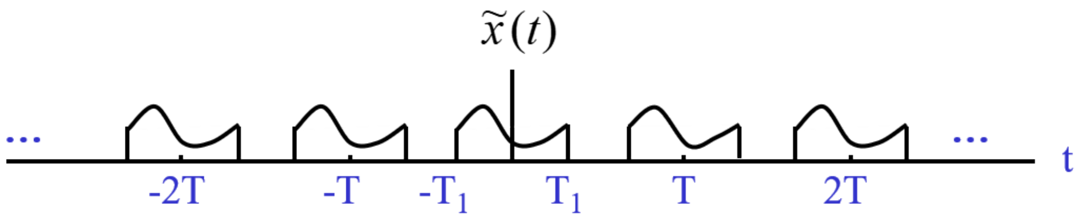
CH_5

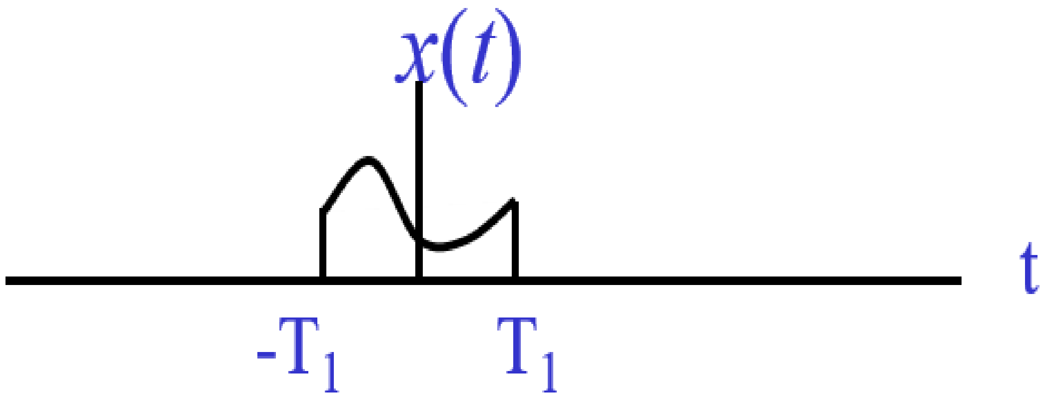
5.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Time Transform

We think of *an aperiodic signal* as the limit of *a periodic signal* as the period **becomes arbitrarily large**, and we examine the limiting behavior of the Fourier series representation for the signal



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}\mathrm{d}t$$

$$Ta_k = \int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}\mathrm{d}t$$



$T \to \infty$

Fourier Transform of x(t)

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t \\
X(j\omega) &= \lim_{T\to\infty} Ta_k = \lim_{\omega_0} 2\pi\frac{a_k}{\omega_0}
\end{aligned} \tag{5.1}
$$

$X(j\omega)$ is actually spectrum-density function and different from a_k, which is the spectrum of periodic signals

$$
\begin{aligned}
\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} &= \sum_{k=-\infty}^{\infty} [\frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}\mathrm{d}t]e^{jk\omega_0 t} \\
&= \frac{\omega_0}{2\pi}\sum_{k=-\infty}^{\infty} [\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}\mathrm{d}t]e^{jk\omega_0 t} \\
x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\mathrm{d}\omega
\end{aligned} \tag{5.2}
$$

the *equation 5.2* is also called as the **inverse fourier transform**

and the pair of *equation 5.1* and *equation 5.2* is called as the **fourier transform pair**

> An useful relationship
> $a_k = \frac{1}{T}X(j\omega)|_{\omega=k\omega_0}$

Fourier Transform Pair

$$
X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t = \mathcal{F}\{x(t)\}
$$

$$
x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\mathrm{d}\omega = \mathcal{F}^{-1}\{X(j\omega)\}
$$

Fourier Transform: $X(j\omega) =< x, e^{j\omega t} >$

Dirichlet Conditions

- must be integrable
- bounded variation
- finite number of discontinuities

5.2 The Fourier Transform For Periodic Signals

$$X(j\omega) = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} \mathrm{d}t = \frac{1}{j(\omega_0 - \omega)} e^{j(\omega_0 - \omega)t} \Big|_{-\infty}^{\infty}$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} \mathrm{d}\omega$$

$$2\pi\delta(t) = \int_{-\infty}^{\infty} e^{j\omega t} \mathrm{d}\omega \qquad 2\pi\delta(-t) = \int_{-\infty}^{\infty} e^{-j\omega t} \mathrm{d}\omega$$

$$2\pi\delta(-\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} \mathrm{d}t$$

$$2\pi\delta(\omega_0 - \omega) = \int_{-\infty}^{\infty} e^{-jt(\omega - \omega_0)} \mathrm{d}t = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} \mathrm{d}t$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

5.3 Properties of The COntinuous-Time Fourier Transform

Linearity

$$\begin{aligned} \text{if} \qquad & x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega) \\ & y(t) \overset{\mathcal{F}}{\leftrightarrow} Y(j\omega) \\ \text{then} \qquad & ax(t) + by(t) \overset{\mathcal{F}}{\leftrightarrow} aX(j\omega) + bY(j\omega) \end{aligned}$$

Time Shifting

$$\begin{aligned} \text{if} \qquad & x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega) \\ \text{then} \qquad & x(t - t_0) \overset{\mathcal{F}}{\leftrightarrow} e^{-j\omega t_0} X(j\omega) \end{aligned}$$

Conjugation and Conjugate Symmetry

$$\begin{aligned} \text{if} \qquad & x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega) \\ \text{then} \qquad & x^*(t) \overset{\mathcal{F}}{\leftrightarrow} X^*(j\omega) \\ \text{if } x(t) \text{ is real, then} \qquad & X(j\omega) = X^*(j\omega) \end{aligned}$$

$$X(j\omega) = |X(j\omega)| e^{j\phi(\omega)} = Re(\omega) + jIm(\omega)$$

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) \overset{\mathcal{F}}{\leftrightarrow} Re(\omega) \qquad x_o(t) \overset{\mathcal{F}}{\leftrightarrow} jIm(\omega)$$

Differentiation and Integration

$$\text{if} \qquad x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega)$$

$$\text{then} \qquad \frac{\mathrm{d}x(t)}{\mathrm{d}t} \overset{\mathcal{F}}{\leftrightarrow} j\omega X(j\omega)$$

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} \overset{\mathcal{F}}{\leftrightarrow} (j\omega)^n X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

$$= X(j\omega) \cdot \left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

Time and Frequency Scaling

$$\text{if} \qquad x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega)$$

$$\text{then} \qquad x(at) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{|a|}X(\frac{j\omega}{a})$$

Duality

$$\text{if} \qquad x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega)$$

$$\text{then} \qquad X(jt) \overset{\mathcal{F}}{\leftrightarrow} 2\pi x(-\omega)$$

Frequency Shifting

$$e^{j\omega_0 t}x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j(\omega - \omega_0))$$

Differentiation in Frequency-Domain

$$-jx(t) \overset{\mathcal{F}}{\leftrightarrow} \frac{\mathrm{d}X(j\omega)}{\mathrm{d}\omega}$$

Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|X(j\omega)\right|^2 \mathrm{d}\omega$$

Zero Point

$$X(0) = \int_{-\infty}^{\infty} x(t)\mathrm{d}t$$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\mathrm{d}\omega$$

5.4 The Convolution Property

$$
\begin{aligned}
Y(j\omega) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau\right]e^{-j\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} e^{-j\omega t}h(t-\tau)\mathrm{d}t\right]\mathrm{d}\tau \\
&= H(j\omega)\int_{-\infty}^{\infty} e^{-j\omega\tau}x(\tau)\mathrm{d}\tau \\
&= H(j\omega)X(j\omega)
\end{aligned}
$$

$$
y(t) = h(t) * x(t) \overset{\mathcal{F}}{\leftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)
$$

5.5 The Multiplication Property

$$
r(t) = s(t)p(t) \overset{\mathcal{F}}{\leftrightarrow} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]
$$