CH_7

Introduction

the z-transform is the discrete-time counterpart of the Laplace transform

the z-transform expand the application in which Fourier analysis can be used

7.1 The z-Transform

Definition

the z-transform of a general discrete-time signal $x[n]$ is defined as

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

Expressing the complex variable z in polar form as $z=re^{j\omega}$

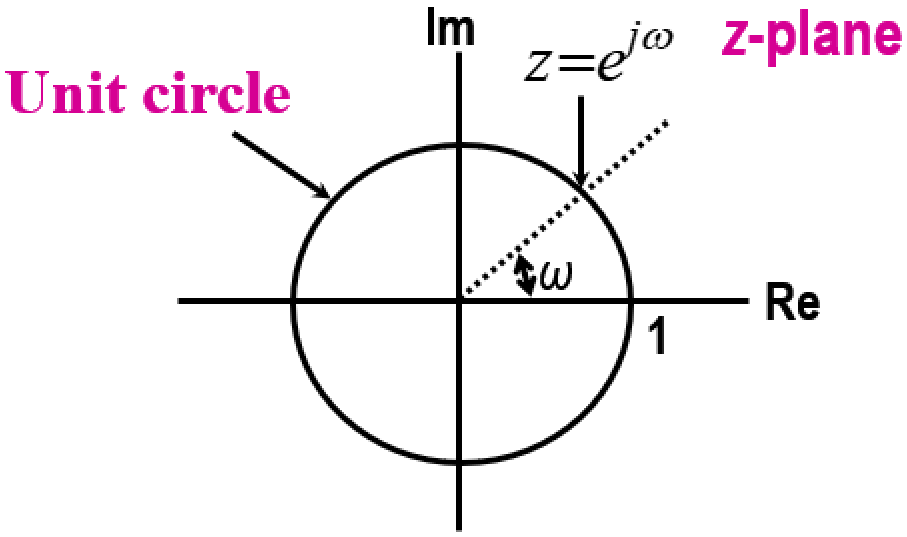
$$X(re^{j\omega})=\sum_{n=-\infty}^{+\infty}x[n](re^{j\omega})^{-n}=\sum_{n=-\infty}^{+\infty}(x[n]r^{-n})e^{-j\omega n}$$

$X(re^{j\omega})$ is the Fourier transform of $x[n]$ multiplied by a real exponential r^{-n}

for $r=1$, or equivalently, $|z|=1$, z-transform equation reduces to the Fourier transform

$$X(z)|_{z=e^{j\omega}}=X(e^{j\omega})=\mathcal{F}\{x[n]\}$$

the z-transform reduces to the Fourier transform for values of z on the unit circle



The Relationship of Laplace Transform and z-Transform

$$x_s(t)=x(t)\cdot\delta_T(t)=x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)=\sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT)$$

$$X(s)=\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT)e^{-st}\mathrm{d}t$$

$$=\sum_{n=-\infty}^{\infty}x(nT)\int_{-\infty}^{\infty}\delta(t-nT)e^{st}\mathrm{d}t=\sum_{n=-\infty}^{\infty}x(nT)e^{-snT}$$

Let $z=e^{sT}$ and $x(nT)\to x[n]$

$$X(s)|_{z=e^{sT}}=\sum_{n=-\infty}^{\infty}x[n]z^{-n}=X(z)$$

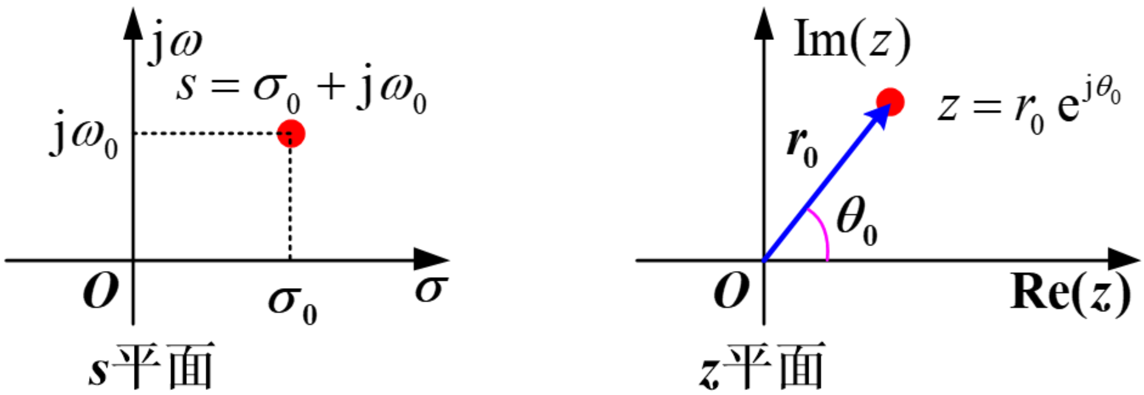
Polar Coordinates

since $s=\sigma+j\omega$ and $z=e^{sT}$

$$z=e^{(\sigma+j\omega)T}=e^{\sigma T}\cdot e^{j\omega T}$$

$$z=re^{j\theta}\Rightarrow\begin{cases}\text{radius: } r=e^{\sigma T}\\ \text{angle: } \theta=\omega T=2\pi\frac{\omega}{\omega_s}\end{cases}$$

where ω_s is the sampling frequency



$$\sigma = 0 \longleftrightarrow r = 1$$

$$\sigma < 0 \longleftrightarrow r < 1$$

$$\sigma > 0 \longleftrightarrow r > 1$$

$$\sigma : -\infty \to \infty \longleftrightarrow r : 0 \to \infty$$

$$\sigma = 0,\ \omega = 0 \longleftrightarrow r = 1,\ \theta = 0$$

$$\omega = 0 \longleftrightarrow \theta = 0$$

$$\omega = \pm\frac{\omega_s}{2} \longleftrightarrow \theta = \pi$$

> if the ROC includes the unit circle, then the Fourier transform also converges

7.2 The Region of Convergence for the z-Transform

Properties

1. The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin
2. The ROC does not contain any poles

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3. if $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$
4. if $x[n]$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC
5. if $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC

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6. if $x[n]$ is two sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$

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7. if the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity
8. if the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is right sided, then the ROC is the region in the z-plane outside the outermost pole
9. if the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is left sided, then the ROC is the region in the z-plane inside the innermost nonzero pole

7.3 The Inverse z-Transform

Partial-Fraction Expansion

$$X(z) = \sum_{i=1}^{m} \frac{A_i}{1 - a_i z^{-1}}$$

- ROC outside the pole: $\frac{A_i}{1-a_i z^{-1}} \leftrightarrow A_i a_i^n u[n]$
- ROC outside the pole: $\frac{A_i}{1-a_i z^{-1}} \leftrightarrow -A_i a_i^n u[-n-1]$

Power-series Expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

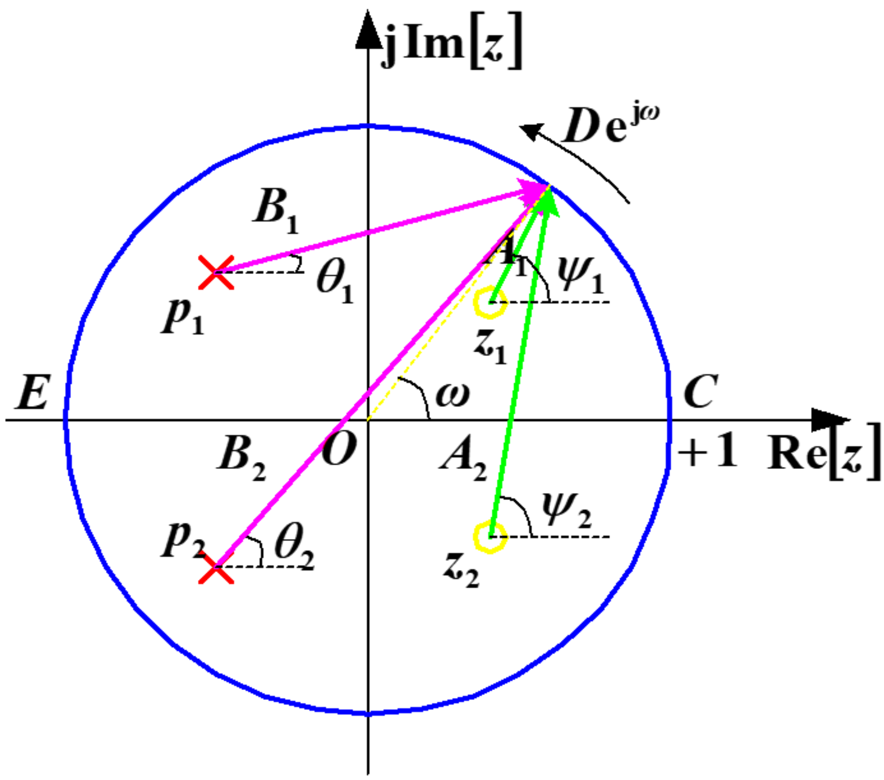
The coefficients in this power series are the sequence values $x[n]$ since $\delta[n+n_0] \leftrightarrow z^{n_0}$

7.4 Geometric Evaluation of Fourier Transform from the Pole-Zero Plot

$$H(z) = \frac{\prod_{r=1}^{M} \ (z - z_r)}{\prod_{k=1}^{N} \ (z - p_k)}$$
$$H(e^{j\omega}) = \frac{\prod_{r=1}^{M} \ (e^{j\omega} - z_r)}{\prod_{k=1}^{N} \ (e^{j\omega} - p_k)}$$

Let $e^{j\omega} - z_r = A - re^{j\psi_r}$, $e^{j\omega} - p_k = B_k e^{j\theta_k}$

- Magnitude Response: $|H(e^{j\omega})| = \frac{\prod_{r=1}^{M} A_r}{\prod_{k=1}^{N} B_k}$
- Phase Response: $\phi(\omega) \ = \sum_{r=1}^{M} \psi_r - \sum_{k=1}^{N} \theta_k$



7.5 Properties of the z-Transform

Linearity

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(z) + bX_2(z)$$

with the ROC containing $R_1 \cap R_2$

Time Shifting

$$z[n - n_0] \longleftrightarrow z^{-n_0} Z(z)$$

with ROC = R, except for the possible addition or deletion of the origin or infinity

> For $n_0 > 0$, poles will be introduced at $z = 0$
> For $n_0 < 0$, zeros will be introduced at $z = 0$

Scaling in the z-Domain

$$z_0^n x[n] \longleftrightarrow X\left(\frac{z}{z_0}\right)$$

with ROC = $|z_0|R$

> Special Case: when z = $e^{j\omega_0}$, $e^{j\omega_0 n}x[n] \longleftrightarrow X(e^{-j\omega_0}z)$ with ROC = R

Time Expansion

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if n is a multiple of k} \\ 0 & \text{if n is not multiple of k} \end{cases}$$

$$x_{(k)}[n] \longleftrightarrow X(z^k)$$

with ROC = $R^{1/k}$

Conjugation

$$x^*[n] \longleftrightarrow X^*(z^*)$$

with ROC = R

> if $x[n]$ is real. $X(z) = X^*(z^*)$
> if X(z) has a pole (or zero) at $z = z_0$, it must also have a pole (or zero) at the complex conjugate point $z = z_0^*$

Convolution Property

$$x_1[n] * x_2[n] \longleftrightarrow X_1(z)X_2(z)$$

with ROC containing $R_1 \cap R_2$

> the ROC may be larger than $R_1 \cap R_2$ if pole-zero cancellation occurs in the product

Difference and Sum Property in Time Domain

$$x[n] - x[n-1] \longleftrightarrow (1 - z^{-1})X(z)$$

$$\sum_{k=-\infty}^{n} x[k] \longleftrightarrow \frac{1}{1 - z^{-1}} X(z)$$

Differentiation in the z-Domain

$$nx[n] \longleftrightarrow -z\frac{\mathrm{d}X(z)}{\mathrm{d}z}$$

with ROC = R

Initial-Value and Final-Value Theorems

if $x[n]$ is a casual sequence, then

$$x[0] = \lim_{z\to\infty} X(z)$$

$$x[n+1] \longleftrightarrow z[X(z) - x(0)]$$

7.6 Analysis and Characterization of LTI System Using z-Transform

$$Y(z) = X(z)H(z)$$

where $H(z)$ is the system function/transfer function

Causality

Definition: if the ROC of its system function is the exterior of a circle, including infinity

A system with rational system with rational system function $H(z)$ is casual if and only if:

- the ROC is the exterior of a circle outside the outermost pole
- for $H(z)$, the order of numerator cannot be greater than the order of the denominator

Stability

Definition: the ROC of its system function $H(z)$ includes the unit circle, $|z| = 1$

A system with rational system function $H(z)$ is stable if and only if all of the poles of $H(z)$ lie inside the i=unit circle

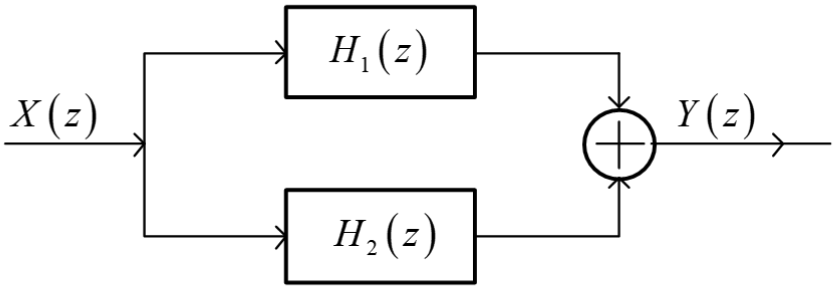
Linear Constant-Coefficient Difference Equations

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

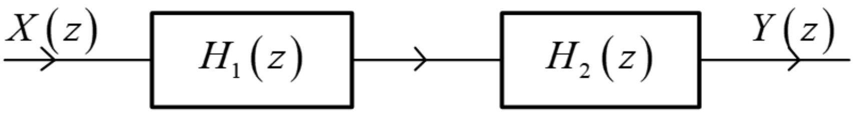
The Relationship between H(z)'s Poles and h[n] Waveform

s-plane	s-plane	z-plane	z-plane
极点位置	$h(t)$特点	极点位置	$h[n]$特点
虚轴上	等辐	单位圆上	等辐
原点处	$u(t) \leftrightarrow \frac{1}{s}$	$\theta = 0\ z = 1$	$u[n] \leftrightarrow \frac{1}{1-z^{-1}}$
左半平面	减幅	单位圆内	减幅
右半平面	增幅	单位圆外	增幅

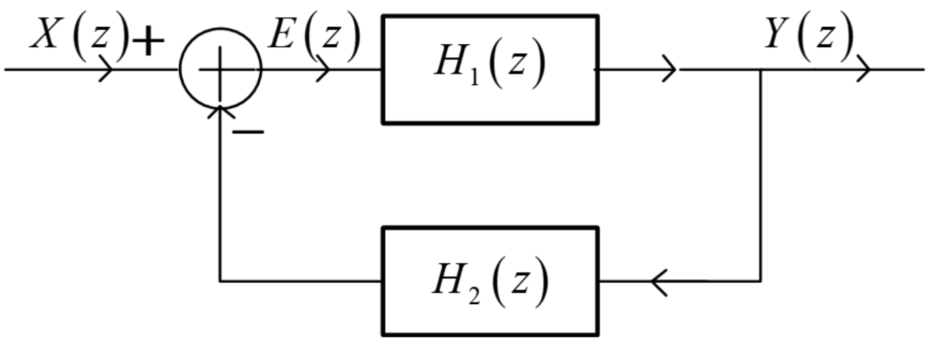
7.7 System Function Algebra and Block Diagram Representations



$$H(z) = H_1(z) + H_2(z)$$



$$H(z) = H_1(z) \cdot H_2(z)$$



$$H(z) = \frac{H_1(z)}{1 + H_1(z) \cdot H_2(z)}$$

7.8 The Unilateral z-Transform

Definition

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

which could be also thought as the bilateral z-transform of $x[n]u[n]$

> the ROC must always be the exterior of a circle

Properties of the Unilateral z-Transform

- time delay

$$x[n-m] \longleftrightarrow z^{-m}\left(X(z)+\sum_{n=-m}^{-1} x[n] z^{-n}\right)$$

- time advance

$$x[n+m] \longleftrightarrow z^{m}\left(X(z)-\sum_{n=0}^{m-1} x[n] z^{-n}\right)$$

Solving Difference Equations Using the Unilateral z-Transform

need more examples

7.9 The Discrete-Time Fourier Transform

Definition

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \mathrm{d}\omega \end{aligned}$$

Relationship between DTFT and z-Transform

$$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$$

periodic with period 2π

7.10 Frequency Response of a Discrete-time System

Definition

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n] e^{-jn\omega} \\ h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) e^{jn\omega} \mathrm{d}\omega \end{aligned}$$

For a casual and stable system, let $x[n] = e^{j\omega n}$

$$\begin{aligned} y[n] = x[n] * h[n] &= \sum_{k=-\infty}^{\infty} h[k] x[n-k] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \end{aligned}$$

Let $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$

then it comes to

$$y[n] = e^{j\omega n} H(e^{j\omega})$$

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = |H(e^{j\omega})| \cdot e^{j\phi(\omega)}$$

- Magnitude Response: $|H(e^{j\omega})| \sim \omega$
- Phase Response: $\phi(\omega) \sim \omega$