TTK4225 - Systems Theory, Autumn 2020

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Introduction to first order systems - the concept of free evolution

Roadmap

- generalization of our examples of first order systems
- the concept of free evolution
- introduction to stability
- free evolution $\neq$ forced response

First order dynamical models

- train velocity

$$\dot{v} = -\frac{k}{m} v(t) + \frac{k}{m} F(t) \tag{1}$$

- RC-circuit

$$\dot{v}_C = -\frac{1}{RC} v_C + \frac{1}{RC} V(t) \tag{2}$$

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In general:

$$\dot{y} = ay + bu \tag{3}$$

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`GitHub/TTK4225/trunk/Jupyter/`
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Overarching question towards model-based control: how does y evolve?

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Qualitative discussions:

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- what happens if $u = 0$, $y_0 \neq 0$, and $a < 0$?
- what happens if $u = 0$, and $a = 0$?

stability = extremely important topic!

will be analysed in more details later on in this course

and much more extensively in others

(feat. Lyapunov, Krasovskii, La-Salle among others)

An important case: free evolution

$$\dot{y} = ay \cancel{+bu} \qquad\qquad y(0) = y_0 \neq 0 \tag{5}$$

Towards the solution of the free evolution starting from a simplified case

$$\dot{y} = y \qquad\qquad y(0) = y_0 \neq 0 \tag{6}$$

(in other words, which function is equal to its derivative?)

The solution of the free evolution in general

$$\dot{y} = ay \qquad\qquad y(0) = y_0 \neq 0 \tag{7}$$

The solution of the free evolution in general

$$\dot{y} = ay \qquad\qquad y(0) = y_0 \neq 0 \tag{7}$$

Alternatively:

$$\dot{y} = \frac{dy}{dt} = ay(t) \tag{8}$$

$$\frac{dy}{y(t)} = a\ dt \tag{9}$$

$$\int_0^t \frac{dy}{y(\tau)} = \int_0^t a\ \mathsf{d}\tau \tag{10}$$

$$\Big[\ln y(\tau)\Big]_0^t = a\Big[\tau\Big]_0^t \tag{11}$$

$$\ln\Big(v(t)\Big) = \ln(y_0) + at \tag{12}$$

$$y(t) = e^{\ln(y_0)at} = e^{\ln(y_0)}e^{at} \tag{13}$$

$$y(t) = y_0 e^{at} \tag{14}$$

And what about the vectorial generalization?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y}$$

$\Longrightarrow$ later in the course, and extensively!

Analysis of first order systems in free evolution

$$\dot{y} = ay \qquad y(0) = y_0 \neq 0 \qquad \implies \qquad y(t) = y_0 e^{at} \tag{15}$$

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Discussion: which case is this one?

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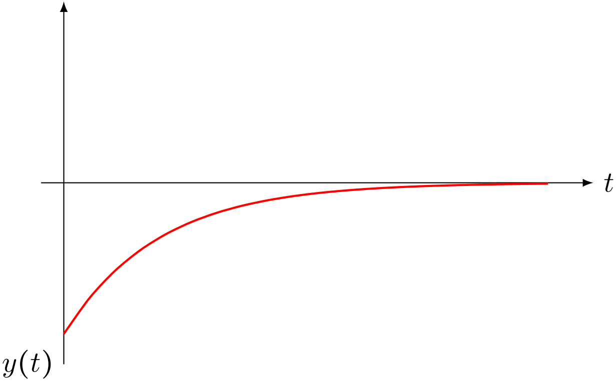

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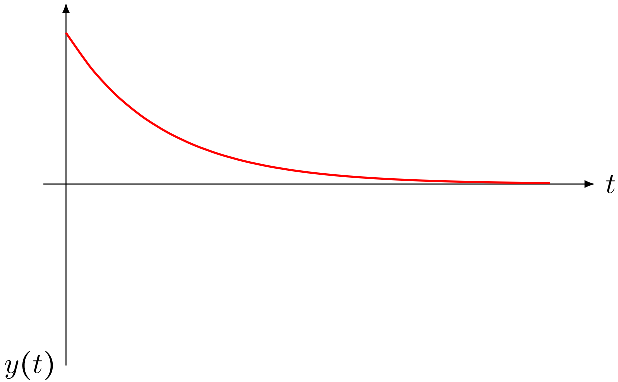

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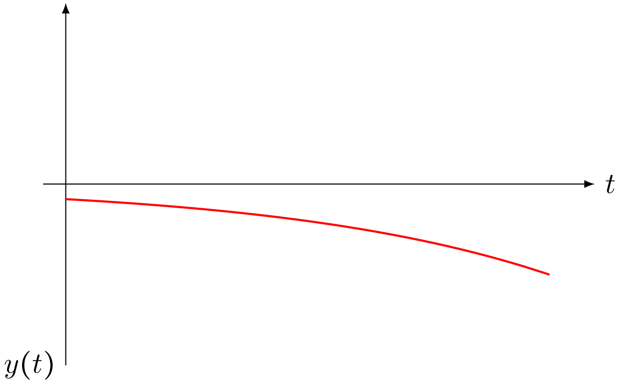

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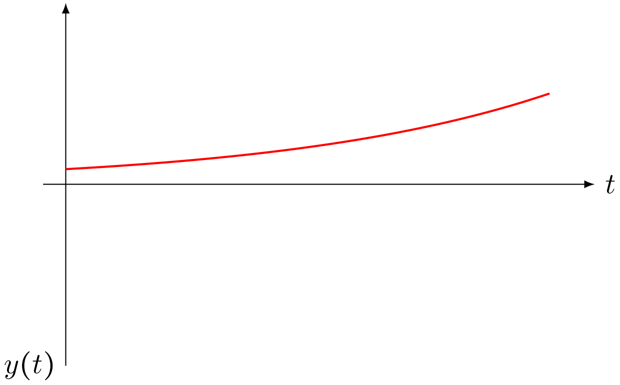

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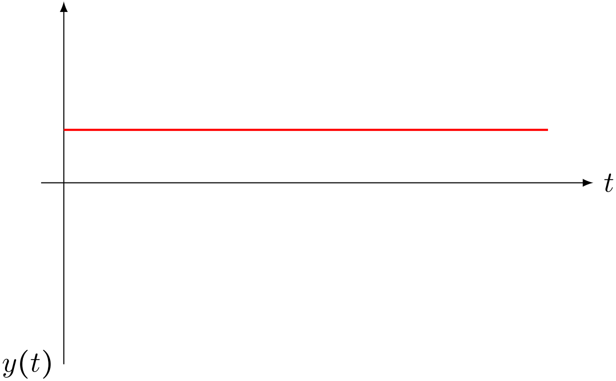

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Introduction to stability

stable system: $y(t) \to 0$ for $t \to +\infty$

unstable system: $|y(t)| \to +\infty$ for $t \to +\infty$

marginally stable system: $y(t) = y_0$ for every t

Introduction to stability

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much more on stability both later on & in later courses!

Are we done with studying the response of first order dynamical models?

$$\dot{y} = ay + bu \tag{16}$$

Forced response

$$\dot{y} = ay + bu \qquad\qquad y(0) = y_0 = 0 \tag{17}$$

(remember: free evolution $=$ *"*$u = 0, y_0 \neq 0$*")*

Forced response

$$\dot{y} = ay + bu \qquad\qquad y(0) = y_0 = 0 \tag{17}$$

(remember: free evolution = "$u = 0, y_0 \neq 0$"*)*

Discussion: does $y(0) = y_0 = 0$ imply that we should study $\dot{y} = bu$?

Try it yourself at home!

`GitHub/TTK4225/trunk/Jupyter/`
`first-order-system-introduction-with-P-controller.ipynb`

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Introduction to first order systems - the concept of forced response

Roadmap

- the concept of “forced response”
- the convolution operator
- some properties of convolution

First order dynamical models

$$\dot{y} = ay + bu \tag{18}$$

`GitHub/TTK4225/trunk/Jupyter/`
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Overarching question towards model-based control: how does y evolve?

$$\dot{y} = ay + bu \tag{19}$$

Free evolution vs. forced response

$$\dot{y} = ay + bu \tag{20}$$

free evolution: $u = 0,\ y_0 \neq 0$

forced response: $u \neq 0,\ y_0 = 0$

Solution to the forced response

(actual solution through equations (2.11)–(2.17) in Balchen's book)

$$\dot{y} = ay + bu, \quad y(0) = y_0 = 0 \qquad \Longrightarrow \qquad y(t) = \int_{-\infty}^{+\infty} h(\tau) bu(t - \tau) d\tau \tag{21}$$

with

$$h(\tau) = \begin{cases} 0 & \text{if } t < 0 \\ e^{a(\tau)} & \text{otherwise} \end{cases}$$

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extremely important definition:

convolution $:= \; h * u(t) \; := \; \int_{-\infty}^{+\infty} h(\tau) u(t - \tau) d\tau$

Visualizing $h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$

Discussions

$$h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

1. is $h * u(t) = u * h(t)$?

Discussions

$$h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

1. is $h * u(t) = u * h(t)$?
2. is $(\alpha h_1 + \beta h_2) * u(t) = \alpha\left(h_1 * u(t)\right) + \beta\left(h_2 * u(t)\right)$?

Discussions

$$h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

1. is $h * u(t) = u * h(t)$?
2. is $(\alpha h_1 + \beta h_2) * u(t) = \alpha\left(h_1 * u(t)\right) + \beta\left(h_2 * u(t)\right)$?
3. assuming

$$h(\tau) = \begin{cases} 0 & \text{if } t < 0 \\ e^{a(\tau)} & \text{otherwise} \end{cases} \qquad \text{and} \qquad u(t) = 0 \text{ if } t < 0,$$

can we simplify

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)bu(t-\tau)d\tau$$

somehow?

Our LTI case

$$\dot{y} = ay + bu, \quad y(0) = y_0 = 0$$

implies

$$y(t) = \int_0^t h(\tau)u(t-\tau)d\tau$$

(or, alternatively and equivalently, $y(t) = \int_0^t h(t-\tau)u(\tau)d\tau$)

Example

$$y(t) = \int_0^t h(\tau)u(t-\tau)d\tau,$$

$$u(t) = \begin{cases} 1 & \text{for } t \in [1,2] \\ 0 & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 2 & \text{for } t \in [0,1) \\ 1 & \text{for } t \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

Discussion:

What is $y(t) = h * u(t)$ for the case

$$u(t) = \begin{cases} 1 & \text{for } t \in [1,2] \text{ and } t \in [2,3] \\ 0 & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 2 & \text{for } t \in [0,2) \\ 0 & \text{otherwise} \end{cases} \quad ?$$

GitHub/TTK4225/trunk/Jupyter/convolution-example.ipynb

Summary

$$\dot{y} = ay + bu \tag{22}$$

free evolution: $u = 0,\ y_0 \neq 0$; implies $y(t) = y_{\text{free}}(t) = y_0 e^{at}$
forced response: $u \neq 0,\ y_0 = 0$; implies $y(t) = y_{\text{forced}}(t) = h * u(t)$ with $h(t) = e^{at}$

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e^{at} = "impulse response"

yet another pillar concept for automatic control
$\rightarrow$ *will be discussed better in a dedicated module*
and re-discussed in ***every*** *control-oriented course*

General response

$$\dot{y} = ay + bu \tag{23}$$

free evolution: $u = 0,\ y_0 \neq 0$; implies $y(t) = y_{\text{free}}(t) = y_0 e^{at}$

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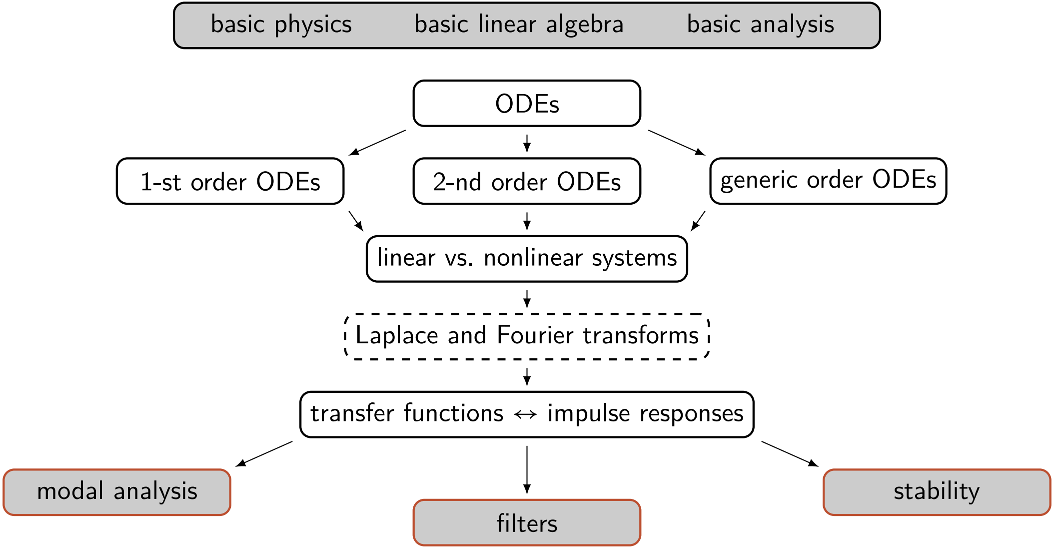
forced response: $u \neq 0$, $y_0 = 0$; implies $y(t) = y_{\text{forced}}(t) = h * u(t)$ with $h(t) = e^{at}$

general response: $u \neq 0$, $y_0 \neq 0$; implies $y(t) = y_{\text{free}}(t) + y_{\text{forced}}(t)$ thanks to the fact that the model is *linear* *(will be seen in more details later on)*

Summary / Roadmap of TTK4225



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Introduction to first order systems - visualizing the system through a block scheme

Roadmap

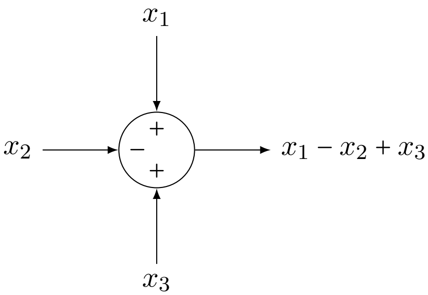
- the most common block schemes
- first order systems as block schemes

Block diagrams - why?

- used very often in companies
- aid visualization *(until a certain complexity is reached...)*
- enable "drag & drop" way of programming

- in this course, primarily used for interpretations

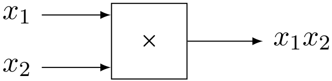
Most common block diagrams - sum of n signals



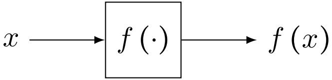
Most common block diagrams - multiplication for a constant

$$x \longrightarrow \boxed{c} \longrightarrow cx$$

Most common block diagrams - multiplication of two signals



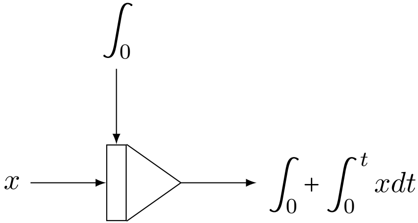
Most common block diagrams - generic functions



Most common block diagrams - derivatives

$$x \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \dot{x}$$

Most common block diagrams - integrals

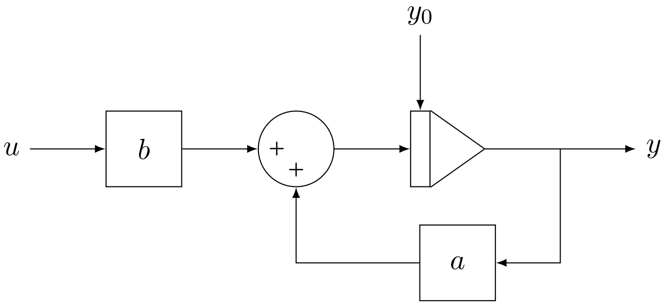


Discussion: how do we represent a first order DE with a block scheme?

$$\dot{y} = ay + bu$$

The solution

$$\dot{y} = ay + bu$$



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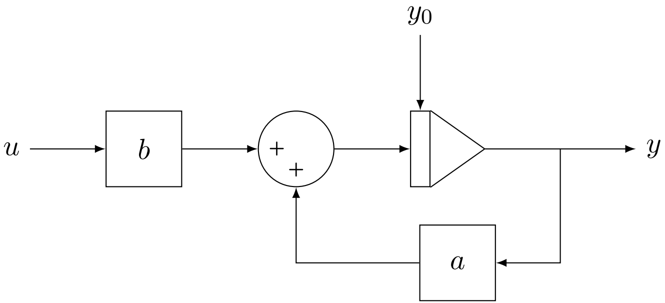

Discussion: do you note the presence of a feedback loop?

Introduction to Simulink

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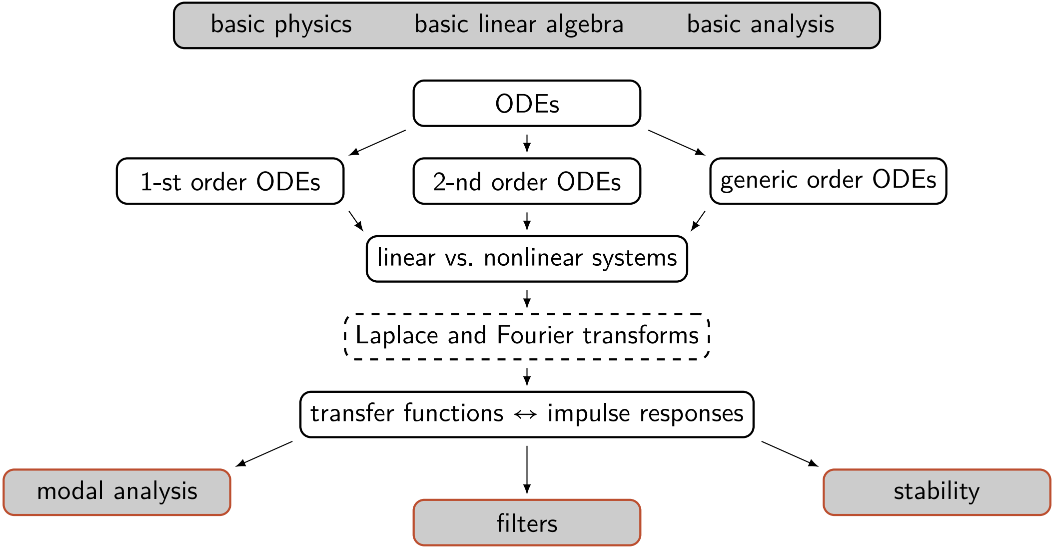
Introduction to generic order systems

Roadmap

- from first order to second order
- generic orders
- classification of the different types of ODEs

Summary / Roadmap of TTK4225



From first order to second order

Discussion: if we can write

$$\dot{y} = ay + bu \; ,$$

can we also write

$$\ddot{y} = a_1\dot{y} + a_0 y + bu \; ?$$

From first order to second order, third order

Discussion: if we can write

$$\dot{y} = ay + bu \ ,$$

can we also write

$$\ddot{y} = a_1\dot{y} + a_0 y + bu \ ?$$

And at this point also

$$\dddot{y} = a_2\ddot{y} + a_1\dot{y} + a_0 y + bu \ ?$$

From first order to second order, third order

Discussion: if we can write

$$\dot{y} = ay + bu \ ,$$

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And at this point also

$$\dddot{y} = a_2\ddot{y} + a_1\dot{y} + a_0y + bu \ ?$$

And why not

$$\dddot{y} = a_2\ddot{y} + a_1\dot{y} + a_0y + b_1\dot{u} + b_0u \ ?$$

ARMA models

$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u \; ?$$

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$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u\ ?$$

Discussion: and why don't we have $a_n y^{(n)}$, instead of only $y^{(n)}$ on the LHS?

Discussion: can we generalize even more?

$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u \,?$$

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time-varying: $\dot{y} = a(t)y + b(t)u$

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time-varying: $\dot{y} = a(t)y + b(t)u$

non-homogeneous: $\dot{y} = a(t)y + b(t)u + c$

non-linear: $\dot{y} = a\sqrt{y} + bu^{2/3}$

Discussion: can we make this linear again?

$$\dot{y} = ay + bu^{2/3}$$

Discussion: can we make this linear again?

$$\dot{y} = ay + bu^{2/3}$$

Another discussion: can we apply the same "linearization trick" to $\dot{y} = a\sqrt{y} + bu$?

main focus in this course:

linear, time invariant, homogeneous (LTI)

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why?

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linear, time invariant, homogeneous (LTI)

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Remember: predictive control of the type

$$\boldsymbol{u}^{\star} = \arg \min_{\boldsymbol{u} \in \mathcal{U}, \boldsymbol{f}(\boldsymbol{u}) \in \mathcal{F}} \mathrm{Cost}\left(\boldsymbol{f}\left(\boldsymbol{u}\right), \boldsymbol{u}\right)$$

requires to be able to:

- define "Cost"
- compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

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linear, time invariant, homogeneous (LTI)

why?

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- define "Cost"
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- be sure that the model does not have "nasty" properties

from the previous lessons:

1-st order LTI models $\mapsto$ impulse responses;

as will be clear later generic order LTI models $\mapsto$ also impulse responses

The question we are trying to answer in this first part of the course

"What will be the solution $y(t)$ to a generic-order LTI ODE starting from a generic initial condition $y(0) = y_0$ and generic input signal $u(t)$?"

(our hope: the solution will be structurally easy, and fast to compute)

?

What is what?

(i.e., a small quiz to see if we are aligned)

	autonomous	linear	homogeneous	time constant
$\frac{\dot{x}}{x} = 1$				
$\pi = \dot{x}x$				
$2\dot{x} + \sin t = 0$				
$\sin(t\dot{x}) - x = 0$				
$t\dot{x} = x \sin t$				
$x - e^{-t} = \ddot{x} + \dot{x}$				

Introduction to generic order systems

Roadmap

- state space representations
- from n-th order to vectorial first order

State space representations - Definition

mathematical model (typically but not limited to of a physical system) as a finite set of inputs, outputs and state variables related by first-order differential equations satisfying the separation principle

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Ingredients

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State space representations - Definition

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- first-order differential equations

State space representations - Definition

<u>mathematical model (typically but not limited to of a physical system) as a finite set of inputs, outputs and state variables related by first-order differential equations satisfying the separation principle</u>

Ingredients

- finite number of inputs, outputs and state variables
- first-order differential equations
- *satisfies the separation principle:* the current value of states contains all the information necessary to forecast the future evolution of the outputs and of the states given the inputs

State space representations - Facts

- the future output depends only on the current state and the future input

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- the future output depends on the past input only through the current state

State space representations - Facts

- the future output depends only on the current state and the future input
- the future output depends on the past input only through the current state
- the state summarizes the effect of past inputs on future output
  (sort of a "memory" of the system)

Example

Rechargeable flashlight:

- state = level of charge of the battery & on / off button
- output = how much light the device is producing

State space representations - Notation

$u_1, \ldots, u_m$ = inputs

$x_1, \ldots, x_n$ = states

$y_1, \ldots, y_p$ = outputs

State space representations - Notation

$$
\begin{aligned}
\dot{x}_1 &= f_1\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
&\vdots \\
\dot{x}_n &= f_n\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
y_1 &= g_1\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
&\vdots \\
y_p &= g_n\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right)
\end{aligned}
$$

State space representations - Notation

$$
\begin{aligned}
\dot{x}_1 &= f_1\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
&\vdots \\
\dot{x}_n &= f_n\left(x_1, \ldots, x_n, u_1, \ldots, u_m\right) \\
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&\vdots \\
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State space representations - Notation

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$$

- $\boldsymbol{f}$ = state transition map
- $\boldsymbol{g}$ = output map

Discussion: is any $\boldsymbol{f}$ and $\boldsymbol{g}$ ok?

$$
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\end{aligned}
$$

No: for every given $\boldsymbol{x}_0$ and $\boldsymbol{u}$ the solution must be:

- unique
- guaranteed to exist for $t \in [0, \bar{t})$ for some $\bar{t} > 0$

(check "existence and uniqueness of solutions of ODEs")

State space representations - Example

$$\begin{cases} \dot{x} = -x^2 \\ y = x \end{cases} \quad \cup \quad x_0 = -1 \qquad \Longrightarrow x(t) = \frac{1}{t-1}$$

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Discussion: *(a bit outside of the current flow of concepts on ODEs, but still relevant and very important)* what happens as $t \to 1$?

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there is a finite escape time!

Discussion: can every physical system be described with a state space representation?

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Noticeable exceptions:

- non-lumped parameters model (i.e., PDEs)
- continuous-time systems with delays
  (will be discussed better later on)

Example: dynamics of sugar and insulin concentrations in blood

- x_1 := sugar concentration
- x_2 := insulin concentration
- u_1 := food intake
- u_2 := insulin intake
- c := sugar concentration in fasting

$$\begin{cases} \dot{x}_2 = a_{21}\left(x_1 - c\right) - a_{22}x_2 + b_2u_2 & x_1 \geq c \\ \dot{x}_2 = -a_{22}x_2 + b_2u_2 & x_1 < c \end{cases}$$

$$\begin{cases} \dot{x}_1 = -a_{11}x_1x_2 + a_{12}\left(c - x_1\right) + b_1u_1 & x_1 \geq c \\ \dot{x}_1 = -a_{11}x_1x_2 + b_1u_1 & x_1 < c \end{cases}$$

?

How can I transform a generic-order ODE into a state space representation (that requires first order DEs)?

$$\dddot{y} = \phi\left(\ddot{y}, \dot{y}, y, \ddot{u}, \dot{u}, u\right);$$

How can I transform a generic-order ODE into a state space representation (that requires first order DEs)?

$$\dddot{y} = \phi\left(\ddot{y}, \dot{y}, y, \ddot{u}, \dot{u}, u\right);$$

change of variables:

$$\boldsymbol{x} = \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix}, \; \boldsymbol{u} = \begin{bmatrix} \ddot{u} \\ \dot{u} \\ u \end{bmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{x}} = \begin{bmatrix} \dddot{y} \\ \ddot{y} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \phi\left(\boldsymbol{x}, \boldsymbol{u}\right) \\ \left[1\ 0\ 0\right]\boldsymbol{x} \\ \left[0\ 1\ 0\right]\boldsymbol{x} \end{bmatrix} = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)$$

Thus, how do I transform a generic order LTI system into a first order LTI system?

$$\ddot{y} = 2\dot{y} + 3y + 4\ddot{u} + 5\dot{u} + 6u$$

Transforming ARMA models

$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots b_0 u \,?$$

Discussion: but then any LTI system is a first order vectorial system?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$$

Discussion: but then any LTI system is a first order vectorial system?

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$$

YES, AND THIS IS WHY
LINEAR ALGEBRA IS ESSENTIAL
FOR CONTROL!

?

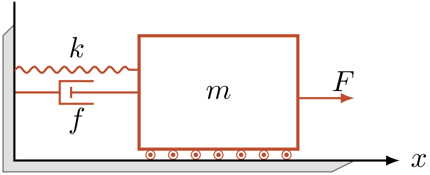
Second order systems

Roadmap

- interesting examples
- interesting responses
- differences between 1st order and 2nd order

Example 1: spring-mass systems

Position of a cart fastened with a spring to a wall and subject to friction:



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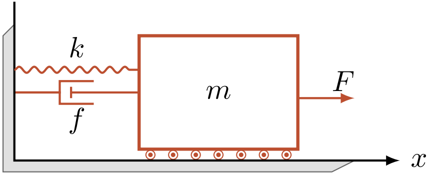

EOM:

- force from the spring: $F_x(t) = -kx(t)$
- friction: $F_f(t) = -f\dot{x}(t)$
- applied force: $F(t)$
- Newton's second law: $\sum F = m\ddot{x}(t)$

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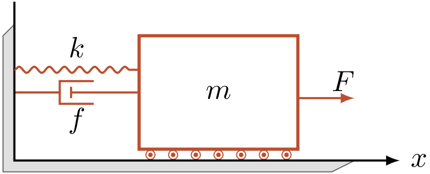

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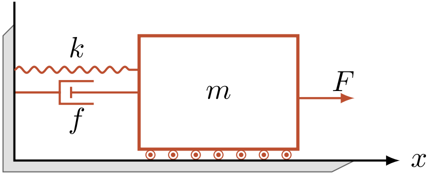

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$$m\ddot{x}(t) = F_x(t) + F_f(t) + F(t) \qquad \mapsto \qquad m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + F(t)$$

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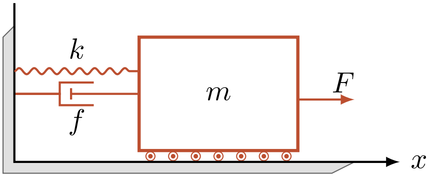

$$\ddot{x}(t) + \frac{f}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{1}{m}F(t) \tag{24}$$

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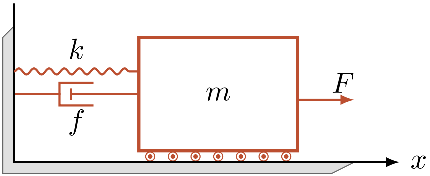

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Generalizing:

$$\ddot{x} = a_1\dot{x} + a_0 x + b_0 u \tag{25}$$

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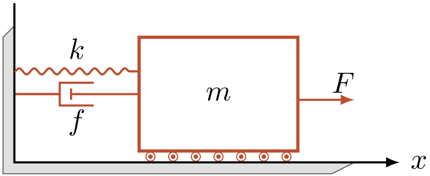

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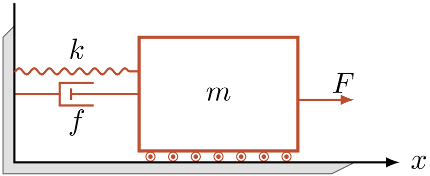

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Discussion: may we write this dynamics using a vectorial first order DE?

Discussion: may we write this dynamics using a scalar first order DE?

Example 1: spring-mass systems

https://youtu.be/lZPtFDXYQRU?t=31

(here there is very little friction, though)

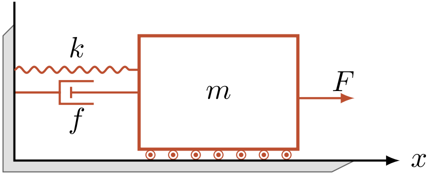
Example 1: spring-mass systems - stability analysis from intuitive perspectives

Discussion: $\dot{x} = a_0 x + b_0 u$ unstable if ...?

Example 1: spring-mass systems - stability analysis from intuitive perspectives

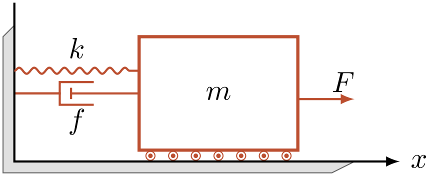
Discussion: $\dot{x} = a_0 x + b_0 u$ unstable if ...? And What about, from intuitive perspectives, when considering

$$\ddot{x} = a_1 \dot{x} + a_0 x + b_0 u \; ? \tag{26}$$



GitHub/TTK4225/trunk/Jupyter/spring-mass-systems-introduction.ipynb

Intuition: asymptotic stability $\leftrightarrow$ dissipation of energy



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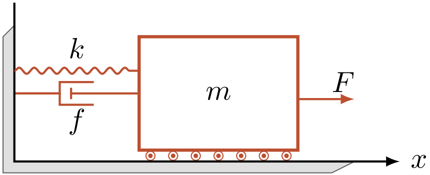
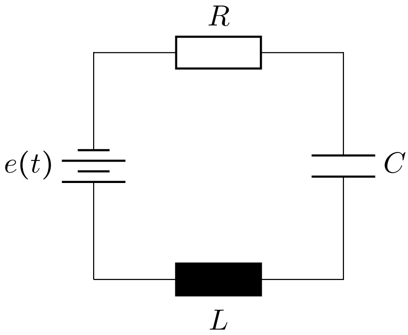


will see this MUCH better in later on courses
when discussing about Lyapunov stability

?

Example 2: RCL-circuit



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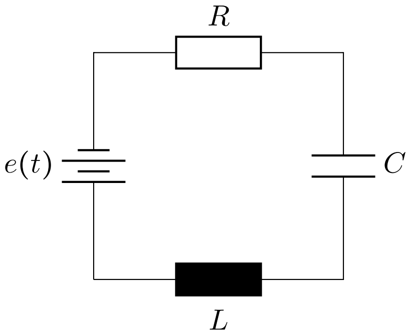

EOM: Kirkhoff laws

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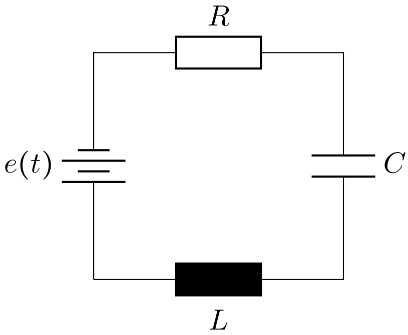

EOM: Kirkhoff laws

$$v_L(t) = L\frac{di(t)}{dt} \qquad v_i(t) = Ri(t) \qquad v_C(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

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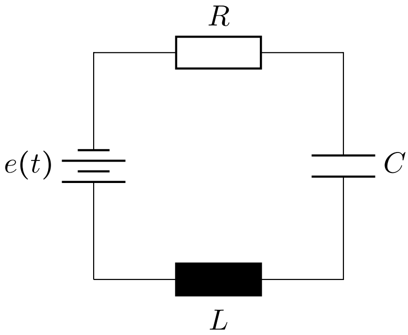

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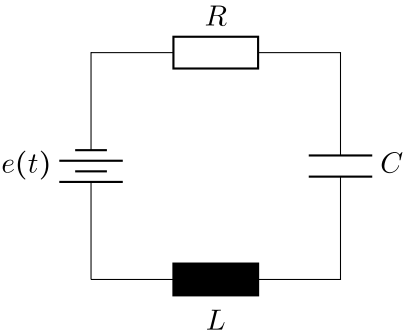

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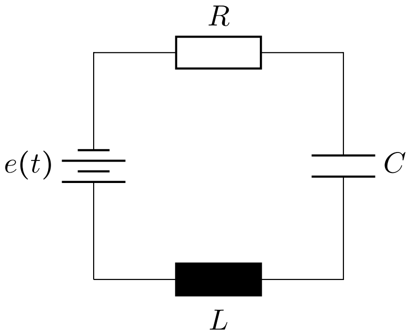

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Discussion: how do we write it in terms of x, $\dot{x}$, $\ldots$?

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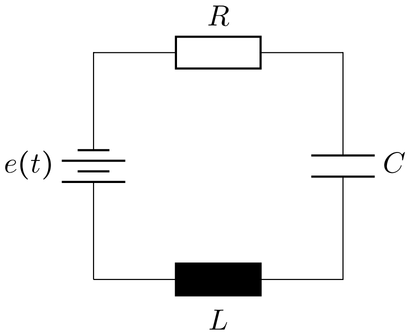

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$$x(t) = i(t), \quad u(t) = \dot{e}(t) \qquad \Longrightarrow \qquad \ddot{x} = -\frac{R}{L}\dot{x} - \frac{1}{LC}x + \frac{1}{L}u \tag{28}$$

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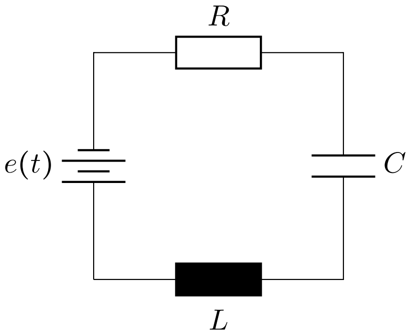

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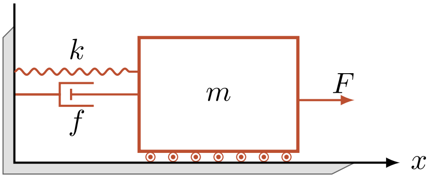
Generalizing:

?

Discussion: what are the potential different ways the cart may move back to its equilibrium?



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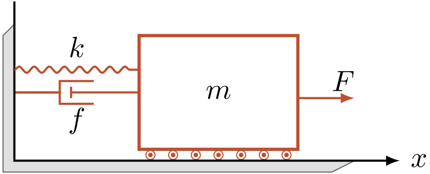
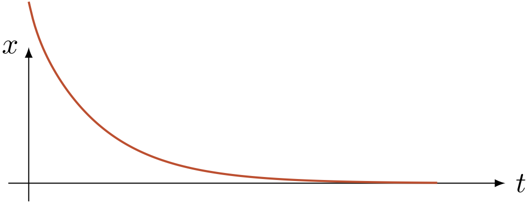


very big friction w.r.t. the spring stiffness $\implies$ overdamping:



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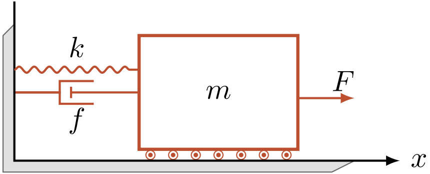

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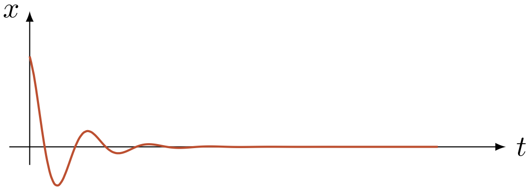

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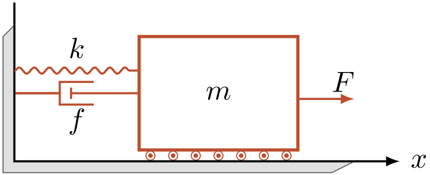
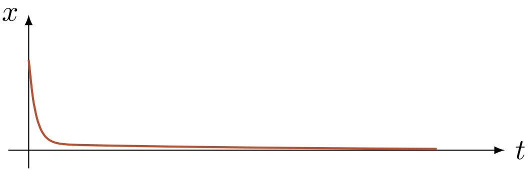


just the right amount of friction w.r.t. the spring stiffness $\implies$ critical damping,
i.e., *the combination that leads to the fastest decay to zero without oscillations*:



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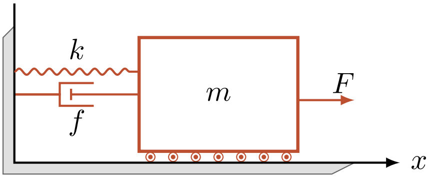
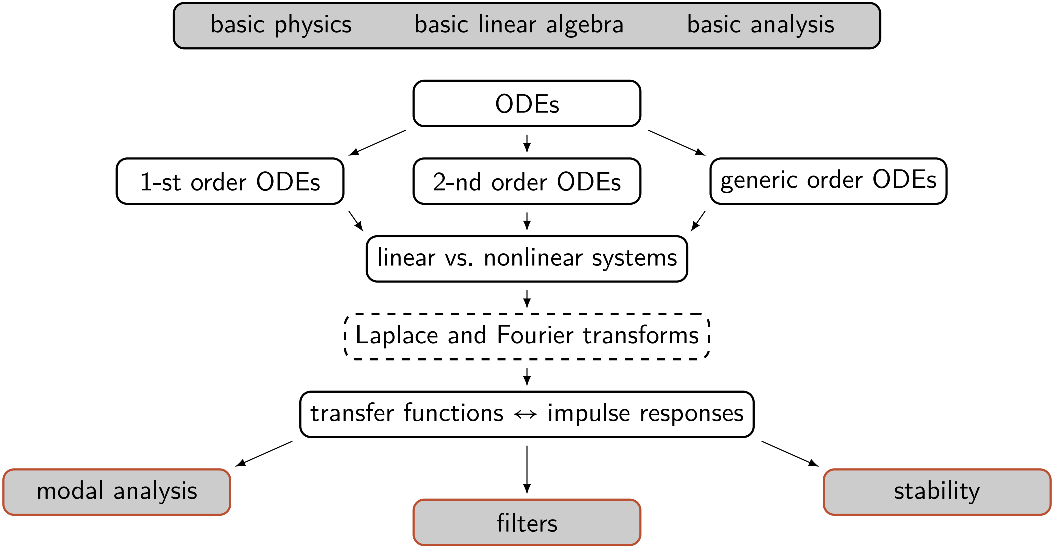


Discussion: can we get these behaviors with a first order system?

How do the free evolutions and forced responses look like numerically?

we will solve this problem with the aid of Laplace transforms

Summary / Roadmap of TTK4225



The essence of TTK4225

A number like 3 is not a particular triplet of things, is about all possible triplets of things. In the same way, a model is not really about a particular system, but is rather about all the possible real-life and physical systems that behave as that model

?