TTK4225 - Systems Theory, Autumn 2020

Damiano Varagnolo

NTNU

The different types of stability properties of an equilibrium

Roadmap

- simple stability
- convergence
- asymptotic stability
- examples

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- convergence
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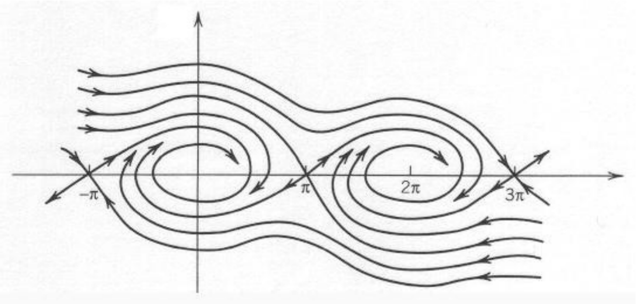
- examples

important assumption in this course: $\boldsymbol{u} = \overline{\boldsymbol{u}} = const.$

The different nature of different equilibria

pendulum with friction: $\ddot{\theta} = -\lambda\dot{\theta} - g\sin(\theta)$



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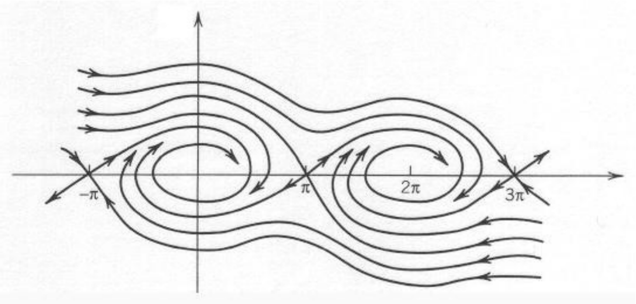
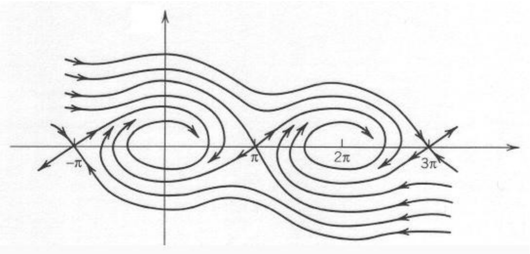


Discussion: why are these equilibria different?

Simply stable equilibrium (continuous time case)



$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const.$ $\qquad$ $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}_e)$ $\qquad$ $(\boldsymbol{y}_e, \boldsymbol{u}_e)$ = equilibrium

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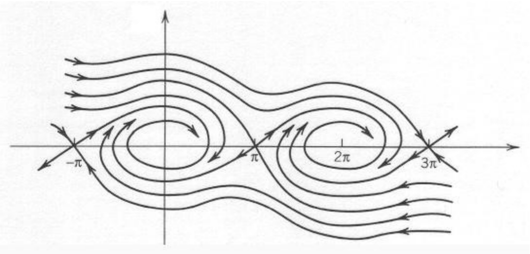

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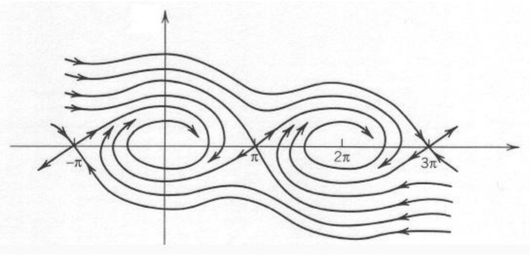
Definition (simply stable equilibrium)

$\boldsymbol{y}_e$ is simply stable if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $\|\boldsymbol{y}_0 - \boldsymbol{y}_e\| \leq \delta$ then $\|\boldsymbol{y}(t) - \boldsymbol{y}_e\| \leq \varepsilon \quad \forall t \geq 0$

Convergent equilibrium



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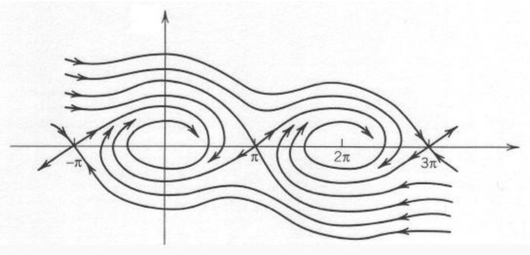

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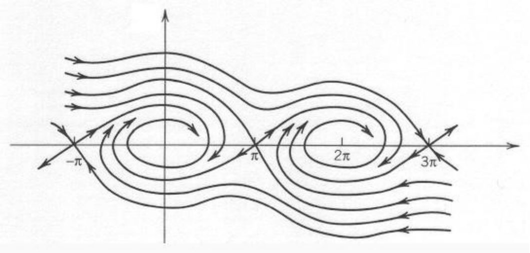
Definition (convergent equilibrium)

$\boldsymbol{y}_e$ = *is convergent if* $\exists \delta > 0$ *s.t. if* $\|\boldsymbol{y}_0 - \boldsymbol{y}_e\| \leq \delta$ *then* $\boldsymbol{y}(t) \xrightarrow{t \to +\infty} \boldsymbol{y}_e$

Important differences



simple stability: I can confine arbitrarily the trajectory by reducing ε opportunely

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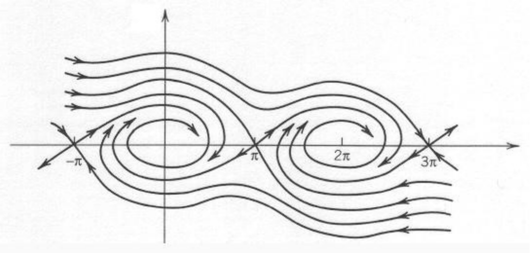


simple stability: I can confine arbitrarily the trajectory by reducing ε opportunely

convergent equilibrium: I *cannot* confine arbitrarily the trajectory, but I know that if I start close enough then *eventually* the distance $\|\boldsymbol{y}(t) - \boldsymbol{y}_e\|$ will go to zero

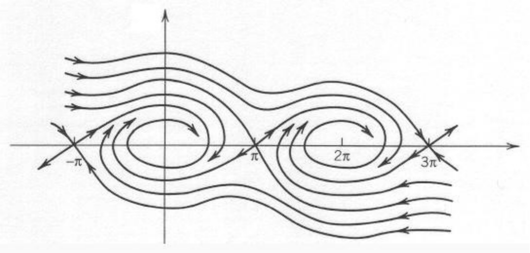
Discussion: Consider the discrete time system

$$x(k+1) = \begin{cases} 2x(k) & \text{if } |x(k)| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Which type of equilibrium is 0? Possibilities:

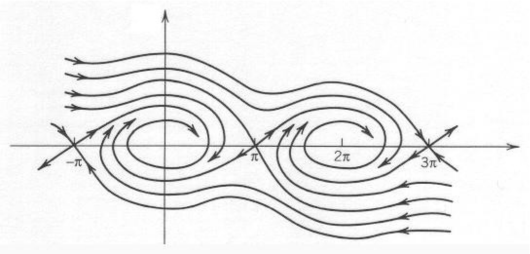
1. a simply stable equilibrium
2. a convergent equilibrium
3. nothing special

Asymptotic stability



$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \overline{\boldsymbol{u}}\right) \qquad \boldsymbol{y}_e = \text{equilibrium}$$

Asymptotic stability



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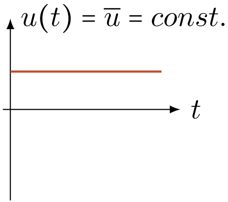
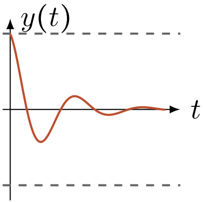
Definition (asymptotically stable equilibrium)

the equilibrium $\boldsymbol{y}_e$ is said to be asymptotically stable if it is simultaneously simply stable & convergent

Asymptotic stability, graphically

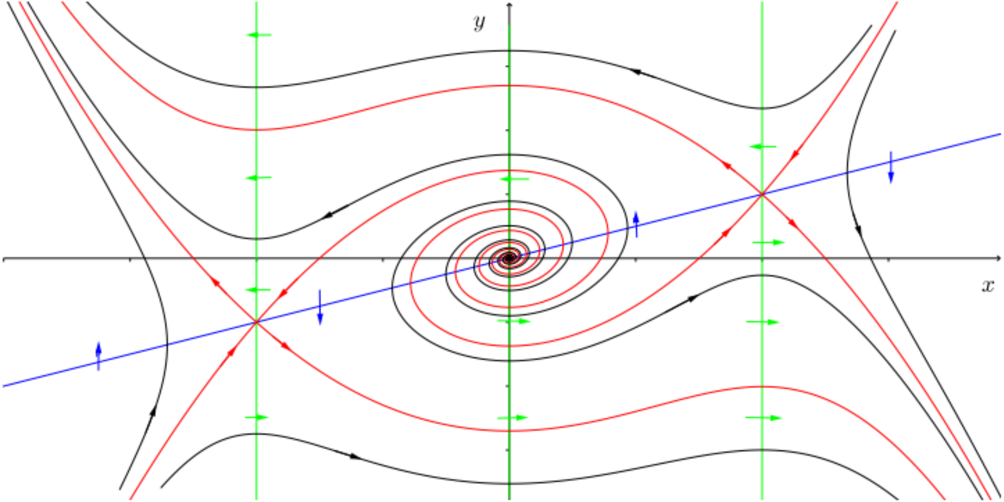




Very important point

to have instability it is enough to have one trajectory that escapes (example: “constrained” flipped pendulum)

Some examples of phase portraits



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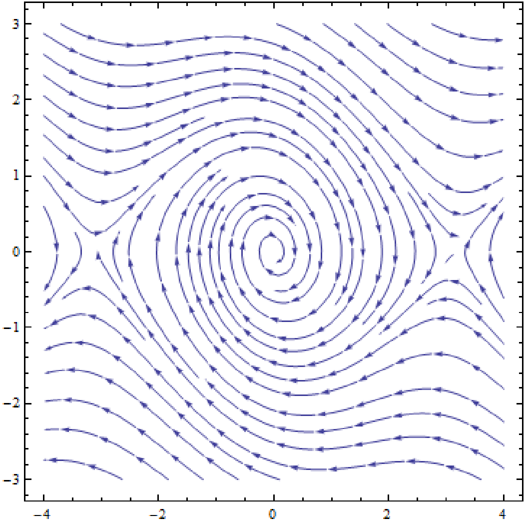

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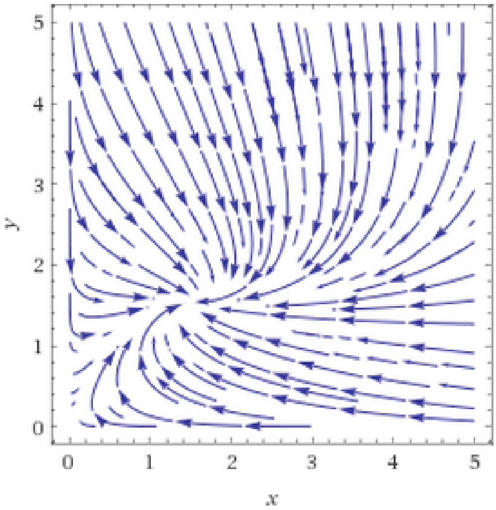

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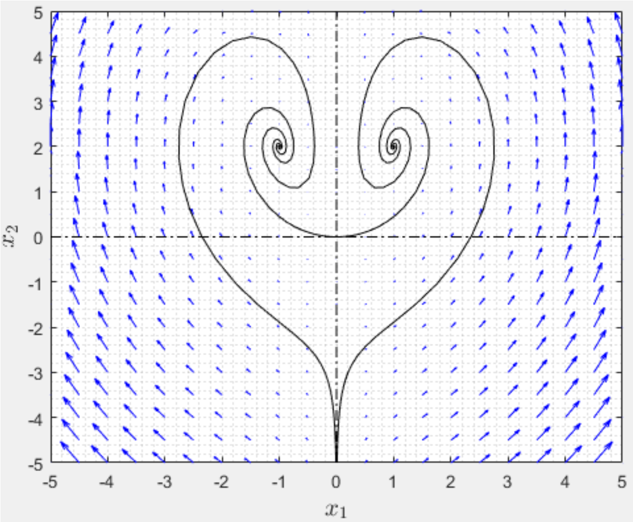

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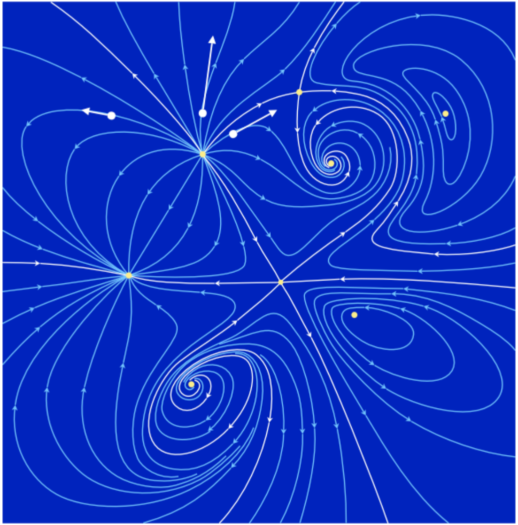

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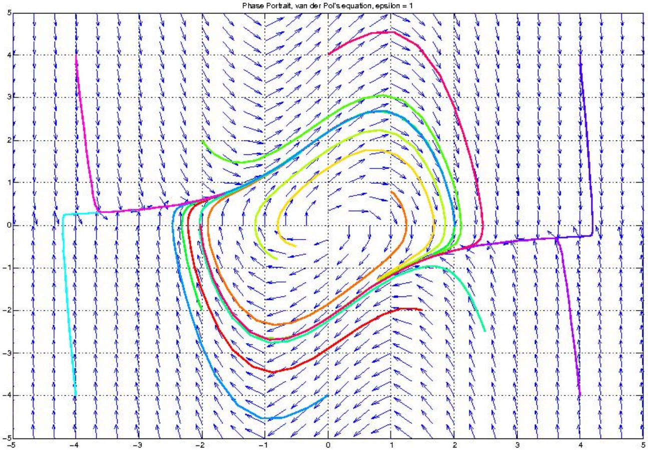

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BIBO stability

Roadmap

- generalizing the concept of stability
- BIBO stability
- connecting BIBO stability with the poles of the transfer functions

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important assumption in this course: $\boldsymbol{u} = \overline{\boldsymbol{u}} = const.$

Important: the term “stability” may refer to specific equilibrium points or specific systems

“Stability” referring to specific equilibria:

1. simply stable equilibrium
2. convergent equilibrium
3. asymptotically stable equilibrium

“Stability” referring to specific systems:

1. Bounded Input Bounded Output (BIBO) stable systems (we will see this now)
2. Input to State Stable (ISS) systems (we will not see this in this course)

BIBO stability

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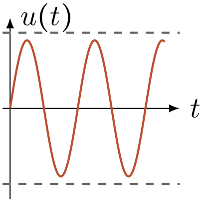
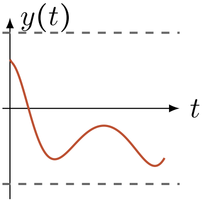
Definition (BIBO stability)

the system $(\boldsymbol{f}, \boldsymbol{g})$ *is said to be Bounded Input Bounded Output (BIBO) stable if* $\|\boldsymbol{u}\| \leq \gamma_u \implies \|\boldsymbol{y}\| \leq \gamma_y$





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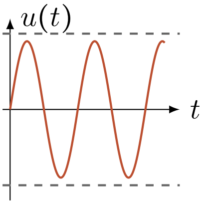
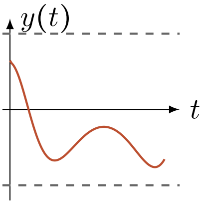
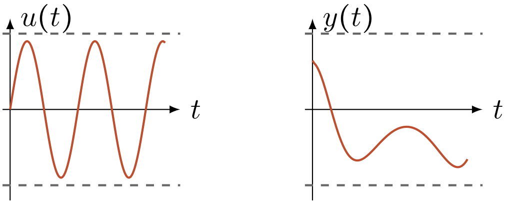




Discussion: is the system $\dot{y} = y^2 u$ BIBO stable?

BIBO stability



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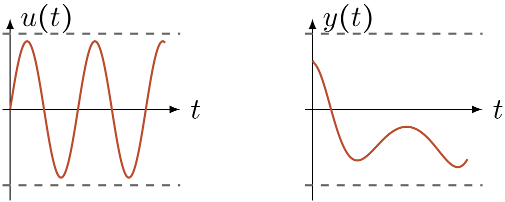
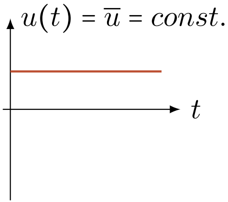
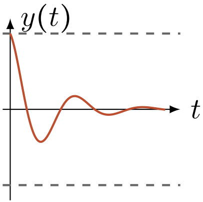


Discussion: is the system $\dot{y} = y^2 u$ BIBO stable? And the system $\dot{y} = -y^2 u$? To check BIBO stability one can use the “small gain theorem”: not in this course! Here we will check the BIBO stability checking either the impulse response or the transfer function

Summarizing

Asymptotic stability





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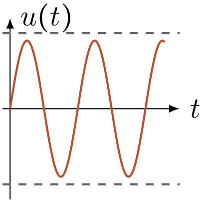

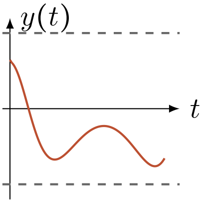

The very important result that we will find now

For general nonlinear systems:

BIBO stable system $\neq$ asymptotically stable equilibria $\neq$ simply stable equilibria

For LTIs:

BIBO stable system $=$ asymptotically stable equilibria $\neq$ simply stable equilibria

Towards coupling the BIBO stability to the system's impulse response or transfer function

$$X(s) = H(s)U(s)$$

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Discussion points:

- $|u(t)| < M_u$ means a "non diverging" $u(t)$. Assuming that $u(t)$ has a rational Laplace transform, where do the poles of $U(s)$ live in this case?

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- what happens if $h(t)$ is non-diverging and non-converging? May I choose some non-diverging $u(t)$ that makes $y(t)$ diverging?

BIBO stability = absolute integrability of the impulse response

BIBO stability:

$$|u(t)| < M_u \quad \Longrightarrow \quad |y(t)| < M_y$$

Impulse response:

$$y(t) = h * u(t) = \int_0^t h(\tau)u(t-\tau)d\tau$$

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Coupling the BIBO stability concept with the poles of a TF

$$\text{if } \int_{-\infty}^{+\infty} |h(\tau)| \, d\tau < +\infty \text{ then BIBO stability}$$

$$H(s) = \frac{N(s)}{D(s)}$$

Discussion: if we want the system be BIBO stable, may $D(s)$ have poles on the imaginary axis?

Important result

BIBO stable LTI system = all the poles have strictly negative real part

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asymptotically stable equilibria $\neq$ simply stable equilibria

?

Nomenclature

BIBO stable LTI system $=$ LTI with all its equilibria asymptotically stable

marginally stable LTI system $=$ LTI with all its equilibria simply stable

unstable LTI system $=$ LTI with unstable equilibria

Examples: are these systems BIBO stable, marginally stable, or unstable?

$$H(s) = \frac{1}{(s+2)(s+1)} \tag{1}$$

$$H(s) = \tag{2}$$

$$H(s) = \tag{3}$$

$$H(s) = \tag{4}$$

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$$H(s) = \frac{1}{s^2(s+5)} \qquad \Longrightarrow \quad A + Bt + Ce^{-5t} \tag{5}$$

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$$H(s) = \frac{1}{(s^2+4)(s-1)} \implies A\cos(2t) + B\sin(2t) + Ce^{t} \tag{6}$$

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$$H(s) = \frac{1}{(s+3+2j)(s+3-2j)} \implies A\sin(2t)e^{-3t} \tag{2}$$

$$H(s) = \frac{1}{(s+2)(s-1)} \implies Ae^{-2t} + Be^{t} \tag{3}$$

$$H(s) = \frac{1}{s(s+1)} \implies A + Be^{-t} \tag{4}$$

$$H(s) = \frac{1}{s^2(s+5)} \implies A + Bt + Ce^{-5t} \tag{5}$$

$$H(s) = \frac{1}{(s^2+4)(s-1)} \implies A\cos(2t) + B\sin(2t) + Ce^{t} \tag{6}$$

$$H(s) = \frac{1}{(s-4)^2(s+1)} \tag{7}$$

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$$H(s) = \frac{1}{(s-4)^2(s+1)} \quad \Longrightarrow \quad Ate^{2t} + Be^{-t} \tag{7}$$

And what about nonlinear systems?

more complicated! Will treat this through Lyapunov theory in more advanced courses

The very important result that we found

For general nonlinear systems:

BIBO stable system $\neq$ asymptotically stable equilibria $\neq$ simply stable equilibria

For LTIs:

BIBO stable system $=$ asymptotically stable equilibria $\neq$ simply stable equilibria

Summarizing, once again

Different types of system stability:

asymptotic input-output (system) stability: independently of $u(t)$, $x(t) \to 0$ when $t \to +\infty$

marginal (or simply input-output) (system) stability: as soon as $|u(t)| < M_u$, $|x(t)| < M_x$ when $t \to +\infty$

(system) instability: there exists at least one signal $u(t)$ for which we cannot do the bound $|x(t)| < M_x$ when $t \to +\infty$

it is necessary to know about potential instabilities, because our control system must stabilize them

?

Where are we now?

characterizing LTI systems

↓

linearization of nonlinear systems

↓

linear algebra as a way
to even better analyse LTI systems

↓

designing filters

Introduction to nonlinear systems

Roadmap

- definitions
- examples
- important differences between linear and nonlinear systems

The case of scalar functions

f is linear if ...

$$f(ax + by) = af(x) + bf(y)$$

for every a, x, b, y

f is nonlinear if ...

there exists at least one a, x, b, y for which

$$f(ax + by) \neq af(x) + bf(y)$$

Example

$$
\begin{aligned}
f(ax+by) &= (ax+by)^2 \\
&= a^2x^2 + 2axby + b^2y^2 \\
&= a^2 f(x) + 2axby + b^2 f(y) \\
&\neq a f(x) + b f(y)
\end{aligned}
$$

The case of vectorial functions

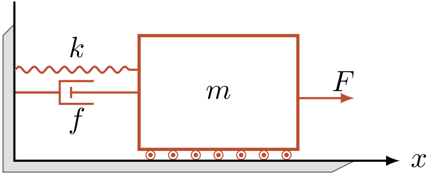
$\boldsymbol{f}$ is linear if ...

all the single components $f_i(\cdot)$ are linear

$\boldsymbol{f}$ is nonlinear if ...

at least one of the single components $f_i(\cdot)$ is nonlinear

Example: spring-mass systems



$$m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + F(t)$$

Remember: ARMA models can be written as vectorial first order systems!

$$y^{(n)} = a_{n-1} y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots + b_0 u \Downarrow$$
$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

Example: Lotka-Volterra

- y_{prey} := prey
- y_{pred} := predator

$$\begin{cases} \dot{y}_{\text{prey}} & = \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} & = -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

Example: Van-der-Pol oscillator

$$\begin{cases} \dot{y}_1 & = \mu \left(y_1 - \frac{y_1^3}{3} - y_2 \right) \\ \dot{y}_2 & = \frac{y_1}{\mu} \end{cases}$$

An extremely important difference between linear and nonlinear systems

no linearity $\Longrightarrow$ *no superposition effects*

An extremely important difference between linear and nonlinear systems

$$\textit{no linearity} \implies \textit{no superposition effects}$$

i.e., without linearity we cannot say that $u = u_1 + u_2$ causes $y = y_1 + y_2$

An extremely important difference between linear and nonlinear systems

$$\textit{no linearity} \implies \textit{no superposition effects}$$

i.e., without linearity we cannot say that $u = u_1 + u_2$ causes $y = y_1 + y_2$ and thus not even

$$y(t) = y_{\text{free}}(t) + y_{\text{forced}}(t)$$

no linearity $\Longrightarrow$ *no modal analysis*

Remember where we want to arrive: Model Predictive Control

$$\boldsymbol{u}^{\star} = \arg \min_{\boldsymbol{u} \in \mathcal{U}, \boldsymbol{f}(\boldsymbol{u}) \in \mathcal{F}} \mathrm{Cost}\left(\boldsymbol{f}\left(\boldsymbol{u}\right), \boldsymbol{u}\right),$$

that requires to:

- define "Cost"
- be able to compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

Remember where we want to arrive: Model Predictive Control

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that requires to:

- define "Cost"
- be able to compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

seems that with nonlinear systems things complicate

Potential approach: linearize through Taylor expansions

we will see this in the next unit

Important differences to always remember, take 1

nonlinear systems admit isolated equilibria,
while LTI systems admit only subspaces of equilibria *(by the way, why?)*

(example: the Lotka Volterra has 2 distinct equilibria)

Important differences to always remember, take 2

linear systems admit exponential bounding,

while nonlinear systems may have finite escape times *(by the way, why?)*

(example: starting $\dot{y} = y^2$ *from* $y_0 = c$ *leads to the trajectory* $y(t) = \dfrac{1}{c-t}$*)*

Important differences to always remember, take 3

nonlinear systems admit limit cycles,
while LTI systems do not (by the way, why?)

?

Roadmap

- recalling the definition of state-space systems
- Taylor approximations, what are they?
- how to linearize a continuous time system
- examples

State space representations - Definition

mathematical model (typically but not limited to of a physical system) as a finite set of inputs, outputs and state variables related by first-order differential equations satisfying the separation principle

Ingredients:

- finite number of inputs, outputs and state variables
- first-order differential equations
- *satisfies the separation principle:* the current value of the state contains all the information necessary to forecast the future evolution of the outputs and of the state. I.e., to compute future $y(t+\tau)$ and $x(t+\tau)$ it is enough to know the current $x(t)$ and the current and future inputs $u(t : t+\tau)$

Example

Rechargeable flashlight:

- state $=$ level of charge of the battery & on / off button
- output $=$ how much light the device is producing

"the current value of the state contains all the information necessary to forecast the future evolution of the outputs and of the state. I.e., to compute future $y(t+\tau)$ and $x(t+\tau)$ it is enough to know the current $x(t)$ and the current and future inputs $u(t:t+\tau)$"

State space representations - Notation

$$u_1, \ldots, u_m = \text{inputs}$$

$$x_1, \ldots, x_n = \text{states}$$

$$y_1, \ldots, y_p = \text{outputs}$$

State space representations - Notation

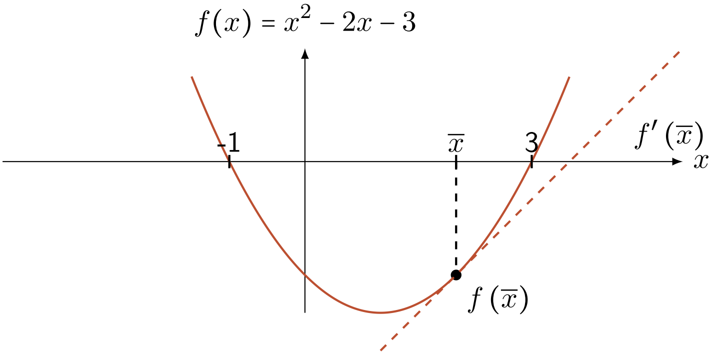
$$
\begin{aligned}
\dot{x}_1 &= f_1\,(\,x_1,\ldots,x_n,\; u_1,\ldots,u_m\,)\\
&\vdots\\
\dot{x}_n &= f_n\,(\,x_1,\ldots,x_n,\; u_1,\ldots,u_m\,)\\
y_1 &= g_1\,(\,x_1,\ldots,x_n,\; u_1,\ldots,u_m\,)\\
&\vdots\\
y_p &= g_p\,(\,x_1,\ldots,x_n,\; u_1,\ldots,u_m\,)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}\,(\boldsymbol{x},\boldsymbol{u})\\
\boldsymbol{y} &= \boldsymbol{g}\,(\boldsymbol{x},\boldsymbol{u})
\end{aligned}
$$

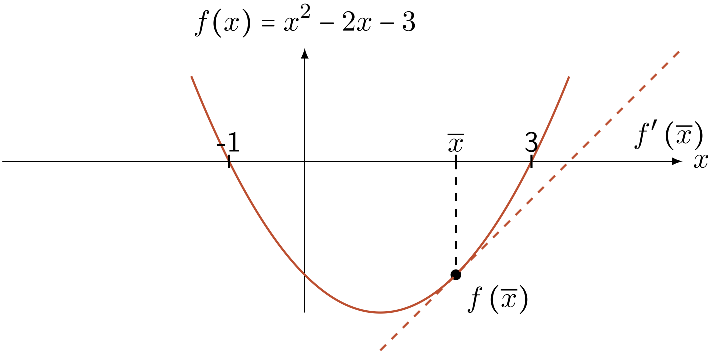
- $\boldsymbol{f}$ = state transition map
- $\boldsymbol{g}$ = output map

?

Linearization - what does it mean?



Linearization - what does it mean?



$$f(x) \approx f(\overline{x}) + \left.\frac{\partial f}{\partial x}\right|_{\overline{x}} (x - \overline{x})$$

(but the approximation is valid only close to the linearization point)

Linearization - what does it mean?

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned} \quad \mapsto \quad \begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} + D\boldsymbol{u} \end{aligned}$$

linearize $\Longrightarrow$ *approximate!*

Discussion: why do we linearize nonlinear systems?

Discussion: where do we linearize nonlinear systems?

Preliminaries: Taylor series

$$f \in C^M(\mathbb{R}) \qquad \Longrightarrow \qquad f(x) \approx \sum_{m=0}^{M} \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m$$

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multivariable extension $=$ less neat formulas, but we will see them!

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multivariable extension $=$ less neat formulas, but we will see them!

the most important case for our purposes:

$$\boldsymbol{f} \in C^1(\mathbb{R}^n) \qquad \Longrightarrow \qquad \boldsymbol{f}(\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}_0) + \nabla \boldsymbol{f}\big|_{\boldsymbol{x}_0}(\boldsymbol{x} - \boldsymbol{x}_0)$$

Discussion (yes, again): where do we linearize nonlinear systems?

Linearization procedure - continuous time systems

$$\left(\boldsymbol{x}_{\mathsf{eq}}, \boldsymbol{u}_{\mathsf{eq}}\right) \text{ equilibrium} \implies \boldsymbol{f}\left(\boldsymbol{x}_{\mathsf{eq}}, \boldsymbol{u}_{\mathsf{eq}}\right) = 0$$

Linearization procedure - continuous time systems

$$(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) \text{ equilibrium} \implies \boldsymbol{f}(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) = 0$$

Procedure (assuming that the Taylor expansion exists):

1. consider $\boldsymbol{x} = \boldsymbol{x}_{\text{eq}} + \Delta\boldsymbol{x}$
2. apply $\boldsymbol{f}(\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}_0) + \nabla\boldsymbol{f}|_{\boldsymbol{x}_0}(\boldsymbol{x} - \boldsymbol{x}_0)$ to the point $\boldsymbol{x}_0 = \boldsymbol{x}_{\text{eq}}$

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$$\implies \qquad \frac{\partial\left(\boldsymbol{x}_{\text{eq}} + \Delta\boldsymbol{x}\right)}{\partial t} \approx \boldsymbol{f}\left(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) + \nabla\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{u} \end{bmatrix}$$

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Linearization procedure - continuous time systems

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- $\nabla\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} = \begin{bmatrix} \nabla_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) & \nabla_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \end{bmatrix}_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}}$

Linearization procedure - continuous time systems

$$(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) \text{ equilibrium} \quad \Longrightarrow$$

$$\Delta \dot{\boldsymbol{x}} \approx \nabla_{\boldsymbol{x}} \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta \boldsymbol{x} + \nabla_{\boldsymbol{u}} \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta \boldsymbol{u}$$

And for $\boldsymbol{y}$?

$$\boldsymbol{y} = \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)$$

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Linearization procedure - continuous time systems

$$\begin{cases} \dot{\boldsymbol{x}} & = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\ \boldsymbol{y} & = \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right) \end{cases}$$

Linearization procedure - continuous time systems

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$$\mapsto$$

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Linearization procedure - continuous time systems

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$$\downarrow$$

$$\begin{cases} \Delta\dot{\boldsymbol{x}} &= A\Delta\boldsymbol{x} + B\Delta\boldsymbol{u} \\ \Delta\boldsymbol{y} &= C\Delta\boldsymbol{x} + D\Delta\boldsymbol{u} \end{cases}$$

Summary

$$\dot{\boldsymbol{x}} = f\left(\boldsymbol{x}, \boldsymbol{u}\right)$$

1. choose an opportune point $\boldsymbol{x}_0, \boldsymbol{u}_0$
2. linearize around $\boldsymbol{x}_0, \boldsymbol{u}_0$:

$$\dot{\boldsymbol{x}}_0 + \Delta\dot{\boldsymbol{x}} \approx \boldsymbol{f}\left(\boldsymbol{x}_0, \boldsymbol{u}_0\right) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_0, \boldsymbol{u}_0} \Delta\boldsymbol{x} + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}_0, \boldsymbol{u}_0} \Delta\boldsymbol{u}$$

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important: if $\boldsymbol{x}_0, \boldsymbol{u}_0$ = *equilibrium then* $\dot{\boldsymbol{x}}_0 = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0) = 0$

Linearization - Example

electrostatic microphone:

- q = capacitor charge
- h = distance of armature from its natural equilibrium
- $\boldsymbol{x} = \left[q, h, \dot{h}\right]$

- R = circuit resistance
- E = voltage generated by the generator (constant)
- C = capacity of the capacitor
- m = mass of the diaphragm + moved air
- k = mechanical spring coefficient
- β = mechanical dumping coefficient
- u_1 = incoming acoustic signal

Linearization - Example

electrostatic microphone:

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- h = distance of armature from its natural equilibrium
- $\boldsymbol{x} = \left[q, h, \dot{h}\right]$

$$\begin{cases} \dot{x}_1 &= -\dfrac{1}{Ra} x_1 \left(L + x_2\right) + \dfrac{E}{R} \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -\dfrac{\beta}{m} x_3 - \dfrac{k}{m} x_2 - \dfrac{x_1^2}{2am} + \dfrac{1}{m} u_1 \end{cases}$$

Linearization - Example

1-st step: compute the equilibria

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2-nd step: compute the matrices

$$A = \nabla_{\boldsymbol{x}} \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \quad B = \nabla_{\boldsymbol{u}} \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \quad C = \nabla_{\boldsymbol{x}} \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \quad D = \nabla_{\boldsymbol{u}} \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}}$$

?

Summarizing, part 1:

- to find the equilibria of a system we need to solve $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = \mathbf{0}$

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- each linearized model $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ is more or less valid only in a neighborhood of $(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}})$. Moreover the size of this neighborhood depends on the curvature of $\boldsymbol{f}$ around that specific equilibrium point

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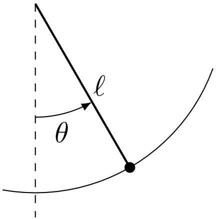
linearization = a very useful tool to do
analysis and design of control systems

Another example: the pendulum



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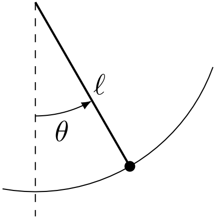

First step: equations of motion:

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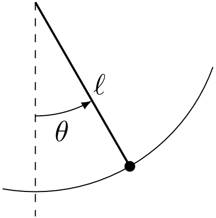

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resulting dynamics: $ml\ddot{\theta} = -mg\sin(\theta) - f\ell\dot{\theta} + \dfrac{u}{\ell}$

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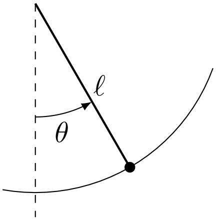

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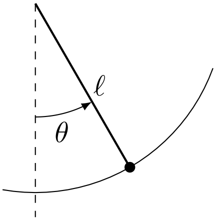

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$$\begin{aligned}
\dot{x}_1 &= x_2 \\
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\end{aligned}$$

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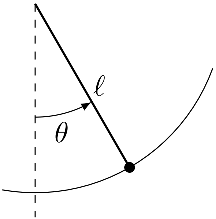

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$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin(x_1) - \frac{f}{m}x_2 + \frac{1}{m\ell^2}u$$

for $u = 0$ (for simplicity)

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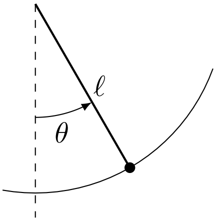

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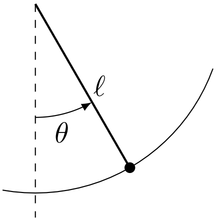

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$$\begin{cases} 0 = x_2 \\ 0 = -\dfrac{g}{\ell}\sin(x_1) - \dfrac{f}{m}x_2 \end{cases} \implies \quad x_{\mathsf{eq}1} = n\pi,\ x_{\mathsf{eq}2} = 0$$

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Equilibrium $\boldsymbol{x}_{\mathsf{eq}\alpha} = \mathbf{0}$, $u = 0$ implies

$$A = \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{\boldsymbol{x}_{\mathsf{eq}\alpha}} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{\ell} & -\dfrac{f}{m} \end{bmatrix}$$

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?

Roadmap

- obvious properties
- simple examples
- understanding through generalizing the simple examples
- some considerations about control of nonlinear systems

Obvious fact: linearizing around an equilibrium keeps that point an equilibrium

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned} \quad \mapsto \quad \begin{aligned} \dot{\widetilde{\boldsymbol{x}}} &= A\widetilde{\boldsymbol{x}} + B\widetilde{\boldsymbol{u}} \\ \widetilde{\boldsymbol{y}} &= C\widetilde{\boldsymbol{x}} + D\widetilde{\boldsymbol{u}} \end{aligned}$$

with

$$\begin{cases} \boldsymbol{x} &= \boldsymbol{x}_{\text{eq}} + \widetilde{\boldsymbol{x}} \\ \boldsymbol{u} &= \boldsymbol{u}_{\text{eq}} + \widetilde{\boldsymbol{u}} \\ \boldsymbol{y} &= \boldsymbol{y}_{\text{eq}} + \widetilde{\boldsymbol{y}} \end{cases}$$

Thus if $\boldsymbol{x}_{\mathsf{eq}}, \boldsymbol{u}_{\mathsf{eq}}$ was an equilibrium for the nonlinear system, it is still an equilibrium for the linearized one. *But if it was a stable one before, will it still be a stable one after?*

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this lesson = answering this question

Starting from the basics: the scalar case

$$\dot{x} = f(x) = x^2 - 2x - 3 = (x-3)(x+1) = 0 \qquad \text{equilibria: } \begin{cases} x_{\mathsf{eq}\alpha} = -1 \\ x_{\mathsf{eq}\beta} = 3 \end{cases}$$

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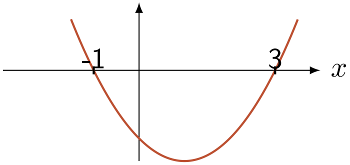

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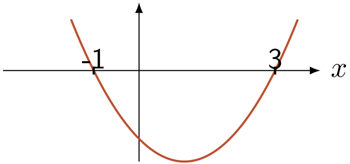

Analysing $x_{\mathsf{eq}\alpha} = -1$:

- $x < -1$ implies $\dot{x} = f(x) > 0$ implies x grows
- $x > -1$ implies $\dot{x} = f(x) < 0$ implies x shrinks *(but only locally)*

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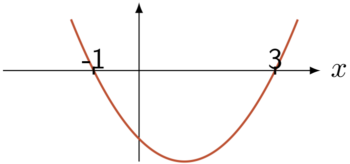

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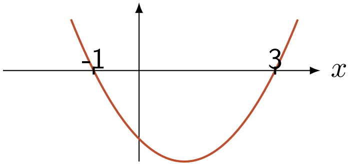

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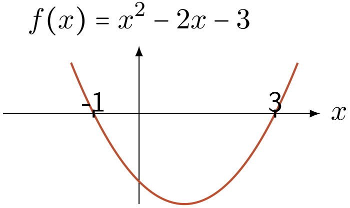

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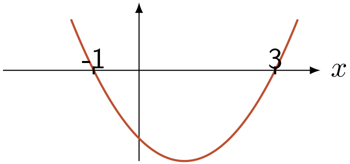

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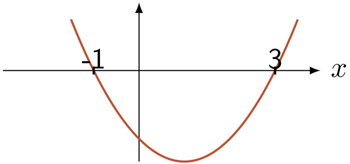

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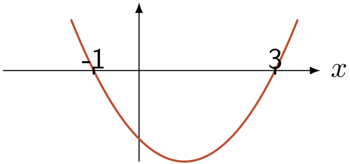

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How do we generalize the previous concepts?

$$\dot{x} = f(x) \quad f(x_{\mathsf{eq}}) = 0, \quad \mapsto \quad \dot{\widetilde{x}} = a_{x_{\mathsf{eq}}} \widetilde{x} \quad \text{with} \quad a_{x_{\mathsf{eq}}} = \left.\frac{\partial f}{\partial x}\right|_{x_{\mathsf{eq}}}$$

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"$a < 0$ implies asymptotically stable"

has been our mantra up to now!

How do we generalize even more?

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with the stability of A something that we will see when we do the linear algebra part of the course

Example

$$\dot{\boldsymbol{x}} = \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} x_2 \\ 1 - x_1^2 - x_2 \end{bmatrix}$$

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Linearization around a generic point:

$$\dot{\widetilde{\boldsymbol{x}}} = A_{\boldsymbol{x}_{\text{eq}}} \widetilde{\boldsymbol{x}} = \begin{bmatrix} 0 & -1 \\ -2x_1 & -1 \end{bmatrix} \widetilde{\boldsymbol{x}}$$

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spoiler (will see this extensively with the "linear algebra" part: the eigenvalues of A will be the poles of the system!

Example (continuation)

<u>“the eigenvalues of $A_{\boldsymbol{x}_{\text{eq}}}$ are the poles of the system”</u>

$$\begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} \implies \text{eigenvalues} = \{-2;\ 1\}$$

$$\begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix} \implies \text{eigenvalues} = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$$

Discussion: how are the modes of the linearized system around equilibrium α? And around equilibrium β?

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Discussion: how are the modes of the linearized system around equilibrium α? And around equilibrium β?

very important message: this implicitly says that studying linearized systems gives information about the nonlinear ones!

Summarizing

- general approach = start with computing the equilibria for the original nonlinear system, get the corresponding $A_{\boldsymbol{x}_{\text{eq}}}$ matrix for each equilibrium $\boldsymbol{x}_{\text{eq}}$, and analyse the stability properties of that $A_{\boldsymbol{x}_{\text{eq}}}$ matrix
- if $A_{\boldsymbol{x}_{\text{eq}}}$ is asymptotically stable, then the original equilibrium $\boldsymbol{x}_{\text{eq}}$ is locally asymptotically stable
- if $A_{\boldsymbol{x}_{\text{eq}}}$ is unstable, then the original equilibrium $\boldsymbol{x}_{\text{eq}}$ is unstable
- if $A_{\boldsymbol{x}_{\text{eq}}}$ is simply stable, then we cannot say anything about the original equilibrium $\boldsymbol{x}_{\text{eq}}$ and we need to do other types of analyses (in later-on courses!)
- in any case the considerations are local considerations, valid only in the neighborhood of $\boldsymbol{x}_{\text{eq}}$

Some philosophical considerations

- sometimes piecewise linearizing systems is a way to deal with nonlinear dynamics, even if this is not the most elegant approach to control
- you will do nonlinear control in later on courses; feedback linearization, one of the approaches, is very powerful
- `https://www.youtube.com/watch?v=uhND7Mvp3f4` ← this is done through classical nonlinear control, not data driven one

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