TTK4225 - Systems Theory, Autumn 2020

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The spaces associated to a matrix

Roadmap

- rank and range
- determinants
- kernel
- connections among the various concepts

Recall:

$\mathrm{span}\,(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle$ = set of all the linear combinations of these vectors

$\mathrm{range}\,(A)$ = span of the columns of A

dimension of a space: max. number of linearly independent vectors

Just to make the importance of the concepts clear:

when does this system have a solution?

$$A\boldsymbol{x} = \begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \boldsymbol{y}$$

(Column) Rank of a matrix

$$\operatorname{rank}(A) = \operatorname{rank}\left(\begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix}\right) = \text{ number of linearly independent columns}$$

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Important result: column-rank $=$ row-rank *(i.e., there are as many linearly independent rows as linearly independent columns)*

$$\operatorname{rank}(A) = \operatorname{rank}\left(A^\top\right) = \operatorname{rank}\left(A^\top A\right) = \operatorname{rank}\left(AA^\top\right)$$

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Example: what is the maximal rank of $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$?

Reconnecting with automatic control

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

$\implies$ structure of A determines how the time derivative $\dot{\boldsymbol{x}}$ is, and how the time derivative is determines the stability and time-evolution properties of the system.

Reconnecting with automatic control

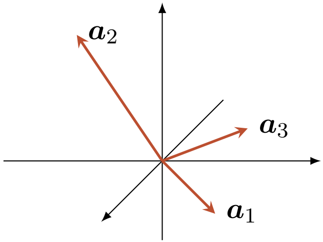
$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

$\Longrightarrow$ structure of A determines how the time derivative $\dot{\boldsymbol{x}}$ is, and how the time derivative is determines the stability and time-evolution properties of the system. E.g.,

$$\mathrm{span}\,(A) = \begin{bmatrix} +1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad \text{if } x_1 \text{ grows then } x_2 \text{ diminishes, and viceversa}$$

Determinant of a square matrix

$$\det(A) = \det\left(\begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix}\right) = \text{(signed) volume of the parallelepiped defined by } \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$$



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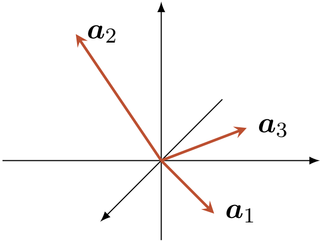
Remember: $\begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix}$ represent where the elements of the basis are mapped into



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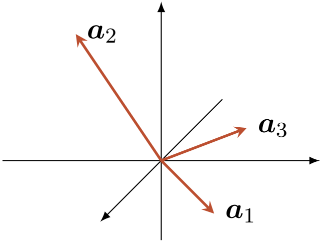

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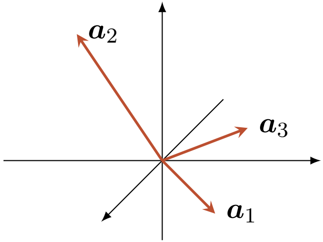
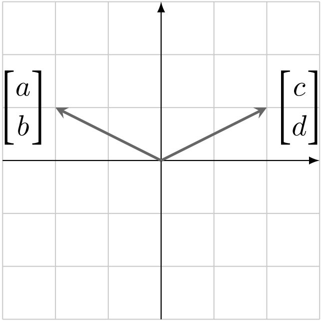


the determinant is a property of the linear transformation $\mathcal{A}$*,*
thus if T *is a change of basis then* $det(A) = det\left(TAT^{-1}\right)$*,*
since changing the basis does not change the underlying transformation

Likely the unique (other) case you should remember on how to compute determinants

$$\det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = ad - bc$$



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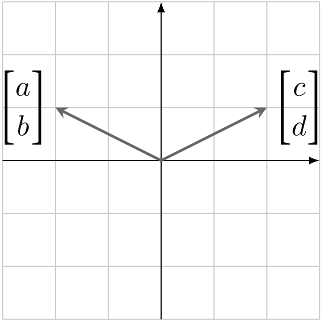

Which other case do you know? (you know for sure already one more)

Determinants and invertibility of linear maps

Immediate implications:

$$\det(A) \neq 0 \quad \Leftrightarrow \quad \mathcal{A} \text{ invertible}$$

$$\det(A) = 0 \quad \Leftrightarrow \quad \mathcal{A} \text{ not-invertible}$$

Why is invertibility important?

because if you want to solve $A\boldsymbol{x} = b$ for generic b then you need A^{-1}

Connections between the determinant and the rank of a square matrix

if $A \in \mathbb{R}^{n \times n}$ then $\mathrm{rank}\,(A) = n$ implies that the columns / rows of A are linearly independent

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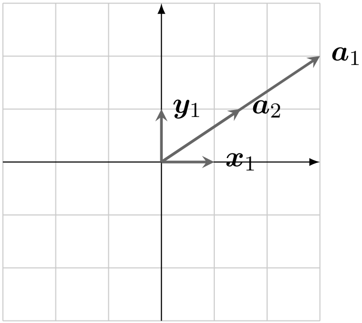
What does it mean that the columns are linearly dependent?



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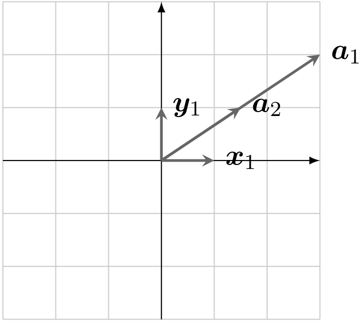
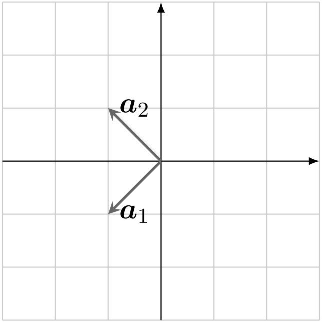


indeed, in this case we cannot "un-map"...

Summary until now and examples: $A \in \mathbb{R}^{2\times 2}$



determinant $=$ *area* spanned by the columns of A

- if $\mathrm{rank}\,(A) = 2$ then the column vectors span an area
- if $\mathrm{rank}\,(A) = 1$ then the column vectors span a line
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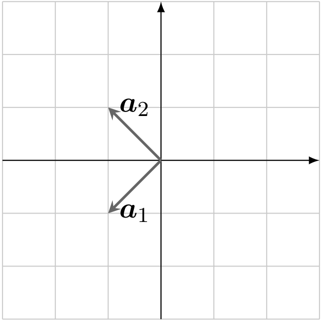

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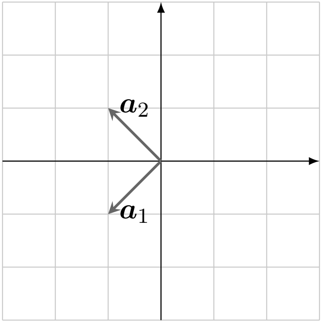

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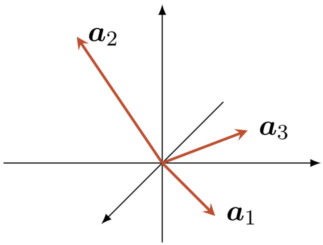
Summary until now and examples: $A \in \mathbb{R}^{3\times 3}$



determinant $=$ *volume* spanned by the columns of A

- if $\operatorname{rank}(A) = 3$ then the column vectors span a volume
- if $\operatorname{rank}(A) = 2$ then the column vectors span an area
- if $\operatorname{rank}(A) = 1$ then the column vectors span a line
- if $\operatorname{rank}(A) = 0$ then the column vectors span nothing

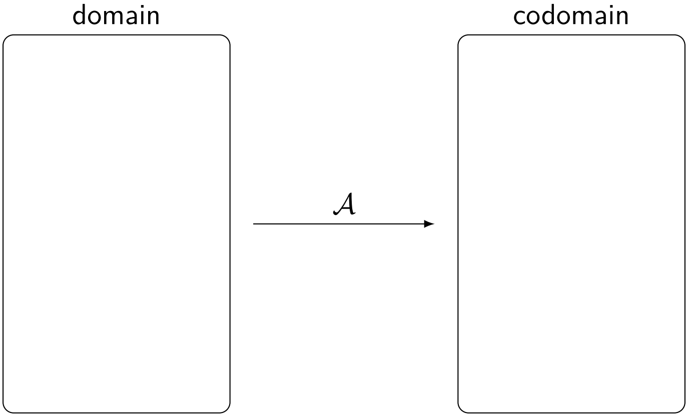
?

Kernel (or null-space) of a matrix $A \in \mathbb{R}^{n \times m}$

$$\ker(A) = \{\boldsymbol{x} \in \mathbb{R}^m \text{ s.t. } A\boldsymbol{x} = \mathbf{0}\}$$

importance: it defines the space of the equilibria of the autonomous system

How do we characterize $\ker(A)$ geometrically?



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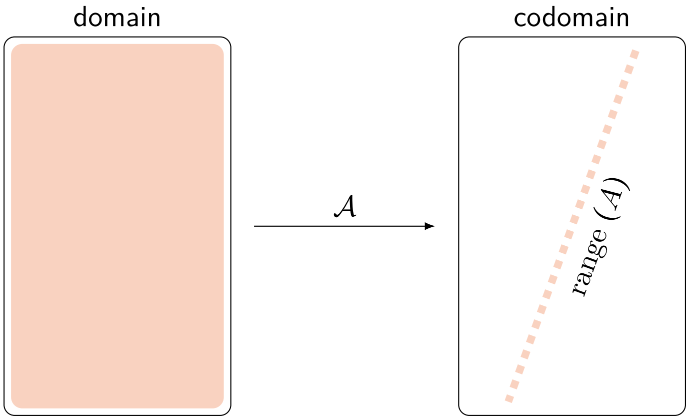

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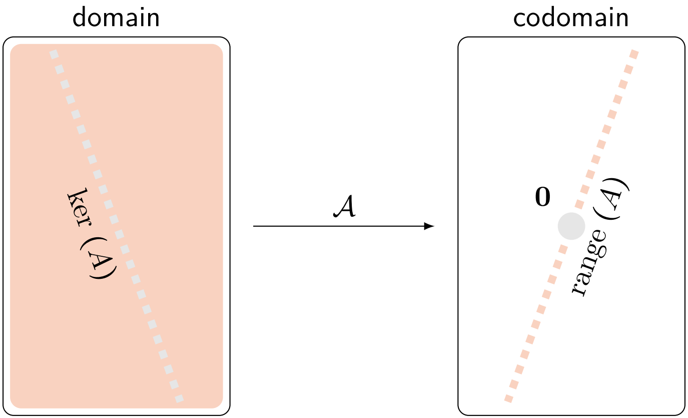
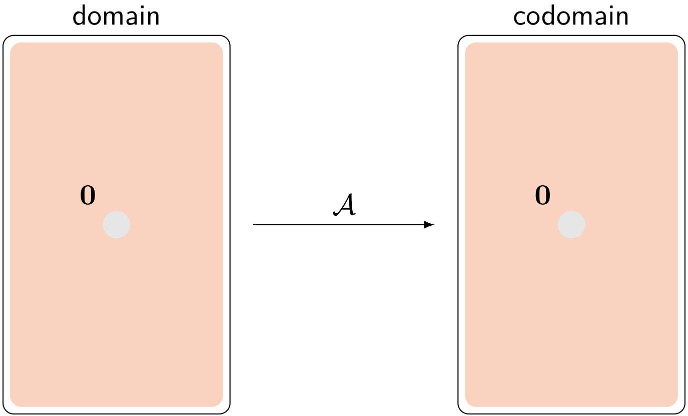
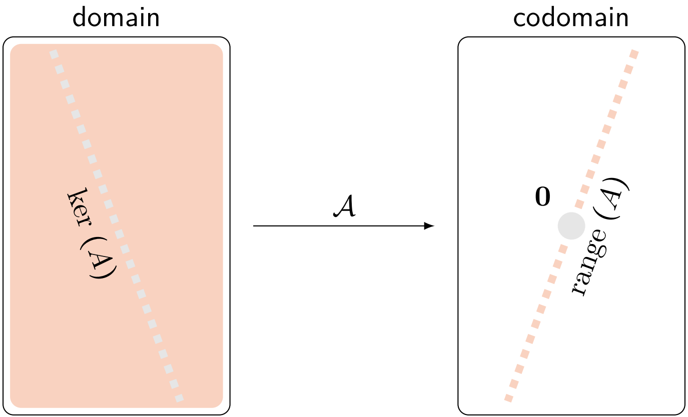


All the possible situations, geometrically



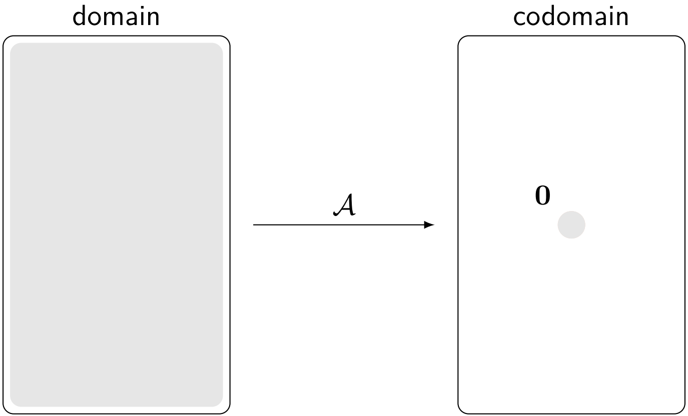
case 1: the kernel is only the $\mathbf{0}$ in the domain

All the possible situations, geometrically



case 2: the kernel is a *subspace* in the domain

All the possible situations, geometrically

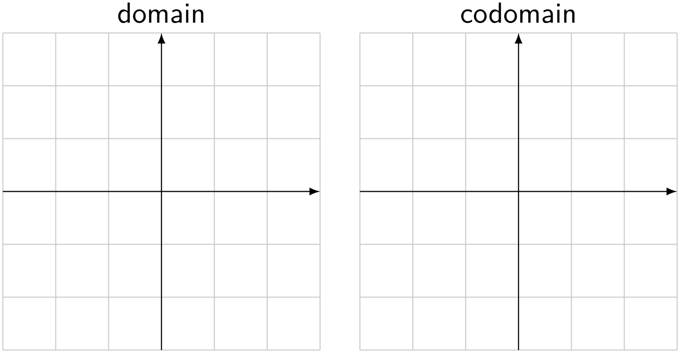


case 3: the kernel is the whole domain

(and this means that $\mathrm{range}\,(A) = \{\mathbf{0}\}$)

Important result: $\ker(A)$ is orthogonal to the rows of A

$$A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$$



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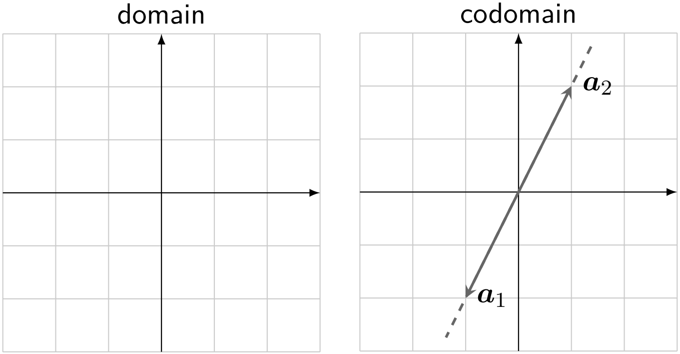

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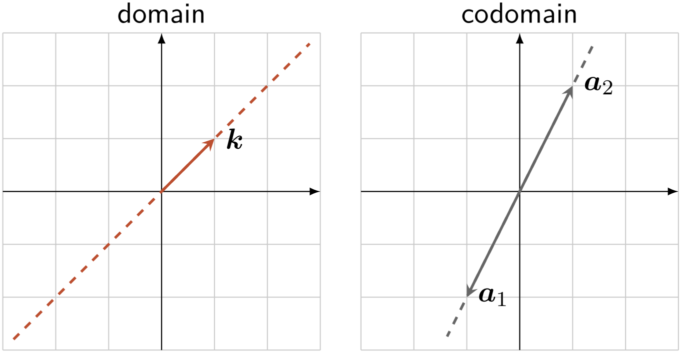

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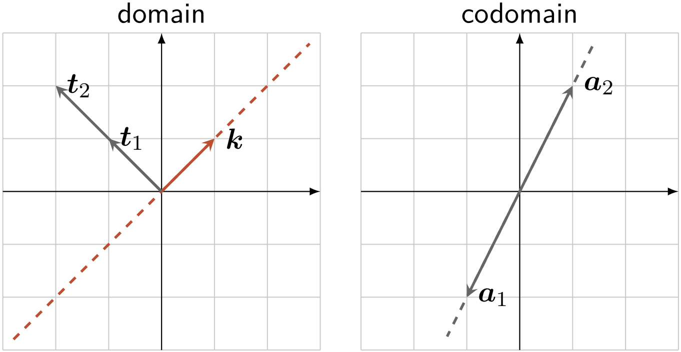
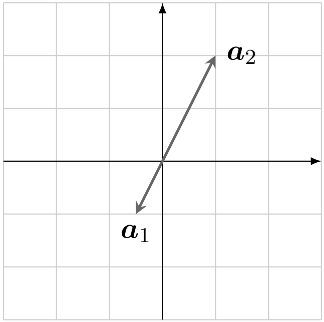


How to visualize the spaces associated to a matrix $A \in \mathbb{R}^{2\times 2}$

$$A = \begin{bmatrix} -0.5 & 1 \\ -1 & 2 \end{bmatrix}$$



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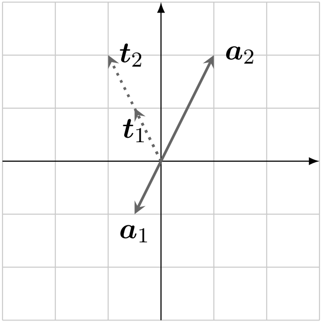

note that in general the fact that $ker\,(A) \neq \{\mathbf{0}\}$ *does not imply* A *to be a projection matrix!*

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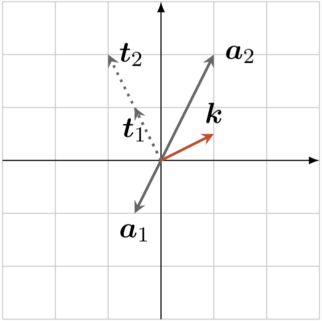

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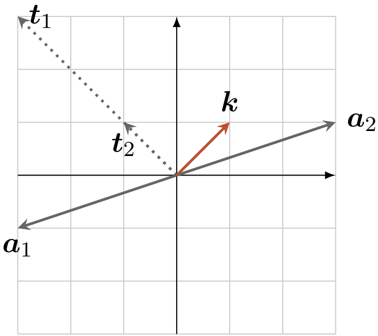
Kernel (or null-space) of a matrix $A \in \mathbb{R}^{n\times m}$: examples

$$A = \begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix}$$



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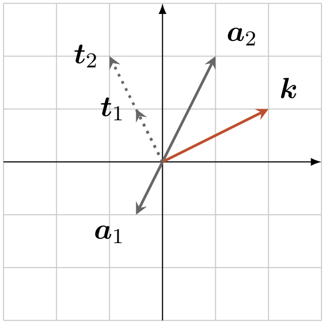
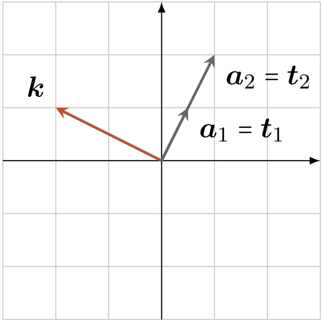


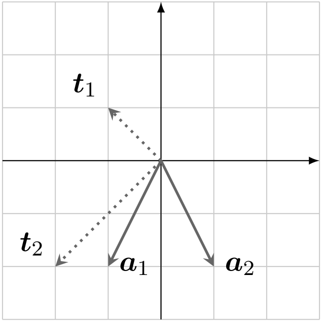
Kernel (or null-space) of a matrix $A \in \mathbb{R}^{n \times m}$: examples

$$A = \begin{bmatrix} 0.5 & 1 \\ 1 & 2 \end{bmatrix}$$ *(symmetric!)*



Kernel (or null-space) of a matrix $A \in \mathbb{R}^{n \times m}$: examples

$$A = \begin{bmatrix} -1 & +1 \\ -2 & -2 \end{bmatrix}$$



Summarizing

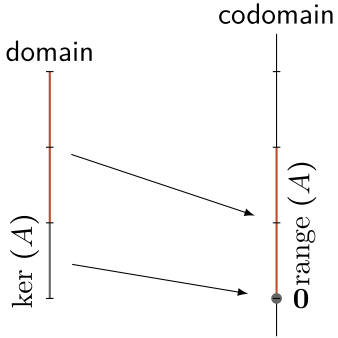
$\ker(A)$ orthogonal to the rows of A

(that implies also $\ker(A) \perp \operatorname{range}\left(A^T\right)$)

An interesting result

(a.k.a. “rank-nullity theorem”)

$$\dim\left(\ker\left(A\right)\right) = \text{number of columns of } A - \operatorname{rank}\left(A\right)$$



(this helps verifying immediately if the kernel is non-trivial)

Alternative viewpoint on the kernel of A

$$A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_m \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{a}_1 \boldsymbol{x} \\ \vdots \\ \boldsymbol{a}_m \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{if and only if} \quad \boldsymbol{a}_i \perp \boldsymbol{x} \ \forall i$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

- $\ker(A) = ?$
- $\operatorname{range}(A) = ?$

bigger matrices, and needing to compute ranges, determinants, or kernels?
$\rightarrow$ use Matlab, python, Wolfram Alpha, whatever

Some useful general rules

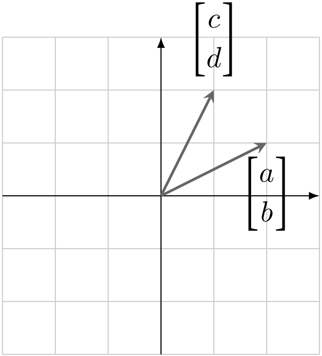
$$
\begin{aligned}
\left(A^\top\right)^\top &= A \\
(A+B)^\top &= A^\top + B^\top \\
(cA)^\top &= cA^\top \\
(AB)^\top &= B^\top A^\top \\
\det\left(A^\top\right) &= \det(A) \\
\left(A^{-1}\right)^\top &= \left(A^\top\right)^{-1}
\end{aligned}
$$

Eigenvectors, eigenspaces, and eigenvalues of a square matrix

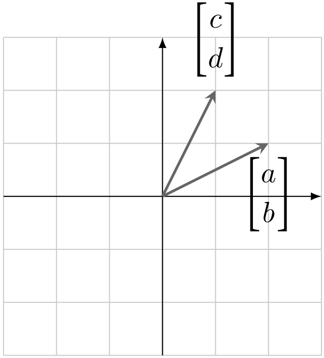
Roadmap

- eigenvectors
- eigenspaces
- eigenvalues
- connections with ranks and determinants

Eigenvectors

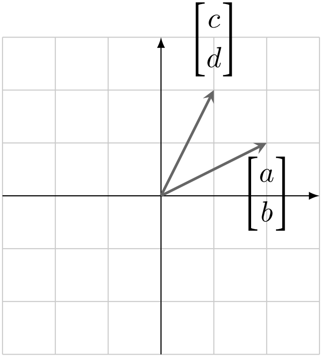


Eigenvectors



are there some directions that get only stretched, i.e., that do not rotate?

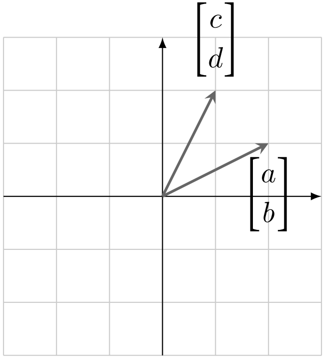
Eigenvectors



are there some directions that get only stretched, i.e., that do not rotate?

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

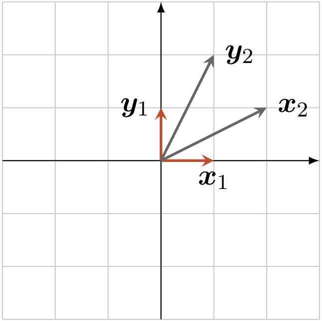
Eigenvectors



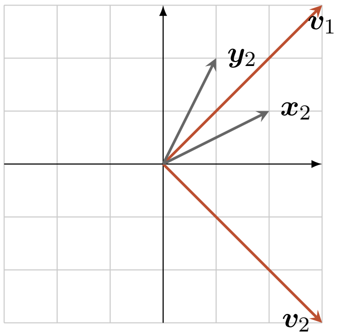
are there some directions that get only stretched, i.e., that do not rotate?

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mapsto \quad \boldsymbol{v}_{\lambda=1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvectors: sometimes you may seem them from the transformation of the hypercube



Why do we like eigenvectors?



because they correspond to situations for which $\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \lambda \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

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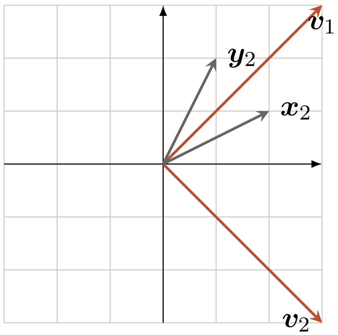

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they are connected to (and actually generalize) the modes of a LTI system

(more information in the next units)

How does one compute eigenvectors?

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- let $\boldsymbol{v} \neq \mathbf{0}$

How does one compute eigenvectors?

- let $\boldsymbol{v} \neq \mathbf{0}$
- verify whether there exists λ such that $A\boldsymbol{v} \stackrel{?}{=} \lambda \boldsymbol{v}$

How does one compute eigenvectors?

- let $\boldsymbol{v} \neq \mathbf{0}$
- verify whether there exists λ such that $A\boldsymbol{v} \stackrel{?}{=} \lambda\boldsymbol{v}$

Important point: if $\boldsymbol{v} \neq \mathbf{0}$ but also $A\boldsymbol{v} = \lambda\boldsymbol{v}$ then $\boldsymbol{v} \in \ker(A - \lambda I)$,

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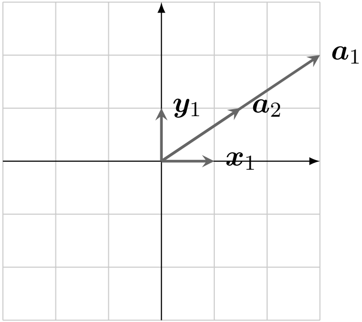
$$\lambda \text{ eigenvalue iff } \det(A - \lambda I) = 0$$

Remember: what does it mean that the determinant of A is zero?

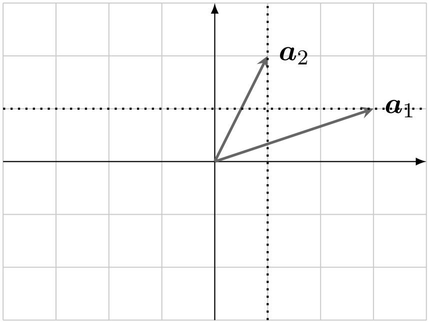
It means that the columns of A are linearly dependent, and this means



What does this mean, geometrically?

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \quad \Longrightarrow \quad (A - \lambda I) = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

question $=$ is $\det(A - \lambda I)$ zero, i.e., searching for stretchings that make *at least* 2 columns of $A - \lambda I$ align



How does one compute eigenvectors, more in general?

$$\lambda \text{ eigenvalue iff } \det(A - \lambda I) = 0$$

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but why is it a polynomial?

In brief, $\det(sI - A)$ is a polynomial because of how determinants are computed

$$\text{thus } A \in \mathbb{R}^{n \times n} \quad \Longrightarrow \quad \det(sI - A) = \prod_{i=1}^{n} (s - \lambda_i) \text{ for opportune } \exists \lambda_1, \ldots, \lambda_n \in \mathbb{C}$$

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note: this is reminiscent of the RTFs in ZPK representation $H(s) = K \dfrac{\prod_j (s - z_j)}{\prod_i (s - p_i)}$

How to find the eigenvalues: numerical example

Definition:

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \quad \Longrightarrow \quad A\boldsymbol{x} - \lambda\boldsymbol{x} = \mathbf{0} \quad \Longrightarrow \quad (A - \lambda I)\,\boldsymbol{x} = \mathbf{0}.$$

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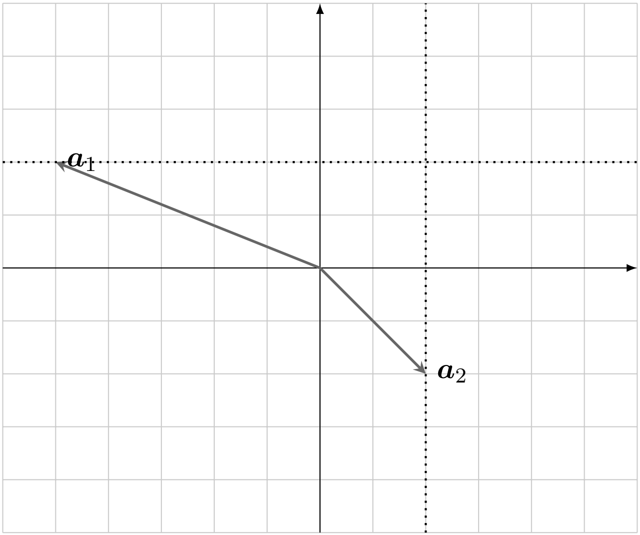
Eigenvalues $= \{-1, -6\}$

The same example, graphically



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How do we find the associated eigenvectors?

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{with} \quad (A - \lambda I) = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \quad \text{and} \quad \lambda_{1,2} = \{-1, -6\}$$

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$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$$\implies x_2 = 2x_1$$

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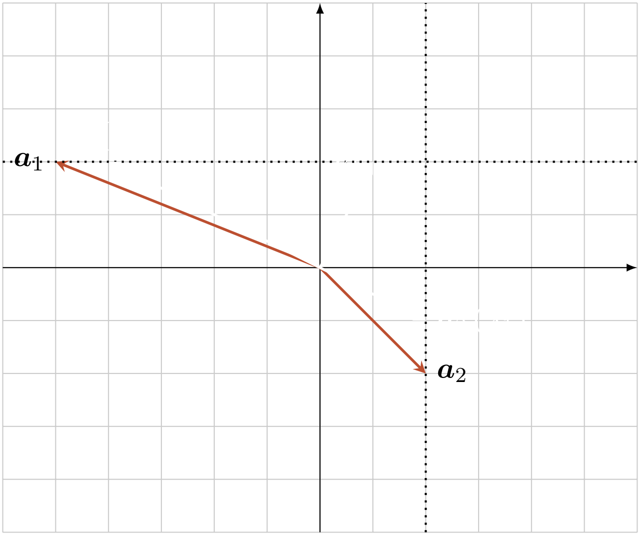

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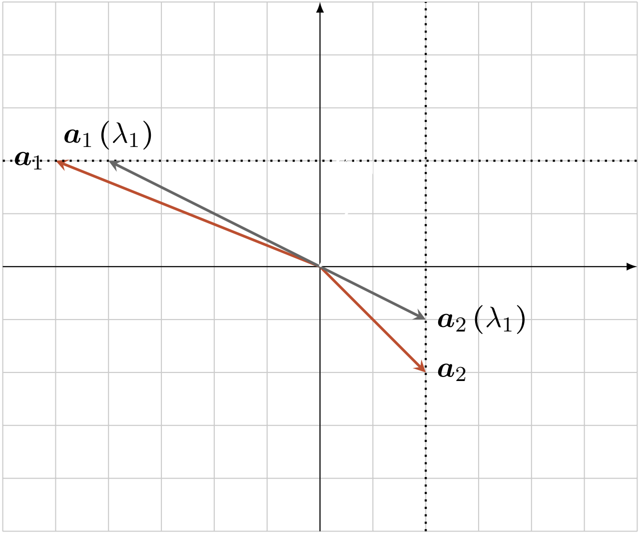

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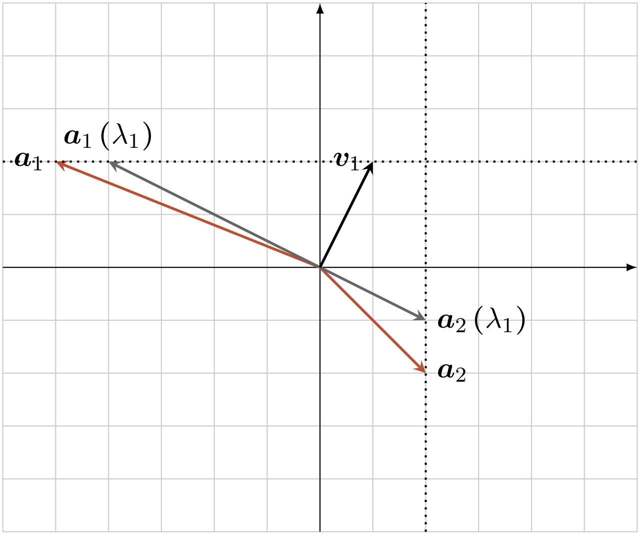

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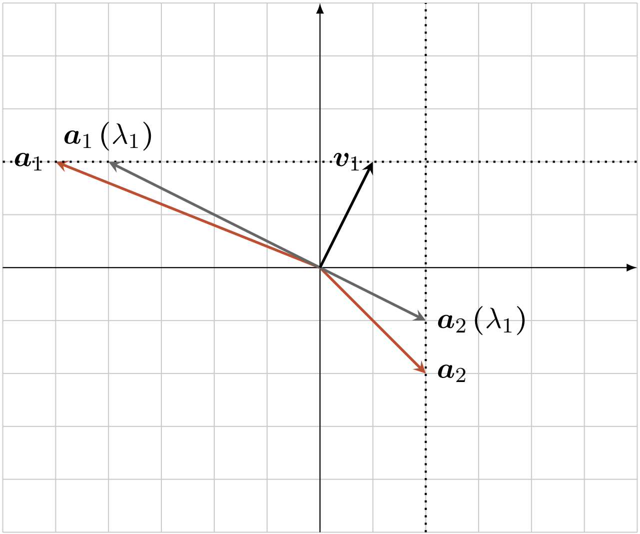

can you see what λ_2 is, and what the corresponding $\boldsymbol{v}_2$ will be?

Summarizing

eigenvectors: directions along which A does not introduce rotations

eigenspaces: set of all the vectors in these directions

eigenvalues: amplification that A causes along the eigenspaces
(remember: along each direction there may be a different amplification!)

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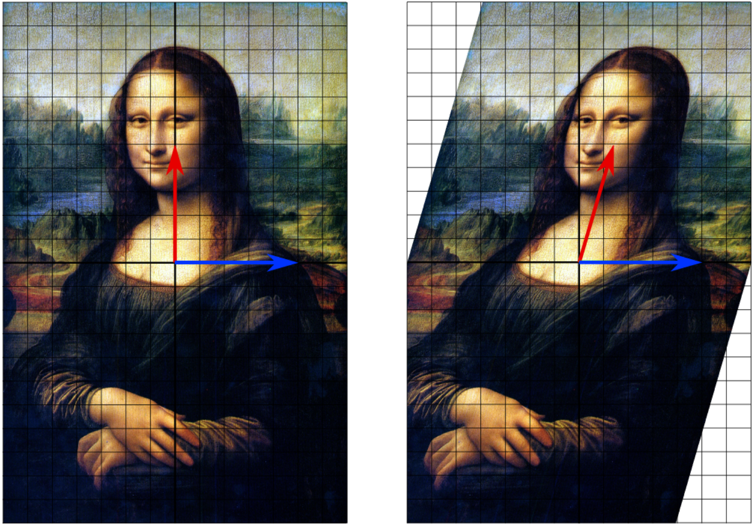
(remember: along each direction there may be a different amplification!)

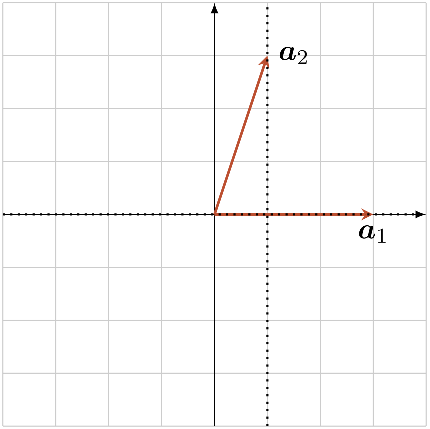
characteristic polynomial: $\det(sI - A)$

algebraic multiplicity of the eigenvalues: the power $\mu(\lambda_i')$ associated to each λ_i in the characteristic polynomial

Discussion: what are the eigenspaces in this case?



Generalizing to the case of a Jordan miniblock (something that will be an extremely important case)



how many “stretchings” can we find so to make the stretched columns align?

Important result

there may be fewer 1-dimensional eigenspaces than columns of A

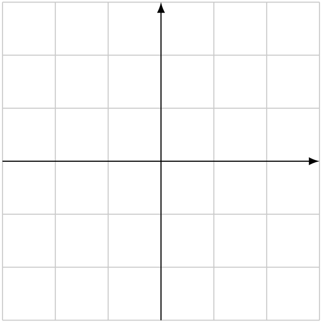
$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

?

Connections between determinant and rank

$$\det(A) = 0 \quad \Leftrightarrow \quad A \text{ rank-deficient}$$

since $\det(A) = 0$ means that the unitary hypercube gets mapped into a degenerate parallelepiped, and this is equivalent to say that A has linearly dependent columns



Connections between determinant and eigenvalues

$$\det(A) = \prod_i \lambda_i$$

i.e., the volume of the mapped parallelepiped is equal to the product of the expansions along the eigenspaces[1]

this is immedate and obvious when considering the Jordan form of A

[1] Kind of imprecise; correct in case the algebraic multiplicity of the eigenvalues is 1. Otherwise they need to be considered with their multiplicities

Connections between determinant and rank

From the previous episodes:

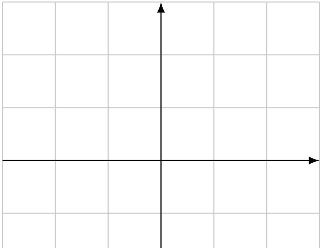
$$\det(A) = 0 \quad \Leftrightarrow \quad A \text{ rank-deficient}$$

and

$$\det(A) = \prod_i \lambda_i$$

thus

$$\exists \lambda_i = 0 \quad \Leftrightarrow \quad A \text{ rank-deficient}$$

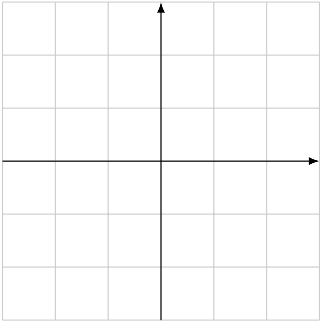


Connections between eigenvalues and invertibility

$$\exists \lambda_i = 0 \quad \Leftrightarrow \quad A \text{ rank-deficient}$$

but also

$$A \text{ rank-deficient} \quad \Leftrightarrow \quad A \text{ not invertible}$$



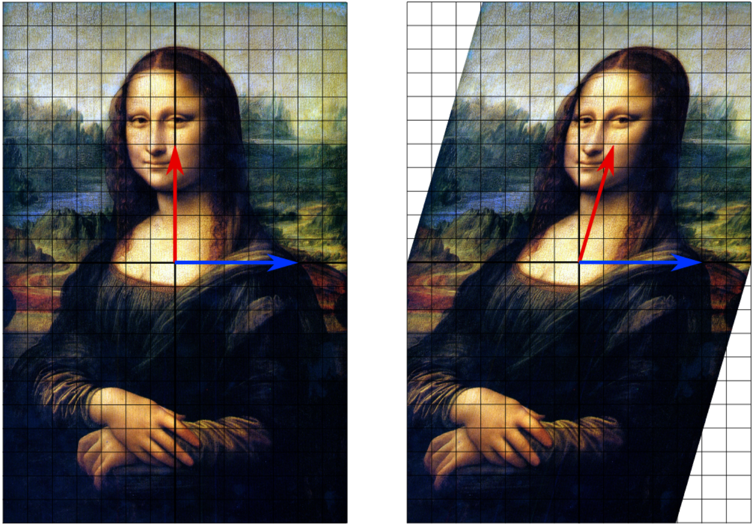
Definitions

singular matrix = non-invertible matrix

non-singular matrix = invertible matrix

?

Do eigenvalues change if we do a change of basis?



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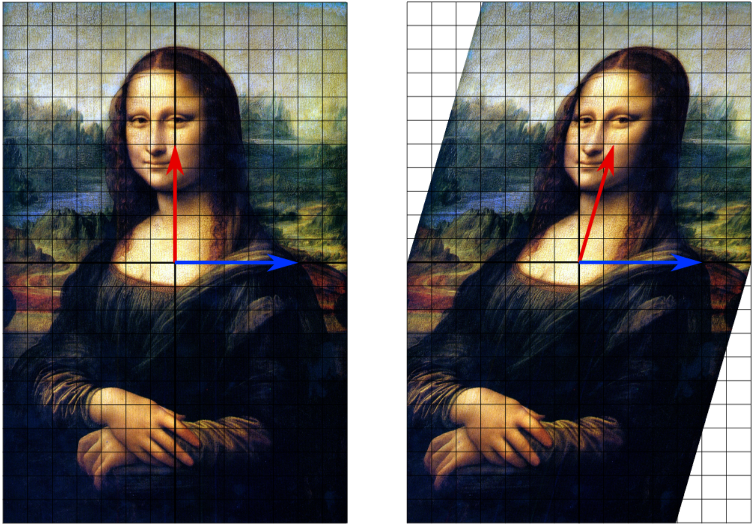

obviously not!

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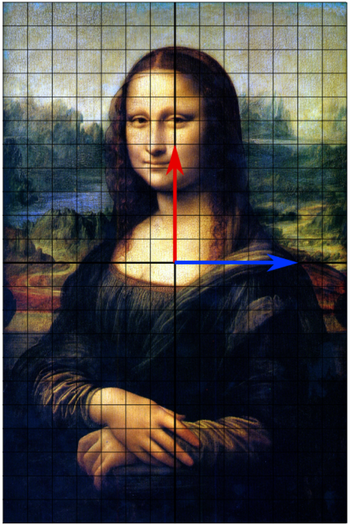

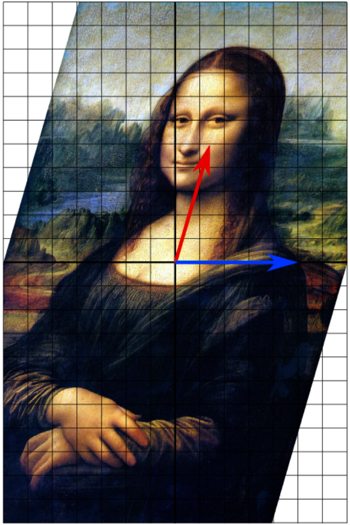

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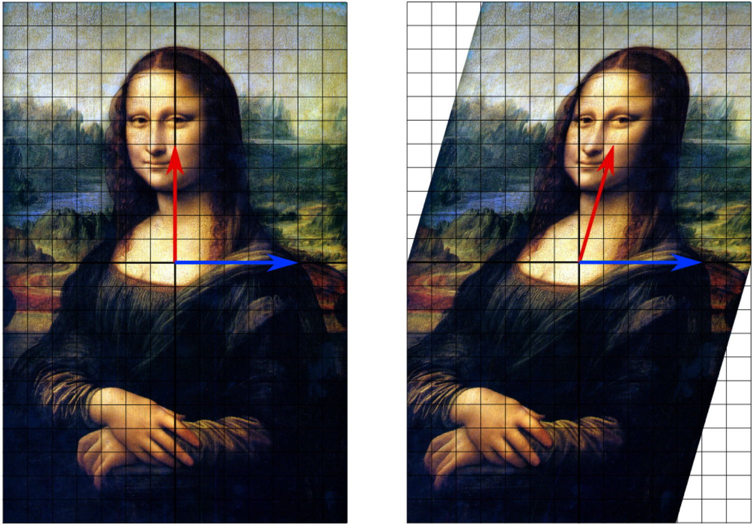

they change "name", but from a physical perspective they are the same object!

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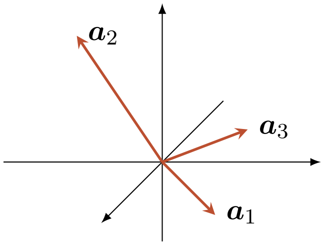
Diagonalization

Roadmap

- what happens if the eigenvectors of A form a basis of $\mathbb{R}^n$?
- what diagonalization means algebraically
- what diagonalization means geometrically
- what diagonalization means in practice

An interesting case: what if the eigenvectors of A form a basis of $\mathbb{R}^n$?



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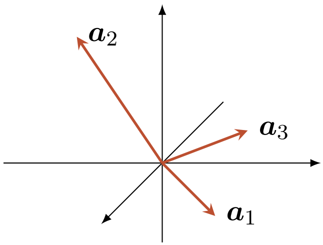

what happens if in this case I choose a new basis formed by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$?

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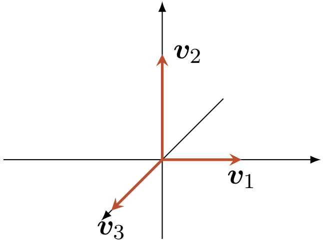

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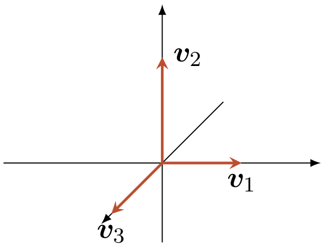

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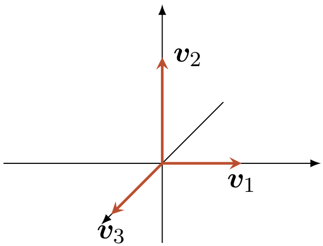


How does $\mathcal{A}$ look like, with respect to this basis?

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

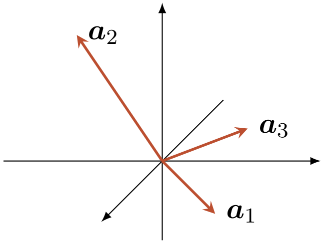
Note that the λ_i's may also be the same! Example:

$$A = \begin{bmatrix} 2.3 & & & \\ & 2.3 & & \\ & & \ddots & \\ & & & 2.3 \end{bmatrix}$$

Diagonalizing a square matrix

hypothesis: A is s.t. there exist $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ linearly independent eigenvectors

thesis: $T = \left[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\right]$ is s.t. $\Lambda = T^{-1}AT = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_n\right)$



Diagonalizing a square matrix: proof that $AT = T\Lambda$

$$AT \overset{(1)}{=} A\,[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n] \overset{(2)}{=} [A\boldsymbol{v}_1, \ldots, A\boldsymbol{v}_n] \overset{(3)}{=} [\lambda_1\boldsymbol{v}_1, \ldots, \lambda_n\boldsymbol{v}_n] \overset{(4)}{=} T\Lambda$$

(1) recall that the columns of T are the eigenvectors

(2) this follows by the geometrical interpretation of matrix-columns multiplications

(3) this is because $\boldsymbol{v}_i$ is an eigenvector

(4) we can rewrite things as a product with a diagonal matrix

What about matrices with multiple eigenvalues?

Example:

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \implies \det(A - sI) = -s^3 - s^2 + 21s + 45 = (s-5)(s+3)^2$$

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$$\Longrightarrow \quad \lambda_1 = 5,\ \lambda_{2,3} = -3$$

Eigenspaces-eigenvectors couples:

$$\left\{\lambda_1, \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right)\right\} \qquad \left\{\lambda_2 = \lambda_3, \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right)\right\}$$

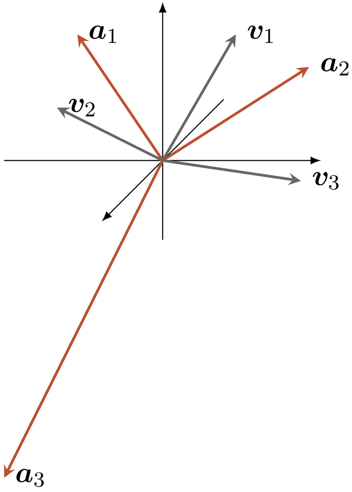
important point: to diagonalize we need n different and linearly independent eigenvectors, not n different eigenvalues

Graphically

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\Lambda = T^{-1}AT = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$



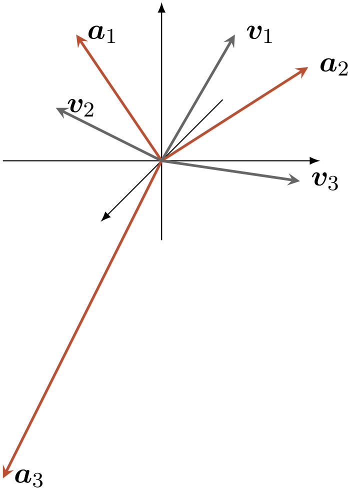
Diagonalization, in numbers

$$A = T\Lambda T^{-1}$$

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0.125 & 0.25 & -0.375 \\ -0.250 & 0.50 & 0.750 \\ 0.125 & 0.25 & 0.625 \end{bmatrix}$$

What does diagonalization mean, graphically?

I look at the world considering as the new axes the eigenspaces



What does diagonalization mean, physically?

Original system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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The same system, but after the change of basis T:

$$\begin{bmatrix} \dot{\widetilde{x}}_1 \\ \dot{\widetilde{x}}_2 \\ \dot{\widetilde{x}}_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \end{bmatrix}$$

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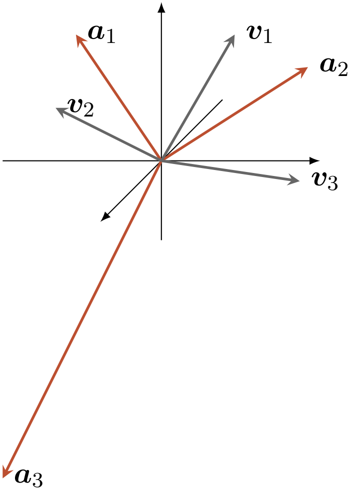
this means that the original system is actually the juxtaposition of 3 independent systems that evolve "ignoring" what is happening in the other ones

Thus diagonalizing = decomposing the dynamics in a set of independent 1-dimensional dynamics

the eigenspaces are where these 1-dimensional dynamics live



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- to be able to diagonalize means to be able to split up a system in independent pieces

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- to be able to diagonalize means to be able to split up a system in independent pieces
- however we can do this diagonalization only if the eigenvectors of A form a basis for $\mathbb{R}^n$, and this is not guaranteed in general

Generalization

Consider

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & & & \\ & \tilde{A}_2 & & \\ & & \ddots & \\ & & & \tilde{A}_k \end{bmatrix};$$

also this means "dividing the system in independent sub-systems"! However "diagonalizing" means finding independent subsystems of dimension 1, while in this general case the dimensions are potentially bigger than 1

?

Towards stranger things: recall that state space representations are ways of expressing LTI systems

$$\ddot{y} + a_1 \dot{y} + a_0 y = b u(t)$$

is equivalent to

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

and thus to

$$\begin{cases} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + Bu \\ y &= C\boldsymbol{x} \end{cases}$$

Towards stranger things: how was this connecting with the first part of the course?

$$\ddot{y} + a_1 \dot{y} + a_0 y = bu \quad \Longrightarrow \quad Y(s) = \frac{bU(s)}{s^2 + a_1 s + a_0} \quad \Longrightarrow \quad \text{modes} = \text{solutions of } s^2 + a_1 s + a_0$$

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with the denominator of the TF equal to $\det(sI - A)$

Towards stranger things: remember this basic fact

$$Y(s) = C\frac{\operatorname{adj}(sI - A)}{\det(sI - A)}BU(s)$$

- changing the basis does not change the characteristic polynomial, thus

$$\det(sI - A) = \det\left(sI - T^{-1}AT\right)$$

(in other words, changing the basis for the state space does not change the poles of the TF, and thus the modes of the LTI system – as it should obviously be)

Stranger things

...but if $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ implies $\det(A - sI) = (s-5)(s+3)^2$ then there is a double pole in -3, corresponding to a mode of the type te^{-3t};

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what is happening here?

Solution (and we will see this in the next unit): the presence or not of the mode te^{-3t} depends on the structure of the eigenspaces of A → we need to study Jordan forms *doing systems theory for LTI systems means studying the inner structure of* $\dot{\boldsymbol{x}} = A\boldsymbol{x}$

?

Jordan forms

Roadmap

- non-diagonalizable matrices
- Jordan forms
- connections with dynamical systems
- summary of the differences between diagonalizable and non-diagonalizable matrices

A small trick, to make things faster

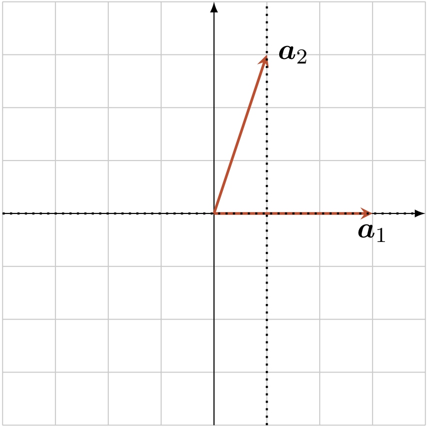
if A is upper triangular or lower triangular then its characteristic polynomial is given by $\prod (s - d_i)$ with the d_i's the elements on the diagonal, i.e.,

$$A = \begin{bmatrix} d_1 & \star & \star & \cdots \\ 0 & d_2 & \star & \cdots \\ 0 & 0 & \ddots & \cdots \\ 0 & 0 & 0 & d_n \end{bmatrix} \implies \det(sI - A) = \prod_i (s - d_i)$$

The case of Jordan miniblocks

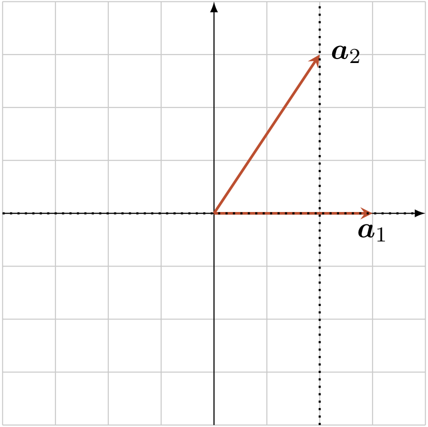
$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \Longrightarrow \quad \text{characteristic polynomial} = (s - \lambda)^2$$

How many 1-dimensional eigenspaces do Jordan miniblocks have?



in this case there is only one "stretching" for which the stretched columns align

Note that this can be generalized to Jordan miniblocks with α instead of 1



(we though like more to write Jordan miniblocks as $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$*)*

"The eigenspaces of Jordan miniblocks have dimension 1": algebraic proof

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad \Longrightarrow \quad \det(sI - A) = (s - \lambda)^3$$

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$$\text{here } (\lambda I - A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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here $\left(\lambda I - A\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

and thus $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \star \\ 0 \\ 0 \end{bmatrix}$

Summarizing

$$A = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda \end{bmatrix}$$

- the eigenspace is 1-dimensional and it is equal to $\ker(\lambda I - A) = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$

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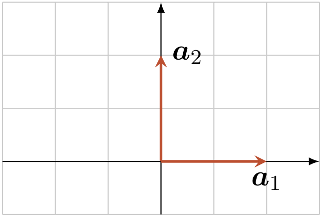
- thus we cannot diagonalize, i.e., we cannot write $A = T \Lambda T^{-1}$
- thus the system $\dot{\boldsymbol{y}} = A\boldsymbol{y}$ cannot be divided into a series of independent 1-dimensional dynamics

?

An example, to make things in practice. System "N":

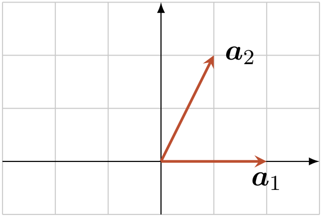
$$
\begin{cases}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \\
y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{cases}
\implies \dim\Big(\ker\left(2I - A\right) \Big) = 2
$$



$\implies$ two independent 1-dimensional systems, each with a mode e^{2t}

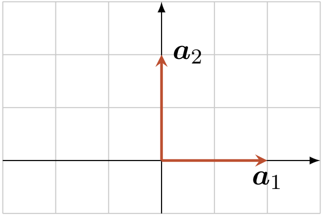
An example, to make things in practice. System "J":

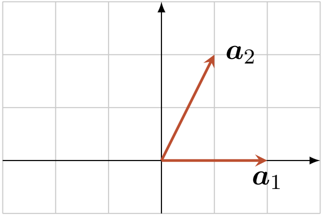
$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \\ y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases} \implies \dim\Big(\ker\left(2I - A\right) \Big) = 1$$



$\implies$ a truly 2-dimensional system, with modes e^{2t} and te^{2t}

Comparing “N” against “J”:





“J” contains an intrinsic shearing that “N” does not contain

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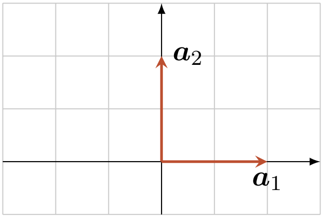
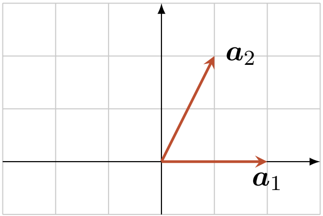


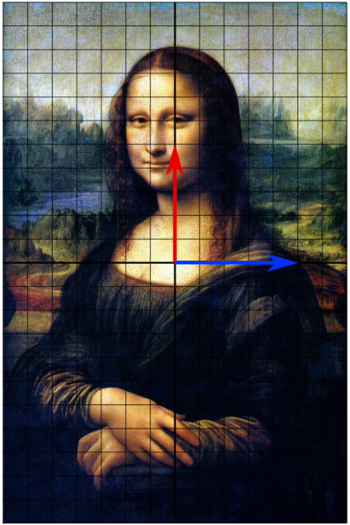


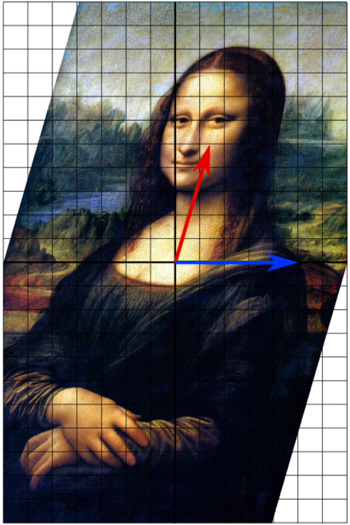
“J” contains an intrinsic shearing that “N” does not contain

(but remember that for the case “N” we are looking at the space through the directions defined by its eigenvectors)

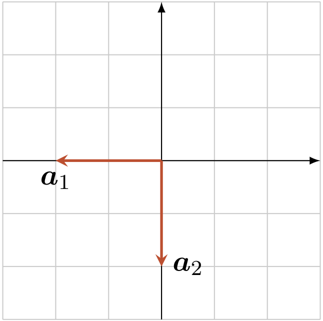
the same applies to $J = \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$ or the higher-dimensions cases

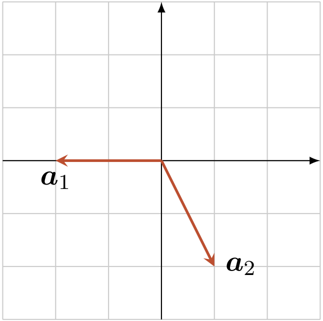
Discussion: is this due to a Jordan map?





Watch out that to have asymptotic stability the eigenvalues must have real part strictly negative!





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Summarizing

$$\det(sI - A) = \prod_{i=1}^{d} (s - \lambda_i)^{\mu(\lambda_i)}$$

- $\det(sI - A) :=$ characteristic polynomial

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Summarizing

$$\det(sI - A) = \prod_{i=1}^{d} (s - \lambda_i)^{\mu(\lambda_i)}$$

- $\det(sI - A)$:= characteristic polynomial
- $\mu(\lambda_i)$:= algebraic multiplicity of λ_i
- $\ker(\lambda_i I - A)$:= eigenspace associated to λ_i

Summarizing

$$\det(sI - A) = \prod_{i=1}^{d} (s - \lambda_i)^{\mu(\lambda_i)}$$

- $\det(sI - A)$:= characteristic polynomial
- $\mu(\lambda_i)$:= algebraic multiplicity of λ_i
- $\ker(\lambda_i I - A)$:= eigenspace associated to λ_i
- $\dim\Big(\ker(\lambda_i I - A)\Big)$:= *geometric multiplicity* of λ_i

$$\boldsymbol{v} \neq \mathbf{0}, \quad A\boldsymbol{v} = \lambda \boldsymbol{v}$$

$$\dim\Big(\ker\left(\lambda_i I - A\right) \Big)$$

$$\det\left(sI - A\right) = \prod_{i=1}^{d} \left(\lambda - \lambda_i'\right)^{\mu\left(\lambda_i'\right)}$$

$$\mu\left(\lambda_i'\right)$$

our aim: understand how these components relate

$\Longrightarrow$ need to go back to the geometric interpretations

(but, before, we need a couple of theoretical results)

Definition (diagonalizable matrix)

A is diagonalizable if $\exists T$ s.t. $T^{-1}AT = \Lambda$ with Λ diagonal

Theorem

A is diagonalizable if and only if A has n linearly independent eigenvectors

Theorem

not all the A's are diagonalizable; e.g., Jordan matrices are not

Theorem (Jordan canonical form)

all the matrices that can not be diagonalized can always be transformed, by using an opportune change of coordinates, to a block diagonal matrix

$$A = \begin{bmatrix} A_1 & & 0 \\ & \dots & \\ 0 & & A_{n'} \end{bmatrix}$$

with $n' < n$ *and at least one block* A_i *of the form*

$$A_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

Example

$$A \quad \mapsto \quad \widetilde{A} = \begin{bmatrix} 2 & & & & & & & & & \\ & 2 & 1 & & & & & & & \\ & & 2 & & & & & & & \\ & & & 2 & 1 & & & & & \\ & & & & 2 & 1 & & & & \\ & & & & & 2 & & & & \\ & & & & & & 3 & & & \\ & & & & & & & 3 & 1 & \\ & & & & & & & & 3 & 1 \\ & & & & & & & & & 3 \end{bmatrix}$$

Example

$$
A \quad \mapsto \quad \widetilde{A} = \begin{bmatrix}
2 & & & & & & & & & \\
& 2 & 1 & & & & & & & \\
& & 2 & & & & & & & \\
& & & 2 & 1 & & & & & \\
& & & & 2 & 1 & & & & \\
& & & & & 2 & & & & \\
& & & & & & 3 & & & \\
& & & & & & & 3 & 1 & \\
& & & & & & & & 3 & 1 \\
& & & & & & & & & 3
\end{bmatrix}
$$

- algebraic multiplicity $=$ dimension of the Jordan block *(since each element on the diagonal adds a term "$(s-\lambda)$" in the characteristic polynomial)*

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- algebraic multiplicity $=$ dimension of the Jordan block *(since each element on the diagonal adds a term "$(s-\lambda)$" in the characteristic polynomial)*
- geometric multiplicity $=$ number of Jordan miniblocks *(since each miniblock adds its own $dim\Big(\ker(2I - A)\Big) = 1$)*

Extremely important facts to remember!!!

Assume T to be a generic change of basis. Then:

1. the eigenvectors and eigenvalues depend only on $\mathcal{A}$, and not on the used basis:
   λ_i eigenvalue of $A \Leftrightarrow \lambda_i$ eigenvalue of $A' = TAT^{-1}$

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   $$\det(\lambda I - A) = \prod_{i=1}^{p} (\lambda - \lambda_i)^{\mu(\lambda_i)} = \det\left(\lambda I - TAT^{-1}\right)$$

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4. the eigenspaces depend only on $\mathcal{A}$, and not on the used basis: $\ker(\lambda_i I - A) = \ker\left(\lambda_i I - TAT^{-1}\right)$
5. the geometric multiplicities depend only on $\mathcal{A}$, and not on the used basis

Recap of the connections

If A is diagonalizable then:

- there exist a basis for $\mathbb{R}^n$ that is composed of eigenvectors of A
- the sum of the geometric multiplicities of the various eigenspaces of A is n
- the various eigenspaces of A span the whole $\mathbb{R}^n$
- the associated system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is actually a series of independent 1-dimensional systems
- the modes of the associated system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ are of the form $e^{\lambda t}$

Recap of the connections

The case "A is not diagonalizable"

- in any case there exists a change of basis that maps A into its Jordan form
- there must be at least one Jordan minibloc, and the effect of this miniblock is to introduce some sort of shearing in some directions
- the eigenvectors of A do not span the entire $\mathbb{R}^n$, but only a part of it
- the sum of the geometric multiplicities of the various eigenspaces of A is smaller than n; actually it is equal to the number of Jordan miniblocks
- the associated system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is actually a series of independent systems, each one corresponding to one of the Jordan miniblocks
- the modes of the associated system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ are not only of the form $e^{\lambda t}$, but there must be some $te^{\lambda t}$ or even higher powers of t

How do we find Jordan forms?

i.e., how can we go from $A = \begin{bmatrix} 3 & 4 & 8 \\ 1 & -5 & 2 \\ -5 & 9 & 1 \end{bmatrix}$ to $J = TAT^{-1}$?

→ needs the concepts of generalized eigenvectors, but this is a bit too much for this course . . . In any case just use numerical tools!