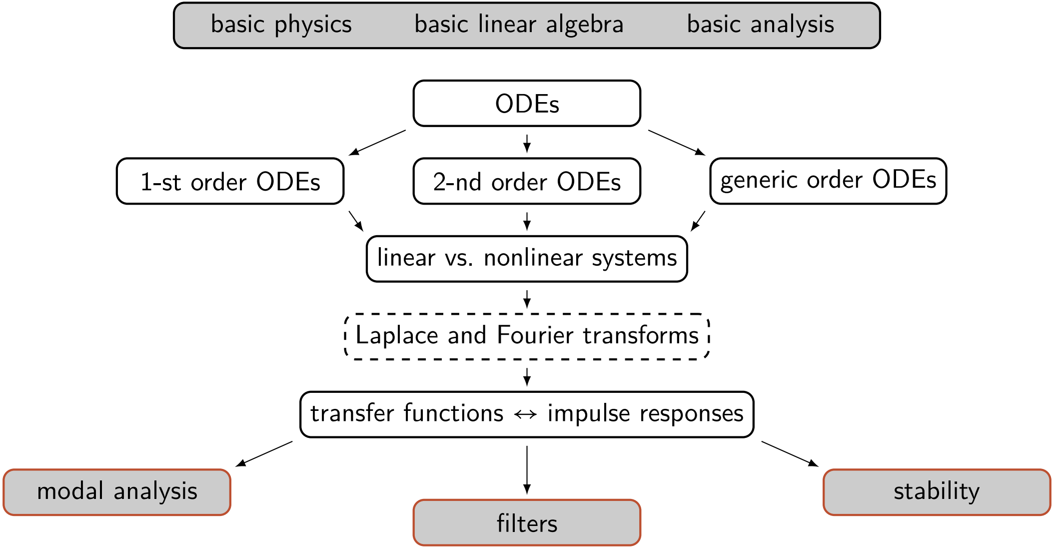
TTK4225 - Systems Theory, Autumn 2020

Damiano Varagnolo

NTNU

Some more insights on the concept of impulse response

Summary / Roadmap of TTK4225



Summary from the previous units

$$\dot{y} = ay + bu \tag{1}$$

free evolution: $u = 0$, $y_0 \neq 0$; implies $y(t) = y_{\text{free}}(t) = y_0 e^{at}$

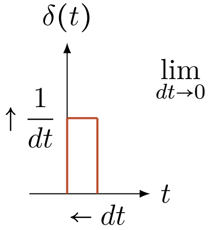
forced response: $u \neq 0$, $y_0 = 0$; implies $y(t) = y_{\text{forced}}(t) = h * u(t)$ with $h(t) = be^{at}$

Roadmap

- some more insights on the concept of “impulse response”

Kronecker's delta

(i.e., informally, "pushing" an unitary mass in an infinitesimal space)



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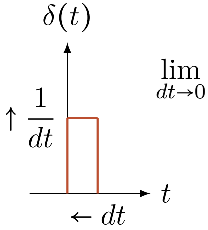

Discussion: $\int_{-\infty}^{+\infty} f(\tau)\delta(10)d\tau = ?$

Impulse response - definition

$h(t) := $ the system's response to an impulse $u(t) = \delta(t)$, since

$$y_\delta(t) = \int_0^t h(\tau)\delta(t-\tau)d\tau = h(t) \tag{2}$$

Impulse responses of first order systems

$$\dot{y} = ay + bu \quad \mapsto \quad h(t) = be^{at}$$

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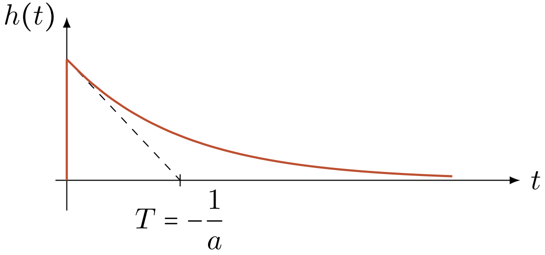

Impulse responses of first order systems

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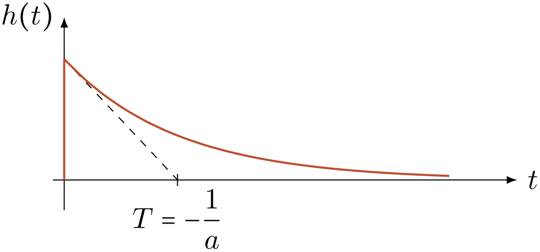

alternative writing: $h(t) = be^{-\frac{1}{T}t} \quad T > 0$

What about the previous examples?

Dynamics of a cart:

$$\dot{v} = -\frac{k}{m}v(t) + \frac{k}{m}F(t) \tag{3}$$

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Discussion: what does it mean to make the cart heavier?

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Discussion: what is the impulse response here? $T = \dfrac{m}{k}$

Discussion: what is the time constant here? $T = \dfrac{m}{k}$

Discussion: what does it mean to make the cart heavier?

And for the RC-circuit?

$$\dot{v}_C = -\frac{1}{RC} v_C + \frac{1}{RC} V(t) \tag{4}$$

Introduction to first order systems - visualizing the system through a block scheme

Roadmap

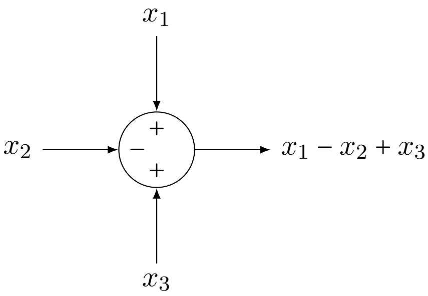
- the most common block schemes
- first order systems as block schemes

Block diagrams - why?

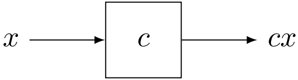
- used very often in companies
- aid visualization *(until a certain complexity is reached. . .)*
- enable "drag & drop" way of programming

- in this course, primarily used for interpretations

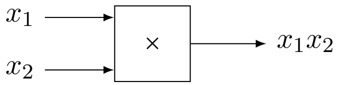
Most common block diagrams - sum of n signals



Most common block diagrams - multiplication for a constant



Most common block diagrams - multiplication of two signals



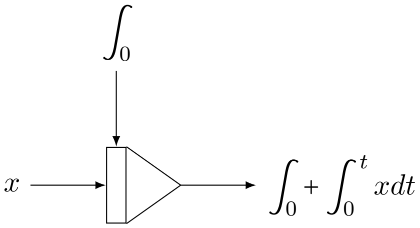
Most common block diagrams - generic functions

$$x \longrightarrow \boxed{f(\cdot)} \longrightarrow f(x)$$

Most common block diagrams - derivatives

$$x \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \dot{x}$$

Most common block diagrams - integrals

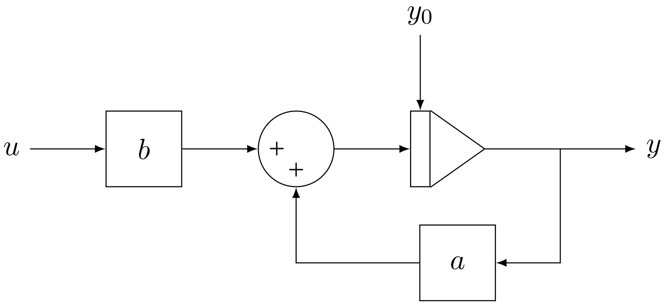


Discussion: how do we represent a first order DE with a block scheme?

$$\dot{y} = ay + bu$$

The solution

$$\dot{y} = ay + bu$$



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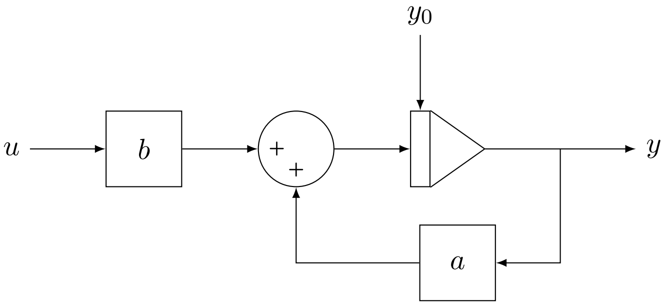


Discussion: do you note the presence of a feedback loop?

Introduction to Simulink

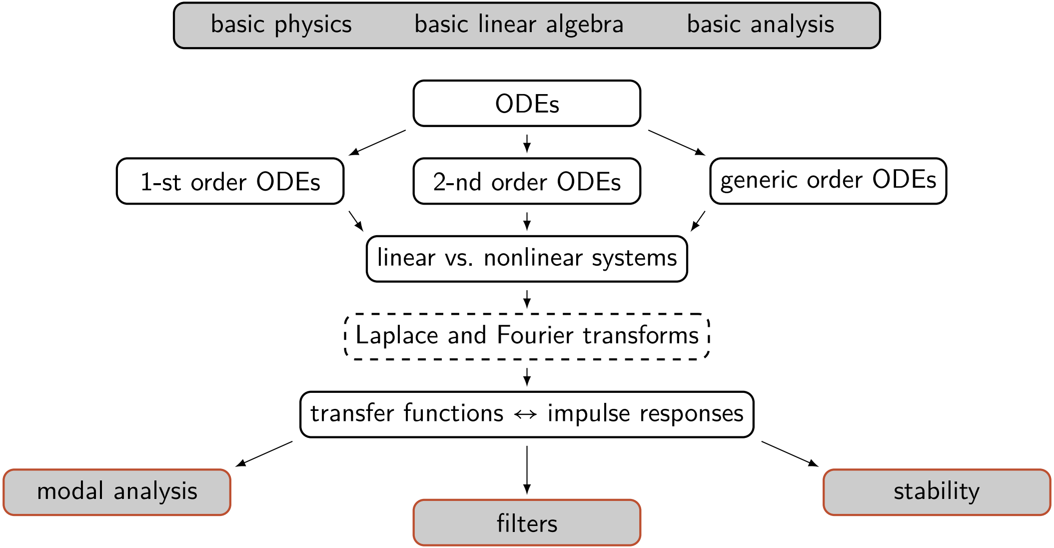
?

Introduction to generic order systems

Roadmap

- from first order to second order
- generic orders
- classification of the different types of ODEs

Summary / Roadmap of TTK4225



From first order to second order

Discussion: if we can write

$$\dot{y} = ay + bu \ ,$$

can we also write

$$\ddot{y} = a_1\dot{y} + a_0 y + bu \ ?$$

From first order to second order, third order

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And at this point also

$$\dddot{y} = a_2\ddot{y} + a_1\dot{y} + a_0 y + bu \ ?$$

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And at this point also

$$\dddot{y} = a_2\ddot{y} + a_1\dot{y} + a_0 y + bu \ ?$$

And why not

$$\dddot{y} = a_2\ddot{y} + a_1\dot{y} + a_0 y + b_1\dot{u} + b_0 u \ ?$$

ARMA models

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Discussion: and why don't we have $a_n y^{(n)}$, instead of only $y^{(n)}$ on the LHS?

Discussion: can we generalize even more?

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time-varying: $\dot{y} = a(t)y + b(t)u$

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time-varying: $\dot{y} = a(t)y + b(t)u$

non-homogeneous: $\dot{y} = a(t)y + b(t)u + c$

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time-varying: $\dot{y} = a(t)y + b(t)u$

non-homogeneous: $\dot{y} = a(t)y + b(t)u + c$

non-linear: $\dot{y} = a\sqrt{y} + bu^{2/3}$

Discussion: can we make this linear again?

$$\dot{y} = ay + bu^{2/3}$$

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Another discussion: can we apply the same "linearization trick" to $\dot{y} = a\sqrt{y} + bu$?

main focus in this course:

linear, time invariant, homogeneous (LTI)

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why?

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Remember: predictive control of the type

$$\boldsymbol{u}^{\star} = \arg \min_{\boldsymbol{u} \in \mathcal{U}, \boldsymbol{f}(\boldsymbol{u}) \in \mathcal{F}} \mathrm{Cost}\left(\boldsymbol{f}\left(\boldsymbol{u}\right), \boldsymbol{u}\right)$$

requires to be able to:

- define "Cost"
- compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

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from the previous lessons:

1-st order LTI models $\mapsto$ impulse responses;

as will be clear later generic order LTI models $\mapsto$ also impulse responses

The question we are trying to answer in this first part of the course

"What will be the solution $y(t)$ to a generic-order LTI ODE starting from a generic initial condition $y(0) = y_0$ and generic input signal $u(t)$?"

(our hope: the solution will be structurally easy, and fast to compute)

?

What is what?

(i.e., a small quiz to see if we are aligned)

	autonomous	linear	homogeneous	time constant
$\frac{\dot{x}}{x} = 1$				
$\pi = \dot{x}x$				
$2\dot{x} + \sin t = 0$				
$\sin(t\dot{x}) - x = 0$				
$t\dot{x} = x \sin t$				
$x - e^{-t} = \ddot{x} + \dot{x}$				

Introduction to generic order systems

Roadmap

- state space representations
- from n-th order to vectorial first order

State space representations - Definition

mathematical model (typically but not limited to of a physical system) as a finite set of inputs, outputs and state variables related by first-order differential equations satisfying the separation principle

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Ingredients

- finite number of inputs, outputs and state variables
- first-order differential equations
- *satisfies the separation principle:* the current value of states contains all the information necessary to forecast the future evolution of the outputs and of the states given the inputs

State space representations - Facts

- the future output depends only on the current state and the future input

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- the future output depends on the past input only through the current state
- the state summarizes the effect of past inputs on future output
  (sort of a "memory" of the system)

Example

Rechargeable flashlight:

- state = level of charge of the battery & on / off button
- output = how much light the device is producing

State space representations - Notation

$$u_1, \ldots, u_m = \text{inputs}$$

$$x_1, \ldots, x_n = \text{states}$$

$$y_1, \ldots, y_p = \text{outputs}$$

State space representations - Notation

$$
\begin{aligned}
\dot{x}_1 &= f_1\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
\vdots \\
\dot{x}_n &= f_n\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
y_1 &= g_1\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
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y_p &= g_n\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right)
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\end{aligned}
$$

- $\boldsymbol{f}$ = state transition map
- $\boldsymbol{g}$ = output map

Discussion: is any $\boldsymbol{f}$ and $\boldsymbol{g}$ ok?

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u})\end{aligned}$$

Discussion: is any $\boldsymbol{f}$ and $\boldsymbol{g}$ ok?

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned}$$

No: for every given $\boldsymbol{x}_0$ and $\boldsymbol{u}$ the solution must be:

- unique
- guaranteed to exist for $t \in \left[0, \bar{t}\right)$ for some $\bar{t} > 0$

(check "existence and uniqueness of solutions of ODEs")

State space representations - Example

$$\begin{cases} \dot{x} = -x^2 \\ y = x \end{cases} \quad \cup \quad x_0 = -1 \qquad \Longrightarrow x(t) = \frac{1}{t-1}$$

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Discussion: *(a bit outside of the current flow of concepts on ODEs, but still relevant and very important)* what happens as $t \to 1$?

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Discussion: *(a bit outside of the current flow of concepts on ODEs, but still relevant and very important)* what happens as $t \to 1$?

there is a finite escape time!

Discussion: can every physical system be described with a state space representation?

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Noticeable exceptions:

- non-lumped parameters model (i.e., PDEs)
- continuous-time systems with delays
  (will be discussed better later on)

Example: dynamics of sugar and insulin concentrations in blood

- x_1 := sugar concentration
- x_2 := insulin concentration
- u_1 := food intake
- u_2 := insulin intake
- c := sugar concentration in fasting

$$\begin{cases} \dot{x}_2 = a_{21}\left(x_1 - c\right) - a_{22}x_2 + b_2u_2 & x_1 \geq c \\ \dot{x}_2 = -a_{22}x_2 + b_2u_2 & x_1 < c \end{cases}$$

$$\begin{cases} \dot{x}_1 = -a_{11}x_1x_2 + a_{12}\left(c - x_1\right) + b_1u_1 & x_1 \geq c \\ \dot{x}_1 = -a_{11}x_1x_2 + b_1u_1 & x_1 < c \end{cases}$$

?

How can I transform a generic-order ODE into a state space representation (that requires first order DEs)?

$$\dddot{y} = \phi\left(\ddot{y}, \dot{y}, y, \ddot{u}, \dot{u}, u\right);$$

How can I transform a generic-order ODE into a state space representation (that requires first order DEs)?

$$\dddot{y} = \phi\left(\ddot{y}, \dot{y}, y, \ddot{u}, \dot{u}, u\right);$$

change of variables:

$$\boldsymbol{x} = \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix}, \ \boldsymbol{u} = \begin{bmatrix} \ddot{u} \\ \dot{u} \\ u \end{bmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{x}} = \begin{bmatrix} \dddot{y} \\ \ddot{y} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \phi\left(\boldsymbol{x}, \boldsymbol{u}\right) \\ \left[1 \ 0 \ 0\right] \boldsymbol{x} \\ \left[0 \ 1 \ 0\right] \boldsymbol{x} \end{bmatrix} = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)$$

Thus, how do I transform a generic order LTI system into a first order LTI system?

$$\ddot{y} = 2\dot{y} + 3y + 4\ddot{u} + 5\dot{u} + 6u$$

Transforming ARMA models

$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u \ ?$$

Discussion: but then any LTI system is a first order vectorial system?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$$

Discussion: but then any LTI system is a first order vectorial system?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$$

YES, AND THIS IS WHY
LINEAR ALGEBRA IS ESSENTIAL
FOR CONTROL!

?

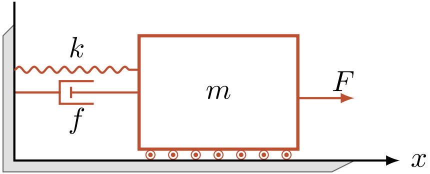
Second order systems

Roadmap

- interesting examples
- interesting responses
- differences between 1st order and 2nd order

Example 1: spring-mass systems

Position of a cart fastened with a spring to a wall and subject to friction:



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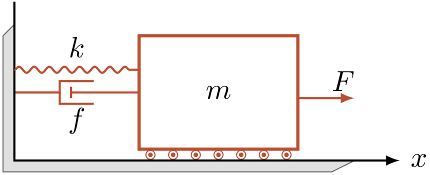

EOM:

- force from the spring: $F_x(t) = -kx(t)$
- friction: $F_f(t) = -f\dot{x}(t)$
- applied force: $F(t)$
- Newton's second law: $\sum F = m\ddot{x}(t)$

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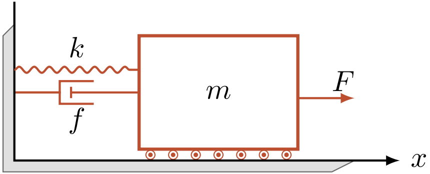

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  $$m\ddot{x}(t) = F_x(t) + F_f(t) + F(t)$$

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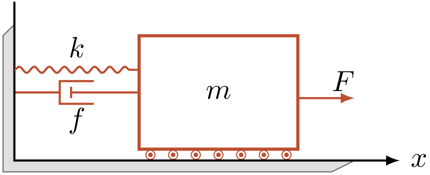

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$$m\ddot{x}(t) = F_x(t) + F_f(t) + F(t) \qquad \mapsto \qquad m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + F(t)$$

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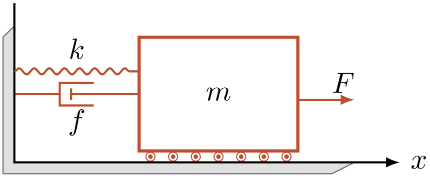

$$\ddot{x}(t) + \frac{f}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{1}{m}F(t) \tag{5}$$

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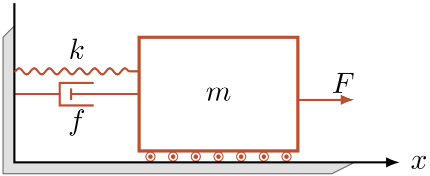


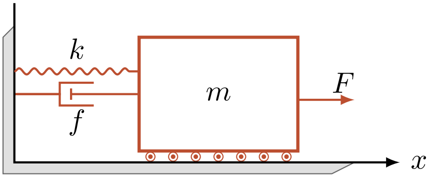
$$\ddot{x}(t) + \frac{f}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{1}{m}F(t) \tag{5}$$

Generalizing:

$$\ddot{x} = a_1\dot{x} + a_0 x + b_0 u \tag{6}$$

Example 1: spring-mass systems

Position of a cart fastened with a spring to a wall and subject to friction:



$$\ddot{x}(t) + \frac{f}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{1}{m}F(t) \tag{5}$$

Generalizing:

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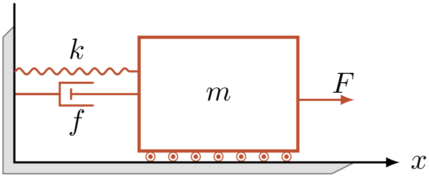

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Discussion: may we write this dynamics using a scalar first order DE?

Example 1: spring-mass systems

`https://youtu.be/lZPtFDXYQRU?t=31`

(here there is very little friction, though)

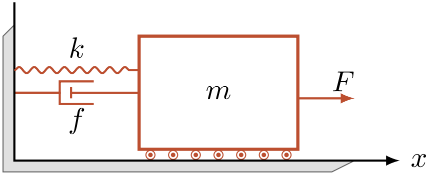
Example 1: spring-mass systems - stability analysis from intuitive perspectives

Discussion: $\dot{x} = a_0 x + b_0 u$ unstable if . . . ?

Example 1: spring-mass systems - stability analysis from intuitive perspectives

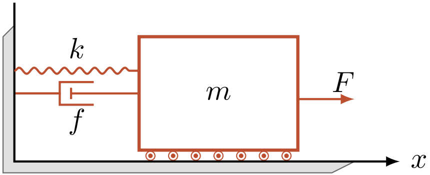
Discussion: $\dot{x} = a_0 x + b_0 u$ unstable if ...? And What about, from intuitive perspectives, when considering

$$\ddot{x} = a_1 \dot{x} + a_0 x + b_0 u\ ? \tag{7}$$



GitHub/TTK4225/trunk/Jupyter/spring-mass-systems-introduction.ipynb

Intuition: asymptotic stability $\leftrightarrow$ dissipation of energy



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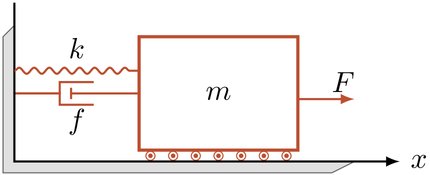
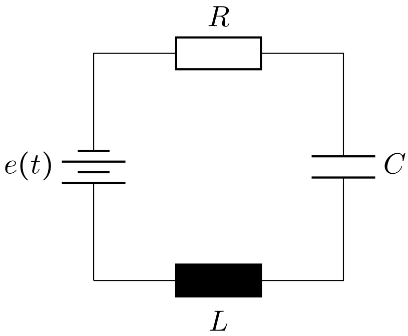


will see this MUCH better in later on courses
when discussing about Lyapunov stability

?

Example 2: RCL-circuit



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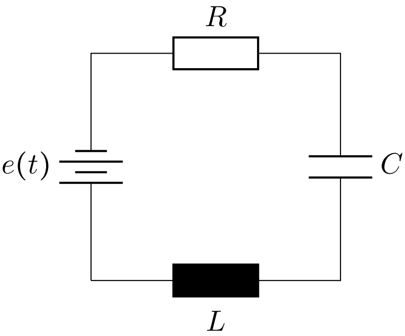

EOM: Kirkhoff laws

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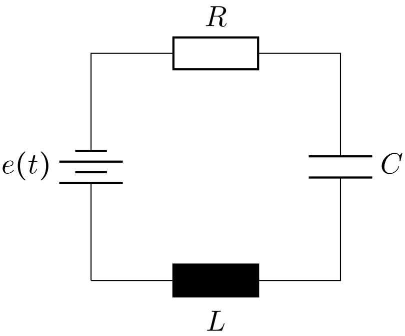

EOM: Kirkhoff laws

$$v_L(t) = L\frac{di(t)}{dt} \qquad v_i(t) = Ri(t) \qquad v_C(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

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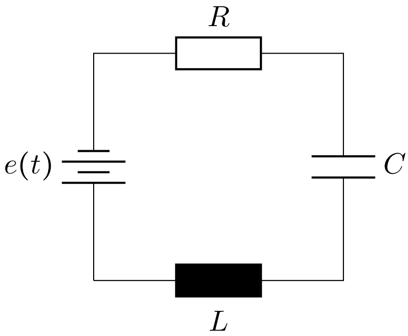

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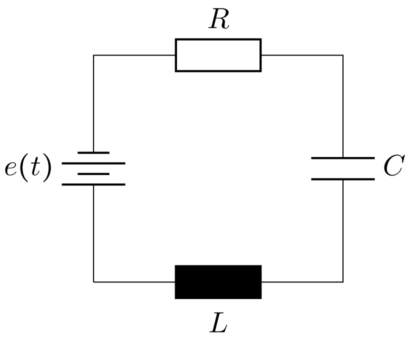

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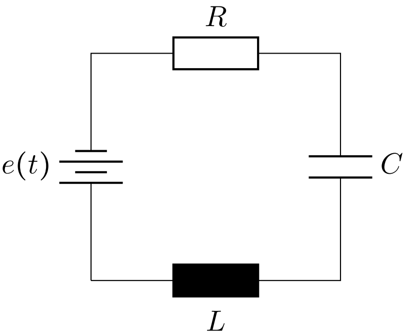

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Discussion: how do we write it in terms of x, $\dot{x}$, ...?

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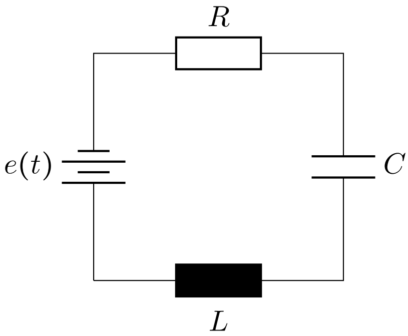

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$$x(t) = i(t), \quad u(t) = \dot{e}(t) \qquad \Longrightarrow \qquad \ddot{x} = -\frac{R}{L}\dot{x} - \frac{1}{LC}x + \frac{1}{L}u \tag{9}$$

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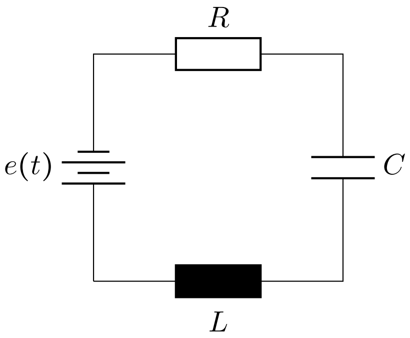
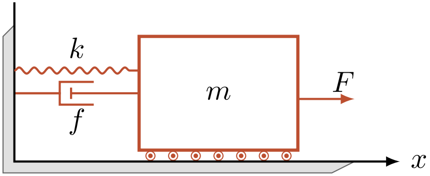


Generalizing:

$$\ddot{x} = -\frac{R}{L}\dot{x} - \frac{1}{LC}x + \frac{1}{L}u \quad \mapsto \quad \ddot{x} = a_1\dot{x} + a_0x + b_0u \tag{10}$$

?

Discussion: what are the potential different ways the cart may move back to its equilibrium?



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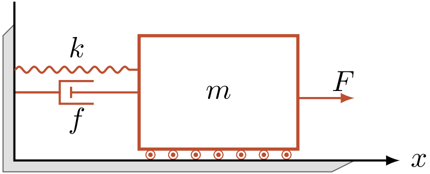
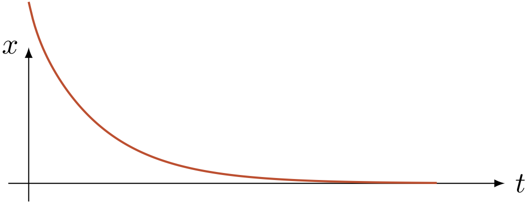


very big friction w.r.t. the spring stiffness $\implies$ overdamping:



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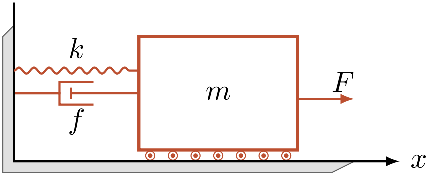

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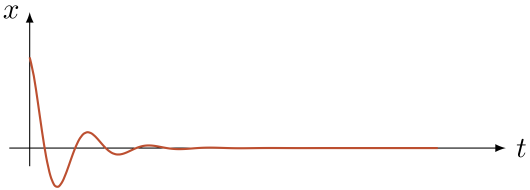

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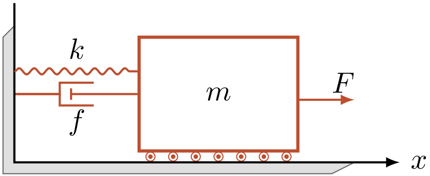
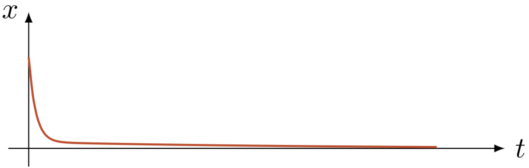


just the right amount of friction w.r.t. the spring stiffness $\Longrightarrow$ critical damping, i.e., *the combination that leads to the fastest decay to zero without oscillations*:



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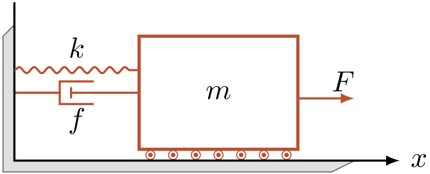
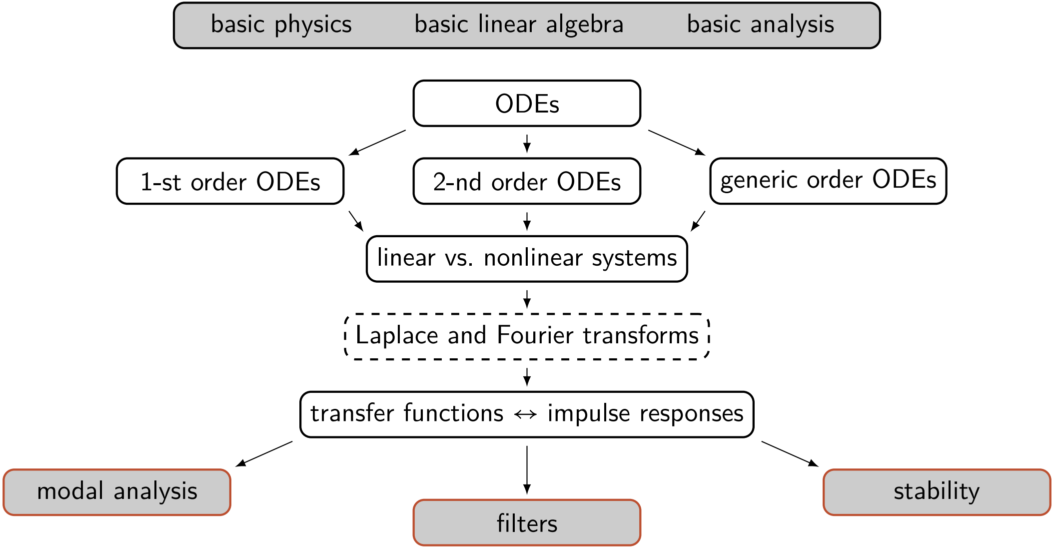


Discussion: can we get these behaviors with a first order system?

How do the free evolutions and forced responses look like numerically?

we will solve this problem with the aid of Laplace transforms

Summary / Roadmap of TTK4225



The essence of TTK4225

A number like 3 is not a particular triplet of things, is about all possible triplets of things. In the same way, a model is not really about a particular system, but is rather about all the possible real-life and physical systems that behave as that model

?

Introduction to Laplace transforms

Laplace who?

Figure: Pierre-Simon Laplace, Beaumont-en-Auge, 23 March 1749 - Paris, 5 March 1827

Roadmap

- why?
- what?
- how, at least intuitively?
- an essential example

Why?

Remember: predictive control of the type

$$\boldsymbol{u}^{\star} = \arg \min_{\boldsymbol{u} \in \mathcal{U}, \boldsymbol{f}(\boldsymbol{u}) \in \mathcal{F}} \mathrm{Cost}\left(\boldsymbol{f}\left(\boldsymbol{u}\right), \boldsymbol{u}\right)$$

requires to be able to:

- define "Cost"
- compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

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Laplace, for how we see it in this course,
helps the "compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly" part

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what will be $y(t)$ starting from y_0 and for a given $u(t)$?

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alternative strategy: solve $g * u(t)$ using Laplace transforms!

Laplace transforms - another pillar of automatic control

in TTK4225: $\mathcal{L}$ used to compute $y(t)$

in TTK4230: $\mathcal{L}$ used to determine $u(t)$

in virtually all the other courses at ITK: $\mathcal{L}$ used for both purposes and more

Laplace transform - definition

Hypothesis: $f(t)$ is a real function that is locally integrable[1] on $[0, +\infty)$ or $(-\infty, +\infty)$

[1] I.e., so that its integral is finite on every compact subset of its domain of definition

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unilateral Laplace transform:

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{+\infty} e^{-st} f(t) dt \qquad \text{for } t \geq 0, s = \sigma + i\omega \in \mathbb{C} \tag{12}$$

for all the $s \in \mathbb{C}$ for which the integral in the RHS converges *(i.e., the ROC)*

[1] I.e., so that its integral is finite on every compact subset of its domain of definition

Laplace transform - definition (2)

Hypothesis: $f(t)$ is a real function that is locally integrable[2] on $[0, +\infty)$ or $(-\infty, +\infty)$

bilateral Laplace transform:

$$F(s) = \mathcal{B}\left\{f(t)\right\} := \int_{-\infty}^{+\infty} e^{-st} f(t) dt \qquad \text{for } t \geq 0, s = \sigma + i\omega \in \mathbb{C} \tag{13}$$

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Laplace transform - ancillary definitions

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$$\Longrightarrow \mathcal{L} : \text{time} \mapsto \text{frequencies}$$

$$\text{we need then } \mathcal{L}^{-1} : \text{frequencies} \mapsto \text{time}$$

Inverse Laplace transform

Hypothesis:

- $F(s)$ complex function

Thesis: there may be a set of piecewise-continuous, exponentially-restricted real functions $f(t)$ which have the property that

$$\mathcal{L}\{f(t)\} = F(s);$$

all the $f(t)$ satisfying this equivalence differ from each other only on a subset having Lebesgue measure zero

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Discussion: is the existence of at least one of such $f(t)$ guaranteed?
Discussion: given any $f(t)$, is the existence of $F(s)$ guaranteed?

Main usefulness: convolution in time transforms into multiplication in Laplace-domain, and viceversa

$$\begin{cases} H(s) = \mathcal{L}\{h(t)\} \\ U(s) = \mathcal{L}\{u(t)\} \end{cases} \implies \mathcal{L}\{h * u(t)\} = H(s)U(s) \tag{15}$$

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$$

Discussion: how then may $\mathcal{L}$ help computing $h * u(t)$?

An intuitive explanation of the usefulness of the Laplace transform in automatic control

initial value problem

$$\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2}$$

An intuitive explanation of the usefulness of the Laplace transform in automatic control

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$$\Big\downarrow$$

$$\boxed{x(t) = 2e^{-t}}$$

solution in the time domain

An intuitive explanation of the usefulness of the Laplace transform in automatic control

initial value problem

algebraic problem

$$
\begin{tikzcd}
\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \\
\boxed{x(t) = 2e^{-t}} &
\end{tikzcd}
$$

solution in the time domain

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solution in the time domain

solution in the frequency domain

An intuitive explanation of the usefulness of the Laplace transform in automatic control

initial value problem | algebraic problem

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$$

solution in the time domain | solution in the frequency domain

?

next few slides:

introduce an utterly important example,

and with it show the “trick” mentioned before

The absolutely (and by far) most important Laplace transform

Discussion: which signal in the time domain is the most important signal you saw up to now?

The absolutely (and by far) most important Laplace transform

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$$\boxed{\dot{y} = ay + bu \qquad \mapsto \qquad y(t) = y_0 e^{at} + \int_0^t e^{a\tau} bu(t-\tau) d\tau}$$

$$e^{at}$$

$$\mathcal{L}\left\{e^{at}\right\} = ?$$

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$$\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \int_0^{+\infty} e^{at} e^{-st} dt \\
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\end{aligned}$$

Remember, though: $F(s)$ is defined for all the $s \in \mathbb{C}$ for which the integral in the RHS converges *(i.e., the ROC)*. Thus, *discussion:* for which s does $F(s)$ exist?

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$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \qquad \text{for } \mathrm{Re}(s) \geq a \tag{16}$$

At the same time ...

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{17}$$

At the same time . . .

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{17}$$

Our game, now: use our new knowledge, i.e., $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$ to compute the solution to $\dot{y} = ay$.

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An important property that we will prove later on

(see 4.2.1 in Balchen's book)

$$\mathcal{L}\left\{\dot{x}\right\} = sX(s) - x_0 \tag{18}$$

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$$\mathcal{L}\left\{\ddot{x}\right\} = s^2 X(s) - s x_0 - \dot{x}_0 \tag{19}$$

An important property that we will prove later on

(see 4.2.1 in Balchen's book)

$$
\begin{aligned}
\mathcal{L}\{\dot{x}\} &= sX(s) - x_0 && (18)\\
\mathcal{L}\{\ddot{x}\} &= s^2X(s) - sx_0 - \dot{x}_0 && (19)\\
\mathcal{L}\{\dddot{x}\} &= s^3X(s) - s^2x_0 - s\dot{x}_0 - \ddot{x}_0 && (20)\\
&\vdots && (21)
\end{aligned}
$$

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\mathcal{L}\left\{\dddot{x}\right\} &= s^3X(s) - s^2x_0 - s\dot{x}_0 - \ddot{x}_0 && (20)\\
&\vdots && (21)
\end{aligned}
$$

$s \quad \Longrightarrow \quad$ derivation; $\dfrac{1}{s} \quad \Longrightarrow \quad$ integration

"mnemonics": $s =$ 'derivasjon'

The "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

(using that we know that $\mathcal{L}\{\dot{y}\} = sY(s) - y_0$ and that $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$)

$$\dot{y} = ay$$

The "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

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$$\dot{y} = ay \qquad \rightarrow \qquad sY - y_0 = aY$$

The "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

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$$\dot{y} = ay \qquad \to \qquad sY - y_0 = aY \qquad \to \qquad Y = y_0 \frac{1}{s-a}$$

initial value problem — algebraic problem

$$
\begin{tikzcd}
\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\
\boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \arrow[l, "\mathcal{L}^{-1}"]
\end{tikzcd}
$$

solution in the time domain — solution in the frequency domain

?

What is a Laplace transform, actually?

Roadmap

- Laplace as a generalization of Fourier
- intuitions behind Fourier transforms
- generalizing sinusoids to exponentially decaying sinusoids

In the previous unit

$$
\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \int_0^{+\infty} e^{at}e^{-st}dt \\
&= \int_0^{+\infty} e^{(a-s)t}dt \\
&= \left[\frac{1}{a-s}e^{(a-s)t}\right]_0^{+\infty} \\
&= \frac{1}{a-s}\left[e^{(a-s)\infty} - e^{(a-s)0}\right] \\
&= \frac{1}{s-a}
\end{aligned}
$$

for $\mathrm{Re}\left(s\right) \geq a$

Main usefulness: convolution in time = multiplication in frequency, and this makes solving ODEs an algebraic problem

initial value problem — algebraic problem

$$
\begin{tikzcd}
\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\
\boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \arrow[l, "\mathcal{L}^{-1}"]
\end{tikzcd}
$$

solution in the time domain — solution in the frequency domain

But what is a Laplace transform, eventually?

$\rightarrow$ an opportune extension of the Fourier transform

But what is a Fourier transform, eventually?

Definition:

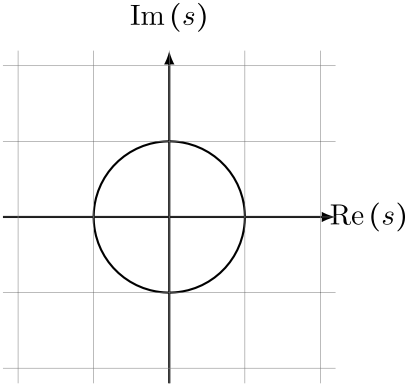
$$\mathcal{F}\left\{f(t)\right\} = \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \tag{22}$$

But what is a Fourier transform, eventually?

Definition:

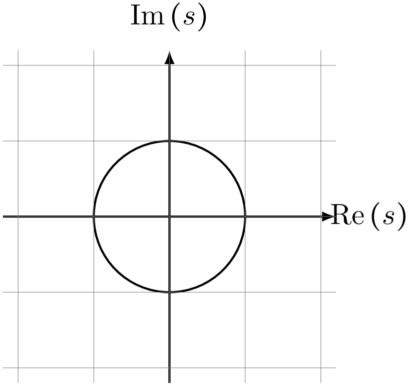
$$\mathcal{F}\{f(t)\} = \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \tag{22}$$

Discussion: where is $e^{-j\omega t}$?



Small recap: angular frequency ≠ ordinary frequency

$$e^{j\omega t} = e^{j2\pi f t}$$



Fourier transforms using Euler formula

$$\begin{aligned}\mathcal{F}\left\{f(t)\right\}(\omega) &= \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t)e^{j\omega t}dt \\ &= \int_{-\infty}^{+\infty} f(t)\cos\left(\omega t\right)dt \quad - \quad j\int_{-\infty}^{+\infty} f(t)\sin\left(\omega t\right)dt\end{aligned} \tag{23}$$

Fourier transforms using Euler formula

$$
\begin{aligned}
\mathcal{F}\left\{f(t)\right\}(\omega) &= \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t) e^{j\omega t} dt \\
&= \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\, dt \quad - \quad j\int_{-\infty}^{+\infty} f(t)\sin(\omega t)\, dt
\end{aligned}
\tag{23}
$$

intuition: for each ω, $\widehat{f}(\omega)$ is a couple of numbers, conveniently put together as a complex number, so that:

- $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ = area of $f(t)\cos(\omega t)$
- $\mathrm{Im}\left(\widehat{f}(\omega)\right)$ = area of $f(t)\sin(\omega t)$

What is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for sinusoidal f's?

$$\star = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt = \int_{-\infty}^{+\infty} \Big(\cos(\alpha t)\Big)\cos(\omega t)\,dt$$

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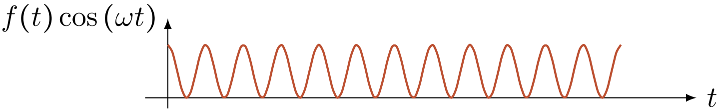
- if $f(t) = \cos(\omega t)$ then $\star = +\infty$



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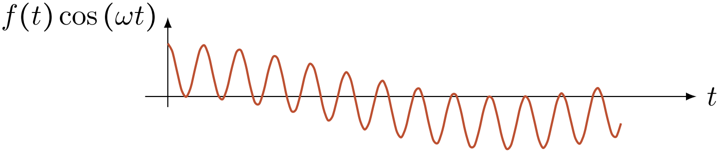
$$\star = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt = \int_{-\infty}^{+\infty} \Big(\cos(\alpha t)\Big)\cos(\omega t)\,dt$$

- if $f(t) = \cos(\omega t)$ then $\star = +\infty$
- if $f(t) = \cos(\alpha t)$ with $\alpha \neq \omega$ then $\star = 0$



What is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for composite sinusoidal f's?

$$\star = \int_{-\infty}^{+\infty} f(t) \cos(\omega t)\, dt = \int_{-\infty}^{+\infty} \left(\sum_k \cos(\alpha_k t) \right) \cos(\omega t)\, dt$$

What is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for composite sinusoidal f's?

$$\star = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt = \int_{-\infty}^{+\infty} \left(\sum_k \cos(\alpha_k t)\right)\cos(\omega t)\,dt$$

as before: if $\omega = \alpha_k$ then $\star = +\infty$, otherwise $\star = 0$!

What is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for composite sinusoidal f's?

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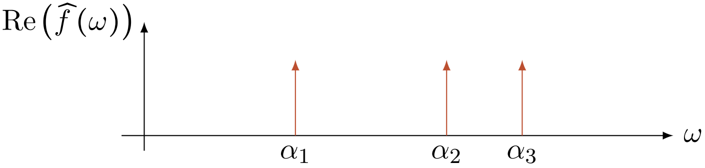
as before: if $\omega = \alpha_k$ then $\star = +\infty$, otherwise $\star = 0$! Thus varying ω makes $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ kind of "comb" for the right frequency:



And what about $\mathrm{Im}\left(\widehat{f}(\omega)\right)$?

$$\star = \int_{-\infty}^{+\infty} f(t) \sin(\omega t) \, dt$$

kind of similar, only scanning for oddity (while $\cos$ scans for even components)
$\rightarrow$ see the additional material linked later for more information

And what is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for generic non-periodic f's?

$$\star = \int_{-\infty}^{+\infty} f(t) \cos(\omega t) \, dt$$

And what is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for generic non-periodic f's?

$$\star = \int_{-\infty}^{+\infty} f(t) \cos(\omega t)\, dt$$

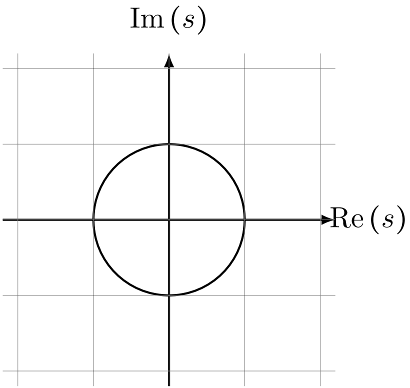
- potentially every ω may contribute to $\star$
- if there is no pure $\cos$ in f then each ω contributes with a finite area

So, what is a Fourier transform, eventually?

So, what is a Fourier transform, eventually?

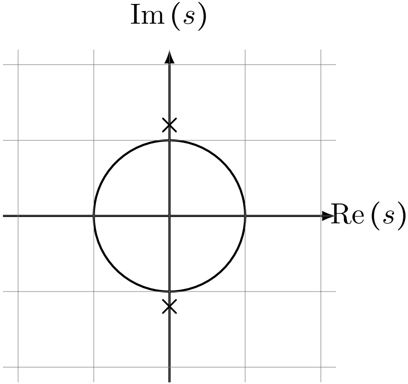
$=$ scanning the imaginary axis for sinusoidal components:

$$
\begin{aligned}
\widehat{f}(\omega) &= \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt \\
&\quad + j\int_{-\infty}^{+\infty} f(t)\sin(\omega t)\,dt
\end{aligned}
$$



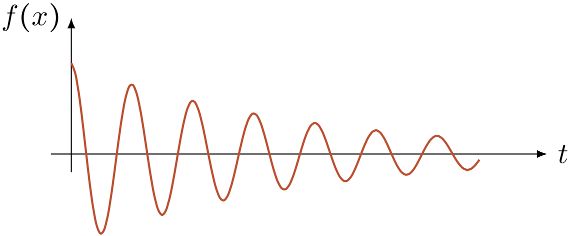
Summarizing, graphically

$$f(t) = \cos(\omega t) = \mathrm{Re}\left(e^{j\omega t}\right) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$



But what about exponentially decaying sinusoids?

$$f(t) = e^{at} \cos(\omega t)$$



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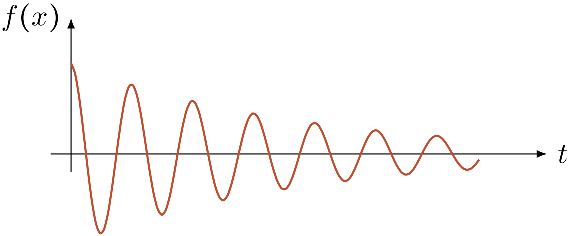


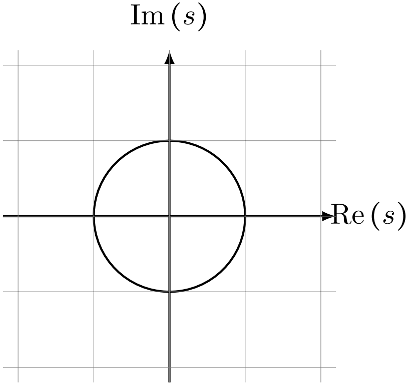
what if we consider $\overline{f}(t) \quad = \quad f(t)e^{-at}$?

Fourier and Laplace, side by side

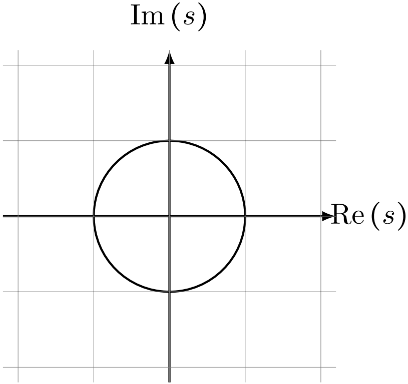
$$\mathcal{F}\{f(t)\} = \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$

$$\mathcal{L}\{f(t)\} := \int_{0}^{+\infty} f(t)e^{-st}dt$$

So, what is a Laplace transform, eventually?



So, what is a Laplace transform, eventually?



beware of the region of convergence!

?

Suggested additional material

Fourier transforms 1:

https://www.youtube.com/watch?v=spUNpyF58BY

Fourier transforms 2 (with some Laplace in it):

https://www.youtube.com/watch?v=3gjJDuCAEQQ

Laplace transforms:

https://www.youtube.com/watch?v=n2y7n6jw5d0

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Laplace = a pillar of automatic control

$\implies$ better to invest time in developing a full understanding of it

Using Laplace transforms with second order systems in free evolution

Recall: the "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

$$\dot{y} = ay \qquad \to \qquad sY - y_0 = aY \qquad \to \qquad Y = y_0 \frac{1}{s-a}$$

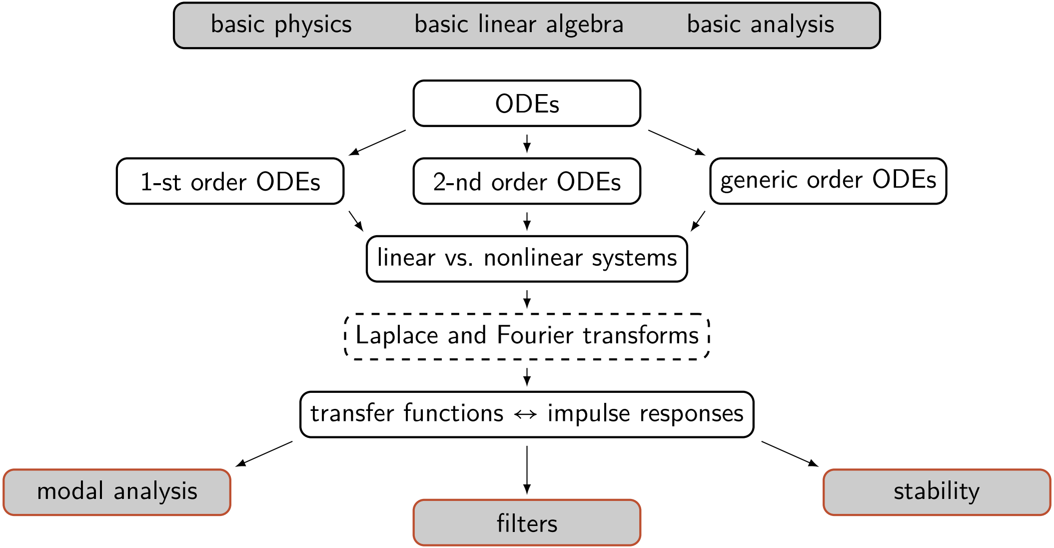
Recall: the "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

initial value problem   algebraic problem

$$\begin{tikzcd}
\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\
\boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \arrow[l, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

solution in the time domain   solution in the frequency domain

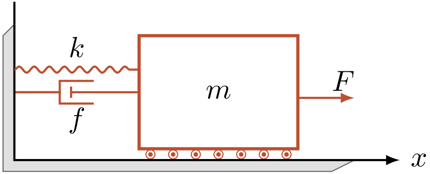
Summary / Roadmap of TTK4225

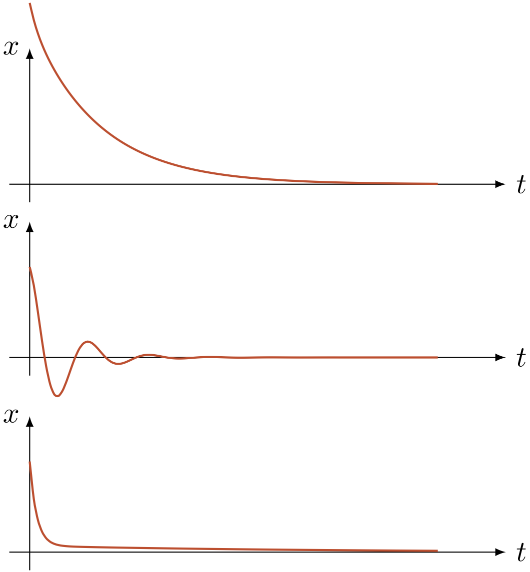


Roadmap

- the same thing, but for second order system
- different algebraic situations $\implies$ different damping properties
- different ways of summarizing the different damping properties

Discussion on a practical example:
how may a second order system free-evolve?




x
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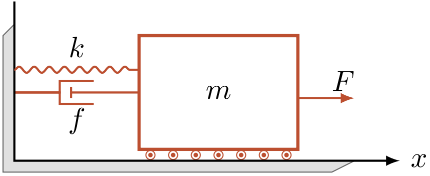
Dissecting the practical example “spring-mass systems”



$$m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + u(t)$$

The properties we shall use

$$\mathcal{L}\left\{\dot{x}\right\} = sX(s) - x_0 \tag{24}$$

$$\mathcal{L}\left\{\ddot{x}\right\} = s^2X(s) - sx_0 - \dot{x}_0 \tag{25}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{26}$$

$$\tag{27}$$

Solution for the free evolution case

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0$$

Solution for the free evolution case

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0 \qquad \Longrightarrow \qquad X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}}$$

Solution for the free evolution case

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0 \qquad \Longrightarrow \qquad X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}}$$

we would like to use $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ to inverse-transform

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we would like to use $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ to inverse-transform

$\Longrightarrow$ <u>we must try to factorize the RHS, i.e., try to write it as</u>

$$X(s) = \frac{A}{s-\alpha} + \frac{B}{s-\beta}$$

(if possible)

Solution for the free evolution case

Next step: factorize

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}}$$

Solution for the free evolution case

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$$X(s) = \frac{x_0 \left(\frac{f}{m} + s \right) + \dot{x}_0}{s^2 + \frac{f}{m} s + \frac{k}{m}}$$

$$s^2 + \frac{f}{m} s + \frac{k}{m} = 0 \qquad \Longrightarrow \qquad s = -\frac{f}{2m} \pm \frac{\sqrt{f^2 - 4mk}}{2m}$$

Solution for the free evolution case

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Discussion: what are the different possibilities?

Solution for the free evolution case

Next step: factorize

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Discussion: what are the different possibilities?

1. $f^2 - 4mk > 0 \implies$ two distinct real solutions *(will imply overdamping)*
2. $f^2 - 4mk = 0 \implies$ one double real solution *(will imply critical damping)*
3. $f^2 - 4mk < 0 \implies$ two conjugate complex solutions *(will imply underdamping)*

$f^2 - 4mk > 0$, i.e., overdamping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

$f^2 - 4mk > 0$, i.e., overdamping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

$$f^2 > 4mk \qquad \Longrightarrow \qquad \begin{cases} \lambda_1 = & -\dfrac{f}{2m} - \dfrac{\sqrt{f^2 - 4mk}}{2m} \in \mathbb{R} \\[2ex] \lambda_2 = & -\dfrac{f}{2m} + \dfrac{\sqrt{f^2 - 4mk}}{2m} \in \mathbb{R} \end{cases}$$

$f^2 - 4mk > 0$, i.e., overdamping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

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$$\Longrightarrow \qquad X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2}$$

Discussion: can we conclude immediately that

$$X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2} \quad \Longrightarrow \quad x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

or do we need to say some property about Laplace transforms that we did not say before?

We need to be sure that Laplace transforms are linear!

i.e., be sure that

$$\mathcal{L}\left\{\alpha x_1 + \beta x_2\right\} = \alpha\mathcal{L}\left\{x_1\right\} + \beta\mathcal{L}\left\{x_2\right\},$$

and that

$$\mathcal{L}^{-1}\left\{\alpha X_1 + \beta X_2\right\} = \alpha\mathcal{L}^{-1}\left\{X_1\right\} + \beta\mathcal{L}^{-1}\left\{X_2\right\}$$

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and that

$$\mathcal{L}^{-1}\left\{\alpha X_1 + \beta X_2\right\} = \alpha \mathcal{L}^{-1}\left\{X_1\right\} + \beta \mathcal{L}^{-1}\left\{X_2\right\}$$

Discussion: why do the previous properties hold? (hint: recall that $\mathcal{L}\left\{x(t)\right\} := \int_0^{+\infty} x(t) e^{-st} dt$)

In conclusion: overdamping in free evolution

$$f^2 > 4mk \quad \Longrightarrow \quad X(s) = \frac{A}{s-\lambda_1} + \frac{B}{s-\lambda_2} \quad \Longrightarrow \quad x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

for opportune A and B.

In conclusion: overdamping in free evolution

$$f^2 > 4mk \quad \Longrightarrow \quad X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2} \quad \Longrightarrow \quad x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

for opportune A and B. *Discussion:* what determines A and B? Hint: recall that

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

important message (will be seen again later on): the denominator of the rational Laplace transform determines the behavior of the system, while the numerator determines which root is dominant and the "amplitude" of each mode

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

Practical example of a overdamped system in free evolution

$$\begin{cases} m & = 10 & \text{kg} \\ f & = 100 & \text{kg/s} \\ d & = 90 & \text{N/s}^2 \end{cases} \implies \begin{cases} \ddot{x} + \dfrac{100}{10}\dot{x} + \dfrac{90}{10}x = 0 \\ \\ \\ \end{cases}$$

Practical example of a overdamped system in free evolution

$$\begin{cases} m & = 10 & \text{kg} \\ f & = 100 & \text{kg/s} \\ d & = 90 & \text{N/s}^2 \end{cases} \implies \begin{cases} \ddot{x} + \frac{100}{10}\dot{x} + \frac{90}{10}x = 0 \\ x(0) = 0.16\text{m} \\ \dot{x}(0) = 0\text{m/s} \end{cases} \implies X(s) = \frac{0.16\,(s+10)}{s^2 + 10s + 9}$$

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`ARMA-continuous-time-example.ipynb`

?

$f^2 - 4mk = 0$, i.e., critical damping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s-\lambda)^2}$$

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Very interesting behavior:

$$x(t) = Ae^{\lambda t} + Bte^{\lambda t}$$

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Very interesting behavior:

$$x(t) = Ae^{\lambda t} + Bte^{\lambda t}$$

Discussion: what contributes to determine A and B?

Discussion: recall the previous "overdamping case", i.e.,

$$\begin{cases} X(s) & = \dfrac{x_0\left(\frac{f}{m}+s\right)+\dot{x}_0}{s^2+\frac{f}{m}s+\frac{k}{m}} & \\ m & = 10 & \text{kg} \\ f & = 100 & \text{kg/s} \\ d & = 90 & \text{N/s}^2 \\ x(0) & = 0.16\text{m} & \\ \dot{x}(0) & = 0\text{m/s} & \end{cases} \Longrightarrow \quad X(s) = \frac{0.16\,(s+10)}{s^2+10s+9}\,.$$

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`ARMA-continuous-time-example.ipynb`

?

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$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

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with λ_1 and λ_2 a complex-conjugate pair

Discussion: did we learn until now in this course how to inverse-Laplace these things?

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"yes, but the corresponding exponentials come out to be complex numbers! How is it possible that a real system produces complex results?"

Solution:

$$\mathcal{L}\left\{e^{at}\sin\omega t\right\} = \frac{\omega}{(s-a)^2 + \omega^2}$$

Discussion: recall the previous "overdamping case", i.e.,

$$\begin{cases} X(s) & = \dfrac{x_0\left(\frac{f}{m}+s\right)+\dot{x}_0}{s^2+\frac{f}{m}s+\frac{k}{m}} \\ m & = 10 & \mathrm{kg} \\ f & = 100 & \mathrm{kg/s} \\ d & = 90 & \mathrm{N/s^2} \\ x(0) & = 0.16\mathrm{m} \\ \dot{x}(0) & = 0\mathrm{m/s} \end{cases} \implies X(s) = \frac{0.16\,(s+10)}{s^2+10s+9}.$$

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`ARMA-continuous-time-example.ipynb`

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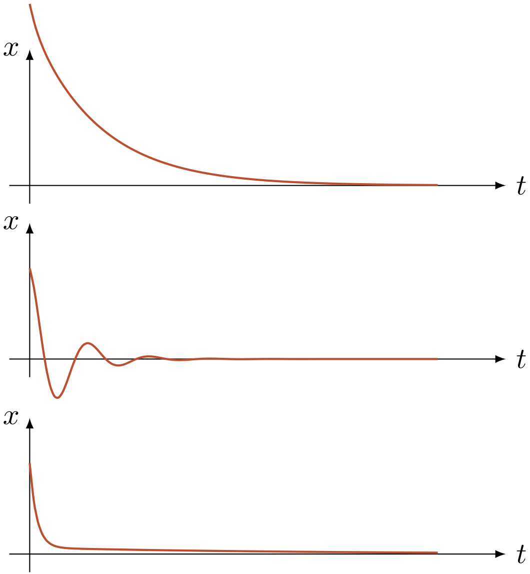
Recap of the possible damping modalities



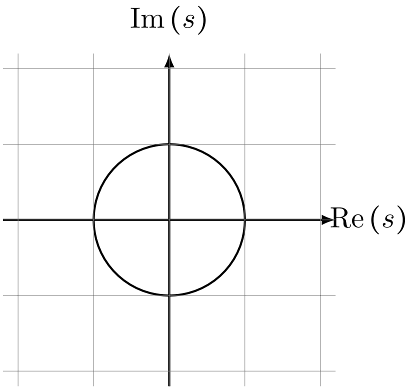
Recap of the corresponding time-frequency pairs

overdamped: $\mathcal{L}\left\{Ae^{\lambda_1 t} + Bte^{\lambda_2 t}\right\} = \dfrac{A}{s-\lambda_1} + \dfrac{B}{s-\lambda_2}, \quad \lambda_1 \neq \lambda_2$

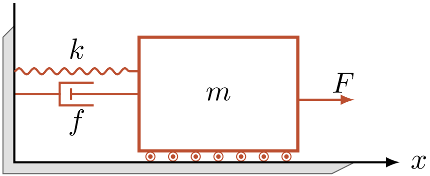
critically damped: $\mathcal{L}\left\{Ae^{\lambda t} + Bte^{\lambda t}\right\} = \dfrac{A}{s-\lambda} + \dfrac{B}{(s-\lambda)^2}$

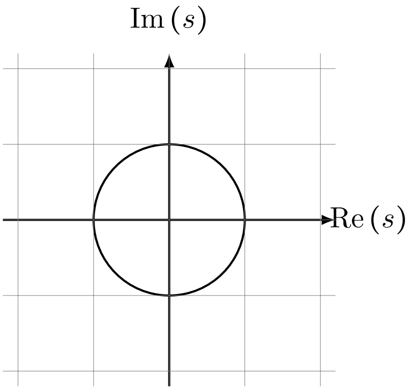
underdamped: $\mathcal{L}\left\{e^{at}\sin\omega t\right\} = \dfrac{\omega}{(s-a)^2+\omega^2}$

Recap using the "poles-zeros plot"



Do you see it?





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Using Laplace transforms with generic n order systems in free evolution

In the last episodes: the "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$, and to do something similar for second order systems in free evolution

$$\dot{y} = ay \qquad \rightarrow \qquad sY - y_0 = aY \qquad \rightarrow \qquad Y = y_0 \frac{1}{s - a}$$

Roadmap

- the same thing, but for generic systems

A small recap

- first oder homogeneous systems:

$$\dot{x} - ax = 0 \quad \mapsto \quad x(t) = x_0 e^{at} \tag{28}$$

- first oder non-homogeneous systems:

$$\dot{x} - ax = bu \quad \mapsto \quad x(t) = x(0)e^{at} + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \tag{29}$$

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The problem: the convolution integral gives us headaches, and we want to use Laplace to ease things up. To understand how, step-by-step path:

1. start from $\dot{x} - ax = 0$
2. move to $\ddot{x} - a_1\dot{x} - a_2 x = 0$
3. move to higher order
4. add the "u" part

Laplace-properties we saw up to now:

(4.2.1 in Balchen's book)

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Linearity

$$\mathcal{L}\left\{af(t) + bg(t)\right\} = a\mathcal{L}\left\{f(t)\right\} + b\mathcal{L}\left\{g(t)\right\} \tag{30}$$

Laplace-properties we saw up to now:

(4.2.1 in Balchen's book)

Derivatives

$$\mathcal{L}\{\dot{x}\} = sX(s) - x(0) \tag{31}$$

$$\mathcal{L}\{\ddot{x}\} = s^2X(s) - sx(0) - \dot{x}(0) \tag{32}$$

$$\vdots \tag{33}$$

Laplace-properties we saw up to now:

(4.2.1 in Balchen's book)

Integration

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s) \tag{34}$$

Laplace-transforms we needed up to now:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
e^{at}	$\dfrac{1}{s-a}$
$e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
te^{at}	$\dfrac{1}{(s-a)^2}$

A more detailed roadmap for this unit:

- make Laplace-transform more matematically ground
- derive some few very important transforms
- generalize how to do partial expansions
- solve the problem: how to use Laplace transforms with generic n order systems in free evolution

?

Laplace transforms, take 2

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{+\infty} e^{-st} f(t) dt \tag{35}$$

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Discussion: assume $f(t)$ to be limited (e.g., $f(t) = 1$). Will $\int_0^{+\infty} e^{-st} f(t) dt$ converge for every s?

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Discussion: assume $f(t)$ to be limited (e.g., $f(t) = 1$). Will $\int_0^{+\infty} e^{-st} f(t) dt$ converge for every s? *no – only for* $s > 0$*!*

Important (but intuitive) teorem

If $f(t)$ is defined and piecewise continuous for $t > 0$,
and there exist two constants M and k so that $|f(t)| \leq Me^{kt}$ for every $t > 0$,
then $\mathcal{L}\{f(t)\}$ is well defined for every $s > k$

Important (but intuitive) property: linearity

Assuming that $f(t)$ and $g(t)$ are so that $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exists, then $\mathcal{L}\{af(t)+bg(t)\}$ exists and is s.t.

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

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Proof

Integrals are linear, thus

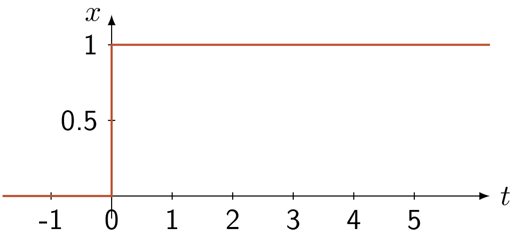
$$\begin{aligned}
\mathcal{L}\{af(t) + bg(t)\} &= \int_0^{+\infty} e^{-st}\left[af(t) + bg(t)\right] dt \\
&= a\int_0^{+\infty} e^{-st} f(t)dt + b\int_0^{+\infty} e^{-st} g(t)dt \\
&= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}
\end{aligned}$$

?

Important example: Laplace-transforming the Heaviside-function

(right now on the "deriving some few very important transforms" of the unit)

$$H(t) := \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$



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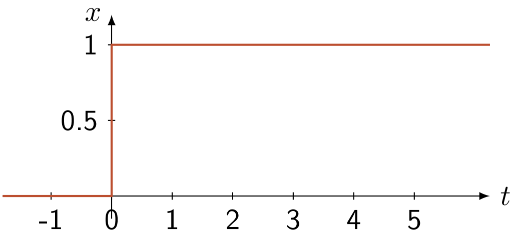


Discussion: is $H(0)$ defined?

Deriving the Heaviside function from the Dirac's impulse

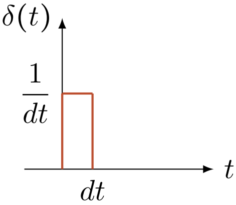
(right now on the "deriving some few very important transforms" of the unit)

$$H(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

with, *very loosely,*

$$\delta(t) := \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{otherwise,} \end{cases}$$

or, better, as the limit of concentrating an unitary point mass at 0:



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&= \left[-\frac{1}{s} e^{-st}\right]_{0}^{+\infty}
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Discussion: what is the ROC?

Back to the important example: Laplace-transforming the Heaviside-function

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\end{aligned}$$

Discussion: what is the ROC? $\mathrm{Re}\left(s\right) > 0$

Another important example: Laplace-transforming the Dirac's delta

(right now on the "deriving some few very important transforms" of the unit)

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{0}^{+\infty} e^{-st}\delta(t)dt$$

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Discussion: what is the ROC? $\forall s$

Likely the most important example: Laplace-transforming an exponential

(right now on the "deriving some few very important transforms" of the unit)

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^{+\infty} e^{at} e^{-st} dt \qquad (36)$$

Likely the most important example: Laplace-transforming an exponential

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On with the important examples: Laplace-transforming polynomials

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$$\int_0^{+\infty} u\dot{v}dt = [uv]_0^{+\infty} - \int_0^{+\infty} \dot{u}vdt \tag{41}$$

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$$= \frac{1}{s^2} \tag{45}$$

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$$\mathcal{L}\left\{t^2\right\} = \int_0^{+\infty} t^2 e^{-st} dt \qquad \begin{aligned} u &= t^2 & v &= e^{-st} \\ \dot{u} &= t & \dot{v} &= -se^{-st} \end{aligned} \tag{46}$$

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$$= \frac{2}{s^3} \tag{49}$$

$$\tag{50}$$

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$$\mathcal{L}\left\{t^n\right\} = \frac{(n-1)!}{s^n} \tag{51}$$

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$$\mathcal{L}\left\{\dot{x}\right\} = \int_0^{+\infty} e^{-st}\dot{x}dt \tag{52}$$

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$$\mathcal{L}\left\{\dot{x}\right\} = \int_0^{+\infty} e^{-st}\dot{x}dt \tag{52}$$

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$$= \left[e^{-st} x(t)\right]_0^{+\infty} - \int_0^{+\infty} \left(-s e^{-st}\right) x(t) dt \tag{53}$$

$$= 0 - x(0) + s \int_0^{+\infty} e^{-st} x(t) dt \tag{54}$$

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\begin{aligned}
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&= 0 - x(0) + s\int_0^{+\infty} e^{-st}x(t)dt && (54)\\
&= sX(s) - x(0) && (55)
\end{aligned}
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$$\mathcal{L}\left\{\ddot{x}\right\} = \int_{0}^{+\infty} e^{-st}\ddot{x}dt \tag{56}$$

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$$= s^2X(s) - sx(0) - \dot{x}(0) \tag{60}$$

$$\tag{61}$$

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$$\mathcal{L}\left\{x^{(n)}(t)\right\} = s^n\mathcal{L}\left\{x(t)\right\} - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - \ldots - x^{(n-1)}(0) \tag{62}$$

?

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The potential points of view of this course:

simply a complex number, i.e., $s = \sigma + j\omega$, that gives rise to a *complex frequency* through the complex exponential $e^{-st} = e^{-\sigma t} e^{-j\omega t}$

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Be careful on how to use s in this course!

Typically, in all the automatic control courses the interest is not too much on the value of $F(s)$ for different s, but rather on the fact that $F(s)$ enables performing algebraic operations to solve ODEs!

Indeed...

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\Downarrow$$

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} \Downarrow$$

$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ (and the other cases)

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$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \text{ (and the other cases)}$$

Discussion: why was it that $\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0$ leads to $X(s) \neq 0$?

what about $\dddot{x} + \alpha_2 \ddot{x} + \alpha_1 \dot{x} + \alpha_0 x = 0$?

what about $\dddot{x} + \alpha_2 \ddot{x} + \alpha_1 \dot{x} + \alpha_0 x = 0$?

and what about higher orders?

Partial fraction decomposition

Partial fraction decomposition

case single poles: if $\dfrac{N(s)}{(s-a)(s-b)(s-c)\cdots}$ is s.t. $a \neq b \neq c \neq \cdots$ then there exist A, B, C, ... s.t.

$$\frac{N(s)}{(s-a)(s-b)(s-c)\cdots} = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots \tag{63}$$

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case repeated poles: if some poles are repeated, then there exist $A_1, \ldots, A_{na}, B_1, \ldots, B_{nb}$, s.t.

$$\frac{N(s)}{(s-a)^{na}(s-b)^{nb}\ldots} = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots \tag{64}$$

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"But how do I compute A, B, etc.?" $\mapsto$
`en.wikipedia.org/wiki/Partial_fraction_decomposition`
(tip: start from `en.wikipedia.org/wiki/Heaviside_cover-up_method`)

Computing the free evolution for generic n order LTI systems

$$\text{start from } x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0 \tag{65}$$

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$$\text{obtain } X(s) = \frac{N(s)}{(s-a)(s-b)(s-c)\cdots} \tag{67}$$

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$$\text{obtain } X(s) = \frac{N(s)}{(s-a)(s-b)(s-c)\cdots} \tag{67}$$

$$\text{do a partial fraction decomposition} \tag{68}$$

Computing the free evolution for generic n order LTI systems

Two main cases:

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$\text{either } X(s) = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots$$

$$\text{or } X(s) = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots$$

Computing the free evolution for generic n order LTI systems

Case single poles

$$x^{(n)} + \alpha_{n-1}x^{(n-1)} + \ldots + \alpha_0 x = 0$$

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Discussion: given $\mathcal{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = e^{\alpha t}$, what is $x(t)$?

Computing the free evolution for generic n order LTI systems

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Discussion: what was $\mathcal{L}^{-1}\left\{\frac{1}{(s-\alpha)^2}\right\}$?

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$$\text{generalization: } \mathcal{L}\left\{\frac{n!}{(s-\alpha)^{n+1}}\right\} = t^n e^{\alpha t}$$

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$$\text{generalization: } \mathcal{L}\left\{\frac{n!}{(s-\alpha)^{n+1}}\right\} = t^n e^{\alpha t}$$

Discussion: given the previous inverse transform, what is $x(t)$?

Extremely important result

a LTI in free evolution behaves as a combination of terms $e^{\alpha t}$, $te^{\alpha t}$, $t^2 e^{\alpha t}$, etc. for a set of different αs and powers of t, called the *modes* of the system

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a LTI in free evolution behaves as a combination
of terms $e^{\alpha t}$, $te^{\alpha t}$, $t^2 e^{\alpha t}$, etc. for a set of different
αs and powers of t, called the *modes* of the system

Discussion: assuming that we have two modes, $e^{-0.3t}$ and $e^{-1.6t}$, so that

$$x(t) = \gamma_1 e^{-0.3t} + \gamma_2 e^{-1.6t}.$$

What determines γ_1 and γ_2?

?