# TTK4225 - Systems Theory, Autumn 2020

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# Introduction to Laplace transforms

# Laplace who?

Figure: Pierre-Simon Laplace, Beaumont-en-Auge, 23 March 1749 - Paris, 5 March 1827

# Roadmap

- why?
- what?
- how, at least intuitively?
- an essential example

# Why?

Remember: predictive control of the type

$$\boldsymbol{u}^{\star} = \arg \min_{\boldsymbol{u} \in \mathcal{U}, \boldsymbol{f}(\boldsymbol{u}) \in \mathcal{F}} \mathrm{Cost}\left(\boldsymbol{f}\left(\boldsymbol{u}\right), \boldsymbol{u}\right)$$

requires to be able to:

- define "$\mathrm{Cost}$"
- compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

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> Laplace, for how we see it in this course,
> helps the "compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly" part

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alternative strategy: solve $g * u(t)$ using Laplace transforms!

# Laplace transforms - another pillar of automatic control

in TTK4225: $\mathcal{L}$ used to compute $y(t)$

in TTK4230: $\mathcal{L}$ used to determine $u(t)$

in virtually all the other courses at ITK: $\mathcal{L}$ used for both purposes and more

# Laplace transform - definition

Hypothesis: $f(t)$ is a real function that is locally integrable[1] on $[0, +\infty)$ or $(-\infty, +\infty)$

[1] I.e., so that its integral is finite on every compact subset of its domain of definition

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**unilateral Laplace transform:**

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{+\infty} e^{-st} f(t) dt \qquad \text{for } t \geq 0, s = \sigma + i\omega \in \mathbb{C} \tag{2}$$

for all the $s \in \mathbb{C}$ for which the integral in the RHS converges *(i.e., the ROC)*

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# Laplace transform - definition (2)

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- $s =$ "[complex] frequency domain" *(will see this better later on)*

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$$\Longrightarrow \mathcal{L} : \text{time} \mapsto \text{frequencies}$$

we need then $\mathcal{L}^{-1}$ : frequencies $\mapsto$ time

# Inverse Laplace transform

Hypothesis:

- $F(s)$ complex function

Thesis: there may be a set of piecewise-continuous, exponentially-restricted real functions $f(t)$ which have the property that

$$\mathcal{L}\{f(t)\} = F(s);$$

all the $f(t)$ satisfying this equivalence differ from each other only on a subset having Lebesgue measure zero

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*Discussion:* is the existence of at least one of such $f(t)$ guaranteed?

*Discussion:* given any $f(t)$, is the existence of $F(s)$ guaranteed?

# Main usefulness: convolution in time transforms into multiplication in Laplace-domain, and viceversa

$$\begin{cases} H(s) = \mathcal{L}\left\{h(t)\right\} \\ U(s) = \mathcal{L}\left\{u(t)\right\} \end{cases} \implies \mathcal{L}\left\{h * u(t)\right\} = H(s)U(s) \tag{5}$$

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*Discussion:* how then may $\mathcal{L}$ help computing $h * u(t)$?

# An intuitive explanation of the usefulness of the Laplace transform in automatic control

initial value problem

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$$\Big\downarrow$$

$$\boxed{x(t) = 2e^{-t}}$$

solution in the time domain

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initial value problem

algebraic problem

$$\dot{x}(t) + x(t) = 0, \quad x(0) = 2 \quad \xrightarrow{\mathcal{L}} \quad sX(s) - x(0) + X(s) = 0$$

$$x(t) = 2e^{-t}$$

solution in the time domain

# An intuitive explanation of the usefulness of the Laplace transform in automatic control

initial value problem | algebraic problem

$$\begin{tikzcd} \boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\ \boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \end{tikzcd}$$

solution in the time domain | solution in the frequency domain

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solution in the time domain      solution in the frequency domain

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next few slides:

introduce an utterly important example,

and with it show the “trick” mentioned before

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*Discussion:* which signal in the time domain is the most important signal you saw up to now?

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$$\boxed{\dot{y} = ay + bu \qquad \mapsto \qquad y(t) = y_0 e^{at} + \int_0^t e^{a\tau} bu(t-\tau)d\tau}$$

$$e^{at}$$

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\mathcal{L}\left\{e^{at}\right\} &= \int_0^{+\infty} e^{at} e^{-st} dt \\
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## $\mathcal{L}\left\{e^{at}\right\} = ?$

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\end{aligned}
$$

Remember, though: $F(s)$ is defined for all the $s \in \mathbb{C}$ for which the integral in the RHS converges *(i.e., the ROC)*. Thus, *discussion:* for which $s$ does $F(s)$ exist?

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$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \qquad \text{for } \mathrm{Re}(s) \geq a \tag{6}$$

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$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{7}$$

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Our game, now: use our new knowledge, i.e., $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$ to compute the solution to $\dot{y} = ay$.

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Our game, now: use our new knowledge, i.e., $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$ to compute the solution to $\dot{y} = ay$. *Discussion:* how did we find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$ before?

# An important property that we will prove later on

(see 4.2.1 in Balchen's book)

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$$\mathcal{L}\left\{\dddot{x}\right\} = s^3 X(s) - s^2 x_0 - s\dot{x}_0 - \ddot{x}_0 \tag{10}$$

$$\vdots \tag{11}$$

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&\vdots && (11)
\end{aligned}
$$

$$\boxed{s \quad \Longrightarrow \quad \text{derivation; } \frac{1}{s} \quad \Longrightarrow \quad \text{integration}}$$

*"mnemonics": $s$ = 'derivasjon'*

# The "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

(using that we know that $\mathcal{L}\{\dot{y}\} = sY(s) - y_0$ and that $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$)

$$\dot{y} = ay$$

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$$\dot{y} = ay \qquad \rightarrow \qquad sY - y_0 = aY$$

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$$\dot{y} = ay \quad \rightarrow \quad sY - y_0 = aY \quad \rightarrow \quad Y = y_0 \frac{1}{s-a}$$

initial value problem — algebraic problem

$$\begin{tikzcd} \boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\ \boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \arrow[l, "\mathcal{L}^{-1}"'] \end{tikzcd}$$

solution in the time domain — solution in the frequency domain

?

What is a Laplace transform, actually?

# Roadmap

- Laplace as a generalization of Fourier
- intuitions behind Fourier transforms
- generalizing sinusoids to exponentially decaying sinusoids

# In the previous unit

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\mathcal{L}\left\{e^{at}\right\} &= \int_0^{+\infty} e^{at}e^{-st}dt \\
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&= \frac{1}{s-a}
\end{aligned}$$

for $\mathrm{Re}\left(s\right) \geq a$

# Main usefulness: convolution in time = multiplication in frequency, and this makes solving ODEs an algebraic problem

$$\begin{tikzcd}
\text{initial value problem} & \text{algebraic problem} \\
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# But what is a Laplace transform, eventually?

$\rightarrow$ an opportune extension of the Fourier transform

# But what is a Fourier transform, eventually?

Definition:

$$\mathcal{F}\left\{f(t)\right\} = \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \tag{12}$$

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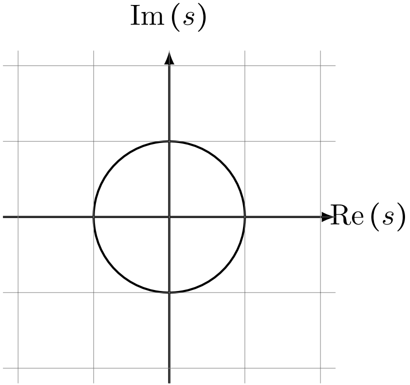
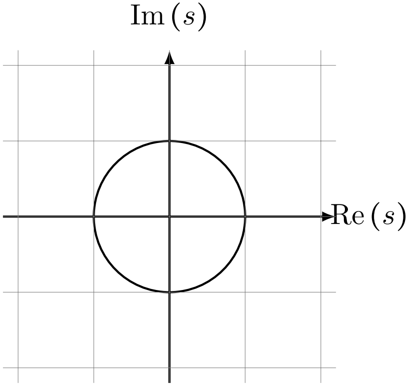
*Discussion:* where is $e^{-j\omega t}$?



# Small recap: angular frequency ≠ ordinary frequency

$$e^{j\omega t} = e^{j2\pi f t}$$



# Fourier transforms using Euler formula

$$\begin{aligned}\mathcal{F}\left\{f(t)\right\}(\omega) &= \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t)e^{j\omega t}dt \\ &= \int_{-\infty}^{+\infty} f(t)\cos\left(\omega t\right)dt \quad - \quad j\int_{-\infty}^{+\infty} f(t)\sin\left(\omega t\right)dt\end{aligned} \tag{13}$$

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*intuition:* for each $\omega$, $\widehat{f}(\omega)$ is a couple of numbers, conveniently put together as a complex number, so that:

- $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ = area of $f(t)\cos(\omega t)$
- $\mathrm{Im}\left(\widehat{f}(\omega)\right)$ = area of $f(t)\sin(\omega t)$

# What is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for sinusoidal $f$'s?

$$\star = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt = \int_{-\infty}^{+\infty} \Big(\cos(\alpha t)\Big)\cos(\omega t)\,dt$$

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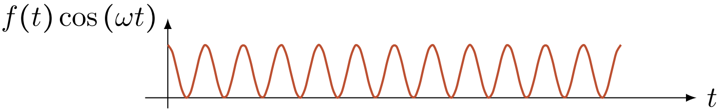

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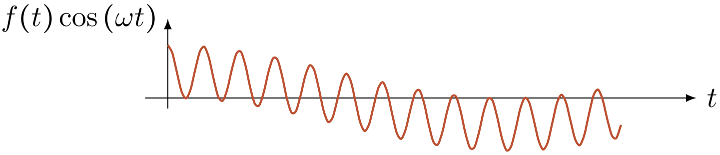
- if $f(t) = \cos(\alpha t)$ with $\alpha \neq \omega$ then $\star = 0$



# What is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for composite sinusoidal $f$'s?

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as before: if $\omega = \alpha_k$ then $\star = +\infty$, otherwise $\star = 0$!

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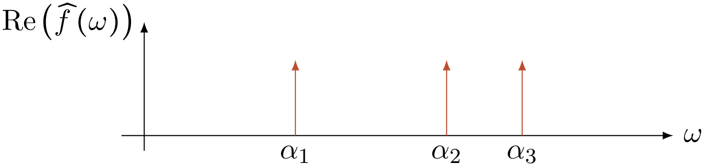
as before: if $\omega = \alpha_k$ then $\star = +\infty$, otherwise $\star = 0$! Thus varying $\omega$ makes $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ kind of "comb" for the right frequency:



# And what about $\mathrm{Im}\left(\widehat{f}(\omega)\right)$?

$$\star = \int_{-\infty}^{+\infty} f(t)\sin(\omega t)\,dt$$

kind of similar, only scanning for oddity (while $\cos$ scans for even components)
$\rightarrow$ see the additional material linked later for more information

## And what is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for generic non-periodic $f$'s?

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# And what is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for generic non-periodic $f$'s?

$$\star = \int_{-\infty}^{+\infty} f(t) \cos(\omega t)\, dt$$

- potentially every $\omega$ may contribute to $\star$
- if there is no pure $\cos$ in $f$ then each $\omega$ contributes with a finite area

# So, what is a Fourier transform, eventually?

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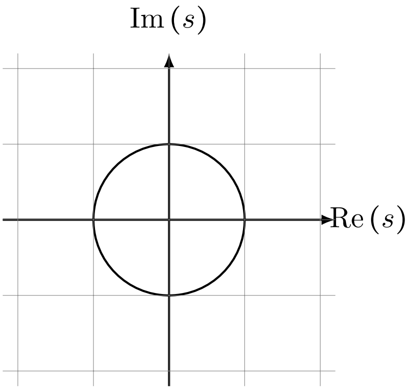
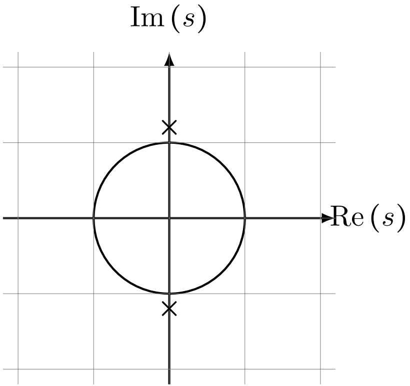
$=$ scanning the imaginary axis for sinusoidal components:

$$\begin{aligned}\widehat{f}(\omega) &= \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt \\ &\quad + j\int_{-\infty}^{+\infty} f(t)\sin(\omega t)\,dt\end{aligned}$$



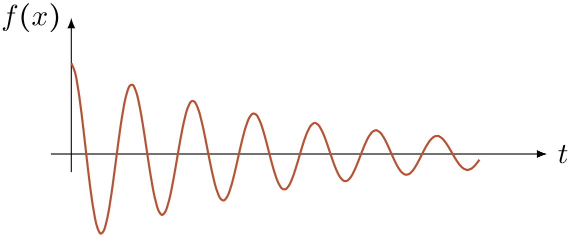
# Summarizing, graphically

$$f(t) = \cos(\omega t) = \mathrm{Re}\left(e^{j\omega t}\right) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$



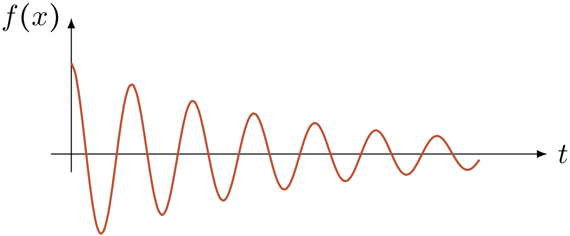
# But what about exponentially decaying sinusoids?

$$f(t) = e^{at} \cos(\omega t)$$



# But what about exponentially decaying sinusoids?

$$f(t) = e^{at} \cos(\omega t)$$



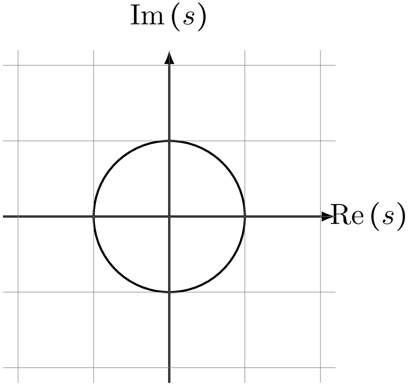

what if we consider $\overline{f}(t) = f(t)e^{-at}$?

# Fourier and Laplace, side by side

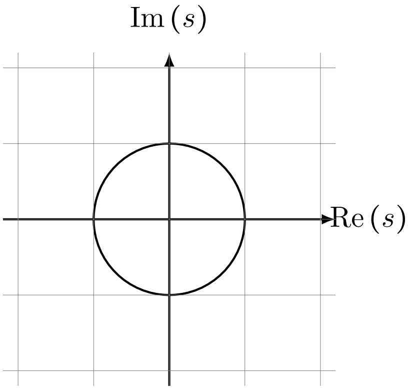
$$\mathcal{F}\left\{f(t)\right\} = \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$

$$\mathcal{L}\left\{f(t)\right\} := \int_{0}^{+\infty} f(t)e^{-st}dt$$

# So, what is a Laplace transform, eventually?



# So, what is a Laplace transform, eventually?




*beware of the region of convergence!*

?

# Suggested additional material

Fourier transforms 1:

https://www.youtube.com/watch?v=spUNpyF58BY

Fourier transforms 2 (with some Laplace in it):

https://www.youtube.com/watch?v=3gjJDuCAEQQ

Laplace transforms:

https://www.youtube.com/watch?v=n2y7n6jw5d0

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Laplace = a pillar of automatic control

$\Longrightarrow$ better to invest time in developing a full understanding of it

## Using Laplace transforms with second order systems in free evolution

# Recall: the "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

$$\dot{y} = ay \qquad \to \qquad sY - y_0 = aY \qquad \to \qquad Y = y_0 \frac{1}{s-a}$$

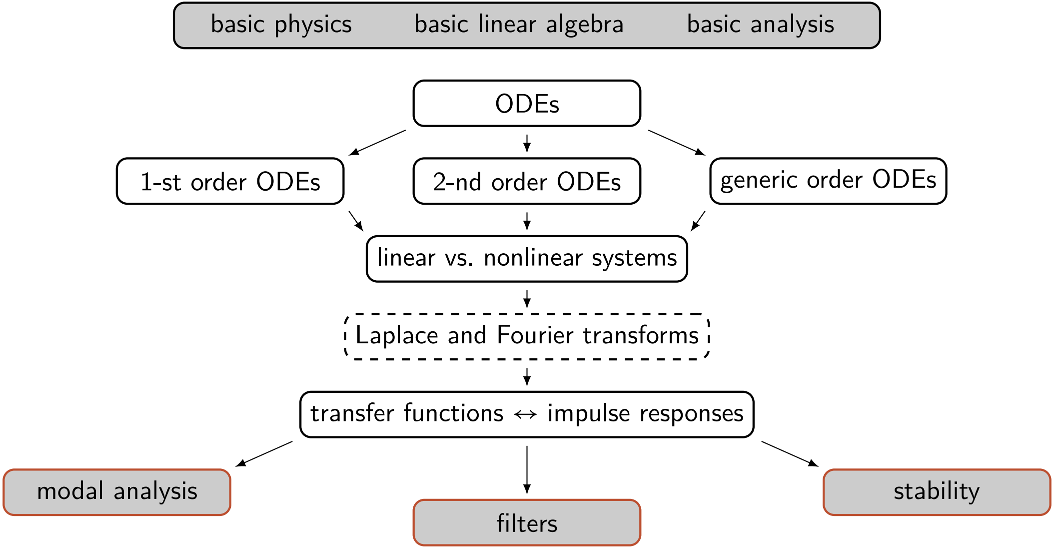
# Recall: the "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

initial value problem    algebraic problem

$$
\begin{tikzcd}
\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\
\boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \arrow[l, "\mathcal{L}^{-1}"]
\end{tikzcd}
$$

solution in the time domain    solution in the frequency domain

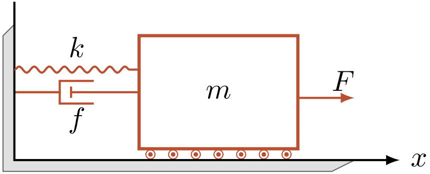
# Summary / Roadmap of TTK4225

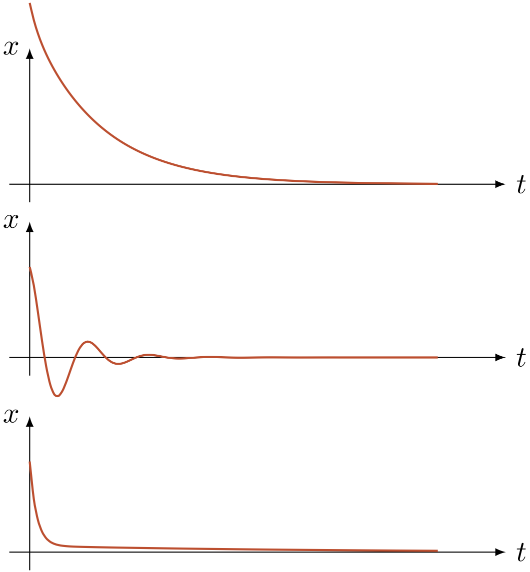


# Roadmap

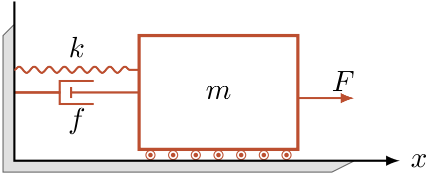
- the same thing, but for second order system
- different algebraic situations $\implies$ different damping properties
- different ways of summarizing the different damping properties

# Discussion on a practical example:
## how may a second order system free-evolve?




x
t
x
t
x
t

# Dissecting the practical example “spring-mass systems”




$$m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + u(t)$$

# The properties we shall use

$$\mathcal{L}\left\{\dot{x}\right\} = sX(s) - x_0 \tag{14}$$

$$\mathcal{L}\left\{\ddot{x}\right\} = s^2X(s) - sx_0 - \dot{x}_0 \tag{15}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{16}$$

$$\tag{17}$$

# Solution for the free evolution case

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0$$

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$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0 \qquad \Longrightarrow \qquad X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}}$$

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we would like to use $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ to inverse-transform

$\Longrightarrow$ we must try to factorize the RHS, i.e., try to write it as

$$X(s) = \frac{A}{s-\alpha} + \frac{B}{s-\beta}$$

(if possible)

# Solution for the free evolution case

Next step: factorize

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}}$$

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*Discussion:* what are the different possibilities?

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*Discussion:* what are the different possibilities?

1. $f^2 - 4mk > 0 \implies$ two distinct real solutions *(will imply overdamping)*
2. $f^2 - 4mk = 0 \implies$ one double real solution *(will imply critical damping)*
3. $f^2 - 4mk < 0 \implies$ two conjugate complex solutions *(will imply underdamping)*

# $f^2 - 4mk > 0$, i.e., overdamping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

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$$f^2 > 4mk \quad \Longrightarrow \quad \begin{cases} \lambda_1 = & -\dfrac{f}{2m} - \dfrac{\sqrt{f^2 - 4mk}}{2m} \in \mathbb{R} \\[2ex] \lambda_2 = & -\dfrac{f}{2m} + \dfrac{\sqrt{f^2 - 4mk}}{2m} \in \mathbb{R} \end{cases}$$

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$$\Longrightarrow \quad X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2}$$

*Discussion:* can we conclude immediately that

$$X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2} \qquad \Longrightarrow \qquad x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

or do we need to say some property about Laplace transforms that we did not say before?

# We need to be sure that Laplace transforms are linear!

i.e., be sure that

$$\mathcal{L}\left\{\alpha x_1 + \beta x_2\right\} = \alpha\mathcal{L}\left\{x_1\right\} + \beta\mathcal{L}\left\{x_2\right\},$$

and that

$$\mathcal{L}^{-1}\left\{\alpha X_1 + \beta X_2\right\} = \alpha\mathcal{L}^{-1}\left\{X_1\right\} + \beta\mathcal{L}^{-1}\left\{X_2\right\}$$

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*Discussion:* why do the previous properties hold? (hint: recall that $\mathcal{L}\left\{x(t)\right\} := \int_0^{+\infty} x(t) e^{-st} dt$)

# In conclusion: overdamping in free evolution

$$f^2 > 4mk \quad \Longrightarrow \quad X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2} \quad \Longrightarrow \quad x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

for opportune $A$ and $B$.

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for opportune $A$ and $B$. *Discussion:* what determines $A$ and $B$? Hint: recall that

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s-\lambda_1)(s-\lambda_2)}$$

important message (will be seen again later on): the denominator of the rational Laplace transform determines the behavior of the system, while the numerator determines which root is dominant and the “amplitude” of each mode

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

# Practical example of a overdamped system in free evolution

$$\begin{cases} m & = 10 & \text{kg} \\ f & = 100 & \text{kg/s} \\ d & = 90 & \text{N/s}^2 \end{cases} \Longrightarrow \begin{cases} \ddot{x} + \dfrac{100}{10}\dot{x} + \dfrac{90}{10}x = 0 \\ \\ \\ \end{cases}$$

# Practical example of a overdamped system in free evolution

$$\begin{cases} m &= 10 \quad \text{kg} \\ f &= 100 \quad \text{kg/s} \\ d &= 90 \quad \text{N/s}^2 \end{cases} \Longrightarrow \begin{cases} \ddot{x} + \frac{100}{10}\dot{x} + \frac{90}{10}x = 0 \\ x(0) = 0.16\text{m} \\ \dot{x}(0) = 0\text{m/s} \end{cases} \Longrightarrow X(s) = \frac{0.16\,(s+10)}{s^2 + 10s + 9}$$

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*note that there are 2 distinct steps!!*

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`ARMA-continuous-time-example.ipynb`

?

## $f^2 - 4mk = 0$, i.e., critical damping in free evolution

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$$\lambda = -\frac{f}{2m} \qquad \Longrightarrow \qquad X(s) = \frac{x_0(1 + s) + \dot{x}_0}{(s - \lambda)^2}$$

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*Discussion:* did we learn until now in this course how to inverse-Laplace these things?

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no, but the solution is: $\mathcal{L}\left\{te^{at}\right\} = \dfrac{1}{(s-a)^2}$

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Very interesting behavior:

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Very interesting behavior:

$$x(t) = Ae^{\lambda t} + Bte^{\lambda t}$$

*Discussion:* what contributes to determine $A$ and $B$?

*Discussion:* recall the previous "overdamping case", i.e.,

$$
\begin{cases}
X(s) & = \dfrac{x_0\left(\frac{f}{m}+s\right)+\dot{x}_0}{s^2+\frac{f}{m}s+\frac{k}{m}} & \\[2ex]
m & = 10 & \mathrm{kg} \\
f & = 100 & \mathrm{kg/s} \\
d & = 90 & \mathrm{N/s^2} \\
x(0) & = 0.16\mathrm{m} & \\
\dot{x}(0) & = 0\mathrm{m/s} &
\end{cases}
\implies \quad X(s) = \frac{0.16\,(s+10)}{s^2+10s+9}\,.
$$

How may we change the parameters and the initial condition so to have a "critical damping" case, i.e.,

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`ARMA-continuous-time-example.ipynb`

?

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$$\Longrightarrow \qquad X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2}$$

with $\lambda_1$ and $\lambda_2$ a complex-conjugate pair

*Discussion:* did we learn until now in this course how to inverse-Laplace these things?

$$X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2} \text{ with } \lambda_1 \text{ and } \lambda_2 \text{ a complex-conjugate pair}$$

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"yes, but the corresponding exponentials come out to be complex numbers! How is it possible that a real system produces complex results?"

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Solution:

$$\mathcal{L}\left\{e^{at}\sin\omega t\right\} = \frac{\omega}{(s-a)^2+\omega^2}$$

*Discussion:* recall the previous "overdamping case", i.e.,

$$\begin{cases} X(s) & = \dfrac{x_0\left(\frac{f}{m}+s\right)+\dot{x}_0}{s^2+\frac{f}{m}s+\frac{k}{m}} \\ m & = 10 \qquad\qquad \mathrm{kg} \\ f & = 100 \qquad\qquad \mathrm{kg/s} \\ d & = 90 \qquad\qquad \mathrm{N/s^2} \\ x(0) & = 0.16\mathrm{m} \\ \dot{x}(0) & = 0\mathrm{m/s} \end{cases} \implies X(s) = \frac{0.16\,(s+10)}{s^2+10s+9}.$$

How may we change the parameters and the initial condition so to have a "critical damping" case, i.e.,

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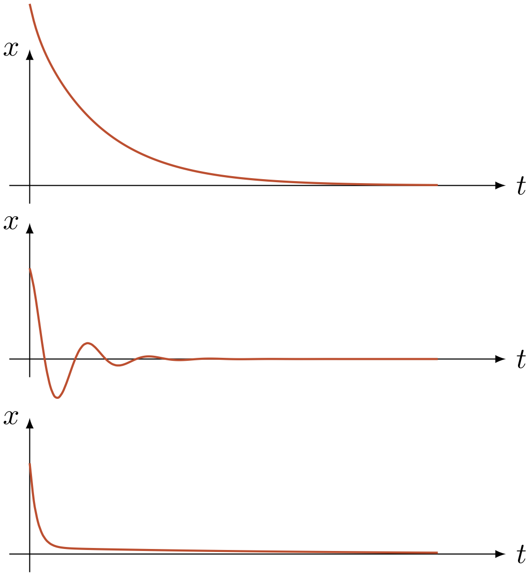
`ARMA-continuous-time-example.ipynb`

?

# Recap of the possible damping modalities



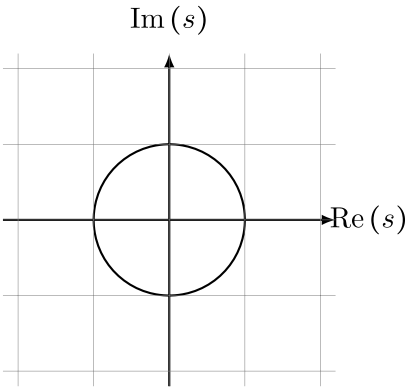
# Recap of the corresponding time-frequency pairs

overdamped: $\mathcal{L}\left\{Ae^{\lambda_1 t} + Bte^{\lambda_2 t}\right\} = \dfrac{A}{s-\lambda_1} + \dfrac{B}{s-\lambda_2}, \quad \lambda_1 \neq \lambda_2$

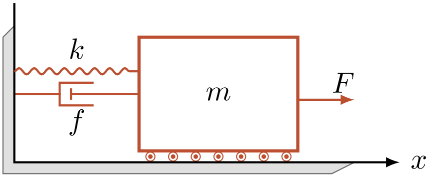
critically damped: $\mathcal{L}\left\{Ae^{\lambda t} + Bte^{\lambda t}\right\} = \dfrac{A}{s-\lambda} + \dfrac{B}{(s-\lambda)^2}$

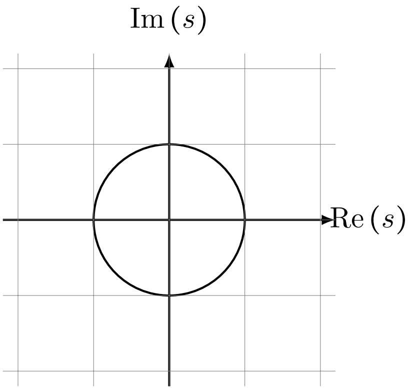
underdamped: $\mathcal{L}\left\{e^{at}\sin\omega t\right\} = \dfrac{\omega}{(s-a)^2+\omega^2}$

# Recap using the “poles-zeros plot”



# Do you see it?



?

Using Laplace transforms with generic $n$ order systems in free evolution

# Roadmap

- make Laplace-transform more matematically ground
- derive some few important examples and properties

# Laplace-properties we saw up to now:

(4.2.1 in Balchen's book)

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## Linearity

$$\mathcal{L}\left\{af(t) + bg(t)\right\} = a\mathcal{L}\left\{f(t)\right\} + b\mathcal{L}\left\{g(t)\right\} \tag{18}$$

# Laplace-properties we saw up to now:

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## Derivatives

$$\mathcal{L}\left\{\dot{x}\right\} = sX(s) - x(0) \tag{19}$$

$$\mathcal{L}\left\{\ddot{x}\right\} = s^2X(s) - sx(0) - \dot{x}(0) \tag{20}$$

$$\vdots \tag{21}$$

# Laplace-properties we saw up to now:

(4.2.1 in Balchen's book)

## Integration

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s) \tag{22}$$

# Laplace-transforms we needed up to now:

| $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|
| $e^{at}$ | $\dfrac{1}{s-a}$ |
| $e^{at}\sin\omega t$ | $\dfrac{\omega}{(s-a)^2+\omega^2}$ |
| $te^{at}$ | $\dfrac{1}{(s-a)^2}$ |

# Laplace transforms, take 2

$$F(s) = \mathcal{L}\left\{f(t)\right\} := \int_{0}^{+\infty} e^{-st} f(t) dt \tag{23}$$

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# Important (but intuitive) teorem

If $f(t)$ is defined and piecewise continuous for $t > 0$,
and there exist two constants $M$ and $k$ so that $|f(t)| \leq Me^{kt}$ for every $t > 0$,
then $\mathcal{L}\{f(t)\}$ is well defined for every $s > k$

# Important (but intuitive) property: linearity

Assuming that $f(t)$ and $g(t)$ are so that $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exists, then $\mathcal{L}\{af(t) + bg(t)\}$ exists and is s.t.

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$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

## Proof

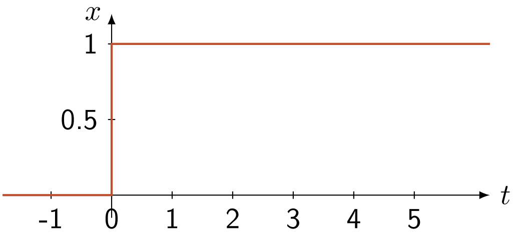
Integrals are linear, thus

$$\begin{aligned}
\mathcal{L}\{af(t)+bg(t)\} &= \int_0^{+\infty} e^{-st}\left[af(t)+bg(t)\right]dt \\
&= a\int_0^{+\infty} e^{-st} f(t)dt + b\int_0^{+\infty} e^{-st} g(t)dt \\
&= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}
\end{aligned}$$

?

# Important example: Laplace-transforming the Heaviside-function

$$H(t) := \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$



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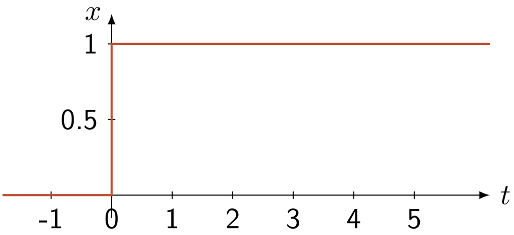



*Discussion:* is $H(0)$ defined?

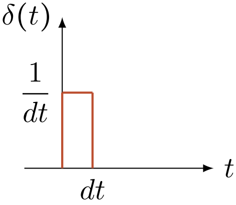
# Deriving the Heaviside function from the Dirac's impulse

$$H(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

with, *very loosely,*

$$\delta(t) := \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{otherwise,} \end{cases}$$

or, better, as the limit of concentrating an unitary point mass at 0:



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*Discussion:* what is the ROC?

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&= \frac{1}{-s} \int_0^{+\infty} t^2 (-se^{-st}) dt && (35) \\
&= -\frac{1}{s} \left( \left[ t^2 e^{-st} \right]_0^{+\infty} - \int_0^{+\infty} 2te^{-st} dt \right) && (36) \\
&= \frac{2}{s^3} && (37) \\
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(this means that often one gets a ramp response and then one has to reconstruct back the impulse response)

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$$\mathcal{L}\left\{\dot{x}\right\} = \int_0^{+\infty} e^{-st}\dot{x}dt \tag{40}$$

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$$\begin{aligned}
\mathcal{L}\left\{\dot{x}\right\} &= \int_0^{+\infty} e^{-st}\dot{x}dt && (40)\\
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&= 0 - x(0) + s\int_0^{+\infty} e^{-st}x(t)dt && (42)
\end{aligned}$$

# On with the important examples: Laplace-transforms of derivatives

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&= \quad - \dot{x}(0) + s\big(sX(s) - x(0)\big) && (47)\\
&= s^2X(s) - sx(0) - \dot{x}(0) && (48)\\
& && (49)
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$$

# On with the important examples: Laplace-transforms of derivatives

$$\mathcal{L}\left\{x^{(n)}(t)\right\} = s^n\mathcal{L}\{x(t)\} - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - \ldots - x^{(n-1)}(0) \tag{50}$$

# Translation theorems

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**Translation in the frequency domain**

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### Translation in the frequency domain

If $f(t)$ admits $F(s)$, then $e^{at}f(t)$ transforms into $F(s-a)$

### Translation in the time domain

If $f(t)$ admits $F(s)$, then $f(t-a)\mu(t-a)$ transforms into $e^{-as}F(s)$

*Discussion:* for which $s$ do these theorems hold?

Answer: only in the intersection of the ROCs of both “original” and “translated” transforms!

## Translation theorems + Laplace-transforming polynomials = **the most useful formula that you may remember**

$$\begin{cases} \mathcal{L}\left\{t^n\right\} & = \dfrac{n!}{s^{n+1}} \\ \mathcal{L}\left\{e^{at} f(t)\right\} & = F(s-a) \end{cases} \quad \Longrightarrow \quad \mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \tag{51}$$

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*Discussion:* what is the ROC?

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*Discussion:* what is the ROC? $\operatorname{Re}(s) > a$

# Laplace-transforms of oscillatory behaviors = **the second most useful formulas that you may remember**

$$\mathcal{L}\left\{e^{at}\sin\omega t\right\} = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\left\{e^{at}\cos\omega t\right\} = \frac{s-a}{(s-a)^2+\omega^2}$$

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Limit cases: $a = 0$, so that

$$\mathcal{L}\left\{\sin\omega t\right\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\left\{\cos\omega t\right\} = \frac{s}{s^2+\omega^2}$$

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The potential points of view of this course:

simply a complex number, i.e., $s = \sigma + j\omega$, that gives rise to a *complex frequency* through the complex exponential $e^{-st} = e^{-\sigma t} e^{-j\omega t}$

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**Be careful on how to use $s$ in this course!**

Typically, in all the automatic control courses the interest is not too much on the value of $F(s)$ for different $s$, but rather on the fact that $F(s)$ enables performing algebraic operations to solve ODEs!

?

# Using Laplace transforms with generic $n$ order systems in free evolution

## In the last episodes: the "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$, and to do something similar for second order systems in free evolution

$$\dot{y} = ay \qquad \rightarrow \qquad sY - y_0 = aY \qquad \rightarrow \qquad Y = y_0 \frac{1}{s-a}$$

# Roadmap

- the same thing, but for generic systems

# A small recap

- first oder homogeneous systems:

$$\dot{x} - ax = 0 \quad \mapsto \quad x(t) = x_0 e^{at} \tag{52}$$

- first oder non-homogeneous systems:

$$\dot{x} - ax = bu \quad \mapsto \quad x(t) = x(0)e^{at} + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \tag{53}$$

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*The problem:* the convolution integral gives us headaches, and we want to use Laplace to ease things up. To understand how, step-by-step path:

1. start from $\dot{x} - ax = 0$
2. move to $\ddot{x} - a_1\dot{x} - a_2 x = 0$
3. move to higher order
4. add the "$u$" part

# A more detailed roadmap for this unit:

- generalize how to do partial expansions
- solve the problem: how to use Laplace transforms with generic $n$ order systems in free evolution

?

# Recalling some past results

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0$$
$$\Downarrow$$
$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} \Downarrow$$
$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \text{ (and the other cases)}$$

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$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \text{ (and the other cases)}$$

*Discussion:* when is it that $\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0$ leads to $X(s) \neq 0$?

what about $\dddot{x} + \alpha_2 \ddot{x} + \alpha_1 \dot{x} + \alpha_0 x = 0$?

what about $\dddot{x} + \alpha_2 \ddot{x} + \alpha_1 \dot{x} + \alpha_0 x = 0$?

and what about higher orders?

# Partial fraction decomposition

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case single poles: if $\dfrac{N(s)}{(s-a)(s-b)(s-c)\cdots}$ is s.t. $a \neq b \neq c \neq \cdots$ then there exist $A$, $B$, $C$, ... s.t.

$$\frac{N(s)}{(s-a)(s-b)(s-c)\cdots} = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots \tag{54}$$

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case repeated poles: if some poles are repeated, then there exist $A_1$, . . . , $A_{na}$, $B_1$, . . . , $B_{nb}$, s.t.

$$\frac{N(s)}{(s-a)^{na}(s-b)^{nb}\ldots} = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots \tag{55}$$

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"But how do I compute $A$, $B$, etc.?" $\mapsto$
`en.wikipedia.org/wiki/Partial_fraction_decomposition`
(tip: start from `en.wikipedia.org/wiki/Heaviside_cover-up_method`)

?

# Computing the free evolution for generic $n$ order LTI systems

$$\text{start from } x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0 \tag{56}$$

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$$\text{obtain } X(s) = \frac{N(s)}{(s-a)(s-b)(s-c)\cdots} \tag{58}$$

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$$\text{do a partial fraction decomposition} \tag{59}$$

# Computing the free evolution for generic $n$ order LTI systems

Two main cases:

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$\text{either } X(s) = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots$$

$$\text{or } X(s) = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots$$

# Computing the free evolution for generic $n$ order LTI systems

## Case single poles

$$x^{(n)} + \alpha_{n-1}x^{(n-1)} + \ldots + \alpha_0 x = 0$$

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*Discussion:* given $\mathcal{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = e^{\alpha t}$, what is $x(t)$?

# Computing the free evolution for generic $n$ order LTI systems

## Case multiple poles

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

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remember: $\mathcal{L}\left\{t^n e^{at}\right\} = \dfrac{n!}{(s-a)^{n+1}}$

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remember: $\mathcal{L}\left\{t^n e^{at}\right\} = \dfrac{n!}{(s-a)^{n+1}}$

*Discussion:* given the previous transform, what is $x(t)$? Solution: a sum of terms of the type $t^n e^{at}$

# Extremely important result

a LTI in free evolution behaves as a combination of terms $e^{\alpha t}$, $te^{\alpha t}$, $t^2e^{\alpha t}$, etc. for a set of different $\alpha$s and powers of $t$, called the *modes* of the system

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of terms $e^{\alpha t}$, $te^{\alpha t}$, $t^2 e^{\alpha t}$, etc. for a set of different
$\alpha$s and powers of $t$, called the *modes* of the system

*Discussion:* assuming that we have two modes, $e^{-0.3t}$ and $e^{-1.6t}$, so that

$$x(t) = \gamma_1 e^{-0.3t} + \gamma_2 e^{-1.6t}.$$

What determines $\gamma_1$ and $\gamma_2$?

`comparison-of-exponentials.ipynb`

?

# Using Laplace transforms with second order systems in free evolution

# In the last episodes: the "Laplace way" to find how LTI systems freely evolve

$$\dot{y} = ay \qquad \rightarrow \qquad sY - y_0 = aY \qquad \rightarrow \qquad Y = y_0 \frac{1}{s - a}$$

# Roadmap

- the same thing, but adding the forced response term

# A small recap

- first oder homgeneous systems:

$$\dot{x} - ax = 0,\ x(0) = x_0 \quad \mapsto \quad x(t) = x_0 e^{at} \tag{60}$$

- first oder non-homgeneous systems:

$$\dot{x} - ax = bu,\ x(0) = x_0 \quad \mapsto \quad x(t) = x(0)e^{at} + \int_0^t e^{a\tau} bu(t-\tau)d\tau \tag{61}$$

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*The problem, once again:* the convolution integral gives us headaches - we want to use Laplace to ease things up

# Laplace-ing everything

$$\dot{x} - ax = bu, \; x(0) = x_0 \tag{62}$$

$$\downarrow \tag{63}$$

$$sX(s) - x(0) - aX(s) = bU(s) \tag{64}$$

$$\downarrow \tag{65}$$

$$X(s) = \frac{x(0)}{s-a} + \frac{bU(s)}{s-a} \tag{66}$$

$$\downarrow \tag{67}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x(0)}{s-a} + \frac{bU(s)}{s-a}\right\} \tag{68}$$

$$\downarrow \tag{69}$$

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(71)

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if $u(t)$ has a rational Laplace transform, then

$$x(t) = \sum \mathcal{L}^{-1} \left\{ \frac{\alpha}{(s-a)^{\beta}} \right\}$$

# Example 1

$$\dot{x} + x = 0, \quad x(0) = 2 \qquad \Longrightarrow \qquad x(t > 0) = ?$$

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$$sX(s) - x(0) + X(s) = 0$$

$$\downarrow$$

$$(s+1)X(s) = x(0)$$

$$\downarrow$$

$$X(s) = \frac{x(0)}{(s+1)} = \frac{2}{s - (-1)}$$

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$$x(t) = 2e^{-t}$$

*Discussion:* what is the time constant in this case?

# Example 2

*Discussion:* $\mathcal{L}\left\{H\left(t\right)\right\} = ?$

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$$\downarrow$$

$$(s+1)X(s) = \frac{1}{s} + 2$$

$$\downarrow$$

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

# Example 2, continued

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

# Example 2, continued

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1)+Bs}{s(s+1)} \tag{72}$$

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$s$: $As + Bs = 0 \implies A = -B$

$1$: $A = 1 \implies B = -1$

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$$x(t) = \mathcal{L}^{-1}\{X(s)\} = e^{-t} + \mu(t) \tag{74}$$

# Example 3

$$\dot{x} + 4x = 3e^{-4t}, \quad x(0) = 1 \qquad \Longrightarrow \qquad x(t > 0) = ?$$

# Example 3

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$$sX(s) - 1 + 4X(s) = \frac{3}{s+4} \tag{75}$$

$$\downarrow \tag{76}$$

$$X(s) = \frac{1}{s+4} + \frac{3}{(s+4)^2} \tag{77}$$

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Remember: $\mathcal{L}\left\{t^n e^{at}\right\} = \dfrac{n!}{(s-a)^{n+1}}$, thus

$$x(t) = e^{-4t} + 3te^{-4t} \tag{79}$$

?