# CH_2



## Intro Linear Time-invariant system

- linear
- time-invariant

## 2.1 Linear Constant-Coefficient Differential Equation

**Linear constant-coefficient differential equation** is the mathematical repression of a continuous-time LTI system.

### Classical Solution of Differential Equations

$$\begin{aligned} &C_0\frac{d^n r(t)}{dt^n}+C_1\frac{d^{n-1}r(t)}{dt^{n-1}}+\cdots+C_{n-1}\frac{dr(t)}{dt}+C_n r(t) \\ &=E_0\frac{d^m e(t)}{dt^m}+E_1\frac{d^{m-1}e(t)}{dt^{m-1}}+\cdots+E_{m-1}\frac{de(t)}{dt}+E_m e(t) \end{aligned} \tag{1}$$

$$C_0\frac{d^n r(t)}{dt^n}+C_1\frac{d^{n-1}r(t)}{dt^{n-1}}+\cdots+C_{n-1}\frac{dr(t)}{dt}+C_n r(t)=0 \tag{2}$$

- Homogeneous solution (natural response)
  1. **Characteristic equation**
     $$C_0\alpha^n+C_1\alpha^{n-1}+\cdots+C_{n-1}\alpha+C_n=0 \tag{3}$$
  2. **Characteristic roots** $\alpha_1,\alpha_2,\cdots,\alpha_n$ satisfy tje characteristic equation
  3. **Homogeneous solution**
     $r_h(t)=\sum_{i=1}^n A_i e^{\alpha_i t}$
     *If $\alpha_1$ is the k-th repeated roots, then there will be k number of $\alpha_1$:$\sum_{i=1}^k A_i t^{k-i}e^{\alpha_1 t}$*

- Particular solution (forced response)
    1. Table look-up

| **Exciting** $e(t)$ | **Response** $r(t)$ |
| --- | --- |
| $t^p$ | $B_1 t^p + B_2 t^{p-1} + L + B_p t + B_{p+1}$ |
| $e^{\alpha t}$ | $Be^{\alpha t}$ |
| $\sin(\omega t)/\cos(\omega t)$ | $B_1 \sin(\omega t) + B_2 \cos(\omega t)$ |
| $t^p e^{\alpha t} \sin(\omega t)/$<br>$t^p e^{\alpha t} \cos(\omega t)$ | $(B_1 t^p + B_2 t^{p-1} + L + B_p t + B_{p+1}) e^{\alpha t} \sin(\omega t) + (D_1 t^p + D_2 t^{p-1} + L + D_p t + D_{p+1}) e^{\alpha t} \cos(\omega t)$ |

- Auxiliary conditions (initial conditions)
    1. Determine the coefficients

$$r(t_0), \frac{dr(t_0)}{dt}, \frac{d^2 r(t_0)}{dt^2}, L, \frac{d^{n-1} r(t_0)}{dt^{n-1}}$$

# 2.2 Partial Initial Conditions

## The Law of Switching

$$v_c(0_-) = v_c(0_+),\ i_L(0_-) = i_L(0_+)$$

## Impulse Function Matching Method

$$\delta(t) \quad \delta'(t) \quad \delta''(t) \quad \cdots$$

## The Laplace Transform

$$X(s) = \int_0^\infty x(t) e^{-st} \mathrm{d}t$$

# 2.3 System Response

$$\begin{aligned} \text{Total Response } &= \\ &= \text{Natural Response} + \text{Forced Response} \\ &= \text{Zero-input Response} + \text{Zero-state Response} \\ &= \text{Transient Response} + \text{Steady-state Response} \end{aligned}$$

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- **Zero-input Response**: results only from the initial state (*energy storage*)
- **Zero-state Response**: results only from the external inputs

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- *Transient Response*: when $t \to \infty$, the response that approaches to 0
- *Steady-state Response*: when $t \to \infty$, the response that keeps

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$$
\begin{aligned}
r(t) = r_{zi}(t) + r_{zs}(t) &= \sum_{k=1}^{n} A_{zik} e^{\alpha_k t} + \sum_{k=1}^{n} A_{zsk} e^{\alpha_k t} + B(t) \\
&= \text{Zero-input Response} + \text{Zero-state Response} \\
&= \sum_{k=1}^{n} A_k e^{\alpha_k t} + B(t) \\
&= \text{Natural Response} + \text{Forced Response}
\end{aligned}
$$

## 2.4 The Unit Impulse Response

$$
\begin{aligned}
& C_0 \frac{\mathrm{d}^n r(t)}{\mathrm{d}t^n} + C_1 \frac{\mathrm{d}^{n-1} r(t)}{\mathrm{d}t^{n-1}} + \cdots + C_{n-1} \frac{\mathrm{d}r(t)}{\mathrm{d}t} + C_n r(t) \\
= & E_0 \frac{\mathrm{d}^m e(t)}{\mathrm{d}t^m} + E_1 \frac{\mathrm{d}^{m-1} e(t)}{\mathrm{d}t^{m-1}} + \cdots + E_{m-1} \frac{\mathrm{d}e(t)}{\mathrm{d}t} + E_m e(t)
\end{aligned} \tag{1}
$$

if $e(t) = \delta(t)$, then $r(t) = h(t)$

$$
\Rightarrow \begin{aligned}
& C_0 h^{(n)}(t) + C_1 h^{(n-1)}(t) + \cdots + C_{n-1} h^{(1)}(t) + C_n h(t) \\
= & E_0 \delta^{(m)}(t) + E_1 \delta^{(m-1)}(t) + \cdots + E_{m-1} \delta^{(1)}(t) + E_m \delta(t)
\end{aligned} \tag{4}
$$

| **Conditions** | $h(t)$ |
| --- | --- |
| $n > m$ | $[\sum_{i=1}^{n} A_i e^{\alpha_i t}] u(t)$ |
| $n = m$ | $[\sum_{i=1}^{n} A_i e^{\alpha_i t}] u(t) + B\delta(t)$ |
| $n < m$ | $[\sum_{i=1}^{n} A_i e^{\alpha_i t}] u(t) + B\delta(t) + C\delta'(t)$ |

## 2.5 The Convolution Integral

$$
y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) \mathrm{d}\tau
$$

- time reversal: $h(\tau) \to h(-\tau)$
- time shift: $h(-\tau) \to h(\tau)$
- multiplication: $x(\tau)h(t - \tau)$
- integrating: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\mathrm{d}\tau$

## 2.6 Properties of Convolution

### The Commutative Property

$$
x(t) * h(t) = h(t) * x(t)
$$

### The Distributive Property

$$
x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)
$$

### The Associative Property

$$
[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]
$$

## The Differential and Integral Property

$$\text{differential}$$

$$y'(t) = x(t) * h'(t) = x'(t) * h(t)$$

$$y^{(n)}(t) = x(t) * h^{(n)}(t) = x^{(n)}(t) * h(t)$$

$$\text{integral}$$

$$y^{(-1)}(t) = x(t) * h^{(-1)}(t) = x^{(-1)}(t) * h(t)$$

$$y^{(-m)}(t) = x(t) * h^{(-m)}(t) = x^{(-m)}(t) * h(t)$$

$$\text{generally}$$

$$y^{(n-m)}(t) = x^{(n)}(t) * h^{(-m)}(t) = x^{(-m)}(t) * h^{(n)}(t)$$

$$\text{specially, } n = 1 \; m = 1$$

$$y(t) = x^{(-1)}(t) * h'(t) = x'(t) * h^{(-1)}(t)$$

# 2.7 System Properties

## Stability

#### Bounded Input Bounded Output (BIBO)

$\int_{-\infty}^{\infty} |h(t)|\mathrm{d}t < \infty$

*Proof:*

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$\text{Let } |x(t)| \leq B$$

$$|y(t)| = |h(t) * x(t)| = |\int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau)\mathrm{d}\tau|$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)| \cdot |x(t-\tau)|\mathrm{d}\tau \leq B \int_{-\infty}^{\infty} |h(\tau)|\mathrm{d}\tau$$

$$\text{If } |y(t)| < \infty \text{, then } \int_{-\infty}^{\infty} |h(\tau)|\mathrm{d}\tau < \infty$$