CH_2

Intro Linear Time-invariant system

- linear
- time-invariant

2.1 Linear Constant-Coefficient Differential Equation

Linear constant-coefficient differential equation is the mathematical repression of a continuous-time LTI system.

Classical Solution of Differential Equations

$$
\begin{aligned}
&C_0\frac{d^n r(t)}{dt^n}+C_1\frac{d^{n-1}r(t)}{dt^{n-1}}+\cdots+C_{n-1}\frac{dr(t)}{dt}+C_n r(t)\\
&=E_0\frac{d^m e(t)}{dt^m}+E_1\frac{d^{m-1}e(t)}{dt^{m-1}}+\cdots+E_{m-1}\frac{de(t)}{dt}+E_m e(t)
\end{aligned}\tag{1}
$$

$$
C_0\frac{d^n r(t)}{dt^n}+C_1\frac{d^{n-1}r(t)}{dt^{n-1}}+\cdots+C_{n-1}\frac{dr(t)}{dt}+C_n r(t)=0\tag{2}
$$

- Homogeneous solution (natural response)
  1. **Characteristic equation**
     $$C_0\alpha^n+C_1\alpha^{n-1}+\cdots+C_{n-1}\alpha+C_n=0\tag{3}$$
  2. **Characteristic roots** $\alpha_1,\alpha_2,\cdots,\alpha_n$ satisfy tje characteristic equation

(**the basic solution to characteristic solutions**)

3. **Homogeneous solution**
   $r_h(t) = \sum_{i=1}^{n} A_i e^{\alpha_i t}$
   If α_1 is the k-th repeated roots, then there will be k number of α_1:$\sum_{i=1}^{k} A_i t^{k-i} e^{\alpha_1 t}$

- Particular solution (forced response)
  1. Table look-up

Exciting $e(t)$	**Response** $r(t)$
t^p	$B_1 t^p + B_2 t^{p-1} + L + B_p t + B_{p+1}$
$e^{\alpha t}$	$Be^{\alpha t}$
$\sin(\omega t)/\cos(\omega t)$	$B_1 \sin(\omega t) + B_2 \cos(\omega t)$
$t^p e^{\alpha t} \sin(\omega t)/$ $t^p e^{\alpha t} \cos(\omega t)$	$(B_1 t^p + B_2 t^{p-1} + L + B_p t + B_{p+1}) e^{\alpha t} \sin(\omega t) + (D_1 t^p + D_2 t^{p-1} + L + D_p t + D_{p+1}) e^{\alpha t} \cos(\omega t)$

- Auxiliary conditions (initial conditions)
  1. Determine the coefficients

$$r(t_0), \frac{dr(t_0)}{dt}, \frac{d^2 r(t_0)}{dt^2}, L, \frac{d^{n-1} r(t_0)}{dt^{n-1}}$$

2.2 Partial Initial Conditions

The Law of Switching

$$v_c(0_-) = v_c(0_+),\ i_L(0_-) = i_L(0_+)$$

Impulse Function Matching Method

$$\delta(t) \quad \delta'(t) \quad \delta''(t) \quad \cdots$$

The Laplace Transform

$$X(s) = \int_0^\infty x(t) e^{-st} \mathrm{d}t$$

2.3 System Response

$$\begin{aligned} \text{Total Response } &= \\ &= \text{Natural Response} + \text{Forced Response} \\ &= \text{Zero-input Response} + \text{Zero-state Response} \\ &= \text{Transient Response} + \text{Steady-state Response} \end{aligned}$$

- **Zero-input Response**: results only from the initial state (*energy storage*)
- **Zero-state Response**: results only from the external inputs
- **零输入响应**: 求系统的通解，由非零的0_-状态值决定的初始值（0_+状态值）求出待定系数
- **零状态响应**: 在激励信号作用下求系统方程的全解，由0_-状态值为零决定的初始值（0_+状态值）求待定系数

- *Transient Response*: when $t \to \infty$, the response that approaches to 0
- *Steady-state Response*: when $t \to \infty$, the response that keeps

$$
\begin{aligned}
r(t) = r_{zi}(t) + r_{zs}(t) &= \sum_{k=1}^{n} A_{zik} e^{\alpha_k t} + \sum_{k=1}^{n} A_{zsk} e^{\alpha_k t} + B(t) \\
&= \text{Zero-input Response} + \text{Zero-state Response} \\
&= \sum_{k=1}^{n} A_k e^{\alpha_k t} + B(t) \\
&= \text{Natural Response} + \text{Forced Response}
\end{aligned}
$$

2.4 The Unit Impulse Response

$$
\begin{aligned}
& C_0 \frac{\mathrm{d}^n r(t)}{\mathrm{d}t^n} + C_1 \frac{\mathrm{d}^{n-1} r(t)}{\mathrm{d}t^{n-1}} + \cdots + C_{n-1} \frac{\mathrm{d}r(t)}{\mathrm{d}t} + C_n r(t) \\
& = E_0 \frac{\mathrm{d}^m e(t)}{\mathrm{d}t^m} + E_1 \frac{\mathrm{d}^{m-1} e(t)}{\mathrm{d}t^{m-1}} + \cdots + E_{m-1} \frac{\mathrm{d}e(t)}{\mathrm{d}t} + E_m e(t)
\end{aligned} \tag{1}
$$

if $e(t) = \delta(t)$, then $r(t) = h(t)$

$$
\Rightarrow \quad
\begin{aligned}
& C_0 h^{(n)}(t) + C_1 h^{(n-1)}(t) + \cdots + C_{n-1} h^{(1)}(t) + C_n h(t) \\
& = E_0 \delta^{(m)}(t) + E_1 \delta^{(m-1)}(t) + \cdots + E_{m-1} \delta^{(1)}(t) + E_m \delta(t)
\end{aligned} \tag{4}
$$

Conditions	$h(t)$
$n > m$	$[\sum_{i=1}^{n} A_i e^{\alpha_i t}] u(t)$
$n = m$	$[\sum_{i=1}^{n} A_i e^{\alpha_i t}] u(t) + B\delta(t)$
$n < m$	$[\sum_{i=1}^{n} A_i e^{\alpha_i t}] u(t) + B\delta(t) + C\delta'(t)$

2.5 The Convolution Integral

$$
y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \mathrm{d}\tau
$$

- time reversal: $h(\tau) \to h(-\tau)$
- time shift: $h(-\tau) \to h(t-\tau)$
- multiplication: $x(\tau)h(t-\tau)$
- integrating: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau$

2.6 Properties of Convolution

The Commutative Property

$$
x(t) * h(t) = h(t) * x(t)
$$

The Distributive Property

$$
x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)
$$

The Associative Property

$$
[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]
$$

The Differential and Integral Property

$$\text{differential}$$

$$y'(t) = x(t) * h'(t) = x'(t) * h(t)$$

$$y^{(n)}(t) = x(t) * h^{(n)}(t) = x^{(n)}(t) * h(t)$$

$$\text{integral}$$

$$y^{(-1)}(t) = x(t) * h^{(-1)}(t) = x^{(-1)}(t) * h(t)$$

$$y^{(-m)}(t) = x(t) * h^{(-m)}(t) = x^{(-m)}(t) * h(t)$$

$$\text{generally}$$

$$y^{(n-m)}(t) = x^{(n)}(t) * h^{(-m)}(t) = x^{(-m)}(t) * h^{(n)}(t)$$

$$\text{specially, } n = 1 \ m = 1$$

$$y(t) = x^{(-1)}(t) * h'(t) = x'(t) * h^{(-1)}(t)$$

Convolution with Unit Impulse Signal

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} f(t)\delta(\tau - t)\mathrm{d}\tau = f(t)$$

$$f(t) * \delta(t - t_0) = f(t - t_0)$$

$$f(t - t_1) * \delta(t - t_2) = f(t - t_1 - t_2)$$

$$f(t) * \delta'(t) = f'(t)$$

$$f(t) * u(t) = \int_{-\infty}^{t} f(\lambda)\mathrm{d}\lambda$$

$$f(t) * \delta^{(k)}(t) = f^{(k)}(t)$$

$$f(t) * \delta^{(k)}(t - t_0) = f^{(k)}(t - t_0)$$

2.7 System Properties

Output Function

$$Output\ Signal = Input\ Signal * Impulse\ Signal$$

Stability

Bounded Input Bounded Output (BIBO)

$\int_{-\infty}^{\infty} |h(t)|\mathrm{d}t < \infty$

Proof:

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

Let $|x(t)| \le B$

$$|y(t)| = |h(t) * x(t)| = |\int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau) \mathrm{d}\tau|$$

$$\le \int_{-\infty}^{\infty} |h(\tau)| \cdot |x(t-\tau)| \mathrm{d}\tau \le B \int_{-\infty}^{\infty} |h(\tau)| \mathrm{d}\tau$$

If $|y(t)| < \infty$, then $\int_{-\infty}^{\infty} |h(\tau)| \mathrm{d}\tau < \infty$