TTK4225 - Systems Theory, Autumn 2020

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Linearizing nonlinear systems

Roadmap

- recalling the definition of state-space systems
- Taylor approximations, what are they?
- how to linearize a continuous time system
- examples

State space representations - Definition

<u>mathematical model (typically but not limited to of a physical system) as a finite set of inputs, outputs and state variables related by first-order differential equations satisfying the separation principle</u>

Ingredients:

- finite number of inputs, outputs and state variables
- first-order differential equations
- *satisfies the separation principle:* the current value of the state contains all the information necessary to forecast the future evolution of the outputs and of the state. I.e., to compute future $y(t+\tau)$ and $x(t+\tau)$ it is enough to know the current $x(t)$ and the current and future inputs $u(t:t+\tau)$

Example

Rechargeable flashlight:

- state = level of charge of the battery & on / off button
- output = how much light the device is producing

"the current value of the state contains all the information necessary to forecast the future evolution of the outputs and of the state. I.e., to compute future $y(t+\tau)$ and $x(t+\tau)$ it is enough to know the current $x(t)$ and the current and future inputs $u(t:t+\tau)$"

State space representations - Notation

$u_1, \ldots, u_m$ = inputs

$x_1, \ldots, x_n$ = states

$y_1, \ldots, y_p$ = outputs

State space representations - Notation

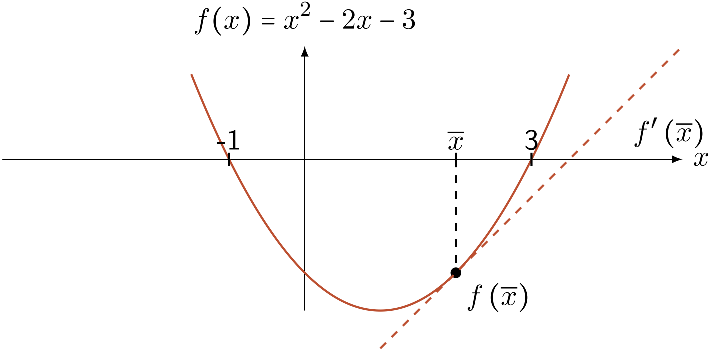
$$
\begin{aligned}
\dot{x}_1 &= f_1\left(x_1, \ldots, x_n,\ u_1, \ldots, u_m \right) \\
&\vdots \\
\dot{x}_n &= f_n\left(x_1, \ldots, x_n,\ u_1, \ldots, u_m \right) \\
y_1 &= g_1\left(x_1, \ldots, x_n,\ u_1, \ldots, u_m \right) \\
&\vdots \\
y_p &= g_p\left(x_1, \ldots, x_n,\ u_1, \ldots, u_m \right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\
\boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)
\end{aligned}
$$

- $\boldsymbol{f}$ = state transition map
- $\boldsymbol{g}$ = output map

?

Linearization - what does it mean?



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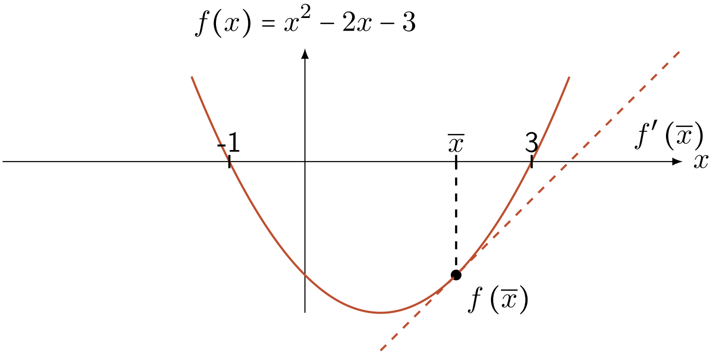

$$f(x) \approx f(\overline{x}) + \left.\frac{\partial f}{\partial x}\right|_{\overline{x}} (x - \overline{x})$$

(but the approximation is valid only close to the linearization point)

Linearization - what does it mean?

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned} \quad \mapsto \quad \begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} + D\boldsymbol{u} \end{aligned}$$

linearize $\Longrightarrow$ *approximate!*

Discussion: why do we linearize nonlinear systems?

Discussion: where do we linearize nonlinear systems?

Preliminaries: Taylor series

$$f \in C^M(\mathbb{R}) \qquad \Longrightarrow \qquad f(x) \approx \sum_{m=0}^{M} \frac{f^{(m)}(x_0)}{m!} (x - x_0)^m$$

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multivariable extension = less neat formulas, but we will see them!

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the most important case for our purposes:

$$\boldsymbol{f} \in C^1(\mathbb{R}^n, \mathbb{R}^m) \qquad \Longrightarrow \qquad \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \approx \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0) + \nabla_{\boldsymbol{x}} \boldsymbol{f}\big|_{\boldsymbol{x}_0}(\boldsymbol{x} - \boldsymbol{x}_0) + \nabla_{\boldsymbol{u}} \boldsymbol{f}\big|_{\boldsymbol{u}_0}(\boldsymbol{u} - \boldsymbol{u}_0)$$

Discussion (yes, again): where do we linearize nonlinear systems?

Linearization procedure - continuous time systems

$$(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) \text{ equilibrium} \quad \Longrightarrow \boldsymbol{f}\left(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) = 0$$

Linearization procedure - continuous time systems

$$(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) \text{ equilibrium} \quad \Longrightarrow \quad \boldsymbol{f}(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) = 0$$

Procedure (assuming that the Taylor expansion exists):

1. consider $\boldsymbol{x} = \boldsymbol{x}_{\text{eq}} + \Delta \boldsymbol{x}$, and $\boldsymbol{u} = \boldsymbol{u}_{\text{eq}} + \Delta \boldsymbol{u}$
2. apply

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \approx \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0) + \nabla_{\boldsymbol{x}} \boldsymbol{f}\big|_{\boldsymbol{x}_0} (\boldsymbol{x} - \boldsymbol{x}_0) + \nabla_{\boldsymbol{u}} \boldsymbol{f}\big|_{\boldsymbol{u}_0} (\boldsymbol{u} - \boldsymbol{u}_0) = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0) + \nabla \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}}$$

to the point $\boldsymbol{x}_0 = \boldsymbol{x}_{\text{eq}}$

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to the point $\boldsymbol{x}_0 = \boldsymbol{x}_{\text{eq}}$

$$\implies \qquad \frac{\partial(\boldsymbol{x}_{\text{eq}} + \Delta\boldsymbol{x})}{\partial t} \approx \boldsymbol{f}(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) + \nabla \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{u} \end{bmatrix}$$

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- $\dfrac{\partial(\boldsymbol{x}_{\text{eq}} + \Delta\boldsymbol{x})}{\partial t} = \Delta\dot{\boldsymbol{x}}$

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- $\dfrac{\partial (\boldsymbol{x}_{\text{eq}} + \Delta \boldsymbol{x})}{\partial t} = \Delta \dot{\boldsymbol{x}}$
- $\boldsymbol{f}(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) = \boldsymbol{0}$

Linearization procedure - continuous time systems

$$\left(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) \ \text{equilibrium} \quad \Longrightarrow$$

$$\Delta \dot{\boldsymbol{x}} \approx \nabla_{\boldsymbol{x}} \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta \boldsymbol{x} + \nabla_{\boldsymbol{u}} \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta \boldsymbol{u}$$

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$$\begin{cases} \dot{\boldsymbol{x}} & = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\ \boldsymbol{y} & = \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right) \end{cases}$$

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$$\downarrow$$

$$\begin{cases} \Delta\dot{\boldsymbol{x}} &= A\Delta\boldsymbol{x} + B\Delta\boldsymbol{u} \\ \Delta\boldsymbol{y} &= C\Delta\boldsymbol{x} + D\Delta\boldsymbol{u} \end{cases}$$

Summary

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u})$$

1. choose an opportune point $\boldsymbol{x}_0, \boldsymbol{u}_0$
2. linearize around $\boldsymbol{x}_0, \boldsymbol{u}_0$:

$$\dot{\boldsymbol{x}}_0 + \Delta\dot{\boldsymbol{x}} \approx \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0) + \nabla_{\boldsymbol{x}} \boldsymbol{f}\big|_{\boldsymbol{x}_0, \boldsymbol{u}_0} \Delta\boldsymbol{x} + \nabla_{\boldsymbol{u}} \boldsymbol{f}\big|_{\boldsymbol{x}_0, \boldsymbol{u}_0} \Delta\boldsymbol{u}$$

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important: if $\boldsymbol{x}_0, \boldsymbol{u}_0$ = *equilibrium then* $\dot{\boldsymbol{x}}_0 = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0) = 0$

Linearization - Example

electrostatic microphone:

- q = capacitor charge
- h = distance of armature from its natural equilibrium
- $\boldsymbol{x} = \left[q, h, \dot{h}\right]$

- R = circuit resistance
- E = voltage generated by the generator (constant)
- C = capacity of the capacitor
- m = mass of the diaphragm $+$ moved air
- k = mechanical spring coefficient
- β = mechanical dumping coefficient
- u_1 = incoming acoustic signal

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$$
\begin{cases}
\dot{x}_1 & = -\dfrac{1}{Ra} x_1 \left(L + x_2\right) + \dfrac{E}{R} \\
\dot{x}_2 & = x_3 \\
\dot{x}_3 & = -\dfrac{\beta}{m} x_3 - \dfrac{k}{m} x_2 - \dfrac{x_1^2}{2am} + \dfrac{1}{m} u_1
\end{cases}
$$

Linearization - Example

1-st step: compute the equilibria

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2-nd step: compute the matrices

$$A = \nabla_{\boldsymbol{x}} \boldsymbol{f} \left(\boldsymbol{x}, \boldsymbol{u} \right) \Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \quad B = \nabla_{\boldsymbol{u}} \boldsymbol{f} \left(\boldsymbol{x}, \boldsymbol{u} \right) \Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \quad C = \nabla_{\boldsymbol{x}} \boldsymbol{g} \left(\boldsymbol{x}, \boldsymbol{u} \right) \Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \quad D = \nabla_{\boldsymbol{u}} \boldsymbol{g} \left(\boldsymbol{x}, \boldsymbol{u} \right) \Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}}$$

?

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- linearizing $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is meaningful only around an equilibrium $(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}})$
- each equilibrium will lead to its "own" corresponding linear model $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$, where A and B thus depend on $(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}})$ and $\boldsymbol{x}$, $\boldsymbol{u}$ in $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ have actually the meaning of $\Delta\boldsymbol{x}$, $\Delta\boldsymbol{u}$ with respect to the equilibrium

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- each linearized model $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ is more or less valid only in a neighborhood of $(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}})$. Moreover the size of this neighborhood depends on the curvature of $\boldsymbol{f}$ around that specific equilibrium point

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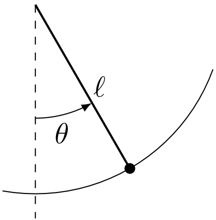
linearization = a very useful tool to do
analysis and design of control systems

Another example: the pendulum



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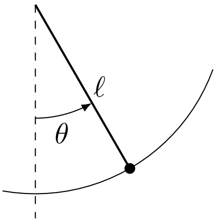

First step: equations of motion:

- gravity: $F_{g,x} = -mg\sin(\theta)$
- friction: $F_f = -fv_x = -f\ell\dot{\theta}$
- input torque: $F_u = u/\ell$

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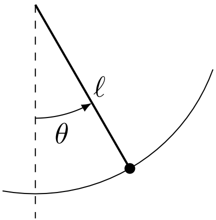

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resulting dynamics: $ml\ddot{\theta} = -mg\sin(\theta) - f\ell\dot{\theta} + \dfrac{u}{\ell}$

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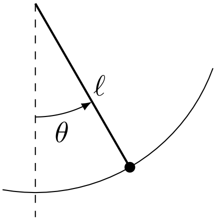

Next step: transform

$$ml\ddot{\theta} = -mg\sin(\theta) - f\ell\dot{\theta} + \frac{u}{\ell}$$

into a state-space form

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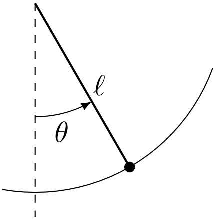

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into a state-space form

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{\ell}\sin(x_1) - \frac{f}{m}x_2 + \frac{1}{m\ell^2}u
\end{aligned}$$

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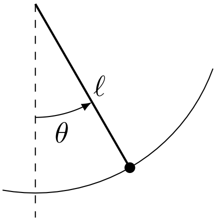

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for $u = 0$ (for simplicity)

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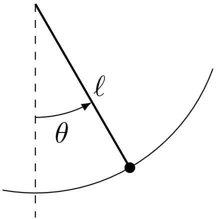

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for $u = 0$ (for simplicity)

$$\begin{cases} 0 = x_2 \\ 0 = -\dfrac{g}{\ell}\sin(x_1) - \dfrac{f}{m}x_2 \end{cases}$$

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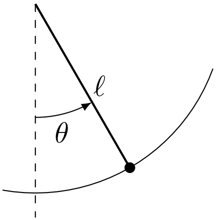

Next step: find the equilibria of

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin(x_1) - \frac{f}{m}x_2 + \frac{1}{m\ell^2}u$$

for $u = 0$ (for simplicity)

$$\begin{cases} 0 = x_2 \\ 0 = -\dfrac{g}{\ell}\sin(x_1) - \dfrac{f}{m}x_2 \end{cases} \implies \quad x_{\mathsf{eq}1} = n\pi,\ x_{\mathsf{eq}2} = 0$$

Another example: the pendulum

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin(x_1) - \frac{f}{m}x_2 + \frac{1}{m\ell^2}u$$

Equilibrium $\boldsymbol{x}_{\mathsf{eq}\alpha} = \mathbf{0}$, $u = 0$ implies

$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{\boldsymbol{x}_{\mathsf{eq}\alpha}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{f}{m} \end{bmatrix}$$

$$B = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\right|_{\boldsymbol{x}_{\mathsf{eq}\alpha}} = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}$$

Another example: the pendulum

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin(x_1) - \frac{f}{m}x_2 + \frac{1}{m\ell^2}u\end{aligned}$$

Equilibrium $\boldsymbol{x}_{\mathsf{eq}\beta} = [\pi, 0]^T$, $u = 0$ implies

$$A = \left.\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix}\right|_{\boldsymbol{x}_{\mathsf{eq}\beta}} = \begin{bmatrix}0 & 1 \\ \frac{g}{\ell} & -\frac{f}{m}\end{bmatrix}$$

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?

Linearizing nonlinear systems - insights on the preservation of the stability properties

Roadmap

- obvious properties
- simple examples
- understanding through generalizing the simple examples
- some considerations about control of nonlinear systems

Obvious fact: linearizing around an equilibrium keeps that point an equilibrium

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned} \qquad \mapsto \qquad \begin{aligned} \dot{\widetilde{\boldsymbol{x}}} &= A\widetilde{\boldsymbol{x}} + B\widetilde{\boldsymbol{u}} \\ \widetilde{\boldsymbol{y}} &= C\widetilde{\boldsymbol{x}} + D\widetilde{\boldsymbol{u}} \end{aligned}$$

with

$$\begin{cases} \boldsymbol{x} &= \boldsymbol{x}_{\text{eq}} + \widetilde{\boldsymbol{x}} \\ \boldsymbol{u} &= \boldsymbol{u}_{\text{eq}} + \widetilde{\boldsymbol{u}} \\ \boldsymbol{y} &= \boldsymbol{y}_{\text{eq}} + \widetilde{\boldsymbol{y}} \end{cases}$$

Thus if $\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}$ was an equilibrium for the nonlinear system, it is still an equilibrium for the linearized one. *But if it was a stable one before, will it still be a stable one after?*

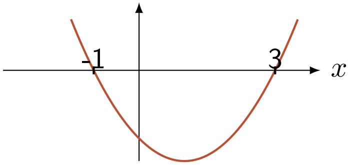
Thus if $\boldsymbol{x}_{\mathsf{eq}}, \boldsymbol{u}_{\mathsf{eq}}$ was an equilibrium for the nonlinear system, it is still an equilibrium for the linearized one. *But if it was a stable one before, will it still be a stable one after?*

this lesson $=$ answering this question

Starting from the basics: the scalar case

$$\dot{x} = f(x) = x^2 - 2x - 3 = (x-3)(x+1) = 0 \qquad \text{equilibria: } \begin{cases} x_{\mathsf{eq}\alpha} = -1 \\ x_{\mathsf{eq}\beta} = 3 \end{cases}$$

$$f(x) = x^2 - 2x - 3$$



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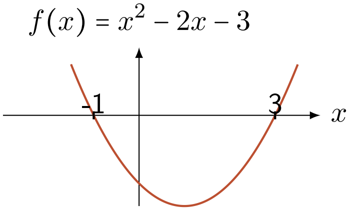

Analysing $x_{\mathsf{eq}\alpha} = -1$:

- $x < -1$ implies $\dot{x} = f(x) > 0$ implies x grows
- $x > -1$ implies $\dot{x} = f(x) < 0$ implies x shrinks *(but only locally)*

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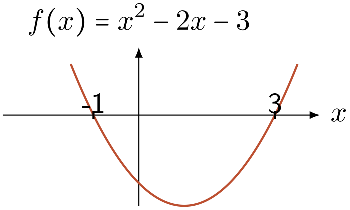

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Discussion: what does this imply?

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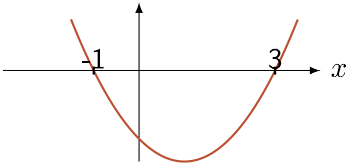

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Discussion: what does this imply? Moving a bit away from $x_{\text{eq}\alpha} = -1$ leads to go back to $x_{\text{eq}\alpha}$, thus this is an asymptotically stable equilibrium!

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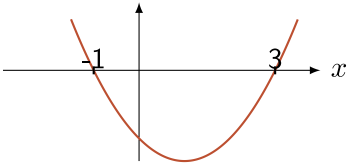

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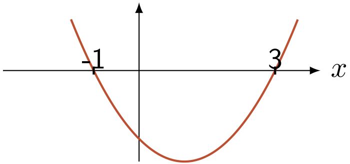

Analysing $x_{\mathsf{eq}\alpha} = 3$:

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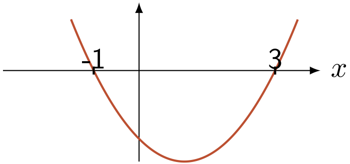

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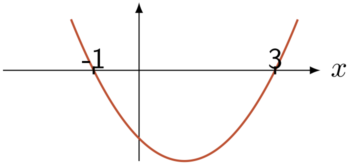

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How do we generalize the previous concepts?

$$\dot{x} = f(x) \qquad f(x_{\mathsf{eq}}) = 0, \qquad \mapsto \qquad \dot{\widetilde{x}} = a_{x_{\mathsf{eq}}} \widetilde{x} \quad \text{with} \quad a_{x_{\mathsf{eq}}} = \left.\frac{\partial f}{\partial x}\right|_{x_{\mathsf{eq}}}$$

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<u>"$a < 0$ implies asymptotically stable"</u>
has been our mantra up to now!

How do we generalize even more?

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\left(\boldsymbol{x}\right) \quad \boldsymbol{f}\left(\boldsymbol{x}_{\text{eq}}\right) = \boldsymbol{0}, \quad \mapsto \quad \dot{\widetilde{\boldsymbol{x}}} = A_{\boldsymbol{x}_{\text{eq}}}\widetilde{\boldsymbol{x}} \quad \text{with} \quad A_{\boldsymbol{x}_{\text{eq}}} = \nabla \boldsymbol{f}\Big|_{\boldsymbol{x}_{\text{eq}}}$$

How do we generalize even more?

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with the stability of A something that we will see when we do the linear algebra part of the course

Example

$$\dot{\boldsymbol{x}} = \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} x_2 \\ 1 - x_1^2 - x_2 \end{bmatrix}$$

Example

$$\dot{\boldsymbol{x}} = \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} x_2 \\ 1 - x_1^2 - x_2 \end{bmatrix}$$

Linearization around a generic point:

$$\dot{\widetilde{\boldsymbol{x}}} = A_{\boldsymbol{x}_{\mathrm{eq}}} \widetilde{\boldsymbol{x}} = \begin{bmatrix} 0 & -1 \\ -2x_1 & -1 \end{bmatrix} \widetilde{\boldsymbol{x}}$$

Example

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$$\boldsymbol{x}_{\mathsf{eq}\alpha} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \implies A_{\boldsymbol{x}_{\mathsf{eq}\alpha}} = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} \qquad \boldsymbol{x}_{\mathsf{eq}\beta} = \begin{bmatrix} +1 \\ 0 \end{bmatrix} \implies A_{\boldsymbol{x}_{\mathsf{eq}\beta}} = \begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix}$$

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spoiler (will see this extensively with the "linear algebra" part: the eigenvalues of A will be the poles of the system!

Example (continuation)

"the eigenvalues of $A_{\boldsymbol{x}_{\mathrm{eq}}}$ are the poles of the system"

$$\begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} \quad \Longrightarrow \quad \text{eigenvalues} = \{-2;\ 1\}$$

$$\begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix} \quad \Longrightarrow \quad \text{eigenvalues} = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$$

Discussion: how are the modes of the linearized system around equilibrium α? And around equilibrium β?

Example (continuation)

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Discussion: how are the modes of the linearized system around equilibrium α? And around equilibrium β?

very important message: this implicitly says that studying linearized systems gives information about the nonlinear ones!

Summarizing

- general approach = start with computing the equilibria for the original nonlinear system, get the corresponding $A_{\boldsymbol{x}_{\text{eq}}}$ matrix for each equilibrium $\boldsymbol{x}_{\text{eq}}$, and analyse the stability properties of that $A_{\boldsymbol{x}_{\text{eq}}}$ matrix
- if $A_{\boldsymbol{x}_{\text{eq}}}$ is asymptotically stable, then the original equilibrium $\boldsymbol{x}_{\text{eq}}$ is locally asymptotically stable
- if $A_{\boldsymbol{x}_{\text{eq}}}$ is unstable, then the original equilibrium $\boldsymbol{x}_{\text{eq}}$ is unstable
- if $A_{\boldsymbol{x}_{\text{eq}}}$ is simply stable, then we cannot say anything about the original equilibrium $\boldsymbol{x}_{\text{eq}}$ and we need to do other types of analyses (in later-on courses!)
- in any case the considerations are local considerations, valid only in the neighborhood of $\boldsymbol{x}_{\text{eq}}$

Some philosophical considerations

- sometimes piecewise linearizing systems is a way to deal with nonlinear dynamics, even if this is not the most elegant approach to control
- you will do nonlinear control in later on courses; feedback linearization, one of the approaches, is very powerful
- `https://www.youtube.com/watch?v=uhND7Mvp3f4` ← this is done through classical nonlinear control, not data driven one

?

Numerically simulating nonlinear systems

Roadmap

- why do we need to numerically simulate?
- Euler methods
- pros and cons
- connections with linearization

Our computers are digital machines, but the ODEs are “analogic” objects

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$$\dot{\boldsymbol{x}} = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)$$

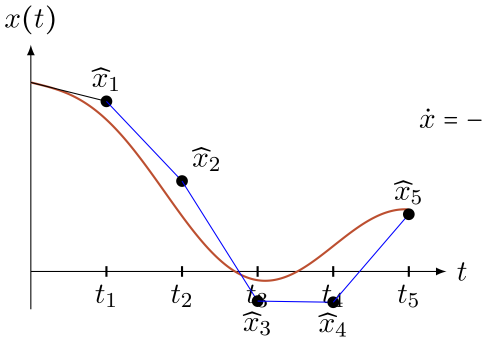
the need is for discretizing these objects, both in time and in space

Simulating nonlinear systems = solving the ODE numerically and in a discrete way

i.e., use the fact that we know that $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$, we know the whole $\boldsymbol{u}$, and we know the initial condition $\boldsymbol{x}(0)$ to compute a series of points

$$\boldsymbol{x}(t_1), \boldsymbol{x}(t_2), \ldots, \boldsymbol{x}(t_N)$$

that approximate the whole trajectory $\boldsymbol{x}(0:T)$:



$$\dot{x} = -\frac{1}{2}x + \cos\left(\frac{3}{2}t\right)$$

The simplest numerical solver: Euler's (forward) method

step 0: $\widehat{\boldsymbol{x}}_0 = \boldsymbol{x}_0$ *(i.e., the initial value)*

The simplest numerical solver: Euler's (forward) method

$$
\begin{aligned}
\text{step 0:} \quad & \widehat{\boldsymbol{x}}_0 &&= \boldsymbol{x}_0 \quad \textit{(i.e., the initial value)} \\
\text{step 1:} \quad & \widehat{\boldsymbol{x}}_1 &&= \widehat{\boldsymbol{x}}_0 + \boldsymbol{f}\left(\widehat{\boldsymbol{x}}_0, \boldsymbol{u}_0\right) \Delta t
\end{aligned}
$$

The simplest numerical solver: Euler's (forward) method

$$
\begin{aligned}
\text{step 0:} \quad \widehat{\boldsymbol{x}}_0 &= \boldsymbol{x}_0 \quad \textit{(i.e., the initial value)} \\
\text{step 1:} \quad \widehat{\boldsymbol{x}}_1 &= \widehat{\boldsymbol{x}}_0 + \boldsymbol{f}\left(\widehat{\boldsymbol{x}}_0, \boldsymbol{u}_0\right) \Delta t \\
\text{step 2:} \quad \widehat{\boldsymbol{x}}_2 &= \widehat{\boldsymbol{x}}_1 + \boldsymbol{f}\left(\widehat{\boldsymbol{x}}_1, \boldsymbol{u}_1\right) \Delta t
\end{aligned}
$$

The simplest numerical solver: Euler's (forward) method

$$
\begin{array}{llll}
\text{step 0:} & \widehat{\boldsymbol{x}}_0 & = \boldsymbol{x}_0 & \textit{(i.e., the initial value)} \\
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\text{step 2:} & \widehat{\boldsymbol{x}}_2 & = \widehat{\boldsymbol{x}}_1 + \boldsymbol{f}\left(\widehat{\boldsymbol{x}}_1, \boldsymbol{u}_1\right) \Delta t & \\
\vdots & \vdots & \vdots \quad \vdots &
\end{array}
$$

Tradeoffs:

- the more "gentle" $\boldsymbol{f}$, the more accurate the results
- the smaller Δt, the more accurate the results & the longer the computational time
- the longer the time horizon T in $\boldsymbol{x}(0:T)$ the less accurate the results

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Known problem: Euler forward may be numerically unstable, especially for "stiff ODEs" (i.e., ODEs for which some terms that can lead to rapid variation in the solution). Will be seen extensively in following courses!

Another example: Euler's backward method

Euler Forward:

$$\widehat{\boldsymbol{x}}_{k+1} = \widehat{\boldsymbol{x}}_k + \boldsymbol{f}\left(\widehat{\boldsymbol{x}}_k, \boldsymbol{u}_k\right)\Delta t$$

Euler Backward:

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Euler Backward:

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can be generalized to Runge-Kutta methods, with better tradeoffs and robustness properties; they will be studied in following courses

Important: Euler's method is another type of linearization

"normal" linearization

$$\boldsymbol{f} \approx \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \Delta \boldsymbol{x} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} \Delta \boldsymbol{u}$$

Euler's method

$$\boldsymbol{x}\,(t + \Delta t) \approx \boldsymbol{x}(t) + \boldsymbol{f}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big)\Delta t$$

What is best, then? Euler or “normal” linearizing?

pros of linearizing

- analytic results usable to design control systems and understand structural properties

pros of Euler

- arbitrarily good accuracy if Δt is sufficiently small
- gives more accurate information about the actual trajectories

cons of linearizing

- results valid only locally
- one may make mistakes in computing the Jacobians

cons of Euler

- computationally heavy
- does not give theoretical insights

?

Where are we now?

characterizing LTI systems

↓

linearization of nonlinear systems

↓

linear algebra as a way
to even better analyse LTI systems

↓

designing filters

Linear algebra - why, if we are doing control?

Roadmap

- why?
- spoilers

Motivations, in very brief

$$\ddot{x} + a_1\dot{x} + a_0x = bu(t) \tag{1}$$

is equivalent to

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t)$$

Motivations, in very brief

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is equivalent to

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$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu,$$

and every analysis problem on the system becomes a linear algebra one

(e.g., computing the equilibria)

How does this connect with the first part of the course?

Via the scalar form:

$$\ddot{x} + a_1\dot{x} + a_0 x = bu \qquad \Longrightarrow \qquad X(s) = \frac{bU(s)}{s^2 + a_1 s + a_0}$$

with the modes defined by the solutions of $s^2 + a_1 s + a_0 = 0$

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Via the matrix form:

$$s\boldsymbol{X}(s) - A\boldsymbol{X}(s) = BU(s)$$

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \quad \Longrightarrow$$

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Via the matrix form:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \quad \Longrightarrow \quad \begin{aligned} s\boldsymbol{X}(s) - A\boldsymbol{X}(s) &= BU(s) \\ (sI - A)\boldsymbol{X}(s) &= BU(s) \end{aligned}$$

How does this connect with the first part of the course?

Via the scalar form:

$$\ddot{x} + a_1\dot{x} + a_0 x = bu \quad \Longrightarrow \quad X(s) = \frac{bU(s)}{s^2 + a_1 s + a_0}$$

with the modes defined by the solutions of $s^2 + a_1 s + a_0 = 0$

Via the matrix form:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \quad \Longrightarrow \quad \begin{aligned} s\boldsymbol{X}(s) - A\boldsymbol{X}(s) &= BU(s) \\ (sI - A)\boldsymbol{X}(s) &= BU(s) \\ \boldsymbol{X}(s) &= (sI - A)^{-1}BU(s) \end{aligned}$$

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with the modes defined by the solutions of $\det(sI - A) = 0$ (and with the formula $(sI - A)^{-1} = \frac{\operatorname{adj}(sI - A)}{\det(sI - A)}$ that will be re-introduced better later on)

Spoilers

- the poles of $\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$ will be the eigenvalues of A
- the structure of A will determine the multiplicity of the poles and much more *(for the brave ones, check the "Rosenbrock's theorem", but only <u>after</u> the course has ended)*

Spoilers

- the poles of $\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$ will be the eigenvalues of A
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> the concepts of eigenvalues, eigenvectors, eigenspaces plus their generalized counterparts are as fundamental as the concepts of modes

List of the knowledge we used in the previous slides

- matrix inverses, i.e., M and M^{-1}
- adjugate of a matrix, i.e., $\operatorname{adj}(M)$
- determinant of a matrix, i.e., $\det(M)$
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Suggested additional resources:

- 3blue1brown: Essence of linear algebra
- Khan Academy: 44 videos on linear algebra
- Khan Academy: Introduction to vectors
- Gilbert Strang: Linear algebra

?

Basic operations

Roadmap

- inner products
- matrix vector products
- matrix matrix products

Notation

$$\text{matrices: } A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

$$\text{column vectors: } \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n}$$

$$\text{row vectors: } \boldsymbol{x} = \begin{bmatrix} x_1 & \ldots & x_m \end{bmatrix} \in \mathbb{R}^{m}$$

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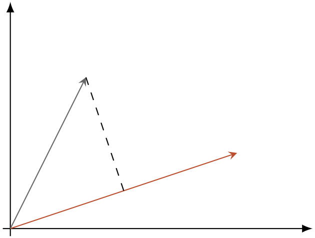
Important: saying "vector" means column vector; to indicate row vectors say "row vectors"!

Transposition

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \implies \boldsymbol{x}^\top = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$$

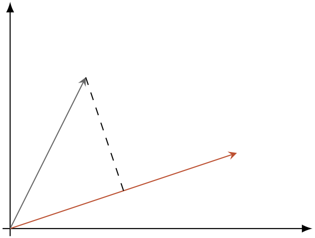
$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \implies A^\top = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1m} & \cdots & a_{nm} \end{bmatrix}$$

Inner product



$$\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \quad \implies \quad \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top \boldsymbol{y} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n$$

Geometrical meaning of inner product, some notes



note: $\boldsymbol{x}$ and $\boldsymbol{y}$ must live in the same space, thus they must have the same length

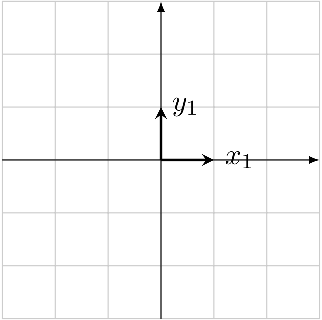
suggested material: 3 blue 1 brown, Dot products and duality, Essence of linear algebra, chapter 9

Matrix-vector product, mathematical definition

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} \star_1 \\ \vdots \\ \star_n \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} \star_1 + \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix} \star_2 + \begin{bmatrix} a_{13} \\ \vdots \\ a_{n3} \end{bmatrix} \star_3 + \ldots$$

Matrix-vector product, geometrically

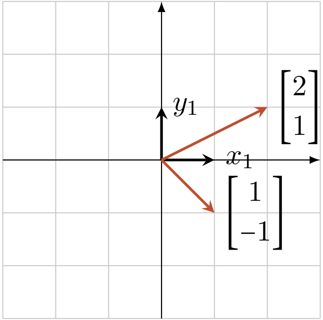
Starting point: canonical basis: $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $y_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



Matrix-vector product, geometrically

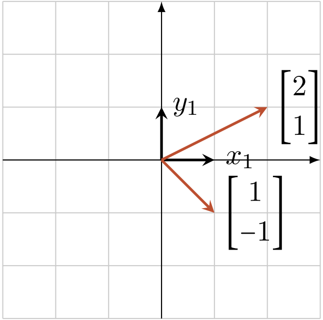
Starting point: canonical basis: $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad y_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

what are then $Ax_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $Ax_2 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?



VERY IMPORTANT INTERPRETATION

the columns of A *are where the elements of the canonical basis are mapped by* A

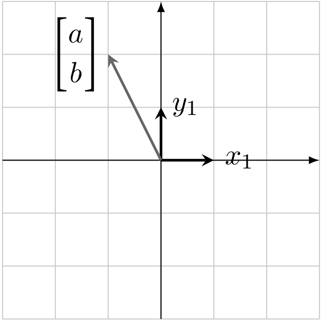


Remember: not all the A's are square

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

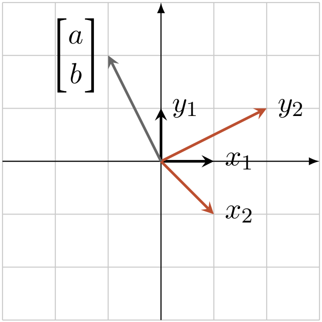
Matrix-vector product, geometrically

$$c = \begin{bmatrix} a \\ b \end{bmatrix} = ax_1 + by_1$$



Matrix-vector product, geometrically

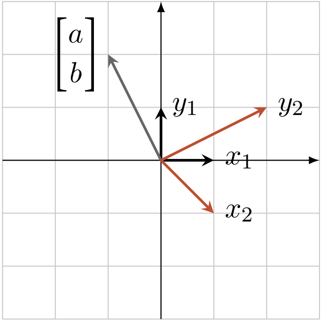
$$c = \begin{bmatrix} a \\ b \end{bmatrix} = a x_1 + b y_1 \qquad A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \qquad \Longrightarrow \qquad Ac = ?$$



Matrix-vector product, geometrically

$$c = \begin{bmatrix} a \\ b \end{bmatrix} = ax_1 + by_1 \qquad A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \qquad \implies \qquad Ac = ?$$

$$x_2 = Ax_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad y_2 = Ay_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



Matrix-vector product, geometrically

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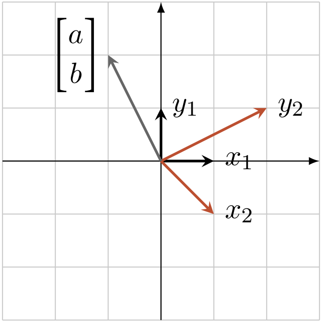
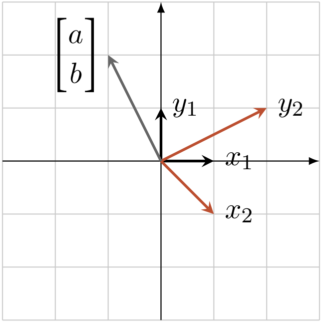
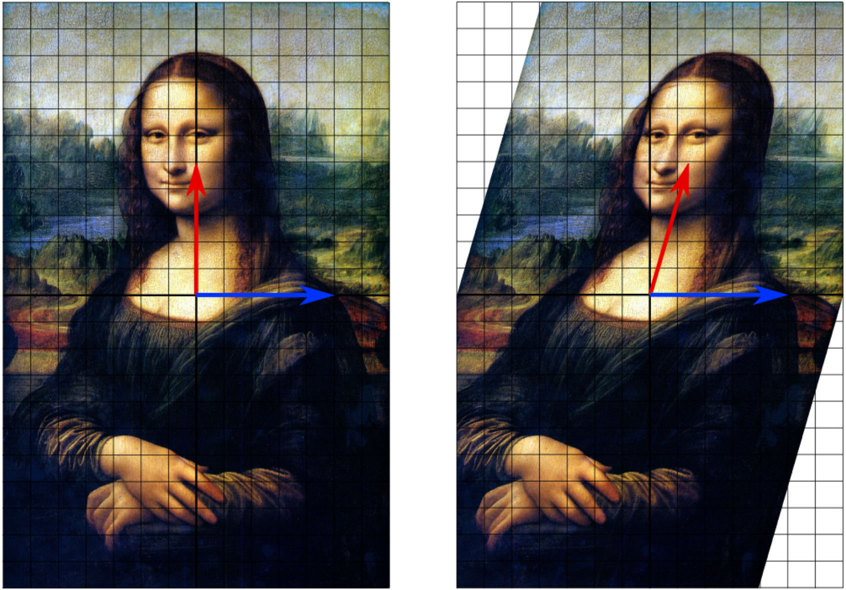


Matrix-vector product, geometrically

$$c = \begin{bmatrix} a \\ b \end{bmatrix} = ax_1 + by_1 \qquad A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \qquad \Longrightarrow \qquad Ac = ?$$

$$x_2 = Ax_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad y_2 = Ay_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad Ac = Aax_1 + Aby_1 = ax_2 + by_2$$





(By TreyGreer62 - Image:Mona Lisa-restored.jpg, CC0, https://commons.wikimedia.org/w/index.php?curid=12768508)

?

How do we go now from

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} \star_1 \\ \vdots \\ \star_n \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} \star_1 + \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix} \star_2 + \begin{bmatrix} a_{13} \\ \vdots \\ a_{n3} \end{bmatrix} \star_3 + \dots$$

to

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} \star_1 & \triangle_1 \\ \vdots & \vdots \\ \star_n & \triangle_n \end{bmatrix} = ?$$

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$$AB = A \begin{bmatrix} \boldsymbol{b}_1 & \ldots & \boldsymbol{b}_p \end{bmatrix} = \begin{bmatrix} A\boldsymbol{b}_1 & \ldots & A\boldsymbol{b}_p \end{bmatrix}$$

Matrix multiplication

$$C = AB$$

Discussion: how must the dimensions of A and B be?

- $A \in \mathbb{R}^{r_A \times c_A}$
- $B \in \mathbb{R}^{r_B \times c_B}$

Matrix multiplication

$$C = AB$$

Discussion: how must the dimensions of A and B be?

- $A \in \mathbb{R}^{r_A \times c_A}$
- $B \in \mathbb{R}^{r_B \times c_B}$
- $c_A = r_B$
- $\implies C \in \mathbb{R}^{r_A \times c_B}$

Examples

$$\begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ 19 & 2 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ 1 & 3 & 0 \\ 3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 11 & 4 \\ 3 & 15 & 4 \end{bmatrix}$$

Examples

$$\begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ 19 & 2 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & 11 \end{bmatrix}$$

Do you see why this does *not* work?

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

In general, $AB \neq BA$

(even if it may actually happen, depending on the eigendecompositions of A and B . . .)

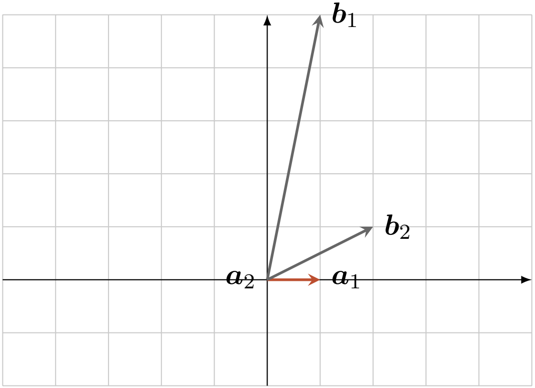
Numerical example:

$$\begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Ok that in general $AB \neq BA$, but why?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix}$$



Important points

- matrix multiplications are not communitative: $AB \neq BA$
- if $AB = BA$ then we say that A and B commute

?

Alternative way of expressing matrix - column multiplications

$$\begin{bmatrix} - & \boldsymbol{a}_1 & - \\ - & \boldsymbol{a}_2 & - \\ & \vdots & \\ - & \boldsymbol{a}_n & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & \dots & \boldsymbol{b}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 & \boldsymbol{a}_1\boldsymbol{b}_2 & \dots & \boldsymbol{a}_1\boldsymbol{b}_n \\ \boldsymbol{a}_2\boldsymbol{b}_1 & \boldsymbol{a}_2\boldsymbol{b}_2 & \dots & \boldsymbol{a}_2\boldsymbol{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a}_n\boldsymbol{b}_1 & \boldsymbol{a}_n\boldsymbol{b}_2 & \dots & \boldsymbol{a}_n\boldsymbol{b}_n \end{bmatrix}$$

Alternative way of expressing matrix - column multiplications

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different interpretations; typically (but not always):

- "columns of the product $=$ linear combinations of the columns of A" more useful when doing control
- "elements of the product $=$ angles between the rows of A and columns of B" more useful when doing data science

?

How to change between bases, and why

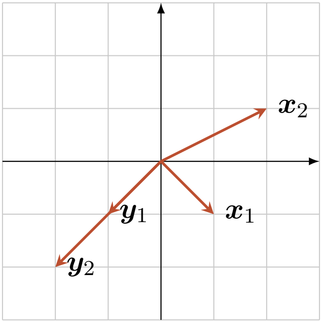
Roadmap

- what is a basis?
- what happens when there are two bases?
- how do I change between the two bases?

Linear independency

$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n$ are said to be *linearly independent* if and only if

$$\sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i = \mathbf{0} \quad \Leftrightarrow \quad \lambda_1 = \ldots = \lambda_m = 0$$



Additional basic definitions

$$\mathrm{span}\,(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle = \text{set of all the linear combinations of } \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$$

Additional basic definitions

$\mathrm{span}\,(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle$ = set of all the linear combinations of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$

dimension of a space: max. number of linearly independent vectors in that space

Basis of a vector space

Definition (basis)

$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^n$ *form a basis for* $\mathbb{R}^n$ *if they are linearly independent vectors*

Basis of a vector space

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Definition (basis of a subspace $\mathcal{B}$)

$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathcal{B} \subset \mathbb{R}^n, m < n$ *are a basis for* $\mathcal{B}$ *if they are linearly independent*

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Definition (basis of a subspace $\mathcal{B}$)

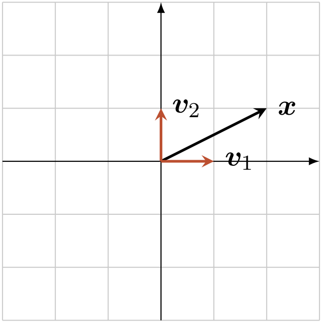
$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathcal{B} \subset \mathbb{R}^n, m < n$ *are a basis for* $\mathcal{B}$ *if they are linearly independent*

important point: they must be as many as there are dimensions in the vectors space we are looking for a basis

How to use a basis

if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ basis of $\mathbb{R}^n$ then

$$\forall \boldsymbol{x} \in \mathbb{R}^n \quad \exists \lambda_1, \ldots, \lambda_n \in \mathbb{R} \quad \text{s.t.} \quad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$



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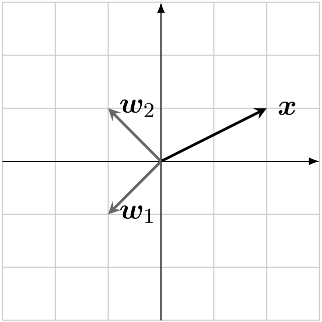
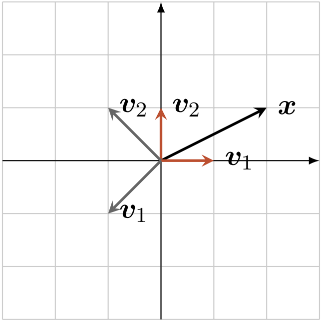
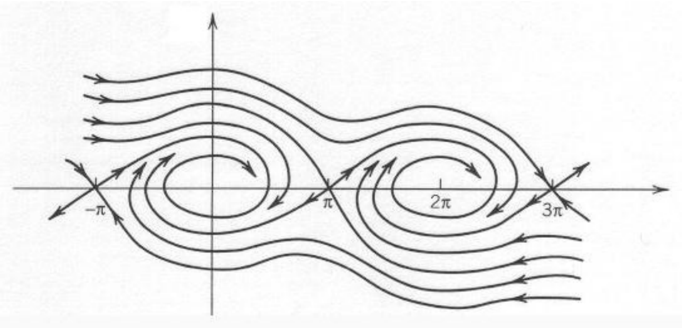


Important message: $\boldsymbol{x}$ is the same object, independently of the basis.

Thus we must be able to “change” between the coordinate systems!



Changing between bases - physical intuitions



the system is the same system, even if I decide to measure things in a different way

Changing between bases

if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ are two separate bases of $\mathbb{R}^n$ then

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}$$

Changing between bases

$$\boldsymbol{v}_1 = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Changing between bases

$$\boldsymbol{v}_1 = \begin{bmatrix} \boldsymbol{v}_1 \ \boldsymbol{v}_2 \cdots \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_1 \ \boldsymbol{w}_2 \cdots \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_{1\to 1} \\ \gamma_{1\to 2} \\ \vdots \\ \gamma_{1\to n} \end{bmatrix}$$

Changing between bases

$$\boldsymbol{v}_1 = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_{1\to 1} \\ \gamma_{1\to 2} \\ \vdots \\ \gamma_{1\to n} \end{bmatrix} \implies \boldsymbol{v}_m = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_{m\to 1} \\ \gamma_{m\to 2} \\ \vdots \\ \gamma_{m\to n} \end{bmatrix}$$

Changing between bases

$$\boldsymbol{v}_1 = \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n\end{bmatrix}\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = \begin{bmatrix}\boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n\end{bmatrix}\begin{bmatrix}\gamma_{1\to1}\\\gamma_{1\to2}\\\vdots\\\gamma_{1\to n}\end{bmatrix} \implies \boldsymbol{v}_m = \begin{bmatrix}\boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n\end{bmatrix}\begin{bmatrix}\gamma_{m\to1}\\\gamma_{m\to2}\\\vdots\\\gamma_{m\to n}\end{bmatrix}$$

$$\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n\end{bmatrix}\begin{bmatrix}\gamma_{1\to1} & \gamma_{2\to1} & \cdots & \gamma_{n\to1}\\\gamma_{1\to2} & \gamma_{2\to2} & \cdots & \gamma_{n\to2}\\\vdots & \vdots & & \vdots\\\gamma_{1\to n} & \gamma_{2\to n} & \cdots & \gamma_{n\to n}\end{bmatrix}$$

or, compactly, $V = W\Gamma_{v\to w}$

Change of basis

$$V = W\Gamma_{v \to w}$$

Change of basis

$$V = W\Gamma_{v\to w} \quad W = V\Gamma_{w\to v}$$

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$$V = W\Gamma_{v\to w} \quad W = V\Gamma_{w\to v} \qquad \Longrightarrow \qquad V = V\Gamma_{w\to v}\Gamma_{v\to w}$$

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Change of basis

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$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

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$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = V\boldsymbol{\lambda}$$

Change of basis

$$V = W\Gamma_{v\to w} \quad W = V\Gamma_{w\to v} \quad \implies \quad V = V\Gamma_{w\to v}\Gamma_{v\to w} \quad \implies \quad \Gamma_{w\to v} = \Gamma_{v\to w}^{-1}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \cdots \ \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = V\boldsymbol{\lambda} = W\Gamma_{v\to w}\boldsymbol{\lambda}$$

Change of basis

$$V = W\Gamma_{v\to w} \quad W = V\Gamma_{w\to v} \quad \Longrightarrow \quad V = V\Gamma_{w\to v}\Gamma_{v\to w} \quad \Longrightarrow \quad \Gamma_{w\to v} = \Gamma_{v\to w}^{-1}$$

$$\boldsymbol{x} = \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_n\end{bmatrix} = V\boldsymbol{\lambda} = W\Gamma_{v\to w}\boldsymbol{\lambda} = W\boldsymbol{\lambda}'$$

Exercise: change the basis of $\boldsymbol{x}$ from V to W

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{w}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \boldsymbol{w}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \boldsymbol{x} = V \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

hint: remember that $\boldsymbol{v}_m = \begin{bmatrix} \boldsymbol{w}_1 \ \boldsymbol{w}_2 \cdots \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_{m\to 1} \\ \gamma_{m\to 2} \\ \vdots \\ \gamma_{m\to n} \end{bmatrix}$ and try to form $\begin{bmatrix} \boldsymbol{v}_1 \ \boldsymbol{v}_2 \cdots \boldsymbol{v}_n \end{bmatrix}$

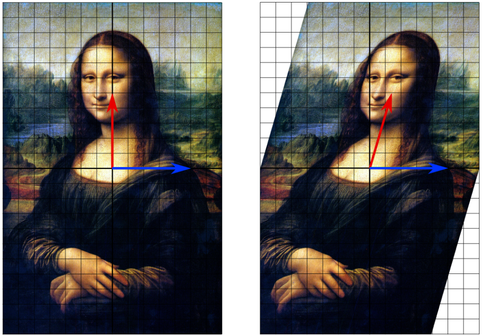
?

Linear transformations

Roadmap

- linear transformations as matrices
- the difference between "linear transformation" and "matrix"
- the effect of changing bases

Linear transformations and matrices



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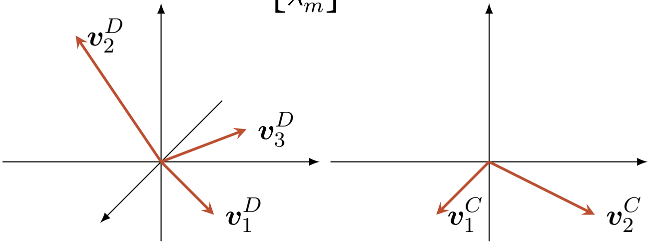
linear transformation $\mathcal{A} \neq$ matrix A

How can I express a linear transformations as a matrix?

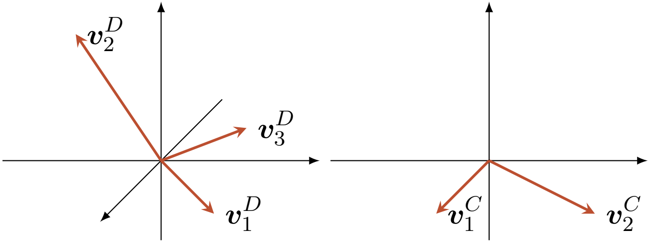
i.e., knowing $\mathcal{A} : D \mapsto C \qquad D = \mathbb{R}^m = \left\langle \boldsymbol{v}_1^D, \ldots, \boldsymbol{v}_m^D \right\rangle \qquad C = \mathbb{R}^n = \left\langle \boldsymbol{v}_1^C, \ldots, \boldsymbol{v}_n^C \right\rangle$

how to go from $\boldsymbol{x} = \left[\boldsymbol{v}_1^D \; \boldsymbol{v}_2^D \cdots \boldsymbol{v}_m^D\right] \begin{bmatrix} \lambda_1^D \\ \lambda_2^D \\ \vdots \\ \lambda_m^D \end{bmatrix}$ to $\boldsymbol{y} = \mathcal{A}\boldsymbol{x} = \left[\boldsymbol{v}_1^C \; \boldsymbol{v}_2^C \cdots \boldsymbol{v}_n^C\right] \begin{bmatrix} \lambda_1^C \\ \lambda_2^C \\ \vdots \\ \lambda_n^C \end{bmatrix}$?



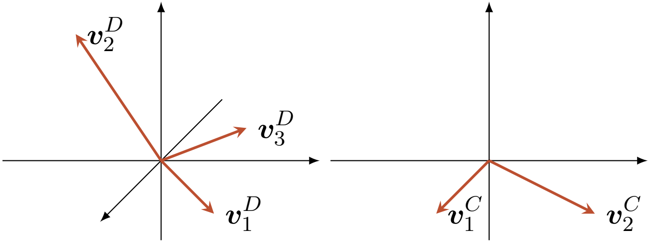
How can I express a linear transformations as a matrix?

$$\mathcal{A}\boldsymbol{v}_1^D = \begin{bmatrix}\boldsymbol{v}_1^C & \boldsymbol{v}_2^C & \cdots & \boldsymbol{v}_n^C\end{bmatrix}\begin{bmatrix}a_{11}\\ a_{12}\\ \vdots\\ a_{1n}\end{bmatrix}$$



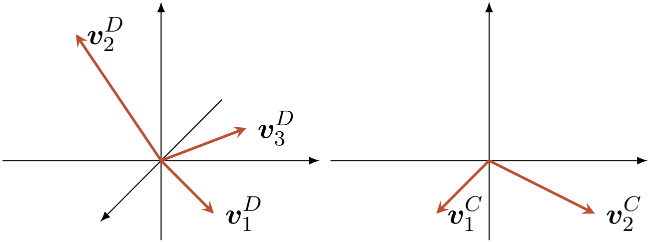
How can I express a linear transformations as a matrix?

$$\mathcal{A}\boldsymbol{v}_2^D = \left[\boldsymbol{v}_1^C \; \boldsymbol{v}_2^C \; \cdots \; \boldsymbol{v}_n^C\right] \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{bmatrix}$$



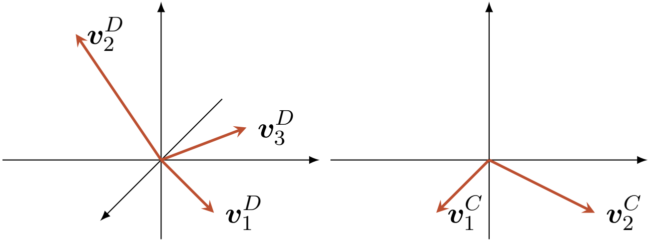
How can I express a linear transformations as a matrix?

$$\mathcal{A}\boldsymbol{v}_3^D = \left[\boldsymbol{v}_1^C\ \boldsymbol{v}_2^C\ \cdots\ \boldsymbol{v}_n^C\right]\begin{bmatrix} a_{31} \\ a_{32} \\ \vdots \\ a_{3n} \end{bmatrix}$$



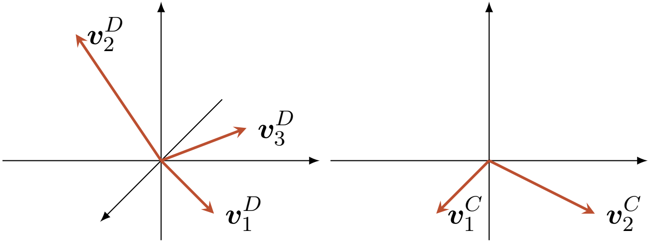
How can I express a linear transformations as a matrix?

$$\left[\mathcal{A}\boldsymbol{v}_1^D \;\; \ldots \;\; \mathcal{A}\boldsymbol{v}_m^D\right] = \left[\boldsymbol{v}_1^C \; \boldsymbol{v}_2^C \; \cdots \; \boldsymbol{v}_n^C\right] \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ a_{12} & \cdots & a_{m2} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}$$



How can I express a linear transformations as a matrix?

$$\mathcal{A}\boldsymbol{x} = \begin{bmatrix} \mathcal{A}\boldsymbol{v}_1^D & \mathcal{A}\boldsymbol{v}_2^D & \cdots & \mathcal{A}\boldsymbol{v}_m^D \end{bmatrix} \begin{bmatrix} \lambda_1^D \\ \lambda_2^D \\ \vdots \\ \lambda_m^D \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1^C & \boldsymbol{v}_2^C & \cdots & \boldsymbol{v}_n^C \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ a_{12} & \cdots & a_{m2} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \lambda_1^D \\ \lambda_2^D \\ \vdots \\ \lambda_m^D \end{bmatrix}$$

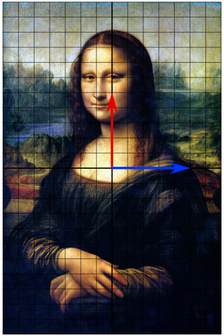


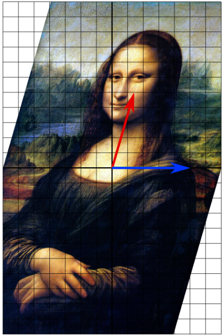
Summary

$$\mathcal{A}\boldsymbol{x} = \left[\boldsymbol{v}_1^C \; \boldsymbol{v}_2^C \cdots \boldsymbol{v}_n^C\right] \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ a_{12} & \cdots & a_{m2} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \lambda_1^D \\ \lambda_2^D \\ \vdots \\ \lambda_m^D \end{bmatrix} \qquad \Longrightarrow \qquad \text{"}A\boldsymbol{x} \mapsto \boldsymbol{y}\text{"}$$

i.e., to go from $\boldsymbol{x}$ to $\boldsymbol{y}$ start from the coordinates of $\boldsymbol{x}$ in the basis of the domain, transform the coordinates through the matrix A transforming the basis in the domain into the basis of the codomain, and consider the new coordinates $\boldsymbol{y}$ as expressed in the basis of the codomain

Linear transformations and matrices





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the transformation is defined by $\mathcal{A}$, *not by* A

And what about *square* matrices?

if $\mathcal{A} : \mathbb{R}^n \mapsto \mathbb{R}^n \quad \Longrightarrow \quad C = D$ we can choose the same basis, i.e.,

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And what about *square* matrices?

if $\mathcal{A} : \mathbb{R}^n \mapsto \mathbb{R}^n$ $\implies$ $C = D$ we can choose the same basis, i.e.,

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solution: $$\mathcal{A}\boldsymbol{x} = \left[\boldsymbol{v}_1 \ \boldsymbol{v}_2 \cdots \boldsymbol{v}_n \right] \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ a_{12} & \cdots & a_{n2} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \implies \text{"}A\boldsymbol{x} \mapsto \boldsymbol{y}\text{"}$$

same concepts as before, just that both $\boldsymbol{x}$ and $\boldsymbol{y}$ are expressed in the the same basis, so that A expresses how the elements *of the given basis* are transformed

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how do A and A' relate?

Changes of bases (summary)

$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ bases of $\mathbb{R}^n$ $\Longrightarrow$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 \; \boldsymbol{v}_2 \cdots \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_1 \; \boldsymbol{w}_2 \cdots \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}$$

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$$\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \gamma_{1\to 1} & \gamma_{2\to 1} & \cdots & \gamma_{n\to 1} \\ \gamma_{1\to 2} & \gamma_{2\to 2} & \cdots & \gamma_{n\to 2} \\ \vdots & \vdots & & \vdots \\ \gamma_{1\to n} & \gamma_{2\to n} & \cdots & \gamma_{n\to n} \end{bmatrix}$$ or, compactly, $V = W\Gamma_{v\to w}$

Effects of changing bases on the representations of $\mathcal{A}$

$$\left[\mathcal{A}\boldsymbol{v}_1 \ \ldots \ \mathcal{A}\boldsymbol{v}_n\right] = \left[\boldsymbol{v}_1 \cdots \boldsymbol{v}_n\right] \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} = VA$$

$$\left[\mathcal{A}\boldsymbol{w}_1 \ \ldots \ \mathcal{A}\boldsymbol{w}_n\right] = \left[\boldsymbol{w}_1 \cdots \boldsymbol{w}_n\right] \begin{bmatrix} a'_{11} & \cdots & a'_{n1} \\ \vdots & & \vdots \\ a'_{1n} & \cdots & a'_{nn} \end{bmatrix} = WA'$$

Effects of changing bases on the representations of $\mathcal{A}$

$$V = W\Gamma_{v\to w} \qquad W = V\Gamma_{w\to v}$$

$$\Downarrow$$

$$A' = \Gamma_{v\to w} A \Gamma_{w\to v}$$

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Convenient notation:

$$\Gamma_{v\to w} = T \qquad A' = TAT^{-1}$$

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Convenient notation:

$$\Gamma_{v\to w} = T \qquad A' = TAT^{-1}$$

operations of the type $A' = TAT^{-1}$ with T full-rank mean changing the basis, i.e., "looking at the linear transformation from a different perspective"

?

The spaces associated to a matrix

Roadmap

- rank and range
- determinants
- kernel
- connections among the various concepts

Recall:

$$\mathrm{span}\,(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle = \text{set of all the linear combinations of these vectors}$$

$$\mathrm{range}\,(A) = \text{span of the columns of } A$$

dimension of a space: max. number of linearly independent vectors

Just to make the importance of the concepts clear:

when does this system have a solution?

$$A\boldsymbol{x} = \begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \boldsymbol{y}$$

(Column) Rank of a matrix

$$\operatorname{rank}(A) = \operatorname{rank}\left(\begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix}\right) = \text{ number of linearly independent columns}$$

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Important result: column-rank $=$ row-rank *(i.e., there are as many linearly independent rows as linearly independent columns)*

$$\operatorname{rank}(A) = \operatorname{rank}\left(A^\top\right) = \operatorname{rank}\left(A^\top A\right) = \operatorname{rank}\left(AA^\top\right)$$

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Example: what is the maximal rank of $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$?

Reconnecting with automatic control

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

$\implies$ structure of A determines how the time derivative $\dot{\boldsymbol{x}}$ is, and how the time derivative is determines the stability and time-evolution properties of the system.

Reconnecting with automatic control

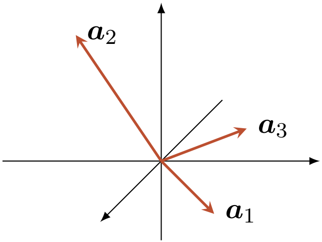
$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

$\implies$ structure of A determines how the time derivative $\dot{\boldsymbol{x}}$ is, and how the time derivative is determines the stability and time-evolution properties of the system. E.g.,

$$\mathrm{span}\,(A) = \begin{bmatrix} +1 \\ -1 \end{bmatrix} \quad \implies \quad \text{if } x_1 \text{ grows then } x_2 \text{ diminishes, and viceversa}$$

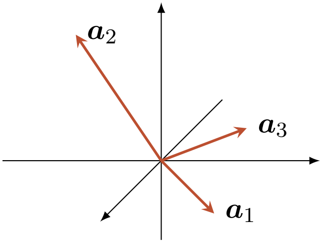
Determinant of a square matrix

$$\det(A) = \det\left(\begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix}\right) =$$ (signed) volume of the parallelepiped defined by $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$



Determinant of a square matrix

Remember: $\begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix}$ represent where the elements of the basis are mapped into



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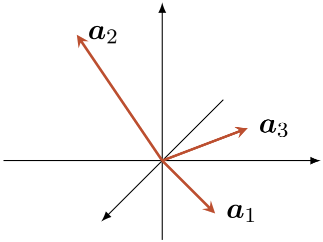

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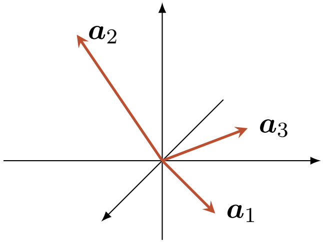
Remember: $\begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix}$ represent where the elements of the basis are mapped into

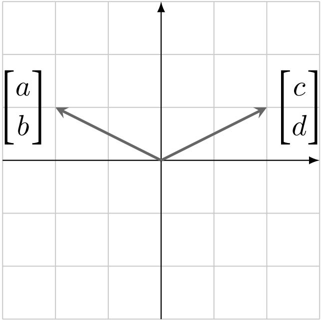
thus "determinant = scaling factor of the linear transformation described by A (and thus defined by the linear transformation $\mathcal{A}$)



the determinant is a property of the linear transformation $\mathcal{A}$*,*
thus if T *is a change of basis then* $det(A) = det\left(TAT^{-1}\right)$*,*
since changing the basis does not change the underlying transformation

Likely the unique (other) case you should remember on how to compute determinants

$$\det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = ad - bc$$



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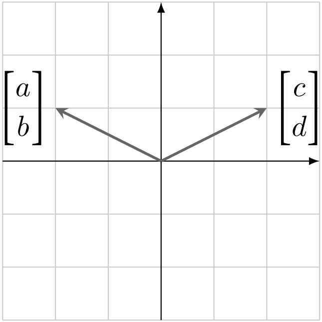

Which other case do you know? (you know for sure already one more)

Determinants and invertibility of linear maps

Immediate implications:

$$\det(A) \neq 0 \quad \Leftrightarrow \quad \mathcal{A} \text{ invertible}$$

$$\det(A) = 0 \quad \Leftrightarrow \quad \mathcal{A} \text{ not-invertible}$$

Why is invertibility important?

because if you want to solve $A\boldsymbol{x} = b$ for generic b then you need A^{-1}

Connections between the determinant and the rank of a square matrix

if $A \in \mathbb{R}^{n \times n}$ then $\mathrm{rank}\,(A) = n$ implies that the columns / rows of A are linearly independent

Connections between the determinant and the rank of a square matrix

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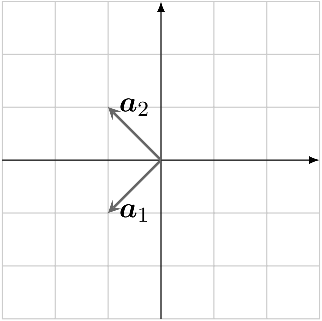
Summary until now and examples: $A \in \mathbb{R}^{2\times 2}$



determinant $=$ *area* spanned by the columns of A

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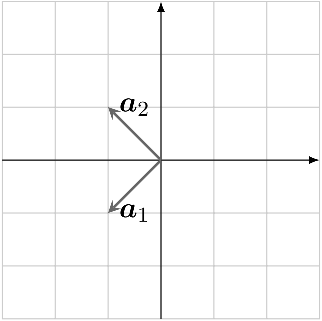

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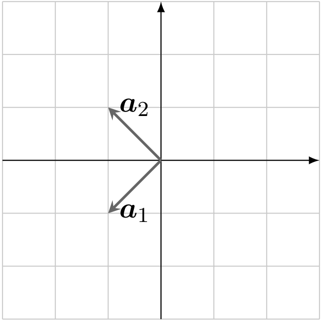

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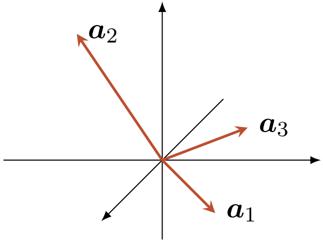
Summary until now and examples: $A \in \mathbb{R}^{3\times3}$



determinant $=$ *volume* spanned by the columns of A

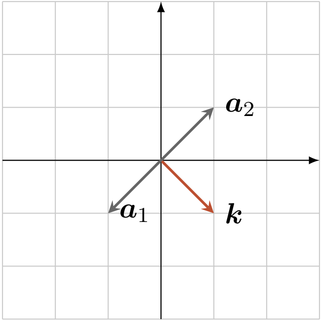
- if $\operatorname{rank}(A) = 3$ then the column vectors span a volume
- if $\operatorname{rank}(A) = 2$ then the column vectors span an area
- if $\operatorname{rank}(A) = 1$ then the column vectors span a line
- if $\operatorname{rank}(A) = 0$ then the column vectors span nothing

?

Kernel (or null-space) of a matrix $A \in \mathbb{R}^{n \times m}$

$$\ker(A) = \{\boldsymbol{x} \in \mathbb{R}^m \text{ s.t. } A\boldsymbol{x} = \mathbf{0}\}$$

Example 1: $\ker(A) = \operatorname{span}(\boldsymbol{k})$ with $\boldsymbol{k} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



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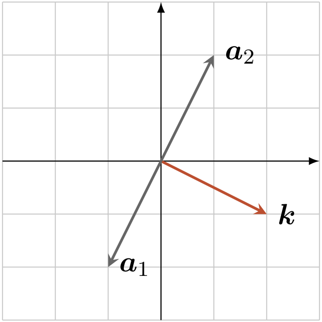
Example 2: $\ker(A) = \operatorname{span}(\boldsymbol{k})$ with $\boldsymbol{k} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$



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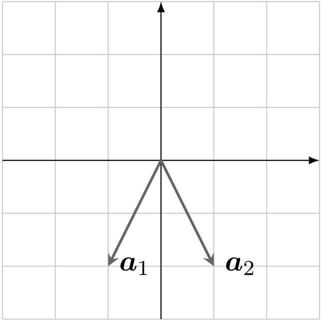
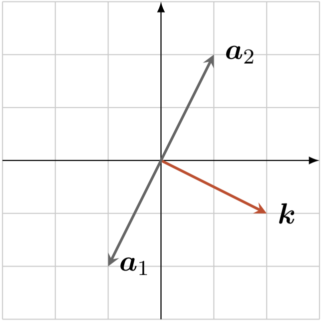
Example 2: $\ker(A) = \mathbf{0}$



Extremely important result: $\ker(A) \perp \operatorname{range}(A)$



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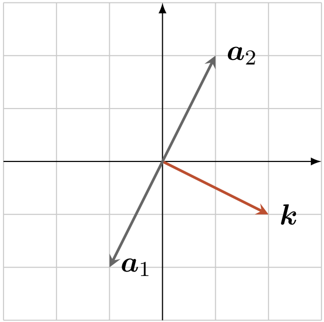

$\implies \operatorname{rank}(A) + \dim(\ker(A))$ = number of columns of A

Alternative viewpoint on the kernel of A

$$A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_m \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{a}_1 \boldsymbol{x} \\ \vdots \\ \boldsymbol{a}_m \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{if and only if} \quad \boldsymbol{a}_i \perp \boldsymbol{x} \; \forall i$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

- $\ker(A) = ?$
- $\text{range}(A) = ?$

bigger matrices, and needing to compute ranges, determinants, or kernels?

$\rightarrow$ use Matlab, python, Wolfram Alpha, whatever

Some useful general rules

$$
\begin{aligned}
\left(A^\top\right)^\top &= A \\
(A+B)^\top &= A^\top + B^\top \\
(cA)^\top &= cA^\top \\
(AB)^\top &= B^\top A^\top \\
\det\left(A^\top\right) &= \det(A) \\
\left(A^{-1}\right)^\top &= \left(A^\top\right)^{-1}
\end{aligned}
$$