dot{y} = ay + bu, \quad y(0) = y_0 = 0$ implies

$$y(t) = \int_{0}^{+\infty} e^{a(\tau)} bu(t-\tau)d\tau$$

and *not*

$$y(t) = \int_{-\infty}^{+\infty} e^{a(\tau)} bu(t-\tau)d\tau \quad ?$$

Example

$$y(t) = \int_0^{+\infty} h(\tau)u(t-\tau)d\tau,$$

$$u(t) = \begin{cases} 1 & \text{for } t \in [1,2] \\ 0 & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 2 & \text{for } t \in [0,1) \\ 1 & \text{for } t \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

Discussion:

What is $y(t) = h * u(t)$ for the case

$$u(t) = \begin{cases} 1 & \text{for } t \in [1,2] \text{ and } t \in [2,3] \\ 0 & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 2 & \text{for } t \in [0,2) \\ 0 & \text{otherwise} \end{cases} \quad ?$$

TODO

`GitHub/TTK4225/trunk/Jupyter/TODO.ipynb`

Summary

$$\dot{y} = ay + bu \tag{34}$$

free evolution: $u = 0,\ y_0 \neq 0$; implies $y(t) = y_{\text{free}}(t) = y_0 e^{at}$

forced response: $u \neq 0,\ y_0 = 0$; implies $y(t) = y_{\text{forced}}(t) = h * u(t)$ with $h(t) = e^{at}$

Summary

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free evolution: $u = 0$, $y_0 \neq 0$; implies $y(t) = y_{\text{free}}(t) = y_0 e^{at}$

forced response: $u \neq 0$, $y_0 = 0$; implies $y(t) = y_{\text{forced}}(t) = h * u(t)$ with $h(t) = e^{at}$

e^{at} = "impulse response"

yet another pillar concept for automatic control

→ *will be discussed better in a dedicated module*

*and re-discussed in **every** control-oriented course*

General response

$$\dot{y} = ay + bu \tag{35}$$

free evolution: $u = 0$, $y_0 \neq 0$; implies $y(t) = y_{\mathrm{free}}(t) = y_0 e^{at}$

forced response: $u \neq 0$, $y_0 = 0$; implies $y(t) = y_{\mathrm{forced}}(t) = h \ast u(t)$ with $h(t) = e^{at}$

General response

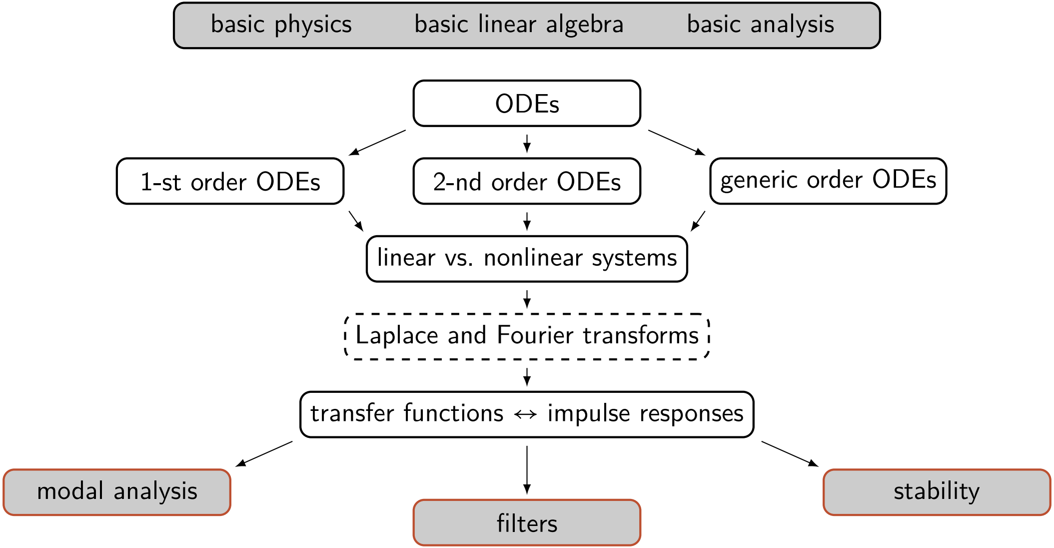
$$\dot{y} = ay + bu \tag{35}$$

free evolution: $u = 0$, $y_0 \neq 0$; implies $y(t) = y_{\text{free}}(t) = y_0 e^{at}$

forced response: $u \neq 0$, $y_0 = 0$; implies $y(t) = y_{\text{forced}}(t) = h * u(t)$ with $h(t) = e^{at}$

general response: $u \neq 0$, $y_0 \neq 0$; implies $y(t) = y_{\text{free}}(t) + y_{\text{forced}}(t)$ thanks to the fact that the model is *linear* *(will be seen in more details later on)*

Summary / Roadmap of TTK4225



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Introduction to first order systems - visualizing the system through a block scheme

Roadmap

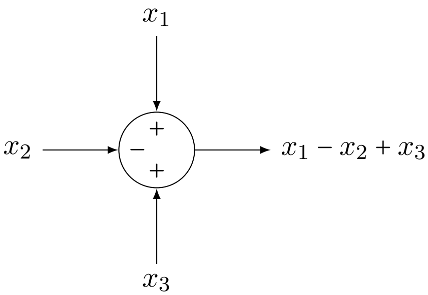
- the most common block schemes
- first order systems as block schemes

Block diagrams - why?

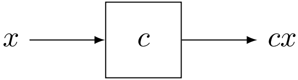
- used very often in companies
- aid visualization *(until a certain complexity is reached...)*
- enable "drag & drop" way of programming

- in this course, primarily used for interpretations

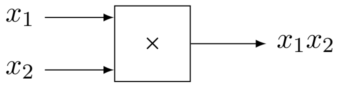
Most common block diagrams - sum of n signals



Most common block diagrams - multiplication for a constant



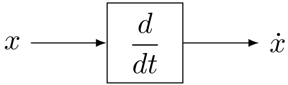
Most common block diagrams - multiplication of two signals



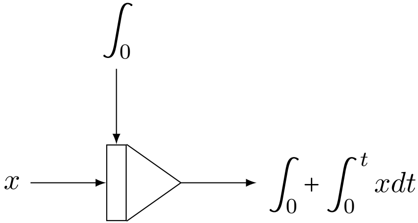
Most common block diagrams - generic functions

$$x \longrightarrow \boxed{f(\cdot)} \longrightarrow f(x)$$

Most common block diagrams - derivatives



Most common block diagrams - integrals

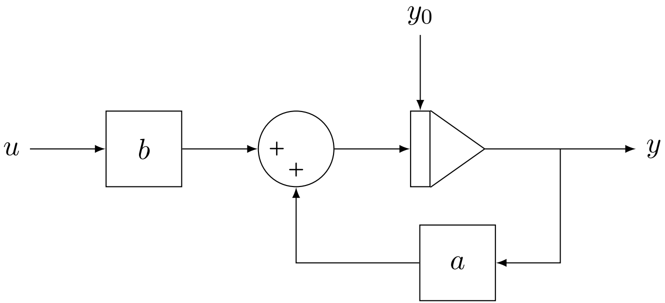


Discussion: how do we represent a first order DE with a block scheme?

$$\dot{y} = ay + bu$$

The solution

$$\dot{y} = ay + bu$$



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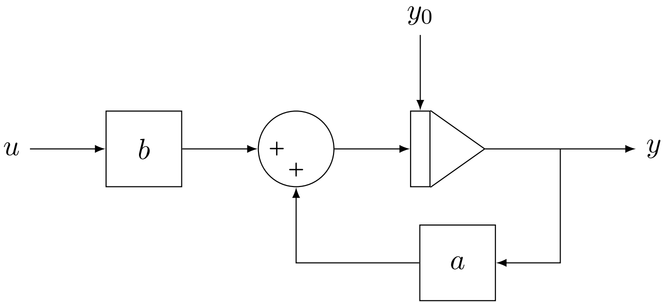


Discussion: do you note the presence of a feedback loop?

Introduction to Simulink

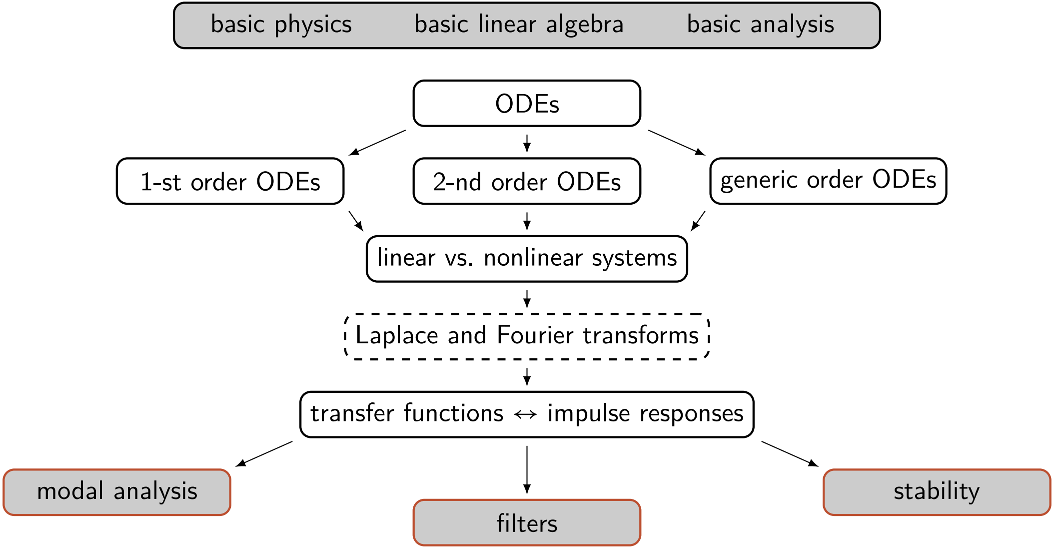
?

Introduction to generic order systems

Roadmap

- from first order to second order
- generic orders
- classification of the different types of ODEs

Summary / Roadmap of TTK4225



From first order to second order

Discussion: if we can write

$$\dot{y} = ay + bu\ ,$$

can we also write

$$\ddot{y} = a_1\dot{y} + a_0y + bu\ ?$$

From first order to second order, third order

Discussion: if we can write

$$\dot{y} = ay + bu \ ,$$

can we also write

$$\ddot{y} = a_1\dot{y} + a_0 y + bu \ ?$$

And at this point also

$$\dddot{y} = a_2\ddot{y} + a_1\dot{y} + a_0 y + bu \ ?$$

From first order to second order, third order

Discussion: if we can write

$$\dot{y} = ay + bu \ ,$$

can we also write

$$\ddot{y} = a_1\dot{y} + a_0 y + bu \ ?$$

And at this point also

$$\dddot{y} = a_2\ddot{y} + a_1\dot{y} + a_0 y + bu \ ?$$

And why not

$$\dddot{y} = a_2\ddot{y} + a_1\dot{y} + a_0 y + b_1\dot{u} + b_0 u \ ?$$

ARMA models

$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u \, ?$$

ARMA models

$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u \;?$$

Discussion: and why don't we have $a_n y^{(n)}$, instead of only $y^{(n)}$ on the LHS?

Discussion: can we generalize even more?

$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u \,?$$

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time-varying: $\dot{y} = a(t) y + b(t) u$

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non-homogeneous: $\dot{y} = a(t)y + b(t)u + c$

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time-varying: $\dot{y} = a(t)y + b(t)u$

non-homogeneous: $\dot{y} = a(t)y + b(t)u + c$

non-linear: $\dot{y} = a\sqrt{y} + bu^{2/3}$

Discussion: can we make this linear again?

$$\dot{y} = ay + bu^{2/3}$$

Discussion: can we make this linear again?

$$\dot{y} = ay + bu^{2/3}$$

Another discussion: can we apply the same "linearization trick" to $\dot{y} = a\sqrt{y} + bu$?

main focus in this course:

linear, time invariant, homogeneous (LTI)

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why?

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linear, time invariant, homogeneous (LTI)

why?

Remember: predictive control of the type

$$\boldsymbol{u}^{\star} = \arg \min_{\boldsymbol{u} \in \mathcal{U}, \boldsymbol{f}(\boldsymbol{u}) \in \mathcal{F}} \mathrm{Cost}\left(\boldsymbol{f}\left(\boldsymbol{u}\right), \boldsymbol{u}\right)$$

requires to be able to:

- define "Cost"
- compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

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from the previous lessons:
1-st order LTI models $\mapsto$ impulse responses;
as will be clear later generic order LTI models $\mapsto$ also impulse responses

The question we are trying to answer in this first part of the course

"What will be the solution $y(t)$ to a generic-order LTI ODE starting from a generic initial condition $y(0) = y_0$ and generic input signal $u(t)$?"

(our hope: the solution will be structurally easy, and fast to compute)

?

What is what?

(i.e., a small quiz to see if we are aligned)

	autonomous	linear	homogeneous	time constant
$\frac{\dot{x}}{x} = 1$				
$\pi = \dot{x}x$				
$2\dot{x} + \sin t = 0$				
$\sin(t\dot{x}) - x = 0$				
$t\dot{x} = x \sin t$				
$x - e^{-t} = \ddot{x} + \dot{x}$				

Introduction to generic order systems

Roadmap

- state space representations
- from n-th order to vectorial first order

State space representations - Definition

mathematical model (typically but not limited to of a physical system) as a finite set of inputs, outputs and state variables related by first-order differential equations satisfying the separation principle

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State space representations - Definition

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Ingredients

- finite number of inputs, outputs and state variables
- first-order differential equations
- *satisfies the separation principle:* the current value of states contains all the information necessary to forecast the future evolution of the outputs and of the states given the inputs

State space representations - Facts

- the future output depends only on the current state and the future input

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- the future output depends on the past input only through the current state
- the state summarizes the effect of past inputs on future output
  (sort of a "memory" of the system)

Example

Rechargeable flashlight:

- state = level of charge of the battery & on / off button
- output = how much light the device is producing

State space representations - Notation

$u_1, \ldots, u_m$ = inputs

$x_1, \ldots, x_n$ = states

$y_1, \ldots, y_p$ = outputs

State space representations - Notation

$$
\begin{aligned}
\dot{x}_1 &= f_1\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
&\vdots \\
\dot{x}_n &= f_n\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
y_1 &= g_1\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
&\vdots \\
y_p &= g_n\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right)
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\dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\
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\dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\
\boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)
\end{aligned}
$$

- $\boldsymbol{f}$ = state transition map
- $\boldsymbol{g}$ = output map

Discussion: is any $\boldsymbol{f}$ and $\boldsymbol{g}$ ok?

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\
\boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)
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Discussion: is any $\boldsymbol{f}$ and $\boldsymbol{g}$ ok?

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\
\boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u})
\end{aligned}
$$

No: for every given $\boldsymbol{x}_0$ and $\boldsymbol{u}$ the solution must be:

- unique
- guaranteed to exist for $t \in \left[0, \bar{t}\right)$ for some $\bar{t} > 0$

(check "existence and uniqueness of solutions of ODEs")

State space representations - Example

$$\begin{cases} \dot{x} = -x^2 \\ y = x \end{cases} \quad \cup \quad x_0 = -1 \qquad \Longrightarrow x(t) = \frac{1}{t-1}$$

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Discussion: *(a bit outside of the current flow of concepts on ODEs, but still relevant and very important)* what happens as $t \to 1$?

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Discussion: *(a bit outside of the current flow of concepts on ODEs, but still relevant and very important)* what happens as $t \to 1$?

there is a finite escape time!

Discussion: can every physical system be described with a state space representation?

Discussion: can every physical system be described with a state space representation?

Noticeable exceptions:

- non-lumped parameters model (i.e., PDEs)
- continuous-time systems with delays
  (will be discussed better later on)

Example: dynamics of sugar and insulin concentrations in blood

- x_1 := sugar concentration
- x_2 := insulin concentration
- u_1 := food intake
- u_2 := insulin intake
- c := sugar concentration in fasting

$$\begin{cases} \dot{x}_2 = a_{21}\left(x_1 - c\right) - a_{22}x_2 + b_2u_2 & x_1 \geq c \\ \dot{x}_2 = -a_{22}x_2 + b_2u_2 & x_1 < c \end{cases}$$

$$\begin{cases} \dot{x}_1 = -a_{11}x_1x_2 + a_{12}\left(c - x_1\right) + b_1u_1 & x_1 \geq c \\ \dot{x}_1 = -a_{11}x_1x_2 + b_1u_1 & x_1 < c \end{cases}$$

?

How can I transform a generic-order ODE into a state space representation (that requires first order DEs)?

$$\dddot{y} = \phi\left(\ddot{y}, \dot{y}, y, \ddot{u}, \dot{u}, u\right);$$

How can I transform a generic-order ODE into a state space representation (that requires first order DEs)?

$$\dddot{y} = \phi\left(\ddot{y}, \dot{y}, y, \ddot{u}, \dot{u}, u\right);$$

change of variables:

$$\boldsymbol{x} = \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix}, \ \boldsymbol{u} = \begin{bmatrix} \ddot{u} \\ \dot{u} \\ u \end{bmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{x}} = \begin{bmatrix} \dddot{y} \\ \ddot{y} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \phi\left(\boldsymbol{x}, \boldsymbol{u}\right) \\ \left[1\ 0\ 0\right] \boldsymbol{x} \\ \left[0\ 1\ 0\right] \boldsymbol{x} \end{bmatrix} = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)$$

Thus, how do I transform a generic order LTI system into a first order LTI system?

$$\ddot{y} = 2\dot{y} + 3y + 4\ddot{u} + 5\dot{u} + 6u$$

Transforming ARMA models

$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u \ ?$$

Discussion: but then any LTI system is a first order vectorial system?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$$

Discussion: but then any LTI system is a first order vectorial system?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$$

YES, AND THIS IS WHY
LINEAR ALGEBRA IS ESSENTIAL
FOR CONTROL!

?

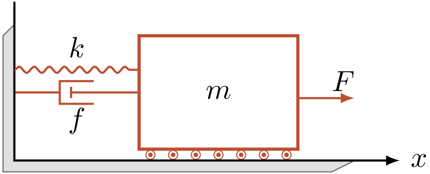
Second order systems

Roadmap

- interesting examples
- interesting responses
- differences between 1st order and 2nd order

Example 1: spring-mass systems

Position of a cart fastened with a spring to a wall and subject to friction:



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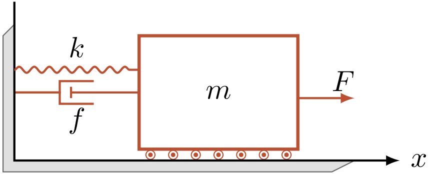

EOM:

- force from the spring: $F_x(t) = -kx(t)$
- friction: $F_f(t) = -f\dot{x}(t)$
- applied force: $F(t)$
- Newton's second law: $\sum F = m\ddot{x}(t)$

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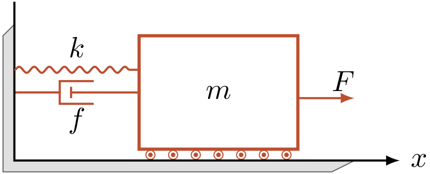

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$$m\ddot{x}(t) = F_x(t) + F_f(t) + F(t)$$

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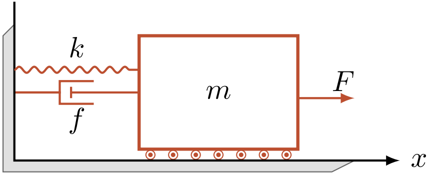

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- friction: $F_f(t) = -f\dot{x}(t)$
- applied force: $F(t)$
- Newton's second law: $\sum F = m\ddot{x}(t)$

$$m\ddot{x}(t) = F_x(t) + F_f(t) + F(t) \qquad \mapsto \qquad m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + F(t)$$

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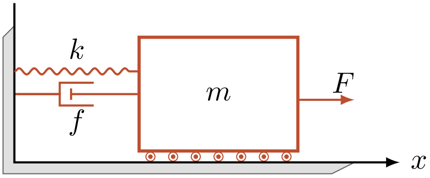

$$\ddot{x}(t) + \frac{f}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{1}{m}F(t) \tag{36}$$

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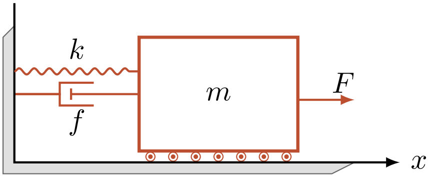

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Generalizing:

$$\ddot{x} = a_1\dot{x} + a_0x + b_0u \tag{37}$$

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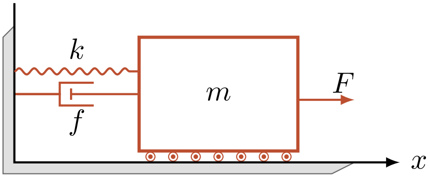

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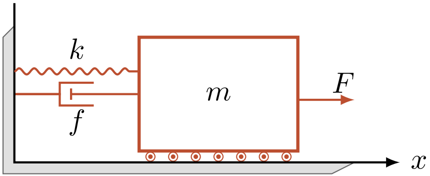

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Discussion: may we write this dynamics using a vectorial first order DE?

Discussion: may we write this dynamics using a scalar first order DE?

Example 1: spring-mass systems

https://youtu.be/lZPtFDXYQRU?t=31

(here there is very little friction, though)

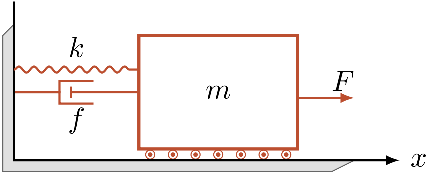
Example 1: spring-mass systems - stability analysis from intuitive perspectives

Discussion: $\dot{x} = a_0 x + b_0 u$ unstable if ...?

Example 1: spring-mass systems - stability analysis from intuitive perspectives

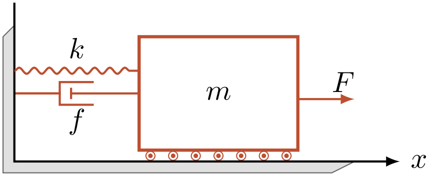
Discussion: $\dot{x} = a_0 x + b_0 u$ unstable if ...? And What about, from intuitive perspectives, when considering

$$\ddot{x} = a_1 \dot{x} + a_0 x + b_0 u \,? \tag{38}$$



`GitHub/TTK4225/trunk/Jupyter/spring-mass-systems-introduction.ipynb`

Intuition: asymptotic stability $\leftrightarrow$ dissipation of energy



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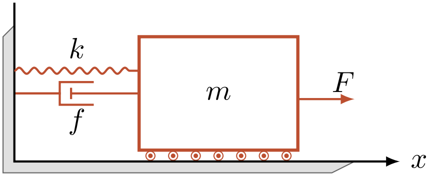
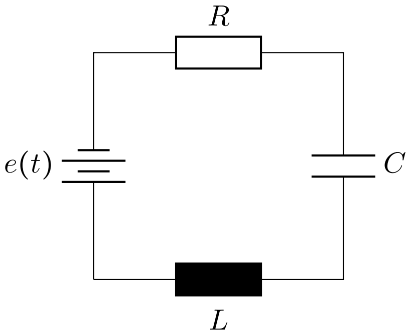


will see this MUCH better in later on courses
when discussing about Lyapunov stability

?

Example 2: RCL-circuit



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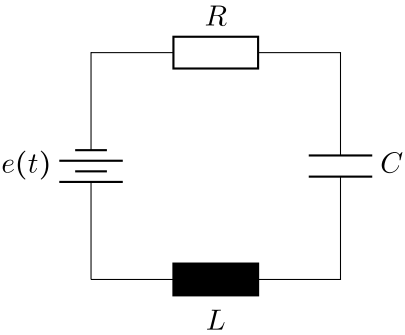

EOM: Kirkhoff laws

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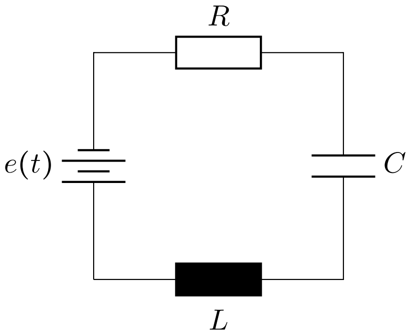
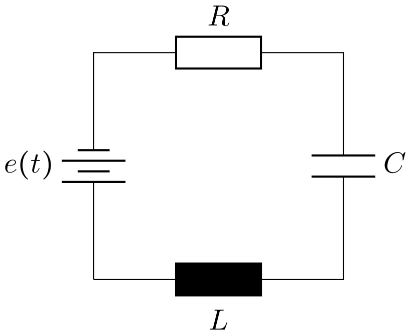


EOM: Kirkhoff laws

$$v_L(t) = L\frac{di(t)}{dt} \qquad v_i(t) = Ri(t) \qquad v_C(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

Example 2: RCL-circuit

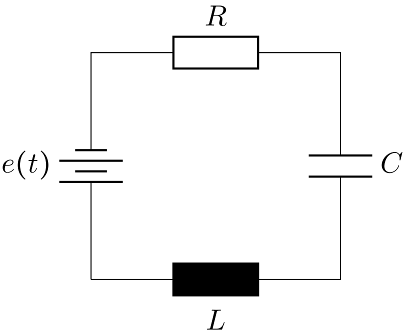


EOM: Kirkhoff laws

$$v_L(t) = L\frac{di(t)}{dt} \qquad v_i(t) = Ri(t) \qquad v_C(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

$$e(t) = v_L(t) + v_R(t) + v_C(t)$$

Example 2: RCL-circuit



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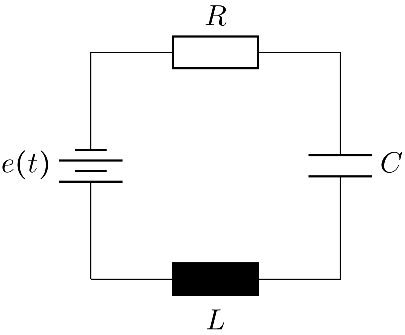

$$e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau$$

Discussion: how do we write it in terms of x, $\dot{x}$, $\ldots$?

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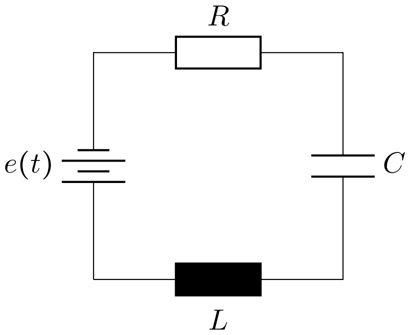

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Discussion: how do we write it in terms of x, $\dot{x}$, ...?

$$x(t) = i(t), \quad u(t) = \dot{e}(t) \qquad \Longrightarrow \qquad \ddot{x} = -\frac{R}{L}\dot{x} - \frac{1}{LC}x + \frac{1}{L}u \tag{40}$$

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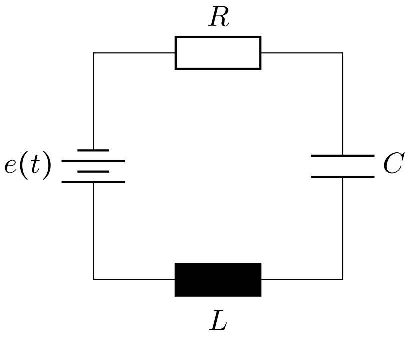

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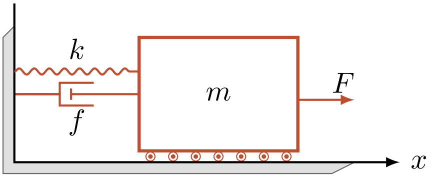
Generalizing:

?

Discussion: what are the potential different ways the cart may move back to its equilibrium?



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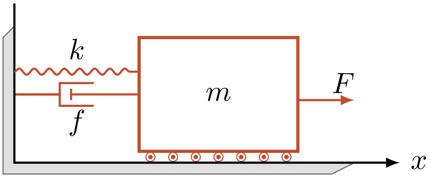
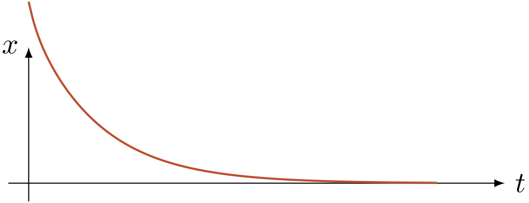


very big friction w.r.t. the spring stiffness $\Longrightarrow$ overdamping:



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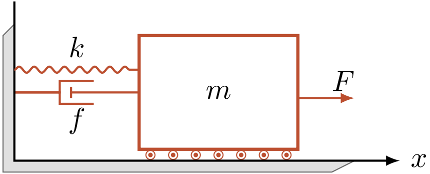
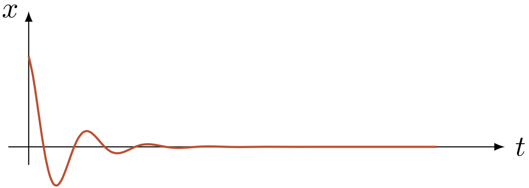


very small friction w.r.t. the spring stiffness $\implies$ underdamping:



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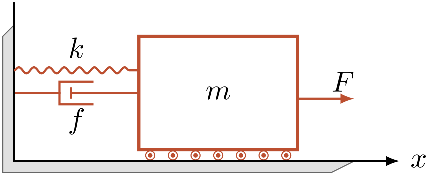
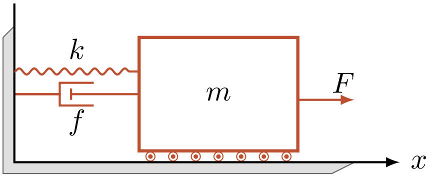


just the right amount of friction w.r.t. the spring stiffness $\implies$ critical damping, i.e., *the combination that leads to the fastest decay to zero without oscillations*:

Discussion: what are the potential different ways the cart may move back to its equilibrium?

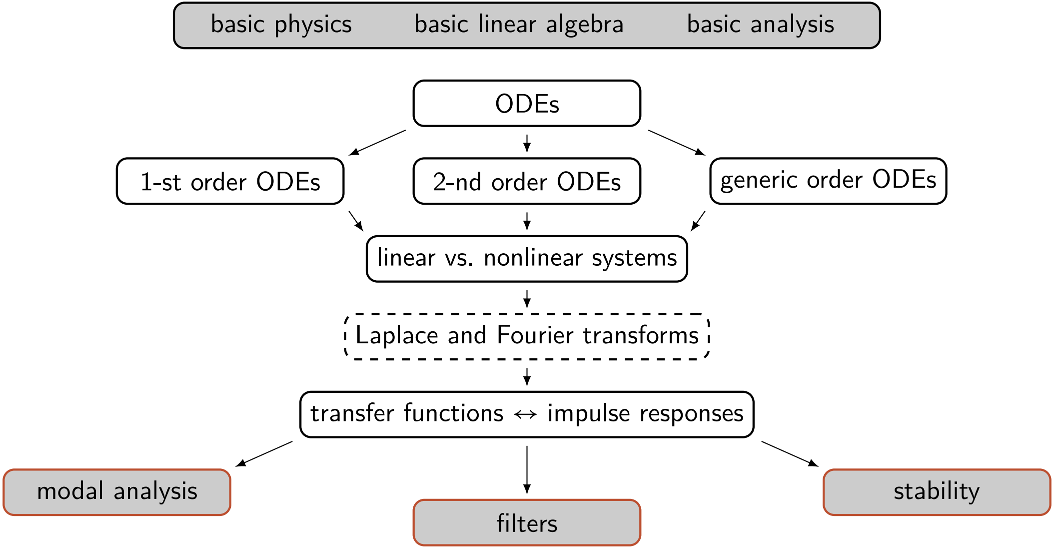


Discussion: can we get these behaviors with a first order system?

How do the free evolutions and forced responses look like numerically?

we will solve this problem with the aid of Laplace transforms

Summary / Roadmap of TTK4225



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