Control Theory

Chapter 1 Introduction

1-1 Basic Terms and Concepts

- **Reference Components**: a reference component generates the reference signal or the input signal
- **Controlled Variable**: the quantity or condition that is measured and controlled
- **Comparison Components**: compare the input with feedback signal and generate the error signal
- **Plant** or Process: any physical object or operation to be controlled
- **Controller**: a compensation component, improves the performance ofg the system
- **Actuator**: acts on that plant directly to adjust the controlled variable
- **Disturbance**: a signal that tends to adversely affect the value of the output of a system
- **Sensor** or measurement component: measure the output or the controlled variable and generate feedback signal

Examples

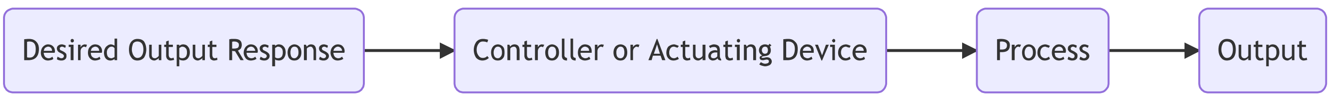
Control System	Plant	Controlled Variable
	water tank	water level
	electric furnace	the furnace temperature

1-2 Types of Control

Open-loop Control Systems

An open-loop system is a system without feed back

- the output of the open-loop system has no effect upon the input signal
- there is only forward action from the input to the output



Advantages

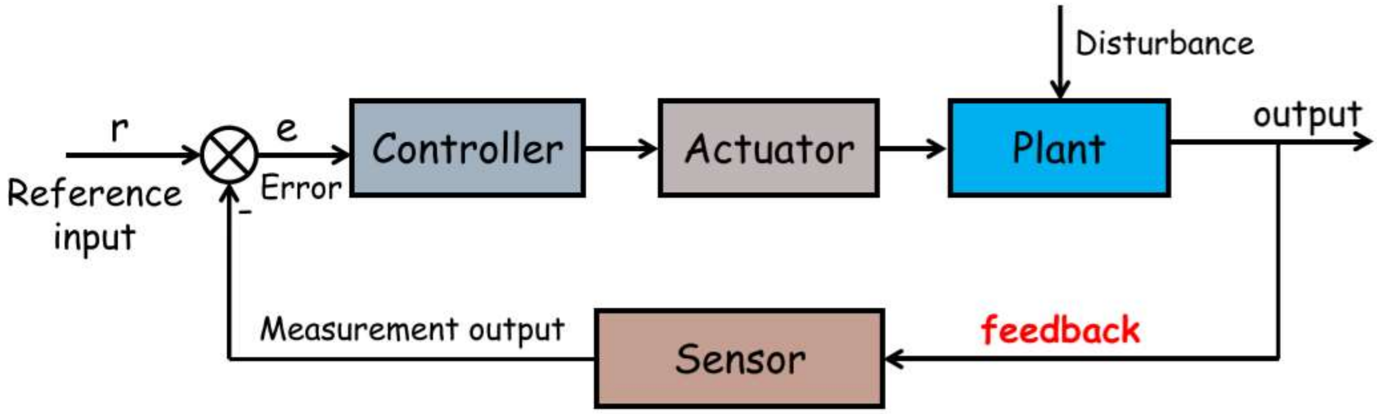
- simple construction and ease of maintenance
- less expensive
- no stability problem
- convenient when output is hard to measure

Disadvantages

- disturbance and changes in calibration cause errors
- to maintain the required quality in the output, recalibration is necessary from time to time

Closed-loop Control System

A closed-loop control system uses a measurement of the output and feedback of this signal to compare it with the desired output



Features

- there are feedbacks in the system so that signals flow through closed loops
- the error signal controls the system

General Requirements for Control Systems

- **Stability**: stability, smooth and steady
- **Swiftness**: peak time, settling time
- **Accuracy**: steady-state error

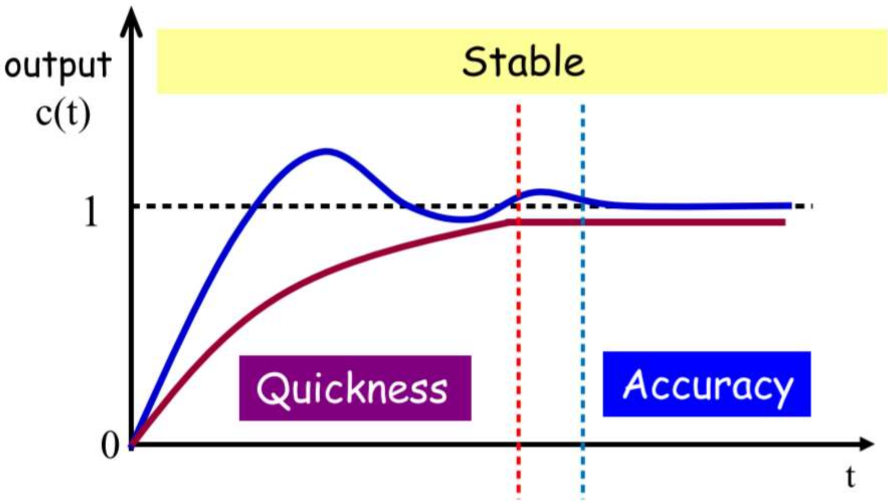
Stable and Unstable

Stable	Unstable

Regulation Process

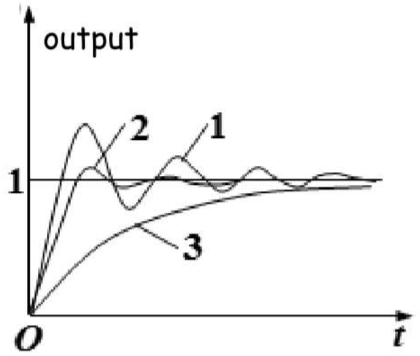
The whole regulation process can be divided into two stages

transient process + **steady-state process**



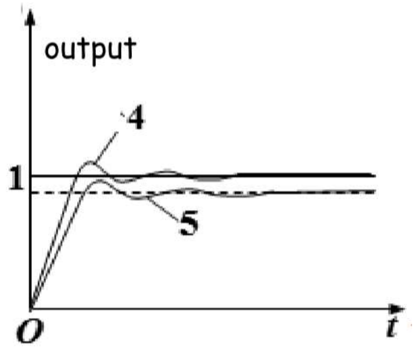
- **transient process** reflects dynamic characteristics
- **steady-state process** reflects steady-state characteristics

Transient Response



- **swiftness**: peak time, settling time

Steady-State Response



- **accuracy**: steady-state error

Chapter 2 Transfer Function and Block Diagram

2-1 Transfer Function and Impulse Response Function

Transfer Function

The *transfer function* of a linear, time-invariant, differential equation system defined as the **ratio** of the **Laplace transform of the output** to the **Laplace transform of the input** under the assumption that all initial conditions are zero

Consider the linear time-invariant system defined by the following differential equation

$$\begin{aligned} a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y \\ = b_0 x^{(m)} + b_1 x^{(m-1)} + \cdots + b_{m-1}\dot{x} + b_m x \end{aligned}$$

where y is the output of the system and x is the input

$$\begin{aligned} \text{Transfer Function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}} \\ = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{N(s)}{D(s)} \end{aligned}$$

> 1. the concept of transfer function is only appropriate to the LTI system
> 2. transfer function is only determined by the structure and parameter of system
> 3. if the highest power of s in the **denominator** of the TF is equal to n, the system is called as **n-th system**

- **characteristic polynomial**: the denominator polynomial $D(s)$
- **characteristic equation**: the formula of $D(s) = 0$
- **zeros**: the roots of the **numerator polynomial** $N(s)$
- **poles**: the roots of the **denominator polynomial** $D(s)$

Example

$$G(s) = \frac{K(2s+1)}{s(3s+1)(T^2s^2 + 2\xi Ts + 1)}$$

which could be rewritten as

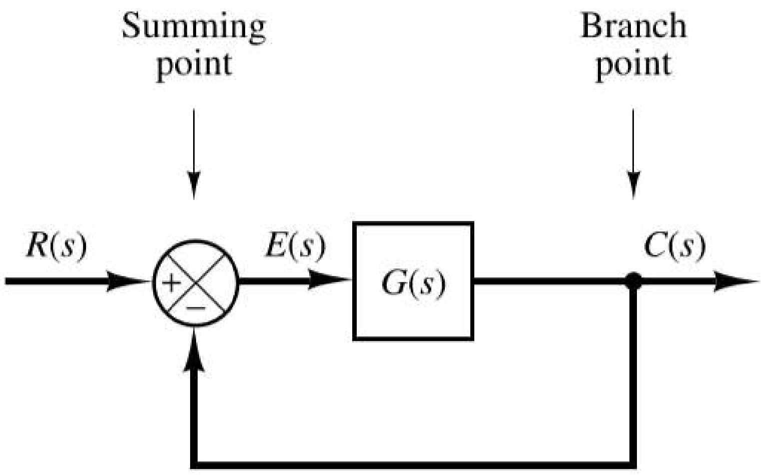
$$G(s) = K \cdot \frac{1}{s} \cdot (2s+1) \cdot \frac{1}{3s+1} \cdot \frac{1}{T^2s^2 + 2\xi Ts + 1}$$

- K: gain factor
- $1/s$: integral factor
- $2s+1$: first-order differential factor (differential factor)
- $1/(3s+1)$: inertial element (reciprocal first-order)
- $1/(T^2s^2 + 2\xi Ts + 1)$: quadratic factor

2-2 Automatic Control Systems

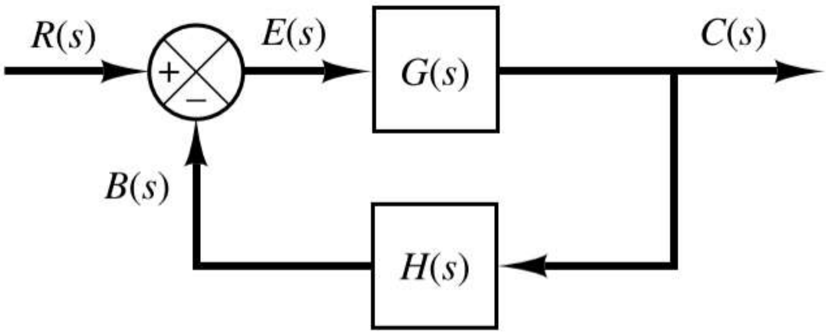
Block Diagrams

A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals. Such a diagram depicts the interrelationships that exist among the various components.



- **signal line**: a line with arrow that indicate the direction of signal transform
- **block**: it expresses the transfer function
- **summing point**: a circle with a cross is the symbol that indicates a summing operation
- **branch point**: a point from which the signal from a block goes concurrently to other blocks or summing points

Block Diagram of a Closed-Loop System



$$C(s) = G(s)E(s)$$

$$E(s) = R(s) - B(s)$$

$$= R(s) - H(s)C(s)$$

Eliminating $E(s)$ from the equations above

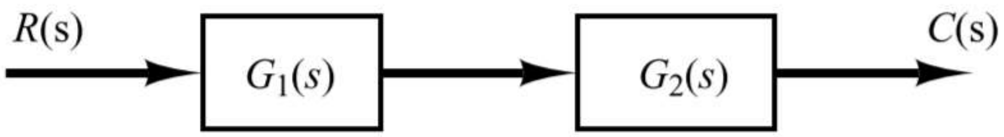
$$C(s) = G(s)[R(s) - H(s)C(s)]$$

or

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

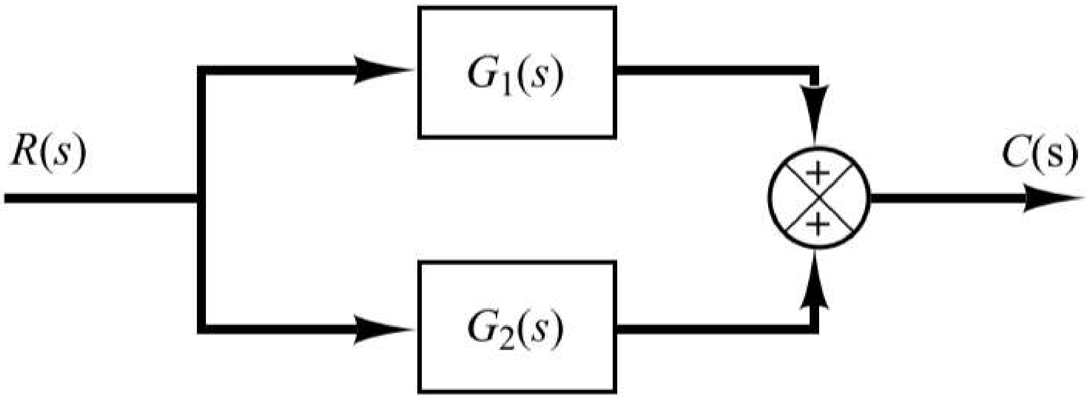
Reduce Block Diagrams

Cascaded Systems



$$\frac{C(s)}{R(s)} = G_1(s)G_2(s)$$

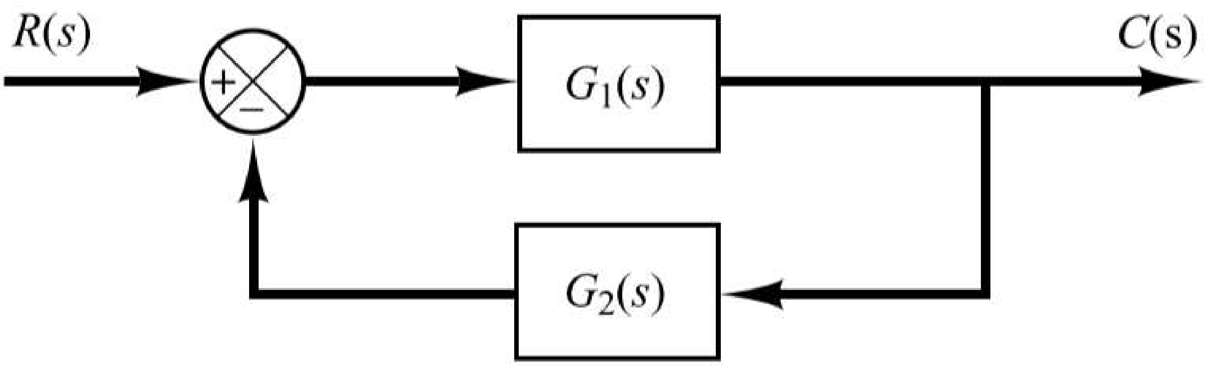
Parallel Connected System



$$\frac{C(s)}{R(s)} = G_1(s) + G_2(s)$$

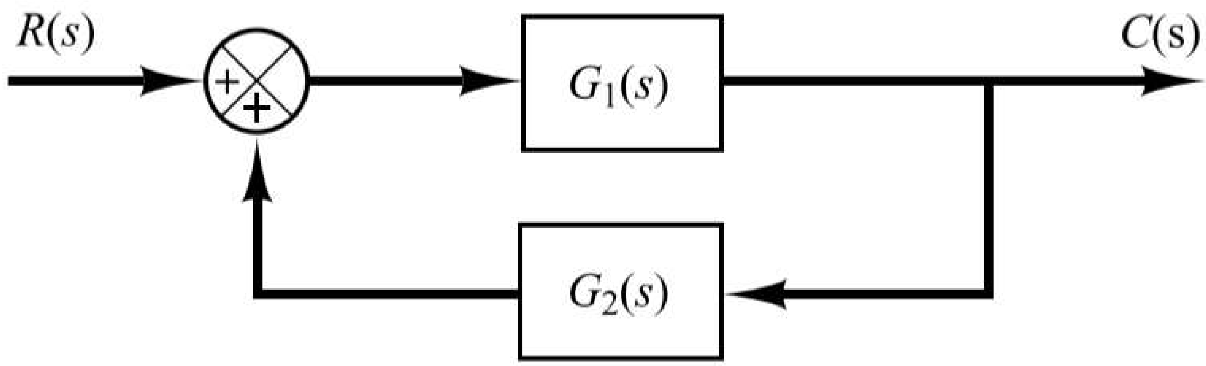
Feedback System

Negative Feedback System



$$\frac{C(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

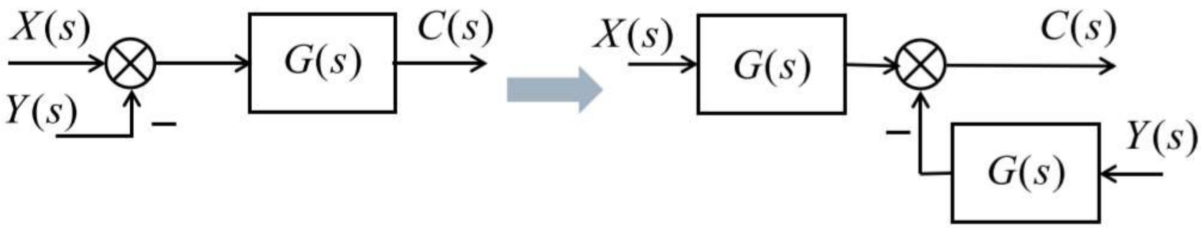
Positive Feedback System



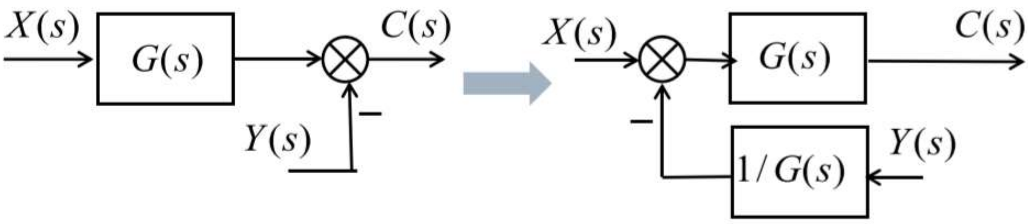
$$\frac{C(s)}{R(s)} = \frac{G_1(s)}{1 - G_1(s)G_2(s)}$$

Slide a Summing Point

Backward

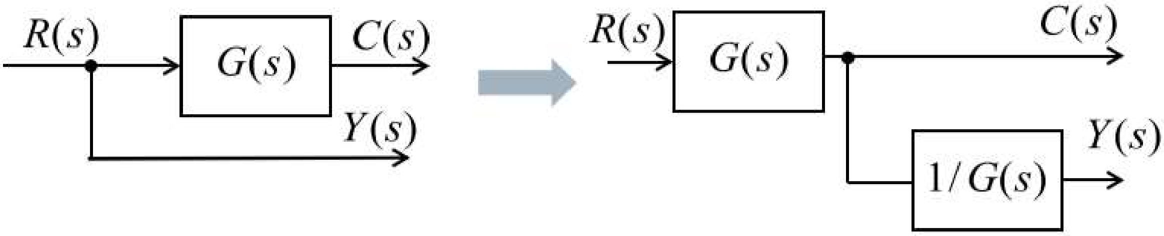


Forward

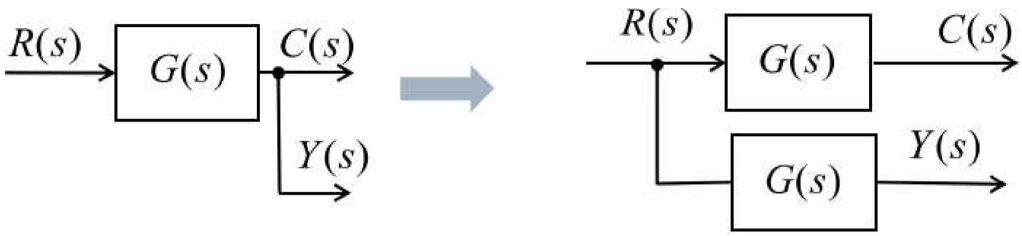


Slide a Branch Point

Backward

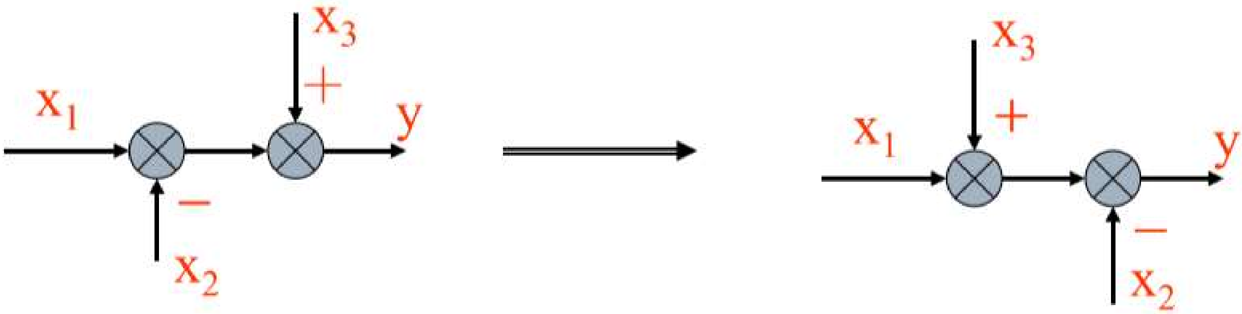


Forward

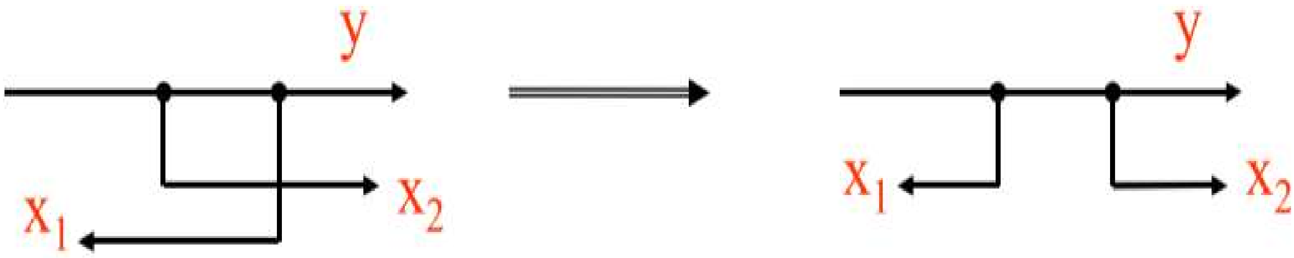


Interchanging the Neighboring

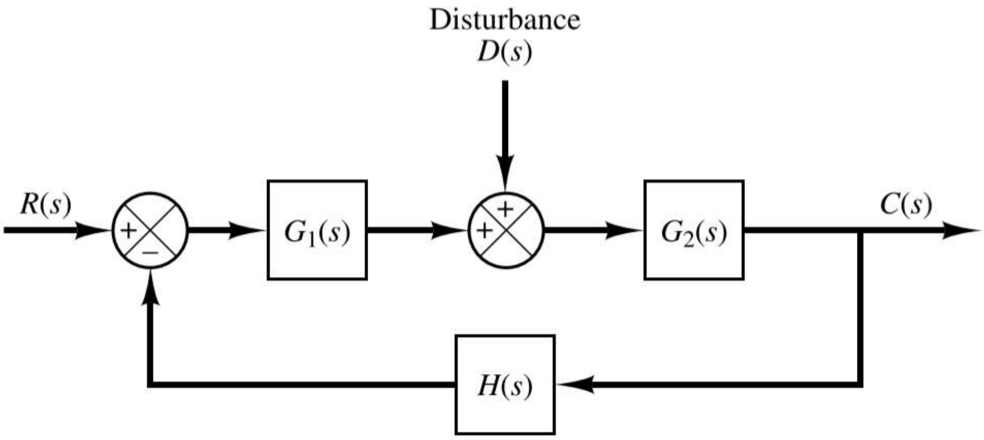
Summing Points



Branch Points



Closed-Loop System Subjected to a Disturbance



We could calculate the response $C_D(s)$ to the disturbance only

$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

and the response to the reference input $R(s)$

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

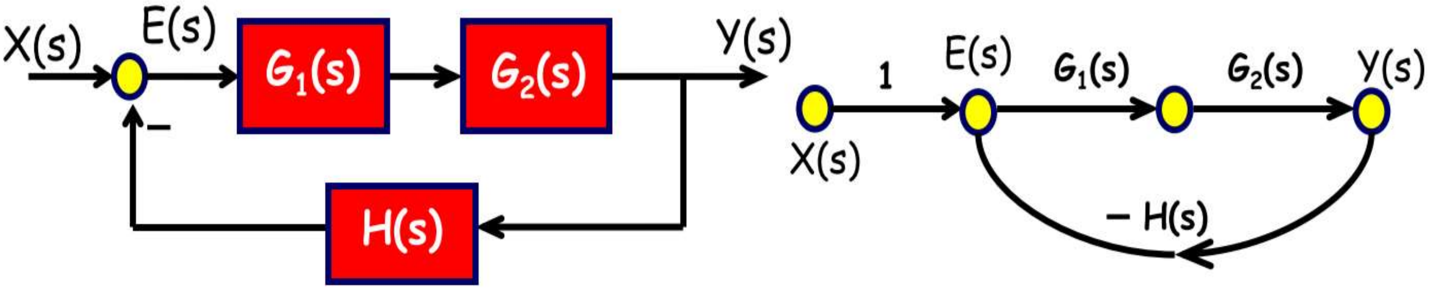
Then the response could be the sum of two responses

$$C(s) = C_D(s) + C_R(s)$$

$$= \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}[G_1(s)R(s) + D(s)]$$

2-3 Signal-Flow Graph Models

A signal-flow graph is a specialized flow graph, a directed graph in which nodes represent system variables, and branches represent functional connections between pairs of nodes.



Basic Components

- **node**: represents a signal
  - input nodes: nodes with only outgoing branches
  - output nodes: nodes with only incoming branches
  - mixed nodes: nodes with both incoming and outgoing branches
- **branch**: directed line segment connecting two nodes
  - signal can only flow along the specified direction
  - each branch is associated with a gain, which is the transfer function
- **path**: a sequence of connected branches
  - forward path: start from an input node and end at and output node
  - forward path gain: product of all branch gains along a forward path
- **loop**: a closed path
  - loop gain: product of all branch gains along a lop
  - nontouching loop: loops that do not have shard nodes
  - touching loop: more than one loop sharing one orr more common nodes

Mason's Gain Formula

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{\Delta}\sum_{k=1}^{n} P_k\Delta_k$$

- Δ: determinant of the graph

$$\Delta = 1 - (\text{sum of all individual loop gains}) + (\text{sum of gain products of all two nontouching loop})$$
$$-(\text{sum of gain products of all three nontouching loop}) + \cdots$$

- P_k: path gain of kth forward path
- Δ_k: factor of the kth forward path

Chapter 3 Steady-State and Stability Analysis

3-1 Transient and Steady-State Response Analysis

The time response of a control system consists of two parts: the transient response and the steady-state response

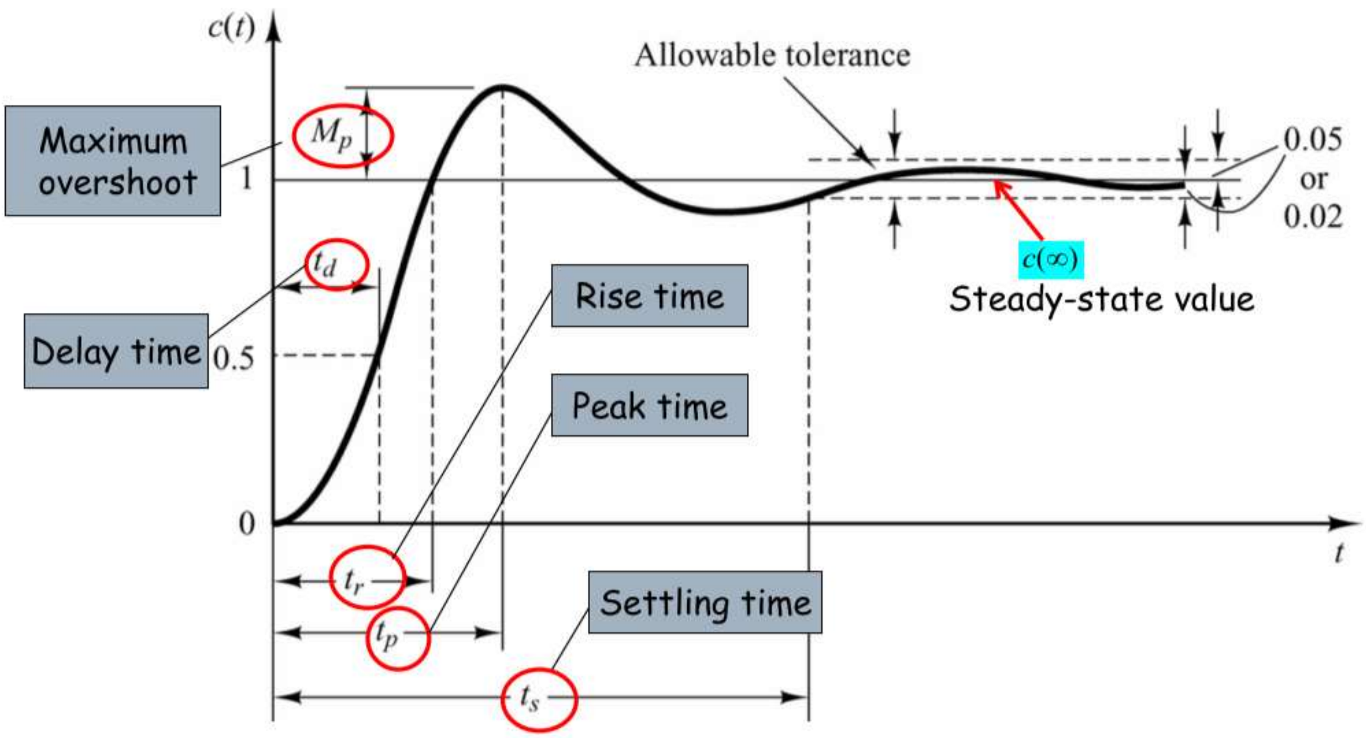
- **transient response**: goes from the initial state to the final state
- **steady-state response**: the manner in which the system output behaves as $t \to \infty$

$$y(t) = \mathscr{L}^{-1}\Big[G(s)H(s)\Big] = y_t(t) + y_s(t)$$

in other words

$$\text{time response} = \text{transient response} + \text{steady-state response}$$

Transient Response



The transient performance could be measured in terms of the transient response to a unit-step input, and there are some specifications

- t_d: **delay time**, the time of the response to reach half the final value

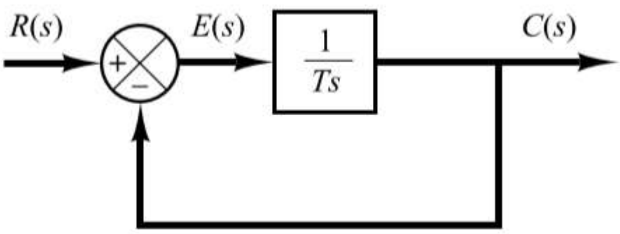
- t_r: **rise time**, the time of the response rise from 10% to 90% or 0% to 100% of its final value
- t_p: **peak time**, the time required for the response to reach the first peak of the overshoot
- t_s: **setting time**, the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 5% or 2%)
- $\sigma\%$: **maximum overshoot**, it is defined by $[c(t_p) - c(\infty)]/c(\infty) \times 100\%$

> **Response speed** is measured by rise time, delay time and setting time
> **Relative stability** is measured by percent overshoot

And there could also be error after the transient response has delayed, leaving only the continuous response

$$e_{ss} = \lim_{t\to\infty} e(t) = e(\infty)$$

3-2 First-Order Systems



Consider the first-order system in the figure above, the input-output relationship is given by

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{Ts}}{1+\frac{1}{Ts}} = \frac{1}{Ts}$$

Unit-Step Response of First-Order Systems

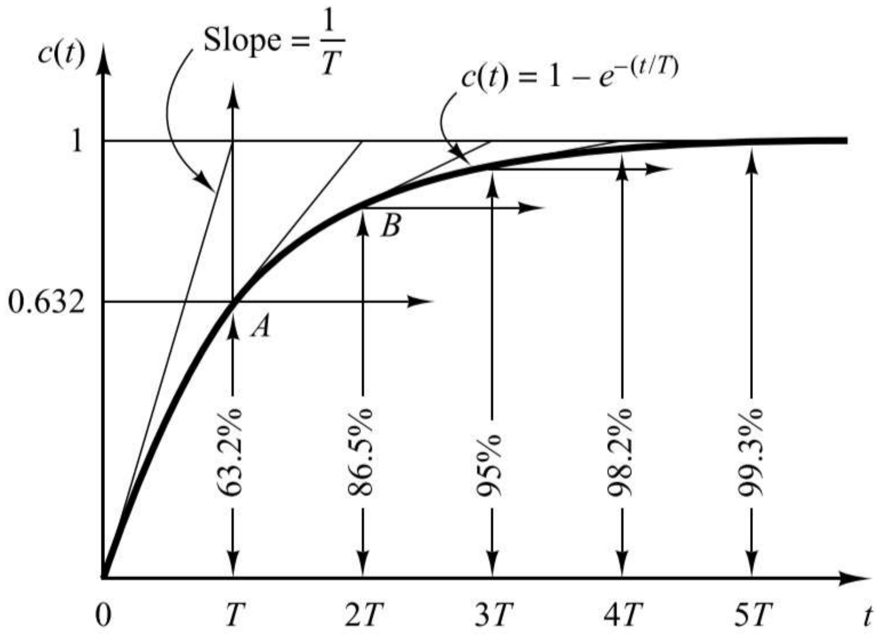
Since the Laplace transform of the unit step function is $1/s$, the unit step response of the system is

$$C(s) = \frac{1}{Ts+1}\frac{1}{s} = \frac{1}{s} - \frac{1}{s+(1/T)}$$

Taking the inverse Laplace transform, we could obtain that

$$c(t) = 1 - e^{-t/T}$$

- 1: steady-state response
- $e^{-t/T}$: transient response

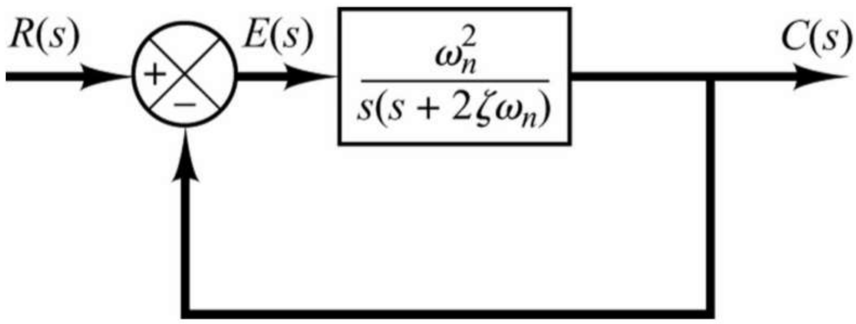


- Steady-State Error is 0
- Setting time is about 3T to 4T
- Smaller the time constant T is, the faster the system response

3-3 Second-Order Systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- ω_n: undamped natural frequency
- ζ: damping ratio



which has the characteristic equation of

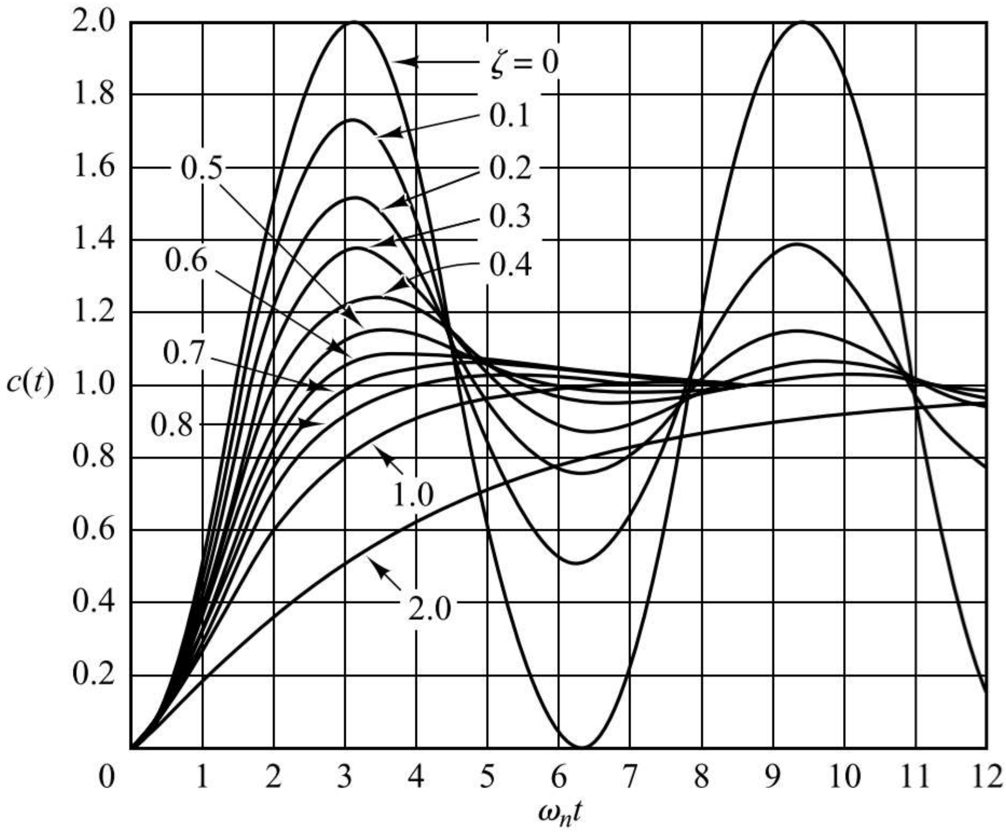
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

and the closed loop poles

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Classifications of second order systems	The number of ζ
Overdamped Case	$\zeta > 1$

Classifications of second order systems	The number of ζ
Critically Damped Case	$\zeta = 1$
Underdamped Case	$0 < \zeta < 1$
Undamped Damped Case	$\zeta = 0$



Overdamped Case

Since the poles are

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta_2 - 1}$$

which means they are all real

And the output gives the following response

$$c(t) = 1 + \frac{1}{\frac{s_1}{s_2} - 1}e^{s_1 t} + \frac{1}{\frac{s_2}{s_1} - 1}e^{s_2 t}$$

which includes two delaying exponential terms

> If s_1 is located very much closer to the $j\omega$ asia than s_2, then the effect of s_2 on the response is much smaller than that of s_1, which response is similar to that of a **first-order** system

Critically Damped Case

The poles become to

$$s_{1,2} = -\omega_n$$

And the output gives the following response

$$C(t) = 1 - (1 + \omega_n t)e^{-\omega_n t}$$

Underdamped Case

There are two complex conjugate poles for this case

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

And the unit-step response becomes

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \beta)$$

- β = $\arccos(\zeta)$
- damped natural frequency: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\sigma\% = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\% = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

and the setting time of it are

$$t_s = \frac{3}{\zeta\omega_n}(\Delta = 0.05)$$
$$t_s = \frac{4}{\zeta\omega_n}(\Delta = 0.02)$$

Undamped Case

There're two poles both on $j\omega$ axis

$$s_{1,2} = \pm j\omega_n$$

And the unit-step response is

$$c(t) = 1 - \cos\omega_n t$$

Remarks

- if two second-order systems have the same ζ but different ω_n, they will exhibit the same overshoot asn the same oscillatory, which is called to have the **same relative stability**
- underdamped system with ζ between 0.5 and 0.8 gets close to the final value more rapid ly than a critically or overdamped system
- overdamped system is always sluggish in responding to any inputs

3-5 Stability Analysis of the n-th Order Systems

$$\frac{C(s)}{R(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{N(s)}{D(s)}$$

For the unit step response

$$C(s)=R(s)G(s)=\frac{1}{s}\frac{N(s)}{D(s)}=\frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{q}(s+p_j)\prod_{k=1}^{r}(s+\alpha_k)^2+\beta_k^2}$$

In the time domain

$$c(t)=a+\sum_{j=1}^{q}b_j e^{-p_j t}+\sum_{k=1}^{r}A_k e^{-\alpha_k t}\sin(\beta_k t+\theta_k)$$

- real poles contribute exponential terms
- complex pair of poles contribute damped oscillations
- magnitude of contribution depends on residues

Stability of System Response

The transient term will converge to zero if and only if **all poles** are on the **left-hand of plains** (LHP), and the further to the left on the LHP for the poles, the faster the convergence.

Method 1: Direct Factorization

Solve for solutions of characteristic equation

$$a_0 s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n=0$$

and check if all of them are on the LHP

Method 2: Routh's Stability Criterion

Determine the locations of roots without having to solve the equation

- step 1: determine if all the coefficients of the characteristic equation **have the same sign** and are **nonzero** or it is unstable
- step 2: if all coefficients are positive, arrange all coefficients in rows and columns to the following pattern, construct the Routh table

power of s	column 1	column 2	column 3	$\cdots$
s^n	a_0	a_2	a_4	$\cdots$
s^{n-1}	a_1	a_3	a_5	$\cdots$
s^{n-2}	$b_1=\frac{a1a2-a0a3}{a1}$	$b_2=\frac{a1a4-a0a5}{a1}$	$\cdots$	$\cdots$
s^{n-3}	$c_1=\frac{b1a3-a1b2}{b1}$	$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
s^2	e_1	e_2		

power of s	column 1	column 2	column 3	$\cdots$
s^1	f_1	0		
s^0	$g = \frac{f_1 e_2 - e_1 \cdot 0}{f_1}$			

Special Case 1

If the first element in any row of Rough's array is zero, but the others are not.

The zero element in the first column should be replaced by small positive number of ϵ, then proceed with Rough's array

$$s^3 + 2s^2 + s + 2 = 0$$

the array of coefficients is

$$\begin{array}{lll} s^4 & 1 & 1 & 1 \\ s^3 & 2 & 2 \\ s^2 & \varepsilon > 0 & 1 \\ s^1 & 2 - \dfrac{2}{\varepsilon} \\ s^0 & 1 \end{array}$$

Special Case 2

An entire row of the Rough array may become zero, which indicates that there are roots of equal magnitude lying radially opposite in the s-plane

The situation can be remedied by forming an **auxiliary polynomial** with the coefficients of the last row.

$$s^6 + 2s^5 + 7s^4 + 12s^3 + 14s^2 + 16s + 8 = 0$$

the array of coefficients is

$$
\begin{array}{lllll}
s^6 & 1 & 7 & 14 & 8 \\
s^5 & 2 & 12 & 16 & 0 \\
s^4 & 1 & 6 & 8 & \\
s^3 & 0 & 0 & 0 & \\
s^3 & 4 & 12 & & \\
s^2 & 3 & 8 & & \\
s^1 & 4/3 & & & \\
s^0 & 8 & & &
\end{array}
$$

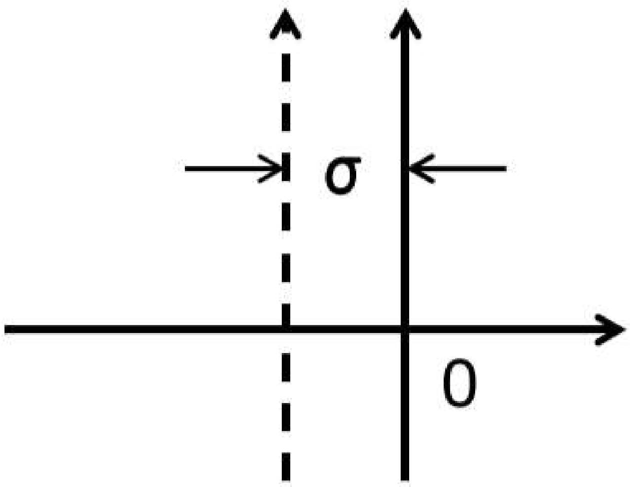
the auxiliary polynomial is

$$p(s) = s^4 + 6s^2 + 8$$

$$\frac{\mathrm{d}p(s)}{\mathrm{d}s} = 4s^3 + 12s$$

Relative Stability Analysis

Apply Rough's stability criterion to the shifted s-palne axis



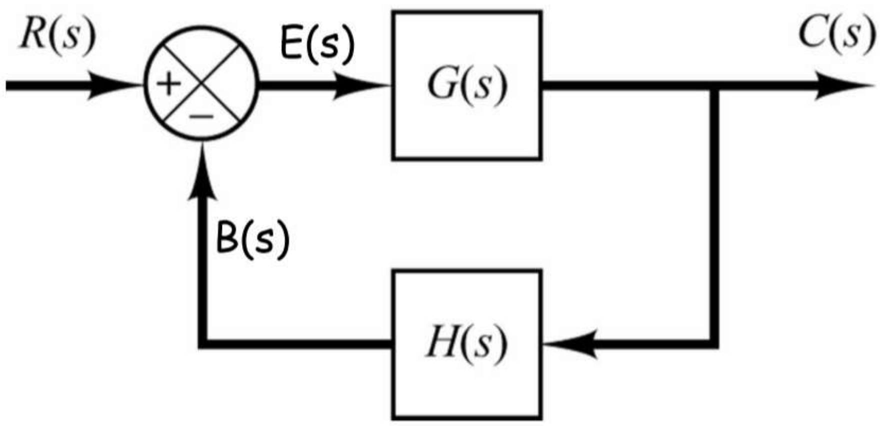
Substitute $s = z - \sigma$ into the Rough's stability criterion

Useful Tips for Stability

System	Equation	Tip
First order system	$a_0 s + a_1 = 0$	a_1 and a_0 have the same sign
Second order system	$a_0 s^2 + a_1 s + a_2 = 0$	a_2, a_1 and a_0 have the same sign

System	Equation	Tip
Third order system	$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$	a_3, a_2, a_1 and a_0 have the same sign and $a_1 a_2 > a_0 a_3$

3-6 Steady-State Errors in Feedback



$$e(t) = r(t) - b(t)$$

- $e(t)$: error
- $r(t)$: reference input
- $b(t)$: feedback signal

Since the transfer function from the input or the disturbance to the error signal is

$$\Phi_e(s) = \frac{E(s)}{R(s)} \qquad \Phi_{en}(s) = \frac{E(s)}{N(s)}$$

Using the final value theorem to obtain the steady-state error

$$e_{ss} = \lim_{s \to 0} s[\Phi_e(s)R(s) + \Phi_{en}(s)N(s)]$$

The **steady-state error** is defined as

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\Phi_e(s)R(s) = \lim_{s \to 0} s \cdot \frac{1}{1 + G(s)H(s)} \cdot R(s)$$

> All roles of $sE(s)$ must lie on the left-half of the s-plane, which means the system is stable

Classification of Control Systems

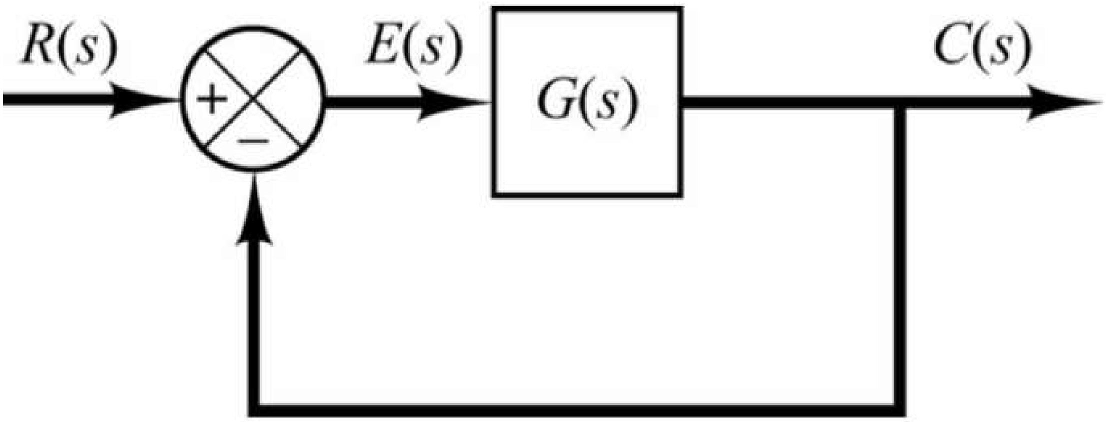
Consider a feed back control system with following open-loop transfer function

$$G(s)H(s) = \frac{K_k \prod_{i=1}^{m}(T_i s + 1)}{s^N \prod_{j=1}^{n-N}(T_j s + 1)}$$

- K_k: open-loop gain
- N: the number of integrations

the number of N	the system integration
0	type 0 system
1	type 1 system
2	type 2 system

Steady-State Errors in Unity-Feedback Control Systems



$$e(t) = r(t) - c(t)$$

The **steady-state error** is defined as

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Unit-Step Input

The input signal is the unit-step signal, where $R(s) = \frac{1}{s}$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\Phi_e(s)R(s) = s \cdot \frac{1}{1+G(s)} \cdot \frac{1}{s} = \frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+K_p}$$

> $K_p = \lim_{s\to 0} G(s)$, which is defined as the static position error constant

Type 0 System

For the type 0 system, there is no difference between static position error constant

$$K_p = K_k$$

Therefore, the steady-state error becomes to

$$e_{ss} = \frac{1}{1+K_k}$$

Type 1 System

According to the definition of the static position error constant

$$K_p = \lim_{s \to 0} \frac{K_k \prod_{i=1}^{m}(T_i s + 1)}{s \prod_{j=1}^{n-N}(T_j s + 1)} = \infty$$

Then the steady-state error becomes to

$$e_{ss} = 0$$

Type 2 System

Similar to the type 1 system

$$K_p = \infty \qquad e_{ss} = 0$$

Unit-Ramp Input

The input signal is the unit-stamp signal, where $R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\Phi_e(s)R(s) = \lim_{s \to 0} s \cdot \frac{1}{1 + G(s)} \cdot \frac{1}{s^2} = \lim_{s \to 0} \frac{1}{sG(s)} = \frac{1}{K_v}$$

> $K_v = \lim_{s \to 0} sG(s)$, which is defined as the static velocity error constant

Type 0 System

For the type 0 system, the steady-state error becomes to

$$K_v = \lim_{s \to 0} sG(s) = 0$$

Therefore, the steady-state error becomes to

$$e_{ss} = \infty$$

Type 1 System

There is no difference between the

$$K_v = K_k$$

Then the steady-state error becomes to

$$e_{ss} = \frac{1}{K_k}$$

Type 2 System

$$K_v = \infty \qquad e_{ss} = 0$$

Unit-Parabolic Input

The input signal is the unit-parabolic signal $r(t) = \frac{1}{2}t^2$, where $R(s) = \frac{1}{s^3}$

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\Phi_e(s)R(s) = \lim_{s \to 0} s \cdot \frac{1}{1 + G(s)} \cdot \frac{1}{s^3} = \lim_{s \to 0} \frac{1}{s^2G(s)} = \frac{1}{K_a}$$

> $K_a = \lim_{s \to 0} s^2G(s)$, which is defined as the static acceleration error constant

Type 0 System

For the type 0 system, the steady-state error becomes to

$$K_a = \lim_{s \to 0} s^2 G(s) = 0$$

Therefore, the steady-state error becomes to

$$e_{ss} = \infty$$

Type 1 System

There is no difference between the

$$K_a = \lim_{s \to 0} s^2 G(s) = 0$$

Then the steady-state error becomes to

$$e_{ss} = \infty$$

Type 2 System

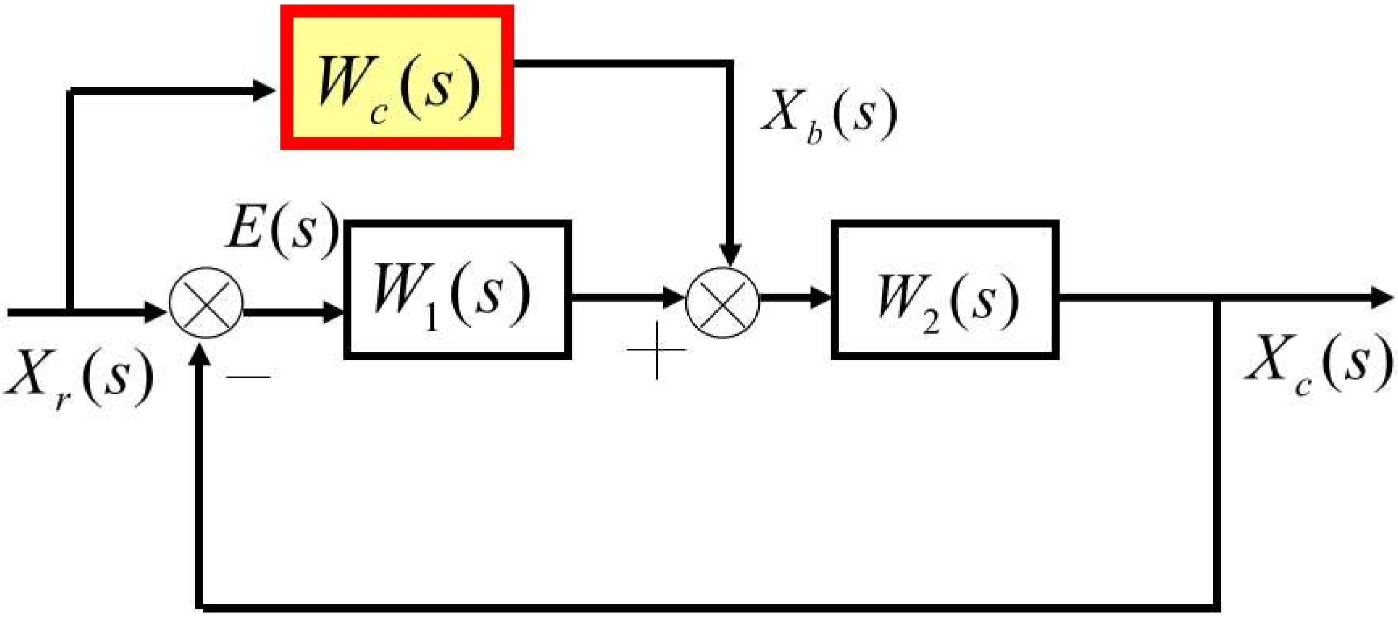
$$K_a = K_k \qquad e_{ss} = \frac{1}{K_k}$$

System Type	Step Input	Ramp Input	Acceleration Input
Type 0	$\frac{1}{K_k+1}$	∞	∞
Type 1	0	$\frac{1}{K_k}$	∞
Type 2	0	0	$\frac{1}{K_k}$

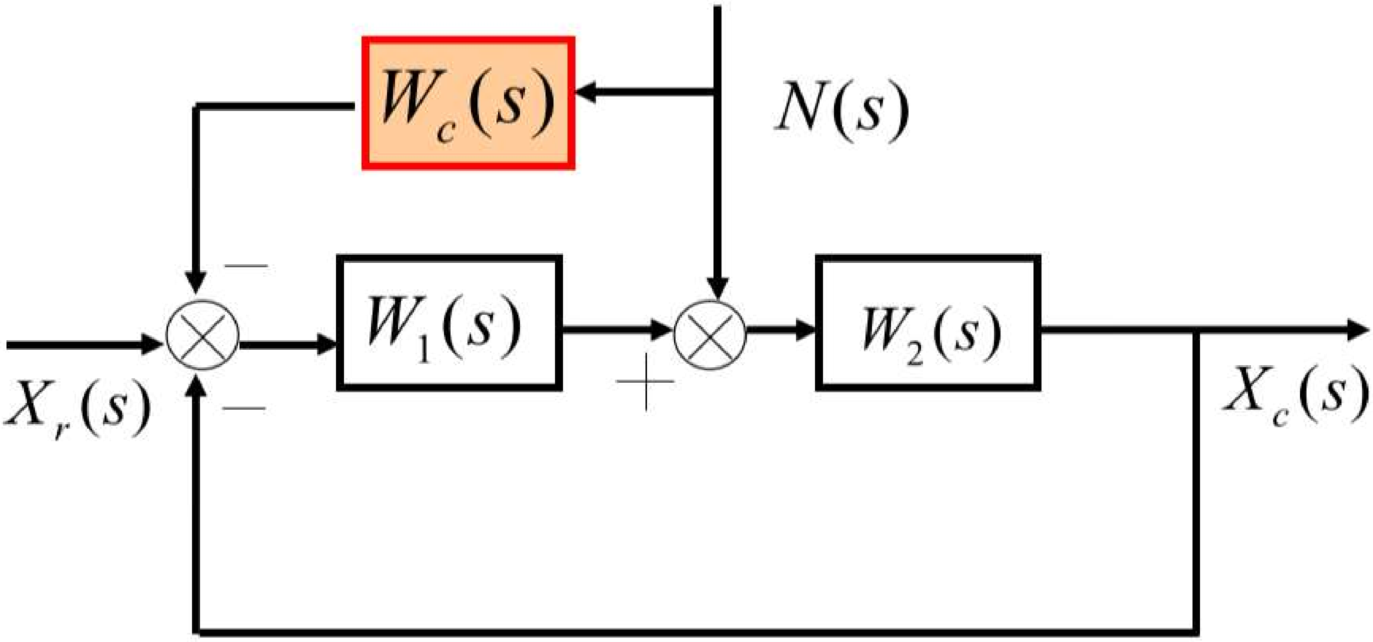
Reduce or Eliminate the Steady State Error

- Method 1: increase the open-loop gain
- Method 2: increase the tpe of the system by adding a integrator or integrators to the feedforward path
- Method 3: feedforward compensation

Feedforward Compensation of Input



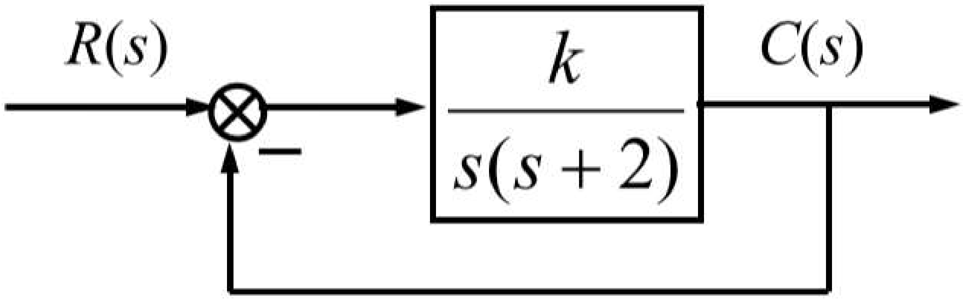
Feedforward Compensation of Disturbance



Chapter 4 Root Locus

4-1 Introduction of the Root Locus Method

Taking the system in the following figure into consideration, we want to figure out that how do the characteristic roots change according to the different values of k



The open-loop transfer function is

$$G(s) = \frac{k}{s(s+2)}$$

And the characteristic equation of the closed-loop system is

$$1 + G(s) = 1 + \frac{k}{s(s+2)} = 0 \Longrightarrow s^2 + 2s + k = 0$$

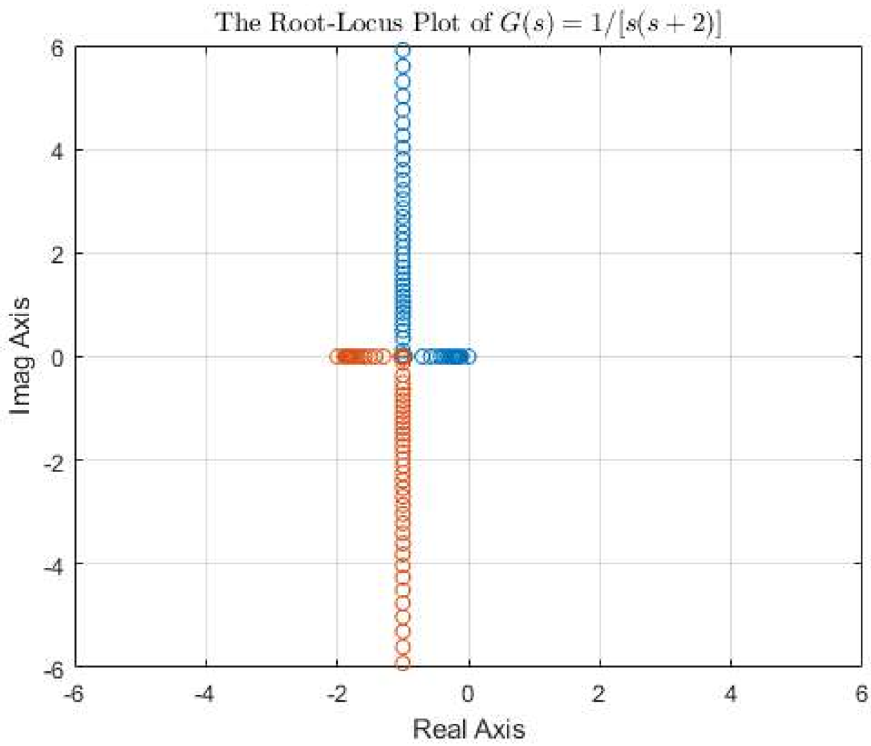
And the roots of the characteristic equation are

$$s_1 = -1 + \sqrt{1-k} \qquad s_2 = -1 - \sqrt{1-k}$$

And the analysis of the roots are

$$s_{1,2} = -1 \pm \sqrt{1-k} \qquad \begin{cases} s_1 = 0 \quad s_2 = -2 & k = 0 \\ \text{Real and different} & 0 < k < 1 \\ s_1 = s_2 = -1 & k = 1 \\ s_{1,2} = -2 \pm j\sqrt{k-1} & k > 1 \end{cases}$$

We could sketch the locus of the roots varying with k from 0 to ∞ on the s-plane



The following words introduce the definition of the **root locus**

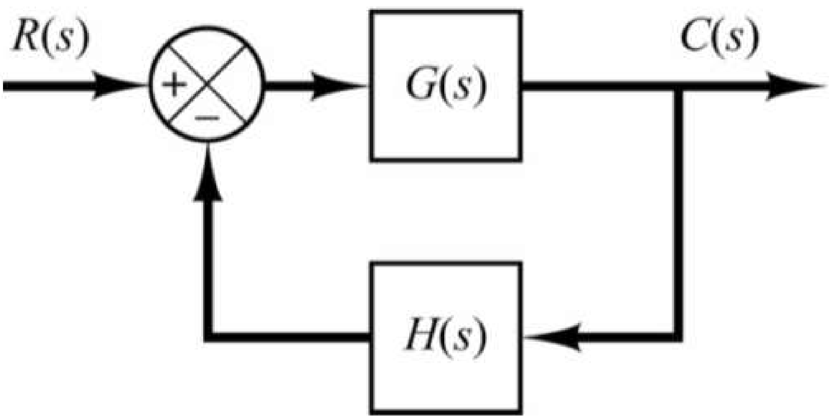
The path of the **roots of the characteristic equation** traced out in the s-plane as a system parameter is changed

We could use the root locus to determine the system performance

- stability: the system is always stable since all of the poles lie in the left-half of s-plane
- transient performance:
  - $0 < k < 1$: overdamped case
  - $k = 1$: critically damped case
  - $k > 1$: underdamped case
- steady-state performance: type 1 system

4-2 The Root-Locus Equation

Consider the following negative feedback system



The characteristic equation of the system is

$$1+G(s)H(s)=0$$

Where $G(s)H(s)$ could be rewritten as

$$G(s)H(s)=\frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}=-1$$

The equation above is also called a the **root locus equation**

and could be expanded into other two conditions

$$\begin{cases} K\frac{\prod_{i=1}^{m}|(s+z_i)|}{\prod_{i=1}^{n}|(s+p_i)|}=1 & \text{Magnitude Condition} \\ \sum_{i=1}^{m}\angle(s+z_i)-\sum_{j=1}^{n}\angle(s+p_j)=\pm(2k+1)\pi & \text{Angle Condition} \end{cases}$$

- Magnitude Condition: the condition is used to determine the value of K for a given root location
- Angle Condition: the condition for a point s which is on the root locus

4-3 General Rules for Sketching Root Locus Plot

Prepare Rules

1. The characteristic equation $1+G(s)H(s)=0$ should be written in the very beginning
2. Factor the equation into the form of poles and zeros
3. Locate the poles and zeros of the equation in s-plane with specified symbols
    - x: poles
    - o: zeros

1 - Beginning and Ending Points

The locus of the roots of the characteristic equation begins at **poles** and ends at **zeros**

If the number of the open-loop zeros m is less than the number of the open-loop poles n, then there're $n-m$ root loci **terminating at the infinity**

2 - Number and Symmetry

The number of loci is **equal to** the number of open loop poles

The root locus is **continuous** and **symmetrical** with respect to the real axis

3 - Loci on Real Axis

The root locus on the real axis always lies in a section of the real axis where there are **odd numbers of poles and zeros to the right**

4 - Asymptotes

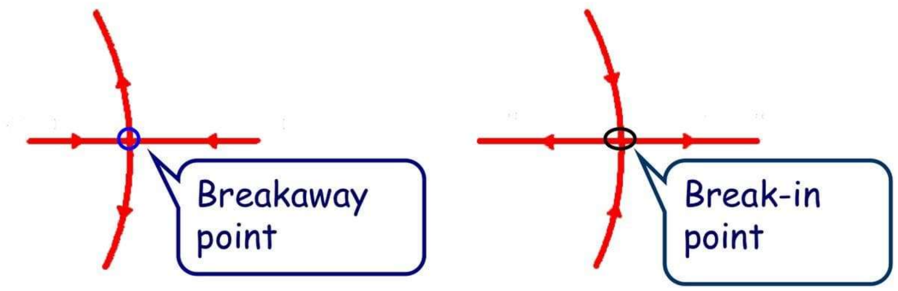
The root locus proceed to the infinity along asymptotes **centered** at σ_A with **angles** of ϕ_A

$$\sigma_A = \frac{\sum_{i=1}^{n}(\text{pole}_i) - \sum_{i=1}^{m}(\text{zero}_i)}{n-m}$$

$$\phi_A = \frac{(2k+1)\pi}{n-m}$$

5 - Breakaway and Break-in Points on the Axis

The breakaway or break-in points correspond to multiple roots of the characteristic equation



$$G(s)H(s) = \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)} = \frac{KN(s)}{D(s)} = -1$$

The points could be find by

$$\begin{cases} \frac{\mathrm{d}K}{\mathrm{d}s} = 0 \\ K = -\frac{D(s)}{N(s)} \end{cases} \Longrightarrow \frac{\mathrm{d}K}{\mathrm{d}s} = -\frac{D'(s)N(s) - D(s)N'(s)}{N(s)^2}$$

6 - Departure Angle and Arrival Angle

According to the angle condition, the angles of departure or arrival could vary in different conditions

- Angle of Departure: $180^\circ - (\sum \text{angles of the vectors from other poles to the pole} - \sum \text{angles of the vectors from other zeros to the pole})$
- Angle of Arrival: $180^\circ - (\sum \text{angles of the vectors from other zeros to the pole} - \sum \text{angles of the vectors from other poles to the pole})$

7 - Crossing Points with the Imaginary Axis

The points where the root loci intersect the imaginary axis can be found by letting $s = j\omega$ in the **characteristic equation**, equating both **real part** and **imaginary part** to **zero**

8 - Sum of the Roots

If the number of poles is larger than the number of the poles, where $n - m \geq 2$, the sum of the roots **remains a constant**

Chapter 5 Frequency Response Analysis

5-1 Frequency Response

Introduction

Consider the low-pass filter built by RC circuit, the transfer function of the filter is

$G(s) = \frac{1}{CRs+1}\Big|_{T=CR} = \frac{1}{Ts+1}$

if the input voltage signal is $A\sin(\omega t)$

$$U_o(t) = \frac{A\omega T}{1+\omega^2T^2}e^{-\frac{t}{T}} + \frac{A}{\sqrt{1+\omega^2T^2}}\sin(\omega t - \arctan\omega T)$$

Where the latter component is called as the **frequency response**

Definition

The ratio of the complex vector of the steady-state output versus sinusoid input for a linear system

$$G(j\omega) = \frac{C(j\omega)}{R(j\omega)}$$

- $C(j\omega)$: complex vector representation of the output
- $R(j\omega)$: complex vector representation of the input

Or it could be rewritten as

$$G(j\omega) = \|G(j\omega)\|\angle G(j\omega)$$

- $A(j\omega)$: magnitude response
- $\phi(j\omega)$: phase response

$$\|G(j\omega)\| = \frac{\|C(j\omega)\|}{\|R(j\omega)\|}$$

$$\angle G(j\omega) = \angle C(j\omega) - \angle R(j\omega)$$