

# Overview

- Overview
  - Part I Sinusoids and Phasors
    - Introduction
    - 1-1 Sinusoids
      - Definition
      - Phase
      - Combining Sinusoids
    - 1-2 Phasors
      - Definition
      - Calculation
      - Phasors and Circuit Elements
        - Inductor
        - Conductor
  - Part II Sinusoidal Steady State Analysis
    - 2-1 Circuit Analysis
      - Node Analysis
      - Mesh Analysis
      - Superposition Theorem
      - Source Transformation
    - 2-2 Operational Amplifier
      - Open-loop Amplifier
      - Closed-loop Amplifier
    - 2-3 Operation Circuits
      - Inverting Amplifier
      - Noninverting Amplifier
      - Summing Amplifier
      - Difference Amplifier
    - 2-4 Comparator Circuit
      - Noninverting Op-Amp
      - Inverting Op-Amp
    - 2-5 Schmitt Trigger
      - Basic Inverting Schmitt Trigger
      - Noninverting Schmitt Trigger
  - Part III Frequency Response and Bode Plots
    - 3-1 Transfer Function
      - Definition
      - Zeros and Poles
    - 3-2 Decibel Scale and Bode Plots
      - Decibel Scale
      - Bode Plots
    - 3-3 Power Considerations
      - Instantaneous Power
      - Average Power
      - Maximum Average Power Transfer

- 3-4 Series & Parallel Resonance
  - Series Resonance
  - Bandwidth
  - Quality Factor
- 3-5 Passive & Active Filters
  - Passive Filters
    - LPF
    - HPF
    - BPF
    - BSF
  - Active Filters
    - First-order Low-pass Filter
    - First-order High-pass Filter
- Part IV Laplace Transform
  - 4-1 The Laplace Transform
  - 4-2 Useful Equations and Properties Table
  - 4-3 The Inverse Laplace Transform
    - Simple Poles
    - Repeated Poles
    - Complex Poles
  - 4-4 Initial and Final Value Theorems
  - 4-5 The Convolution Integral
  - 4-6 Applications of Laplace Transform
- Part V Fourier Transform
  - 5-1 Fourier Series
    - Magnitude-Phase Form
    - Effects of Symmetry on Fourier coefficients
  - 5-2 Circuits and Fourier Series
  - 5-3 Average Power
  - 5-4 Exponential form of the Fourier Series
  - 5-5 Fourier Transform
    - Transform Pairs and Properties Table
- Reference

## Part I Sinusoids and Phasors

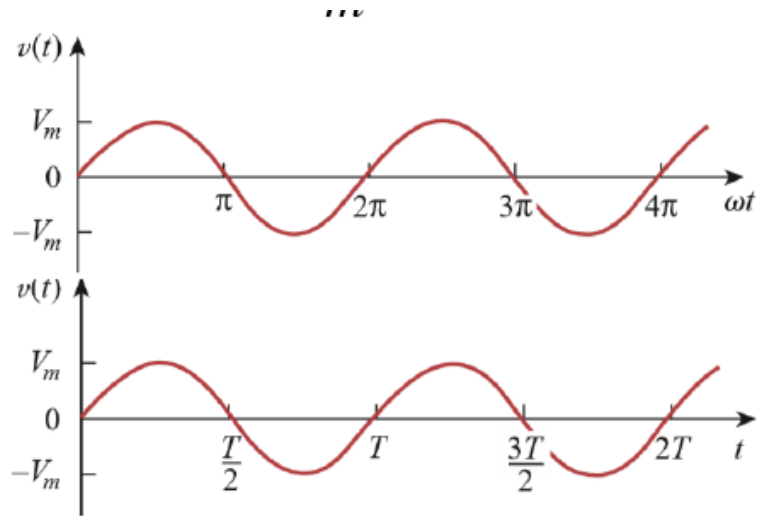
### Introduction

A **sinusoid** is a signal that has the form of the sine or cosine function

### 1-1 Sinusoids

#### Definition

$$v(t) = V_m \sin \omega t$$

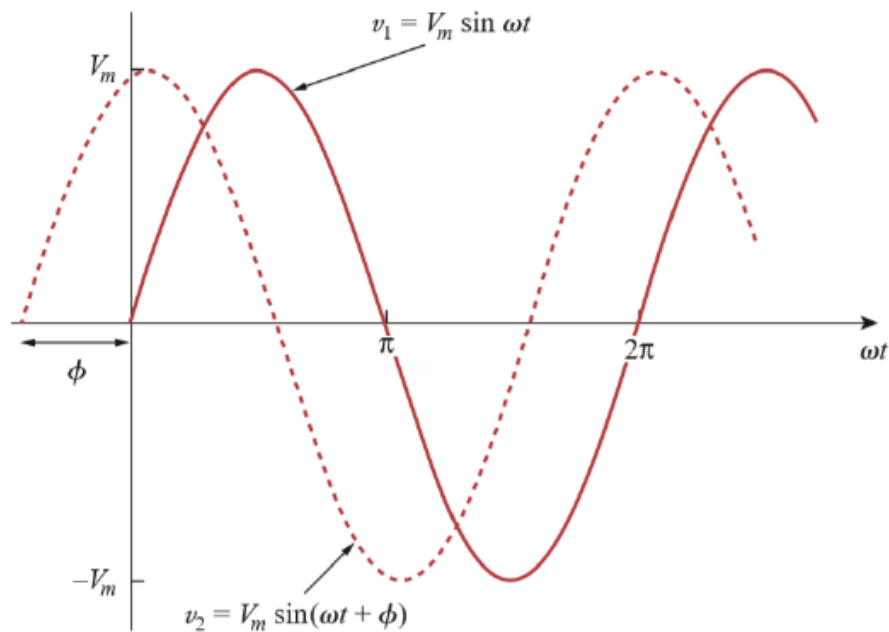


- $\omega = \frac{2\pi}{T}$
- $T = \frac{2\pi}{\omega}$
- $f = \frac{1}{T}$
- $\omega = 2\pi f$

sinusoids are periodic, which implies  $v(t + T) = v(t)$

## Phase

$$v_1(t) = V_m \sin \omega t \quad v_2(t) = V_m \sin(\omega t + \phi)$$



where  $\phi$  is the **phase**

## Combining Sinusoids

$$A \cos \omega t + B \sin \omega t = \sqrt{A^2 + B^2} \cos(\omega t - \theta)$$

where  $\theta = \arctan \frac{B}{A}$

## 1-2 Phasors

### Definition

A **phasor** is a complex number that represents the amplitude and phase of a sinusoid

$$z = x + jy = r\angle\phi = re^{j\phi} = r(\cos\phi + j\sin\phi)$$

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

## Calculation

$$\frac{dv}{dt} \longleftrightarrow j\omega V$$

$$\int v dt \longleftrightarrow \frac{V}{j\omega}$$

## Phasors and Circuit Elements

### Inductor

if the current in the inductor is  $i(t) = I_m \cos(\omega t + \phi)$  then the voltage across it will be

$$\begin{aligned} v &= L \frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) \\ &= \omega L I_m \cos(\omega t + \phi + 90^\circ) \end{aligned}$$

transforming as a phasor and take  $I_m e^{j\phi}$  as  $I$

$$V = \omega L I_m e^{j(\phi+90^\circ)} = j\omega L I$$

### Conductor

similarly since the current on the conductor is  $C \frac{dv}{dt}$

$$V = \frac{I}{j\omega C}$$

## Part II Sinusoidal Steady State Analysis

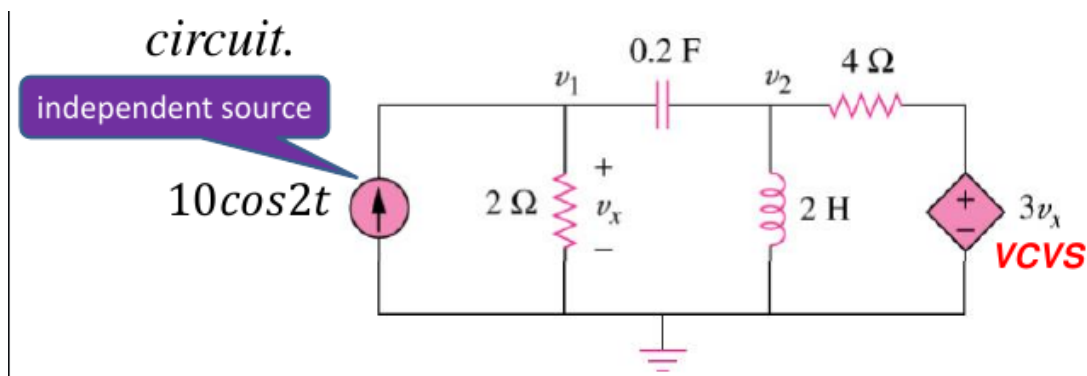
### 2-1 Circuit Analysis

#### Node Analysis

- define node voltage
- define node current ( $I_{in} = I_{out}$ )

#### EX 2-1

Using nodal analysis, find  $v_1$  and  $v_2$  in the following circuit



$$\begin{cases} 10 = \frac{V_1 - V_2}{-2.5j} + \frac{V_1}{2} \\ \frac{V_2}{4j} = \frac{V_1 - V_2}{-2.5j} + \frac{3V_1 - V_2}{4} \end{cases} \implies \begin{cases} (-2 + 2.5j)V_1 + 2V_2 = 50j \\ (-4 + 7.5j)V_1 + (1.5 - 2.5j)V_2 = 0 \end{cases}$$

solve the equations by using Carmer's Rule

Consider a system of  $n$  linear equations for  $n$  unknowns, represented in matrix multiplication form as follows:

$$Ax = b$$

where the  $n \times n$  matrix  $A$  has a nonzero determinant, and the vector  $x = (x_1, \dots, x_n)^T$  is the column vector of the variables. Then the theorem states that in this case the system has a unique solution, whose individual values for the unknowns are given by:

$$x_i = \frac{\det(A_i)}{\det(A)} \quad i = 1, \dots, n$$

where  $A_i$  is the matrix formed by replacing the  $i$ -th column of  $A$  by the column vector  $b$ .

$$\det A = \begin{vmatrix} -2 + 2.5j & 2 \\ -4 + 7.5j & 1.5 - 2.5j \end{vmatrix} = 12.87 \angle -29.05^\circ$$

$$\det A_1 = \begin{vmatrix} 50j & 2 \\ 0 & 1.5 - 2.5j \end{vmatrix} = 145.77 \angle 30.96^\circ$$

$$\det A_2 = \begin{vmatrix} -2 + 2.5j & 50j \\ -4 + 7.5j & 0 \end{vmatrix} = 425 \angle 28.07^\circ$$

$$V_1 = \frac{\det A_1}{\det A} = 11.33 \angle 60.01^\circ V$$

$$V_2 = \frac{\det A_2}{\det A} = 33.02 \angle 57.12^\circ V$$

$$v_1 = 11.33 \cos(2t + \angle 60.01^\circ) V$$

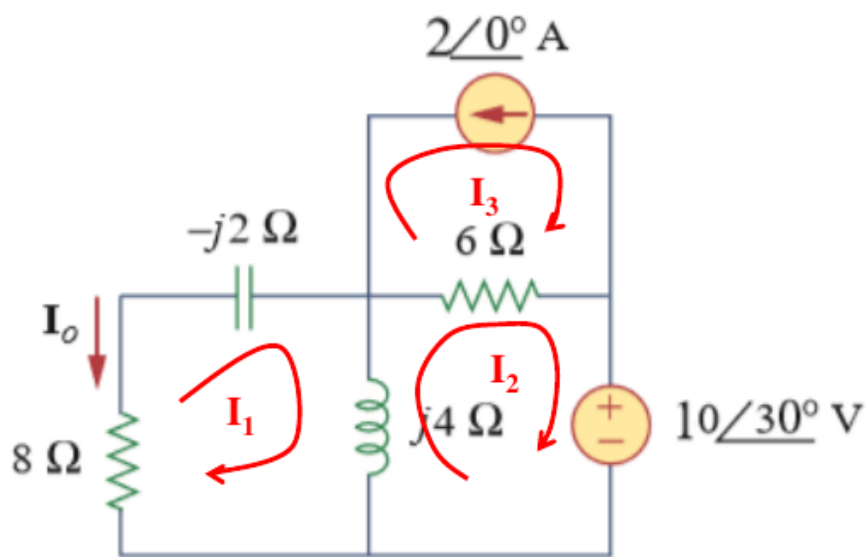
$$v_2 = 33.02 \cos(2t + \angle 57.12^\circ) V$$

## Mesh Analysis

mesh analysis also works well for phasors

### EX 2-2

Finding  $I_0$  using mesh analysis



$$\begin{cases} -2j \times I_1 + 4j \times (I_1 - I_2) + 8 \times I_1 = 0 \\ 4j \times (I_2 - I_1) + 6(I_2 + I_3) = -10\angle 30^\circ \implies I_1 = 1.194\angle -114.55^\circ \\ I_3 = 2 \end{cases}$$

$$I_0 = -I_1 = 1.194\angle 65.45^\circ$$

## Superposition Theorem

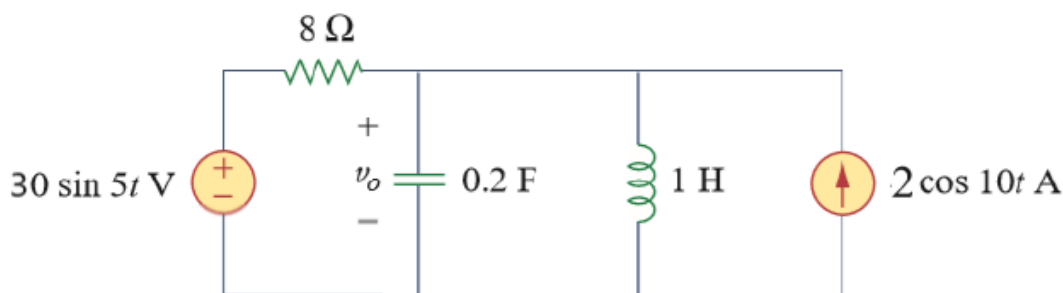
since as circuits are **linear**, the superposition applies to ac circuits the same way it applies to dc circuits

this theorem becomes important if the circuit has sources operating at different frequencies

the total response must be obtained by adding the individual responses in the time domain

### EX2-3

Using superposition theorem to calculate  $v_0$



Considering the circuit without voltage source

$$V_1 = 2 \times \frac{1}{\frac{1}{8} + \frac{1}{-0.5j} + \frac{1}{10j}} = 1.05\angle -86.24^\circ$$

$$v_1 = 1.05 \cos(10t - 86.24^\circ)$$

Considering the circuit without current source

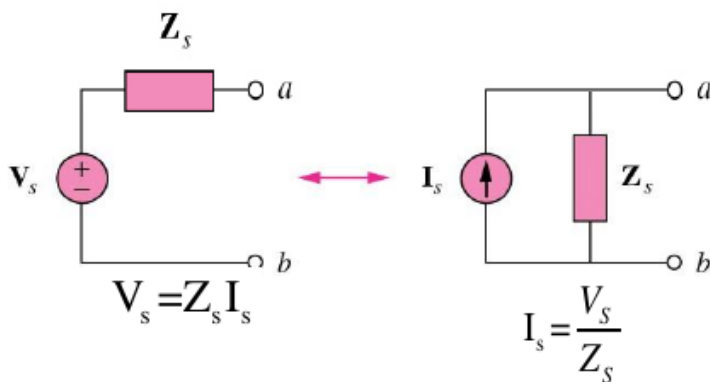
$$V_2 = 30 \times \frac{-1.25j}{8 - 1.25j} = 4.63 \angle -81.12^\circ$$

$$v_2 = 4.63 \sin(5t - 81.12^\circ)$$

Therefore the voltage is  $1.05 \cos(10t - 86.24^\circ) + 4.63 \sin(5t - 81.12^\circ)$

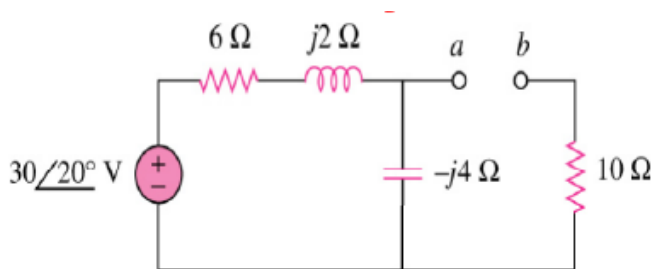
## Source Transformation

- A linear circuit is replaced by a voltage source in series with an impedance
- A linear circuit is replaced by a current source in parallel with an impedance



### EX2-4

Find the Thevenin equivalent at terminals a-b in the following circuit

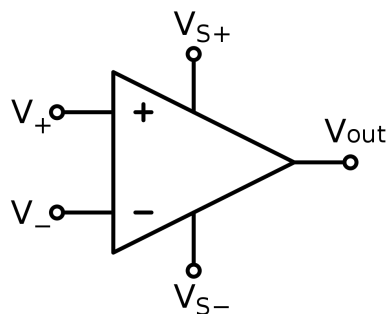


$$Z_{th} = 10 + (-4j) \parallel (6 + 2j)$$

$$= 12.81 \angle -14.47^\circ$$

$$V_{ab} = \frac{-4j}{6 - 2j} \times 30 \angle 20^\circ = 18.97 \angle -51.57^\circ$$

## 2-2 Operational Amplifier



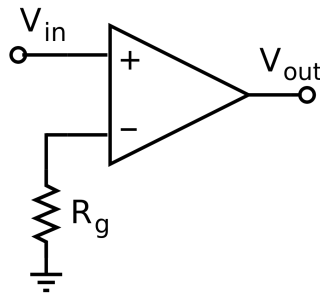
the amplifier's differential inputs consist of a non-inverting input(+) with voltage  $V_+$  and an inverting input(-) with voltage  $V_-$

ideally the op amp amplifies only the difference in voltage between the two and the output voltage of the op amp  $V_{out}$  is given by the equation

$$V_{out} = A_{OL}(V_+ - V_-)$$

where  $A_{OL}$  is the open-loop gain of the amplifier <sup>1</sup>

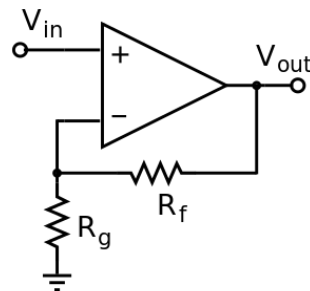
## Open-loop Amplifier



Since the magnitude of  $A_{OL}$  is typically very large, the op amp without negative feedback will work as a comparator. <sup>2</sup>

$$V_{out} = \begin{cases} +\infty & V_{in} > 0 \\ -\infty & V_{in} < 0 \end{cases}$$

## Closed-loop Amplifier

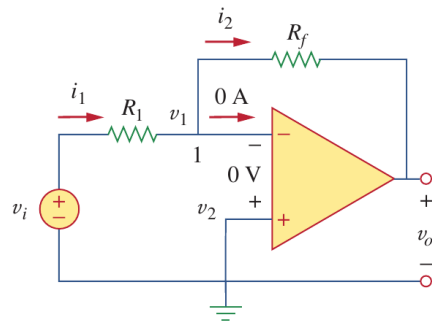


- $V_+ = V_-$ : when an op amp operates in linear mode, the **difference in voltage** between the non-inverting (+) pin and the inverting (-) pin is **negligibly small**
- $I_{in} = 0$ : the **input impedance** between (+) and (-) pins is **much larger** than other resistances in the circuit <sup>3</sup>

## 2-3 Operation Circuits

### Inverting Amplifier

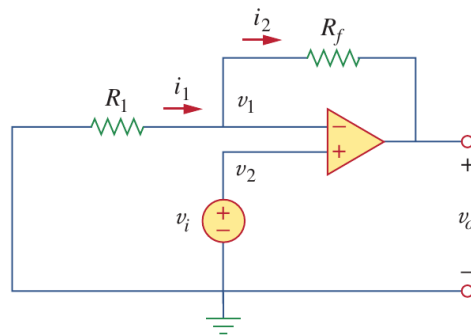




$$\frac{v_i - v_1}{R_1} = \frac{v_1 - v_o}{R_f}$$

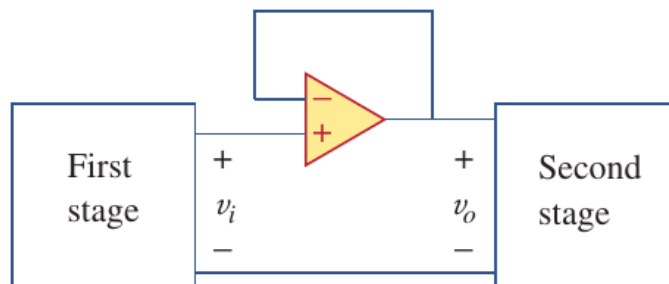
$$v_o = -\frac{R_f}{R_1}v_i$$

## Noninverting Amplifier



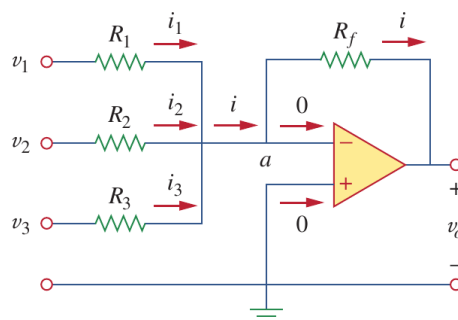
$$\frac{0 - v_i}{R_1} = \frac{v_i - v_o}{R_f}$$

$$v_o = \left(1 + \frac{R_f}{R_1}\right)v_i$$



$$v_o = v_i$$

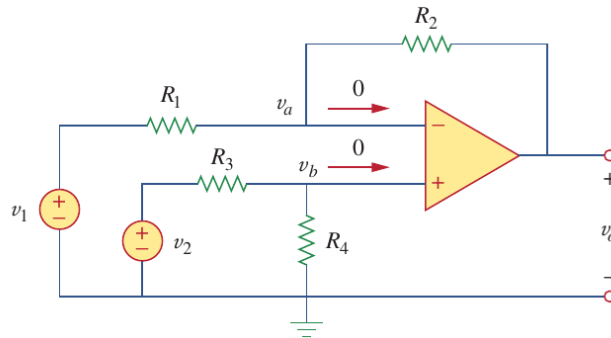
## Summing Amplifier



$$\frac{0 - v_0}{R_f} = \frac{v_1 - 0}{R_1} + \frac{v_2 - 0}{R_2} + \frac{v_3 - 0}{R_3}$$

$$v_0 = -\left(\frac{R_f}{R_1}v_1 + \frac{R_f}{R_2}v_2 + \frac{R_f}{R_3}v_3\right)$$

## Difference Amplifier

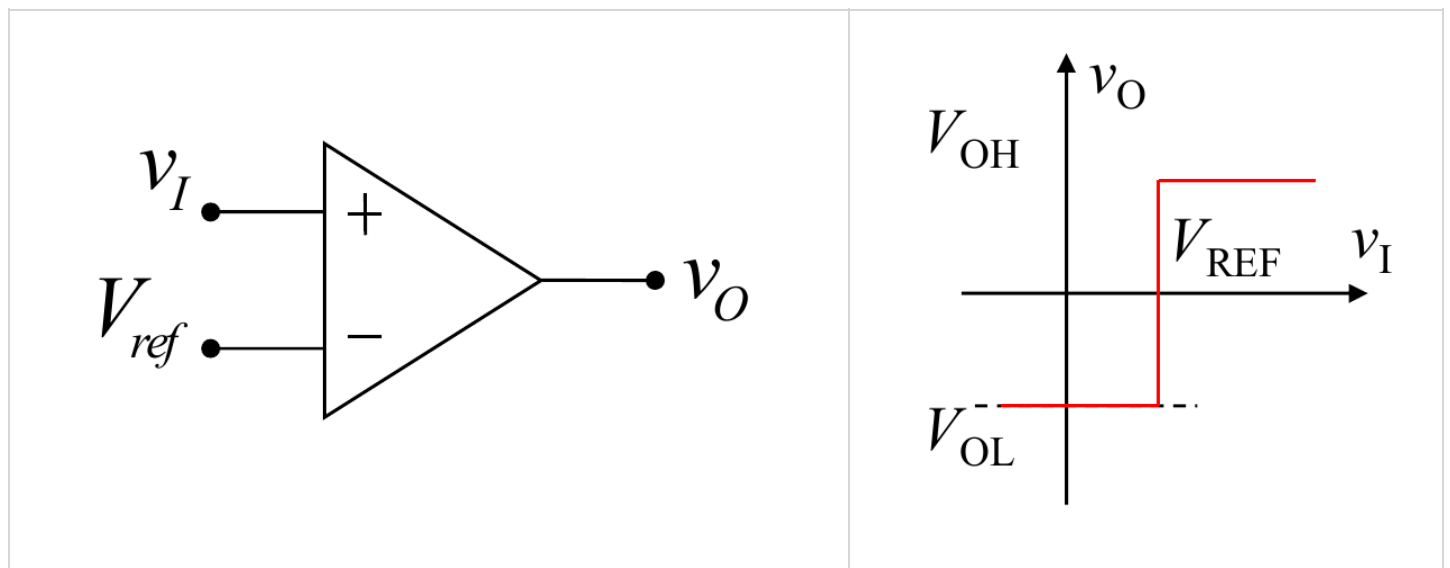


$$\begin{cases} \frac{v_1 - v_a}{R_1} = \frac{v_a - v_0}{R_2} \\ v_a = v_b = \frac{R_4}{R_3 + R_4} v_2 \end{cases} \Rightarrow v_0 = \frac{R_2(1 + R_1/R_2)}{R_1(1 + R_3/R_4)} v_2 - \frac{R_2}{R_1} v_1$$

## 2-4 Comparator Circuit

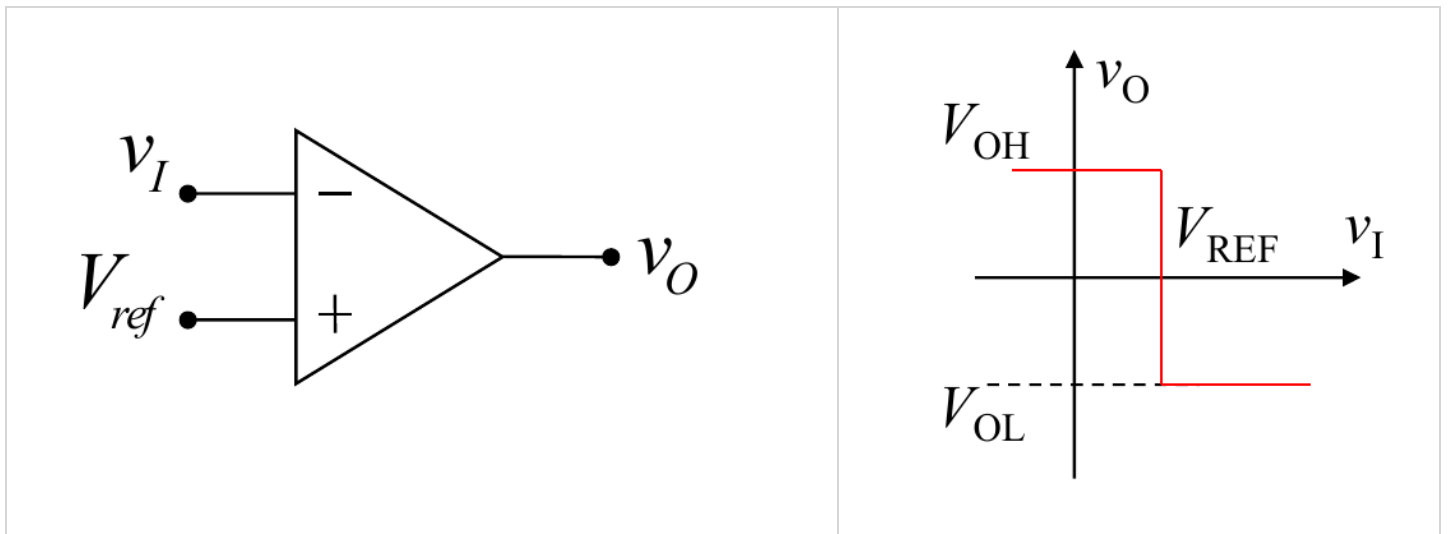
If we use operation amplifier as the comparator, there come two cases

### Noninverting Op-Amp



$$V_O = \begin{cases} +V_{\text{SAT}} & V_i > V_{\text{ref}} \\ -V_{\text{SAT}} & V_i < V_{\text{ref}} \end{cases}$$

### Inverting Op-Amp

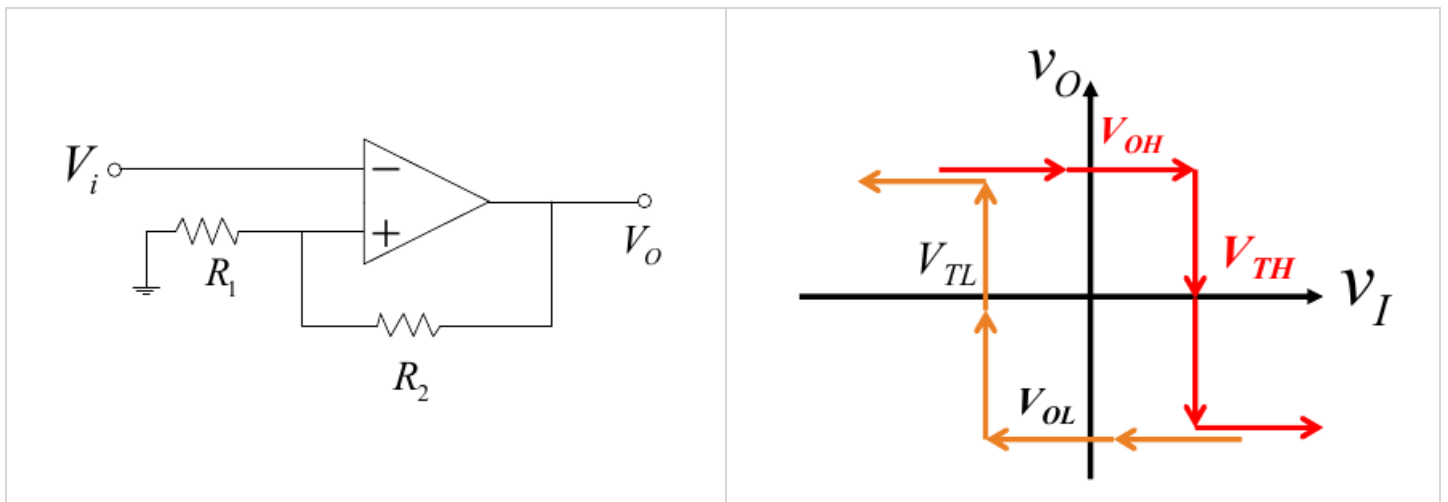


$$V_O = \begin{cases} -V_{SAT} & V_i > V_{ref} \\ +V_{SAT} & V_i < V_{ref} \end{cases}$$

## 2-5 Schmitt Trigger

similarly, there two cases for the Schmitt Trigger

### Basic Inverting Schmitt Trigger



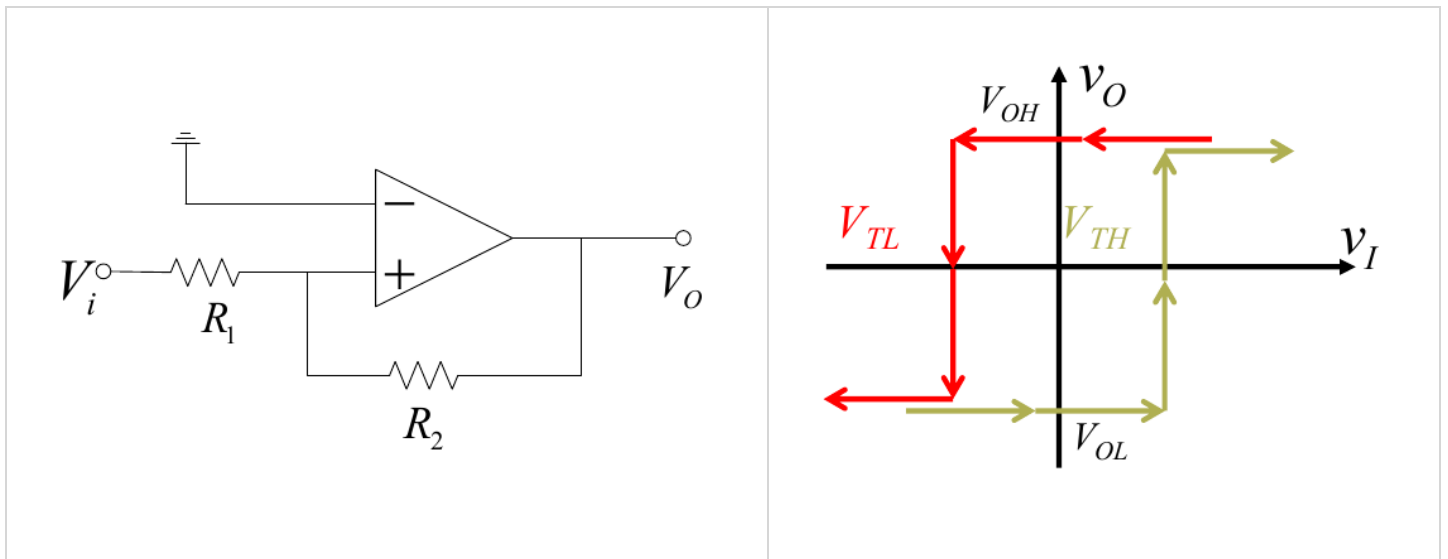
- output voltage is high

$$V_O = \begin{cases} V_{OH} & V_I < \frac{R_1}{R_1+R_2} V_{OH} \\ V_{OL} & V_I > \frac{R_1}{R_1+R_2} V_{OH} \end{cases}$$

- output voltage is low

$$V_O = \begin{cases} V_{OL} & V_I > \frac{R_1}{R_1+R_2} V_{OL} \\ V_{OH} & V_I < \frac{R_1}{R_1+R_2} V_{OL} \end{cases}$$

### Noninverting Schmitt Trigger



- output voltage is high

$$V_O = \begin{cases} V_{OH} & V_I > -\frac{R_1}{R_2} V_{OH} \\ V_{OL} & V_I < -\frac{R_1}{R_2} V_{OH} \end{cases}$$

- output voltage is low

$$V_O = \begin{cases} V_{OL} & V_I < -\frac{R_1}{R_2} V_{OL} \\ V_{OH} & V_I > -\frac{R_1}{R_2} V_{OL} \end{cases}$$

## Part III Frequency Response and Bode Plots

### 3-1 Transfer Function

#### Definition

$$\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}$$

the transfer function  $\mathbf{H}(\omega)$  of a circuit is the frequency-dependent ratio of a phasor output  $\mathbf{Y}(\omega)$  to a phasor input  $\mathbf{X}(\omega)$

since the input and output can be either voltage or current at any place in the circuit, there are four possible transfer functions

$$H(\omega) = \text{Voltage Gain} = \frac{V_O(\omega)}{V_i(\omega)}$$

$$H(\omega) = \text{Current Gain} = \frac{I_O(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer Impedance} = \frac{V_O(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer Admittance} = \frac{I_O(\omega)}{V_i(\omega)}$$

## Zeros and Poles

the transfer function  $\mathbf{H}(\omega)$  can also be expressed in terms of its numerator polynomial  $\mathbf{N}(\omega)$  and denominator polynomial  $\mathbf{D}(\omega)$

$$\mathbf{H}(\omega) = \frac{\mathbf{N}(\omega)}{\mathbf{D}(\omega)}$$

- **zeros:** the *roots* of the numerator polynomial  $\mathbf{N}(\omega)$
- **poles:** the *roots* of the denominator polynomial  $\mathbf{D}(\omega)$

## 3-2 Decibel Scale and Bode Plots

### Decibel Scale

$$\begin{aligned} G_{dB} &= 10 \log_{10} \frac{P_2}{P_1} = 10 \log_{10} \frac{V_2^2/R_2}{V_1^2/R_1} \\ &= 10 \log_{10} \left( \frac{V_2}{V_1} \right)^2 - 10 \log_{10} \frac{R_1}{R_2} \\ &= 20 \log_{10} \frac{V_1}{V_2} - 10 \log_{10} \frac{R_1}{R_2} \end{aligned}$$

for the case when  $R_1 = R_2$ , we can get the basic equation

$$G_{dB} = 20 \log_{10} \frac{V_1}{V_2}$$

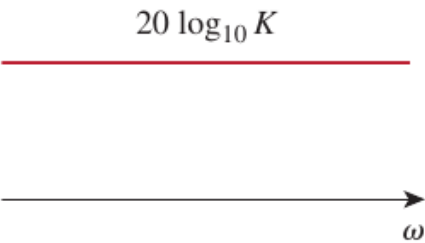
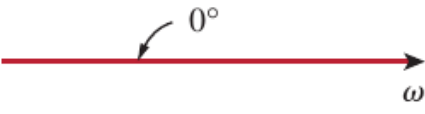
### Bode Plots

one transfer function could be written in terms of factors that have real and imaginary parts

$$\mathbf{H}(j\omega) = \frac{K(j\omega)^{\pm 1} (1 + j\omega/z_1) [1 + j2\zeta\omega/\omega_k + (j\omega/\omega_k)^2] \cdots}{(1 + j\omega/p_1) [1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2] \cdots}$$

which combines several types of factors:

- gain  $K$
- zero  $(j\omega)$  or pole  $(j\omega)^{-1}$  at the origin
- simple zero  $(1 + j\omega/z_1)$  or pole  $1/(1 + j\omega/p_1)$
- quadratic zero  $[1 + j2\zeta_2\omega/\omega_k + (j\omega/\omega_k)^2]$  or pole  $1/[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]$

Factor	Magnitude	Phase
$K$		

Factor	Magnitude	Phase
$(j\omega)^N$	<p>20N dB/decade</p>	<p><math>90N^\circ</math></p>
$\frac{1}{(j\omega)^N}$	<p>-20N dB/decade</p>	<p><math>-90N^\circ</math></p>
$\left(1 + \frac{j\omega}{z}\right)^N$	<p>20N dB/decade</p>	<p><math>0^\circ</math> <math>90N^\circ</math></p>
$\frac{1}{(1+j\omega/p)^N}$	<p>-20N dB/decade</p>	<p><math>0^\circ</math> <math>-90N^\circ</math></p>
$\left[1 + \frac{2j\omega\zeta}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right]^N$	<p>40N dB/decade</p>	<p><math>0^\circ</math> <math>180N^\circ</math></p>
$\frac{1}{[1+2j\omega\zeta/\omega_k+(j\omega/\omega_k)^2]^N}$	<p>-40N dB/decade</p>	<p>20N dB/decade</p>

### 3-3 Power Considerations

#### Instantaneous Power

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

where we can see that the first part is constant while the second part is periodic

#### Average Power

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

#### Maximum Average Power Transfer

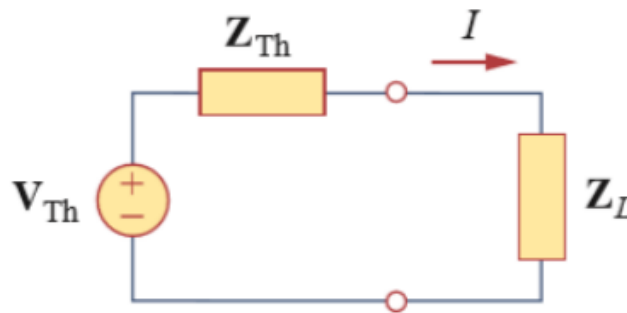


Fig. Circuit with a load and its Thevenin equivalent.

$$P = \frac{1}{2}|I|^2 R_L = \frac{|V_{Th}|^2 R_L / 2}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}$$

How to adjust the load parameters  $R_L$  and  $X_L$  so that **P is maximum**

$$\begin{cases} X_L = -X_{Th} \\ R_L = \sqrt{R_{Th}^2 + (X_{Th} + X_L)^2} \end{cases}$$

Therefore,  $Z_L = R_{Th} - jX_{Th} = Z_{Th}^*$

### 3-4 Series & Parallel Resonance

#### Series Resonance

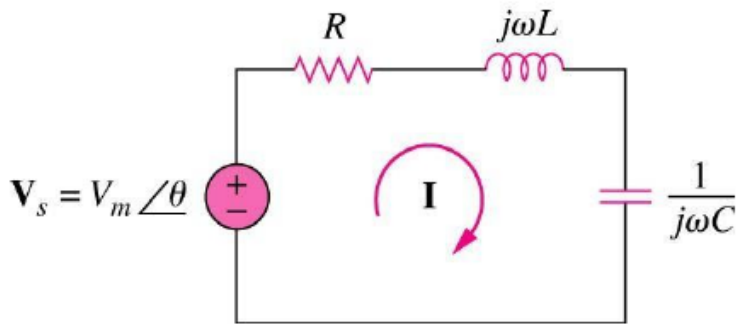


Fig. The series resonant circuit.

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

Resonance is a condition in an RLC circuit in which the **capacitive and inductive reactances are equal in magnitude** thereby resulting in **a purely resistive impedance**

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad f_0 = \frac{1}{2\pi\sqrt{LC}}$$

### Bandwidth

Half-power frequencies  $\omega_1$  and  $\omega_2$  are frequencies at which the dissipated power is **half the maximum value**

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \quad \omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

where we can find

$$\omega_0 = \sqrt{\omega_1 \omega_2} \quad B = \omega_2 - \omega_1$$

### Quality Factor

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}$$

and the relationship between the  $B$ ,  $Q$  and  $\omega_0$

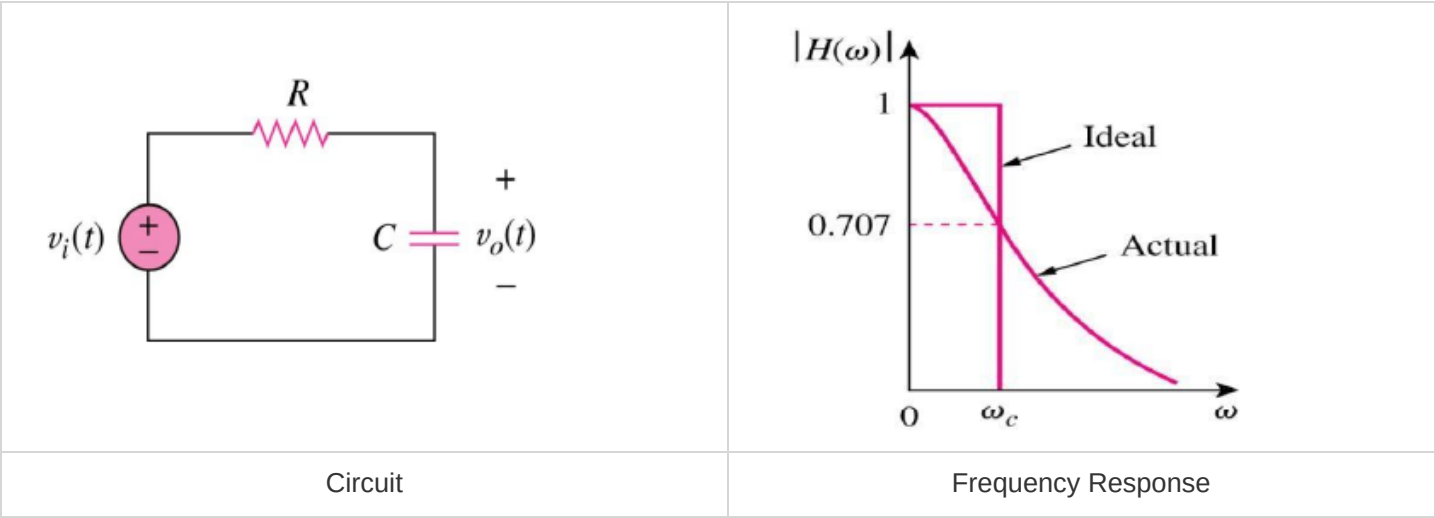
$$B = \frac{R}{L} = \frac{\omega_0}{Q} = \omega_0^2 C R$$

## 3-5 Passive & Active Filters

### Passive Filters

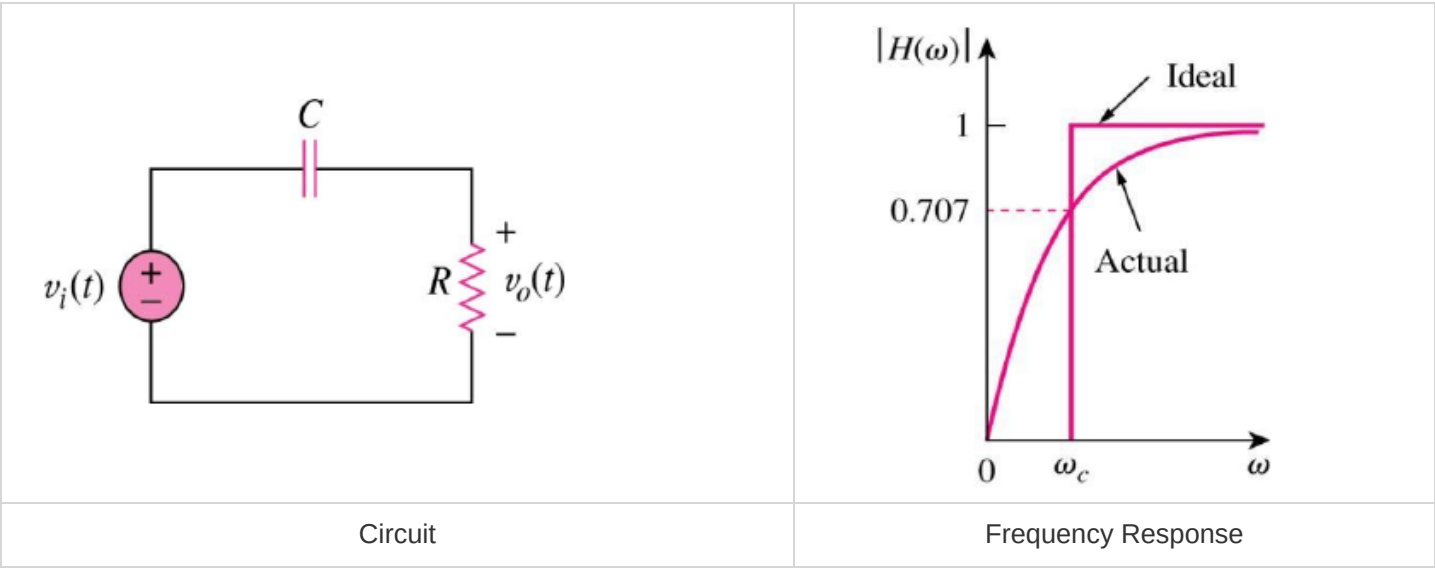
LPF





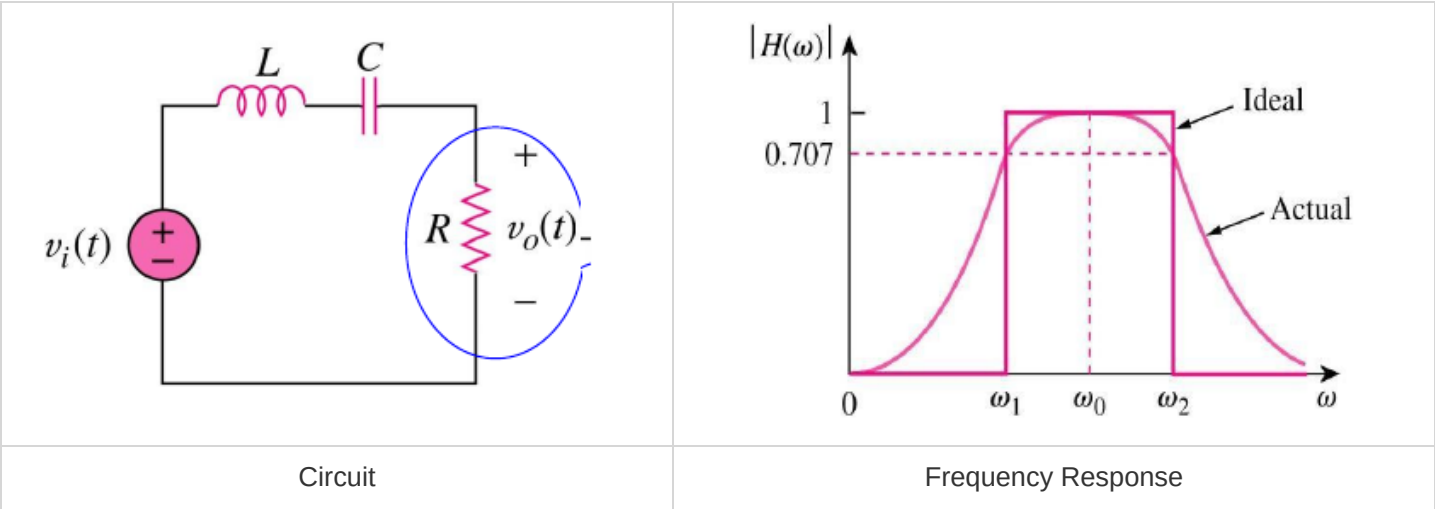
$$\omega_c = \frac{1}{RC}$$

**HPF**

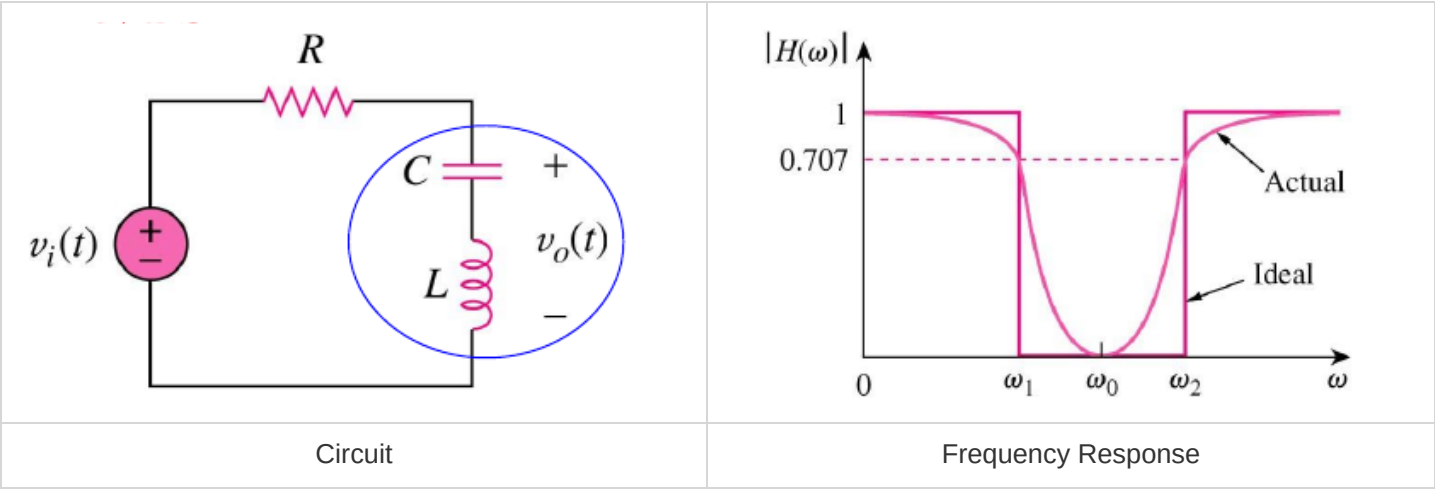


$$\omega_c = \frac{1}{RC}$$

**BPF**



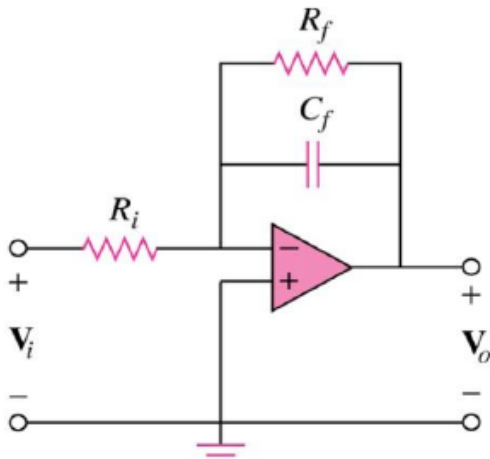
$$\omega_0 = \frac{1}{\sqrt{LC}}$$



$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Active Filters

First-order Low-pass Filter



$$\omega_C = \frac{1}{R_f C_f}$$

First-order High-pass Filter

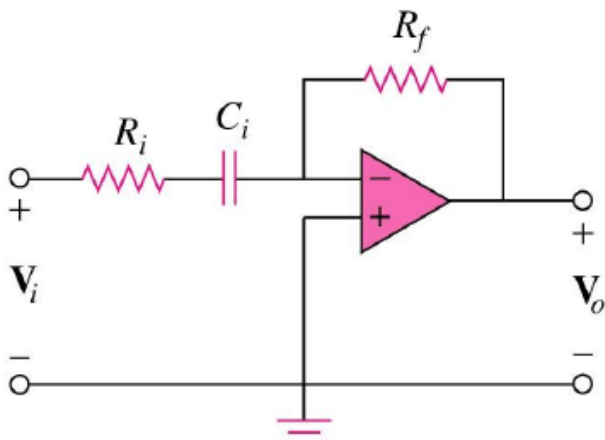


Fig. first-order high pass filter

$$\omega_C = \frac{1}{R_i C_i}$$

## Part IV Laplace Transform

### 4-1 The Laplace Transform

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st} dt$$

which is called as the one-sided Laplace Transform

### 4-2 Useful Equations and Properties Table

$$\frac{d}{dt}x(t) \longleftrightarrow sX(s) - x(0^-)$$

$$\frac{d^2}{dt^2}x(t) \longleftrightarrow s^2X(s) - sx(0^-) - x'(0^-)$$

Aperiodic Signal	Fourier Signal	Signal	Transform	ROC
$ax(t) + by(t)$	$aX(s) + bY(s)$	$\delta(t)$	1	All $s$
$x(t - t_0)$	$e^{-st_0} X(s)$	$u(t)$	$\frac{1}{s}$	$s > 0$
$e^{s_0 t} x(t)$	$X(s - s_0)$	$-u(-t)$	$\frac{1}{s}$	$s < 0$
$x^*(t)$	$X^*(s^*)$	$\frac{t^{n-1}}{(n-1)!} u(t)$	$\frac{1}{s^n}$	$s > 0$
$x(at)$	$\frac{1}{ a } X\left(\frac{s}{a}\right)$	$-\frac{t^{n-1}}{(n-1)!} u(-t)$	$\frac{1}{s^n}$	$s < 0$
$x(-t)$	$X(-s)$	$e^{-\alpha t} u(t)$	$\frac{1}{s+\alpha}$	$s > -\alpha$
$x(t) * y(t)$	$X(s)Y(s)$	$-e^{-\alpha t} u(-t)$	$\frac{1}{s+\alpha}$	$s < -\alpha$
$\frac{d}{dt}x(t)$	$sX(s)$	$\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(t)$	$\frac{1}{(s+\alpha)^n}$	$s > -\alpha$
$\int_{-\infty}^t x(t) dt$	$\frac{1}{s} X(s)$	$-\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(-t)$	$\frac{1}{(s+\alpha)^n}$	$s < -\alpha$
$-tx(t)$	$\frac{d}{ds} X(s)$	$\delta(t - T)$	$e^{-sT}$	All $s$
		$[\cos \omega_0 t] u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$s > 0$
		$[\sin \omega_0 t] u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$s > 0$
		$[e^{-\alpha t} \cos \omega_0 t] u(t)$	$\frac{s+\alpha}{(s+\alpha)^2 + \omega_0^2}$	$s > \alpha$
		$[e^{-\alpha t} \sin \omega_0 t] u(t)$	$\frac{\omega_0}{(s+\alpha)^2 + \omega_0^2}$	$s > \alpha$

### 4-3 The Inverse Laplace Transform

Given  $F(s)$ , how do we transform it back to the time domain and obtain the corresponding  $f(t)$

$$X(s) = \frac{N(s)}{D(s)}$$

- the roots of  $N(s)$  are defined as the zeros
- the roots of  $S(s)$  are defined as the poles

$F(s)$  can always be reduced to a ratio with  $m < n$  by polynomial division

## Simple Poles

if the degree of  $N(s)$  is less than the degree of  $D(s)$

$$F(s) = \frac{k_1}{s + p_1} + \frac{k_2}{s + p_2} + \cdots + \frac{k_n}{s + p_n}$$

where  $k_i = (s + p_i)F(s) \Big|_{s=-p_i}$

## Repeated Poles

$$F(s) = \frac{k_n}{(s + p)^n} + \frac{k_{n-1}}{(s + p)^{n-1}} + \cdots + \frac{k_1}{s + p}$$

## Complex Poles

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} = \frac{A_1(s + \alpha)}{(s + \alpha)^2 + \beta^2} + \frac{B_1 \beta}{(s + \alpha)^2 + \beta^2}$$

## 4-4 Initial and Final Value Theorems

- Initial Value:  $f(0) = \lim_{s \rightarrow \infty} sF(s)$
- Final Value:  $f(\infty) = \lim_{s \rightarrow 0} sF(s)$

Final value theorem **fails** if the real part of pole is **non-negative**

## 4-5 The Convolution Integral

$$y(t) = \int x(\tau)h(t - \tau)d\tau \xLeftrightarrow{L} Y(s) = X(s)H(s)$$

## 4-6 Applications of Laplace Transform

- Resistor:  $Z(s) = R$
- Inductor:  $Z(s) = sL$
- Capacitor:  $Z(s) = \frac{1}{sC}$

# Part V Fourier Transform

## 5-1 Fourier Series

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

where  $a_0$  is the dc term, and the latter one is the ac components

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega_0 t dt$$

## Magnitude-Phase Form

$$f(t) = a_0 + \sum_{n=1}^{\infty} (A_n \cos(n\omega_0 t + \phi_n))$$

- $A_n: \sqrt{a_n^2 + b_n^2}$
- $\phi_n: -\tan^{-1}(b_n/a_n)$

## Effects of Symmetry on Fourier coefficients

Symmetry	$a_0$	$a_n$	$b_n$
Even	/	/	0
Odd	0	0	/
Half-wave	0	$a_{2n} = 0$	$b_{2n} = 0$
Quarter-wave	0	0	$b_{2n} = 0$

## 5-2 Circuits and Fourier Series

Fourier series approaches can be used where a **repetitive non-sinusoidal** forcing function occurs and the Natural Response is not important

1. Decide on the upper limit of  $n$ :  $N_{max}$
2. Find the FS of the **forcing function** for  $n \in [0, N_{max}]$
3. Find the **transfer function** of the circuit
4. Solve for each component of the Forced Response using phasor method
5. Combine the response to each harmonic frequency to give the overall response

## 5-3 Average Power

$$v(t) = V_{dc} + \sum_{n=1}^{\infty} \cos(n\omega_0 t - \theta_n) \quad i(t) = I_{dc} + \sum_{n=1}^{\infty} I_n \cos(m\omega_0 t - \phi_m)$$

then substitute them into

$$\begin{aligned}
 P &= \int_0^T v(t)i(t)dt \\
 &= V_{dc}I_{dc} + \frac{1}{2} \sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n)
 \end{aligned}$$

## 5-4 Exponential form of the Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

the function could also be written as the pattern above, where

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt$$

## 5-5 Fourier Transform

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

### Transform Pairs and Properties Table

Aperiodic Signal	Fourier Signal	Signal	Fourier Transform
$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$	$\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$
$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$	$e^{j\omega_0 t}$	$2\pi \delta(\omega - \omega_0)$
$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$	$\cos \omega_0 t$	$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$x^*(t)$	$X^*(-j\omega)$	$\sin \omega_0 t$	$\frac{\pi}{j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$x(-t)$	$X(-j\omega)$	$x(t) = 1$	$2\pi \delta(\omega)$
$x(at)$	$\frac{1}{ a } X(\frac{j\omega}{a})$	$x(t) = \begin{cases} 1 &   t   < T_1 \\ 0 &   t   > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$
$x(t) * y(t)$	$X(j\omega) Y(j\omega)$	$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1 &   \omega   < W \\ 0 &   \omega   > W \end{cases}$
$x(t)y(t)$	$\frac{1}{2\pi} X(j\omega) * Y(j\omega)$	$\delta(t)$	1
$\frac{d}{dt} x(t)$	$j\omega X(j\omega)$	$u(t)$	$\frac{1}{j\omega} + \pi \delta(\omega)$
$\int_{-\infty}^t x(t) dt$	$\frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega)$	$\delta(t - t_0)$	$e^{j\omega t_0}$
$tx(t)$	$j \frac{d}{d\omega} X(j\omega)$	$e^{-at} u(t)$	$\frac{1}{a + j\omega}$
$X(jt)$	$2\pi x(-\omega)$	$te^{-at} u(t)$	$\frac{1}{(a + j\omega)^2}$

## Reference

- [1] [Operational amplifier - 1 Operation - Wikipedia](#)
- [2] [Operational amplifier - 1.1 Open-loop amplifier - Wikipedia](#)
- [3] [Operational amplifier - 1.2 Closed-loop amplifier - Wikipedia](#)