TTK4225 - Systems Theory, Autumn 2020

Damiano Varagnolo

NTNU

Introduction to Laplace transforms

Laplace who?

Figure: Pierre-Simon Laplace, Beaumont-en-Auge, 23 March 1749 - Paris, 5 March 1827

Roadmap

- why?
- what?
- how, at least intuitively?
- an essential example

Why?

Remember: predictive control of the type

$$\boldsymbol{u}^\star = \arg \min_{\boldsymbol{u} \in \mathcal{U}, \boldsymbol{f}(\boldsymbol{u}) \in \mathcal{F}} \mathrm{Cost}\left(\boldsymbol{f}\left(\boldsymbol{u}\right), \boldsymbol{u}\right)$$

requires to be able to:

- define "Cost"
- compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

Why?

Remember: predictive control of the type

$$\boldsymbol{u}^{\star} = \arg \min_{\boldsymbol{u} \in \mathcal{U}, \boldsymbol{f}(\boldsymbol{u}) \in \mathcal{F}} \mathrm{Cost}\left(\boldsymbol{f}\left(\boldsymbol{u}\right), \boldsymbol{u}\right)$$

requires to be able to:

- define "Cost"
- compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly
- be sure that the model does not have "nasty" properties

> Laplace, for how we see it in this course,
> helps the "compute $\boldsymbol{f}\left(\boldsymbol{u}\right)$ rapidly" part

The essential question we want to answer:

what will be $y(t)$ starting from y_0 and for a given $u(t)$?

The essential question we want to answer:

what will be $y(t)$ starting from y_0 and for a given $u(t)$?

First order systems:

$$y(t) = y_0 e^{at} + \int_0^t e^{a(t-\tau)} b u(\tau) d\tau, \quad t > 0 \tag{1}$$

The essential question we want to answer: what will be $y(t)$ starting from y_0 and for a given $u(t)$?

First order systems:

$$y(t) = y_0 e^{at} + \int_0^t e^{a(t-\tau)} b u(\tau) d\tau, \quad t > 0 \tag{1}$$

Problem: computing by hand the convolution integral $g * u(t)$ may be complicated; computing it via numerical integration may require some time

The essential question we want to answer:
what will be $y(t)$ starting from y_0 and for a given $u(t)$?

First order systems:

$$y(t) = y_0 e^{at} + \int_0^t e^{a(t-\tau)} b u(\tau) d\tau, \quad t > 0 \tag{1}$$

Problem: computing by hand the convolution integral $g * u(t)$ may be complicated; computing it via numerical integration may require some time

alternative strategy: solve $g * u(t)$ using Laplace transforms!

Laplace transforms - another pillar of automatic control

in TTK4225: $\mathcal{L}$ used to compute $y(t)$

in TTK4230: $\mathcal{L}$ used to determine $u(t)$

in virtually all the other courses at ITK: $\mathcal{L}$ used for both purposes and more

Laplace transform - definition

Hypothesis: $f(t)$ is a real function that is locally integrable[1] on $[0, +\infty)$ or $(-\infty, +\infty)$

[1] I.e., so that its integral is finite on every compact subset of its domain of definition

Laplace transform - definition

Hypothesis: $f(t)$ is a real function that is locally integrable[1] on $[0, +\infty)$ or $(-\infty, +\infty)$

unilateral Laplace transform:

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{+\infty} e^{-st} f(t) dt \qquad \text{for } t \geq 0, s = \sigma + i\omega \in \mathbb{C} \tag{2}$$

for all the $s \in \mathbb{C}$ for which the integral in the RHS converges *(i.e., the ROC)*

[1] I.e., so that its integral is finite on every compact subset of its domain of definition

Laplace transform - definition (2)

Hypothesis: $f(t)$ is a real function that is locally integrable[2] on $[0, +\infty)$ or $(-\infty, +\infty)$

bilateral Laplace transform:

$$F(s) = \mathcal{B}\{f(t)\} := \int_{-\infty}^{+\infty} e^{-st} f(t) dt \qquad \text{for } t \geq 0, s = \sigma + i\omega \in \mathbb{C} \tag{3}$$

for all the $s \in \mathbb{C}$ for which the integral in the RHS converges *(i.e., the ROC)*

[2] I.e., so that its integral is finite on every compact subset of its domain of definition

Laplace transform - ancillary definitions

unilateral Laplace transform:

$$F(s) = \mathcal{L}\left\{f(t)\right\} := \int_0^{+\infty} e^{-st} f(t) dt \qquad \text{for } t \geq 0, s = \sigma + i\omega \in \mathbb{C} \tag{4}$$

Laplace transform - ancillary definitions

unilateral Laplace transform:

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{+\infty} e^{-st} f(t) dt \qquad \text{for } t \geq 0, s = \sigma + i\omega \in \mathbb{C} \tag{4}$$

- t = "time domain"
- s = "[complex] frequency domain" *(will see this better later on)*

Laplace transform - ancillary definitions

unilateral Laplace transform:

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{+\infty} e^{-st} f(t) dt \qquad \text{for } t \geq 0, s = \sigma + i\omega \in \mathbb{C} \tag{4}$$

- $t =$ "time domain"
- $s =$ "[complex] frequency domain" *(will see this better later on)*

$$\Longrightarrow \mathcal{L} : \text{time} \mapsto \text{frequencies}$$

Laplace transform - ancillary definitions

unilateral Laplace transform:

$$F(s) = \mathcal{L}\left\{f(t)\right\} := \int_0^{+\infty} e^{-st} f(t) dt \qquad \text{for } t \geq 0, s = \sigma + i\omega \in \mathbb{C} \tag{4}$$

- $t =$ "time domain"
- $s =$ "[complex] frequency domain" *(will see this better later on)*

$$\Longrightarrow \mathcal{L} : \text{time} \mapsto \text{frequencies}$$

$$\text{we need then } \mathcal{L}^{-1} : \text{frequencies} \mapsto \text{time}$$

Inverse Laplace transform

Hypothesis:

- $F(s)$ complex function

Thesis: there may be a set of piecewise-continuous, exponentially-restricted real functions $f(t)$ which have the property that

$$\mathcal{L}\{f(t)\} = F(s);$$

all the $f(t)$ satisfying this equivalence differ from each other only on a subset having Lebesgue measure zero

Inverse Laplace transform

Hypothesis:

- $F(s)$ complex function

Thesis: there may be a set of piecewise-continuous, exponentially-restricted real functions $f(t)$ which have the property that

$$\mathcal{L}\{f(t)\} = F(s);$$

all the $f(t)$ satisfying this equivalence differ from each other only on a subset having Lebesgue measure zero

Discussion: is the existence of at least one of such $f(t)$ guaranteed?

Inverse Laplace transform

Hypothesis:

- $F(s)$ complex function

Thesis: there may be a set of piecewise-continuous, exponentially-restricted real functions $f(t)$ which have the property that

$$\mathcal{L}\{f(t)\} = F(s);$$

all the $f(t)$ satisfying this equivalence differ from each other only on a subset having Lebesgue measure zero

Discussion: is the existence of at least one of such $f(t)$ guaranteed?

Discussion: given any $f(t)$, is the existence of $F(s)$ guaranteed?

Main usefulness: convolution in time transforms into multiplication in Laplace-domain, and viceversa

$$\begin{cases} H(s) = \mathcal{L}\{h(t)\} \\ U(s) = \mathcal{L}\{u(t)\} \end{cases} \implies \mathcal{L}\{h * u(t)\} = H(s)U(s) \tag{5}$$

Main usefulness: convolution in time transforms into multiplication in Laplace-domain, and viceversa

$$\begin{cases} H(s) = \mathcal{L}\left\{h(t)\right\} \\ U(s) = \mathcal{L}\left\{u(t)\right\} \end{cases} \implies \mathcal{L}\left\{h * u(t)\right\} = H(s)U(s) \tag{5}$$

Discussion: how then may $\mathcal{L}$ help computing $h * u(t)$?

An intuitive explanation of the usefulness of the Laplace transform in automatic control

initial value problem

$$\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2}$$

An intuitive explanation of the usefulness of the Laplace transform in automatic control

initial value problem

$$\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2}$$

$$\Big\downarrow$$

$$\boxed{x(t) = 2e^{-t}}$$

solution in the time domain

An intuitive explanation of the usefulness of the Laplace transform in automatic control

$$
\begin{tikzcd}[column sep=large]
\text{initial value problem} & \text{algebraic problem} \\
\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \\
\boxed{x(t) = 2e^{-t}} & \\
\text{solution in the time domain} &
\end{tikzcd}
$$

An intuitive explanation of the usefulness of the Laplace transform in automatic control

initial value problem

algebraic problem

$$\begin{tikzcd} \boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\ \boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \end{tikzcd}$$

solution in the time domain

solution in the frequency domain

An intuitive explanation of the usefulness of the Laplace transform in automatic control

initial value problem | algebraic problem

$$
\begin{tikzcd}
\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\
\boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \arrow[l, "\mathcal{L}^{-1}"]
\end{tikzcd}
$$

solution in the time domain | solution in the frequency domain

?

next few slides:
introduce an utterly important example,
and with it show the “trick” mentioned before

The absolutely (and by far) most important Laplace transform

Discussion: which signal in the time domain is the most important signal you saw up to now?

The absolutely (and by far) most important Laplace transform

Discussion: which signal in the time domain is the most important signal you saw up to now?

$$\boxed{\dot{y} = ay + bu \qquad \mapsto \qquad y(t) = y_0 e^{at} + \int_0^t e^{a\tau} bu(t-\tau) d\tau}$$

$$e^{at}$$

$$\mathcal{L}\left\{e^{at}\right\} = ?$$

$\mathcal{L}\left\{e^{at}\right\} = ?$

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^{+\infty} e^{at} e^{-st} dt$$

$\mathcal{L}\left\{e^{at}\right\} = ?$

$$\begin{aligned}\mathcal{L}\left\{e^{at}\right\} &= \int_0^{+\infty} e^{at}e^{-st}dt \\ &= \int_0^{+\infty} e^{(a-s)t}dt\end{aligned}$$

$\mathcal{L}\left\{e^{at}\right\} = ?$

$$
\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \int_0^{+\infty} e^{at} e^{-st} dt \\
&= \int_0^{+\infty} e^{(a-s)t} dt \\
&= \left[\frac{1}{a-s} e^{(a-s)t} \right]_0^{+\infty}
\end{aligned}
$$

$\mathcal{L}\left\{e^{at}\right\} = ?$

$$\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \int_0^{+\infty} e^{at} e^{-st} dt \\
&= \int_0^{+\infty} e^{(a-s)t} dt \\
&= \left[\frac{1}{a-s} e^{(a-s)t}\right]_0^{+\infty} \\
&= \frac{1}{a-s}\left[e^{(a-s)\infty} - e^{(a-s)0}\right]
\end{aligned}$$

Remember, though: $F(s)$ is defined for all the $s \in \mathbb{C}$ for which the integral in the RHS converges *(i.e., the ROC)*. Thus, *discussion:* for which s does $F(s)$ exist?

$\mathcal{L}\left\{e^{at}\right\} = ?$

$$\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \int_0^{+\infty} e^{at} e^{-st} dt \\
&= \int_0^{+\infty} e^{(a-s)t} dt \\
&= \left[\frac{1}{a-s} e^{(a-s)t}\right]_0^{+\infty} \\
&= \frac{1}{a-s}\left[e^{(a-s)\infty} - e^{(a-s)0}\right]
\end{aligned}$$

Remember, though: $F(s)$ is defined for all the $s \in \mathbb{C}$ for which the integral in the RHS converges *(i.e., the ROC)*. Thus, *discussion:* for which s does $F(s)$ exist?

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \qquad \text{for } \mathrm{Re}(s) \geq a \tag{6}$$

At the same time . . .

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{7}$$

At the same time ...

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{7}$$

Our game, now: use our new knowledge, i.e., $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$ to compute the solution to $\dot{y} = ay$.

At the same time . . .

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{7}$$

Our game, now: use our new knowledge, i.e., $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$ to compute the solution to $\dot{y} = ay$. *Discussion:* how did we find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$ before?

An important property that we will prove later on

(see 4.2.1 in Balchen's book)

$$\mathcal{L}\left\{\dot{x}\right\} = sX(s) - x_0 \tag{8}$$

An important property that we will prove later on

(see 4.2.1 in Balchen's book)

$$\mathcal{L}\left\{\dot{x}\right\} = sX(s) - x_0 \tag{8}$$

$$\mathcal{L}\left\{\ddot{x}\right\} = s^2 X(s) - s x_0 - \dot{x}_0 \tag{9}$$

An important property that we will prove later on

(see 4.2.1 in Balchen's book)

$$\mathcal{L}\left\{\dot{x}\right\} = sX(s) - x_0 \tag{8}$$

$$\mathcal{L}\left\{\ddot{x}\right\} = s^2X(s) - sx_0 - \dot{x}_0 \tag{9}$$

$$\mathcal{L}\left\{\dddot{x}\right\} = s^3X(s) - s^2x_0 - s\dot{x}_0 - \ddot{x}_0 \tag{10}$$

$$\vdots \tag{11}$$

An important property that we will prove later on

(see 4.2.1 in Balchen's book)

$$\mathcal{L}\{\dot{x}\} = sX(s) - x_0 \tag{8}$$

$$\mathcal{L}\{\ddot{x}\} = s^2X(s) - sx_0 - \dot{x}_0 \tag{9}$$

$$\mathcal{L}\{\dddot{x}\} = s^3X(s) - s^2x_0 - s\dot{x}_0 - \ddot{x}_0 \tag{10}$$

$$\vdots \tag{11}$$

$s \quad \Longrightarrow \quad$ derivation; $\dfrac{1}{s} \quad \Longrightarrow \quad$ integration

"mnemonics": s = 'derivasjon'

The "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

(using that we know that $\mathcal{L}\{\dot{y}\} = sY(s) - y_0$ and that $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$)

$$\dot{y} = ay$$

The "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

(using that we know that $\mathcal{L}\{\dot{y}\} = sY(s) - y_0$ and that $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$)

$$\dot{y} = ay \qquad \rightarrow \qquad sY - y_0 = aY$$

The "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

(using that we know that $\mathcal{L}\{\dot{y}\} = sY(s) - y_0$ and that $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$)

$$\dot{y} = ay \quad \rightarrow \quad sY - y_0 = aY \quad \rightarrow \quad Y = y_0 \frac{1}{s-a}$$

initial value problem $\qquad\qquad$ algebraic problem

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\
\boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \arrow[l, "\mathcal{L}^{-1}"]
\end{tikzcd}
$$

solution in the time domain $\qquad\qquad$ solution in the frequency domain

?

What is a Laplace transform, actually?

Roadmap

- Laplace as a generalization of Fourier
- intuitions behind Fourier transforms
- generalizing sinusoids to exponentially decaying sinusoids

In the previous unit

$$\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \int_0^{+\infty} e^{at}e^{-st}dt \\
&= \int_0^{+\infty} e^{(a-s)t}dt \\
&= \left[\frac{1}{a-s}e^{(a-s)t}\right]_0^{+\infty} \\
&= \frac{1}{a-s}\left[e^{(a-s)\infty} - e^{(a-s)0}\right] \\
&= \frac{1}{s-a}
\end{aligned}$$

for $\mathrm{Re}\left(s\right) \geq a$

Main usefulness: convolution in time = multiplication in frequency, and this makes solving ODEs an algebraic problem

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\text{initial value problem} & \text{algebraic problem} \\
\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\
\boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \arrow[l, "\mathcal{L}^{-1}"] \\
\text{solution in the time domain} & \text{solution in the frequency domain}
\end{tikzcd}
$$

But what is a Laplace transform, eventually?

$\rightarrow$ an opportune extension of the Fourier transform

But what is a Fourier transform, eventually?

Definition:

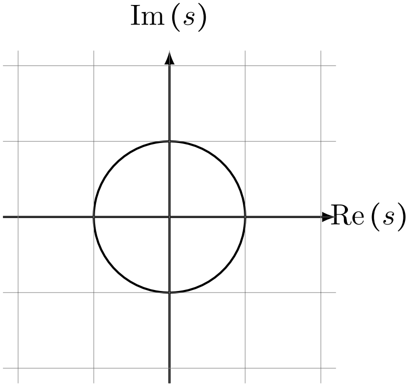
$$\mathcal{F}\left\{f(t)\right\} = \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \qquad (12)$$

But what is a Fourier transform, eventually?

Definition:

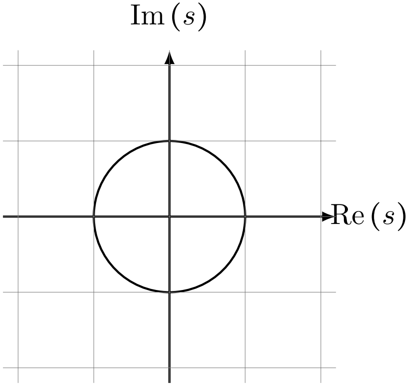
$$\mathcal{F}\left\{f(t)\right\} = \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \tag{12}$$

Discussion: where is $e^{-j\omega t}$?



Small recap: angular frequency ≠ ordinary frequency

$$e^{j\omega t} = e^{j2\pi f t}$$



Fourier transforms using Euler formula

$$
\begin{aligned}
\mathcal{F}\left\{f(t)\right\}(\omega) &= \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t) e^{j\omega t} dt \\
&= \int_{-\infty}^{+\infty} f(t) \cos(\omega t)\, dt \quad - \quad j \int_{-\infty}^{+\infty} f(t) \sin(\omega t)\, dt
\end{aligned} \tag{13}
$$

Fourier transforms using Euler formula

$$\begin{aligned}\mathcal{F}\left\{f(t)\right\}(\omega) &= \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t)e^{j\omega t}dt \\ &= \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt \quad - \quad j\int_{-\infty}^{+\infty} f(t)\sin(\omega t)\,dt\end{aligned} \tag{13}$$

intuition: for each ω, $\widehat{f}(\omega)$ is a couple of numbers, conveniently put together as a complex number, so that:

- $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ = area of $f(t)\cos(\omega t)$
- $\mathrm{Im}\left(\widehat{f}(\omega)\right)$ = area of $f(t)\sin(\omega t)$

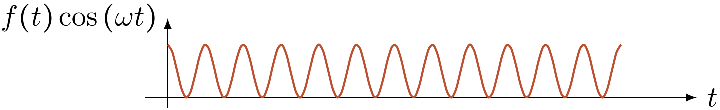
What is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for sinusoidal f's?

$$\star = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt = \int_{-\infty}^{+\infty} \Big(\cos(\alpha t)\Big)\cos(\omega t)\,dt$$

What is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for sinusoidal f's?

$$\star = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt = \int_{-\infty}^{+\infty} \Big(\cos(\alpha t)\Big)\cos(\omega t)\,dt$$

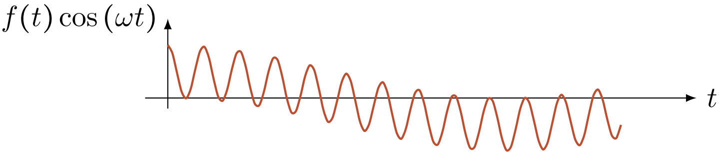
- if $f(t) = \cos(\omega t)$ then $\star = +\infty$



What is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for sinusoidal f's?

$$\star = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\, dt = \int_{-\infty}^{+\infty} \Big(\cos(\alpha t)\Big)\cos(\omega t)\, dt$$

- if $f(t) = \cos(\omega t)$ then $\star = +\infty$
- if $f(t) = \cos(\alpha t)$ with $\alpha \neq \omega$ then $\star = 0$



What is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for composite sinusoidal f's?

$$\star = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt = \int_{-\infty}^{+\infty} \left(\sum_k \cos(\alpha_k t)\right)\cos(\omega t)\,dt$$

What is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for composite sinusoidal f's?

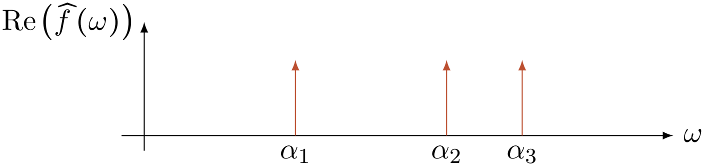
$$\star = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt = \int_{-\infty}^{+\infty} \left(\sum_k \cos(\alpha_k t)\right)\cos(\omega t)\,dt$$

as before: if $\omega = \alpha_k$ then $\star = +\infty$, otherwise $\star = 0$!

What is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for composite sinusoidal f's?

$$\star = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt = \int_{-\infty}^{+\infty} \left(\sum_k \cos(\alpha_k t)\right)\cos(\omega t)\,dt$$

as before: if $\omega = \alpha_k$ then $\star = +\infty$, otherwise $\star = 0$! Thus varying ω makes $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ kind of "comb" for the right frequency:



And what about $\operatorname{Im}\left(\widehat{f}(\omega)\right)$?

$$\star = \int_{-\infty}^{+\infty} f(t) \sin(\omega t)\, dt$$

kind of similar, only scanning for oddity (while $\cos$ scans for even components)

$\rightarrow$ see the additional material linked later for more information

And what is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for generic non-periodic f's?

$$\star = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt$$

And what is $\mathrm{Re}\left(\widehat{f}(\omega)\right)$ for generic non-periodic f's?

$$\star = \int_{-\infty}^{+\infty} f(t) \cos(\omega t)\, dt$$

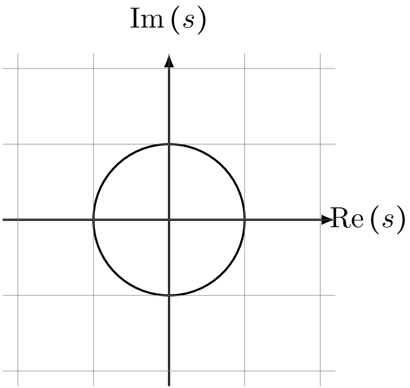
- potentially every ω may contribute to $\star$
- if there is no pure $\cos$ in f then each ω contributes with a finite area

So, what is a Fourier transform, eventually?

So, what is a Fourier transform, eventually?

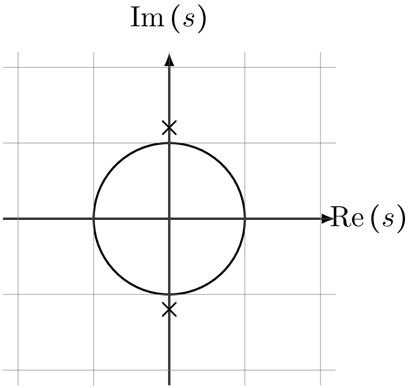
$=$ scanning the imaginary axis for sinusoidal components:

$$
\begin{aligned}
\widehat{f}(\omega) &= \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\,dt \\
&\quad + j\int_{-\infty}^{+\infty} f(t)\sin(\omega t)\,dt
\end{aligned}
$$



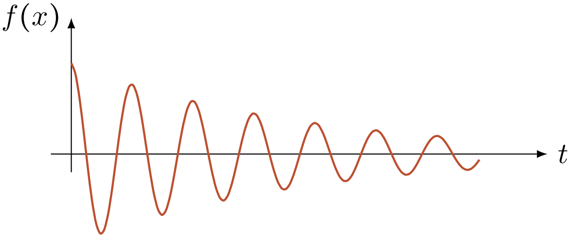
Summarizing, graphically

$$f(t) = \cos(\omega t) = \mathrm{Re}\left(e^{j\omega t}\right) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$



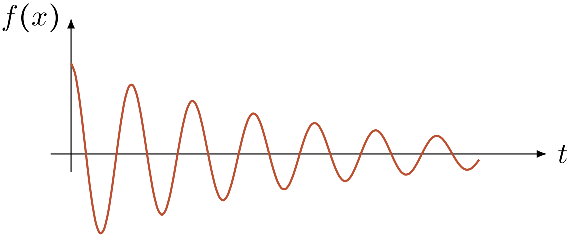
But what about exponentially decaying sinusoids?

$$f(t) = e^{at}\cos(\omega t)$$



But what about exponentially decaying sinusoids?

$$f(t) = e^{at}\cos(\omega t)$$



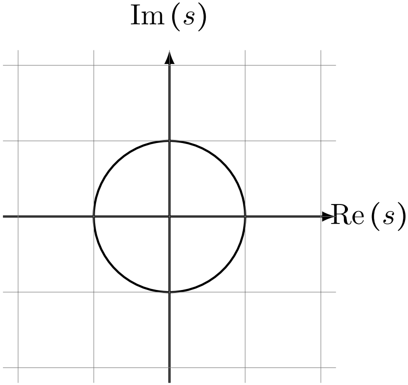
what if we consider $\overline{f}(t) = f(t)e^{-at}$?

Fourier and Laplace, side by side

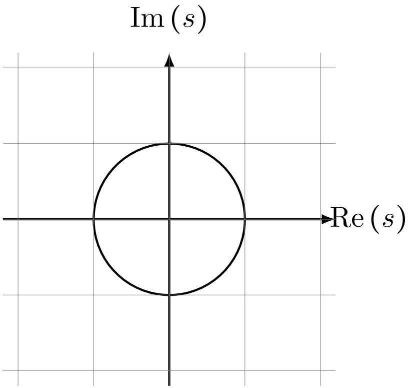
$$\mathcal{F}\left\{f(t)\right\} = \widehat{f}(\omega) := \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$

$$\mathcal{L}\left\{f(t)\right\} := \int_{0}^{+\infty} f(t)e^{-st}dt$$

So, what is a Laplace transform, eventually?



So, what is a Laplace transform, eventually?



beware of the region of convergence!

?

Suggested additional material

Fourier transforms 1:

https://www.youtube.com/watch?v=spUNpyF58BY

Fourier transforms 2 (with some Laplace in it):

https://www.youtube.com/watch?v=3gjJDuCAEQQ

Laplace transforms:

https://www.youtube.com/watch?v=n2y7n6jw5d0

Suggested additional material

Fourier transforms 1:

`https://www.youtube.com/watch?v=spUNpyF58BY`

Fourier transforms 2 (with some Laplace in it):

`https://www.youtube.com/watch?v=3gjJDuCAEQQ`

Laplace transforms:

`https://www.youtube.com/watch?v=n2y7n6jw5d0`

Laplace = a pillar of automatic control

$\Longrightarrow$ better to invest time in developing a full understanding of it

Using Laplace transforms with second order systems in free evolution

Recall: the "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

$$\dot{y} = ay \qquad \rightarrow \qquad sY - y_0 = aY \qquad \rightarrow \qquad Y = y_0 \frac{1}{s-a}$$

Recall: the "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$

initial value problem

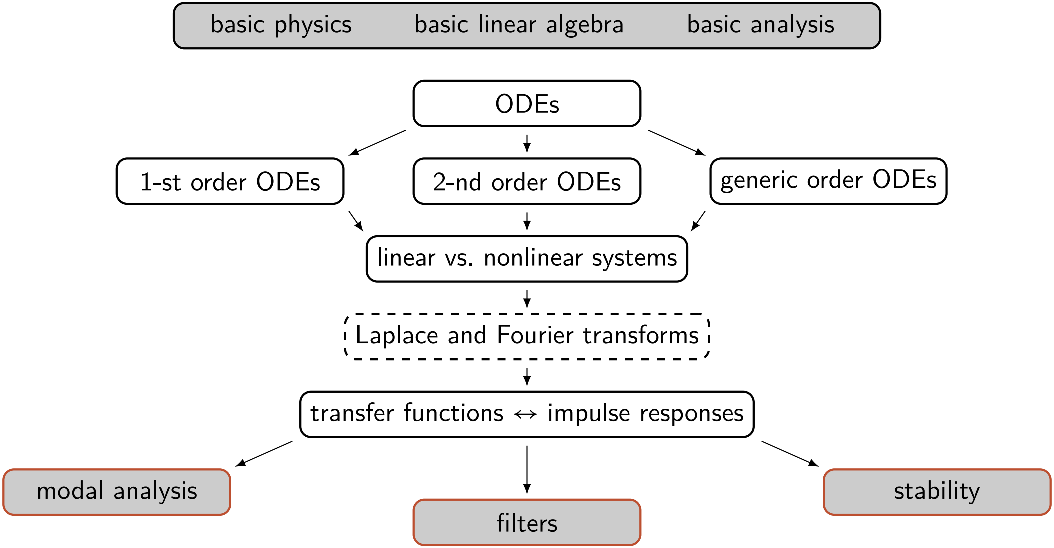
algebraic problem

$$\begin{tikzcd}
\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\
\boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \arrow[l, "\mathcal{L}^{-1}"]
\end{tikzcd}$$

solution in the time domain

solution in the frequency domain

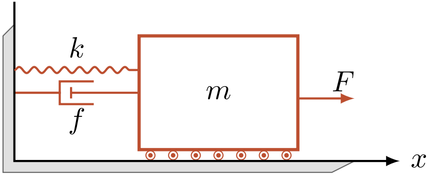
Summary / Roadmap of TTK4225

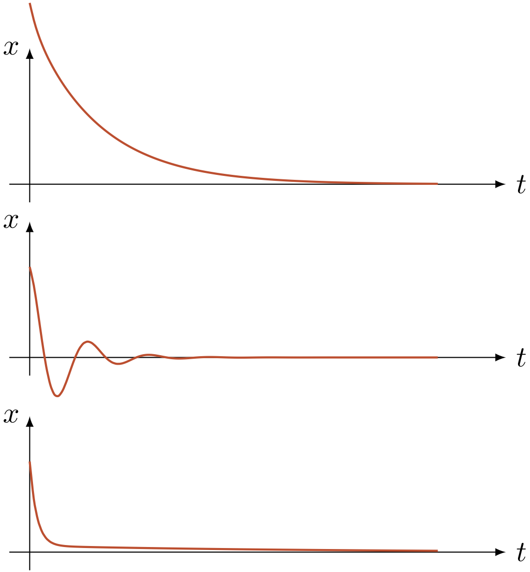


Roadmap

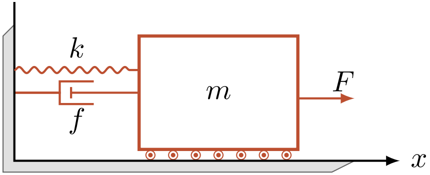
- the same thing, but for second order system
- different algebraic situations $\Longrightarrow$ different damping properties
- different ways of summarizing the different damping properties

Discussion on a practical example:
how may a second order system free-evolve?




x
t
x
t
x
t

Dissecting the practical example “spring-mass systems”



$$m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + u(t)$$

The properties we shall use

$$\mathcal{L}\left\{\dot{x}\right\} = sX(s) - x_0 \tag{14}$$

$$\mathcal{L}\left\{\ddot{x}\right\} = s^2X(s) - sx_0 - \dot{x}_0 \tag{15}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{16}$$

$$\tag{17}$$

Solution for the free evolution case

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0$$

Solution for the free evolution case

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0 \qquad \Longrightarrow \qquad X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}}$$

Solution for the free evolution case

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0 \qquad \Longrightarrow \qquad X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}}$$

we would like to use $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$ to inverse-transform

Solution for the free evolution case

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0 \qquad \Longrightarrow \qquad X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}}$$

we would like to use $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ to inverse-transform

$\Longrightarrow$ <u>we must try to factorize the RHS, i.e., try to write it as</u>

$$X(s) = \frac{A}{s-\alpha} + \frac{B}{s-\beta}$$

(if possible)

Solution for the free evolution case

Next step: factorize

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}}$$

Solution for the free evolution case

Next step: factorize

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}}$$

$$s^2 + \frac{f}{m}s + \frac{k}{m} = 0 \qquad \Longrightarrow \qquad s = -\frac{f}{2m} \pm \frac{\sqrt{f^2 - 4mk}}{2m}$$

Solution for the free evolution case

Next step: factorize

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}}$$

$$s^2 + \frac{f}{m}s + \frac{k}{m} = 0 \qquad \Longrightarrow \qquad s = -\frac{f}{2m} \pm \frac{\sqrt{f^2 - 4mk}}{2m}$$

Discussion: what are the different possibilities?

Solution for the free evolution case

Next step: factorize

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}}$$

$$s^2 + \frac{f}{m}s + \frac{k}{m} = 0 \qquad \Longrightarrow \qquad s = -\frac{f}{2m} \pm \frac{\sqrt{f^2 - 4mk}}{2m}$$

Discussion: what are the different possibilities?

1. $f^2 - 4mk > 0 \implies$ two distinct real solutions *(will imply overdamping)*
2. $f^2 - 4mk = 0 \implies$ one double real solution *(will imply critical damping)*
3. $f^2 - 4mk < 0 \implies$ two conjugate complex solutions *(will imply underdamping)*

$f^2 - 4mk > 0$, i.e., overdamping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

$f^2 - 4mk > 0$, i.e., overdamping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

$$f^2 > 4mk \quad \Longrightarrow \quad \begin{cases} \lambda_1 = & -\dfrac{f}{2m} - \dfrac{\sqrt{f^2 - 4mk}}{2m} \in \mathbb{R} \\[2ex] \lambda_2 = & -\dfrac{f}{2m} + \dfrac{\sqrt{f^2 - 4mk}}{2m} \in \mathbb{R} \end{cases}$$

$f^2 - 4mk > 0$, i.e., overdamping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

$$f^2 > 4mk \quad \Longrightarrow \quad \begin{cases} \lambda_1 = & -\dfrac{f}{2m} - \dfrac{\sqrt{f^2 - 4mk}}{2m} \in \mathbb{R} \\[2ex] \lambda_2 = & -\dfrac{f}{2m} + \dfrac{\sqrt{f^2 - 4mk}}{2m} \in \mathbb{R} \end{cases}$$

$$\Longrightarrow \quad X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2}$$

Discussion: can we conclude immediately that

$$X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2} \quad \Longrightarrow \quad x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

or do we need to say some property about Laplace transforms that we did not say before?

We need to be sure that Laplace transforms are linear!

i.e., be sure that

$$\mathcal{L}\left\{\alpha x_1 + \beta x_2\right\} = \alpha\mathcal{L}\left\{x_1\right\} + \beta\mathcal{L}\left\{x_2\right\},$$

and that

$$\mathcal{L}^{-1}\left\{\alpha X_1 + \beta X_2\right\} = \alpha\mathcal{L}^{-1}\left\{X_1\right\} + \beta\mathcal{L}^{-1}\left\{X_2\right\}$$

We need to be sure that Laplace transforms are linear!

i.e., be sure that

$$\mathcal{L}\left\{\alpha x_1 + \beta x_2\right\} = \alpha \mathcal{L}\left\{x_1\right\} + \beta \mathcal{L}\left\{x_2\right\},$$

and that

$$\mathcal{L}^{-1}\left\{\alpha X_1 + \beta X_2\right\} = \alpha \mathcal{L}^{-1}\left\{X_1\right\} + \beta \mathcal{L}^{-1}\left\{X_2\right\}$$

Discussion: why do the previous properties hold? (hint: recall that $\mathcal{L}\left\{x(t)\right\} := \int_0^{+\infty} x(t) e^{-st} dt$)

In conclusion: overdamping in free evolution

$$f^2 > 4mk \quad \Longrightarrow \quad X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2} \quad \Longrightarrow \quad x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

for opportune A and B.

In conclusion: overdamping in free evolution

$$f^2 > 4mk \qquad \Longrightarrow \qquad X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2} \qquad \Longrightarrow \qquad x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

for opportune A and B. *Discussion:* what determines A and B? Hint: recall that

$$X(s) = \frac{x_0 \left(\frac{f}{m} + s \right) + \dot{x}_0}{s^2 + \frac{f}{m} s + \frac{k}{m}} = \frac{x_0 \left(\frac{f}{m} + s \right) + \dot{x}_0}{\left(s - \lambda_1 \right) \left(s - \lambda_2 \right)}$$

important message (will be seen again later on): the denominator of the rational Laplace transform determines the behavior of the system, while the numerator determines which root is dominant and the "amplitude" of each mode

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

Practical example of a overdamped system in free evolution

$$\begin{cases} m & = 10 & \text{kg} \\ f & = 100 & \text{kg/s} \\ k & = 90 & \text{N/s}^2 \end{cases} \implies \begin{cases} \ddot{x} + \dfrac{100}{10}\dot{x} + \dfrac{90}{10}x = 0 \\ \\ \\ \end{cases}$$

Practical example of a overdamped system in free evolution

$$\begin{cases} m & = 10 & \text{kg} \\ f & = 100 & \text{kg/s} \\ k & = 90 & \text{N/s}^2 \end{cases} \implies \begin{cases} \ddot{x} + \dfrac{100}{10}\dot{x} + \dfrac{90}{10}x = 0 \\ x(0) = 0.16\text{m} \\ \dot{x}(0) = 0\text{m/s} \end{cases} \implies X(s) = \frac{0.16\,(s+10)}{s^2 + 10s + 9}$$

Practical example of a overdamped system in free evolution

$$
\begin{cases}
m & = 10 & \text{kg} \\
f & = 100 & \text{kg/s} \\
k & = 90 & \text{N/s}^2
\end{cases}
\implies
\begin{cases}
\ddot{x} + \dfrac{100}{10}\dot{x} + \dfrac{90}{10}x = 0 \\
x(0) = 0.16\text{m} \\
\dot{x}(0) = 0\text{m/s}
\end{cases}
\implies
X(s) = \frac{0.16\,(s+10)}{s^2 + 10s + 9}
$$

note that there are 2 distinct steps!!

Practical example of a overdamped system in free evolution

$$\begin{cases} m &= 10 & \text{kg} \\ f &= 100 & \text{kg/s} \\ k &= 90 & \text{N/s}^2 \end{cases} \Longrightarrow \begin{cases} \ddot{x} + \dfrac{100}{10}\dot{x} + \dfrac{90}{10}x = 0 \\ x(0) = 0.16\text{m} \\ \dot{x}(0) = 0\text{m/s} \end{cases} \Longrightarrow X(s) = \frac{0.16\,(s+10)}{s^2 + 10s + 9}$$

note that there are 2 distinct steps!!

`ARMA-continuous-time-example.ipynb`

?

$f^2 - 4mk = 0$, i.e., critical damping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda)^2}$$

$f^2 - 4mk = 0$, i.e., critical damping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s-\lambda)^2}$$

$$f^2 = 4mk,\ \lambda = -\frac{f}{2m} \qquad \Longrightarrow \qquad X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s-\lambda)^2}$$

$f^2 - 4mk = 0$, i.e., critical damping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s-\lambda)^2}$$

$$f^2 = 4mk,\ \ \lambda = -\frac{f}{2m} \qquad \Longrightarrow \qquad X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s-\lambda)^2}$$

$$\Longrightarrow \qquad X(s) = \frac{A}{s-\lambda} + \frac{B}{(s-\lambda)^2}$$

Discussion: did we learn until now in this course how to inverse-Laplace these things?

$$X(s) = \frac{A}{s-\lambda} + \frac{B}{(s-\lambda)^2}$$

Discussion: did we learn until now in this course how to inverse-Laplace these things?

$$X(s) = \frac{A}{s-\lambda} + \frac{B}{(s-\lambda)^2}$$

no, but the solution is: $\mathcal{L}\left\{te^{at}\right\} = \dfrac{1}{(s-a)^2}$

Discussion: did we learn until now in this course how to inverse-Laplace these things?

$$X(s) = \frac{A}{s-\lambda} + \frac{B}{(s-\lambda)^2}$$

no, but the solution is: $\mathcal{L}\left\{te^{at}\right\} = \dfrac{1}{(s-a)^2}$

Very interesting behavior:

$$x(t) = Ae^{\lambda t} + Bte^{\lambda t}$$

Discussion: did we learn until now in this course how to inverse-Laplace these things?

$$X(s) = \frac{A}{s-\lambda} + \frac{B}{(s-\lambda)^2}$$

no, but the solution is: $\mathcal{L}\left\{te^{at}\right\} = \dfrac{1}{(s-a)^2}$

Very interesting behavior:

$$x(t) = Ae^{\lambda t} + Bte^{\lambda t}$$

Discussion: what contributes to determine A and B?

Discussion: recall the previous "overdamping case", i.e.,

$$\left\{\begin{array}{lll} X(s) & = \dfrac{x_0\left(\frac{f}{m}+s\right)+\dot{x}_0}{s^2+\frac{f}{m}s+\frac{k}{m}} & \\ m & = 10 & \mathrm{kg} \\ f & = 100 & \mathrm{kg/s} \\ k & = 90 & \mathrm{N/s^2} \\ x(0) & = 0.16\mathrm{m} & \\ \dot{x}(0) & = 0\mathrm{m/s} & \end{array}\right. \Longrightarrow \quad X(s) = \frac{0.16\,(s+10)}{s^2+10s+9}\,.$$

How may we change the parameters and the initial condition so to have a "critical damping" case, i.e.,

$$X(s) = \frac{A}{s-\lambda} + \frac{B}{(s-\lambda)^2}\ ?$$

Discussion: recall the previous "overdamping case", i.e.,

$$\begin{cases} X(s) & = \dfrac{x_0\left(\frac{f}{m}+s\right)+\dot{x}_0}{s^2+\frac{f}{m}s+\frac{k}{m}} \\ m & = 10 & \mathrm{kg} \\ f & = 100 & \mathrm{kg/s} \\ k & = 90 & \mathrm{N/s^2} \\ x(0) & = 0.16\mathrm{m} \\ \dot{x}(0) & = 0\mathrm{m/s} \end{cases} \implies X(s) = \frac{0.16\,(s+10)}{s^2+10s+9}.$$

How may we change the parameters and the initial condition so to have a "critical damping" case, i.e.,

$$X(s) = \frac{A}{s-\lambda} + \frac{B}{(s-\lambda)^2} \; ?$$

`ARMA-continuous-time-example.ipynb`

?

$f^2 - 4mk < 0$, i.e., underdamping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

$f^2 - 4mk < 0$, i.e., underdamping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

$$f^2 < 4mk \quad \Longrightarrow \quad \begin{cases} \lambda_1 = & -\dfrac{f}{2m} - j\dfrac{\sqrt{-f^2 + 4mk}}{2m} \in \mathbb{C} \\ \lambda_2 = & -\dfrac{f}{2m} + j\dfrac{\sqrt{-f^2 + 4mk}}{2m} \in \mathbb{C} \end{cases}$$

$f^2 - 4mk < 0$, i.e., underdamping in free evolution

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{(s - \lambda_1)(s - \lambda_2)}$$

$$f^2 < 4mk \quad \Longrightarrow \quad \begin{cases} \lambda_1 = & -\frac{f}{2m} - j\frac{\sqrt{-f^2 + 4mk}}{2m} \in \mathbb{C} \\ \lambda_2 = & -\frac{f}{2m} + j\frac{\sqrt{-f^2 + 4mk}}{2m} \in \mathbb{C} \end{cases}$$

$$\Longrightarrow \qquad X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2}$$

with λ_1 and λ_2 a complex-conjugate pair

Discussion: did we learn until now in this course how to inverse-Laplace these things?

$$X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2} \text{ with } \lambda_1 \text{ and } \lambda_2 \text{ a complex-conjugate pair}$$

Discussion: did we learn until now in this course how to inverse-Laplace these things?

$$X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2} \text{ with } \lambda_1 \text{ and } \lambda_2 \text{ a complex-conjugate pair}$$

"yes, but the corresponding exponentials come out to be complex numbers!
How is it possible that a real system produces complex results?"

Discussion: did we learn until now in this course how to inverse-Laplace these things?

$$X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2} \text{ with } \lambda_1 \text{ and } \lambda_2 \text{ a complex-conjugate pair}$$

"yes, but the corresponding exponentials come out to be complex numbers! How is it possible that a real system produces complex results?"

Solution:

$$\mathcal{L}\left\{e^{at}\sin\omega t\right\} = \frac{\omega}{(s-a)^2 + \omega^2} \qquad \mathcal{L}\left\{e^{at}\cos\omega t\right\} = \frac{s-a}{(s-a)^2 + \omega^2}$$

Discussion: recall the previous "overdamping case", i.e.,

$$\begin{cases} X(s) & = \dfrac{x_0\left(\frac{f}{m}+s\right)+\dot{x}_0}{s^2+\frac{f}{m}s+\frac{k}{m}} \\ m & = 10 & \text{kg} \\ f & = 100 & \text{kg/s} \\ k & = 90 & \text{N/s}^2 \\ x(0) & = 0.16\text{m} \\ \dot{x}(0) & = 0\text{m/s} \end{cases} \implies X(s) = \frac{0.16\,(s+10)}{s^2+10s+9}.$$

How may we change the parameters and the initial condition so to have a "underdamping" case, i.e.,

$$X(s) = \frac{A\omega}{(s-a)^2+\omega^2} + \frac{B\,(s-a)}{(s-a)^2+\omega^2}\ ?$$

Discussion: recall the previous "overdamping case", i.e.,

$$\begin{cases} X(s) &= \dfrac{x_0\left(\frac{f}{m}+s\right)+\dot{x}_0}{s^2+\frac{f}{m}s+\frac{k}{m}} \\ m &= 10 & \mathrm{kg} \\ f &= 100 & \mathrm{kg/s} \\ k &= 90 & \mathrm{N/s^2} \\ x(0) &= 0.16\mathrm{m} \\ \dot{x}(0) &= 0\mathrm{m/s} \end{cases} \implies X(s) = \frac{0.16\,(s+10)}{s^2+10s+9}.$$

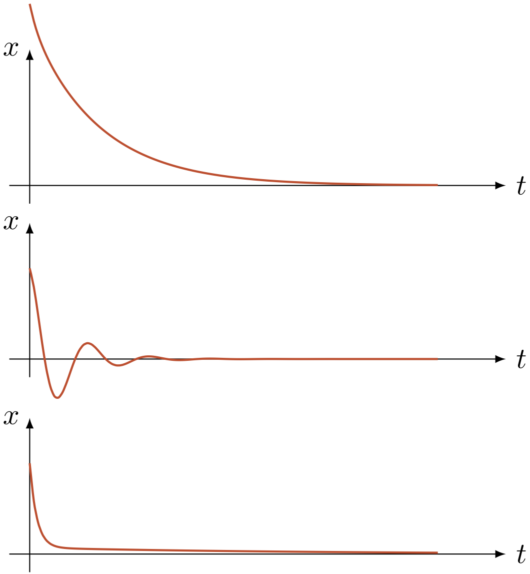
How may we change the parameters and the initial condition so to have a "underdamping" case, i.e.,

$$X(s) = \frac{A\omega}{(s-a)^2+\omega^2} + \frac{B\,(s-a)}{(s-a)^2+\omega^2}\ ?$$

`ARMA-continuous-time-example.ipynb`

?

Recap of the possible damping modalities



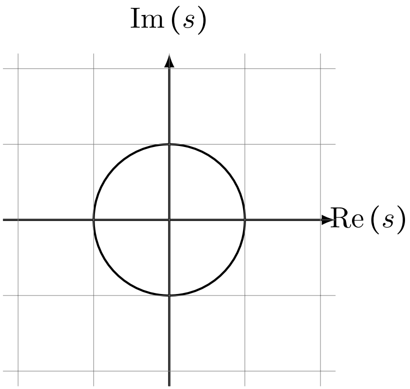
Recap of the corresponding time-frequency pairs

overdamped: $\mathcal{L}\left\{Ae^{\lambda_1 t} + Be^{\lambda_2 t}\right\} = \dfrac{A}{s-\lambda_1} + \dfrac{B}{s-\lambda_2}, \quad \lambda_1 \neq \lambda_2$

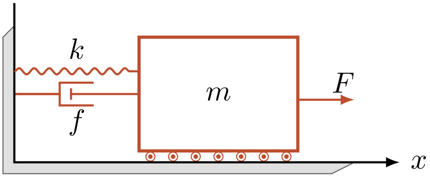
critically damped: $\mathcal{L}\left\{Ae^{\lambda t} + Bte^{\lambda t}\right\} = \dfrac{A}{s-\lambda} + \dfrac{B}{(s-\lambda)^2}$

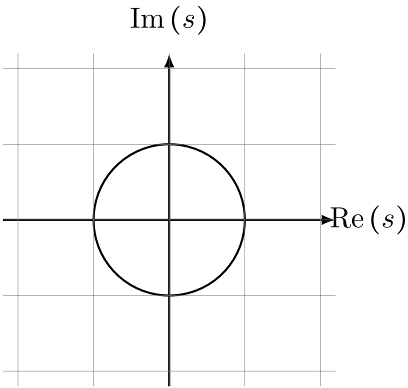
underdamped: $\mathcal{L}\left\{e^{at}\left(A\sin\omega t + B\cos\omega t\right)\right\} = \dfrac{A\omega}{(s-a)^2+\omega^2} + \dfrac{B(s-a)}{(s-a)^2+\omega^2}$

Recap using the “poles-zeros plot”



Do you see it?





?

The most important Laplace transforms properties for control people

Roadmap

- make Laplace-transform more matematically ground
- derive some few important examples and properties

Laplace-properties we saw up to now:

(4.2.1 in Balchen's book)

Laplace-properties we saw up to now:

(4.2.1 in Balchen's book)

Linearity

$$\mathcal{L}\left\{af(t) + bg(t)\right\} = a\mathcal{L}\left\{f(t)\right\} + b\mathcal{L}\left\{g(t)\right\} \tag{18}$$

Laplace-properties we saw up to now:

(4.2.1 in Balchen's book)

Derivatives

$$\mathcal{L}\left\{\dot{x}\right\} = sX(s) - x(0) \tag{19}$$

$$\mathcal{L}\left\{\ddot{x}\right\} = s^2X(s) - sx(0) - \dot{x}(0) \tag{20}$$

$$\vdots \tag{21}$$

Laplace-properties we saw up to now:

(4.2.1 in Balchen's book)

Integration

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s) \tag{22}$$

Laplace-transforms we needed up to now:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
e^{at}	$\dfrac{1}{s-a}$
te^{at}	$\dfrac{1}{(s-a)^2}$
$e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos\omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$

Laplace transforms, take 2

$$F(s) = \mathcal{L}\left\{f(t)\right\} := \int_0^{+\infty} e^{-st} f(t) dt \tag{23}$$

Laplace transforms, take 2

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{+\infty} e^{-st} f(t) dt \tag{23}$$

Discussion: does this integral converge always?

Laplace transforms, take 2

$$F(s) = \mathcal{L}\left\{f(t)\right\} := \int_0^{+\infty} e^{-st} f(t) dt \tag{23}$$

Discussion: does this integral converge always? For which $f(t)$ will we have convergence problems?

Laplace transforms, take 2

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{+\infty} e^{-st} f(t) dt \tag{23}$$

Discussion: does this integral converge always? For which $f(t)$ will we have convergence problems?

> if $|f(t)|$ grows faster than e^{st} then we have a problem

Laplace transforms, take 2

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{+\infty} e^{-st} f(t) dt \tag{23}$$

Discussion: does this integral converge always? For which $f(t)$ will we have convergence problems?

> if $|f(t)|$ grows faster than e^{st} then we have a problem

Discussion: assume $f(t)$ to be limited (e.g., $f(t) = 1$). Will $\int_0^{+\infty} e^{-st} f(t) dt$ converge for every s?

Laplace transforms, take 2

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{+\infty} e^{-st} f(t) dt \tag{23}$$

Discussion: does this integral converge always? For which $f(t)$ will we have convergence problems?

if $|f(t)|$ grows faster than e^{st} then we have a problem

Discussion: assume $f(t)$ to be limited (e.g., $f(t) = 1$). Will $\int_0^{+\infty} e^{-st} f(t) dt$ converge for every s? *no – only for* $s > 0$*!*

Important (but intuitive) teorem

If $f(t)$ is defined and piecewise continuous for $t > 0$,
and there exist two constants M and k so that $|f(t)| \leq Me^{kt}$ for every $t > 0$,
then $\mathcal{L}\{f(t)\}$ is well defined for every $s > k$

Important (but intuitive) property: linearity

Assuming that $f(t)$ and $g(t)$ are so that $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exists, then $\mathcal{L}\{af(t) + bg(t)\}$ exists and is s.t.

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Important (but intuitive) property: linearity

Assuming that $f(t)$ and $g(t)$ are so that $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exists, then $\mathcal{L}\{af(t)+bg(t)\}$ exists and is s.t.

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Proof

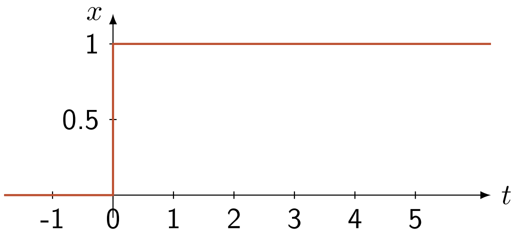
Integrals are linear, thus

$$\begin{aligned}
\mathcal{L}\{af(t)+bg(t)\} &= \int_0^{+\infty} e^{-st}\left[af(t)+bg(t)\right]dt \\
&= a\int_0^{+\infty} e^{-st} f(t)dt + b\int_0^{+\infty} e^{-st} g(t)dt \\
&= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}
\end{aligned}$$

?

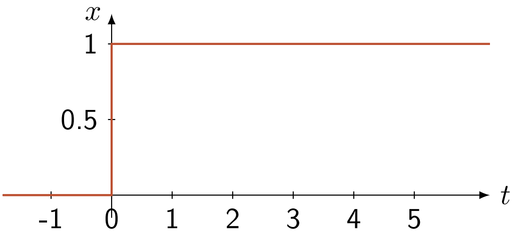
Important example: Laplace-transforming the Heaviside-function

$$H(t) := \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$



Important example: Laplace-transforming the Heaviside-function

$$H(t) := \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$



Discussion: is $H(0)$ defined?

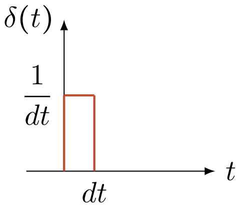
Deriving the Heaviside function from the Dirac's impulse

$$H(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

with, *very loosely,*

$$\delta(t) := \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{otherwise,} \end{cases}$$

or, better, as the limit of concentrating an unitary point mass at 0:



Back to the important example: Laplace-transforming the Heaviside-function

$$\mathcal{L}\left\{H\left(t\right)\right\} = \int_0^{+\infty} e^{-st}dt$$

Back to the important example: Laplace-transforming the Heaviside-function

$$\begin{aligned}\mathcal{L}\left\{H\left(t\right)\right\} &= \int_0^{+\infty} e^{-st} dt \\ &= \left[-\frac{1}{s} e^{-st}\right]_0^{+\infty}\end{aligned}$$

Back to the important example: Laplace-transforming the Heaviside-function

$$
\begin{aligned}
\mathcal{L}\left\{H\left(t\right)\right\} &= \int_0^{+\infty} e^{-st}dt \\
&= \left[-\frac{1}{s}e^{-st}\right]_0^{+\infty} \\
&= -\frac{1}{s}e^{-s\cdot\infty} - \left(-\frac{1}{s}e^{-s\cdot 0}\right)
\end{aligned}
$$

Back to the important example: Laplace-transforming the Heaviside-function

$$
\begin{aligned}
\mathcal{L}\left\{H\left(t\right)\right\} &= \int_0^{+\infty} e^{-st}dt \\
&= \left[-\frac{1}{s}e^{-st}\right]_0^{+\infty} \\
&= -\frac{1}{s}e^{-s\cdot\infty} - \left(-\frac{1}{s}e^{-s\cdot 0}\right) \\
&= 0 - \left(-\frac{1}{s}\right)
\end{aligned}
$$

Back to the important example: Laplace-transforming the Heaviside-function

$$
\begin{aligned}
\mathcal{L}\left\{H(t)\right\} &= \int_0^{+\infty} e^{-st} dt \\
&= \left[-\frac{1}{s} e^{-st}\right]_0^{+\infty} \\
&= -\frac{1}{s} e^{-s\cdot\infty} - \left(-\frac{1}{s} e^{-s\cdot 0}\right) \\
&= 0 - \left(-\frac{1}{s}\right) \\
&= \frac{1}{s}
\end{aligned}
$$

Back to the important example: Laplace-transforming the Heaviside-function

$$
\begin{aligned}
\mathcal{L}\left\{H\left(t\right)\right\} &= \int_0^{+\infty} e^{-st} dt \\
&= \left[-\frac{1}{s}e^{-st}\right]_0^{+\infty} \\
&= -\frac{1}{s}e^{-s\cdot\infty} - \left(-\frac{1}{s}e^{-s\cdot 0}\right) \\
&= 0 - \left(-\frac{1}{s}\right) \\
&= \frac{1}{s}
\end{aligned}
$$

Discussion: what is the ROC?

Back to the important example: Laplace-transforming the Heaviside-function

$$
\begin{aligned}
\mathcal{L}\left\{H\left(t\right)\right\} &= \int_0^{+\infty} e^{-st} dt \\
&= \left[-\frac{1}{s}e^{-st}\right]_0^{+\infty} \\
&= -\frac{1}{s}e^{-s\cdot\infty} - \left(-\frac{1}{s}e^{-s\cdot 0}\right) \\
&= 0 - \left(-\frac{1}{s}\right) \\
&= \frac{1}{s}
\end{aligned}
$$

Discussion: what is the ROC? $\mathrm{Re}\left(s\right) > 0$

Another important example: Laplace-transforming the Dirac's delta

$$\mathcal{L}\left\{\delta(t)\right\} = \int_0^{+\infty} e^{-st}\delta(t)dt$$

Another important example: Laplace-transforming the Dirac's delta

$$
\begin{aligned}
\mathcal{L}\left\{\delta(t)\right\} &= \int_0^{+\infty} e^{-st}\delta(t)dt \\
&= e^{-s0}
\end{aligned}
$$

Another important example: Laplace-transforming the Dirac's delta

$$\begin{aligned}
\mathcal{L}\left\{\delta(t)\right\} &= \int_0^{+\infty} e^{-st}\delta(t)dt \\
&= e^{-s0} \\
&= 1
\end{aligned}$$

Another important example: Laplace-transforming the Dirac's delta

$$
\begin{aligned}
\mathcal{L}\left\{\delta(t)\right\} &= \int_0^{+\infty} e^{-st}\delta(t)dt \\
&= e^{-s0} \\
&= 1
\end{aligned}
$$

Discussion: what is the ROC? $\forall s$

Likely the most important example: Laplace-transforming an exponential

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^{+\infty} e^{at} e^{-st} dt \tag{24}$$

Likely the most important example: Laplace-transforming an exponential

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^{+\infty} e^{at} e^{-st} dt \tag{24}$$

$$= \int_0^{+\infty} e^{(a-s)t} dt \tag{25}$$

Likely the most important example: Laplace-transforming an exponential

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^{+\infty} e^{at}e^{-st}dt \tag{24}$$

$$= \int_0^{+\infty} e^{(a-s)t}dt \tag{25}$$

$$= \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^{+\infty} \tag{26}$$

Likely the most important example: Laplace-transforming an exponential

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^{+\infty} e^{at} e^{-st} dt \tag{24}$$

$$= \int_0^{+\infty} e^{(a-s)t} dt \tag{25}$$

$$= \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^{+\infty} \tag{26}$$

$$= \frac{1}{a-s}\left(e^{(a-s)\infty} - e^{(a-s)0}\right) \tag{27}$$

Likely the most important example: Laplace-transforming an exponential

$$
\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \int_0^{+\infty} e^{at} e^{-st} dt && (24)\\
&= \int_0^{+\infty} e^{(a-s)t} dt && (25)\\
&= \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^{+\infty} && (26)\\
&= \frac{1}{a-s}\left(e^{(a-s)\infty} - e^{(a-s)0}\right) && (27)\\
&= \frac{1}{s-a} && (28)
\end{aligned}
$$

Likely the most important example: Laplace-transforming an exponential

$$
\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \int_0^{+\infty} e^{at} e^{-st} dt && (24)\\
&= \int_0^{+\infty} e^{(a-s)t} dt && (25)\\
&= \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^{+\infty} && (26)\\
&= \frac{1}{a-s}\left(e^{(a-s)\infty} - e^{(a-s)0}\right) && (27)\\
&= \frac{1}{s-a} && (28)
\end{aligned}
$$

Discussion: what is the ROC?

Likely the most important example: Laplace-transforming an exponential

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^{+\infty} e^{at} e^{-st} dt \tag{24}$$

$$= \int_0^{+\infty} e^{(a-s)t} dt \tag{25}$$

$$= \frac{1}{a-s} \left[e^{(a-s)t}\right]_0^{+\infty} \tag{26}$$

$$= \frac{1}{a-s} \left(e^{(a-s)\infty} - e^{(a-s)0}\right) \tag{27}$$

$$= \frac{1}{s-a} \tag{28}$$

Discussion: what is the ROC? $\mathrm{Re}\,(s) > a$

On with the important examples: Laplace-transforming polynomials

On with the important examples: Laplace-transforming polynomials

Remember: integration by parts

$$\int_0^{+\infty} u\dot{v}dt = [uv]_0^{+\infty} - \int_0^{+\infty} \dot{u}vdt \tag{29}$$

On with the important examples: Laplace-transforming polynomials

Remember: integration by parts

$$\int_0^{+\infty} u\dot{v}dt = [uv]_0^{+\infty} - \int_0^{+\infty} \dot{u}vdt \tag{29}$$

$$\mathcal{L}\{t\} = \int_0^{+\infty} te^{-st}dt = \star$$

On with the important examples: Laplace-transforming polynomials

Remember: integration by parts

$$\int_0^{+\infty} u\dot{v}dt = [uv]_0^{+\infty} - \int_0^{+\infty} \dot{u}vdt \tag{29}$$

$$\mathcal{L}\{t\} = \int_0^{+\infty} te^{-st}dt = \star \qquad \begin{matrix} u = t & v = e^{-st} \\ \dot{u} = 1 & \dot{v} = -se^{-st} \end{matrix} \tag{30}$$

On with the important examples: Laplace-transforming polynomials

Remember: integration by parts

$$\int_0^{+\infty} u\dot{v}dt = [uv]_0^{+\infty} - \int_0^{+\infty} \dot{u}vdt \tag{29}$$

$$\mathcal{L}\{t\} = \int_0^{+\infty} te^{-st}dt = \star \qquad \begin{matrix} u = t & v = e^{-st} \\ \dot{u} = 1 & \dot{v} = -se^{-st} \end{matrix} \tag{30}$$

$$\star = \frac{1}{-s}\int_0^{+\infty} t\left(-se^{-st}\right)dt \tag{31}$$

On with the important examples: Laplace-transforming polynomials

Remember: integration by parts

$$\int_0^{+\infty} u\dot{v}dt = [uv]_0^{+\infty} - \int_0^{+\infty} \dot{u}vdt \tag{29}$$

$$\mathcal{L}\left\{t\right\} = \int_0^{+\infty} te^{-st}dt = \star \qquad \begin{array}{ll} u = t & v = e^{-st} \\ \dot{u} = 1 & \dot{v} = -se^{-st} \end{array} \tag{30}$$

$$\star = \frac{1}{-s}\int_0^{+\infty} t\left(-se^{-st}\right)dt \tag{31}$$

$$= -\frac{1}{s}\left(\left[te^{-st}\right]_0^{+\infty} - \int_0^{+\infty} e^{-st}dt\right) \tag{32}$$

On with the important examples: Laplace-transforming polynomials

Remember: integration by parts

$$\int_0^{+\infty} u\dot{v}dt = [uv]_0^{+\infty} - \int_0^{+\infty} \dot{u}v dt \tag{29}$$

$$\mathcal{L}\{t\} = \int_0^{+\infty} te^{-st}dt = \star \qquad \begin{matrix} u = t & v = e^{-st} \\ \dot{u} = 1 & \dot{v} = -se^{-st} \end{matrix} \tag{30}$$

$$\star = \frac{1}{-s}\int_0^{+\infty} t\left(-se^{-st}\right)dt \tag{31}$$

$$= -\frac{1}{s}\left(\left[te^{-st}\right]_0^{+\infty} - \int_0^{+\infty} e^{-st}dt\right) \tag{32}$$

$$= \frac{1}{s^2} \tag{33}$$

On with the important examples: Laplace-transforming polynomials

$$\mathcal{L}\left\{t^2\right\} = \int_0^{+\infty} t^2 e^{-st} dt$$

On with the important examples: Laplace-transforming polynomials

$$\mathcal{L}\left\{t^2\right\} = \int_0^{+\infty} t^2 e^{-st} dt \qquad \begin{aligned} u &= t^2 & v &= e^{-st} \\ \dot{u} &= t & \dot{v} &= -se^{-st} \end{aligned} \tag{34}$$

On with the important examples: Laplace-transforming polynomials

$$\mathcal{L}\left\{t^2\right\} = \int_0^{+\infty} t^2 e^{-st} dt \qquad \begin{matrix} u = t^2 & v = e^{-st} \\ \dot{u} = t & \dot{v} = -se^{-st} \end{matrix} \tag{34}$$

$$= \frac{1}{-s} \int_0^{+\infty} t^2 (-se^{-st}) dt \tag{35}$$

On with the important examples: Laplace-transforming polynomials

$$\mathcal{L}\left\{t^2\right\} = \int_0^{+\infty} t^2 e^{-st} dt \qquad \begin{matrix} u = t^2 & v = e^{-st} \\ \dot{u} = t & \dot{v} = -se^{-st} \end{matrix} \tag{34}$$

$$= \frac{1}{-s} \int_0^{+\infty} t^2 (-se^{-st}) dt \tag{35}$$

$$= -\frac{1}{s} \left(\left[t^2 e^{-st} \right]_0^{+\infty} - \int_0^{+\infty} 2te^{-st} dt \right) \tag{36}$$

On with the important examples: Laplace-transforming polynomials

$$
\begin{aligned}
\mathcal{L}\left\{t^2\right\} &= \int_0^{+\infty} t^2 e^{-st} dt & \begin{matrix} u = t^2 & v = e^{-st} \\ \dot{u} = t & \dot{v} = -se^{-st} \end{matrix} \qquad (34) \\
&= \frac{1}{-s} \int_0^{+\infty} t^2 (-se^{-st}) dt & (35) \\
&= -\frac{1}{s} \left(\left[t^2 e^{-st} \right]_0^{+\infty} - \int_0^{+\infty} 2te^{-st} dt \right) & (36) \\
&= \frac{2}{s^3} & (37) \\
& & (38)
\end{aligned}
$$

On with the important examples: Laplace-transforming polynomials

$$\mathcal{L}\left\{t^n\right\} = \frac{(n-1)!}{s^n} \tag{39}$$

On with the important examples: Laplace-transforming polynomials

$$\mathcal{L}\left\{t^n\right\} = \frac{(n-1)!}{s^n} \tag{39}$$

Discussion: what is the ROC?

On with the important examples: Laplace-transforming polynomials

$$\mathcal{L}\left\{t^n\right\} = \frac{(n-1)!}{s^n} \tag{39}$$

Discussion: what is the ROC? $\mathrm{Re}\left(s\right) > 0$

An important digression:

- Dirac's delta $\implies$ impulse response; $\delta(t) \mapsto 1$

An important digression:

- Dirac's delta $\implies$ impulse response; $\delta(t) \mapsto 1$
- Heaviside $=$ integration of Dirac's delta $\implies$ step response; $H\left(t\right) \mapsto \dfrac{1}{s}$

An important digression:

- Dirac's delta $\implies$ impulse response; $\delta(t) \mapsto 1$
- Heaviside $=$ integration of Dirac's delta $\implies$ step response; $H\left(t\right) \mapsto \dfrac{1}{s}$
- (linear) ramp $=$ integration of Heaviside $\implies$ (linear) ramp response; $tH\left(t\right) \mapsto \dfrac{1}{s^2}$

An important digression:

- Dirac's delta $\implies$ impulse response; $\delta(t) \mapsto 1$
- Heaviside = integration of Dirac's delta $\implies$ step response; $H(t) \mapsto \frac{1}{s}$
- (linear) ramp = integration of Heaviside $\implies$ (linear) ramp response; $tH(t) \mapsto \frac{1}{s^2}$
- quadratic ramp = integration of linear ramp $\implies$ quadratic ramp response; $t^2 H(t) \mapsto \frac{1}{s^3}$

An important digression:

- Dirac's delta $\implies$ impulse response; $\delta(t) \mapsto 1$
- Heaviside = integration of Dirac's delta $\implies$ step response; $H(t) \mapsto \dfrac{1}{s}$
- (linear) ramp = integration of Heaviside $\implies$ (linear) ramp response; $tH(t) \mapsto \dfrac{1}{s^2}$
- quadratic ramp = integration of linear ramp $\implies$ quadratic ramp response; $t^2 H(t) \mapsto \dfrac{1}{s^3}$ ...

ramp responses are often used to do system identification;
it is easier to implement quadratic ramps than lower order ones

ramp responses are often used to do system identification;
it is easier to implement quadratic ramps than lower order ones

(this means that often one gets a ramp response and then one has to reconstruct back the impulse response)

On with the important examples: Laplace-transforms of derivatives

$$\mathcal{L}\left\{\dot{x}\right\} = \int_0^{+\infty} e^{-st}\dot{x}dt \tag{40}$$

On with the important examples: Laplace-transforms of derivatives

$$\mathcal{L}\left\{\dot{x}\right\} = \int_0^{+\infty} e^{-st}\dot{x}dt \tag{40}$$

$$= \left[e^{-st}x(t)\right]_0^{+\infty} - \int_0^{+\infty} \left(-se^{-st}\right)x(t)dt \tag{41}$$

On with the important examples: Laplace-transforms of derivatives

$$\mathcal{L}\left\{\dot{x}\right\} = \int_0^{+\infty} e^{-st}\dot{x}dt \tag{40}$$

$$= \left[e^{-st}x(t)\right]_0^{+\infty} - \int_0^{+\infty} \left(-se^{-st}\right) x(t)dt \tag{41}$$

$$= 0 - x(0) + s\int_0^{+\infty} e^{-st}x(t)dt \tag{42}$$

On with the important examples: Laplace-transforms of derivatives

$$\begin{aligned}
\mathcal{L}\left\{\dot{x}\right\} &= \int_0^{+\infty} e^{-st}\dot{x}dt && (40)\\
&= \left[e^{-st}x(t)\right]_0^{+\infty} - \int_0^{+\infty} \left(-se^{-st}\right)x(t)dt && (41)\\
&= 0 - x(0) + s\int_0^{+\infty} e^{-st}x(t)dt && (42)\\
&= sX(s) - x(0) && (43)
\end{aligned}$$

On with the important examples: Laplace-transforms of derivatives

$$\mathcal{L}\left\{\ddot{x}\right\} = \int_0^{+\infty} e^{-st}\ddot{x}dt \tag{44}$$

On with the important examples: Laplace-transforms of derivatives

$$\mathcal{L}\left\{\ddot{x}\right\} = \int_0^{+\infty} e^{-st}\ddot{x}dt \quad (44)$$

$$= \left[e^{-st}\dot{x}(t)\right]_0^{+\infty} - \int_0^{+\infty} -se^{-st}\dot{x}(t)dt \quad (45)$$

On with the important examples: Laplace-transforms of derivatives

$$\mathcal{L}\left\{\ddot{x}\right\} = \int_0^{+\infty} e^{-st}\ddot{x}dt \tag{44}$$

$$= \left[e^{-st}\dot{x}(t)\right]_0^{+\infty} - \int_0^{+\infty} -se^{-st}\dot{x}(t)dt \tag{45}$$

$$= 0 - \dot{x}(0) + s\int_0^{+\infty} e^{-st}\dot{x}(t)dt \tag{46}$$

On with the important examples: Laplace-transforms of derivatives

$$\begin{aligned}
\mathcal{L}\{\ddot{x}\} &= \int_0^{+\infty} e^{-st}\ddot{x}dt && (44)\\
&= \left[e^{-st}\dot{x}(t)\right]_0^{+\infty} - \int_0^{+\infty} -se^{-st}\dot{x}(t)dt && (45)\\
&= 0 - \dot{x}(0) + s\int_0^{+\infty} e^{-st}\dot{x}(t)dt && (46)\\
&= \quad - \dot{x}(0) + s\Big(sX(s) - x(0)\Big) && (47)
\end{aligned}$$

On with the important examples: Laplace-transforms of derivatives

$$
\begin{aligned}
\mathcal{L}\{\ddot{x}\} &= \int_0^{+\infty} e^{-st}\ddot{x}dt && (44)\\
&= \left[e^{-st}\dot{x}(t)\right]_0^{+\infty} - \int_0^{+\infty} -se^{-st}\dot{x}(t)dt && (45)\\
&= 0 - \dot{x}(0) + s\int_0^{+\infty} e^{-st}\dot{x}(t)dt && (46)\\
&= \quad - \dot{x}(0) + s\big(sX(s) - x(0)\big) && (47)\\
&= s^2X(s) - sx(0) - \dot{x}(0) && (48)\\
& && (49)
\end{aligned}
$$

On with the important examples: Laplace-transforms of derivatives

$$\mathcal{L}\left\{x^{(n)}(t)\right\} = s^n\mathcal{L}\left\{x(t)\right\} - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - \ldots - x^{(n-1)}(0) \tag{50}$$

Translation theorems

Translation theorems

Translation in the frequency domain

If $f(t)$ admits $F(s)$, then $e^{at}f(t)$ transforms into $F(s-a)$

Translation theorems

Translation in the frequency domain

If $f(t)$ admits $F(s)$, then $e^{at}f(t)$ transforms into $F(s-a)$

Translation in the time domain

If $f(t)$ admits $F(s)$, then $f(t-a)\mu(t-a)$ transforms into $e^{-as}F(s)$

Discussion: for which s do these theorems hold?

Answer: only in the intersection of the ROCs of both "original" and "translated" transforms!

Translation theorems + Laplace-transforming polynomials = **the most useful formula that you may remember**

$$\begin{cases} \mathcal{L}\left\{t^n\right\} & = \dfrac{n!}{s^{n+1}} \\ \mathcal{L}\left\{e^{at} f(t)\right\} & = F(s-a) \end{cases} \implies \mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \tag{51}$$

Translation theorems + Laplace-transforming polynomials = **the most useful formula that you may remember**

$$\begin{cases} \mathcal{L}\left\{t^n\right\} & = \dfrac{n!}{s^{n+1}} \\ \mathcal{L}\left\{e^{at} f(t)\right\} & = F(s-a) \end{cases} \implies \mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \tag{51}$$

Discussion: what is the ROC?

Translation theorems + Laplace-transforming polynomials = **the most useful formula that you may remember**

$$\begin{cases} \mathcal{L}\left\{t^n\right\} & = \dfrac{n!}{s^{n+1}} \\ \mathcal{L}\left\{e^{at} f(t)\right\} & = F(s-a) \end{cases} \implies \mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \tag{51}$$

Discussion: what is the ROC? $\mathrm{Re}\,(s) > a$

Laplace-transforms of oscillatory behaviors = **the second most useful formulas that you may remember**

$$\mathcal{L}\left\{e^{at}\sin\omega t\right\} = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\left\{e^{at}\cos\omega t\right\} = \frac{s-a}{(s-a)^2+\omega^2}$$

Laplace-transforms of oscillatory behaviors = **the second most useful formulas that you may remember**

$$\mathcal{L}\left\{e^{at}\sin\omega t\right\} = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\left\{e^{at}\cos\omega t\right\} = \frac{s-a}{(s-a)^2+\omega^2}$$

Limit cases: $a = 0$, so that

$$\mathcal{L}\left\{\sin\omega t\right\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\left\{\cos\omega t\right\} = \frac{s}{s^2+\omega^2}$$

But what is s, eventually?

But what is s, eventually?

The potential points of view of this course:

simply a complex number, i.e., $s = \sigma + j\omega$, that gives rise to a *complex frequency* through the complex exponential $e^{-st} = e^{-\sigma t} e^{-j\omega t}$

But what is s, eventually?

The potential points of view of this course:

simply a complex number, i.e., $s = \sigma + j\omega$, that gives rise to a *complex frequency* through the complex exponential $e^{-st} = e^{-\sigma t} e^{-j\omega t}$

a "derivation" operator, since derivation in the time domain means multiplication times s in the frequency domain

But what is s, eventually?

The potential points of view of this course:

simply a complex number, i.e., $s = \sigma + j\omega$, that gives rise to a *complex frequency* through the complex exponential $e^{-st} = e^{-\sigma t} e^{-j\omega t}$

a "derivation" operator, since derivation in the time domain means multiplication times s in the frequency domain

Discussion: which "unit of measurement" shall s have?

But what is s, eventually?

The potential points of view of this course:

simply a complex number, i.e., $s = \sigma + j\omega$, that gives rise to a *complex frequency* through the complex exponential $e^{-st} = e^{-\sigma t} e^{-j\omega t}$

a "derivation" operator, since derivation in the time domain means multiplication times s in the frequency domain

Discussion: which "unit of measurement" shall s have? e^{-st} is adimensional, thus st is adimensional, thus s is "time^{-1}"

But what is s, eventually?

The potential points of view of this course:

simply a complex number, i.e., $s = \sigma + j\omega$, that gives rise to a *complex frequency* through the complex exponential $e^{-st} = e^{-\sigma t} e^{-j\omega t}$

a "derivation" operator, since derivation in the time domain means multiplication times s in the frequency domain

Discussion: which "unit of measurement" shall s have? e^{-st} is adimensional, thus st is adimensional, thus s is "time^{-1}"

Be careful on how to use s in this course!

Typically, in all the automatic control courses the interest is not too much on the value of $F(s)$ for different s, but rather on the fact that $F(s)$ enables performing algebraic operations to solve ODEs!

?

Using Laplace transforms with generic n order systems in free evolution

In the last episodes: the "Laplace way" to find that $\dot{y} = ay \mapsto y(t) = y_0 e^{at}$, and to do something similar for second order systems in free evolution

$$\dot{y} = ay \qquad \to \qquad sY - y_0 = aY \qquad \to \qquad Y = y_0 \frac{1}{s - a}$$

Roadmap

- the same thing, but for generic systems

A small recap

- first oder homogeneous systems:

$$\dot{x} - ax = 0 \quad \mapsto \quad x(t) = x_0 e^{at} \tag{52}$$

- first oder non-homogeneous systems:

$$\dot{x} - ax = bu \quad \mapsto \quad x(t) = x(0)e^{at} + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \tag{53}$$

A small recap

- first oder homogeneous systems:

$$\dot{x} - ax = 0 \quad \mapsto \quad x(t) = x_0 e^{at} \tag{52}$$

- first oder non-homogeneous systems:

$$\dot{x} - ax = bu \quad \mapsto \quad x(t) = x(0)e^{at} + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \tag{53}$$

The problem: the convolution integral gives us headaches, and we want to use Laplace to ease things up. To understand how, step-by-step path:

1. start from $\dot{x} - ax = 0$
2. move to $\ddot{x} - a_1\dot{x} - a_2 x = 0$
3. move to higher order
4. add the "u" part

A more detailed roadmap for this unit:

- generalize how to do partial expansions
- solve the problem: how to use Laplace transforms with generic n order systems in free evolution

?

Recalling some past results

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\Downarrow$$

$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} \Downarrow$$

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \text{ (and the other cases)}$$

Recalling some past results

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0$$
$$\Downarrow$$
$$X(s) = \frac{x_0\left(\frac{f}{m} + s\right) + \dot{x}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} \Downarrow$$
$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ (and the other cases)

Discussion: when is it that $\ddot{x} + \dfrac{f}{m}\dot{x} + \dfrac{k}{m}x = 0$ leads to $X(s) \neq 0$?

what about $\dddot{x} + \alpha_2 \ddot{x} + \alpha_1 \dot{x} + \alpha_0 x = 0$?

what about $\dddot{x} + \alpha_2 \ddot{x} + \alpha_1 \dot{x} + \alpha_0 x = 0$?

and what about higher orders?

Partial fraction decomposition

Partial fraction decomposition

case single poles: if $\dfrac{N(s)}{(s-a)(s-b)(s-c)\cdots}$ is s.t. $a \neq b \neq c \neq \cdots$ then there exist A, B, C, . . . s.t.

$$\frac{N(s)}{(s-a)(s-b)(s-c)\cdots} = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots \tag{54}$$

Partial fraction decomposition

case single poles: if $\dfrac{N(s)}{(s-a)(s-b)(s-c)\cdots}$ is s.t. $a \neq b \neq c \neq \cdots$ then there exist A, B, C, ... s.t.

$$\frac{N(s)}{(s-a)(s-b)(s-c)\cdots} = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots \tag{54}$$

case repeated poles: if some poles are repeated, then there exist A_1, ..., A_{na}, B_1, ..., B_{nb}, s.t.

$$\frac{N(s)}{(s-a)^{na}(s-b)^{nb}\ldots} = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots \tag{55}$$

Partial fraction decomposition

case single poles: if $\frac{N(s)}{(s-a)(s-b)(s-c)\cdots}$ is s.t. $a \neq b \neq c \neq \cdots$ then there exist A, B, C, ... s.t.

$$\frac{N(s)}{(s-a)(s-b)(s-c)\cdots} = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots \tag{54}$$

case repeated poles: if some poles are repeated, then there exist $A_1, \dots, A_{na}, B_1, \dots, B_{nb}$, s.t.

$$\frac{N(s)}{(s-a)^{na}(s-b)^{nb}\dots} = \frac{A_1}{s-a} + \dots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \dots + \frac{B_{nb}}{(s-b)^{nb}} + \dots \tag{55}$$

"But how do I compute A, B, etc.?" $\mapsto$

`en.wikipedia.org/wiki/Partial_fraction_decomposition`

(tip: start from `en.wikipedia.org/wiki/Heaviside_cover-up_method`)

?

Computing the free evolution for generic n order LTI systems

$$\text{start from } x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0 \tag{56}$$

Computing the free evolution for generic n order LTI systems

$$\text{start from } x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0 \tag{56}$$

$$\text{for each } m \text{ apply } \mathcal{L}\left\{x^{(m)}(t)\right\} = s^m \mathcal{L}\left\{x(t)\right\} - s^{m-1} x(0) - s^{m-2} \dot{x}(0) - \ldots - x^{(m-1)}(0) \tag{57}$$

Computing the free evolution for generic n order LTI systems

$$\text{start from } x^{(n)} + \alpha_{n-1}x^{(n-1)} + \ldots + \alpha_0 x = 0 \tag{56}$$

$$\text{for each } m \text{ apply } \mathcal{L}\left\{x^{(m)}(t)\right\} = s^m \mathcal{L}\left\{x(t)\right\} - s^{m-1}x(0) - s^{m-2}\dot{x}(0) - \ldots - x^{(m-1)}(0) \tag{57}$$

$$\text{obtain } X(s) = \frac{N(s)}{(s-a)(s-b)(s-c)\cdots} \tag{58}$$

Computing the free evolution for generic n order LTI systems

$$\text{start from } x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0 \tag{56}$$

$$\text{for each } m \text{ apply } \mathcal{L}\left\{x^{(m)}(t)\right\} = s^m \mathcal{L}\left\{x(t)\right\} - s^{m-1} x(0) - s^{m-2} \dot{x}(0) - \ldots - x^{(m-1)}(0) \tag{57}$$

$$\text{obtain } X(s) = \frac{N(s)}{(s-a)(s-b)(s-c)\cdots} \tag{58}$$

$$\text{do a partial fraction decomposition} \tag{59}$$

Computing the free evolution for generic n order LTI systems

Two main cases:

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$\text{either } X(s) = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots$$

$$\text{or } X(s) = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots$$

Computing the free evolution for generic n order LTI systems

Case single poles

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$X(s) = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots$$

Computing the free evolution for generic n order LTI systems

Case single poles

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$X(s) = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \cdots$$

Discussion: given $\mathcal{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = e^{\alpha t}$, what is $x(t)$?

Computing the free evolution for generic n order LTI systems

Case multiple poles

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$X(s) = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots$$

Computing the free evolution for generic n order LTI systems

Case multiple poles

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$X(s) = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots$$

Discussion: what was $\mathcal{L}^{-1}\left\{\frac{1}{(s-\alpha)^2}\right\}$?

Computing the free evolution for generic n order LTI systems

Case multiple poles

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$X(s) = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots$$

Discussion: what was $\mathcal{L}^{-1}\left\{\frac{1}{(s-\alpha)^2}\right\}$?

remember: $\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}$

Computing the free evolution for generic n order LTI systems

Case multiple poles

$$x^{(n)} + \alpha_{n-1} x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$X(s) = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots$$

Discussion: what was $\mathcal{L}^{-1}\left\{\frac{1}{(s-\alpha)^2}\right\}$?

remember: $\mathcal{L}\left\{t^n e^{at}\right\} = \dfrac{n!}{(s-a)^{n+1}}$

Discussion: given the previous transform, what is $x(t)$?

Computing the free evolution for generic n order LTI systems

Case multiple poles

$$x^{(n)} + \alpha_{n-1}x^{(n-1)} + \ldots + \alpha_0 x = 0$$

$$\downarrow$$

$$X(s) = \frac{A_1}{s-a} + \ldots + \frac{A_{na}}{(s-a)^{na}} + \frac{B_1}{s-b} + \ldots + \frac{B_{nb}}{(s-b)^{nb}} + \ldots$$

Discussion: what was $\mathcal{L}^{-1}\left\{\frac{1}{(s-\alpha)^2}\right\}$?

$$\text{remember: } \mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}$$

Discussion: given the previous transform, what is $x(t)$? Solution: a sum of terms of the type $t^n e^{at}$

Extremely important result

a LTI in free evolution behaves as a combination of terms $e^{\alpha t}$, $te^{\alpha t}$, $t^2e^{\alpha t}$, etc. for a set of different αs and powers of t, called the *modes* of the system

Extremely important result

> a LTI in free evolution behaves as a combination of terms $e^{\alpha t}$, $te^{\alpha t}$, $t^2 e^{\alpha t}$, etc. for a set of different αs and powers of t, called the *modes* of the system

Discussion: assuming that we have two modes, $e^{-0.3t}$ and $e^{-1.6t}$, so that

$$x(t) = \gamma_1 e^{-0.3t} + \gamma_2 e^{-1.6t}.$$

What determines γ_1 and γ_2?

`comparison-of-exponentials.ipynb`

?

Using Laplace transforms with second order systems in free evolution

In the last episodes: the "Laplace way" to find how LTI systems freely evolve

$$\dot{y} = ay \qquad \rightarrow \qquad sY - y_0 = aY \qquad \rightarrow \qquad Y = y_0 \frac{1}{s-a}$$

Roadmap

- the same thing, but adding the forced response term

A small recap

- first oder homgeneous systems:

$$\dot{x} - ax = 0,\ x(0) = x_0 \quad \mapsto \quad x(t) = x_0 e^{at} \tag{60}$$

- first oder non-homgeneous systems:

$$\dot{x} - ax = bu,\ x(0) = x_0 \quad \mapsto \quad x(t) = x(0)e^{at} + \int_0^t e^{a\tau} bu(t-\tau)d\tau \tag{61}$$

A small recap

- first oder homgeneous systems:

$$\dot{x} - ax = 0,\ x(0) = x_0 \quad \mapsto \quad x(t) = x_0 e^{at} \tag{60}$$

- first oder non-homgeneous systems:

$$\dot{x} - ax = bu,\ x(0) = x_0 \quad \mapsto \quad x(t) = x(0)e^{at} + \int_0^t e^{a\tau} bu(t-\tau)d\tau \tag{61}$$

The problem, once again: the convolution integral gives us headaches - we want to use Laplace to ease things up

Laplace-ing everything

$$\dot{x} - ax = bu, \; x(0) = x_0 \tag{62}$$

$$\downarrow \tag{63}$$

$$sX(s) - x(0) - aX(s) = bU(s) \tag{64}$$

$$\downarrow \tag{65}$$

$$X(s) = \frac{x(0)}{s-a} + \frac{bU(s)}{s-a} \tag{66}$$

$$\downarrow \tag{67}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x(0)}{s-a} + \frac{bU(s)}{s-a}\right\} \tag{68}$$

$$\downarrow \tag{69}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x(0)}{s-a}\right\} + \mathcal{L}^{-1}\left\{\frac{bU(s)}{s-a}\right\} \tag{70}$$

(71)

Laplace-ing everything

$$\dot{x} - ax = bu, \; x(0) = x_0 \tag{62}$$

$$\downarrow \tag{63}$$

$$sX(s) - x(0) - aX(s) = bU(s) \tag{64}$$

$$\downarrow \tag{65}$$

$$X(s) = \frac{x(0)}{s-a} + \frac{bU(s)}{s-a} \tag{66}$$

$$\downarrow \tag{67}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x(0)}{s-a} + \frac{bU(s)}{s-a}\right\} \tag{68}$$

$$\downarrow \tag{69}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x(0)}{s-a}\right\} + \mathcal{L}^{-1}\left\{\frac{bU(s)}{s-a}\right\} \tag{70}$$

(71)

Laplace-ing everything

$$\dot{x} - ax = bu, \ x(0) = x_0 \tag{62}$$

$$\downarrow \tag{63}$$

$$sX(s) - x(0) - aX(s) = bU(s) \tag{64}$$

$$\downarrow \tag{65}$$

$$X(s) = \frac{x(0)}{s-a} + \frac{bU(s)}{s-a} \tag{66}$$

$$\downarrow \tag{67}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x(0)}{s-a} + \frac{bU(s)}{s-a}\right\} \tag{68}$$

$$\downarrow \tag{69}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x(0)}{s-a}\right\} + \mathcal{L}^{-1}\left\{\frac{bU(s)}{s-a}\right\} \tag{70}$$

(71)

if $u(t)$ has a rational Laplace transform, then

$$x(t) = \sum \mathcal{L}^{-1}\left\{\frac{\alpha}{(s-a)^{\beta}}\right\}$$

Example 1

$$\dot{x} + x = 0, \quad x(0) = 2 \qquad \Longrightarrow \qquad x(t > 0) = ?$$

Example 1

$$\dot{x} + x = 0, \quad x(0) = 2 \qquad \Longrightarrow \qquad x(t > 0) = ?$$

$$sX(s) - x(0) + X(s) = 0$$

$$\downarrow$$

$$(s+1)X(s) = x(0)$$

$$\downarrow$$

$$X(s) = \frac{x(0)}{(s+1)} = \frac{2}{s - (-1)}$$

Example 1

$$\dot{x} + x = 0, \quad x(0) = 2 \qquad \Longrightarrow \qquad x(t > 0) = ?$$

$$sX(s) - x(0) + X(s) = 0$$

$$\downarrow$$

$$(s + 1)X(s) = x(0)$$

$$\downarrow$$

$$X(s) = \frac{x(0)}{(s + 1)} = \frac{2}{s - (-1)}$$

$$x(t) = 2e^{-t}$$

Example 1

$$\dot{x} + x = 0, \quad x(0) = 2 \qquad \Longrightarrow \qquad x(t > 0) = ?$$

$$sX(s) - x(0) + X(s) = 0$$

$$\downarrow$$

$$(s+1)X(s) = x(0)$$

$$\downarrow$$

$$X(s) = \frac{x(0)}{(s+1)} = \frac{2}{s-(-1)}$$

$$x(t) = 2e^{-t}$$

Discussion: what is the time constant in this case?

Example 2

Discussion: $\mathcal{L}\left\{H\left(t\right)\right\} = ?$

Example 2

Discussion: $\mathcal{L}\left\{H\left(t\right)\right\} =?$ Solution: $\dfrac{1}{s}$

Example 2

Discussion: $\mathcal{L}\left\{H\left(t\right)\right\} =?$ Solution: $\dfrac{1}{s}$

$$\dot{x} + x = H\left(t\right), \quad x(0) = 2 \qquad \Longrightarrow \qquad x(t > 0) =?$$

Example 2

Discussion: $\mathcal{L}\{H(t)\} = ?$ Solution: $\dfrac{1}{s}$

$$\dot{x} + x = H(t), \quad x(0) = 2 \qquad \Longrightarrow \qquad x(t > 0) = ?$$

$$sX(s) - x(0) + X(s) = \frac{1}{s}$$

$$\downarrow$$

$$(s+1)X(s) = \frac{1}{s} + 2$$

$$\downarrow$$

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

Example 2, continued

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

Example 2, continued

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1) + Bs}{s(s+1)} \tag{72}$$

Example 2, continued

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1)+Bs}{s(s+1)} \tag{72}$$

s: $As + Bs = 0 \implies A = -B$

1: $A = 1 \implies B = -1$

Example 2, continued

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1)+Bs}{s(s+1)} \tag{72}$$

s: $As + Bs = 0 \implies A = -B$

1: $A = 1 \implies B = -1$

$$X(s) = \frac{2}{s+1} + \frac{1}{s(s+1)} = \frac{2}{s+1} + \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s+1} + \frac{1}{s} \tag{73}$$

Example 2, continued

$$X(s) = \frac{2}{s+1} + \frac{1}{s(1+s)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1)+Bs}{s(s+1)} \tag{72}$$

s: $As + Bs = 0 \implies A = -B$

1: $A = 1 \implies B = -1$

$$X(s) = \frac{2}{s+1} + \frac{1}{s(s+1)} = \frac{2}{s+1} + \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s+1} + \frac{1}{s} \tag{73}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = e^{-t} + \mu(t) \tag{74}$$

Example 3

$$\dot{x} + 4x = 3e^{-4t}, \quad x(0) = 1 \qquad \Longrightarrow \qquad x(t > 0) = ?$$

Example 3

$$\dot{x} + 4x = 3e^{-4t}, \quad x(0) = 1 \qquad \Longrightarrow \qquad x(t > 0) = ?$$

$$sX(s) - 1 + 4X(s) = \frac{3}{s+4} \tag{75}$$

$$\downarrow \tag{76}$$

$$X(s) = \frac{1}{s+4} + \frac{3}{(s+4)^2} \tag{77}$$

$$\tag{78}$$

Example 3

$$\dot{x} + 4x = 3e^{-4t}, \quad x(0) = 1 \quad \Longrightarrow \quad x(t > 0) = ?$$

$$sX(s) - 1 + 4X(s) = \frac{3}{s+4} \tag{75}$$

$$\downarrow \tag{76}$$

$$X(s) = \frac{1}{s+4} + \frac{3}{(s+4)^2} \tag{77}$$

$$\tag{78}$$

Remember: $\mathcal{L}\left\{t^n e^{at}\right\} = \dfrac{n!}{(s-a)^{n+1}}$, thus

$$x(t) = e^{-4t} + 3te^{-4t} \tag{79}$$

?