5x 20pts = 100pts

stochastic vol & local vol.

1. $\frac{dS_t}{S_t} = \sigma_L(S_t, t)\, dZ_t$.   local vol.

undiscounted price of European option. $C(S_0, K, T) = \int_K^\infty dS_T\, \varphi(S_T, T; S_0)(S_T - K)$.

Focker-Planck equation: $\frac{1}{2}\frac{\partial^2}{\partial S_T^2}(\sigma_L^2(S_T, T) S_T^2 \varphi) = \frac{\partial \varphi}{\partial T}$

(a) Derive Dupire equation. $\frac{\partial C}{\partial T} = \sigma_L^2(K, T)\frac{K^2}{2}\frac{\partial^2 C}{\partial K^2}$.

Solution: $\frac{\partial C}{\partial T} = \int_0^\infty dS_T \left\{ \frac{1}{2}\frac{\partial^2}{\partial S_T^2}(\sigma_L^2(S_T, T) S_T^2 \varphi) \right\} (S_T - K)^+$.

integral by part twice.

$= \frac{1}{2}\sigma_L^2 K^2 \varphi(K) = \frac{1}{2}\sigma_L^2 K^2 \frac{\partial^2 C}{\partial K^2}$.

(b) $\frac{dS_t}{S_t} = \sqrt{V_t}\, dZ_t$.

Show that $\sigma_L^2(K, T) = E[V_T | S_T = K]$.

Solution: Terminal payoff $= (S_T - K)^+$

Ito's lemma: $d(S_T - K)^+ = \theta(S_T - K)\, dS_T + \frac{1}{2} V_T S_T^2\, \delta(S_T - K)\, dT$.

Conditional exp: $dC = dE[(S_T - K)^+] = \frac{1}{2}E[V_T | S_T = K]\, K^2\, \underbrace{E[\delta(S_T - K)]}_{\text{density}}\, dT$

$= \frac{1}{2}K^2 E[V_T | S_T = K]\frac{\partial^2 C}{\partial K^2}\, dT$

$\therefore \frac{\partial C}{\partial T} = \frac{1}{2}K^2 E[V_T | S_T = K]\frac{\partial^2 C}{\partial K^2}$.

compared with Dupire equation: $\sigma_L^2(K, T) = E[V_T | S_T = K]$.

2. $\frac{dS_t}{S_t} = \sqrt{V_t}\, dZ_t.$

$dV_t = -k(V_t - \bar{V})\, dt + \alpha V_t\, dW_t.$

(a) $\xi_t(u) = E[V_u | F_t]$

$dE[V_t] = -k(E[V_t] - \bar{V})dt$

$$\frac{dE[V_t]}{dt} = -k(E[V_t] - \bar{V}).$$

$E_t[V_u] = (V_t - \bar{V})e^{-k(u-t)} + \bar{V}.$

$d\xi_t(u) = e^{-k(u-t)} \alpha V_t dW_t.$ (martingale).

(b). Bergomi: $\sigma_{BS}(k,\tau) = \hat{\sigma}_T + S_T k + O(\varepsilon^2).$

$$S_T = \sqrt{\frac{w}{T}} \frac{1}{2w^2} C^{x\xi}$$

$$C^{x\xi} = \int_0^T dt \int_t^T du \frac{E[dx_t\, d\xi_t(u)]}{dt}.$$

$E_0[dx_t\, d\xi_t(u)] = E_0[\sqrt{V_t} dZ_t.\ e^{-k(u-t)} \alpha V_t dW_t]$

$= E_0[V_t^{\frac{3}{2}} dt\, \rho \alpha e^{-k(u-t)}] = \xi_0(t)^{\frac{3}{2}} \rho \alpha e^{-k(u-t)} dt + O(\alpha^2).$

Given $E[V_t^{\frac{3}{2}}] = \xi_0(t)^{\frac{3}{2}} + O(\alpha^2)$

$$C^{x\xi} = \xi_0(t)^{\frac{3}{2}} \rho\alpha. \int_0^T -\frac{1}{k}(e^{-k(T-t)} - 1).\ dt.$$

$$= \underbrace{\xi_0(t)^{\frac{3}{2}}}_{\bar{V}} \rho\alpha\left[-\frac{1}{k^2}(1-e^{-kT}) + \frac{1}{k}T\right] + O(\varepsilon^2)$$

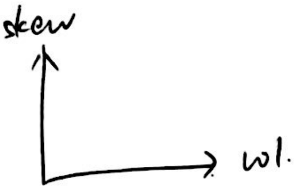
(c). B. Term structure of ATM vol skew.

$$\psi_L(T) = \frac{\partial}{\partial k}\sigma_{BS}(k,T)\Big|_{k=0}.$$

$$\psi_L(T) = S_T = \frac{1}{2T^2}\frac{1}{v^{\frac{3}{2}}}\bar{v}^{\frac{3}{2}}\frac{\rho\alpha}{k}\left\{T-\frac{1-e^{-kT}}{k}\right\}.$$

$$= \frac{\rho\alpha}{2}\frac{1}{kT}\left\{1-\frac{1-e^{-kT}}{kT}\right\} \quad \text{(log normal)}.$$

(d) $$\psi_H(T) = \frac{\rho\eta}{2\sqrt{\bar{v}}}\frac{1}{kT}\left\{1-\frac{1-e^{-kT}}{kT}\right\} \quad \text{(Heston)}$$



Log normal: no pattern.

Heston: (log-log plot)

3. SSVI. $w(k,T) = \sigma_{BS}^2(k,T)\,T.$

$$w(k,T) = \frac{\theta(T)}{2}\left\{1+\rho\phi(T)k+\sqrt{(\phi(T)k+\rho)^2+(1-\rho^2)}\right\}.$$

$\theta(T)$: ATM vol.

$\phi(T)$: skew.

(a). $w = \theta(1-\phi k)^+.$ $\quad \rho = -1.$

(b). ATM skew and curvature.

$w(0,T) = \theta.$ $\quad \partial_k w(0,T) = -\theta\phi\, 1_{k<\frac{1}{\phi}}$ $\quad \partial_{k,k} w(0,T) = 0.$

(c). $$v_L(k,T) = \frac{\frac{\partial w}{\partial T}}{\left(1-\frac{1}{2}\frac{k}{w}\frac{\partial w}{\partial k}\right)^2-\frac{1}{4}\left(\frac{1}{4}+\frac{1}{w}\right)\left(\frac{\partial w}{\partial k}\right)^2+\frac{1}{2}\frac{\partial^2 w}{\partial k^2}}$$ $\quad \phi(T) = \frac{\alpha}{T^\gamma}.\ \alpha > 0.$

show $\gamma \geq 1$, T large. $\gamma \leq \frac{1}{2}$. T small

Only consider butterfly:

large T: $f(T) \sim 1-\frac{1}{4}\left(\frac{1}{4}+\frac{1}{w}\right)\left(\frac{\partial w}{\partial k}\right)^2 \sim 1-\frac{1}{16}(\theta\phi)^2 \sim 1-\frac{1}{16}\alpha^2 T^{2-2\gamma}.$

$\gamma \geq 1$

small T:

$$f(T) \sim 1-\frac{T}{4}\left(\frac{T}{4}+\frac{1}{w}\right)\left(\frac{\partial w}{\partial k}\right)^2 \sim 1-\frac{1}{16}(\theta\varphi)^2 \sim 1-\frac{1}{16}\alpha^2 T^{1-2\gamma} \quad \gamma \le \frac{1}{2}$$

(d). NO. arbitrage-free.

(e). $\frac{1}{T^\gamma}$ $\frac{1}{(T_0+T)^{1-\gamma}}$ $\gamma \le \frac{1}{2}$ both ~~long~~ large and short.

4. $1+\max\left[0, \text{MaxCoupon} + \sum_{i=1}^{N} \min(0, r_i)\right]$ $\quad r_i = \frac{S_i - S_{i-1}}{S_{i-1}}$

(a) principle guarantee: 1.

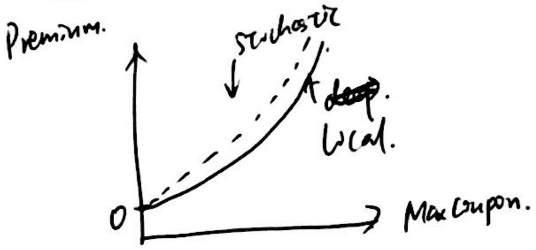
(b) Maximum redemption: 1+ MaxCoupon.

~~(c)~~ ~~Max~~ Under what condition: $r_i > 0 \quad \forall i.$

(c) principle not guaranteed: $1 + \text{MaxCoupon} + \underbrace{\sum_{i=1}^{N} \min(0, r_i)}_{\text{short puts. (forward starting)}}$

(d) MaxCoupon $= 0$. payoff $= 1$.

(e) stochastic vol & local vol:



short put spreads. (Loss is capped).

forward skew

use deep under local vol.

5. swaps. variance & gamma.

$$dS_t = \sqrt{v_t}\, S_t\, dZ_t.$$

$$dv_t = -\lambda(v_t - \bar{v})\, dt + \eta\sqrt{v_t}\, dW_t.$$

(a) forward variance ODE: $\hat{v}(t) = E[v_t]$ $\quad \frac{d\hat{v}}{dt} = -\lambda(\hat{v}-\bar{v})$ $\quad \hat{v}(t) = (v_0 - \bar{v})\, e^{-\lambda t} + \bar{v}.$

(b). fair value of variance swap. $\quad V(T) = \frac{1}{T} E\left[\int_0^T v_t\, dt\right] = \frac{1}{T}\left\{(v_0-\bar{v})\frac{1-e^{-\lambda T}}{\lambda} + \bar{v}T\right\}$

(c). ODE $u(t) = E[S_t v_t]$

$$d(S_t v_t) = S_t dv_t + v_t dS_t + dS_t dv_t$$

$$du = -\lambda(u - S_0\bar{v})dt + \rho\eta u\,dt$$

$$= [(\rho\eta - \lambda)u + \lambda S_0 \bar{v}]dt$$

$$\frac{du}{dt} = [\lambda S_0 \bar{v} - (\lambda - \rho\eta)u]dt$$

$$= \lambda'(u - \bar{u}')dt$$

$$\lambda' = \lambda - \rho\eta$$

$$\bar{u}' = \frac{\lambda S_0 \bar{v}}{\lambda - \rho\eta}$$

(d) gamma swap: $G(T) = \frac{1}{T}E\left[\int_0^T \frac{S_t}{S_0} v_t dt\right] = \frac{1}{T} \left\{ (v_0 - \bar{v}')\frac{1-e^{-\lambda' T}}{\lambda'} + \bar{v}'T \right\}$

$$\hat{u}(t) = (u_0 - \bar{u}')e^{-\lambda' t} + \bar{u}'$$

$$\bar{v}' = \frac{\lambda \bar{v}}{\lambda - \rho\eta}$$

(e) $\rho = 0$. $L(T) = G(T) - V(T) = 0$.

(f). $L(T) = \frac{1}{S_0}\int_0^T Cov[S_t, v_t]dt$ quadratic covariation.

$$L(T) = G(T) - V(T) = -\frac{1}{T}E\left[\int_0^T v_t dt\right] + \frac{1}{TS_0}E\left[\int_0^T \frac{S_t}{S_0} v_t dt\right]$$

$$= \frac{-1}{TS_0}E\left[\int_0^T S_0 v_t dt\right] + \frac{1}{TS_0}E\left[\int_0^T S_t v_t dt\right]$$

$$= \frac{-1}{TS_0}\int_0^T S_0 E[v_t]dt + \frac{1}{TS_0}\int_0^T E[S_t v_t]dt$$

$$= \frac{1}{TS_0}\int_0^T (E[S_t]E[v_t] + E[S_t v_t])dt$$