Overview

Part I Sampling

1-1 Sampling of Analog Signal

Consider a continuous-time signal $x_a(t)$ and a discrete-time signal $x[n]$ is obtained by periodic sampling.

Let T be the **sampling period**, then we define $x[n] = x_a(nT)$

If $x_a(t) = A\cos(2\pi Ft + \phi)$ and the signal is sampled at a rate of $F_s = \frac{1}{T}$

then $x[n] = A\cos(2\pi FTn + \phi) = A\cos(2\pi nf + \phi)$

Normalized Frequency

the signal with an analog frequency of F is sampled at a rate of F_s

then the relationship between the **normalized frequency** f and the **analog frequency** F is

$$f = \frac{F}{F_s}$$

1-2 Sampling Theorem

- If maximum frequency of the signal is known to be F_{max}, the **minimum** sampling rate would be

$$\text{Nyquist Rate} = 2 \times F_{max}$$

- If the sampling rate is know to be F_s, the **maximum** frequency in the **signal** must not exceed

$$\text{Nyquist Rate} = \frac{1}{2}F_S$$

Part II Discrete-Time Signals and Systems

2-1 Classification of Discrete-Time Signal and System

Energy and Power Signals

$$E = \sum_{-\infty}^{\infty} ||x[n]||^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{x=-N}^{N} ||x[n]||^2$$

Periodic Signals

$$x(n+N) = x(n)$$

Even and Odd Signals

$$\text{Even}: \quad x[-n] = x[n]$$

$$\text{Odd}: \quad x[-n] = -x[n]$$

Systems

- **FIR**: **finite** impulse response system
- **IIR**: **infinite** impulse response system

> $x[n] \to \delta[n] \qquad y[n] \to h[n]$

2-2 Properties of Discrete-Time Signals

Linear Property

$$x_1[n] \to y_1[n] \qquad x_2[n] \to y_2[n]$$

$$ax_1[n] + bx_2[n] \longrightarrow ay_1[n] + by_2[n]$$

Displacement Invariant Property

$$x[n] \to y[n]$$

$$x[n-n_0] \to y[n-n_0]$$

Causality

The output of the system at some time instant only depends on the time before.

In other words,

$$h(n) = 0 \qquad n < 0$$

Stability

The input is bounded as well as the output is bounded

$$||x[n]|| \leq M_x < \infty$$

$$||y[n]|| \leq M_y < \infty$$

2-3 Convolution of Discrete-Time Systems

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[n]h[n-k]$$

2-4 Correlation Sequences

the **cross correlation** of two energy $x(n)$ and $y(n)$ is defined as

$$r_{xy}(l)=\sum_{n=-\infty}^{\infty}x(n)y(n-l)$$

Properties

- $r_{yx}(l)=r_{xy}(-l)$
- $r_{xx}(l)=\sum_{n=-\infty}^{\infty}x(n)x(n-l)$
- $r_{xx}(l)=r_{xx}(-l)$
- $r_{xy}(l)=x(l)*y(-l)$

Normalized Correlation Sequences

the correlation sequences are often normalized to the range $[-1,1]$

$$\rho_{xy}(l)=\frac{r_{xy}(l)}{\sqrt{r_{xx}(0)r_{yy}(0)}}$$

Correlation of Periodic Sequences

$$r_{xy}(l)=\frac{1}{N}\sum_{n=0}^{N-1}x(n)y(n-l)$$
$$r_{xx}(l)=\frac{1}{N}\sum_{n=0}^{N-1}x(n)x(n-l)$$

Detection of Hidden Periodicity

For a periodic signal $s(n)$, suppose that the signal is buried in noise. Let $\omega(n)$ be the zero-mean white noise. What can we observe is the noisy signal $x(n)$

$$x(n)=s(n)+\omega(n)$$

Calculate the autocorrelation sequence of $x(n)$

$$r_{xx}(l)=r_{ss}(l)+r_{s\omega}+r_{\omega s}+r_{\omega\omega}(l)$$

Since $\omega(n)$ is a zero-mean white noise signal

$$r_{s\omega}(l)=0\qquad r_{\omega s}(l)=0\qquad r_{s\omega}(l)=0\qquad r_{\omega\omega}(l)=\sigma^2\delta(l)$$

Then we get

$$r_{xx}(l)=r_{ss}(l)+\sigma^2\delta(l)$$

Part III The z-Transform

3-1 The Direct z-Transform

the z-transform of a general discrete-time signal $x[n]$ is defined as

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

3-2 The Inverse z-Transform

$$X(z) = \sum_{i=1}^{m} \frac{A_i}{1 - a_i z^{-1}}$$

- ROC outside the pole: $\frac{A_i}{1-a_i z^{-1}} \leftrightarrow A_i a_i^n u[n]$
- ROC outside the pole: $\frac{A_i}{1-a_i z^{-1}} \leftrightarrow -A_i a_i^n u[-n-1]$

Power-series Expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

The coefficients in this power series are the sequence values $x[n]$ since $\delta[n+n_0] \leftrightarrow z^{n_0}$

3-3 Properties of the z-Transform

Linearity

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(z) + bX_2(z)$$

with the ROC containing $R_1 \cap R_2$

Time Shifting

$$x[n-k] \longleftrightarrow z^{-k} X(z)$$

with ROC = R, except for the possible addition or deletion of the origin or infinity

> For $n_0 > 0$, poles will be introduced at $z = 0$
> For $n_0 < 0$, zeros will be introduced at $z = 0$

Scaling in the z-Domain

$$a^n x[n] \longleftrightarrow X(a^{-1} z)$$

with ROC = $|a|R$

Time Reversal

$$x[-n] \longleftrightarrow X(z^{-1})$$

with ROC = $\frac{1}{R}$

Differentiation in the z-Domain

$$nx[n] \longleftrightarrow -z \frac{\mathrm{d}X(z)}{\mathrm{d}z}$$

with ROC = R

Convolution Property

$$x_1[n] * x_2[n] \longleftrightarrow X_1(z) X_2(z)$$

Initial-Value and Final-Value Theorems

if $x[n]$ is a casual sequence, then

$$x[0] = \lim_{z \to \infty} X(z)$$

Time-Domain	z-Domain	ROC
$\delta[n]$	1	all z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|\|z\|\| > 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|\|z\|\| > \|\|a\|\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|\|z\|\| < \|\|a\|\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|\|z\|\| > \|\|a\|\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|\|z\|\| < \|\|a\|\|$
$\cos(\omega_0 n)u[n]$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|\|z\|\| > 1$
$\sin(\omega_0 n)u[n]$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|\|z\|\| > 1$
$a^n \cos(\omega_0 n)u[n]$	$\frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|\|z\|\| > 1$
$\sin(\omega_0 n)u[n]$	$\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|\|z\|\| > 1$

3-4 Analysis of LTI System Using z-Transform

$$Y(z) = X(z)H(z)$$

where $H(z)$ is the system function/transfer function

Causality

Definition: if the ROC of its system function is the exterior of a circle, including infinity

A system with rational system with rational system function $H(z)$ is casual if and only if:

- the ROC is the exterior of a circle outside the outermost pole
- for $H(z)$, the order of numerator cannot be greater than the order of the denominator

Stability

Definition: the ROC of its system function $H(z)$ includes the unit circle, $|z| = 1$

A system with rational system function $H(z)$ is stable if and only if all of the poles of $H(z)$ lie inside the i=unit circle

3-5 The Inverse z-Transform Methods

- Inspection Method
  - using the properties and transform paris to get the inverse transform directly
- Power Series Method
  - has only few terms
  - can be found easily using Taylor series expansion/long division
- **Partial Fraction Expansion Method**
  - Simple poles

- Repeated poles

Part IV The Discrete Fourier Transform

4-1 N-Point DFT

The formula for the DFT is given by

$$X(k)=\sum_{n=0}^{N-1}x(n)W_N^{nk}\quad W_N=e^{-\frac{2j\pi}{N}}$$

and also could be expressed as the matrix form

$$\mathbf{x}_N=\begin{bmatrix}x[0]\\x[1]\\\cdots\\x[N-1]\end{bmatrix}\quad \mathbf{W}_N=\begin{bmatrix}1&1&1&\cdots&1\\1&W_N^1&W_N^2&\cdots&W_N^{N-1}\\1&W_N^2&W_N^4&\cdots&W_N^{2(N-1)}\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&W_N^{N-1}&W_N^{2(N-1)}&\cdots&W_N^{(N-1)(N-1)}\end{bmatrix}$$

where the N-point DFT could be expressed in the matrix form as

$$\mathbf{X}_N=\mathbf{W}_N\mathbf{x}_N$$

4-2 Circular or Linear Convolution Using DFT

For example, do the circular convolution of the following teo sequences

$$x_1[n]=\{2,1,2,1\}\qquad x_2[n]=\{1,2,3,4\}$$

Then we can get x_3 by

$$x_3[n]=\begin{bmatrix}1&4&3&2\\2&1&4&3\\3&2&1&4\\4&3&2&1\end{bmatrix}\begin{bmatrix}2\\1\\2\\1\end{bmatrix}=\{14,16,14,16\}$$

in other way

$$X_1(k)=2+W_4^k+2W_4^{2k}+W_4^{3k}\qquad X_2(k)=1+2W_4^k+3W_4^{2k}+4W_4^{3k}$$

then according to $Y(k)=X_1(k)X_2(k)$

$$Y(k)=2+5W_4^k+10W_4^{2k}+16W_4^{3k}$$

$$12W_4^{4k}+11W_4^{5k}+4W_4^{6k}=14+16W_4^k+14W_4^{2k}+16W_4^{3k}$$

Part V The Design of Filter

5-1 FIR Filter Design

Filter Type	Impulse Response
Lowpass	$\frac{\sin(\omega_c n)}{\pi n}$
Highpass	$\delta(n) - \frac{\sin(\omega_c n)}{\pi n}$
Bandpass	$\frac{\sin(\omega_2 n) - \sin(\omega_1 n)}{\pi n}$
Bandstop	$\frac{\sin(\omega_1 n) - \sin(\omega_2 n)}{\pi n}$

Given

- Passband ripple: $A_p = 20\log(1+\delta_p)$
- Passband edge: $f_p = \omega_p/2\pi$
- Stopband attenuation: $A_s = -20\log\delta_s$
- Stopband edge: $f_s = \omega_s/2\pi$
- Sample rate: f

Window Method Steps

1. Get δ_p and δ_s using A_p and A_s
2. Calculate $A = -20\log(\min(\delta_s, \delta_p))$
3. Find the appropriate window type which $A_w \geq A$
4. The transition bandwidth $\Delta\omega = 2\pi(f_s - f_p)/f \geq \alpha\pi/N$ to get integer N
5. Get cut-off frequency $\omega_c = 2\pi(f_s + f_p)/f$ then the ideal lowpass filter $h_s(n) = \sin(\omega_c n)/(\pi n)$
6. For window function $N_1 = N/2$ or $(N-1)/2$
7. The windowed filter equals the multiply of the window and the filter
8. Delay the impulse response by N_1 to obtain the final filter

Kaiser Window Steps

1. Get δ_p and δ_s using A_p and A_s
2. Calculate $A = -20\log(\min(\delta_s, \delta_p))$
3. Determine β for the Kaiser window using A
4. Compute the filter length $N + 1 \geq (A - 7.95)/(2.285\Delta\omega)$
5. Get cut-off frequency $\omega_c = 2\pi(f_s + f_p)/f$ then the ideal lowpass filter $h_s(n) = \sin(\omega_c n)/(\pi n)$
6. For window function $N_1 = N/2$ or $(N-1)/2$
7. The windowed filter equals the multiply of the window and the filter
8. Delay the impulse response by N_1 to obtain the final filter

Band Filter Design

1. Get δ_p and δ_s using A_p and A_s
2. Calculate $A = -20\log(\min(\delta_s, \delta_p))$
3. Find the appropriate window type which $A_w \geq A$
4. Calculate $\Delta f = \min(f'_{p_1} - f_s, f'_s - f_{p_2})$
5. The transition bandwidth $\Delta\omega = 2\pi\Delta f/f \geq \alpha\pi/N$ to get integer N
6. Get cut-off frequency $\omega_{c_1} = 2\pi(f'_{p_1} - \Delta f/2)/f$ and $\omega_{c_2} = 2\pi(f_{p_2} + \Delta f/2)/f$
7. Then the ideal bandpass filter has the impulse response $h_d(n)$
8. For window function $N_1 = N/2$ or $(N-1)/2$

9. The windowed filter equals the multiply of the window and the filter
10. Delay the impulse response by N_1 to obtain the final filter

5-2 IIR Filter Design

N	**Normalized Butterworth LPF** $H(s) = 1/D(s) \quad \Omega_C = 1$
1	$D(s) = s + 1$
2	$D(s) = s^2 + \sqrt{2}s + 1$
3	$D(s) = s^3 + 2s^2 + 2s + 1$
4	$D(s) = (s^2 + 0.7653s + 1)(s^2 + 1.6180s + 1)$
5	$D(s) = (s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)$

Given

- Passband ripple: $A_p = 20\log(1 + \delta_p)$
- Passband edge: $f_p = \omega_p/2\pi$
- Stopband attenuation: $A_s = -20\log\delta_s$
- Stopband edge: $f_s = \omega_s/2\pi$
- Sample rate: f

Butterworth Approximation Steps

1. Specify the filter requirements and get Ω_p and Ω_s by equation ω/T

> using $T = 1$ for impulse invariant method

2. From $\|H(j\omega)\|^2 = (1 + (\Omega/\Omega_c)^{2N})^{-1} = (1 + \varepsilon^2(\Omega/\Omega_p)^{2N})^{-1}$
3. we get $\|H(j\Omega_p)\| = (1 - \delta_p)^2 = 10^{-\frac{A_p}{10}}$
4. and $\|H(j\Omega_s)\| = \delta_s^2 = 10^{-\frac{A_s}{10}}$ to solve for N and Ω_c
5. Find the transfer function of the analog filter from the table and get roots by partial fraction expression
6. Find the impulse response $h(t)$ and transfer function $H(z)$

Bilinear Transform Steps

1. Get $\omega_{p,s} = 2\pi(f_{p,s}/f)$ then use bilinear transform to get $\Omega_{p,s} = \frac{2}{T}\tan(\omega_{p,s}/2)$

> using $T = 1$ for impulse invariant method

2. Using $\|H(j\omega)\|^2 = (1 + (\Omega/\Omega_c)^{2N})^{-1} = (1 + \varepsilon^2(\Omega/\Omega_p)^{2N})^{-1}$
3. We get $\varepsilon^2 \leq 10^{-\frac{A_p}{10}} - 1$
4. and $(\Omega_s/\Omega_p)^{2N} \geq \frac{10^{\frac{A_s}{10}} - 1}{\varepsilon^2}$ solve for ε^2 and N
5. The analog filter is mapped into the desired digital filter $H(z) = H_a(s)|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$, where $H_a(s) = \frac{1}{s+1}$

Type of Transform	Transformation	Parameters (Prototype low pass filter has band edge frequency ω_p
Lowpass	$z^{-1} \rightarrow \frac{z^{-1}-a}{1-az^{-1}}$	ω'_p = band edge frequency of newfilter $a = \frac{\sin((\omega_p - \omega'_p)/2)}{\sin((\omega_p + \omega'_p)/2)}$
Highpass	$z^{-1} \rightarrow \frac{z^{-1}+a}{1+az^{-1}}$	ω'_p = band edge frequency of newfilter $a = \frac{\cos((\omega_p + \omega'_p)/2)}{\cos((\omega_p - \omega'_p)/2)}$
Bandpass	$z^{-1} \rightarrow -\frac{z^{-2}-a_1 z^{-1}+a_2}{a_2 z^{-2}-a_1 z^{-1}+1}$	ω_l = lower band edge frequency ω_u = upper band edge frequency $a_1 = 2\alpha K/(K+1)$ $a_2 = (K-1)/(K+1)$ $\alpha = \frac{\cos((\omega_u + \omega_l)/2)}{\cos((\omega_u - \omega_l)/2)}$ $K = \cot[(\omega_u - \omega_l)/2]\tan(\omega_p/2)$
Bandstop	$z^{-1} \rightarrow -\frac{z^{-2}-a_1 z^{-1}+a_2}{a_2 z^{-2}-a_1 z^{-1}+1}$	ω_l = lower band edge frequency ω_u = upper band edge frequency $a_1 = 2\alpha/(K+1)$ $a_2 = (1-K)/(1+K)$ $\alpha = \frac{\cos((\omega_u + \omega_l)/2)}{\cos((\omega_u - \omega_l)/2)}$ $K = \tan[(\omega_u - \omega_l)/2]\tan(\omega_p/2)$