# TTK4225 - Systems Theory, Autumn 2020

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# BIBO stability

# Roadmap

- generalizing the concept of stability
- BIBO stability
- connecting BIBO stability with the poles of the transfer functions

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- BIBO stability
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*important assumption in this course:* $\boldsymbol{u} = \overline{\boldsymbol{u}} = const.$

# Important: the term “stability” may refer to specific equilibrium points or specific systems

“Stability” referring to specific equilibria:

1. simply stable equilibrium
2. convergent equilibrium
3. asymptotically stable equilibrium

“Stability” referring to specific systems:

1. Bounded Input Bounded Output (BIBO) stable systems (we will see this now)
2. Input to State Stable (ISS) systems (we will not see this in this course)

# BIBO stability

$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right)$$

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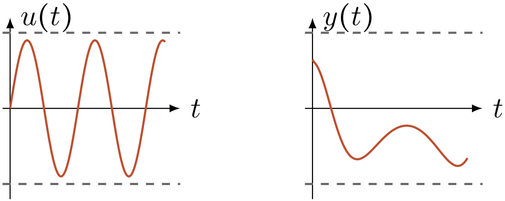
## Definition (BIBO stability)

*the system* $(\boldsymbol{f}, \boldsymbol{g})$ *is said to be Bounded Input Bounded Output (BIBO) stable if*
$\|\boldsymbol{u}\| \leq \gamma_u \implies \|\boldsymbol{y}\| \leq \gamma_y$



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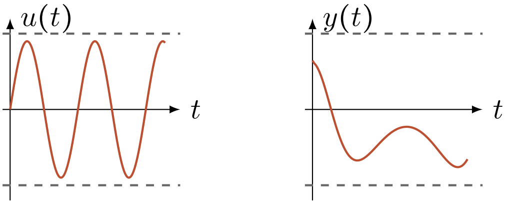


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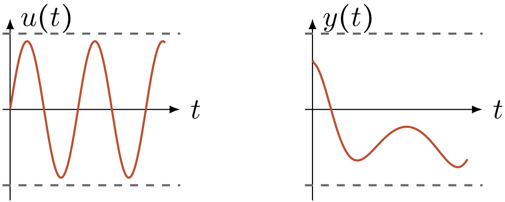


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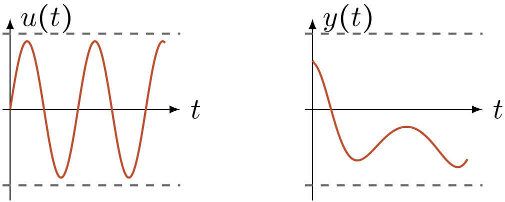
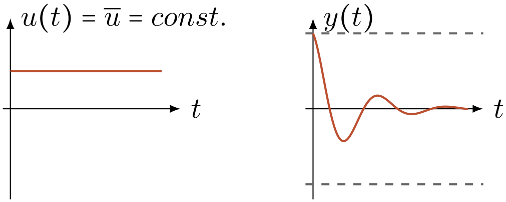


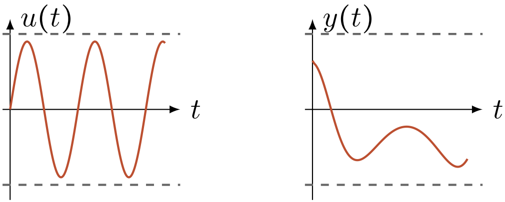

*Discussion:* is the system $\dot{y} = y^2 u$ BIBO stable? And the system $\dot{y} = -y^2 u$? To check BIBO stability one can use the "small gain theorem": not in this course! Here we will check the BIBO stability checking either the impulse response or the transfer function

# Summarizing

## Asymptotic stability




## BIBO stability



# The very important result that we will find now

For general nonlinear systems:

BIBO stable system $\neq$ asymptotically stable equilibria $\neq$ simply stable equilibria

For LTIs:

BIBO stable system = asymptotically stable equilibria $\neq$ simply stable equilibria

# Towards coupling the BIBO stability to the system's impulse response or transfer function

$$X(s) = H(s)U(s)$$

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- $|u(t)| < M_u$ means a "non diverging" $u(t)$. Assuming that $u(t)$ has a rational Laplace transform, where do the poles of $U(s)$ live in this case?

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- what happens if $h(t)$ is diverging? Where are its poles?
- what happens if $h(t)$ is converging to 0? Where are the poles of the associated $H(s)$? And may I choose some non-diverging $u(t)$ that makes $y(t)$ diverging?

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- what happens if $h(t)$ is converging to 0? Where are the poles of the associated $H(s)$? And may I choose some non-diverging $u(t)$ that makes $y(t)$ diverging?
- what happens if $h(t)$ is non-diverging and non-converging? May I choose some non-diverging $u(t)$ that makes $y(t)$ diverging?

# BIBO stability = absolute integrability of the impulse response

BIBO stability:

$$|u(t)| < M_u \quad \Longrightarrow \quad |y(t)| < M_y$$

Impulse response:

$$y(t) = h * u(t) = \int_0^t h(\tau)u(t-\tau)d\tau$$

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if $\displaystyle\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < +\infty$ then BIBO stability

# Coupling the BIBO stability concept with the poles of a TF

if $\int_{-\infty}^{+\infty} |h(\tau)| \, d\tau < +\infty$ then BIBO stability

$$H(s) = \frac{N(s)}{D(s)}$$

*Discussion:* if we want the system be BIBO stable, may $D(s)$ have poles on the imaginary axis?

# Important result

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asymptotically stable equilibria $\neq$ simply stable equilibria

?

# Nomenclature

BIBO stable LTI system $=$ LTI with all its equilibria asymptotically stable

marginally stable LTI system $=$ LTI with all its equilibria simply stable

unstable LTI system $=$ LTI with unstable equilibria

# Examples: are these systems BIBO stable, marginally stable, or unstable?

$$H(s) = \frac{1}{(s+2)(s+1)} \tag{1}$$

$$H(s) = \tag{2}$$

$$H(s) = \tag{3}$$

$$H(s) = \tag{4}$$

$$H(s) = \tag{5}$$

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# And what about nonlinear systems?

*more complicated! Will treat this through Lyapunov theory in more advanced courses*

# The very important result that we found

For general nonlinear systems:

BIBO stable system $\neq$ asymptotically stable equilibria $\neq$ simply stable equilibria

For LTIs:

BIBO stable system $=$ asymptotically stable equilibria $\neq$ simply stable equilibria

# Summarizing, once again

Different types of system stability:

asymptotic input-output (system) stability: independently of $u(t)$, $x(t) \to 0$ when $t \to +\infty$

marginal (or simply input-output) (system) stability: as soon as $|u(t)| < M_u$, $|x(t)| < M_x$ when $t \to +\infty$

(system) instability: there exists at least one signal $u(t)$ for which we cannot do the bound $|x(t)| < M_x$ when $t \to +\infty$

it is necessary to know about potential instabili-
ties, because our control system must stabilize them

?

# Where are we now?

characterizing LTI systems

↓

linearization of nonlinear systems

↓

linear algebra as a way
to even better analyse LTI systems

↓

designing filters

# Introduction to nonlinear systems

# Roadmap

- definitions
- examples
- important differences between linear and nonlinear systems

# The case of scalar functions

### $f$ is linear if ...

$$f(ax + by) = af(x) + bf(y)$$

for every $a$, $x$, $b$, $y$

### $f$ is nonlinear if ...

there exists at least one $a$, $x$, $b$, $y$ for which

$$f(ax + by) \neq af(x) + bf(y)$$

# Example

$$
\begin{aligned}
f(ax+by) &= (ax+by)^2 \\
&= a^2x^2 + 2axby + b^2y^2 \\
&= a^2 f(x) + 2axby + b^2 f(y) \\
&\neq af(x) + bf(y)
\end{aligned}
$$

# The case of vectorial functions

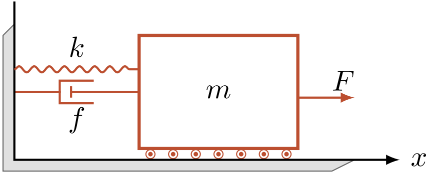
**$\boldsymbol{f}$ is linear if ...**

all the single components $f_i(\cdot)$ are linear

**$\boldsymbol{f}$ is nonlinear if ...**

at least one of the single components $f_i(\cdot)$ is nonlinear

# Example: spring-mass systems




$$m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + F(t)$$

# Remember: ARMA models can be written as vectorial first order systems!

$$y^{(n)} = a_{n-1} y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots + b_0 u \Downarrow$$
$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

# Example: Lotka-Volterra

- $y_{\text{prey}}$ := prey
- $y_{\text{pred}}$ := predator

$$\begin{cases} \dot{y}_{\text{prey}} & = \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} & = -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

# Example: Van-der-Pol oscillator

$$\begin{cases} \dot{y}_1 & = \mu\left(y_1 - \dfrac{y_1^3}{3} - y_2\right) \\ \dot{y}_2 & = \dfrac{y_1}{\mu} \end{cases}$$

# An extremely important difference between linear and nonlinear systems

$$\textit{no linearity} \Longrightarrow \textit{no superposition effects}$$

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$$\textit{no linearity} \implies \textit{no superposition effects}$$

i.e., without linearity we cannot say that $u = u_1 + u_2$ causes $y = y_1 + y_2$

# An extremely important difference between linear and nonlinear systems

*no linearity* $\Longrightarrow$ *no superposition effects*

i.e., without linearity we cannot say that $u = u_1 + u_2$ causes $y = y_1 + y_2$ and thus not even

$$y(t) = y_{\text{free}}(t) + y_{\text{forced}}(t)$$

*no linearity* $\Longrightarrow$ *no modal analysis*

# Remember where we want to arrive: Model Predictive Control

$$\boldsymbol{u}^{\star} = \arg \min_{\boldsymbol{u} \in \mathcal{U}, \boldsymbol{f}(\boldsymbol{u}) \in \mathcal{F}} \mathrm{Cost}\left(\boldsymbol{f}\left(\boldsymbol{u}\right), \boldsymbol{u}\right),$$

that requires to:

- define "$\mathrm{Cost}$"
- be able to compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

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- be sure that the model does not have "nasty" properties

seems that with nonlinear systems things complicate

# Potential approach: linearize through Taylor expansions

we will see this in the next unit

# Important differences to always remember, take 1

*nonlinear systems admit isolated equilibria,*
*while LTI systems admit only subspaces of equilibria (by the way, why?)*

*(example: the Lotka Volterra has 2 distinct equilibria)*

# Important differences to always remember, take 2

*linear systems admit exponential bounding,*

*while nonlinear systems may have finite escape times* *(by the way, why?)*

*(example: starting* $\dot{y} = y^2$ *from* $y_0 = c$ *leads to the trajectory* $y(t) = \dfrac{1}{c - t}$*)*

# Important differences to always remember, take 3

*nonlinear systems admit limit cycles,*
*while LTI systems do not (by the way, why?)*

?

# Roadmap

- recalling the definition of state-space systems
- Taylor approximations, what are they?
- how to linearize a continuous time system
- examples

# State space representations - Definition

mathematical model (typically but not limited to of a physical system) as a finite set of inputs, outputs and state variables related by first-order differential equations satisfying the separation principle

Ingredients:

- finite number of inputs, outputs and state variables
- first-order differential equations
- *satisfies the separation principle:* the current value of the state contains all the information necessary to forecast the future evolution of the outputs and of the state. I.e., to compute future $y(t+\tau)$ and $x(t+\tau)$ it is enough to know the current $x(t)$ and the current and future inputs $u(t : t+\tau)$

# Example

Rechargeable flashlight:

- state $=$ level of charge of the battery & on / off button
- output $=$ how much light the device is producing

"the current value of the state contains all the information necessary to forecast the future evolution of the outputs and of the state. I.e., to compute future $y(t+\tau)$ and $x(t+\tau)$ it is enough to know the current $x(t)$ and the current and future inputs $u(t:t+\tau)$"

# State space representations - Notation

$$u_1, \ldots, u_m \; = \text{inputs}$$

$$x_1, \ldots, x_n \; = \text{states}$$

$$y_1, \ldots, y_p \; = \text{outputs}$$

# State space representations - Notation

$$
\begin{aligned}
\dot{x}_1 &= f_1\left( x_1, \dots, x_n,\ u_1, \dots, u_m \right) \\
&\vdots \\
\dot{x}_n &= f_n\left( x_1, \dots, x_n,\ u_1, \dots, u_m \right) \\
y_1 &= g_1\left( x_1, \dots, x_n,\ u_1, \dots, u_m \right) \\
&\vdots \\
y_p &= g_p\left( x_1, \dots, x_n,\ u_1, \dots, u_m \right)
\end{aligned}
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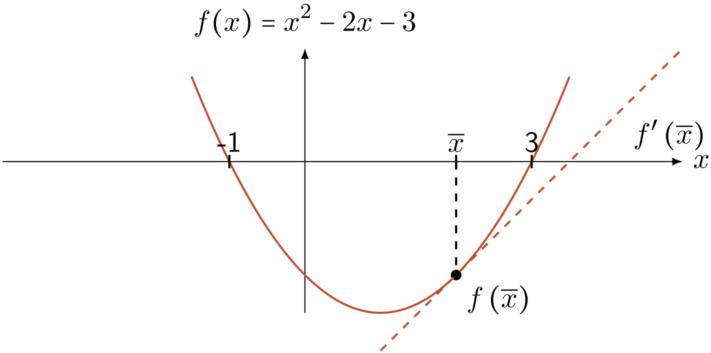
$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\
\boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)
\end{aligned}
$$

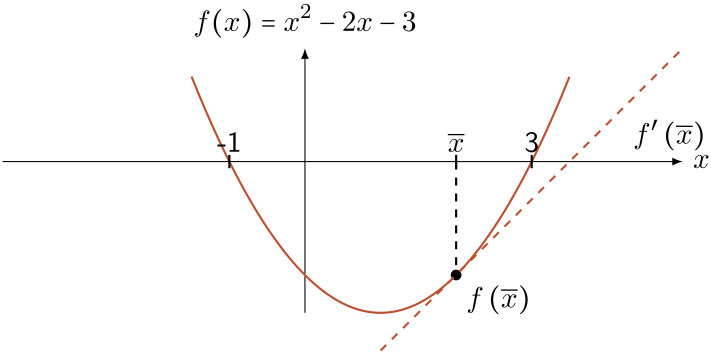
- $\boldsymbol{f}$ = state transition map
- $\boldsymbol{g}$ = output map

?

# Linearization - what does it mean?



# Linearization - what does it mean?




$$f(x) \approx f(\overline{x}) + \left.\frac{\partial f}{\partial x}\right|_{\overline{x}} (x - \overline{x})$$

(but the approximation is valid only close to the linearization point)

# Linearization - what does it mean?

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned} \quad \mapsto \quad \begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} + D\boldsymbol{u} \end{aligned}$$

*linearize* $\Longrightarrow$ *approximate!*

*Discussion:* why do we linearize nonlinear systems?

*Discussion:* where do we linearize nonlinear systems?

# Preliminaries: Taylor series

$$f \in C^M(\mathbb{R}) \qquad \Longrightarrow \qquad f(x) \approx \sum_{m=0}^{M} \frac{f^{(m)}(x_0)}{m!} (x - x_0)^m$$

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the most important case for our purposes:

$$\boldsymbol{f} \in C^1(\mathbb{R}^n) \qquad \Longrightarrow \qquad \boldsymbol{f}(\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}_0) + \nabla \boldsymbol{f}|_{\boldsymbol{x}_0} (\boldsymbol{x} - \boldsymbol{x}_0)$$

*Discussion (yes, again):* where do we linearize nonlinear systems?

# Linearization procedure - continuous time systems

$$(\boldsymbol{x}_{\mathsf{eq}}, \boldsymbol{u}_{\mathsf{eq}}) \text{ equilibrium} \quad \Longrightarrow \boldsymbol{f}\left(\boldsymbol{x}_{\mathsf{eq}}, \boldsymbol{u}_{\mathsf{eq}}\right) = 0$$

# Linearization procedure - continuous time systems

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Procedure (assuming that the Taylor expansion exists):

1. consider $\boldsymbol{x} = \boldsymbol{x}_{\text{eq}} + \Delta\boldsymbol{x}$
2. apply $\boldsymbol{f}(\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}_0) + \nabla \boldsymbol{f}|_{\boldsymbol{x}_0} (\boldsymbol{x} - \boldsymbol{x}_0)$ to the point $\boldsymbol{x}_0 = \boldsymbol{x}_{\text{eq}}$

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$$\Longrightarrow \qquad \frac{\partial\left(\boldsymbol{x}_{\text{eq}} + \Delta\boldsymbol{x}\right)}{\partial t} \approx \boldsymbol{f}\left(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) + \nabla \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{u} \end{bmatrix}$$

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- $\dfrac{\partial(\boldsymbol{x}_{\text{eq}} + \Delta\boldsymbol{x})}{\partial t} = \Delta\dot{\boldsymbol{x}}$

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- $\dfrac{\partial \left(\boldsymbol{x}_{\text{eq}} + \Delta \boldsymbol{x}\right)}{\partial t} = \Delta \dot{\boldsymbol{x}}$
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- $\dfrac{\partial (\boldsymbol{x}_{\text{eq}} + \Delta \boldsymbol{x})}{\partial t} = \Delta \dot{\boldsymbol{x}}$
- $\boldsymbol{f}(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) = \boldsymbol{0}$
- $\nabla \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} = \begin{bmatrix} \nabla_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) & \nabla_{\boldsymbol{u}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \end{bmatrix}_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}}$

# Linearization procedure - continuous time systems

$$(\boldsymbol{x}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}) \text{ equilibrium} \quad \Longrightarrow$$

$$\Delta \dot{\boldsymbol{x}} \approx \nabla_{\boldsymbol{x}} \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}} \Delta \boldsymbol{x} + \nabla_{\boldsymbol{u}} \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}} \Delta \boldsymbol{u}$$

# And for $\boldsymbol{y}$?

$$\boldsymbol{y} = \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)$$

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$$\downarrow$$

$$\boldsymbol{y}_{\text{eq}} + \Delta\boldsymbol{y} \approx \boldsymbol{g}\left(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) + \nabla\boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{u} \end{bmatrix}$$

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$$\Delta \boldsymbol{y} \approx \nabla_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u})\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta \boldsymbol{x} + \nabla_{\boldsymbol{u}} \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u})\Big|_{u_{\text{eq}}} \Delta \boldsymbol{u}$$

# Linearization procedure - continuous time systems

$$\begin{cases} \dot{\boldsymbol{x}} & = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\ \boldsymbol{y} & = \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right) \end{cases}$$

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$$\begin{cases} \Delta\dot{\boldsymbol{x}} &= \nabla_{\boldsymbol{x}} \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta\boldsymbol{x} + \nabla_{u} \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta\boldsymbol{u} \\ \Delta\boldsymbol{y} &= \nabla_{\boldsymbol{x}} \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta\boldsymbol{x} + \nabla_{u} \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)\Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta\boldsymbol{u} \end{cases}$$

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$$\downarrow$$

$$\begin{cases} \Delta\dot{\boldsymbol{x}} &= A\Delta\boldsymbol{x} + B\Delta\boldsymbol{u} \\ \Delta\boldsymbol{y} &= C\Delta\boldsymbol{x} + D\Delta\boldsymbol{u} \end{cases}$$

# Summary

$$\dot{\boldsymbol{x}} = f\left(\boldsymbol{x}, \boldsymbol{u}\right)$$

1. choose an opportune point $\boldsymbol{x}_0, \boldsymbol{u}_0$
2. linearize around $\boldsymbol{x}_0, \boldsymbol{u}_0$:

$$\dot{\boldsymbol{x}}_0 + \Delta\dot{\boldsymbol{x}} \approx \boldsymbol{f}\left(\boldsymbol{x}_0, \boldsymbol{u}_0\right) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_0, \boldsymbol{u}_0} \Delta\boldsymbol{x} + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}_0, \boldsymbol{u}_0} \Delta\boldsymbol{u}$$

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*important: if* $\boldsymbol{x}_0, \boldsymbol{u}_0 =$ *equilibrium then* $\dot{\boldsymbol{x}}_0 = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0) = 0$

# Linearization - Example

electrostatic microphone:

- $q$ = capacitor charge
- $h$ = distance of armature from its natural equilibrium
- $\boldsymbol{x} = \left[q, h, \dot{h}\right]$
- $R$ = circuit resistance
- $E$ = voltage generated by the generator (constant)
- $C$ = capacity of the capacitor
- $m$ = mass of the diaphragm $+$ moved air
- $k$ = mechanical spring coefficient
- $\beta$ = mechanical dumping coefficient
- $u_1$ = incoming acoustic signal

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$$
\begin{cases}
\dot{x}_1 & = -\dfrac{1}{Ra} x_1 \left(L + x_2\right) + \dfrac{E}{R} \\
\dot{x}_2 & = x_3 \\
\dot{x}_3 & = -\dfrac{\beta}{m} x_3 - \dfrac{k}{m} x_2 - \dfrac{x_1^2}{2am} + \dfrac{1}{m} u_1
\end{cases}
$$

# Linearization - Example

1-st step: compute the equilibria

$$\begin{cases} \dot{x}_1 &= -\dfrac{1}{Ra} x_1 \left(L + x_2\right) + \dfrac{E}{R} \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -\dfrac{\beta}{m} x_3 - \dfrac{k}{m} x_2 - \dfrac{x_1^2}{2am} + \dfrac{1}{m} u_1 \end{cases}$$

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1-st step: compute the equilibria

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2-nd step: compute the matrices

$$A = \nabla_{\boldsymbol{x}} \boldsymbol{f} \left( \boldsymbol{x}, \boldsymbol{u} \right) \Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \quad B = \nabla_{\boldsymbol{u}} \boldsymbol{f} \left( \boldsymbol{x}, \boldsymbol{u} \right) \Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \quad C = \nabla_{\boldsymbol{x}} \boldsymbol{g} \left( \boldsymbol{x}, \boldsymbol{u} \right) \Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \quad D = \nabla_{\boldsymbol{u}} \boldsymbol{g} \left( \boldsymbol{x}, \boldsymbol{u} \right) \Big|_{\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}}$$

?

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- each equilibrium will lead to its "own" corresponding linear model $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$, where $A$ and $B$ thus depend on $(\boldsymbol{x}_{\mathsf{eq}}, \boldsymbol{u}_{\mathsf{eq}})$ and $\boldsymbol{x}$, $\boldsymbol{u}$ in $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ have actually the meaning of $\Delta\boldsymbol{x}$, $\Delta\boldsymbol{u}$ with respect to the equilibrium

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- each linearized model $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ is more or less valid only in a neighborhood of $(\boldsymbol{x}_{\text{eq}}, \boldsymbol{u}_{\text{eq}})$. Moreover the size of this neighborhood depends on the curvature of $\boldsymbol{f}$ around that specific equilibrium point

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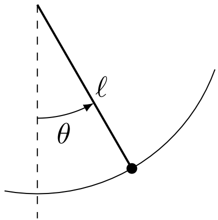
linearization = a very useful tool to do
analysis and design of control systems

# Another example: the pendulum



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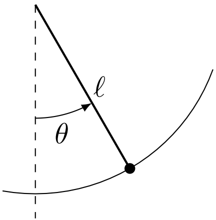


First step: equations of motion:

- gravity: $F_{g,x} = -mg\sin(\theta)$
- friction: $F_f = -fv_x = -f\ell\dot{\theta}$
- input torque: $F_u = u/\ell$

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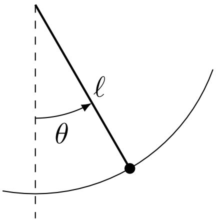


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resulting dynamics: $ml\ddot{\theta} = -mg\sin(\theta) - f\ell\dot{\theta} + \dfrac{u}{\ell}$

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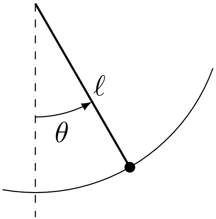


Next step: transform

$$ml\ddot{\theta} = -mg\sin(\theta) - f\ell\dot{\theta} + \frac{u}{\ell}$$

into a state-space form

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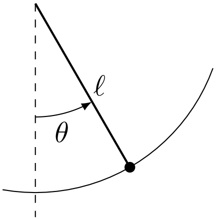


Next step: transform

$$m l \ddot{\theta} = -mg\sin(\theta) - f\ell\dot{\theta} + \frac{u}{\ell}$$

into a state-space form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin(x_1) - \frac{f}{m}x_2 + \frac{1}{m\ell^2}u \end{aligned}$$

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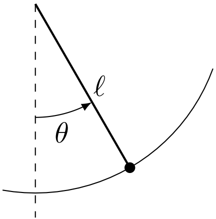


Next step: find the equilibria of

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for $u = 0$ (for simplicity)

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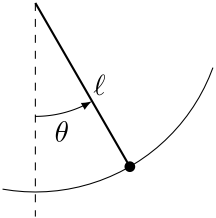


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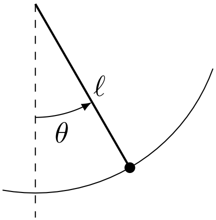


Next step: find the equilibria of

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin(x_1) - \frac{f}{m}x_2 + \frac{1}{m\ell^2}u$$

for $u = 0$ (for simplicity)

$$\begin{cases} 0 = x_2 \\ 0 = -\dfrac{g}{\ell}\sin(x_1) - \dfrac{f}{m}x_2 \end{cases} \implies \quad x_{\mathsf{eq}1} = n\pi,\ x_{\mathsf{eq}2} = 0$$

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$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{\ell}\sin(x_1) - \frac{f}{m}x_2 + \frac{1}{m\ell^2}u\end{aligned}$$

Equilibrium $\boldsymbol{x}_{\mathsf{eq}\alpha} = \mathbf{0}$, $u = 0$ implies

$$A = \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{\boldsymbol{x}_{\mathsf{eq}\alpha}} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{\ell} & -\dfrac{f}{m} \end{bmatrix}$$

$$B = \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_2}{\partial u} \end{bmatrix}\right|_{\boldsymbol{x}_{\mathsf{eq}\alpha}} = \begin{bmatrix} 0 \\ \dfrac{1}{m\ell^2} \end{bmatrix}$$

# Another example: the pendulum

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin(x_1) - \frac{f}{m}x_2 + \frac{1}{m\ell^2}u$$

Equilibrium $\boldsymbol{x}_{\text{eq}\beta} = [\pi, 0]^T$, $u = 0$ implies

$$A = \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{\boldsymbol{x}_{\text{eq}\beta}} = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{\ell} & -\dfrac{f}{m} \end{bmatrix}$$

$$B = \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_2}{\partial u} \end{bmatrix}\right|_{\boldsymbol{x}_{\text{eq}\beta}} = \begin{bmatrix} 0 \\ \dfrac{1}{m\ell^2} \end{bmatrix}$$

?

# Roadmap

- obvious properties
- simple examples
- understanding through generalizing the simple examples
- some considerations about control of nonlinear systems

# Obvious fact: linearizing around an equilibrium keeps that point an equilibrium

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\
\boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u})
\end{aligned}
\quad \mapsto \quad
\begin{aligned}
\dot{\widetilde{\boldsymbol{x}}} &= A\widetilde{\boldsymbol{x}} + B\widetilde{\boldsymbol{u}} \\
\widetilde{\boldsymbol{y}} &= C\widetilde{\boldsymbol{x}} + D\widetilde{\boldsymbol{u}}
\end{aligned}
$$

with

$$
\begin{cases}
\boldsymbol{x} &= \boldsymbol{x}_{\text{eq}} + \widetilde{\boldsymbol{x}} \\
\boldsymbol{u} &= \boldsymbol{u}_{\text{eq}} + \widetilde{\boldsymbol{u}} \\
\boldsymbol{y} &= \boldsymbol{y}_{\text{eq}} + \widetilde{\boldsymbol{y}}
\end{cases}
$$

Thus if $\boldsymbol{x}_{\mathsf{eq}}, \boldsymbol{u}_{\mathsf{eq}}$ was an equilibrium for the nonlinear system, it is still an equilibrium for the linearized one. *But if it was a stable one before, will it still be a stable one after?*

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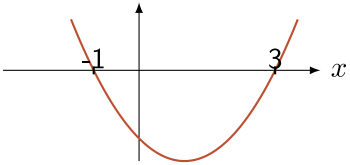
this lesson $=$ answering this question

# Starting from the basics: the scalar case

$$\dot{x} = f(x) = x^2 - 2x - 3 = (x-3)(x+1) = 0 \qquad \text{equilibria: } \begin{cases} x_{\mathsf{eq}\alpha} = -1 \\ x_{\mathsf{eq}\beta} = 3 \end{cases}$$

$$f(x) = x^2 - 2x - 3$$



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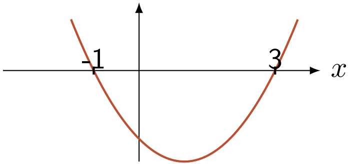


Analysing $x_{\mathsf{eq}\alpha} = -1$:

- $x < -1$ implies $\dot{x} = f(x) > 0$ implies $x$ grows
- $x > -1$ implies $\dot{x} = f(x) < 0$ implies $x$ shrinks *(but only locally)*

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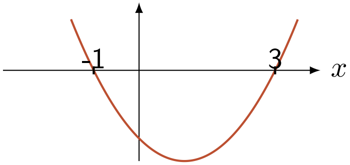


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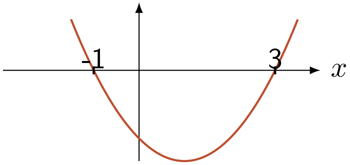


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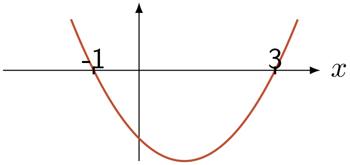

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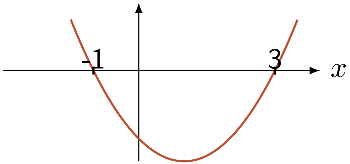


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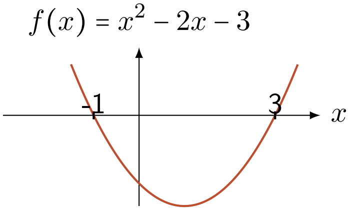


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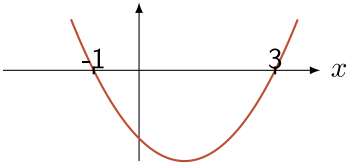


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# How do we generalize the previous concepts?

$$\dot{x} = f(x) \qquad f(x_{\mathsf{eq}}) = 0, \qquad \mapsto \qquad \dot{\widetilde{x}} = a_{x_{\mathsf{eq}}} \widetilde{x} \quad \text{with} \quad a_{x_{\mathsf{eq}}} = \left. \frac{\partial f}{\partial x} \right|_{x_{\mathsf{eq}}}$$

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<u>"$a < 0$ implies asymptotically stable"</u>

has been our mantra up to now!

# How do we generalize even more?

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \quad \boldsymbol{f}(\boldsymbol{x}_{\text{eq}}) = \mathbf{0}, \quad \mapsto \quad \dot{\widetilde{\boldsymbol{x}}} = A_{\boldsymbol{x}_{\text{eq}}}\widetilde{\boldsymbol{x}} \quad \text{with} \quad A_{\boldsymbol{x}_{\text{eq}}} = \nabla \boldsymbol{f}\Big|_{\boldsymbol{x}_{\text{eq}}}$$

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*with the stability of $A$ something that we will see when we do the linear algebra part of the course*

# Example

$$\dot{\boldsymbol{x}} = \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} x_2 \\ 1 - x_1^2 - x_2 \end{bmatrix}$$

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Linearization around a generic point:

$$\dot{\widetilde{\boldsymbol{x}}} = A_{\boldsymbol{x}_{\mathsf{eq}}} \widetilde{\boldsymbol{x}} = \begin{bmatrix} 0 & -1 \\ -2x_1 & -1 \end{bmatrix} \widetilde{\boldsymbol{x}}$$

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$$\boldsymbol{x}_{\text{eq}\alpha} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \implies A_{\boldsymbol{x}_{\text{eq}\alpha}} = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} \qquad \boldsymbol{x}_{\text{eq}\beta} = \begin{bmatrix} +1 \\ 0 \end{bmatrix} \implies A_{\boldsymbol{x}_{\text{eq}\beta}} = \begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix}$$

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spoiler (will see this extensively with the “linear algebra” part: the eigenvalues of $A$ will be the poles of the system!

# Example (continuation)

"the eigenvalues of $A_{\boldsymbol{x}_{\mathsf{eq}}}$ are the poles of the system"

$$\begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} \quad \Longrightarrow \quad \text{eigenvalues} = \{-2;\ 1\}$$

$$\begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix} \quad \Longrightarrow \quad \text{eigenvalues} = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$$

*Discussion:* how are the modes of the linearized system around equilibrium $\alpha$? And around equilibrium $\beta$?

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***very important message: this implicitly says that studying linearized systems gives information about the nonlinear ones!***

# Summarizing

- general approach = start with computing the equilibria for the original nonlinear system, get the corresponding $A_{\boldsymbol{x}_{\text{eq}}}$ matrix for each equilibrium $\boldsymbol{x}_{\text{eq}}$, and analyse the stability properties of that $A_{\boldsymbol{x}_{\text{eq}}}$ matrix
- if $A_{\boldsymbol{x}_{\text{eq}}}$ is asymptotically stable, then the original equilibrium $\boldsymbol{x}_{\text{eq}}$ is locally asymptotically stable
- if $A_{\boldsymbol{x}_{\text{eq}}}$ is unstable, then the original equilibrium $\boldsymbol{x}_{\text{eq}}$ is unstable
- if $A_{\boldsymbol{x}_{\text{eq}}}$ is simply stable, then we cannot say anything about the original equilibrium $\boldsymbol{x}_{\text{eq}}$ and we need to do other types of analyses (in later-on courses!)
- in any case the considerations are local considerations, valid only in the neighborhood of $\boldsymbol{x}_{\text{eq}}$

# Some philosophical considerations

- sometimes piecewise linearizing systems is a way to deal with nonlinear dynamics, even if this is not the most elegant approach to control
- you will do nonlinear control in later on courses; feedback linearization, one of the approaches, is very powerful
- `https://www.youtube.com/watch?v=uhND7Mvp3f4` ← this is done through classical nonlinear control, not data driven one

?

# Roadmap

- why do we need to numerically simulate?
- Euler methods
- pros and cons
- connections with linearization

# Our computers are digital machines, but the ODEs are “analogic” objects

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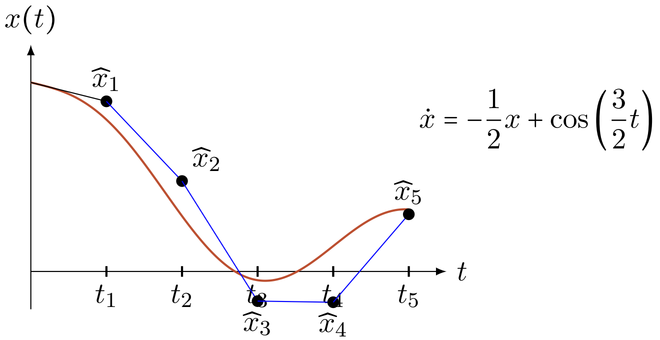
*the need is for discretizing these objects, both in time and in space*

# Simulating nonlinear systems = solving the ODE numerically and in a discrete way

i.e., use the fact that we know that $\dot{\boldsymbol{x}} = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)$, we know the whole $\boldsymbol{u}$, and we know the initial condition $\boldsymbol{x}(0)$ to compute a series of points

$$\boldsymbol{x}(t_1), \boldsymbol{x}(t_2), \ldots, \boldsymbol{x}(t_N)$$

that approximate the whole trajectory $\boldsymbol{x}(0:T)$:



# The simplest numerical solver: Euler's (forward) method

step 0: $\widehat{\boldsymbol{x}}_0 = \boldsymbol{x}_0$ *(i.e., the initial value)*

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$$
\begin{aligned}
\text{step 0:} \quad \widehat{\boldsymbol{x}}_0 &= \boldsymbol{x}_0 \quad \textit{(i.e., the initial value)} \\
\text{step 1:} \quad \widehat{\boldsymbol{x}}_1 &= \widehat{\boldsymbol{x}}_0 + \boldsymbol{f}\left(\widehat{\boldsymbol{x}}_0, \boldsymbol{u}_0\right) \Delta t
\end{aligned}
$$

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$$
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&\text{step 2:} \quad \widehat{\boldsymbol{x}}_2 &&= \widehat{\boldsymbol{x}}_1 + \boldsymbol{f}\left(\widehat{\boldsymbol{x}}_1, \boldsymbol{u}_1\right) \Delta t
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&\vdots && \vdots && \vdots \quad \vdots
\end{aligned}
$$

# Tradeoffs:

- the more "gentle" $\boldsymbol{f}$, the more accurate the results
- the smaller $\Delta t$, the more accurate the results & the longer the computational time
- the longer the time horizon $T$ in $\boldsymbol{x}(0:T)$ the less accurate the results

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*Known problem:* Euler forward may be numerically unstable, especially for "stiff ODEs" (i.e., ODEs for which some terms that can lead to rapid variation in the solution). Will be seen extensively in following courses!

# Another example: Euler's backward method

## Euler Forward:

$$\widehat{\boldsymbol{x}}_{k+1} = \widehat{\boldsymbol{x}}_k + \boldsymbol{f}\left(\widehat{\boldsymbol{x}}_k, \boldsymbol{u}_k\right) \Delta t$$

## Euler Backward:

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can be generalized to Runge-Kutta methods, with better tradeoffs and robustness properties; they will be studied in following courses

# Important: Euler's method is another type of linearization

## "normal" linearization

$$\boldsymbol{f} \approx \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \Delta \boldsymbol{x} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} \Delta \boldsymbol{u}$$

## Euler's method

$$\boldsymbol{x}\left(t + \Delta t\right) \approx \boldsymbol{x}(t) + \boldsymbol{f}\Big(\boldsymbol{x}(t), \boldsymbol{u}(t)\Big) \Delta t$$

# What is best, then? Euler or "normal" linearizing?

### pros of linearizing

- analytic results usable to design control systems and understand structural properties

### pros of Euler

- arbitrarily good accuracy if $\Delta t$ is sufficiently small
- gives more accurate information about the actual trajectories

### cons of linearizing

- results valid only locally
- one may make mistakes in computing the Jacobians

### cons of Euler

- computationally heavy
- does not give theoretical insights

?