CH_4

4.1 Fourier Series Representation of Continuous-time Periodic Signals

Trigonometric Fourier Series

Periodic Signal $x(t)$, with period T_1

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_1 t) + b_n \sin(n\omega_1 t)$$

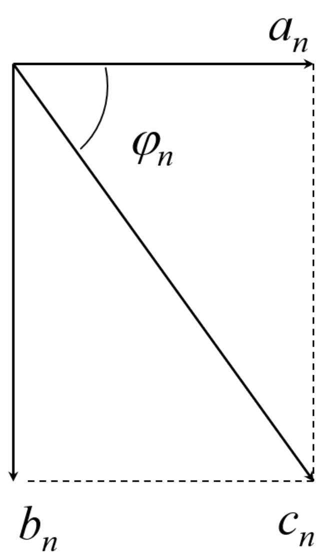
- Constant Component: $a_0 = \frac{1}{T_1}\int_{t_0}^{t_0+T_1} x(t)\mathrm{d}t$
- Cosine Coefficients: $a_n = \frac{2}{T_1}\int_{t_0}^{t_0+T_1} x(t)\cos(n\omega_1 t)\mathrm{d}t$
- Sine Coefficients: $b_n = \frac{2}{T_1}\int_{t_0}^{t_0+T_1} x(t)\sin(n\omega_1 t)\mathrm{d}t$

An Alternative Form for the Fourier Series

$$x(t) = c_0 + \sum_{n=0}^{\infty} c_n \cos(n\omega_1 t + \phi_n)$$

$$c_0 = a_0 \qquad c_n = \sqrt{a_n^2 + b_n^2} \qquad \phi_n = \arctan(-\frac{b_n}{a_n})$$

$$a_n = c_n \cos(\phi) \qquad b_n = -c_n \sin(\phi_n)$$



- x(t) is real and even

$$b_n = 0 \qquad c_n = a_n$$

$$\phi_n = \begin{cases} 0 & a_n > 0 \\ \pm\pi & a_n < 0 \end{cases}$$

- x(t) is real and odd

$$a_n = 0 \qquad \phi_n = \pm\frac{\pi}{2}$$

Complex Exponentially Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_1 t) + b_n \sin(n\omega_1 t)$$

$$= a_0 + \sum_{n=1}^{\infty} [\frac{a_n - jb_n}{2} e^{jn\omega_1 t} + \frac{a_n + jb_n}{2} e^{-jn\omega_1 t}]$$

Let

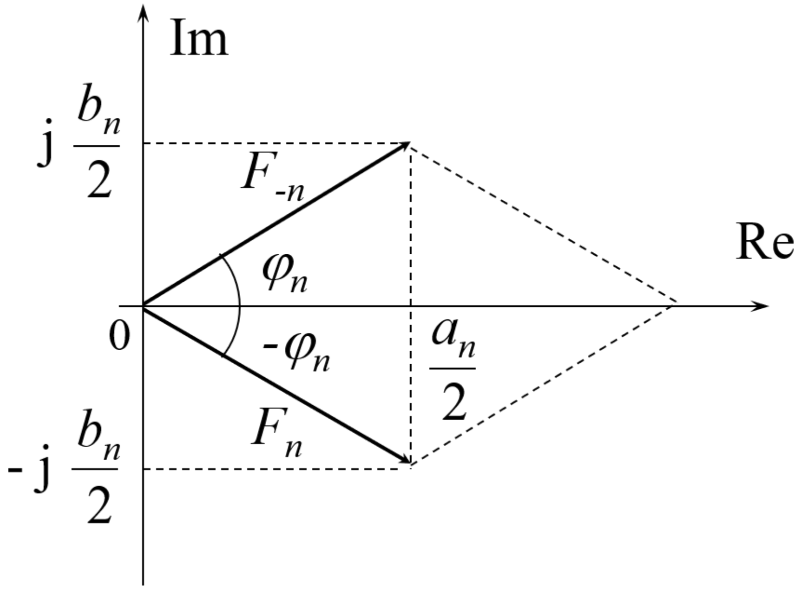
$$F(n\omega_1) = \frac{a_n - jb_n}{2}$$

$$F(-n\omega_1) = \frac{a_n + jb_n}{2}$$

$$\because \sum_{n=1}^{\infty} F(-n\omega_1) e^{-jn\omega_1 t} = \sum_{n=-1}^{-\infty} F(n\omega_1) e^{jn\omega_1 t}$$

$$\therefore x(t) = \sum_{n=-\infty}^{\infty} F(n\omega_1) e^{jn\omega_1 t}$$

$$F(n\omega_1) = \frac{1}{T_1} \int_0^{T_1} x(t) e^{-jn\omega_1 t} \mathrm{d}t$$



$$F(n\omega_1) = |F(n\omega_1)|e^{j\phi_n}$$

$$|F(n\omega_1)| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} = \frac{1}{2}c_n \qquad \phi_n = \arctan(-\frac{b_n}{a_n})$$

$$|F_n| = |F_{-n}| = \frac{1}{2}c_n \quad |F_n| + |F_{-n}| = c_n \quad F_n + F_{-n} = a_n$$

since the complex exponential Fourier series is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

- $a_1 e^{\pm j\omega_0 t}$: fundamental components (first harmonic components)
- $a_n e^{\pm jn\omega_0 t}$: Nth harmonic components

Names of the Equations and Coefficients

Given that $\omega_0 = \frac{2\pi}{T}$

- Synthesis Equation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- Analysis Equation:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\mathrm{d}t$$

- Fourier Series Coefficients (Spectral Coefficients): $\{a_k\}$
- Magnitude Spectrum: $|a_k|$
- Phase Spectrum: $\angle a_k$

> the magnitude spectrum and phase spectrum all have **discrete property**, **harmonic property** and **convergence property**

- Constant Component:

$$a_0 = \frac{1}{T}\int_T x(t)\mathrm{d}t$$

The Effect of Symmetry

Even Function

The Fourier Series only contains **constant coefficients** and **cosine coefficients**, and $F(n\omega)$ is the **real function**

$$b_n = 0 \qquad a_n \neq 0$$

$$F_n = \frac{1}{2}a_n \qquad \phi_n = 0$$

Odd Function

The Fourier Series function only contains **sine coefficients**, and $F(n\omega_1)$ is the **virtual function**

$$a_0 = 0 \qquad a_n = 0$$

$$b_n \neq 0 \qquad F_n = -\frac{1}{2}jb_n$$

Odd Harmonic Function

$$x(t) = -x(t \pm \frac{T}{2})$$

$$\begin{cases} a_0 = 0 \\ a_n = b_n = 0 \end{cases} \qquad n = 2k$$

$$\begin{cases} a_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos(n\omega_1 t) \\ b_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\sin(n\omega_1 t) \end{cases} \qquad n = 2k-1$$

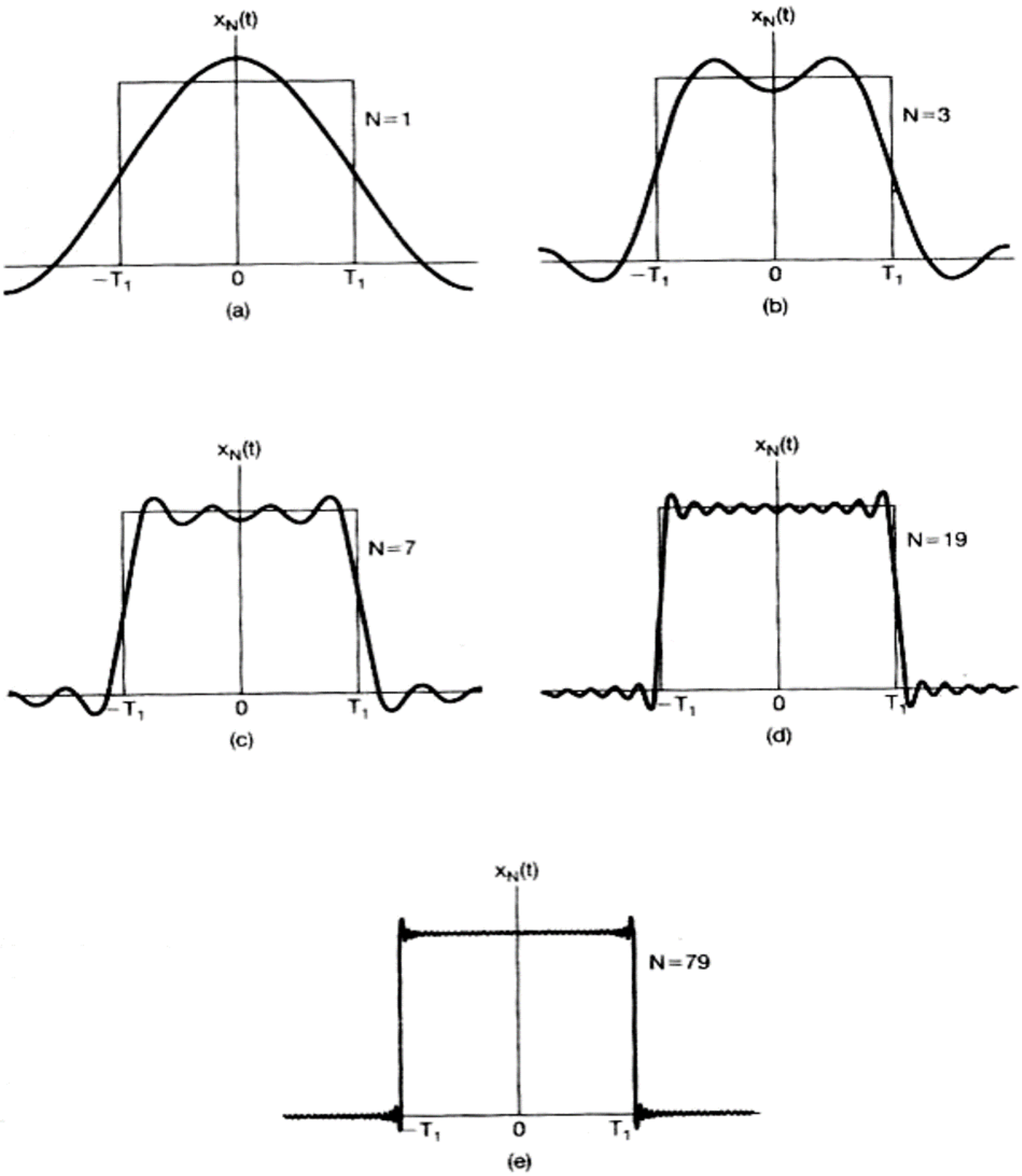
4.2 Convergence of the Fourier Series

Dirichlet Conditions

- must be integrable
- bounded variation
- finite number of discontinuities

Gibbs Phenomenon

the truncated Fourier series approximation $x_N(t)$ of a discontinuous signal $x(t)$ will in general exhibit high-frequency ripples and overshoot $x(t)$ near the discontinuities



4.3 Parseval's Relation

the **total average power** in a periodic signal equals the sum of the average powers in **all of its harmonic components**