TTK4225 - Systems Theory, Autumn 2020

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A complete example, from the beginning to the end

Roadmap

- Lotka-Volterra, but this time from the beginning to the end

Lotka-Volterra

- y_{prey} := prey
- y_{pred} := predator

$$\begin{cases} \dot{y}_{\text{prey}} &= \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} &= -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

`GitHub/TTK4225/trunk/Jupyter/Lotka-Volterra-introduction.ipynb`

“Our” Lotka-Volterra

$$\begin{cases} \dot{y}_{\text{prey}} &= 10 y_{\text{prey}} - 1 y_{\text{prey}} y_{\text{pred}} - u_{\text{prey}} \\ \dot{y}_{\text{pred}} &= -0.1 y_{\text{pred}} + y_{\text{prey}} y_{\text{pred}} - u_{\text{pred}} \end{cases}$$

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Steps:

1. find the equilibria

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1. find the equilibria
2. linearize the system around the equilibria

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4. characterize the trajectories of the system in the neighborhood of these equilibria
5. understand the controllability and observability properties of the system *(not in this course)*
6. design minimal human interventions that make the system behave as desired *(not in this course)*

?

Complex numbers - introduction

Roadmap

- definition
- sum, subtraction, multiplication, division
- why is this important?

What is a complex number, and why did we introduce them?

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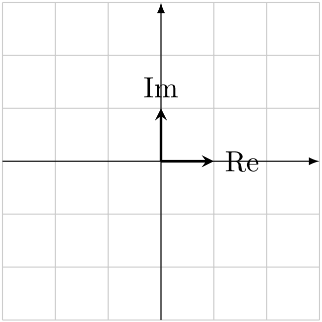
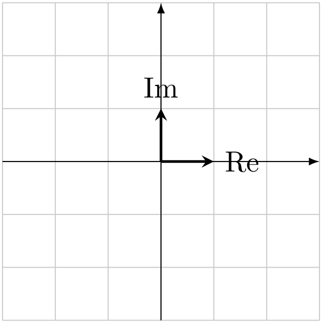
In essence:

1. a point in the Cartesian plane
2. to be sure to find all the roots of polynomials *(i.e., be able to write polynomials in convenient forms)*

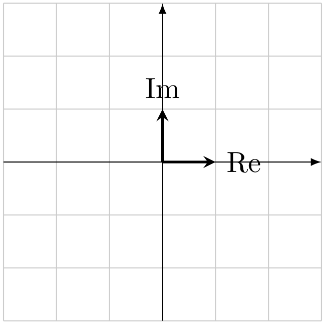


The “imaginary unit”



$$i \quad : \quad i^2 = -1$$

Simple operations with complex numbers: sums

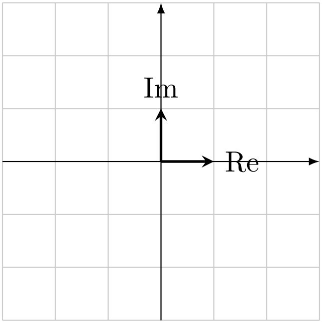


$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

implies

$$z_1 + z_2 = \left(a_1 + ib_1\right) + \left(a_2 + ib_2\right) = \left(a_1 + a_2\right) + i\left(b_1 + b_2\right)$$

Simple operations with complex numbers: subtractions

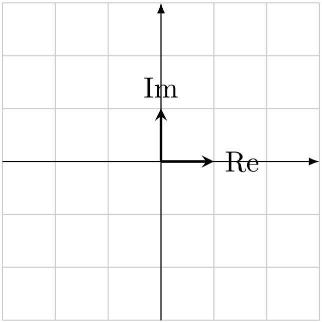


$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

implies

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i\,(b_1 - b_2)$$

Simple operations with complex numbers: multiplication

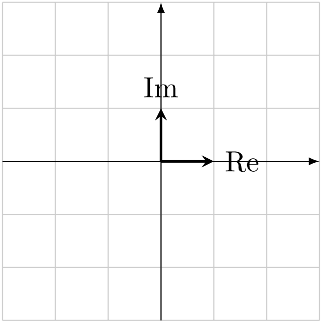


$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

implies

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Simple operations with complex numbers: inversion

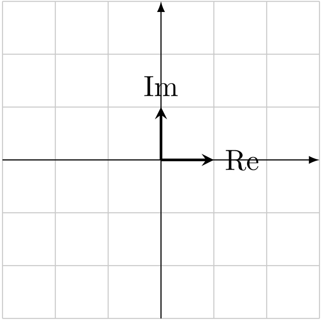


$$z_1 = a_1 + ib_1$$

implies

$$z_1^{-1} = \frac{a_1}{a_1^2 + b_1^2} - i\frac{b_1}{a_1^2 + b_1^2}$$

Simple operations with complex numbers: division

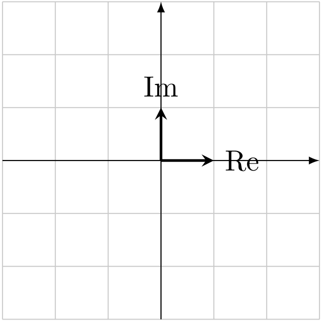


$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

implies

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}$$

Simple operations with complex numbers: conjugation



$$z_1 = a_1 + ib_1$$

implies

$$\overline{z_1} = a_1 - ib_1$$

Conjugacy: a good way of simplifying the previous operations

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- addition: $z + \overline{z} = a + ib + a - ib = 2a$, thus $\mathrm{Re}\,(z) = \dfrac{1}{2}\,(z + \overline{z})$

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- subtraction: $z - \overline{z} = a + ib - a + ib = 2ib$, thus $\mathrm{Im}\,(z) = \dfrac{1}{2i}\,(z + \overline{z})$

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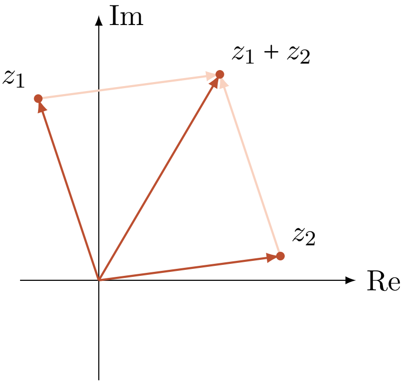
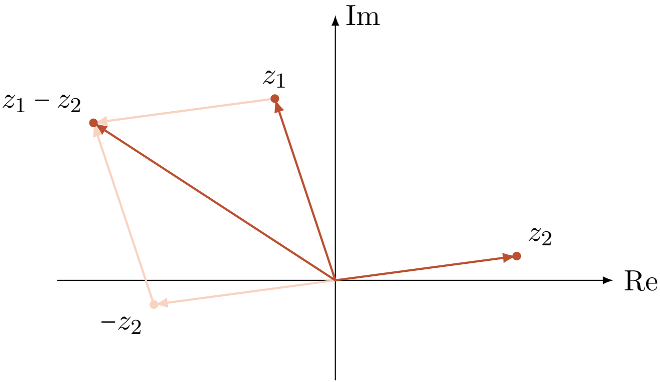
- subtraction: $z - \overline{z} = a + ib - a + ib = 2ib$, thus $\mathrm{Im}\left(z\right) = \dfrac{1}{2i}\left(z + \overline{z}\right)$
- multiplication: $z\overline{z} = \left(a + ib\right)\left(a - ib\right) = a^2 + b^2$, thus $\left|z\right|^2 = z\overline{z}$

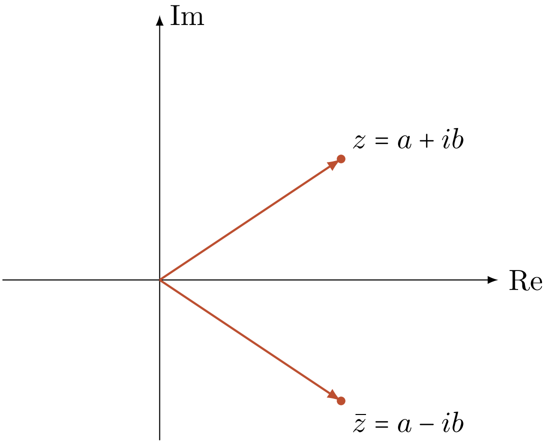
Once again, graphically: addition



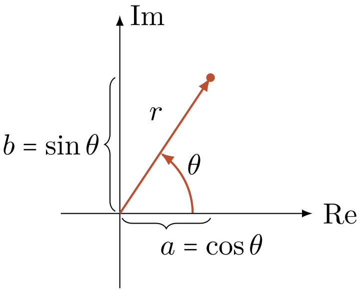
Once again, graphically: subtraction



Once again, graphically: conjugation



Polar coordinates



$$z = a + ib$$

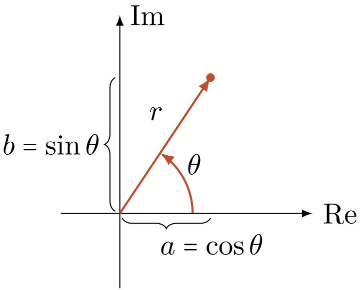
can be rewritten through r and θ so that

$$a = r\cos\theta \text{ and } b = r\sin\theta$$

so that

$$z = r\left(\cos\theta + i\sin\theta\right)$$

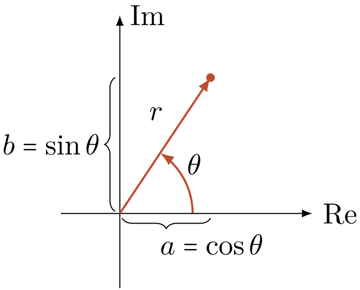
Polar coordinates



Equations:

$$
\begin{aligned}
r &= |z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}} \\
\theta &= \arg z = \operatorname{atan}(b, a) = \tan^{-1}\left(\frac{b}{a}\right)
\end{aligned}
$$

Polar coordinates



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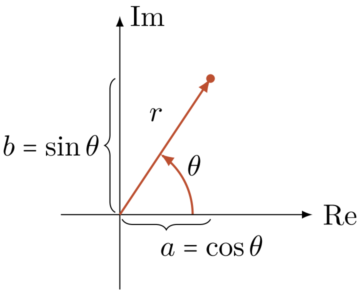
Notation:

- $r =$ absolute value or modulus of z
- $\theta =$ argument, angle, or phase of z

Problem: different θ's lead to the same z



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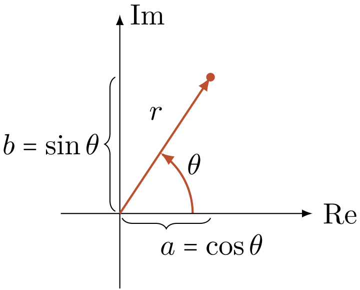
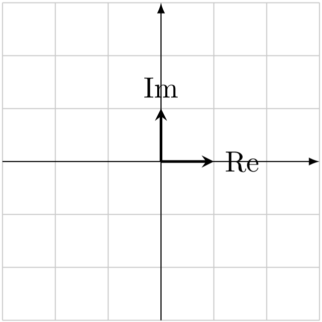


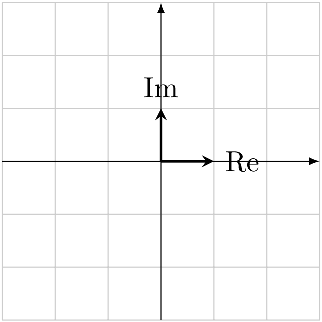
Definition: principal value of z = that value of θ that is in $[-\pi, \pi]$

Usefulness of polar forms: the multiplication is immediate



$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i \sin\left(\theta_1 + \theta_2\right)\right]$$

Implication: the division is immediate



$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$$

Usefulness of the multiplication: it enables Taylor expansions!

Taylor expansions: a tool to do not underestimate

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Taylor expansions: a tool to do not underestimate

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E.g., thus

$$e^z = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \dots$$

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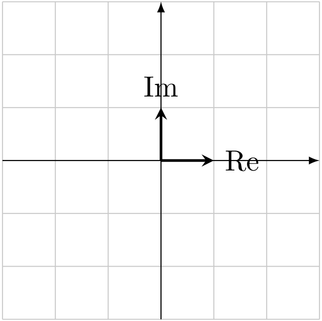
The absolute value of a complex number



Meaning: Euclidean length of the vector. Very important for control, since very often we compute the absolute value of a transfer function at a specific $s = i\omega$ *(and very very often the transfer function is rational)*

Thus, chain of implications highlighting the importance of complex numbers for control perspectives

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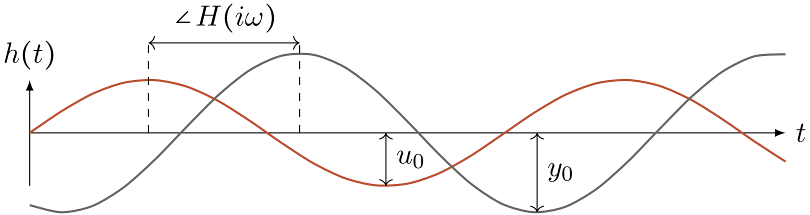
1: LTI systems have sinusoidal fidelity

$$u(t) = u_0 \sin(\omega t) \quad \Longrightarrow \quad y(t) = u_0 |H(i\omega)| \sin(\omega t + \angle H(i\omega))$$



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2: we very often consider rational H's

$$\Longrightarrow \quad K \frac{\prod (s - z_i)}{\prod (s - p_i)}$$

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$$\Longrightarrow \qquad K \frac{\prod (s - z_i)}{\prod (s - p_i)}$$

3: we very often consider rational H's

this means that to compute $|H(i\omega)|$ we need to do multiplications and divisions among the complex numbers $s - \star_i$

Example: $H(s) = \dfrac{1+2s}{1+2s+s^2}$. What is $|H(i\omega)|$?

$$
\begin{aligned}
|H(i\omega)| &= \sqrt{H(i\omega)\overline{H(i\omega)}} \\
&= \sqrt{\frac{1+2i\omega}{1+2i\omega-\omega^2}\cdot\frac{1-2i\omega}{1-2i\omega-\omega^2}} \\
&= \sqrt{\frac{1+4\omega^2}{(1-\omega^2)^2+4\omega^2}} \\
&= \sqrt{\frac{1+4\omega^2}{\omega^4+2\omega^2+1}}
\end{aligned}
$$

?

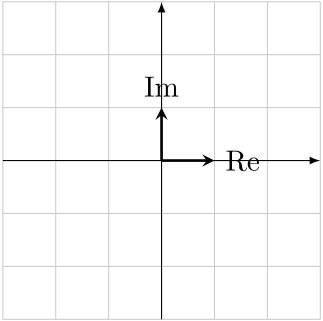
Complex functions

Roadmap

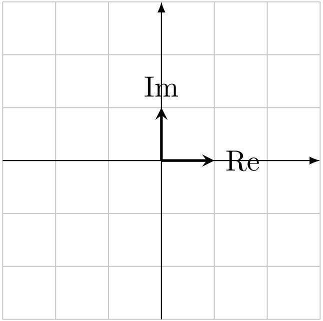
- definition
- why are they important?

Complex function: definition

$$f : \mathbb{C} \mapsto \mathbb{C}$$



$\mapsto$



Complex function: caveats

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is so that

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thus, also in polar representations, $(r,\theta) \mapsto \left(r',\theta'\right)$ with in general both r' and θ' functions of both r and θ

Example: if $f(z) = z^2 + 3z$ then what is $f(1+3j)$?

$$
\begin{aligned}
f(z) &= (x+iy)(x+iy) + 3x + 3iy \\
&= x^2 + 2ixy - y^2 + 3x + 3iy \\
&= x^2 - y^2 + 3x + i(2xy + 3y)
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thus

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thus

$$\begin{aligned} f(1+3j) &= u(1,3) + iv(1,3) \\ &= 1^3 - 3^2 + 3 + i(2 \cdot 1 \cdot 3 + 3 \cdot 3) \\ &= -5 + 15i \end{aligned}$$

Complex function: why are they important?

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Recall: the forced evolution is given by

$$Y(s) = H(s)U(s)$$

with $H(s)$ very often rational, i.e., ratio of polynomials.

Complex function: why are they important?

Recall: the forced evolution is given by

$$Y(s) = H(s)U(s)$$

with $H(s)$ very often rational, i.e., ratio of polynomials. *Essential tool for automatic control people: roots of complex polynomials*

Roots of complex functions

Roots of complex functions

Primary definition: root of a complex number

if $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then the n complex roots of z are the n complex numbers $z_0, \ldots, z_{n-1}$ for which $z_k^n = z$, i.e.,

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \quad \text{for} \quad k = 0, 1, \ldots, n-1$$

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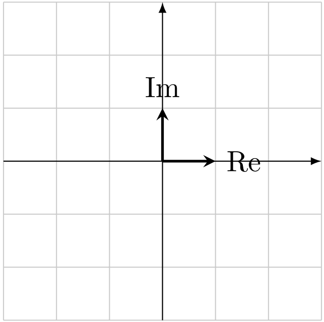
The intuition on how to get them follows from:

$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right]$$

Roots of complex functions

Geometrically:



Roots of complex functions

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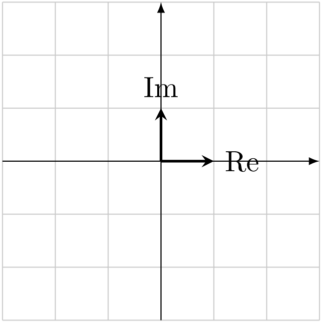
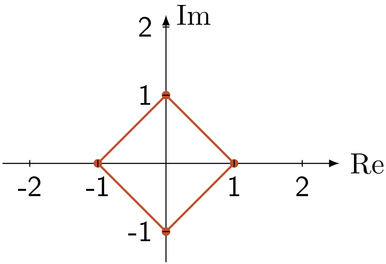


these n roots always exist

Roots of complex functions, example: quartic roots of 1

$$\sqrt[4]{1} = \{1, i, -1, -i\}$$

(note that only two of them are in $\mathbb{R}$)



IMPORTANT: ONE SHOULD CONSIDER THE PRINCIPAL VALUE

. . . otherwise one may artificially add $2\pi k$ to the phase of $w = \sqrt[n]{z}$ and have an infinite number of roots . . .

Why are we using so much time on this?

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Because we often have to do with objects of the type $z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$, thus we need to know what we are dealing with!

Why are we using so much time on this?

Because we often have to do with objects of the type $z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$, thus we need to know what we are dealing with! *Essential results:*

- n-order polynomials have always from 0 to n real roots
- n-order polynomials have always n complex roots

Example

$$z^4 - 6iz^2 + 16 = 0$$

implies

$$z_1 = 2 + 2i \quad z_2 = -2 - 2i \quad z_3 = -1 + i \quad z_4 = 1 - i$$

(to get the solution let $y = z^2$, and then do a bit of massaging)

?

Complex exponentials

Roadmap

- intuitions
- definition
- Euler's identities
- complex logarithms

In the previous episodes . . .

- complex sums and multiplications
- complex roots
- complex polynomials

In the previous episodes ...

- complex sums and multiplications
- complex roots
- complex polynomials

→ generalizing everything, even the functions

Discussion

why are exponentials important in control?

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why are exponentials important in control?

Because they are the essence of the modes of LTI systems with rational transfer functions, and LTI systems are often good approximations of nonlinear systems around their equilibria

First usefulness of complex exponentials: simplify notation even further

Question: can we write $z = r\,(\cos\theta + i\sin\theta)$ in a more complex way?

First usefulness of complex exponentials: simplify notation even further

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with $\sin$ and $\cos$? Of course, Euler's formula!

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thus

$$e^z = e^x (\cos y + i \sin y)$$

The new representation given by Euler's formula and polar representations

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$$\begin{aligned} z &= x + iy = r(\cos\theta + i\sin\theta) \\ z &= re^{i\theta} \end{aligned}$$

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$$\begin{aligned} ze^{i\alpha} &= re^{i\theta}e^{i\alpha} = re^{i(\theta+\alpha)} \\ zi &= re^{i\theta}e^{i\frac{\pi}{2}} = re^{i\left(\theta+\frac{\pi}{2}\right)} \end{aligned}$$

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that, by the way, implies $(x + iy)\,i = -y + ix$, i.e., a 90-degrees rotation

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thus

$$\sin y = \frac{1}{2i}\left(e^{iy} - e^{-iy}\right) \qquad \cos y = \frac{1}{2}\left(e^{iy} + e^{-iy}\right)$$

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- exponentials are never equal to 0, i.e., $e^z \neq 0$ independently of z
- exponentials are periodic, i.e., $e^{z+2\pi i} = e^z$

Notation: "fundamental region of the exponential"

$-\pi < \mathrm{Im}\,(z) \leq \pi$

Multiplications and divisions through the complex functions

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- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Roots through the complex functions

$w = z^n$ is s.t. $w = re^{i\theta + 2\pi k}$ and is equal to

$$z_k = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}$$

(note that besides $k = 0, 1, \ldots, n-1$*, for other* k*'s we get the same roots as before)*

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Starting point: real logarithms

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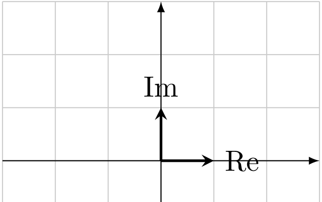
Graphically:



Very important difference

as soon as $z = re^{i\theta}$ is s.t. $r > 0$ then

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Example

$$\ln(-10) = 2.30259 + i\pi.$$

LTI filters - motivations

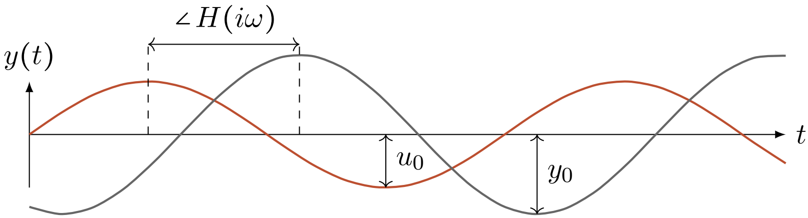
Roadmap

- how to use Fourier to analyse LTI systems
- Bode plots

Recall: why are Fourier transforms important for control people?

Among others: $H(i\omega)$ says how to apply the sinusoidal fidelity property, i.e., the fact that

$$u(t) = u_0 \sin(\omega t) \quad \Longrightarrow \quad y(t) = u_0 |H(i\omega)| \sin(\omega t + \angle H(i\omega))$$



A motivating example: sinusoidal fidelity

$$\dot{y} = ay + u \quad u(t) = \sin \omega t \qquad \Longrightarrow \qquad y(t) = y_0 e^{at} + e^{at} \int_0^t e^{-a\tau} \sin(\omega\tau)\, d\tau$$

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$$\int e^{-a\tau} \sin(\omega t) \, d\tau = \frac{e^{-a\tau}}{a^2 + \omega^2} (-a \sin \omega \tau - \omega \cos \omega \tau) + C$$

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Sinusoidal fidelity of LTIs, in details

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Notation:

- first term $=$ free evolution
- second term $=$ transient response *(part of the forced response)*
- third term $=$ stationary response *(part of the forced response)*

Why do we have free evolution, plus transient & stationary response?

Example: spring-mass system:

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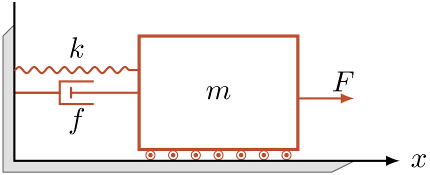

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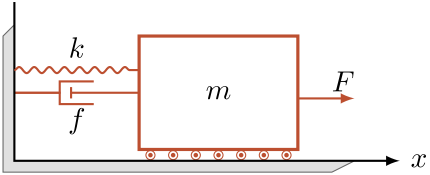
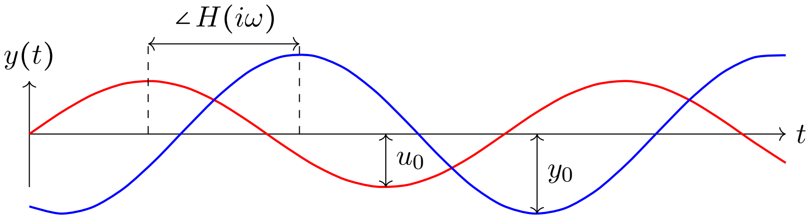


Answer: even if we have $y_0 = 0$, the machine needs to "warm up"

LTI means sinusoidal fidelity, and sinusoidal fidelity for every sine means LTI

Idea: we can estimate H by repeating experiments with different sinusoidal u's!

$$y(t) \approx u_0 |H(i\omega)| \sin(\omega t + \angle H(i\omega))$$



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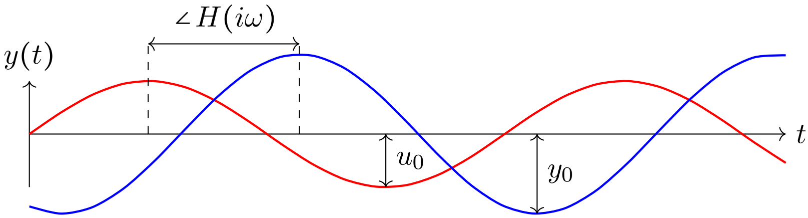
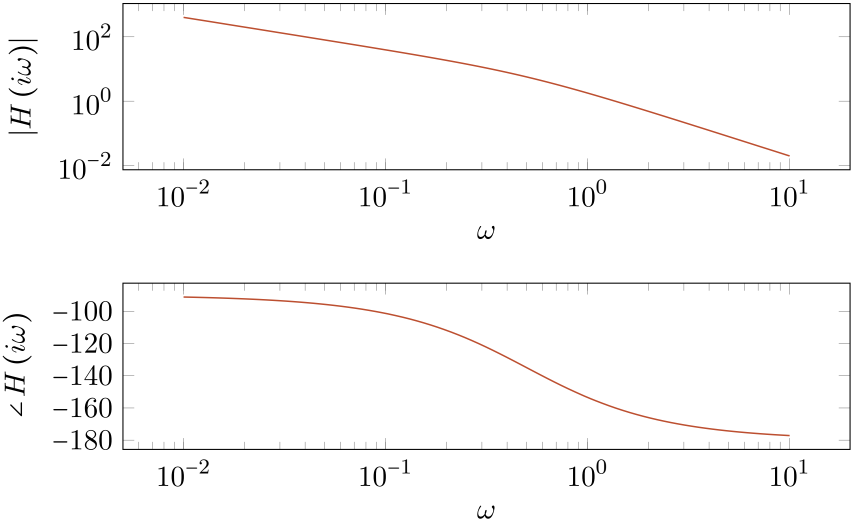


Good idea: check the "ETFE" system identification approach if you want to know more

Idea: we can estimate h by first estimating H, and then inverse-Laplacing!

...actually not. There are better ways of doing this → will be seen in courses that deal with system identification

Our goal: arrive at Bode plots



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