CH_6

6.1 The Laplace Transform

For some signals which have no Fourier transforms, we can preprocess them like

$$X(j\omega) = \int_{-\infty}^{\infty} \left[x(t)e^{-\sigma t}\right] e^{-j\omega t}\mathrm{d}t$$

define $s = \sigma + j\omega$, and using $X(s)$ to denote this integral

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\mathrm{d}t$$

which is a two-sided (bilateral) Laplace Transform

And we can say that the Laplace Transform is an extension of the Fourier

> **The Fourier transform is a special case of the Laplace transform when $\sigma = 0$**

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\mathrm{d}t$$

which is a one-sided (unilateral) Laplace Transform

some useful LT pairs

> $\cos(\omega t)u(t) \overset{LT}{\longleftrightarrow} \frac{s}{s^2+\omega^2}, \Re\{s\} > 0$ > $\sin(\omega t)u(t) \overset{LT}{\longleftrightarrow} \frac{\omega}{s^2+\omega^2}, \Re\{s\} > 0$

Generally, the Laplace transform is rational

$$X(s) = \frac{N(s)}{D(s)}$$

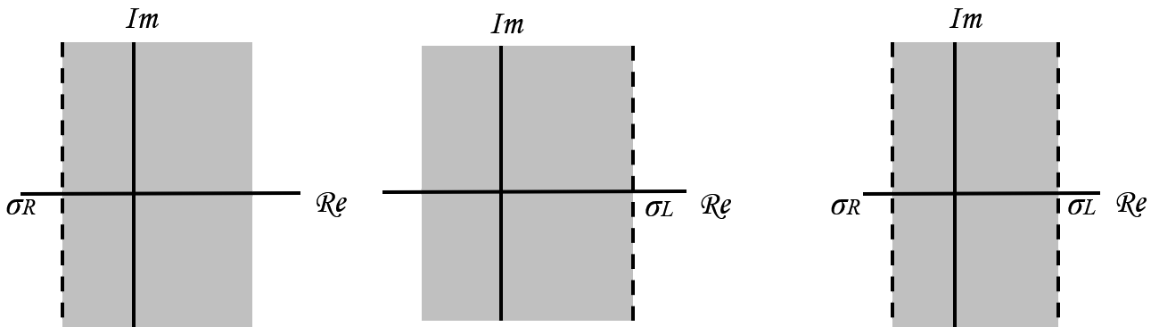
- the roots of $N(s)$ are defined as the zeros
- the roots of $S(s)$ are defined as the poles

the representation of $X(s)$ through its poles an zeros in the s-plane is referred to a as the pole-zero plot of $X(s)$

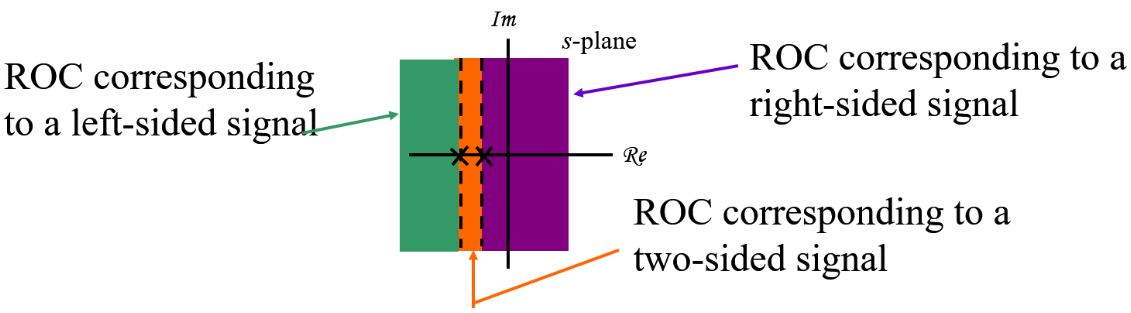
6.2 Region of Convergence for Laplace Transform

- The ROC of $X(s)$ consists of stripes parallel to the $j\omega$-axis in the s-plane
- For rational Laplace transforms, the ROC does not contain any poles
- If $x(t)$ is of finite duration and is absolutely integrable, the the ROC id the entire s-plane

the line $\Re\{s\} = \sigma_0$ is in the ROC



- If $x(t)$ is **right sided**, then all values of s for which $\Re\{s\} > \sigma_0$ will also be in the ROC; and the ROC of a right-sided signal is a right-half plane
- If $x(t)$ is **left sided**, then all values of s for which $\Re\{s\} < \sigma_0$ will also be in the ROC; and the ROC of a left-sided signal is a left-half plane
- If $x(t)$ is **two sided**, then the ROC will consist of a strip in the s-plane that includes the line $\Re\{s\} = \sigma_0$



- If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC
- If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the region in the s-plane to the left of the leftmost pole

6.3 The Inverse Laplace Transform

from the inverse Fourier transform, we could conclude that

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{j\omega t}\mathrm{d}\omega$$

multiplying both sides both by $e^{\sigma t}$, we can obtain

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{st}\mathrm{d}\omega$$

and obtain that $\mathrm{d}s = j\mathrm{d}\omega$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}\mathrm{d}s$$

we only discuss the class of **rational transforms**, the inverse Laplace transform can be determine by using the technique of **partial-fraction expansion**

$$\begin{aligned} X(s) = \frac{N(s)}{D(s)} &= \frac{a_m s^m + a_{m-1}s^{m-a} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \\ &= \frac{a_m(s-z_1)(s-z_2)\cdots(s-z_m)}{b_n(s-p_1)(s-p_2)\cdots(s-p_n)} \end{aligned}$$

- $m < n$ and no repeated roots in poles: $X(s) = \sum_{i=1}^{n} \frac{A_i}{s+a_i}$

6.4 Geometric Evaluation of the Fourier Transform from the Pole-Zero Plot

a general rational Laplace transform has the form

$$X(s) = \frac{N(s)}{D(s)}$$

which can be factored into the form

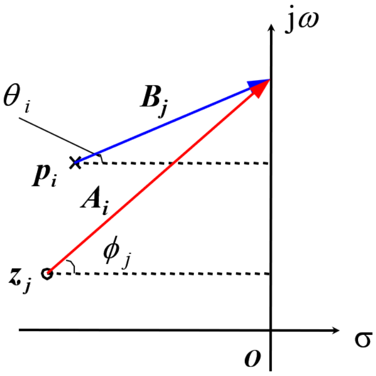
$$X(s) = M\frac{\prod_{i=1}^{R}(s-z_i)}{\prod_{j=1}^{P}(s-p_j)}$$

where z_i, p_j are zeros and poles of $X(s)$

then denote **zero vectors** and **pole vectors**:

$$j\omega - z_i = A_i e^{j\phi_i}$$

$$j\omega - p_j = B_j e^{j\phi_i}$$



- Magnitude:

$$|X(j\omega)| = M\frac{\prod_{i=1}^{R}(j\omega - z_i)}{\prod_{j=1}^{R}(j\omega - p_j)}$$

- Phase:

$$\angle X(j\omega) = \sum_{i=1}^{R}\phi_i - \sum_{j=1}^{P}\theta_j$$

6.5 Properties of the Laplace Transform

Linearity

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(s) + bX_2(s) \qquad \mathrm{ROC} = R_1 \cap R_2$$

Time shifting

$$x(t) \longleftrightarrow X(s) \qquad \mathrm{ROC} = R$$
$$x(t - t_0) \longleftrightarrow e^{-st_0}X(s) \qquad \mathrm{ROC} = R$$

Shifting in the s-domain

$$x(t) \longleftrightarrow X(s) \qquad \mathrm{ROC} = R$$
$$e^{s_0 t}x(t) \longleftrightarrow X(s - s_0) \qquad \mathrm{ROC} = R + \Re\{s_0\}$$

Time scaling

$$x(t) \longleftrightarrow X(s) \qquad \mathrm{ROC} = R$$
$$x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|}X\left(\frac{s}{\alpha}\right) \qquad \mathrm{ROC} = \alpha R$$
$$x(-t) \longleftrightarrow X(-s) \qquad \mathrm{ROC} = -R$$

Conjugation

$$x^*(t) \longleftrightarrow X^*(s^*) \qquad \mathrm{ROC} = R$$

when $x(t)$ is real: $X(s) = X^*(s^*)$

> if $x(t)$ is real and if $X(s)$ has a pole or zero at $s = s_0$, then $X(s)$ also has a pole or zero at the complex conjugate point $s = s_0^*$.

Convolution property

$$x_1(t) * bx_2(t) \longleftrightarrow X_1(s)X_2(s) \qquad \mathrm{ROC} = R_1 \cap R_2$$

Differentiation in the time domain

$$x(t) \longleftrightarrow X(s) \qquad \mathrm{ROC} = R$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \longleftrightarrow sX(s) \qquad \mathrm{ROC} = R$$

Differentiation in the s-domain

$$-tx(t) \longleftrightarrow \frac{\mathrm{d}X(s)}{\mathrm{d}s} \qquad \mathrm{ROC} = R$$

Integration in the time domain

$$\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau \longleftrightarrow \frac{X(s)}{s} \qquad \mathrm{ROC} = R \cap \{\Re\{s\} > 0\}$$

The initial-value and final-value theorems

Initial-value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Final-value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

6.6 The Unilateral Laplace Transform

> the bilateral transform and the unilateral transform of a casual signal are identical
> the ROC for the unilateral transform is always a right-half plane
> unilateral transform can be recognized as the bilateral transform of $x(t)u(t)$

Differentiation in the Time Domain

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \longleftrightarrow sX(s) - x(0^-)$$

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}x(t) \longleftrightarrow s^nX(s) - \sum_{k=0}^{n-1} s^{n-k-1}x^{(k)}(0^-)$$

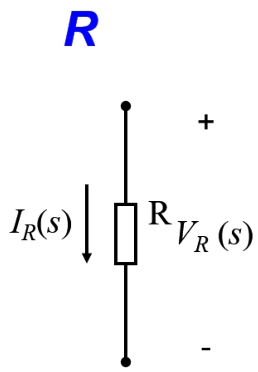
Solving Differential Equations Using the Unilateral Laplace Transform

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}$$

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Representation of Circuits in s-domain

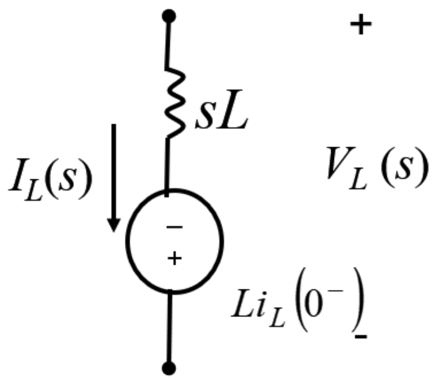
Resistor

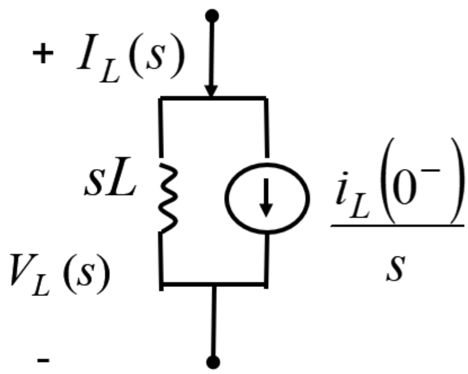


$$v_R(t) = R i_R(t)$$

$$V_R(s) = R I_R(s)$$

Inductor



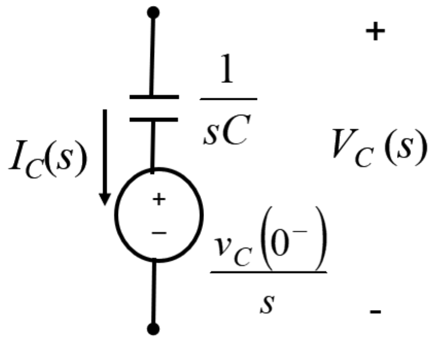


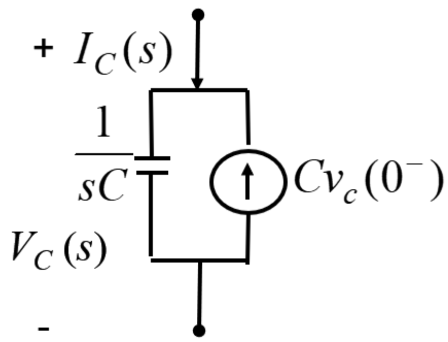
$$v_L(t) = L \frac{\mathrm{d}i_L(t)}{\mathrm{d}t}$$

$$V_L(s) = sL I_L(s) - L i_L(0^-)$$

$$I_L(s) = \frac{1}{sL} V_L(s) = \frac{1}{s} i_L(0^-)$$

Conductor



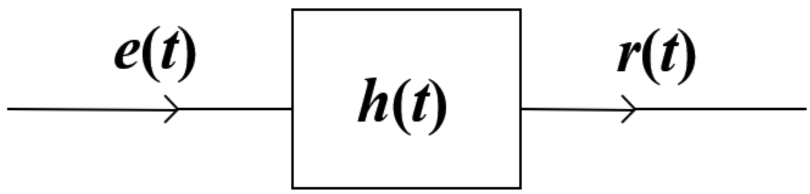


$$i_c(t) = C\frac{\mathrm{d}v_c(t)}{\mathrm{d}t}$$

$$V_C(s) = \frac{1}{sC}I_C(s) + \frac{1}{s}v_C(0^-)$$

$$I_C(s) = sCV_C(s) - Cv_C(0^-)$$

6.7 System Function

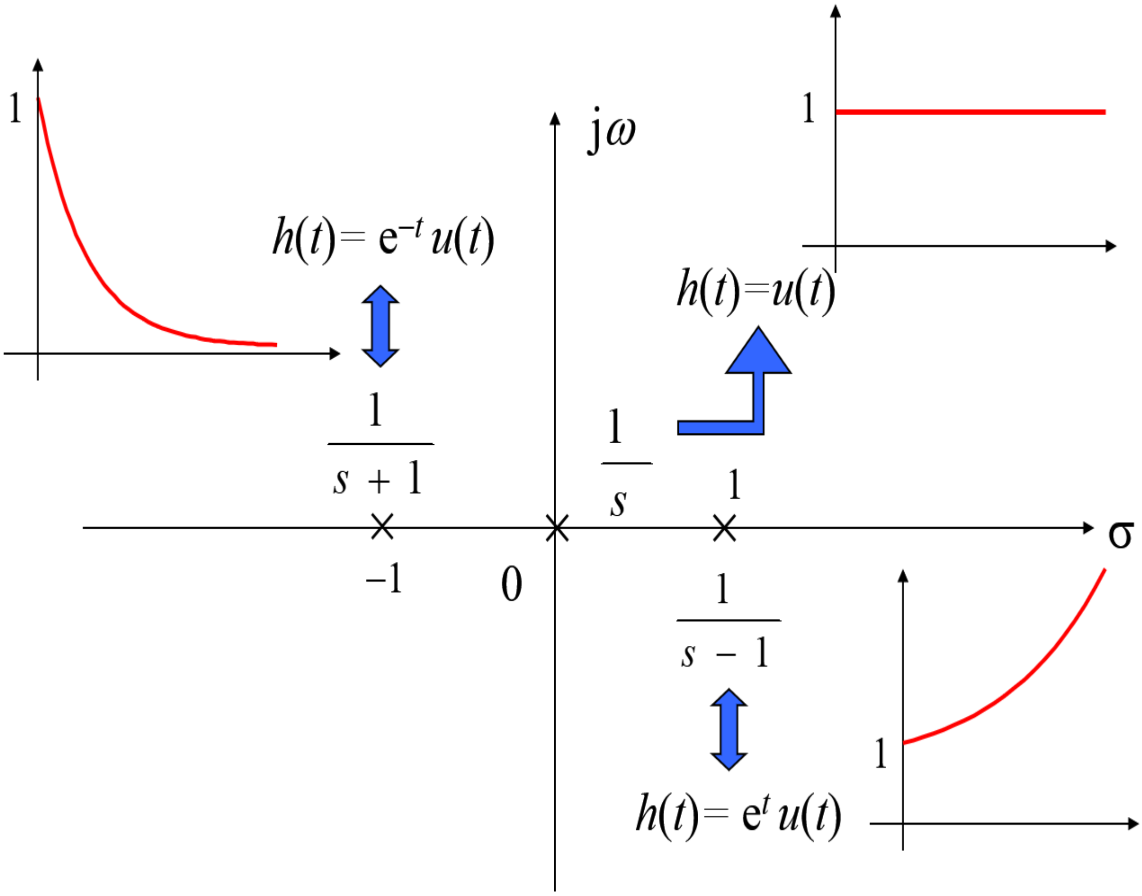


$$r(t) = e(t) * h(t) \longleftrightarrow R(s) = E(s) \cdot H(s)$$

$$H(s) = \frac{R(s)}{E(s)}$$

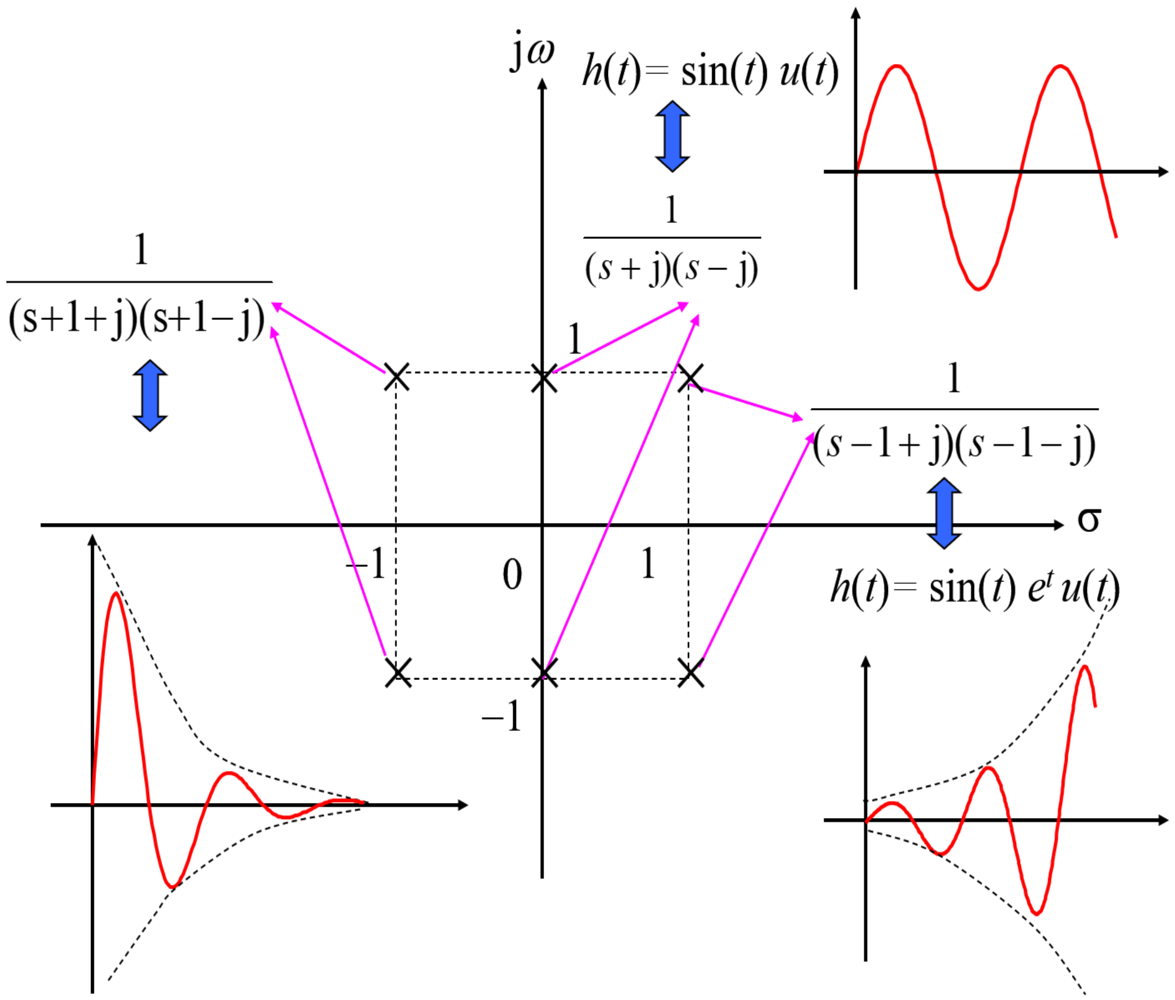
$H(s)$ Poles

- Single Pole



- **Positive Real Number**: increase exponentially
- **Negative Real Number**: decrease exponentially

- Conjugation Poles



- **Poles on Imaginary Axis**: Constant amplitude shock
- **Poles on Right Part of ROC**: Shock increase
- **Poles on Left Part of ROC**: Shock decrease

6.8 Analysis and Characterization of LTI system Using the Laplace Transform

Causality

> **The ROC associated with the system function for a causal system is a right-half plane**

Stability

> **An LTI system is stable if and only if the ROC of its system function $H(s)$ includes the $j\omega$-axis**

> A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane —i.e. all of the poles have negative real parts.

6.9 Frequency Response of An LTI System