Chapter-2

1 - Descriptions: Positions, Orientations, and Frames

Position

Any point could be located in the universe with a 3×1 **position vector** once a coordinate system is established. And the notation for the position of A is ${}^{A}P$, which is defined by

$$
{}^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}
$$

Orientations

In order to describe the orientation of body, we will *attach a coordinate system to the body* and then give a description of this coordinate system **relative to** the reference system.

One way to describe the body-attached coordinate system $\{B\}$, is to write the unit vectors of its three principal axes in terms of the coordinate system $\{A\}$

The principal directions of $\{B\}$ is defined as $\hat{X}_B$, $\hat{Y}_B$ and $\hat{Z}_B$. If the terms of coordinate system $\{A\}$, they are called ${}^{A}\hat{X}_B$, ${}^{A}\hat{Y}_B$ and ${}^{A}\hat{Z}_B$, whose combination will be called as a **rotation matrix**, where $\{B\}$ is relative to $\{A\}$

$$
{}^A_BR = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z} \end{bmatrix} = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}
$$

since ${}^A_BR = {}^B_AR^T$ and ${}^A_BR \cdot {}^B_A R^T = I$, we can know that the rotation matrix is **a orthogonal matrix**

Frames

A frame is defined as the set of four vectors giving position position and orientation information

2 - Mappings

Translation

according to the spacial relationship, translation could be simply calculated by addition

$$
{}^AP = {}^BP + {}^AP_{BORG}
$$

Rotation

according to the definition of the rotation matrix, rotation could be simply calculated by multiply

$$
{}^AP = {}^A_BR\, {}^BP
$$

Axis

- X axis

$$
R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}
$$

- Y axis

$$
R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}
$$

- Z axis

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

General Frames

according to the translation and rotation, the general transform could be written into

$$\begin{bmatrix} {}^A P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^A_B R & & {}^A P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \begin{bmatrix} {}^B P \\ 1 \end{bmatrix}$$

Algorithm

- if the tool frame is to be rotated or translated about the k_{th} unit vector of the **reference frame**, then **pre-multiply** it by the general frames
- if the tool frame is to be rotated or translated about **its own k_{th} unit vector**, then **post-multiply** it by the general frame