Chapter 2

2-1 Transfer Function and Impulse Response Function

Transfer Function

The *transfer function* of a linear, time-invariant, differential equation system defined as the **ratio** of the **Laplace transform of the output** to the **Laplace transform of the input** under the assumption that all initial conditions are zero

Consider the linear time-invariant system defined by the following differential equation

$$\begin{aligned} a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y \\ = b_0 x^{(m)} + b_1 x^{(m-1)} + \cdots + b_{m-1}\dot{x} + b_m x \end{aligned}$$

where y is the output of the system and x is the input

$$\begin{aligned} \text{Transfer Function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}} \\ = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{N(s)}{D(s)} \end{aligned}$$

> 1. the concept of transfer function is only appropriate to the LTI system
> 2. transfer function is only determined by the structure and parameter of system
> 3. if the highest power of s in the **denominator** of the TF is equal to n, the system is called as **n-th system**

- **characteristic polynomial**: the denominator polynomial $D(s)$
- **characteristic equation**: the formula of $D(s) = 0$
- **zeros**: the roots of the **numerator polynomial** $N(s)$
- **poles**: the roots of the **denominator polynomial** $D(s)$

Example

$$G(s) = \frac{K(2s+1)}{s(3s+1)(T^2s^2+2\xi Ts+1)}$$

which could be rewritten as

$$G(s) = K \cdot \frac{1}{s} \cdot (2s+1) \cdot \frac{1}{3s+1} \cdot \frac{1}{T^2s^2+2\xi Ts+1}$$

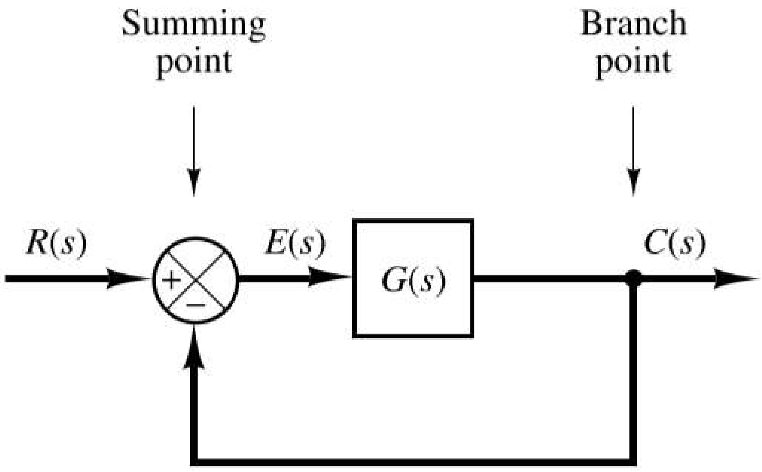
- K: gain factor
- $1/s$: integral factor
- $2s+1$: first-order differential factor (differential factor)

- $1/(3s+1)$: inertial element (reciprocal first-order)
- $1/(T^2s^2+2\xi Ts+1)$: quadratic factor

2-2 Automatic Control Systems

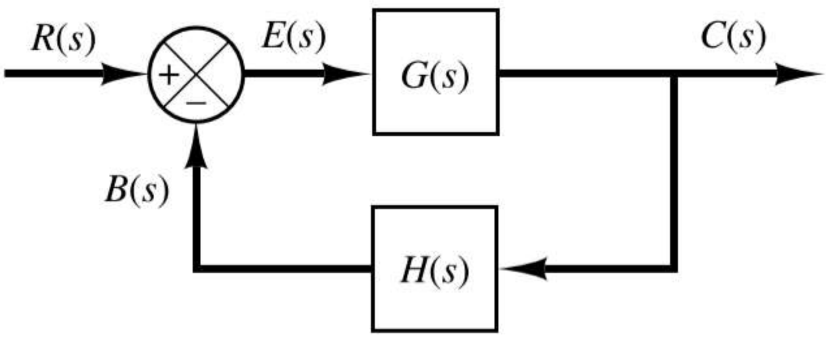
Block Diagrams

A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals. Such a diagram depicts the interrelationships that exist among the various components.



- **signal line**: a line with arrow that indicate the direction of signal transform
- **block**: it expresses the transfer function
- **summing point**: a circle with a cross is the symbol that indicates a summing operation
- **branch point**: a point from which the signal from a block goes concurrently to other blocks or summing points

Block Diagram of a Closed-Loop System



$$C(s) = G(s)E(s)$$

$$\begin{aligned} E(s) &= R(s) - B(s) \\ &= R(s) - H(s)C(s) \end{aligned}$$

Eliminating $E(s)$ from the equations above

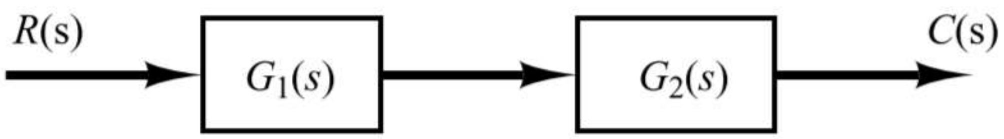
$$C(s) = G(s)[R(s) - H(s)C(s)]$$

or

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

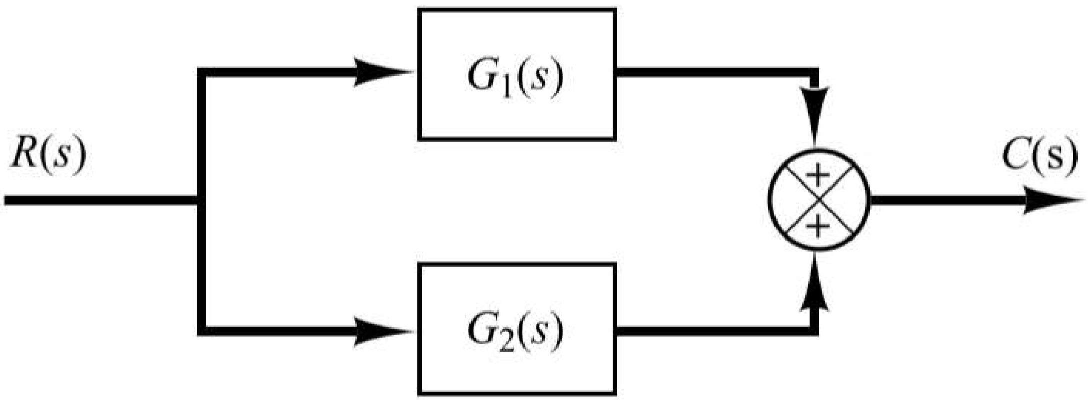
Reduce Block Diagrams

Cascaded Systems



$$\frac{C(s)}{R(s)} = G_1(s)G_2(s)$$

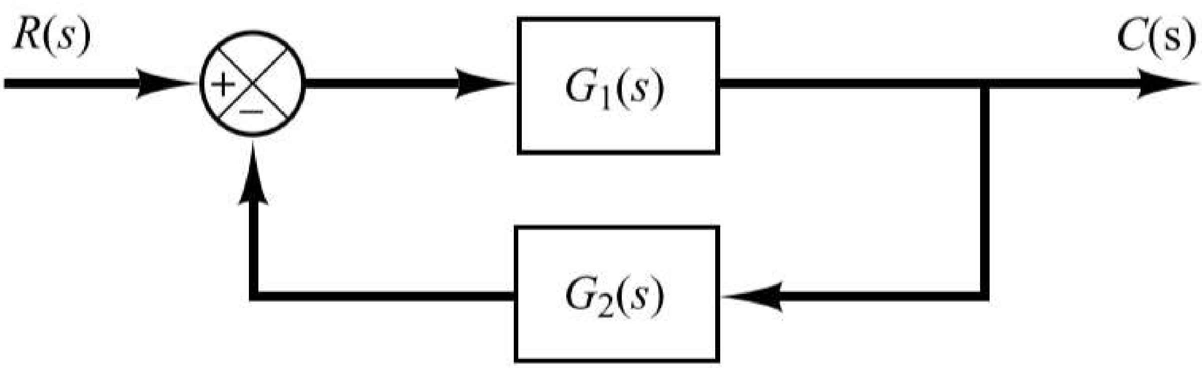
Parallel Connected System



$$\frac{C(s)}{R(s)} = G_1(s) + G_2(s)$$

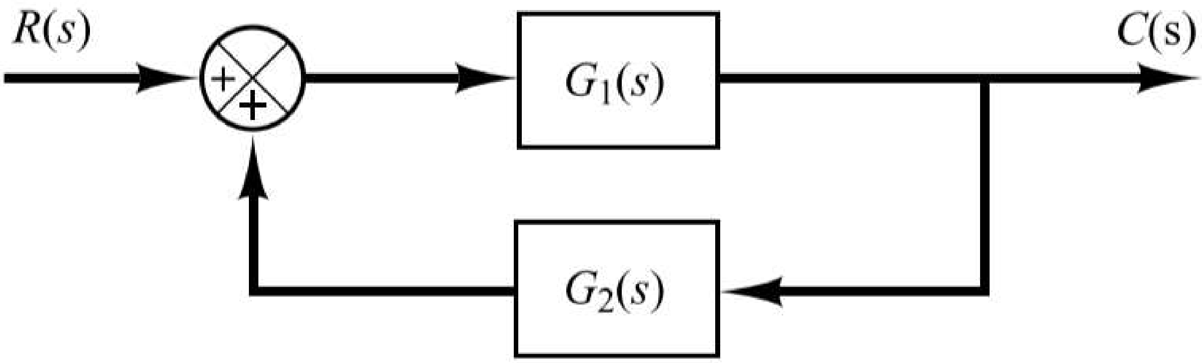
Feedback System

Negative Feedback System



$$\frac{C(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

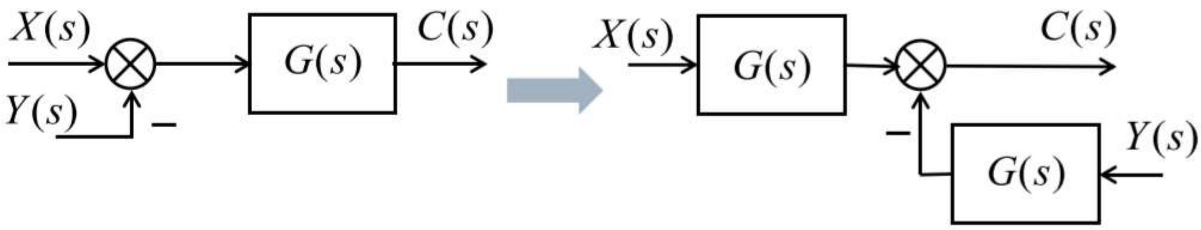
Positive Feedback System



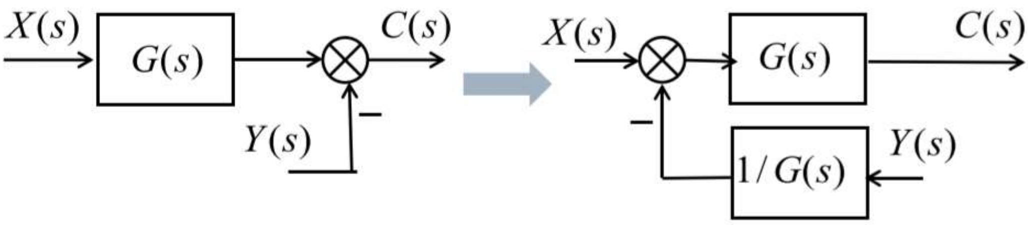
$$\frac{C(s)}{R(s)} = \frac{G_1(s)}{1 - G_1(s)G_2(s)}$$

Slide a Summing Point

Backward

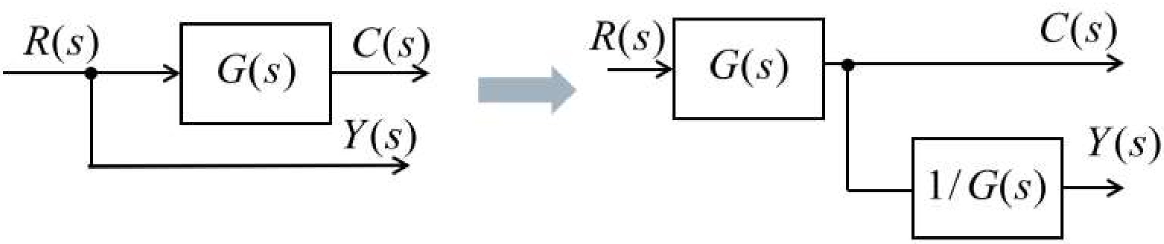


Forward

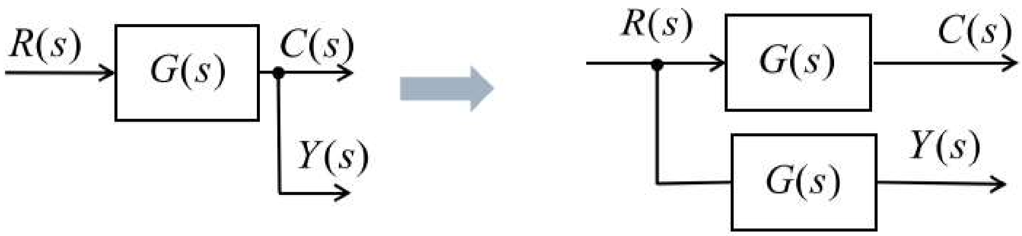


Slide a Branch Point

Backward

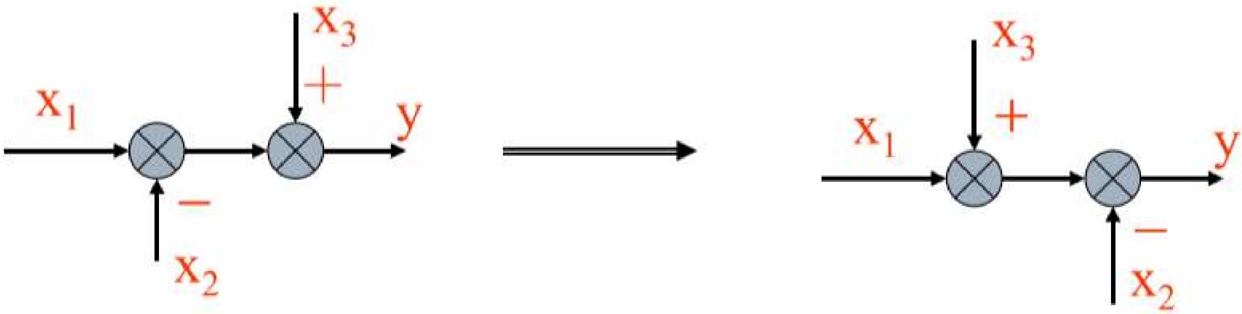


Forward

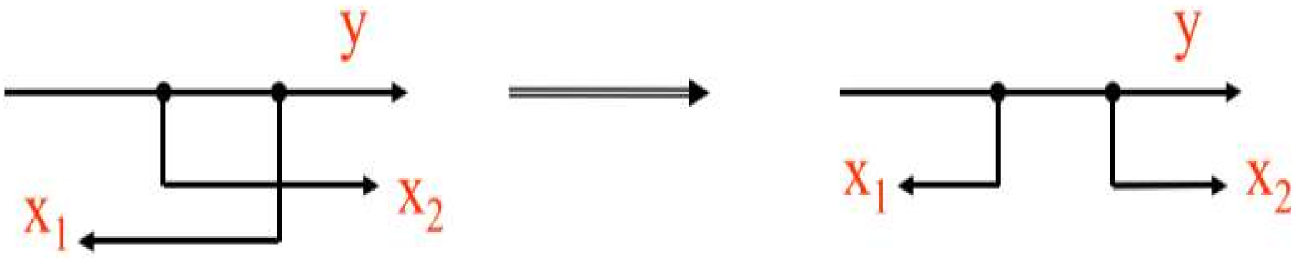


Interchanging the Neighboring

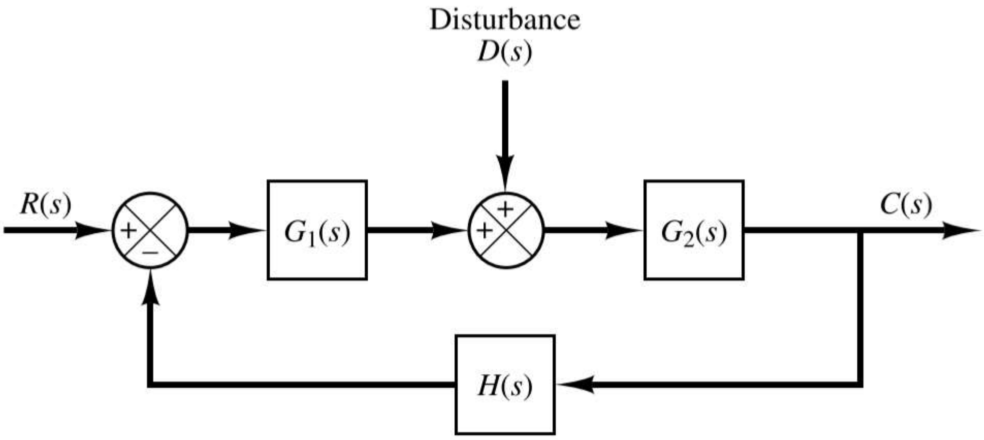
Summing Points



Branch Points



Closed-Loop System Subjected to a Disturbance



We could calculate the response $C_D(s)$ to the disturbance only

$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

and the response to the reference input $R(s)$

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

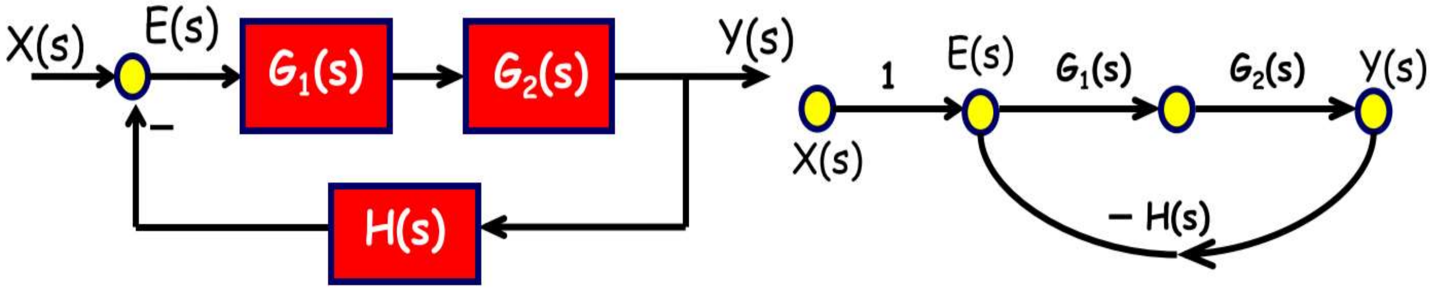
Then the response could be the sum of two responses

$$C(s) = C_D(s) + C_R(s)$$

$$= \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}[G_1(s)R(s) + D(s)]$$

2-3 Signal-Flow Graph Models

A signal-flow graph is a specialized flow graph, a directed graph in which nodes represent system variables, and branches represent functional connections between pairs of nodes.



Basic Components

- **node**: represents a signal
  - input nodes: nodes with only outgoing branches
  - output nodes: nodes with only incoming branches
  - mixed nodes: nodes with both incoming and outgoing branches
- **branch**: directed line segment connecting two nodes
  - signal can only flow along the specified direction
  - each branch is associated with a gain, which is the transfer function
- **path**: a sequence of connected branches
  - forward path: start from an input node and end at and output node
  - forward path gain: product of all branch gains along a forward path
- **loop**: a closed path
  - loop gain: product of all branch gains along a lop
  - nontouching loop: loops that do not have shard nodes
  - touching loop: more than one loop sharing one orr more common nodes

Mason's Gain Formula

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{\Delta}\sum_{k=1}^{n} P_k \Delta_k$$

- Δ: determinant of the graph

$$\Delta = 1 - (\text{sum of all individual loop gains}) + (\text{sum of gain products of all two nontouching loop})$$
$$-(\text{sum of gain products of all three nontouching loop}) + \cdots$$

- P_k: path gain of kth forward path
- Δ_k: factor of the kth forward path