TTK4225 - Systems Theory, Autumn 2020

Damiano Varagnolo

NTNU

Recent feedback

- it is generally felt that there are too many hours of lecture
  - note: currently we do approx. 3 of lecture per week, and approx. 3 hours of (hopefully) conceptual questions together
  - question for me: how much time are the assignments taking?
- it is generally felt that there is a lack of structure in the lecture (especially in the middle part) → fair, will try to highlight connections more
- "we are given too little time to answer the questions in class" → kind of a feedback that you may easily give on the spot, no?
- it is desired to have entire lectures dedicated to do exercises → do you realize that procedural exercises have very little sense? Anyway, if you prefer to have 10 mins to answer 4 questions and then see them in a batch ok, and if you want one entire lesson to do exercises ok

“we also realise that you rely very much on learning strategies and models, but as engineers we both know that there is one problem with the theory: it doesn’t really work in practice. We need to be flexible with it.”

- active learning is not theory, it is actually practice; the problem is that I have not been able to make you active until now

Where are we now?

generic ODEs

$\downarrow$

LTI systems

$\downarrow$

convolution + I.R. $\mapsto y$

$\downarrow$

Laplace of I.R. = T.F.

$\downarrow$

modes and stability

What are the zeros of a system? Some more details

Roadmap

- a more formal definition
- sinusoidal fidelity
- algebraic and numeric examples

Preliminary fact

complex exponentials are eigenfunctions of LTI systems:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{t} h(\tau) e^{-j\omega(t-\tau)} d\tau \\
&= \int_{-\infty}^{t} h(\tau) e^{-j\omega t} e^{j\omega\tau} d\tau \\
&= e^{-j\omega t} \int_{-\infty}^{t} h(\tau) e^{j\omega\tau} d\tau
\end{aligned}
$$

Preliminary fact

complex exponentials are eigenfunctions of LTI systems:

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} h(\tau) e^{-j\omega(t-\tau)} d\tau \\ &= \int_{-\infty}^{t} h(\tau) e^{-j\omega t} e^{j\omega\tau} d\tau \\ &= e^{-j\omega t} \int_{-\infty}^{t} h(\tau) e^{j\omega\tau} d\tau \end{aligned}$$

Note: the integral goes from $-\infty$ so to ignore transient effects!

Important consequence: LTIs show “sinusoidal fidelity”

$$u(t) = \cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\Downarrow$$

$$y(t) = |H(j\omega)| \cos(\omega t + \angle H(j\omega))$$

Definition (Zero of a function)

$$z_i = \textit{ zero of } H(\cdot) \;\; \textit{if } H(z_i) = 0$$

Physical meaning of a zero of a transfer function

$$H \text{ LTI:} \qquad u(t) = e^{\overline{s}t} \qquad \Longrightarrow \qquad y(t) = H(\overline{s})e^{\overline{s}t}$$

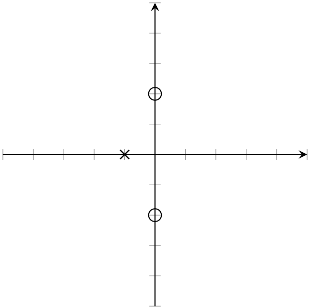
Physical meaning of a zero of a transfer function

$$H \text{ LTI:} \qquad u(t) = e^{\bar{s}t} \qquad \Longrightarrow \qquad y(t) = H(\bar{s})e^{\bar{s}t}$$

$$\text{thus} \quad y(0) = 0, \quad H(s) = K\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)}, \quad \text{and} \quad u(t) \propto e^{z_i t} \qquad \Longrightarrow \qquad y(t) = 0$$

Physical meaning of a zero - Algebraic example

$$H(s) = K\frac{s^2+4}{s^3+3s^2+3s+1} = K\frac{(s+2j)(s-2j)}{(s+1)^3}$$



Thus $\quad u(t) = 2\cos(2t) = \left(e^{j2t}+e^{-j2t}\right) \quad \Longrightarrow \quad y(t) = H(2j)e^{2jt} + H(-2j)e^{-2jt} = 0$

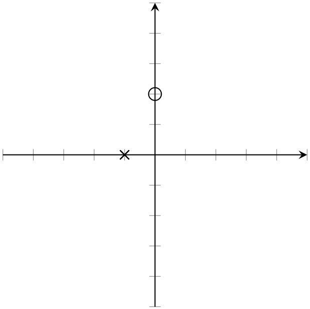
Discussion

Is it that

$$H(s) = K\frac{(s-2j)}{(s+1)^3}$$

implies

$$u(t) = 2\cos(2t) = \left(e^{j2t} + e^{-j2t}\right) \quad \Longrightarrow \quad y(t) = 0 \quad ?$$



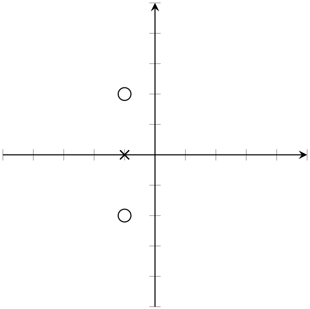
Discussion

Is it that

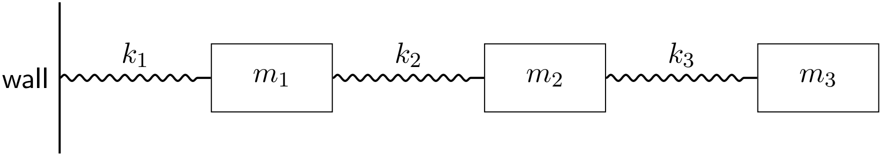
$$H(s) = K\frac{(s+1-2j)(s+1+2j)}{(s+1)^3}$$

implies

$$u(t) = 2\cos(2t) = \left(e^{j2t} + e^{-j2t}\right) \implies y(t) = 0 \quad ?$$



Physical meaning of a zero - Physical example

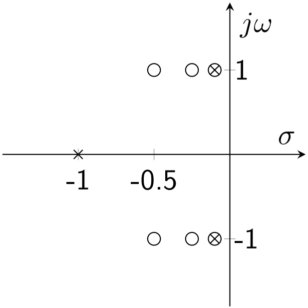


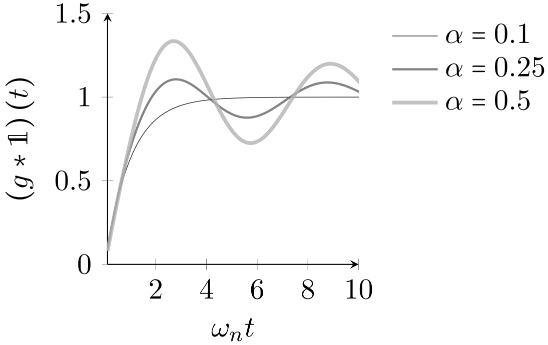
Effects of the position of the zeros: example

$$H(s) = \frac{(s+\alpha)^2 + 1}{(s+1)\left((s+0.1)^2 + 1\right)}$$

Effects of the position of the zeros: example

$$H(s) = \frac{(s+\alpha)^2 + 1}{(s+1)\left((s+0.1)^2+1\right)}$$





?

How many zeros may we have?

Discussion: s is a rational transfer function, with zero poles and one zero. May a physical system admit this specific transfer function?

How many zeros may we have?

Discussion: s is a rational transfer function, with zero poles and one zero. May a physical system admit this specific transfer function? No, because it would imply that the system knows the future!

How many zeros may we have?

Discussion: s is a rational transfer function, with zero poles and one zero. May a physical system admit this specific transfer function? No, because it would imply that the system knows the future! Nomenclature:

strictly proper transfer function: number of poles $>$ number of zeros

proper transfer function: number of poles $\geq$ number of zeros

improper transfer function: number of poles $<$ number of zeros

How many zeros may we have?

Discussion: s is a rational transfer function, with zero poles and one zero. May a physical system admit this specific transfer function? No, because it would imply that the system knows the future! Nomenclature:

strictly proper transfer function: number of poles $>$ number of zeros

proper transfer function: number of poles $\geq$ number of zeros

improper transfer function: number of poles $<$ number of zeros

all the physically realizable systems are at
least proper, and very often strictly improper

?

Connections with step responses

Roadmap

- definition of step response
- properties for first order systems
- other important examples

Definition: step response

$$Y(s) = H(s)U(s), \qquad U(s) = \frac{1}{s}$$

Definition: step response

$$Y(s) = H(s)U(s), \qquad U(s) = \frac{1}{s}$$

Discussion: may we easily implement step responses in real systems? What are the limitations?

Step responses for first order systems

Example: $\dot{x}(t) = ax(t) + bu(t)$, with $u(t) = H(t)$, implies

$$sX(s) - x_0 = aX(s) + bU(s) \qquad \Longrightarrow \qquad X(s) = \frac{1}{s-a}x_0 + \frac{b}{s(s-a)}$$

and thus $x(t) = e^{at}x_0 + \dfrac{b}{a}\left(e^{at} - H(t)\right)$.

Step responses for first order systems

Example: $\dot{x}(t) = ax(t) + bu(t)$, with $u(t) = H(t)$, implies

$$sX(s) - x_0 = aX(s) + bU(s) \qquad \Longrightarrow \qquad X(s) = \frac{1}{s-a}x_0 + \frac{b}{s(s-a)}$$

and thus $x(t) = e^{at}x_0 + \frac{b}{a}\left(e^{at} - H(t)\right)$. So, how does this look like, if $x_0 = 0$?

Step responses for first order systems

Example: $\dot{x}(t) = ax(t) + bu(t)$, with $u(t) = H(t)$, implies

$$sX(s) - x_0 = aX(s) + bU(s) \qquad \Longrightarrow \qquad X(s) = \frac{1}{s-a}x_0 + \frac{b}{s(s-a)}$$

and thus $x(t) = e^{at}x_0 + \frac{b}{a}\left(e^{at} - H(t)\right)$. So, how does this look like, if $x_0 = 0$?

$$\boxed{\kappa\left(H(t) - e^{at}\right) \text{ = generic step response of first order systems}}$$

Step responses for first order systems

Example: $\dot{x}(t) = ax(t) + bu(t)$, with $u(t) = H(t)$, implies

$$sX(s) - x_0 = aX(s) + bU(s) \qquad \Longrightarrow \qquad X(s) = \frac{1}{s-a}x_0 + \frac{b}{s(s-a)}$$

and thus $x(t) = e^{at}x_0 + \dfrac{b}{a}\left(e^{at} - H(t)\right)$. So, how does this look like, if $x_0 = 0$?

$$\boxed{\kappa\left(H(t) - e^{at}\right) \text{ = generic step response of first order systems}}$$

Discussion:

- does the system converge somewhere?

Step responses for first order systems

Example: $\dot{x}(t) = ax(t) + bu(t)$, with $u(t) = H(t)$, implies

$$sX(s) - x_0 = aX(s) + bU(s) \qquad \Longrightarrow \qquad X(s) = \frac{1}{s-a}x_0 + \frac{b}{s(s-a)}$$

and thus $x(t) = e^{at}x_0 + \frac{b}{a}\left(e^{at} - H(t)\right)$. So, how does this look like, if $x_0 = 0$?

$$\boxed{\kappa\left(H(t) - e^{at}\right) = \text{generic step response of first order systems}}$$

Discussion:

- does the system converge somewhere?
- may the system response pass this value?

Step responses for first order systems

Example: $\dot{x}(t) = ax(t) + bu(t)$, with $u(t) = H(t)$, implies

$$sX(s) - x_0 = aX(s) + bU(s) \qquad \Longrightarrow \qquad X(s) = \frac{1}{s-a}x_0 + \frac{b}{s(s-a)}$$

and thus $x(t) = e^{at}x_0 + \dfrac{b}{a}\left(e^{at} - H(t)\right)$. So, how does this look like, if $x_0 = 0$?

$$\boxed{\kappa\left(H(t) - e^{at}\right) \text{ = generic step response of first order systems}}$$

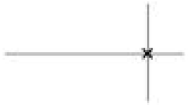
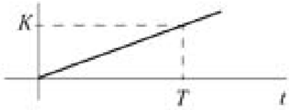
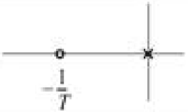
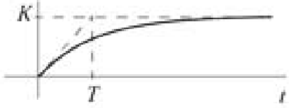
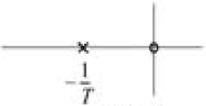
Discussion:

- does the system converge somewhere?
- may the system response pass this value?
- may we compute the time constant of the system in this case?

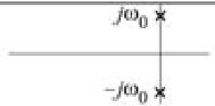
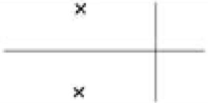
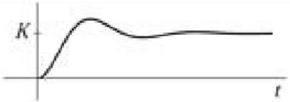
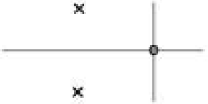
Examples for zeroth order systems

K		

Examples for first order systems

$K\frac{1}{Ts}$		
$K\frac{1+Ts}{Ts}$		
$K\frac{1}{1+Ts}$		
$K\frac{Ts}{1+Ts}$		

Examples for second order systems

$\dfrac{K}{1+\left(\dfrac{s}{\omega_0}\right)^2}$		
$\dfrac{K}{1+2\zeta\dfrac{s}{\omega_0}+\left(\dfrac{s}{\omega_0}\right)^2}$		
$\dfrac{Ks}{1+2\zeta\dfrac{s}{\omega_0}+\left(\dfrac{s}{\omega_0}\right)^2}$		

Important message

if the input is $\dot{u}$ then the step response gives the impulse response that we would get if the input were u, instead

Discussion

What about $\dfrac{K(s+a)}{1+2\xi\frac{s}{\omega}+\left(\frac{s}{\omega}\right)^2}$?

?

Block diagrams

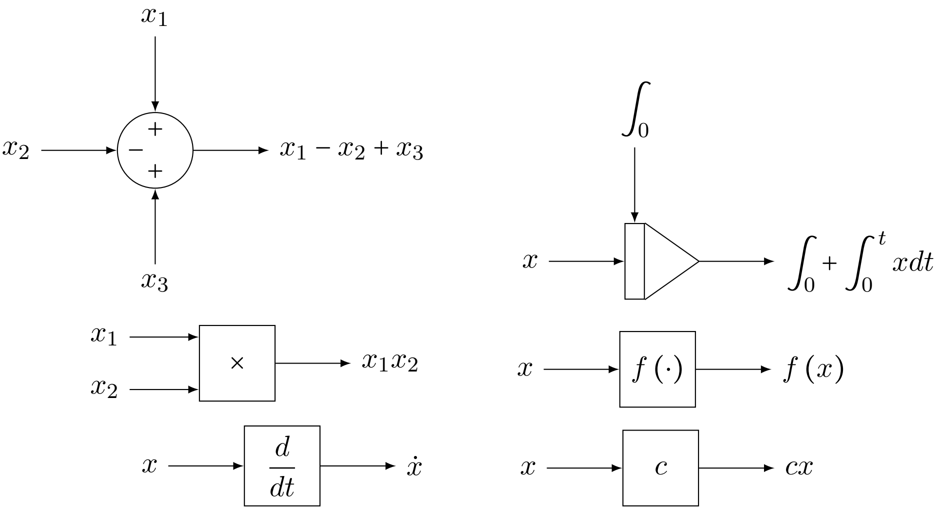
Roadmap

- recap of the diagrams in the time domain
- recap of the diagrams in the frequency domain
- rules for how to transform the diagrams
- examples

Block diagrams - why?

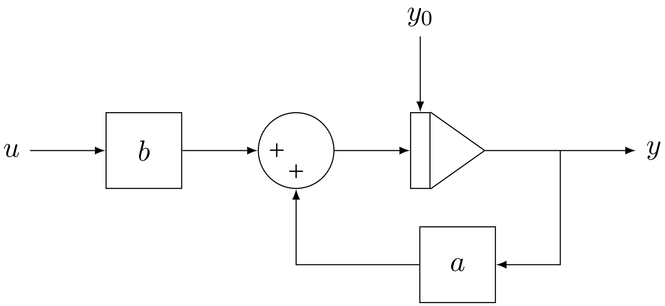
- used very often in companies
- aid visualization *(until a certain complexity is reached...)*
- enable "drag & drop" way of programming
- here primarily used for interpretations

Most common block diagrams in the time domain



Representing a first order DE with a block scheme

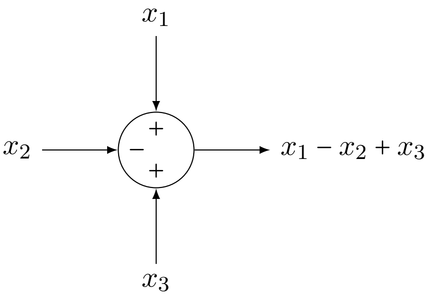
$$\dot{y} = ay + bu$$

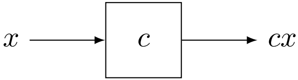


Discussion: how do we represent $\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}u$?

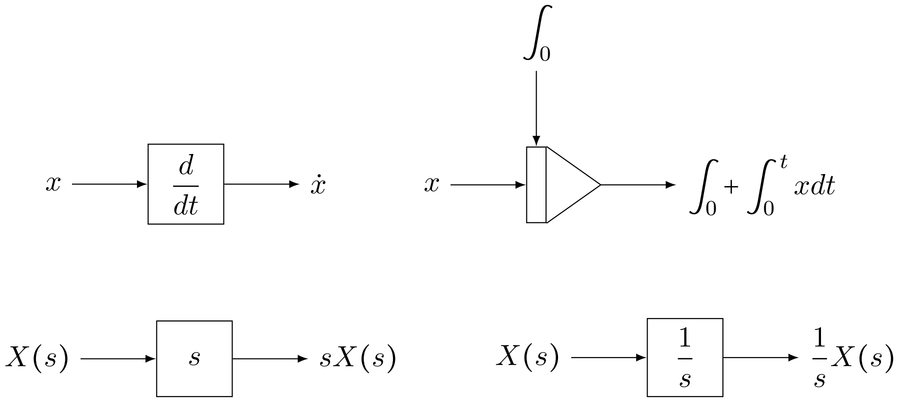
?

Block diagrams that are equal in both time and frequency domains

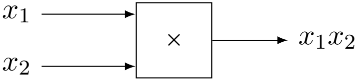


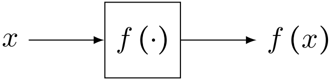


Block diagrams that are logically the same in both time and frequency domains



Block diagrams that do not exist in the frequency domain





Discussion: why?

rules for manipulating block diagrams in the frequency domain

Series of transfer functions

$$U(s) \longrightarrow \boxed{H_a(s)} \longrightarrow \boxed{H_b(s)} \longrightarrow Y(s)$$

is equivalent to

$$U(s) \longrightarrow \boxed{H_a(s)H_b(s)} \longrightarrow Y(s)$$

Series of transfer functions

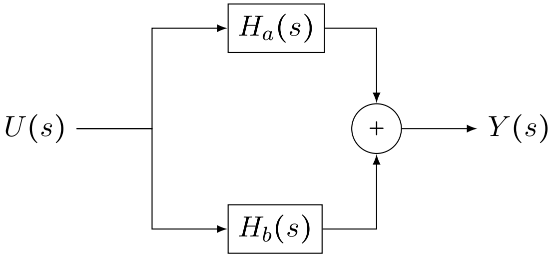
$$U(s) \longrightarrow \boxed{H_a(s)} \longrightarrow \boxed{H_b(s)} \longrightarrow Y(s)$$

is equivalent to

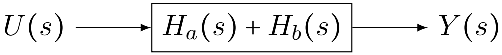
$$U(s) \longrightarrow \boxed{H_a(s)H_b(s)} \longrightarrow Y(s)$$

Discussion: why?

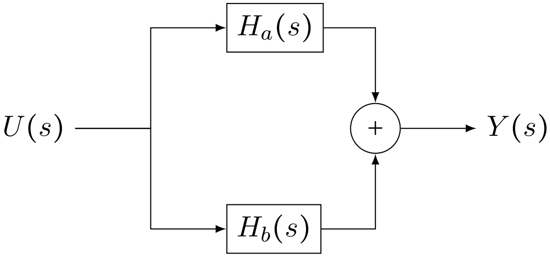
Parallel of transfer functions



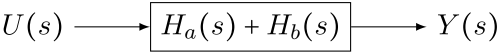
is equivalent to



Parallel of transfer functions

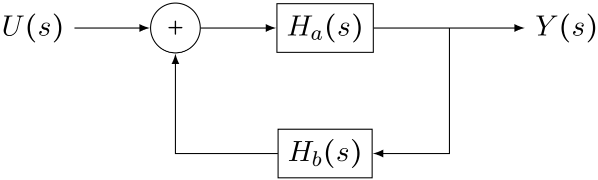


is equivalent to

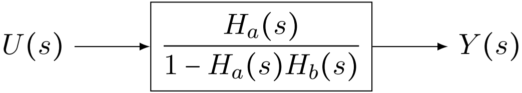


Discussion: why?

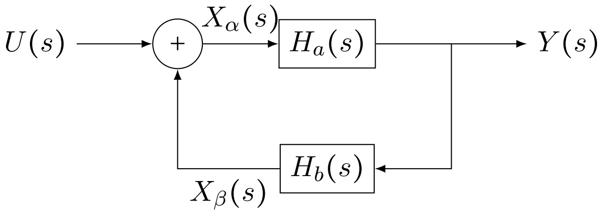
Elimination of feedback loops



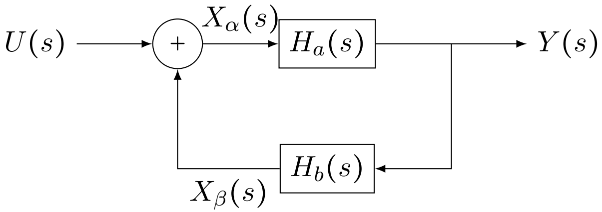
is equivalent to



Elimination of feedback loops: how to remember the formula

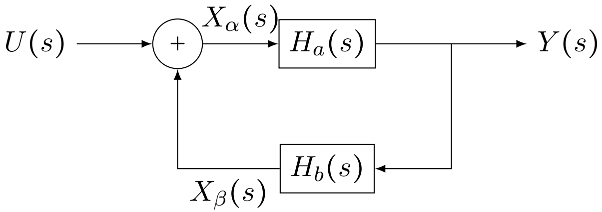


Elimination of feedback loops: how to remember the formula



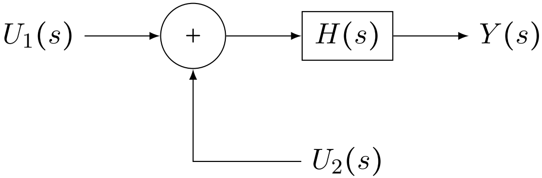
- $Y = H_a X_\alpha$
- $X_\alpha = U + X_\beta$
- $X_\beta = H_b Y$

Elimination of feedback loops: how to remember the formula

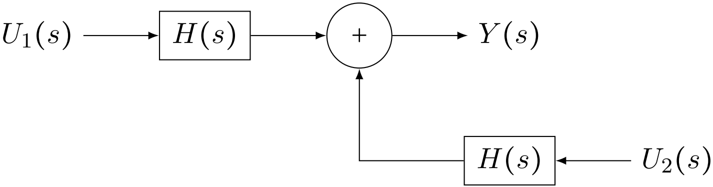


- $Y = H_a X_\alpha$
- $X_\alpha = U + X_\beta$
- $X_\beta = H_b Y$
- $\implies Y = H_a\left(U + H_b Y\right)$

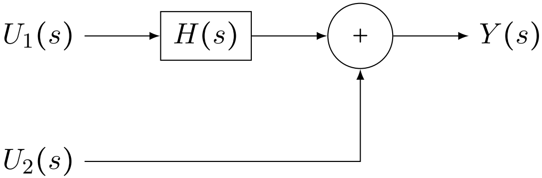
Moving blocks around sum operators



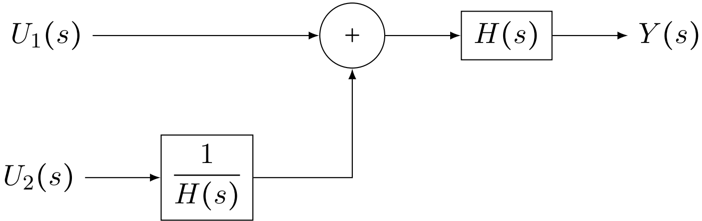
is equivalent to



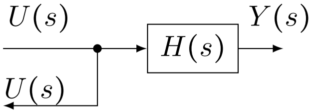
Moving blocks around sum operators



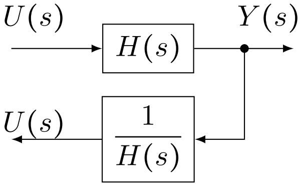
is equivalent to



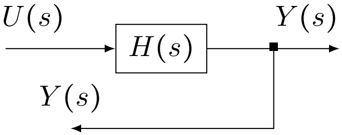
Moving blocks around connections



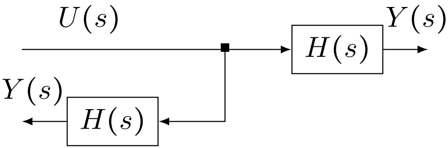
is equivalent to



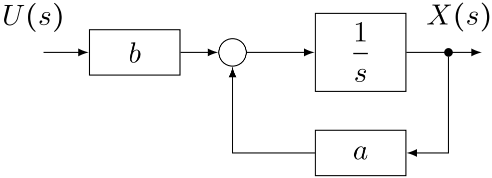
Moving blocks around connections



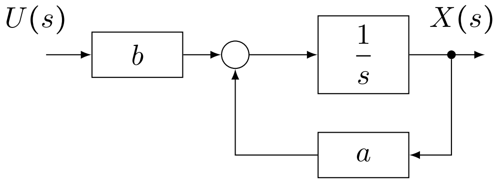
is equivalent to

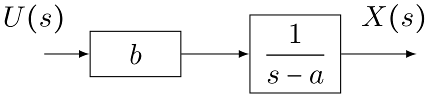


Example: $\dot{x} = ax + bu$

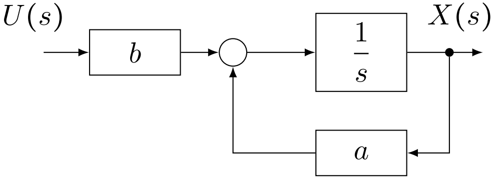


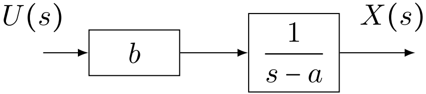
Example: $\dot{x} = ax + bu$

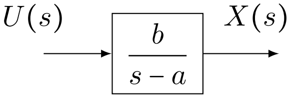




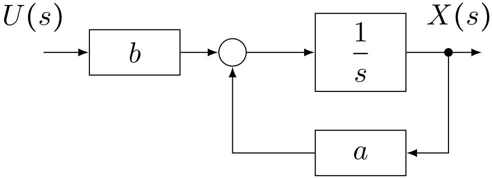
Example: $\dot{x} = ax + bu$

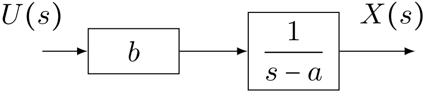


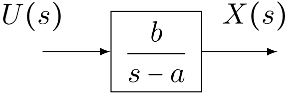




Example: $\dot{x} = ax + bu$





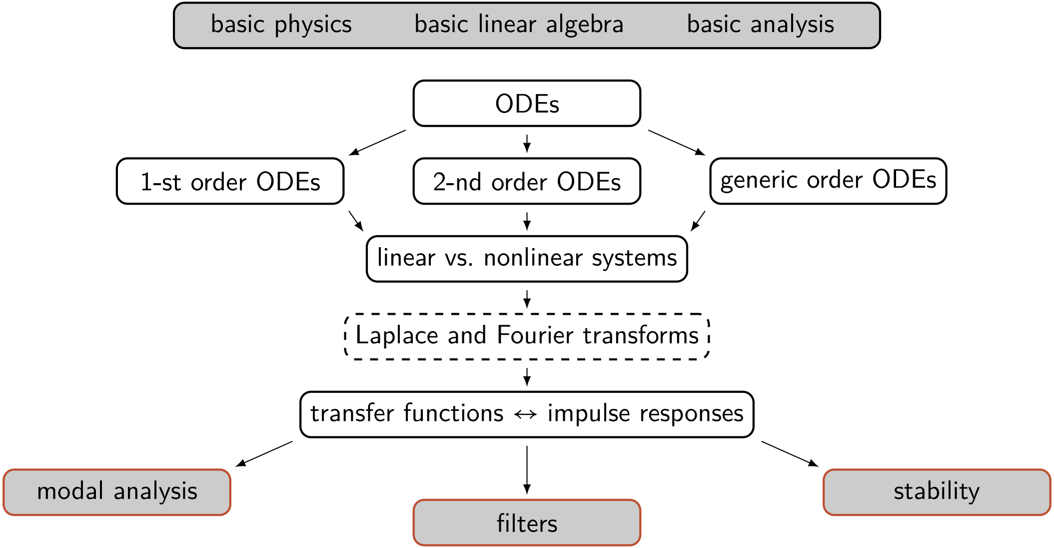


Discussion: what about the stability of the feedback loop?

?

Equilibria

Summary / Roadmap of TTK4225



Roadmap

- what does “equilibrium” mean?
- examples
- equilibria in LTI systems

Equilibrium, what does it mean?

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})$$

Example: exponential growth

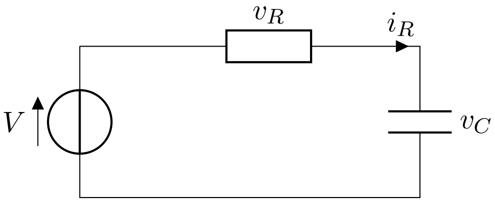
Scalar version:

$$\dot{y} = \alpha y + \beta u \tag{1}$$

Matricial version:

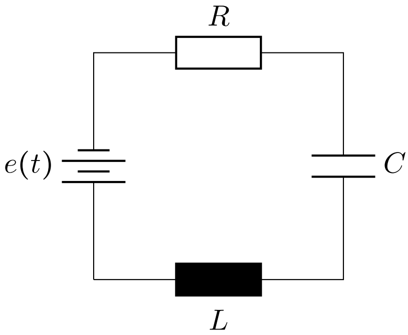
$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u} \tag{2}$$

Example: RC-circuit



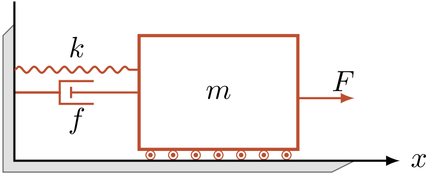
$$\dot{v}_C = -\frac{1}{RC}v_C + \frac{1}{RC}V(t)$$

Example: RCL-circuit



$$e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau$$

Example: spring-mass systems



$$m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + F(t)$$

Example: Lotka-Volterra

- y_{prey} := prey
- y_{pred} := predator

$$\begin{cases} \dot{y}_{\text{prey}} &= \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} &= -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

Example: Van-der-Pol oscillator

$$\begin{cases} \dot{y}_1 & = \mu\left(y_1 - \dfrac{y_1^3}{3} - y_2\right) \\ \dot{y}_2 & = \dfrac{y_1}{\mu} \end{cases}$$

Example: balancing robot

$$
\begin{aligned}
\left(I_b + m_b l_b^2\right)\ddot{\theta}_b &= +m_b l_b g \sin\left(\theta_b\right) - m_b l_b \ddot{x}_w \cos\left(\theta_b\right) - \frac{K_t}{R_m} v_m + \left(\frac{K_e K_t}{R_m} + b_f\right)\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \\
\left(\frac{I_w}{l_w} + l_w m_b + l_w m_w\right)\ddot{x}_w &= -m_b l_b l_w \ddot{\theta}_b \cos\left(\theta_b\right) + m_b l_b l_w \dot{\theta}_b^2 \sin\left(\theta_b\right) + \frac{K_t}{R_m} v_m - \left(\frac{K_e K_t}{R_m} + b_f\right)\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right)
\end{aligned}
$$

Example: insulin concentration

- $x_1 :=$ sugar concentration
- $x_2 :=$ insulin concentration
- $u_1 :=$ food intake
- $u_2 :=$ insulin intake
- $c :=$ sugar concentration in fasting

$$\begin{cases} \dot{x}_2 = a_{21}\left(x_1 - c\right) - a_{22}x_2 + b_2 u_2 & x_1 \geq c \\ \dot{x}_2 = -a_{22}x_2 + b_2 u_2 & x_1 < c \end{cases}$$

$$\begin{cases} \dot{x}_1 = -a_{11}x_1 x_2 - a_{12}\left(x_1 - c\right) + b_1 u_1 & x_1 \geq c \\ \dot{x}_1 = -a_{11}x_1 x_2 + b_1 u_1 & x_1 < c \end{cases}$$

Very important case: what are the equilibria here?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y}$$

Very important case: what are the equilibria here?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y}$$

for linear systems $\mathbf{0}$ is always an equilibrium, and if $\overline{\boldsymbol{y}} \neq \mathbf{0}$ is also an equilibrium then every $\alpha\overline{\boldsymbol{y}}$ is an equilibrium too

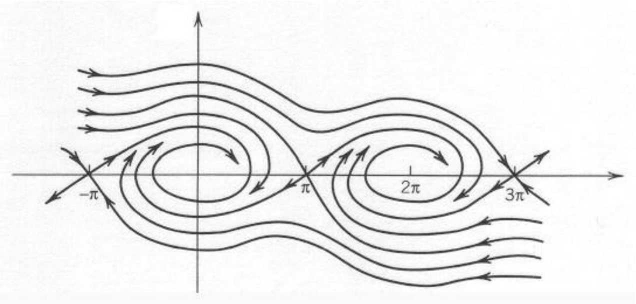
Stability

Roadmap

- what does “equilibrium” mean?
- examples
- equilibria in LTI systems

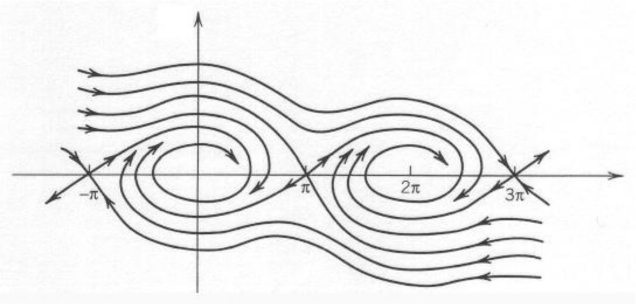
The different nature of different equilibria

pendulum with friction: $\ddot{\theta} = -\lambda\dot{\theta} - g\sin(\theta)$



The different nature of different equilibria

pendulum with friction: $\ddot{\theta} = -\lambda\dot{\theta} - g\sin(\theta)$



Discussion: do you find different types of equilibria?

Different notions of equilibria and stability

For specific equilibria:

1. simply stable equilibrium
2. convergent equilibrium
3. asymptotically stable equilibrium

For systems:

1. Bounded Input Bounded Output (BIBO) stable systems
2. Input to State Stable (ISS) systems (not in this course)

Different notions of equilibria and stability

For specific equilibria:

1. simply stable equilibrium
2. convergent equilibrium
3. asymptotically stable equilibrium

For systems:

1. Bounded Input Bounded Output (BIBO) stable systems
2. Input to State Stable (ISS) systems (not in this course)

important assumption for 1, 2 & 3: $\boldsymbol{u} = \overline{\boldsymbol{u}} = const.$

Simply stable equilibrium (continuous time case)

$$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const. \qquad \begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_e) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}_e) \end{aligned} \qquad (\boldsymbol{x}_e, \boldsymbol{u}_e) = \text{equilibrium}$$

Definition (simply stable equilibrium)

an equilibrium $\boldsymbol{x}_e$ *is called simply stable if* $\forall \varepsilon > 0 \; \exists \delta > 0$ *s.t. if* $\|\boldsymbol{x}_0 - \boldsymbol{x}_e\| \leq \delta$ *then* $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \leq \varepsilon$ *for every* $t \geq 0$

Convergent equilibrium

$$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const.$$

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}_e\right) \\ \boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}_e\right) \end{aligned}$$

$\left(\boldsymbol{x}_e, \boldsymbol{u}_e\right)$ = equilibrium

Convergent equilibrium

$$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const.$$

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}_e\right) \\ \boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}_e\right) \end{aligned}$$

$\left(\boldsymbol{x}_e, \boldsymbol{u}_e\right)$ = equilibrium

Definition (convergent equilibrium)

the equilibrium $\boldsymbol{x}_e$ = *is said to be convergent if* $\exists \delta > 0$ *s.t. if* $\left\|\boldsymbol{x}_0 - \boldsymbol{x}_e\right\| \leq \delta$ *then* $\boldsymbol{x}(t) \xrightarrow{t \to +\infty} \boldsymbol{x}_e$

Important differences

simple stability: I can confine arbitrarily the trajectory by reducing ε opportunely

Important differences

simple stability: I can confine arbitrarily the trajectory by reducing ε opportunely

convergent equilibrium: I *cannot* confine arbitrarily the trajectory, but I know that if I start close enough then the more I wait the smaller the $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\|$ will be

Important differences

simple stability: I can confine arbitrarily the trajectory by reducing ε opportunely

convergent equilibrium: I *cannot* confine arbitrarily the trajectory, but I know that if I start close enough then the more I wait the smaller the $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\|$ will be

Discussion: Consider the system

$$x(k+1) = \begin{cases} 2x(k) & \text{if } |x(k)| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is 0? Possibilities:

1. a simply stable equilibrium
2. a convergent equilibrium
3. nothing special

Asymptotic stability

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \overline{\boldsymbol{u}}\right) \\ \boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \overline{\boldsymbol{u}}\right) \end{aligned}$$

$\boldsymbol{x}_e$ = equilibrium

Asymptotic stability

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \overline{\boldsymbol{u}}\right) \\ \boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \overline{\boldsymbol{u}}\right) \end{aligned} \qquad \boldsymbol{x}_e = \text{equilibrium}$$

Definition (asymptotically stable equilibrium)

the equilibrium $\boldsymbol{x}_e$ is said to be asymptotically stable if it is simultaneously simply stable & convergent

BIBO stability

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\
\boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u})
\end{aligned}
$$

BIBO stability

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned}$$

Definition (BIBO stability)

the system $(\boldsymbol{f}, \boldsymbol{g})$ *is said to be Bounded Input Bounded Output (BIBO) stable if*

$\|\boldsymbol{u}\| \leq \gamma_u \implies \|\boldsymbol{y}\| \leq \gamma_y$

BIBO stability

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\ \boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right) \end{aligned}$$

Definition (BIBO stability)

the system $\left(\boldsymbol{f}, \boldsymbol{g}\right)$ is said to be Bounded Input Bounded Output (BIBO) stable if
$\|\boldsymbol{u}\| \leq \gamma_u \implies \|\boldsymbol{y}\| \leq \gamma_y$

Discussion: is the system $\dot{x} = -x^2 u$ BIBO stable?

BIBO stability

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\ \boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right) \end{aligned}$$

Definition (BIBO stability)

the system $(\boldsymbol{f}, \boldsymbol{g})$ *is said to be Bounded Input Bounded Output (BIBO) stable if*
$\|\boldsymbol{u}\| \leq \gamma_u \implies \|\boldsymbol{y}\| \leq \gamma_y$

Discussion: is the system $\dot{x} = -x^2 u$ BIBO stable? to check BIBO stability one can use

the "small gain theorem" (not in this course – in this course we will check the BIBO stability checking either the impulse response or the transfer function)

Noticeable differences

In general:

$$\text{BIBO stable} \neq \text{asymptotically stable} \neq \text{simply stable}$$

For LTIs:

$$\text{BIBO stable} = \text{asymptotically stable} \neq \text{simply stable}$$

but with the "$=$" valid only under certain conditions

Simple stability - continuous time case

$$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const.$$

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}_e\right) \\ \boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}_e\right) \end{aligned}$$

$\left(\boldsymbol{x}_e, \boldsymbol{u}_e\right)$ = equilibrium

Simple stability - continuous time case

$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const.$

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}_e\right) \\ \boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}_e\right) \end{aligned}$$

$\left(\boldsymbol{x}_e, \boldsymbol{u}_e\right)$ = equilibrium

Discussion: consider $\dot{x} = -\tan(x) + \sqrt{u}$, and $u = u_e = const.$ Then $x_e = \ldots$

- $\arctan\left(u_e^2\right)$?
- $\arctan\left(\sqrt{u_e}\right)$?
- $\sqrt{u_e}$?

Very important point

to have instability it is enough to have one trajectory that escapes (example: “constrained” flipped pendulum)

Remember: simple stability

$$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const.$$

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_e) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}_e) \end{aligned}$$

$(\boldsymbol{x}_e, \boldsymbol{u}_e)$ = equilibrium

Definition (simply stable equilibrium)

an equilibrium $\boldsymbol{x}_e$ is called simply stable if $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. if $\|\boldsymbol{x}_0 - \boldsymbol{x}_e\| \leq \delta$ then $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \leq \varepsilon$ for every $t \geq 0$

Some examples of phase portraits

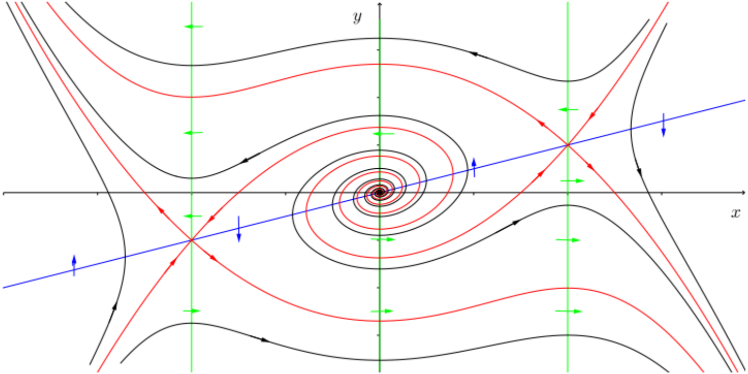
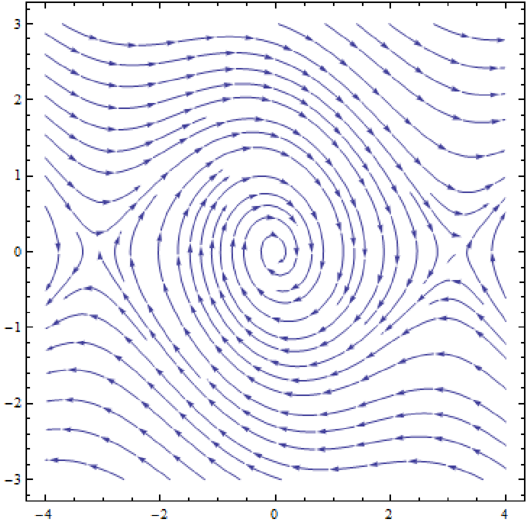
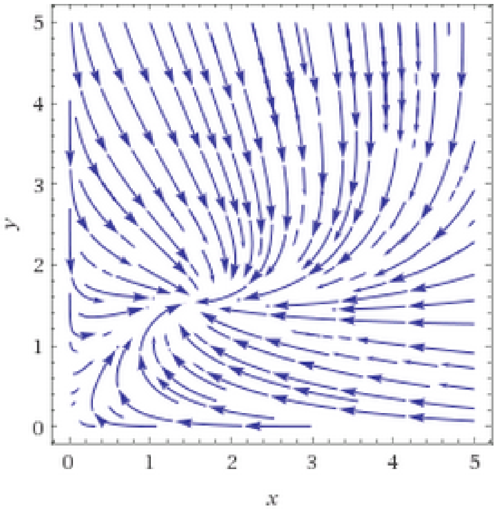


Figure 14: Phase portrait of system (43). Note the horizontal isoclines are coloured green, while the vertical isocline is coloured blue.

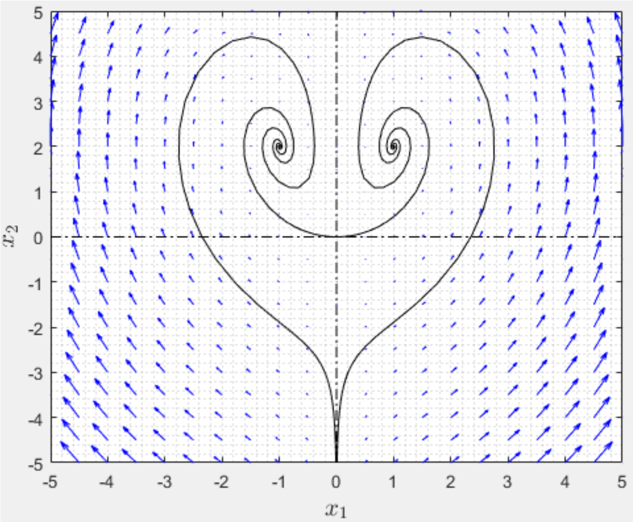
Some examples of phase portraits



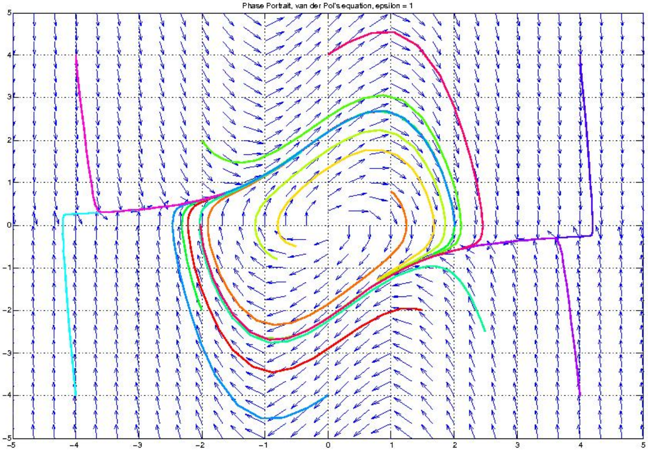
Some examples of phase portraits



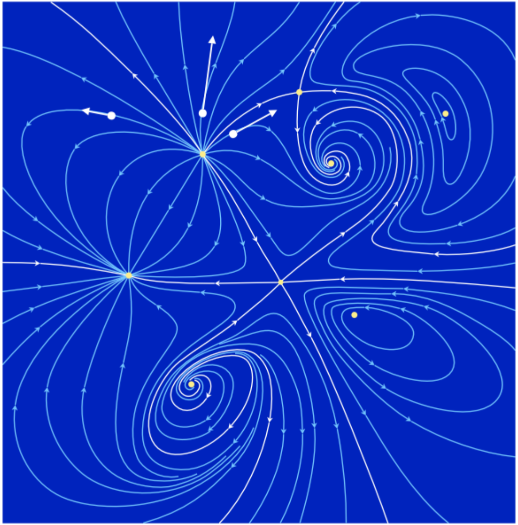
Some examples of phase portraits



Some examples of phase portraits



Some examples of phase portraits



The concepts above, in words

asymptotic input-output stability: independently of $u(t)$, $x(t) \to 0$ when $t \to +\infty$

marginal (or simply input-output) stability: as soon as $|u(t)| < M_u$, $|x(t)| < M_x$ when $t \to +\infty$

instability: there exists at least one signal $u(t)$ for which we cannot do the bound $|x(t)| < M_x$ when $t \to +\infty$

The concepts above, in words

asymptotic input-output stability: independently of $u(t)$, $x(t) \to 0$ when $t \to +\infty$

marginal (or simply input-output) stability: as soon as $|u(t)| < M_u$, $|x(t)| < M_x$ when $t \to +\infty$

instability: there exists at least one signal $u(t)$ for which we cannot do the bound $|x(t)| < M_x$ when $t \to +\infty$

now = connect all these concepts with what we saw before in the course

?

Coupling the various stabilities to the impulse response and transfer function

$$x(t) = h * u(t) = \int_0^t h(\tau)u(t-\tau)d\tau$$

$$X(s) = H(s)U(s)$$

Coupling the various stabilities to the impulse response and transfer function

$$x(t) = h * u(t) = \int_0^t h(\tau)u(t-\tau)d\tau$$

$$X(s) = H(s)U(s)$$

Discussion points:

- $|u(t)| < M_u$ means a "simply stable" $u(t)$. Assuming that $u(t)$ has a rational Laplace transform, where do the poles of $U(s)$ live in this case?

Coupling the various stabilities to the impulse response and transfer function

$$x(t) = h * u(t) = \int_0^t h(\tau)u(t-\tau)d\tau$$

$$X(s) = H(s)U(s)$$

Discussion points:

- $|u(t)| < M_u$ means a "simply stable" $u(t)$. Assuming that $u(t)$ has a rational Laplace transform, where do the poles of $U(s)$ live in this case?
- what happens if h is unstable? Where are its poles?

Coupling the various stabilities to the impulse response and transfer function

$$x(t) = h * u(t) = \int_0^t h(\tau)u(t-\tau)d\tau$$

$$X(s) = H(s)U(s)$$

Discussion points:

- $|u(t)| < M_u$ means a "simply stable" $u(t)$. Assuming that $u(t)$ has a rational Laplace transform, where do the poles of $U(s)$ live in this case?
- what happens if h is unstable? Where are its poles?
- what happens if h is strictly stable? Where are its poles? And may I choose some stable $u(t)$ that makes the system unstable?

Coupling the various stabilities to the impulse response and transfer function

$$x(t) = h * u(t) = \int_0^t h(\tau) u(t - \tau) d\tau$$

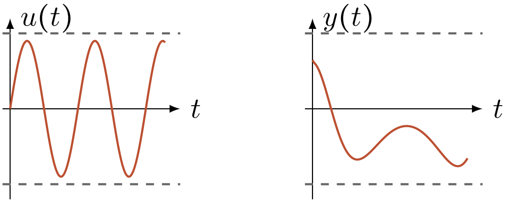
$$X(s) = H(s) U(s)$$

Discussion points:

- $|u(t)| < M_u$ means a "simply stable" $u(t)$. Assuming that $u(t)$ has a rational Laplace transform, where do the poles of $U(s)$ live in this case?
- what happens if h is unstable? Where are its poles?
- what happens if h is strictly stable? Where are its poles? And may I choose some stable $u(t)$ that makes the system unstable?
- what happens if h is simply stable? Where are its poles? And may I choose some stable $u(t)$ that makes the system unstable?

Summarizing

BIBO stability



Asymptotic stability