Dipole Model of Maggy

H.A.E

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1 Introduction

TODO: Add some illustration and explanation of the system/model here.

2 Nomenclature

Position and Geometry

$\mathbf{r}_i$	Position of magnet i in the base [m]
$\mathbf{r}_j$	Position of sensor j in the base [m]
$\mathbf{r}$	Position of levitating magnet [m]
$\mathbf{d}_i := \mathbf{r} - \mathbf{r}_i$	Vector from magnet i to levitating magnet [m]
$d_i := \|\mathbf{d}_i\|$	Distance from magnet i to levitating magnet [m]
$\mathbf{d}_j := \mathbf{r}_j - \mathbf{r}$	Vector from levitating magnet to sensor j [m]
$d_j := \|\mathbf{d}_j\|$	Distance from levitating magnet to sensor j [m]

Orientation and Motion

$\boldsymbol{\eta} := [\phi, \theta, \psi]^\mathsf{T}$	Euler angles (ZYX convention) [rad]
$\boldsymbol{\omega}$	Angular velocity of levitating magnet [rad/s]
$\mathbf{v}$	Linear velocity of levitating magnet [m/s]
$\mathbf{T}$	Euler angle rate matrix (ZYX) [–]
$\mathbf{R}$	Rotation matrix (ZYX) [–]

Magnetic Properties

m_i	Magnitude of magnetic moment in base magnet i [$\mathrm{A} \cdot \mathrm{m}^2$]
m	Magnitude of levitating magnet's magnetic moment [$\mathrm{A} \cdot \mathrm{m}^2$]
$\mathbf{m}_i := [0, 0, m_i]^\mathsf{T}$	Magnetic moment vector of base magnet i [$\mathrm{A} \cdot \mathrm{m}^2$]
$\mathbf{m} := \mathbf{R}[0, 0, m]^\mathsf{T}$	Magnetic moment vector of levitating magnet [$\mathrm{A} \cdot \mathrm{m}^2$]
μ_0	Magnetic permeability of free space ($4\pi \cdot 10^{-7}$) [H/m]
$\mathbf{B}_j$	Magnetic field at sensor j [T]

System Parameters

M	Mass of the levitating magnet [kg]
$\mathbf{I}$	Inertia matrix of the levitating magnet [$\mathrm{kg} \cdot \mathrm{m}^2$]
g	Gravitational acceleration constant [$\mathrm{m/s}^2$]
$\mathbf{g} := [0, 0, -g]^\mathsf{T}$	Gravity vector [$\mathrm{m/s}^2$]
I_i	Current in solenoid i [A]
n_w	Number of windings per solenoid [–]

States and Dimensions

$\mathbf{x}$	State vector [mixed units]
$\mathbf{y}$	Output/measurement vector [T]
n_y	Number of sensors [–]
n_u	Number of solenoids/magnets in the base [–]

3 Equations

3.1 Rotational matrices

$$\mathbf{R} = \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}, \tag{1a}$$

$$\mathbf{T} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \end{bmatrix}. \tag{1b}$$

3.2 Magnetic field

The magnetic field measured in sensor j from the levitating magnet is

$$\mathbf{B}_j = \frac{\mu_0}{4\pi d_j^3}\left[3\frac{(\mathbf{m}\cdot\mathbf{d}_j)\mathbf{d}_j}{d_j^2} - \mathbf{m}\right]. \tag{2}$$

3.3 Force and torque

The magnetic force and torque from magnet i acting on the levitating magnet are

$$\mathbf{F}_i = \frac{3\mu_0}{4\pi d_i^5}\left[(\mathbf{m}_i\cdot\mathbf{d}_i)\,\mathbf{m} + (\mathbf{m}\cdot\mathbf{d}_i)\,\mathbf{m}_i + (\mathbf{m}_i\cdot\mathbf{m})\,\mathbf{d}_i - \frac{5\,(\mathbf{m}_i\cdot\mathbf{d}_i)\,(\mathbf{m}\cdot\mathbf{d}_i)}{d_i^2}\mathbf{d}_i\right], \tag{3a}$$

$$\boldsymbol{\tau}_i = \frac{\mu_0}{4\pi d_i^3}\left[\mathbf{m}\times\left(\frac{3\,(\mathbf{m}_i\cdot\mathbf{d}_i)\,\mathbf{d}_i}{d_i^2} - \mathbf{m}_i\right)\right]. \tag{3b}$$

3.4 Equations of motion

The motion of the levitating magnet is described by

$$M\dot{\mathbf{v}} = \sum_{i=1}^{n_u}\mathbf{F}_i + M\mathbf{g}, \tag{4a}$$

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \sum_{i=1}^{n_u}\boldsymbol{\tau}_i - \boldsymbol{\omega}\times(\mathbf{I}\boldsymbol{\omega}). \tag{4b}$$

4 Complete state-space model

$$\dot{x} = \begin{bmatrix} \mathbf{v} \\ \mathbf{T}^{-1}\boldsymbol{\omega} \\ \frac{1}{M}\left(\sum_{i=1}^{n_u}\mathbf{F}_i + M\mathbf{g}\right) \\ \mathbf{I}^{-1}\left(\sum_{i=1}^{n_u}\boldsymbol{\tau}_i - \boldsymbol{\omega}\times(\mathbf{I}\boldsymbol{\omega})\right) \end{bmatrix}, \tag{5a}$$

$$\mathbf{y} = \begin{bmatrix} B_1 \\ \vdots \\ B_{n_y} \end{bmatrix}. \tag{5b}$$