CH_2

Intro Linear Time-invariant system

- linear
- time-invariant

2.1 Linear Constant-Coefficient Differential Equation

Linear constant-coefficient differential equation is the mathematical repression of a continuous-time LTI system.

Classical Solution of Differential Equations

$$\begin{aligned} &C_0\frac{d^n r(t)}{dt^n}+C_1\frac{d^{n-1}r(t)}{dt^{n-1}}+\cdots+C_{n-1}\frac{dr(t)}{dt}+C_n r(t) \\ &= E_0\frac{d^m e(t)}{dt^m}+E_1\frac{d^{m-1}e(t)}{dt^{m-1}}+\cdots+E_{m-1}\frac{de(t)}{dt}+E_m e(t) \end{aligned} \tag{1}$$

$$C_0\frac{d^n r(t)}{dt^n}+C_1\frac{d^{n-1}r(t)}{dt^{n-1}}+\cdots+C_{n-1}\frac{dr(t)}{dt}+C_n r(t)=0 \tag{2}$$

- Homogeneous solution (natural response)
  1. **Characteristic equation**
     $$C_0\alpha^n+C_1\alpha^{n-1}+\cdots+C_{n-1}\alpha+C_n=0 \tag{3}$$
  2. **Characteristic roots** $\alpha_1,\alpha_2,\cdots,\alpha_n$ satisfy tje characteristic equation
  3. **Homogeneous solution**
     $r_h(t)=\sum_{i=1}^{n}A_i e^{\alpha_i t}$
     If α_1 is the k-th repeated roots, then there will be k number of α_1:$\sum_{i=1}^{k}A_i t^{k-i}e^{\alpha_1 t}$
- Particular solution (forced response)
  1. Table look-up

Exciting $e(t)$	**Response** $r(t)$
t^p	$B_1 t^p + B_2 t^{p-1} + L + B_p t + B_{p+1}$
$e^{\alpha t}$	$B e^{\alpha t}$
$\sin(\omega t)/\cos(\omega t)$	$B_1 \sin(\omega t) + B_2 \cos(\omega t)$
$t^p e^{\alpha t} \sin(\omega t)/$ $t^p e^{\alpha t} \cos(\omega t)$	$(B_1 t^p + B_2 t^{p-1} + L + B_p t + B_{p+1}) e^{\alpha t} \sin(\omega t) + (D_1 t^p + D_2 t^{p-1} + L + D_p t + D_{p+1}) e^{\alpha t} \cos(\omega t)$

- Auxiliary conditions (initial conditions)
    1. Determine the coefficients

$$r(t_0), \frac{dr(t_0)}{dt}, \frac{d^2 r(t_0)}{dt^2}, L, \frac{d^{n-1} r(t_0)}{dt^{n-1}}$$

2.2 Partial Initial Conditions

The Law of Switching

$$v_c(0_-) = v_c(0_+),\ i_L(0_-) = i_L(0_+)$$

Impulse Function Matching Method

$$\delta(t) \quad \delta'(t) \quad \delta''(t) \quad \cdots$$

The Laplace Transform

$$X(s) = \int_0^{\infty} x(t) e^{-st} \mathrm{dt}$$

2.3 System Response

$$\begin{aligned} \text{Total Response } &= \\ &= \text{Natural Response} + \text{Forced Response} \\ &= \text{Zero-input Response} + \text{Zero-state Response} \\ &= \text{Transient Response} + \text{Steady-state Response} \end{aligned}$$

- **Zero-input Response**: results only from the initial state (*energy storage*)
- **Zero-state Response**: results only from the external inputs

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- *Transient Response*: when $t \to \infty$, the response that approaches to 0
- *Steady-state Response*: when $t \to \infty$, the response that keeps

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$$\begin{aligned} r(t) = r_{zi}(t) + r_{zs}(t) &= \sum_{k=1}^{n} A_{zik} e^{\alpha_k t} + \sum_{k=1}^{n} A_{zsk} e^{\alpha_k t} + B(t) \\ &= \text{Zero-input Response} + \text{Zero-state Response} \\ &= \sum_{k=1}^{n} A_k e^{\alpha_k t} + B(t) \\ &= \text{Natural Response} + \text{Forced Response} \end{aligned}$$

2.4 The Unit Impulse Response

$$
\begin{aligned}
&C_0\frac{\mathrm{d}^n r(t)}{\mathrm{d}t^n}+C_1\frac{\mathrm{d}^{n-1} r(t)}{\mathrm{d}t^{n-1}}+\cdots+C_{n-1}\frac{\mathrm{d}r(t)}{\mathrm{d}t}+C_n r(t)\\
&=E_0\frac{\mathrm{d}^m e(t)}{\mathrm{d}t^m}+E_1\frac{\mathrm{d}^{m-1} e(t)}{\mathrm{d}t^{m-1}}+\cdots+E_{m-1}\frac{\mathrm{d}e(t)}{\mathrm{d}t}+E_m e(t)
\end{aligned}
\tag{1}
$$

if $e(t)=\delta(t)$, then $r(t)=h(t)$

$$
\Rightarrow \quad
\begin{aligned}
&C_0h^{(n)}(t)+C_1h^{(n-1)}(t)+\cdots+C_{n-1}h^{(1)}(t)+C_nh(t)\\
&=E_0\delta^{(m)}(t)+E_1\delta^{(m-1)}(t)+\cdots+E_{m-1}\delta^{(1)}(t)+E_m\delta(t)
\end{aligned}
\tag{4}
$$

Conditions	$h(t)$
$n>m$	$[\sum_{i=1}^n A_i e^{\alpha_i t}]u(t)$
$n=m$	$[\sum_{i=1}^n A_i e^{\alpha_i t}]u(t)+B\delta(t)$
$n<m$	$[\sum_{i=1}^n A_i e^{\alpha_i t}]u(t)+B\delta(t)+C\delta'(t)$