# TTK4225 - Systems Theory, Autumn 2020

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# matrix exponentials and ODEs

# Roadmap

- definition of matrix exponentials
- which role they play in solving ODEs
- different ways of computing matrix exponentials
- the Cayley-Hamilton theorem
- Jordan forms: *the* way of seeing LTIs

# Looking back: general solution of 1-st order differential equations

$$\dot{x} = ax + bu \quad \Longrightarrow \quad x(t) = e^{at}x_0 + \int_0^t e^{a\tau} bu(t-\tau)d\tau$$

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*but what is* $e^{At}$*?*

# What is $e^{At}$?

WRONG way of doing it:

$$A = \begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix} \qquad \Longrightarrow \qquad e^{At} := \begin{bmatrix} e^{a_{11}t} & e^{a_{12}t} \\ e^{a_{21}t} & e^{a_{22}t} \end{bmatrix}$$

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This is wrong because exponentials must be s.t. $e^{as}e^{at} = e^{a(s+t)}$, and the definition above does not hold:

$$\begin{aligned} e^{As}e^{At} &= \begin{bmatrix} e^{a_{11}s} & e^{a_{12}s} \\ e^{a_{21}s} & e^{a_{22}s} \end{bmatrix} \begin{bmatrix} e^{a_{11}t} & e^{a_{12}t} \\ e^{a_{21}t} & e^{a_{22}t} \end{bmatrix} \\ &= \begin{bmatrix} e^{a_{11}s+a_{11}t} + e^{a_{12}s+a_{21}t} & e^{a_{11}s+a_{12}t} + e^{a_{12}s+a_{22}t} \\ e^{a_{21}s+a_{11}t} + e^{a_{22}s+a_{21}t} & e^{a_{21}s+a_{12}t} + e^{a_{22}s+a_{22}t} \end{bmatrix} \neq e^{A(s+t)} \end{aligned}$$

# What is $e^{At}$?

Good definition $=$ through Taylor expansions:

$$e^{at} = \sum_{n=0}^{+\infty} \frac{(at)^n}{n!} = 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \dots$$

implies

$$e^{At} = \sum_{n=0}^{+\infty} \frac{(At)^n}{n!} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots$$

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Implications

- $\frac{d}{dt}e^{At} = Ae^{At}$ *(i.e., as expected and desired)*
- $e^{As}e^{At} = e^{A(s+t)}$ *(i.e., as expected and desired)*

# Additional results

| | |
|---|---|
| derivation | $\frac{d}{dt}e^{At} = Ae^{At}$ |
| product of same exponentials | $e^{As}e^{At} = e^{A(s+t)}$ |
| preservation of commutativity | $AB = BA \quad \Longleftrightarrow \quad e^{A}e^{B} = e^{B}e^{A}$ |
| exponential of zero | $e^{\mathbf{0}} = I$ |
| non-null determinant | $\det(e^{A}) \neq \mathbf{0}$ |
| inversion | $(e^{At})^{-1} = e^{-At}$ |
| decomposition | $e^{\boldsymbol{P}B\boldsymbol{P}^{-1}} = \boldsymbol{P}e^{B}\boldsymbol{P}^{-1}$ |

?

# Going back to the original problem: solving first-order ODEs

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \quad \Longrightarrow \quad \boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}_0 + \int_{t_0}^{t} e^{A\tau} B\boldsymbol{u}(t-\tau)d\tau$$

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3. inserting this in the original ODE implies

$$e^{At}\dot{\boldsymbol{y}}(t) = B\boldsymbol{u} \quad \Longleftrightarrow \quad \dot{\boldsymbol{y}}(t) = e^{-At}B\boldsymbol{u} \quad \Longleftrightarrow \quad \boldsymbol{y}(t) = \int_{t_0}^{t} e^{-A\tau} B\boldsymbol{u}(\tau) d\tau + \boldsymbol{k}$$

with $\boldsymbol{k}$ an opportune constant

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4. plugging in this result back gives then

   $$\boldsymbol{x}(t) = e^{At}\left(\int_{t_0}^{t} e^{-A\tau}B\boldsymbol{u}(\tau)d\tau + \boldsymbol{k}\right) = \int_{t_0}^{t} e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau + e^{At}\boldsymbol{k} \Longrightarrow \boldsymbol{k} = \boldsymbol{x}_0 e^{-At_0}$$

# $e^{At}$ := transition matrix

**Notation**

$e^{At}$ := transition matrix $= \Phi(t)$

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> computers will do the computations. You though need to know what are the concepts

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# Finding the transition matrix $\Phi(t) = e^{\boldsymbol{At}}$ using Taylor expansions

$$e^{\boldsymbol{At}} \approx I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!}$$

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Note that this approximation may actually be exact if there exists $m$ s.t. $A^m = \mathbf{0}$, since in this case

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^{(m-1)}t^{(m-1)}}{(m-1)!}$$

# Example

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\implies \quad e^{At} = I + At + \frac{A^2 t^2}{2!} = \begin{bmatrix} 1 & t & t + \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

# Practical example: spring-mass system (with $m = 1$)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

## Practical example: very particular spring-mass system (with $m = 1$, $f = 0$, and $k = 0$)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \implies A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

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this means that no friction and no spring implies that if

$$\boldsymbol{x}(0) = \begin{bmatrix} p_0 \\ v_0 \end{bmatrix}$$

leads to

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} p_0 + v_0 t \\ v_0 \end{bmatrix}$$

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Meaning: the velocity keeps constant, the position grows linearly

?

# And what about a generic spring-mass system (with $m = 1$)?

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

need is to compute $e^{At}$ explicitly!

# Conceptual tool: $e^{At}$ when $A$ is diagonalizable

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Obvious result: product of diagonal matrices = diagonal matrix:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} ad & 0 & 0 \\ 0 & eb & 0 \\ 0 & 0 & cf \end{bmatrix}$$

# Conceptual tool: $e^{\Lambda t}$ when $\Lambda$ is diagonal

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&= \begin{bmatrix} \sum_{k=0}^{+\infty} \frac{1}{k!} \lambda_1^k t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{k=0}^{+\infty} \frac{1}{k!} \lambda_n^k t \end{bmatrix}
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&= \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}
\end{aligned}
$$

$\Longrightarrow$ the transition matrix of a diagonal system can be computed immediately

# Summarizing: dynamics of the system when $\Lambda$ is diagonal

$$\dot{\boldsymbol{x}} = \Lambda \boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

implies

$$\boldsymbol{x}(t) = e^{\Lambda(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{\Lambda\tau} B\boldsymbol{u}(t-\tau)d\tau$$

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Then choosing

$$\widetilde{\boldsymbol{x}} = T^{-1}\boldsymbol{x} \qquad \widetilde{B} = T^{-1}B$$

leads to

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and thus to

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## Building on top of the practical example before: spring-mass system with $m = 1$, $f = 0$, and $k = 1$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

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implies

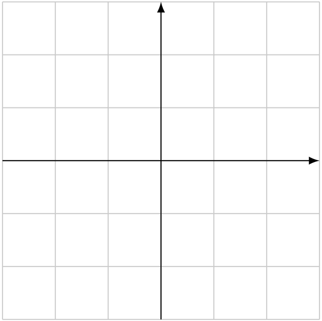
$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A\tau} B\boldsymbol{u}(t-\tau)d\tau$$

with

$$e^{At} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

# Once again

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boldsymbol{x} \qquad \Longrightarrow \qquad \boldsymbol{x}(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \boldsymbol{x}_0$$



?

*and what about non-diagonalizable $A$'s?*

## A first tool to compute $\Phi(t)$ for non-diagonalizable $A$'s: the Cayley-Hamilton's teorem

Starting ingredient: characteristic polynomial:

$$\det(sI - A) = \prod_{i=1}^{d} (s - \lambda_i)^{\mu(\lambda_i)}$$

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**Cayley-Hamilton's teorem:**

$$A^n + \alpha_{n-1} A^{n-1} + \ldots + \alpha_1 A + \alpha_0 I = \mathbf{0}$$

# Cayley-Hamilton's teorem: implications

$$A \in \mathbb{R}^{n \times n} \quad \Longrightarrow \quad A^n + \alpha_{n-1} A^{n-1} + \ldots + \alpha_1 A + \alpha_0 I = \mathbf{0}$$

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but since $A^{n-1} A = A^n$ is a linear combination of $A^{n-1}, \ldots, A, I$ then the claim follows

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*Discussion:* do you think that the following claim is true?

$$\textit{all the } A^{n+k} \textit{ is a linear combination of } A^{n-1}, \ldots, A, I, \textit{ for every } k \in \mathbb{N}_+$$

# The actual meaning of Cayley-Hamilton

Remember:

$$e^{At} = \sum_{n=0}^{+\infty} \frac{(At)^n}{n!} = I + (At) + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

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$$(At)^{n+k} = c_{n-1,k,t}\,(At)^{n-1} + \ldots + c_{1,k,t}\,(At) + c_{0,k,t} I \qquad \forall k \geq 0$$

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thus

$$e^{At} = c_0(t) I + c_1(t) A + \dots + c_{n-1}(t) A^{n-1}$$

# Evaluating $e^{At}$ via Cayley-Hamilton

Cayley-Hamilton implies

$$e^{At} = c_0(t)I + c_1(t)A + \ldots + c_{n-1}(t)A^{n-1} ,$$

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1. if $\lambda$ is an eigenvalue of $A$, then $\lambda t$ is an eigenvalue of $At$

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3. if I do the same trick above for several $\lambda$'s, then I may get a system in $c_0, \ldots, c_{n-1}t^{n-1}$

# Example of evaluating $e^{At}$ via Cayley-Hamilton: spring-mass system with $m = 1$, $k = 0$, $f = 1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

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Cayley-Hamilton then says

$$A^2 + A = 0 \quad \Longrightarrow \quad A^2 = -A$$

and

$$e^{At} = c_0(t) I + c_1(t) At \qquad \text{for opportune } c_0, c_1$$

and

$$e^{\lambda t} = c_0(t) I + c_1(t) \lambda t \qquad \text{if } \lambda \text{ is an eigenvalue}$$

## Example of evaluating $e^{At}$ via Cayley-Hamilton: spring-mass system with $m = 1$, $k = 0$, $f = 1$

$\lambda_1 = 0$, $\lambda_2 = -1$ and $e^{\lambda t} = c_0 I + c_1 \lambda t$ imply then

$$\begin{cases} e^{0t} = & c_0(t) + c_1(t)(0t) \\ e^{-1t} = & c_0(t) + c_1(t)(-1t) \end{cases}$$

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and thus

$$e^{At} = c_0(t)I + c_1(t)At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(1 - e^{-t}\right)\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

Damiano's personal view on this way of computing transition matrices:
useless from a practical point of view,
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In other words:

- to compute $e^{At}$ use Matlab, Mathematica, whatever. Don't do things by hand
- remember though the concepts, and what the previous ingredients gave us in terms of understanding

# Summary of the messages from Cayley-Hamilton

- square matrices are annihilate their characteristic polynomial
- $A$ is a linear combination of $I, A, \ldots, A^{n-1}$
- $e^{At}$ is a time-varying linear combination of $I, A, \ldots, A^{n-1}$
- $e^{\lambda t}$ with $\lambda$ eigenvalue of $A$ is a time-varying linear combination of $1, \lambda, \ldots, \lambda^{n-1}$

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Cayley-Hamilton is essential to understand controllability and observability

?

# Jordan forms: for sure the best way of understanding the structure of the the modes of LTI systems

If $A = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$ then

$$e^{At} = I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \dots$$

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$$\begin{aligned} e^{At} &= I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \dots \\ &= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & t^2/2!e^{\lambda t} & t^3/3!e^{\lambda t} & \cdots \\ & e^{\lambda t} & te^{\lambda t} & t^2/2!e^{\lambda t} & \cdots \\ & & & \ddots & \\ 0 & & & & e^{\lambda t} \end{bmatrix} \end{aligned}$$

Proof:

$$A = \lambda I + N \qquad \Longrightarrow \qquad e^{At} = e^{\lambda I t + N t} = e^{\lambda I t} e^{N t}$$

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$$e^{\lambda I t} = e^{\lambda t} I$$

$$\begin{aligned} e^{Nt} &= I + Nt + N^2 \frac{t^2}{2!} + N^3 \frac{t^3}{3!} + \dots \\ &= \begin{bmatrix} 1 & t & t^2/2! & t^3/3! & \cdots \\ & 1 & t & t^2/2! & \cdots \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} \end{aligned}$$

# Extremely important message

$$e^{Jt} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & t^2/2!e^{\lambda t} & t^3/3!e^{\lambda t} & \cdots \\ & e^{\lambda t} & te^{\lambda t} & t^2/2!e^{\lambda t} & \cdots \\ & & & \ddots & \\ 0 & & & & e^{\lambda t} \end{bmatrix}$$

# Example

$$A = \begin{bmatrix} 2 & 1 & & & & & & \\ & 2 & & & & & & \\ & & 2 & & & & & \\ & & & 0 & 1 & & & \\ & & & & 0 & 1 & & \\ & & & & & 0 & 1 & \\ & & & & & & 0 & \\ & & & & & & & -3 \end{bmatrix}$$

# And if $A$ is not in its Jordan form?

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \qquad \text{with} \qquad A = TJT^{-1}$$

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Then choosing

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leads to

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and thus to

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and thus to

$$\boldsymbol{x}(t) = T\widetilde{\boldsymbol{x}}(t)$$

# Jordan form = making the modes explicit

$$y = \widetilde{C}\widetilde{x}$$

with

$$\widetilde{\boldsymbol{x}}(t) = e^{J(t-t_0)}\widetilde{\boldsymbol{x}}(t_0) + \int_{t_0}^{t} e^{J\tau}\widetilde{B}\boldsymbol{u}(t-\tau)d\tau$$

and $e^{J(t-t_0)}$ showing all the modes explicitly

?

# Final summary: all the methods to compute $e^{At}$

## Direct computation

- use the definition $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \dots$
- if there exists a $k$ for which $A^k = \mathbf{0}$ then you can find the exact solution, otherwise it will be approximate

## By diagonalizing / reducing to a Jordan form

- change the basis of the system so to go from $A$ to $\Lambda$ or $J$
- find $e^{\Lambda t}$ or $e^{Jt}$ immediately

## Cayley-Hamilton

- find the coefficients $c_0(t), \dots, c_{n-1}(t)$ that give $e^{At} = c_0(t)I + \dots + c_{n-1}(t)A^{n-1}$ using the eigenvalues of $A$
- compute $e^{At}$ in that way

## (we did not see this, and we won't)Inverse Laplace-transforming

?

## A matrix: a strange collection of numbers, or a precise way of defining the dynamics of a system?

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

doing systems theory for LTI systems means
studying the inner structure of $\dot{\boldsymbol{x}} = A\boldsymbol{x}$.
Indeed the Jordan form of $A$ says precisely how the state evolves,
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state-space representations are "richer" than input-output ones $\Longrightarrow$ one can design control systems better when working with state-space representations

# Where are we now?

characterizing LTI systems

↓

linearization of nonlinear systems

↓

linear algebra as a way
to even better analyse LTI systems

↓

designing filters

?

# A complete example, from the beginning to the end

# Roadmap

- Lotka-Volterra, but this time from the beginning to the end

# Lotka-Volterra

- $y_{\text{prey}}$ := prey
- $y_{\text{pred}}$ := predator

$$\begin{cases} \dot{y}_{\text{prey}} & = \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} & = -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

`GitHub/TTK4225/trunk/Jupyter/Lotka-Volterra-introduction.ipynb`

# “Our” Lotka-Volterra

$$\begin{cases} \dot{y}_{\text{prey}} & = 10 y_{\text{prey}} - 1 y_{\text{prey}} y_{\text{pred}} - u_{\text{prey}} \\ \dot{y}_{\text{pred}} & = -0.1 y_{\text{pred}} + y_{\text{prey}} y_{\text{pred}} - u_{\text{pred}} \end{cases}$$

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Steps:

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Steps:

1. find the equilibria
2. linearize the system around the equilibria

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Steps:

1. find the equilibria
2. linearize the system around the equilibria
3. find the stability properties of these equilibria

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5. understand the controllability and observability properties of the system *(not in this course)*

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1. find the equilibria
2. linearize the system around the equilibria
3. find the stability properties of these equilibria
4. characterize the trajectories of the system in the neighborhood of these equilibria
5. understand the controllability and observability properties of the system *(not in this course)*
6. design minimal human interventions that make the system behave as desired *(not in this course)*

?