MTH 9875 The Volatility Surface: Fall 2019

Lecture 12: The VIX index and its derivatives

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Outline of lecture

- Volatility derivatives
  - Model-independence: bounds, conjectures and counterexample
- The VIX index
- VIX futures and options
- The VVIX index

- Variance curve models
  - The Double mean reverting (DMR) model
  - Dynamics consistent with VIX options prices
- VIX futures under rough volatility
- VIX-based ETNs
  - Backtessting VIX strategies

Valuation of volatility derivatives given vanilla prices

- In some Lecture 5, we saw that if volatility and stock price moves are uncorrelated, volatility derivatives may in principle be valued given just the prices of European options on stock.
  - The zero-correlation assumption is completely unrealistic but the connection between European option prices and volatility derivative values is suggestive.
- How tightly do the prices of European options (which are assumed to be known) constrain the fair value of a volatility derivative?

A simple lognormal model

Assume that $\log\left(\sqrt{\langle x\rangle_T}\right)$ is normally distributed with mean μ and variance s^2. Then $\log(\langle x\rangle_T)$ is also normally distributed with mean $2\,\mu$ and variance $4\,s^2$.

Volatility and variance swap values are given by respectively

$$\mathbb{E}[\sqrt{\langle x\rangle_T}] = e^{\mu+s^2/2};\ \mathbb{E}\left[\langle x\rangle_T\right] = e^{2\mu+2s^2}.$$

Solving for μ and s^2 gives

$$s^2 = 2\log\left(\frac{\sqrt{\mathbb{E}\left[\langle X\rangle_T\right]}}{\mathbb{E}\left[\sqrt{\langle X\rangle_T}\right]}\right);\ \ \mu = \log\left(\frac{\mathbb{E}\left[\sqrt{\langle X\rangle_T}\right]^2}{\sqrt{\mathbb{E}\left[\langle X\rangle_T\right]}}\right)$$

and the convexity adjustment is given by

$$\sqrt{\mathbb{E}\left[\langle X\rangle_T\right]} - \mathbb{E}\left[\sqrt{\langle X\rangle_T}\right] = \left(e^{s^2/2} - 1\right)\,\mathbb{E}\left[\sqrt{\langle X\rangle_T}\right].$$

Black-Scholes-style formula

Under this lognormal assumption, calls on variance may be valued using a Black-Scholes style formula:

(1)

$$\mathbb{E}\left[\langle x\rangle_T - K\right]^+ = e^{2\mu+2s^2}\,N(\tilde{d}_+) - K\,N(\tilde{d}_-)$$

with

$$\begin{aligned}\tilde{d}_+ &= \frac{-\frac{1}{2}\log K + \mu + 2s^2}{s}\\ \tilde{d}_- &= \frac{-\frac{1}{2}\log K + \mu}{s}.\end{aligned}$$

- This simple lognormal assumption seems not too unreasonable
  - The empirical distribution of implied volatility changes looks lognormal.
  - Also, the dynamics of the volatility skew are consistent with approximately lognormal volatility dynamics.
- In this model, the variance and volatility swap values (or equivalently the variance swap value plus the convexity adjustment) are all that is required to fix the values of all options on variance.

- Moreover, at least for the major equity indices, there is a tight market in variance swaps and a somewhat less liquid market in the convexity adjustment.

A Heston example

Suppose the true dynamics of the underlying were Heston (recall that lognormal volatility is a much more realistic assumption) with BCC parameters:

$$\lambda = 1.15, \rho = 0, \sigma_0^2 = \bar{\sigma}^2 = 0.04, \eta = 0.39.$$

- We showed earlier how to compute the value of a volatility swap in terms of the Heston parameters.
- We can also easily compute the values of calls on variance in terms of the Heston parameters.

- Given the values of the variance and the volatility swap, we can also apply equation (1) to value the same calls on variance.
- How big is the error from applying the lognormal variance call formula?

Lognormal formula vs exact Heston computation

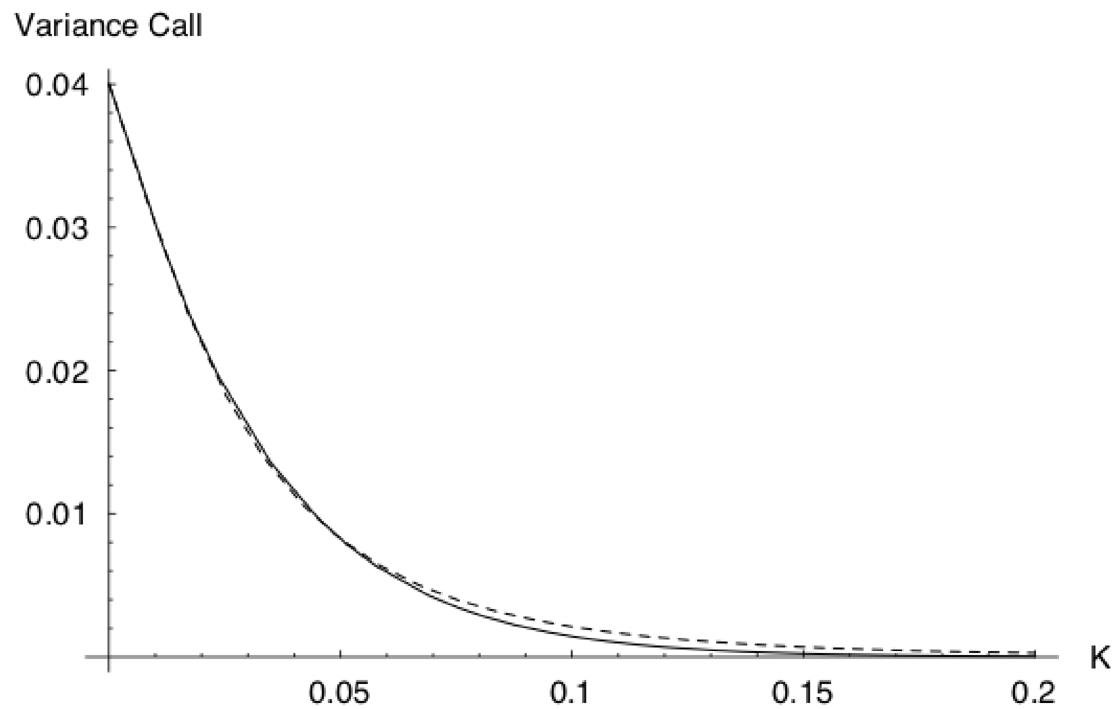


Figure 1: Value of one-year variance call vs variance strike K with the BCC parameters. The solid line is a numerical Heston solution; the dashed line comes from our lognormal approximation.

Remarks

- Obviously, the two approaches must agree at zero strike because then we have just a variance swap.
- It's not clear how big the valuation error really is away from zero strike.
- To get a better sense for the error, in Figure 2, we compare the density we get by inverting the Heston characteristic function numerically and the approximate lognormal density with mean $2\,\mu$ and variance $4\,s^2$.

Comparison of densities

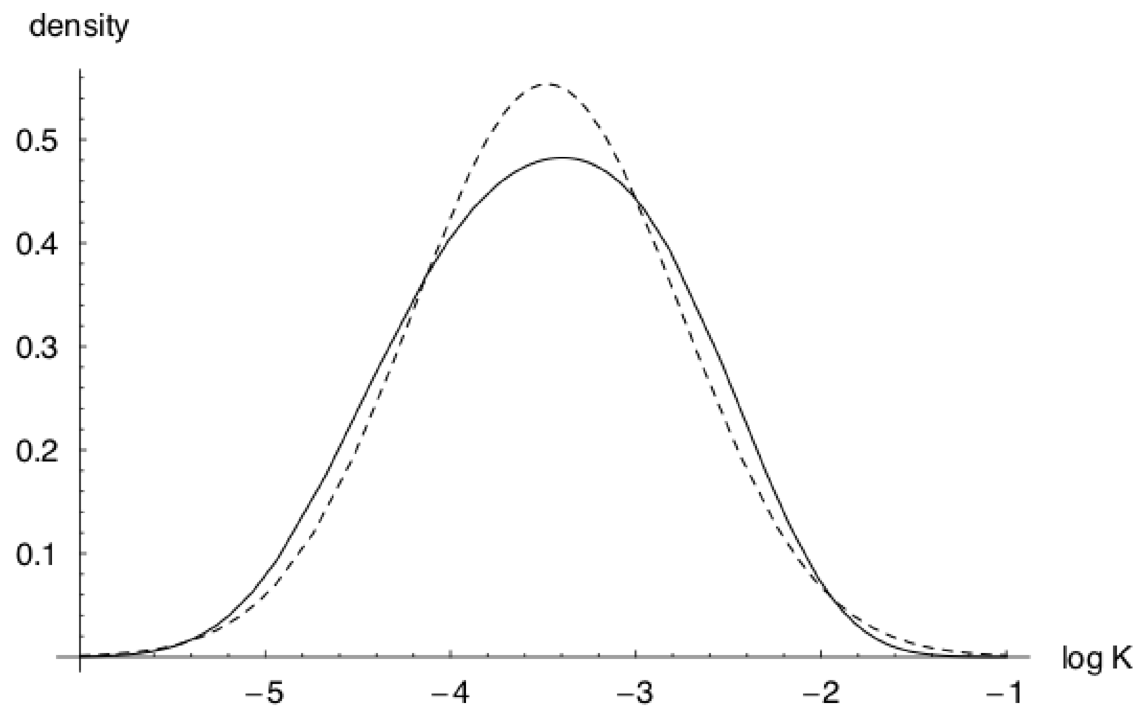


Figure 2: The pdf of the log of one-year quadratic variation with BCC parameters. The solid line comes from an exact numerical Heston computation; the dashed line comes from our lognormal approximation.

Constraints on volatility derivative pricing

- We see that agreement is pretty close.
  - And if agreement is so close in the Heston case where we believe the dynamics to be unrealistic, how much better should the agreement be when the underlying volatility dynamics are lognormal (the more realistic case)?
- We conclude that the values of T-maturity volatility derivatives are in practice tightly constrained by the prices of expiration-T European options.

Another potential recipe

- To make the argument even more explicit, we fit our preferred stochastic volatility model (lognormal let's say) to the prices of European options, and we set the correlation ρ to zero (this doesn't affect the value of volatility derivatives).
- Now recall the following exact formula for the volatility swap under zero correlation:

(2)
$$\mathbb{E}\left[\sqrt{\langle x\rangle_T}\right] = \sqrt{2\,\pi}\,\hat{c}(0) + \sqrt{\frac{2}{\pi}}\int_0^\infty e^{k/2}\,I_1\left(\frac{k}{2}\right)\,c(k)\,dk$$

where $I_n(\cdot)$ represents a modified Bessel function of the first kind.

- Compute the value of the volatility swap using (2) and apply our lognormal variance call option formula (1), confident that the result will be robust to the specific parameters.
  - The only constraint is that the fit to expiration-T European prices should be good.

Options on volatility: More on model-independence

- We have argued so far that given the prices of European options of all strikes and expirations, the values of options on variance should be tightly constrained.
  - The extent to which this is true and even how this claim should be precisely stated are as yet unresolved questions.

A model-independent lower bound

- [Dupire][8] shows how a model-independent lower bound for calls on realized variance may be obtained by considering the delta hedging of a portfolio of European call options.

- The idea is to hedge in business time defined as the timescale of realized quadratic variation.
  - Options are rehedged each time quadratic variation increases by a given amount.
  - Unlike conventional delta hedging in (atomic) clock time, assuming no jumps, this hedging strategy guarantees profits on a short option position if realized volatility is below implied volatility (the volatility implied by the initial sale price).
  - Conversely the hedged option position is guaranteed to lose if realized volatility is higher than implied at inception.

- Losses are insured against by buying an option on quadratic variation.
- The hedger ends up short a portfolio of European options and long an option on realized variance with only positive payoffs: this portfolio must be worth at least zero and a model-independent lower bound on the price of an option on variance follows.

- In the same work, Dupire shows (without formally proving) how solutions to the Skorokhod Embedding problem can generate effective upper and lower bounds on the price of an option on variance constrained by known prices of European options.

A wrong conjecture

- In [The Volatility Surface][9], it was conjectured that the minimum possible value of an option on variance is the one generated from a local volatility model fitted to the volatility surface.
  - Options on variance have value even in a local volatility model because realized variance depends on the realized path of the stock price from inception to expiration.

- Given that local variance is a risk-neutral conditional expectation of instantaneous variance, it seemed obvious that any other model would generate extra fluctuations of the local volatility surface relative to its initial state.

A counterexample due to Beiglböck et al.

Consider the following toy model from [Beiglböck et al.][4] where the volatility path can be one of

$$v_+ = \begin{cases} 2 & \text{if } t \in [0,1) \\ 3 & \text{if } t \in [1,2) \\ 1 & \text{if } t \in [2,3] \end{cases} \quad v_- = \begin{cases} 2 & \text{if } t \in [0,1) \\ 1 & \text{if } t \in [1,2) \\ 3 & \text{if } t \in [2,3] \end{cases}$$

depending on the toss of a coin.

Independent of the result of the coin-toss, the total variance

$$W_3 = \int_0^3 v_t \, dt = 6$$

so $\mathbb{E}[(W_3 - 6)^+] = 0$.

Now define

$$\Theta(x) = \partial_T C_{BS};\ \Psi(x) = K^2\, \partial_{K,K} C_{BS};$$

Then local variance is given by

$$\begin{aligned} v_\ell(x,t) = 2\, \frac{\partial_T C}{K^2\, \partial_{K,K} C} &= 2\, \frac{\Theta_+(x) + \Theta_-(x)}{\Psi_+(x) + \Psi_-(x)} \\ &= \frac{v_+(t)\, \Psi_+(x) + v_-(t)\, \Psi_-(x)}{\Psi_+(x) + \Psi_-(x)} \end{aligned}$$

which is obviously a nonconstant function of x and t.

One can verify with a Monte Carlo computation that

$\mathbb{E}[(\tilde{W}_3 - 6)^+] \approx 0.026 > 0.$

In words, the value of the option on variance under local volatility is greater than the value of the option on variance under stochastic volatility in this case.

Carr and Lee upper and lower bounds

- Given European options prices, [Carr and Lee][5] derive model-independent upper and lower bounds for both spot- and forward-starting options on quadratic variation assuming only that the underlying is a positive continuous semimartingale.
- Carr and Lee explicitly construct sub- and super-hedges.
  - The sub-hedge (and hence lower bound) is due to Dupire: Delta hedge a short call position in business time, making money if the realized variance is less than the implied variance at the time of sale.
  - The super-hedge involves replicating in business time a double knockout with a carefully chosen payoff at the barrier.

Conclusion

- Between these model-independent upper and lower bounds, it seems reasonable to suppose that the fair value of an option on variance is closest to that given by a stochastic volatility model.
  - It's easy to reject local volatility.

The VIX index

- In 2004, the CBOE listed futures on the VIX - an implied volatility index.
- Originally, the VIX computation was designed to mimic the implied volatility of an at-the-money 1 month option on the OEX index. It did this by averaging volatilities from 8 options (puts and calls from the closest to ATM strikes in the nearest and next to nearest months).
- The CBOE changed the VIX computation: "CBOE is changing VIX to provide a more precise and robust measure of expected market volatility and to create a viable underlying index for tradable volatility products."

The VIX formula

Here is the revised VIX definition (converted to our notation) as specified in the CBOE white paper:

(3)

$$VIX^2 = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2}\, Q_i(K_i) \,-\, \frac{1}{T} \left[\frac{F}{K_0} - 1 \right]^2$$

where Q_i is the price of the out-of-the-money option with strike K_i and K_0 is the highest strike below the forward price F .

We recognize (3) as a straightforward discretization of the log-strip and makes clear the reason why the CBOE implies that the new index permits replication of volatility.

Specifically, (with obvious notation)

$$\begin{aligned}
\frac{VIX^2\,T}{2} &= \int_0^F \frac{dK}{K^2}\,P(K) + \int_F^\infty \frac{dK}{K^2}\,C(K) \\
&= \int_0^{K_0} \frac{dK}{K^2}\,P(K) + \int_{K_0}^\infty \frac{dK}{K^2}\,C(K) + \int_{K_0}^F \frac{dK}{K^2}\,(P(K) - C(K)) \\
&=: \int_0^\infty \frac{dK}{K^2}\,Q(K) + \int_{K_0}^F \frac{dK}{K^2}\,(K - F) \\
&\approx \int_0^\infty \frac{dK}{K^2}\,Q(K) + \frac{1}{K_0^2}\int_{K_0}^F dK\;(K - F) \\
&= \int_0^\infty \frac{dK}{K^2}\,Q(K) - \frac{1}{K_0^2}\,\frac{(K_0 - F)^2}{2}.
\end{aligned}$$

One possible discretization of this last expression is

$$VIX^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2}\,Q_i(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2$$

as in the VIX specification (3).

VIX futures and options

A time-T VIX future is valued at time t as

$$\mathbb{E}_t\left[\sqrt{\frac{1}{\Delta}\mathbb{E}_T\left[\int_T^{T+\Delta} v_s\,ds\right]}\right]$$

where Δ is around one month (or $\Delta \approx 1/12$).

A VIX option expiring at time T with strike K_{VIX} is valued at time t as

$$\mathbb{E}_t\left[\left(\sqrt{\frac{1}{\Delta}\mathbb{E}_T\left[\int_T^{T+\Delta} v_s\,ds\right]} - K_{VIX}\right)^+\right].$$

Remarks on VIX futures and options

- Historically, the options have been much more liquid and actively traded than the futures.
- Both futures and synthetic futures (from put-call parity) used to trade at a substantial premium to the forward-starting variance swap.
  - This arbitrage has come and gone over time.
- Apparently, since the second quarter of 2013, there has been more vega traded in VIX than on SPX, which is now only the second biggest global market for volatility.

VIX futures and options

- Note that we can span the payoff of a forward starting variance swap $\mathbb{E}_t\left[\int_T^{T+\Delta} v_s\,ds\right]$ using VIX options.
- Recall the spanning formula:

$$\begin{aligned}
\mathbb{E}\left[g(S_T)\right] = g(F) &+ \int_0^F dK\,\tilde{P}(K)\,g''(K) \\
&+ \int_F^\infty dK\,\tilde{C}(K)\,g''(K).
\end{aligned}$$

- In this case, $g(x) = x^2$ so

$$\mathbb{E}_t \left[\int_T^{T+\Delta} v_s \, ds \right] = F_{VIX}^2 + 2 \int_0^{F_{VIX}} \tilde{P}(K) \, dK \\ + 2 \int_{F_{VIX}}^{\infty} \tilde{C}(K) \, dK.$$

  - F_{VIX} can be computed using put-call parity or observed directly as the VIX futures price.
  - We need to interpolate and extrapolate out-of-the-money option prices to get the *convexity adjustment*.

Martingale optimal transport

- The idea is to bound the expected value of the option over all martingale measures that have a given marginal at maturity.
  - Typically, super- and sub-replicating portfolios are also given.

- Matthias Beiglböck is a key player, albeit with very abstract contributions.

- A 2015 paper by [De Marco and Henry-Labordère][7] applies martingale optimal transport to SPX and VIX options.

  - They claim that on July 12, 2013, "market prices of VIX options are well above our analytical upper bound for $K/VIX > 1$."

Upper bound on VIX futures

[De Marco and Henry-Labordère][7] also present the following nice argument showing that the VIX future should be bounded above by its value in a local volatility model.

- For Δ/T sufficiently small

$$\begin{aligned} VIX_t(T) &= \mathbb{E}\left[\left. \sqrt{\frac{1}{\Delta} \int_T^{T+\Delta} \xi_T(s) \, ds} \right| \mathcal{F}_t \right] \\ &\approx \mathbb{E}\left[\left. \sqrt{v_T} \right| \mathcal{F}_t \right] \\ &\leq \mathbb{E}\left[\left. \sqrt{\mathbb{E}\left[v_T | S_T \right]} \right| \mathcal{F}_t \right] \text{ (Jensen)} \\ &= \mathbb{E}\left[\left. \sqrt{v_\ell(S_T, T)} \right| \mathcal{F}_t \right] \\ &\approx VIX_t^{LV}(T). \end{aligned}$$

- According to the authors "market prices of VIX futures typically lie in practice above prices given by the local volatility model", *i.e.* above the upper bound.

Graphic of claimed arbitrage

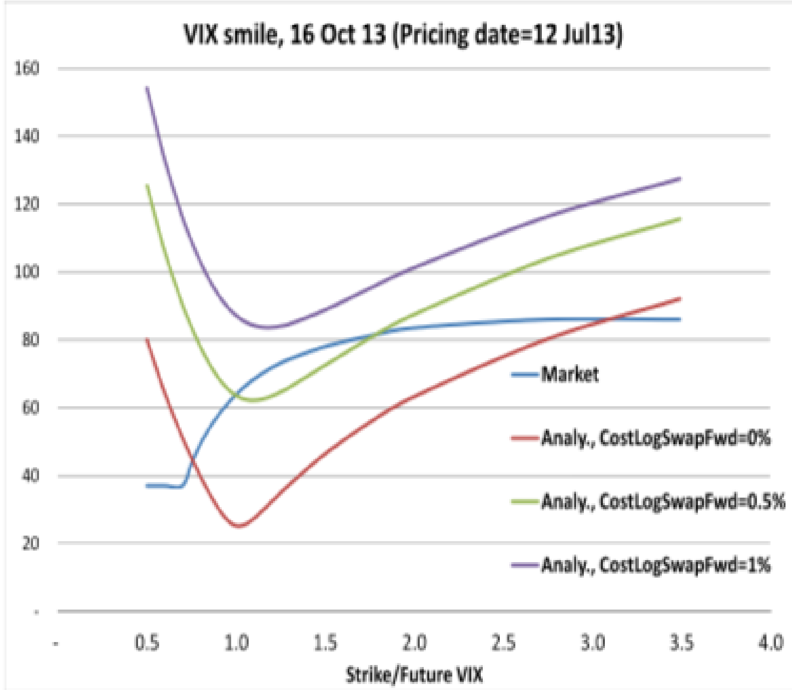


FIGURE 2. Analytical upper bound versus market values for VIX smile t_1 =16 Oct. 13 (pricing date = 12 Jul. 2013). $\sigma_{1,2}$ = 18.15% (VIX = 18.05%).

A real example: 15-Sep-2011

- We compute the fair value of variance swaps from SVI fits to the SPX smiles using the formula
$$\mathbb{E}[VIX^2] = \int_0^\infty dz\, N'(z)\, \sigma_{BS}^2(z).$$
with $z = d_2$.
- We compare with market variance swap quotes obtained from a friendly investment bank for the same date.

Load some R-code and data

In [1]:
```
download.file(url="http://mfe.baruch.cuny.edu/wp-content/uploads/2018/11/9875-12.zip", destfile
unzip(zipfile="9875-12.zip")

library(quantmod)
library(PerformanceAnalytics)
library(repr)
```

```
Loading required package: xts
Loading required package: zoo

Attaching package: 'zoo'

The following objects are masked from 'package:base':

    as.Date, as.Date.numeric

Loading required package: TTR
Version 0.4-0 included new data defaults. See ?getSymbols.

Attaching package: 'PerformanceAnalytics'

The following object is masked from 'package:graphics':

    legend
```

In [2]:
```
source("BlackScholes.R")
source("Lewis.R")
source("Heston.R")
source("plotIvols.R")
source("svi.R")
source("sviVarSwap.R")
```

In [3]:
```
# Read in SPX and VIX options data from 15-Sep-2011
load("spxVix20110915.rData")
load("fitQR110915.rData")
```

In [4]:
```
options(repr.plot.width=10,repr.plot.height=7)
```

SPX volatility smiles as of 15-Sep-2011

In [5]:
```
res <- plotIvols(spxOptData,sviMatrix=fitQR )
texp <- res$expiries
```

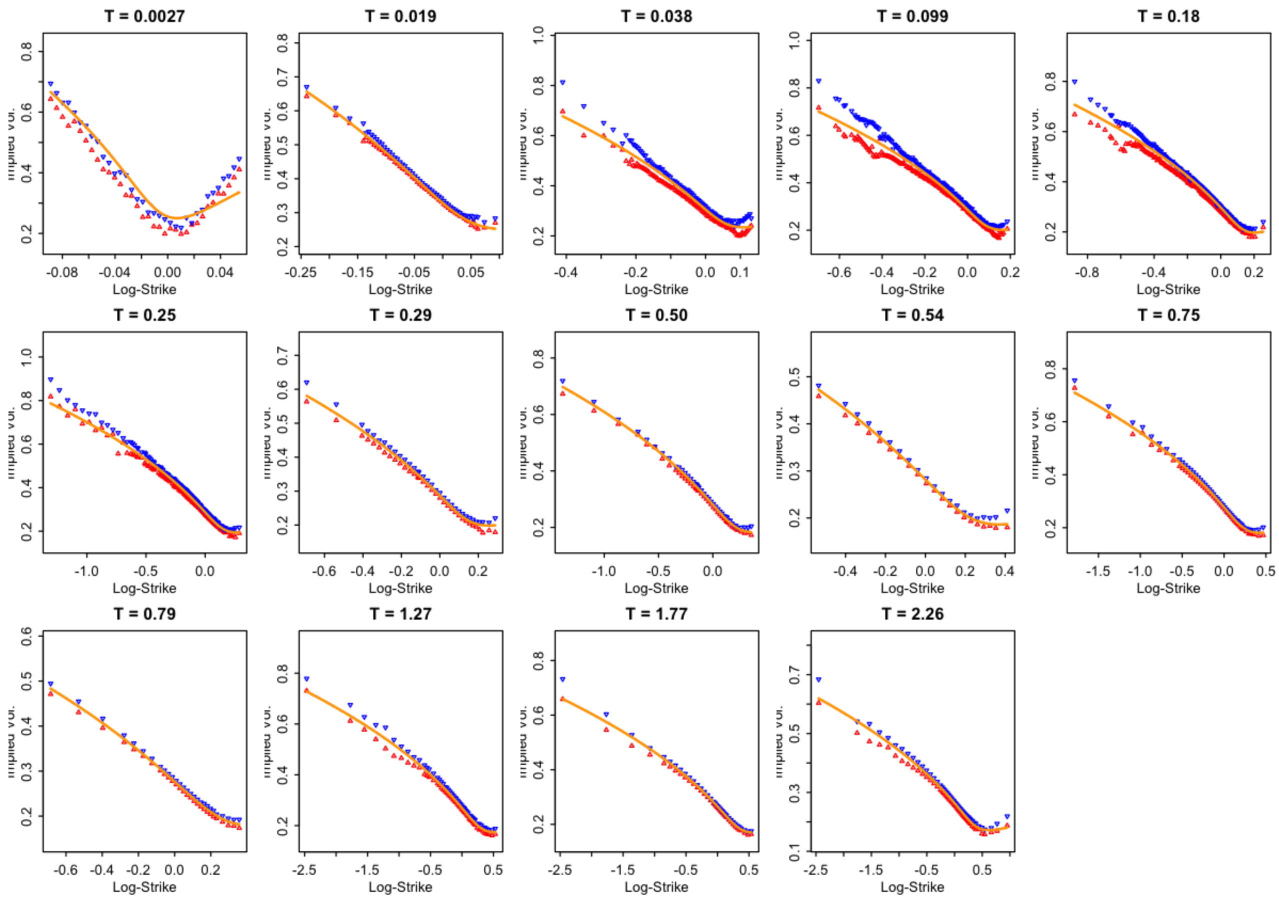


Figure 3: Implied volatility smiles of SPX options as of 15-Sep-2011. SVI fits are in orange.

Variance swaps as of 15-Sep-2011

In [6]:
```
sviVS <- sqrt(sviVarSwap(fitQR,texp=res$expiries))
plot(texp,sviVS,ylab="Variance swap level", xlab="Expiry",type="p",pch=20,cex=2,col="dark green
points(mktVarSwapData$texp,mktVarSwapData$vsBid,col="blue",type="b",lty=2, pch=20,cex=1)
points(mktVarSwapData$texp,mktVarSwapData$vsAsk,col="red",type="b",lty=2,pch=20,cex=1)
```

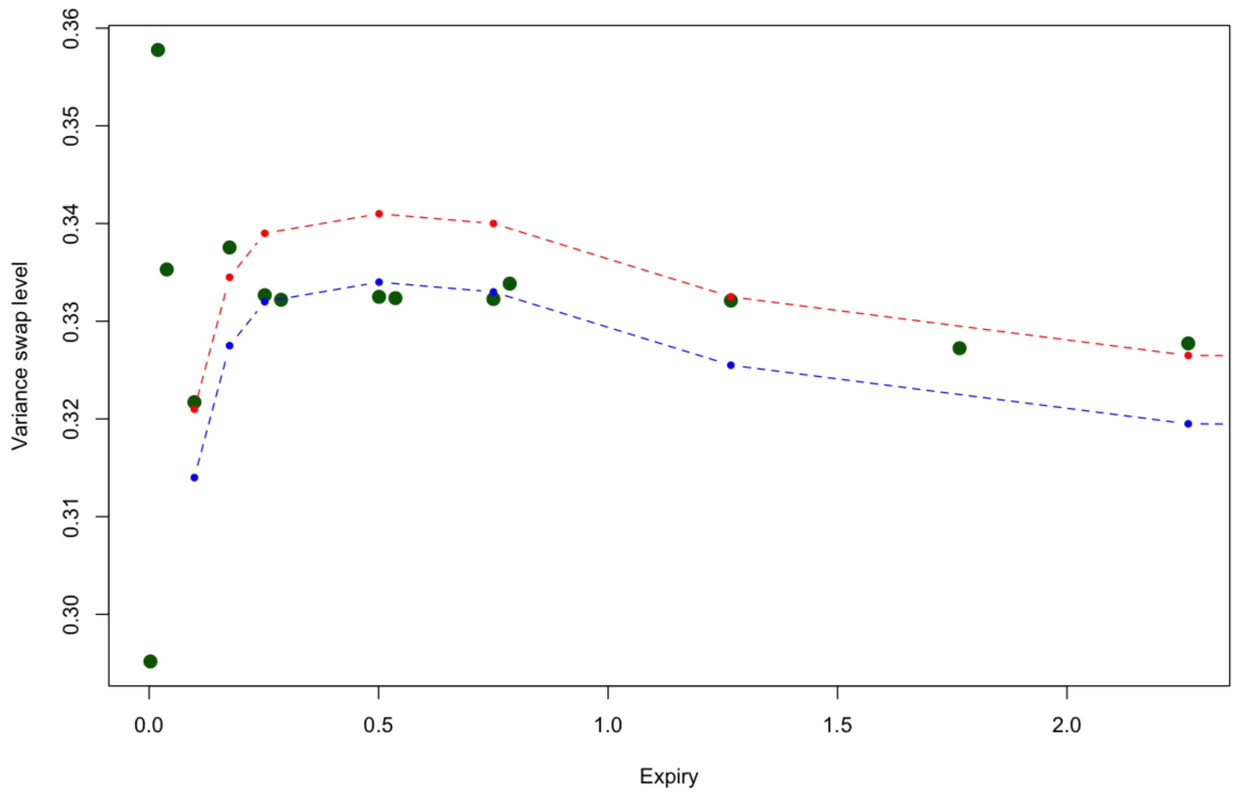


Figure 4: Green dots are computed using the log-strip of SPX options. Blue and red points are bid and ask variance swap quotes from a friendly investment bank.

Forward variance swaps from SPX and VIX options

- Once again, we can compute the fair value of forward starting variance swaps in three ways:
  - Using variance swaps from the SPX log-strip
    - We just computed these variance swaps.
  - From the linear strip of VIX options.
  - By interpolating market variance swap quotes.
- We now compute the linear VIX strip and compare the three computations as of September 15, 2011.

The VIX volatility smile as of 15-Sep-2011

In [7]:
```
options(repr.plot.width=7,repr.plot.height=5)
```

```
In [8]: res <- plotIvols(vixOptData)
```

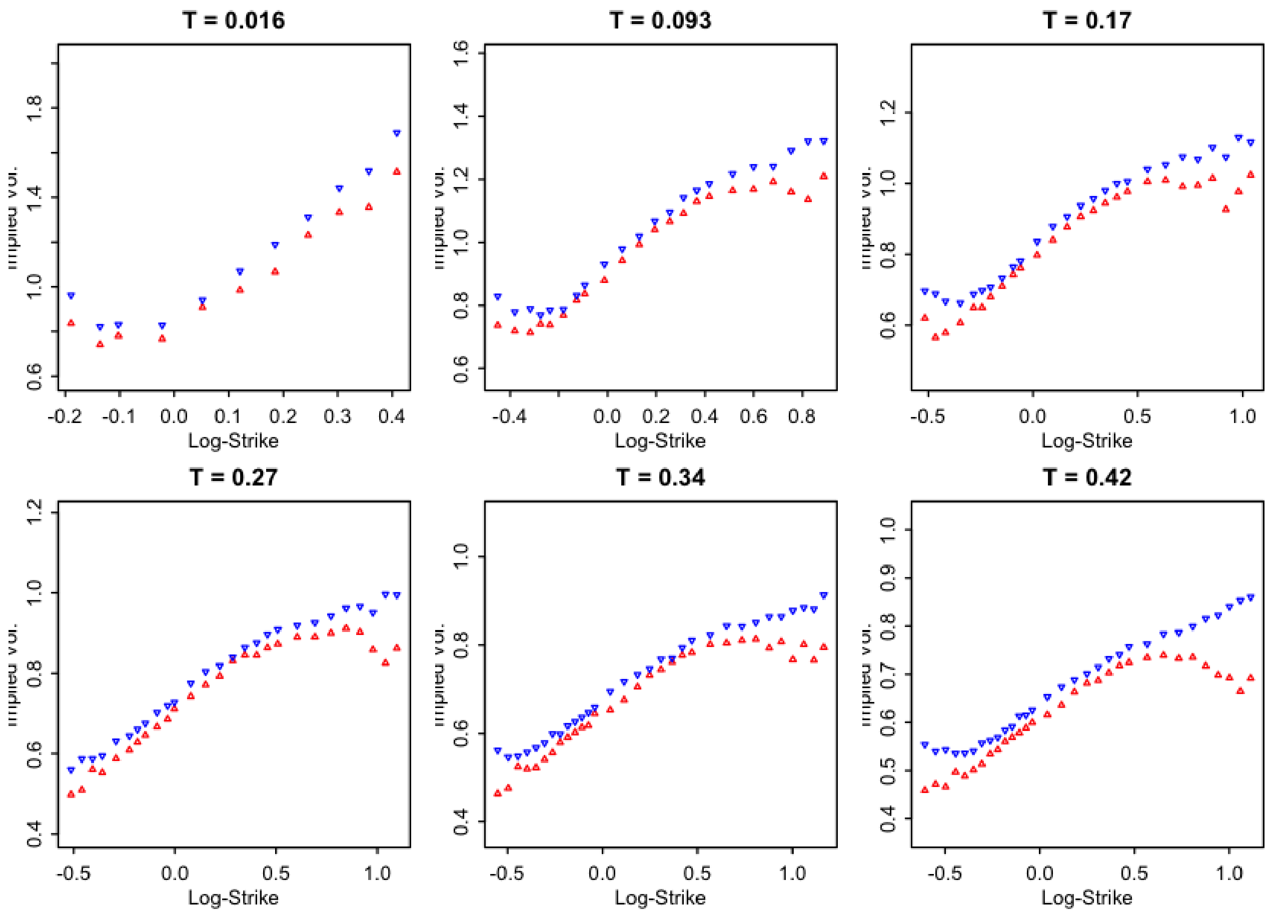


Figure 5: Implied volatility smiles of VIX options as of 15-Sep-2011. Note that all smiles are upward sloping.

Interpolation and extrapolation of VIX smiles

- We can't use SVI because VIX smiles are often concave.
- We adopt the Fukasawa robust interpolation/ extrapolation scheme.
  - Monotonic spline interpolation of mid-vols.
  - Extrapolation at constant level.

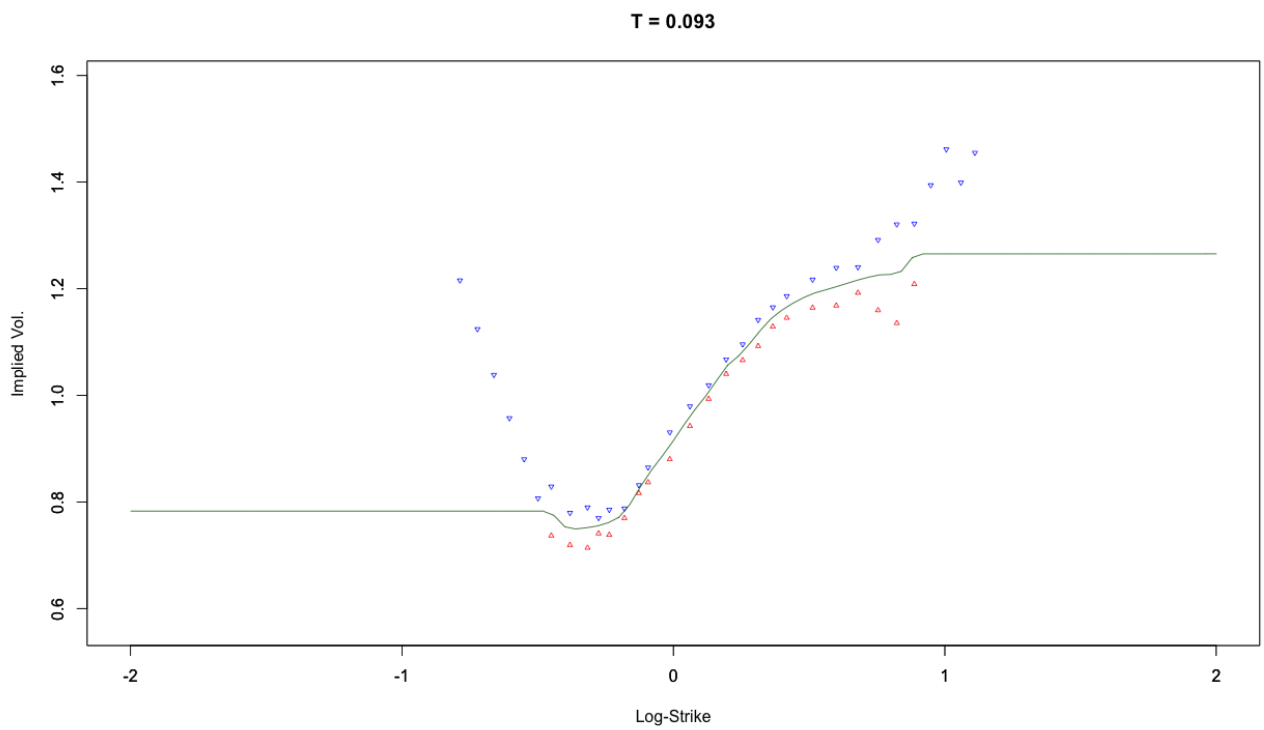


Figure 6: The VIX one month smile with monotonic spline interpolation/ extrapolation.

Forward variance swaps as of 15-Sep-2011

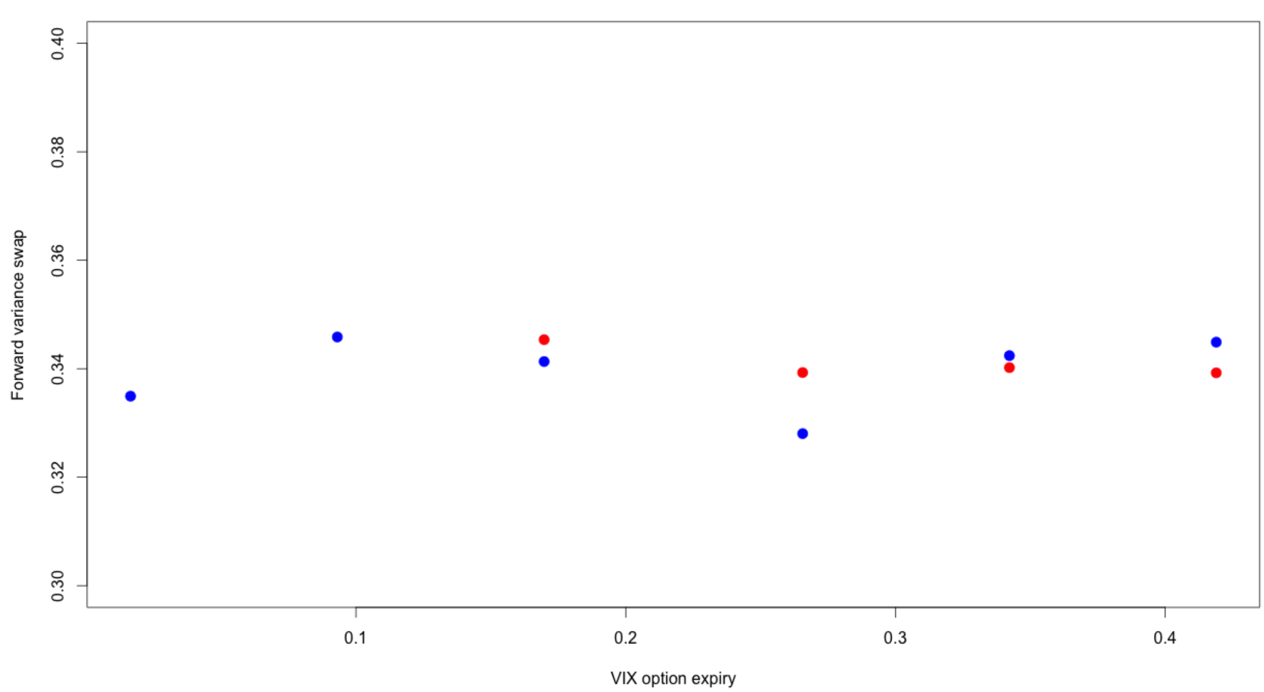


Figure 7: Red dots are forward variance swap estimates from SPX variance swaps; Green dots are interpolation of the SPX log-strip; Blue dots are forward variance swap estimates from the linear VIX option strip.

Consistency of forward variance swap estimates

- Forward variance swap estimates from SPX and VIX are very consistent on this date.
- Sometimes, VIX futures trade at a premium to the forward-starting variance swap.
  - This arbitrage has come and gone over time.

- Taking advantage as a proprietary trader is difficult because you need to cross the bid-ask so often.
  - Buy the long dated variance swap, sell the shorter-dated variance swap.
  - Sell the linear strip of VIX options.
- However, the practical consequence is that buyers of volatility should typically buy variance swaps, sellers should sell VIX.

The VVIX index

- Recall that we may compute the fair value of quadratic variation of $\log S$ from the log-strip of SPX options:

$$\mathbb{E}[\langle \log S \rangle_T] = -2\,\mathbb{E}[\log S_T].$$

- We may also compute the fair value of quadratic variation of $\log VIX$ from the log-strip of VIX options:

$$\mathbb{E}[\langle \log VIX \rangle_T] = -2\,\mathbb{E}[\log VIX_T].$$

- The VVIX index is computed from the log-strip of VIX options in exactly the same way that VIX is computed from SPX options.
  - Roughly speaking, the VVIX index is the fair value of one-month quadratic variation of VIX.
- We then, again roughly speaking, also have a measure of the volatility of volatility.

The fair value of VIX futures

As before, we have $F_{VIX} = \mathbb{E}[VIX]$ and

$$\mathbb{E}[VIX^2] - (\mathbb{E}[VIX])^2 = \mathbb{E}[VIX^2] - F_{VIX}^2 = Var(VIX).$$

Recall that $\mathbb{E}[VIX^2]$ may be computed from a constant strip of VIX options. As for $Var(VIX)$, we have

$$Var(VIX) \approx F_{VIX}^2\, VVIX^2\, T$$

so

$$\mathbb{E}[VIX^2] \approx F_{VIX}^2\,(1 + VVIX^2\, T)$$

which gives yet another way of assessing the fair value of VIX futures.

Why model SPX and VIX jointly

- We had so much success with model-free computations, why should we model SPX and VIX jointly?
  - We may want to value exotic options that are sensitive to the precise dynamics of the underlying such as barrier options, lookbacks or cliquets.
- But is it possible to come up with a parsimonious, realistic model that fits SPX and VIX jointly?
  - And even if we could, would it be possible to calibrate such a model efficiently?
- We will now see that it is indeed possible to do this!

VIX smiles

- As with options on the underlying, knowledge of the following is equivalent:
  - The VIX implied volatility smile
  - VIX option prices
  - The VIX density
  - The VIX characteristic function

- The VIX option smile thus encodes information about the dynamics of volatility in a stochastic volatility model.
  - For example, under stochastic volatility, a very long-dated VIX smile would give us the stable distribution of VIX.

Affine forward variance (AFV) models

Recall the generic form of an affine forward variance model from [Gatheral and Keller-Ressel][9]

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \sqrt{v_t}\, dZ_t \\
d\xi_t(u) &= \sqrt{v_t}\, \kappa(u-t)\, dW_t
\end{aligned}
$$

for some deterministic, non-negative decreasing kernel κ, which satisfies $\int_0^T \kappa(r)dr < \infty$ for all $T > 0$.

Joint characteristic function of X_T and VIX_T^2 in AFV models

Proposition 4.4 of [Gatheral and Keller-Ressel][9] states that

(4)
$$
\mathbb{E}\left[\exp\left\{u\,X_T + \int_T^{T'} h(T'-s)\,\xi_T(s)\,ds\right\}\middle|\mathcal{F}_t\right] = \exp\left\{u\,X_t + \int_t^{T'} \xi_t(s)\,g(T'-s;u,h)\,ds\right\} =: \exp\{u\,X_t + G_t\}
$$

where $T' = T + \Delta$.

Moreover, $g(.\,,u,h) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\leqslant 0}$ is the unique global continuous solution of the convolution Riccati equation

$$
g(t;u,h) = R\Big(u, \int_0^t \kappa(t-s)g(s;u,h)ds\Big) = R\Big(u, (\kappa \star g)(t;u,h)\Big), \quad t \geq \Delta
$$

where

$$
R(u,w) = \frac{1}{2}(u^2 - u) + \rho\, u\, w + \frac{1}{2}\, w^2
$$

and with initial condition

$$
g(t;u,h) = h(t), \quad t \in [0,\Delta).
$$

Derivation of the Riccati equation

Following closely the same derivation as in Lecture 3,

$$
\begin{aligned}
M_t &= \mathbb{E}\left[\exp\left\{u\,X_T + \int_T^{T'} h(T'-s)\,\xi_T(s)\,ds\right\}\middle|\mathcal{F}_t\right]\\
&= \exp\left\{u\,X_t + \int_t^{T'} \xi_t(s)\,g(T'-s;u,h)\,ds\right\} =: \exp\{u\,X_t + G_t\}
\end{aligned}
$$

is a conditional expectation and thus a martingale in t.

Applying Itô's Lemma to M gives

$$
\frac{dM_t}{M_t} = u\,dX_t + dG_t + \frac{u^2}{2}\,d\langle X\rangle_t + \frac{1}{2}\,d\langle G\rangle_t + u\,d\langle X,G\rangle_t.
$$

Now, dropping the explicit dependence of g on u and h for ease of notation,

$$
\begin{aligned}
dX_t &= -\frac{1}{2}\,v_t\,dt + \sqrt{v_t}\,dZ_t\\
dG_t &= -\xi_t(t)\,g(T-t)\,dt + \int_t^T d\xi_t(s)\,g(T-s)\,ds\\
&= -v_t\,g(T-t)\,dt + \int_t^T \kappa(s-t)\,\sqrt{v_t}\,dW_t\,g(T-s)\,ds.
\end{aligned}
$$

We compute

$$
\begin{aligned}
d\langle X\rangle_t &= v_t\,dt\\
d\langle G\rangle_t &= v_t\,dt\left(\int_t^T \kappa(s-t)\,g(T-s)\,ds\right)^2\\
d\langle X,G\rangle_t &= \rho\,v_t\,dt\int_t^T \kappa(s-t)\,g(T-s)\,ds.
\end{aligned}
$$

Imposing $\mathbb{E}\left[dM_t\right] = 0$ and letting $\tau = T - t$ gives

$$
0 = v_t\,dt\,\left\{-\frac{1}{2}\,u + \frac{1}{2}\,u^2 - g(\tau;u) + \rho\,u\,(\kappa\star g)(\tau) + \frac{1}{2}\,(\kappa\star g)(\tau)^2\right\}
$$

where the convolution integral is given by

$$
(\kappa\star g)(\tau) = \int_0^\tau \kappa(\tau-s)\,g(s)\,ds.
$$

The convolution Riccati equation

Rearranging gives

$$
g(\tau;u,h) = \frac{1}{2}\,u(u-1) + \rho\,u\,(\kappa\star g)(\tau;u,h) + \frac{1}{2}\,(\kappa\star g)(\tau;u,h)^2,
$$

as required.

The initial condition

As for the initial condition, we must have that for $s \in [T, T+\Delta]$,

$$
g(T'-s;u,h) = h(T'-s).
$$

Putting $\tau = T' - s = T + \Delta - s$, we obtain

$$
g(\tau;u,h) = h(\tau), \qquad \tau \in [0,\Delta].
$$

The joint characteristic function

Putting $u = \mathrm{i}a$ and $h(\cdot) = \mathrm{i}b$ in (4) gives

$$\mathbb{E}\left[\exp\left\{\mathrm{i}a\,X_T + \mathrm{i}b\int_T^{T'}\xi_T(s)\,ds\right\}\middle|\mathcal{F}_t\right] = \exp\left\{\mathrm{i}a\,X_t + \int_t^{T'}\xi_t(s)\,g(T'-s;u,h)\,ds\right\}$$

where $T' = T + \Delta$, and g satisfies the Riccati Volterra equation with initial condition $g(\tau) = \mathrm{i}b,\ \tau \in [0, \Delta]$.

- We identify $\frac{1}{\Delta}\int_T^{T+\Delta}\xi_T(s)\,ds$ as the forward variance swap VIX^2.

Applications

- European claims involving both VIX_T and X_T may in principle be easily valued in any affine model.
  - In particular, the classical and rough Heston models.
  - The characteristic function of VIX is the special case with $a = 0$.
- The quasi-closed from joint characteristic function requires the solution of a Riccati-Volterra integral equation.
  - This becomes an ODE in the classical Heston case and
  - A fractional ODE in the rough Heston case.

The Double Mean Reverting (DMR) model

One of the earliest attempts to jointly model SPX and VIX was the DMR[10]. model with dynamics

(5)

$$\begin{aligned}\frac{dS}{S} &= \sqrt{v}\,dW\\ dv &= -\kappa\,(v - v')\,dt + \eta_1\,v^{\alpha}\,dZ_1\\ dv' &= -c\,(v' - z_3)\,dt + \eta_2\,{v'}^{\beta}\,dZ_2\end{aligned}$$

for any choice of $\alpha,\ \beta \in [1/2, 1]$. The short term variance level v reverts to a moving level v' at rate κ. v' reverts to the long-term level z_3 at the slower rate $c < \kappa$.

- We call the case $\alpha = \beta = 1/2$ *Double Heston*,
- the case $\alpha = \beta = 1$ *Double Lognormal*,
- and the general case *DMR*.

Discriminating between SV models using VIX options

- VIX options allow us to discriminate between models.
- Let's compare Double Lognormal and Double Heston fits

Fit of Double Lognormal model to VIX options

As of 03-Apr-2007, from Monte Carlo simulation with parameters

$$z_1 = 0.0137;\ z_2 = 0.0208;\ z_3 = 0.0421;\ \kappa = 12;\ \xi_1 = 7;\ c = 0.34;\ \xi_2 = 0.94;$$

we get the following fits (orange lines):

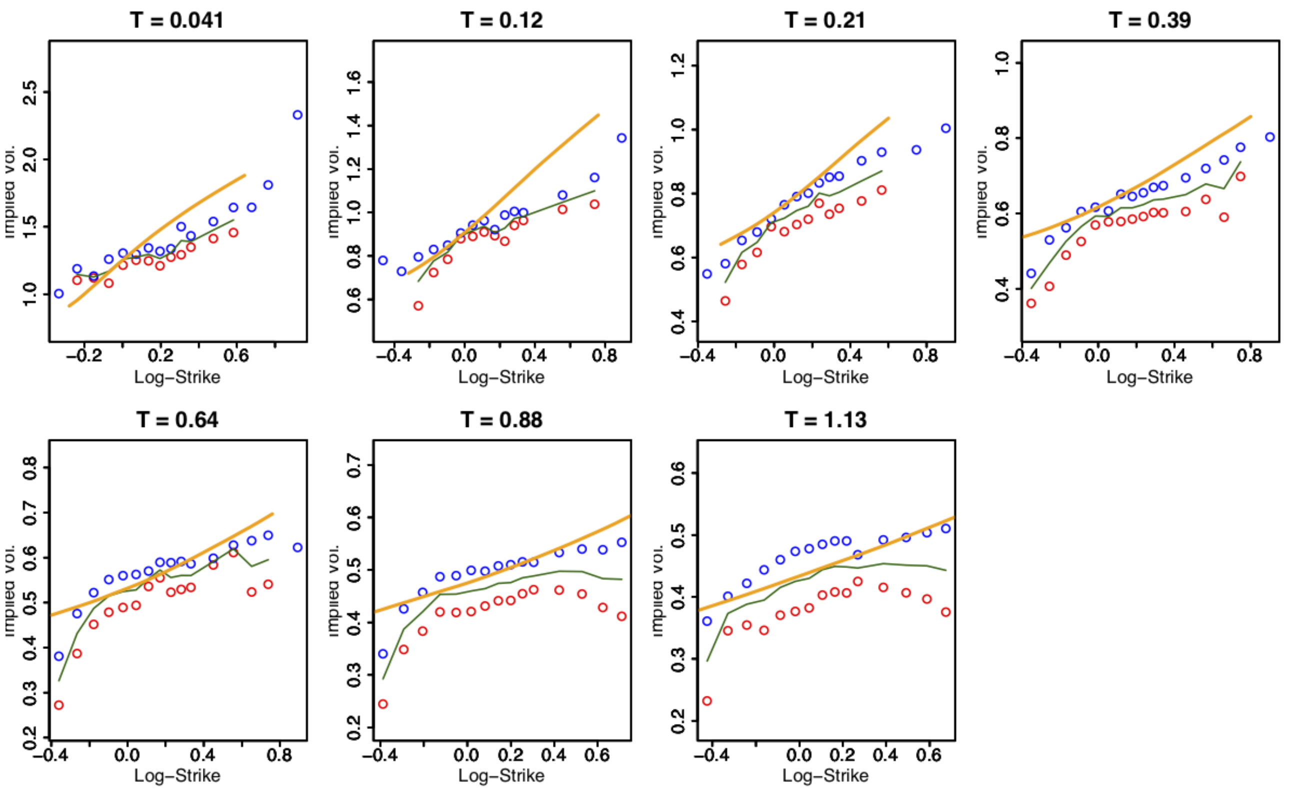


Figure 8: Fit of the Double Lognormal model to the volatility surface as of 03-Apr-2007.

Fit of Double Heston model to VIX options

As of 03-Apr-2007, from Monte Carlo simulation with parameters

$$z_1 = 0.0137;\ z_2 = 0.0208;\ z_3 = 0.0421;\ \kappa = 12;\ \xi_1 = 0.7;\ c = 0.34;\ \xi_2 = 0.14;$$

we get the following fits (orange lines):

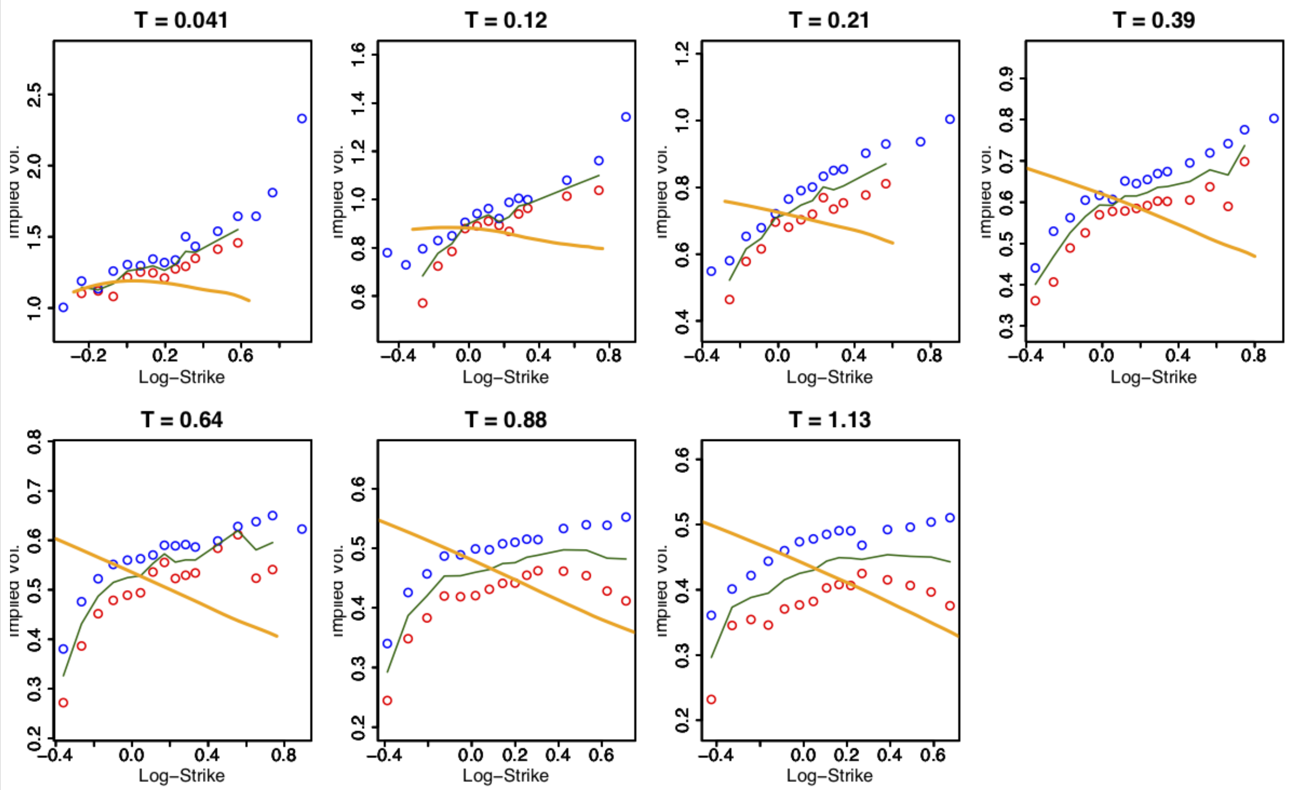


Figure 9: Fit of the Double Heston model to the volatility surface as of 03-Apr-2007.

In terms of densities of VIX

- When we draw the densities of VIX for the last expiration ($T = 1.13$) under each of the two modeling assumptions, we see what's happening:

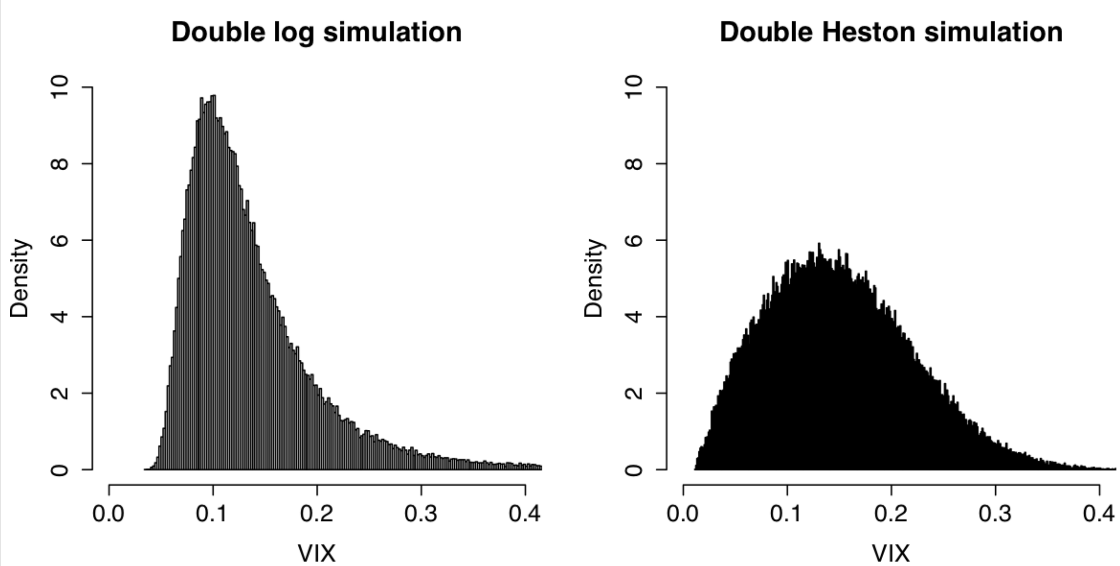


Figure 10: In the (double) Heston model, v_t spends too much time in the neighborhood of $v = 0$ and too little time at high volatilities.

Efficient fitting of the DMR model to SPX and VIX options

- No closed-form solution for European options exists so finite difference or Monte Carlo methods are needed for pricing.
- In [Bayer, Gatheral and Karlsmark][3], the DMR model is calibrated using the Ninomiya-Victoir Monte Carlo scheme.
  - Joint calibration of the model to SPX and VIX options is possible in less than 5 seconds.

"More tractable" models

- To compute option prices, and so for calibration, models such as DMR require Monte Carlo simulation.
  - This is not such a big deal in practice as noted earlier.
- We now explore two tractable alternatives:
  - Sepp (2008)
  - Cont and Kokholm (2013)

The Sepp model

The model of [Sepp][13] assumes the following dynamics:

$$\begin{aligned}\frac{dS_t}{S_t} &= \sigma(t)\,\sqrt{v_t}\,dZ_t \\ dv_t &= -\kappa\,(v_t - 1)\,dt + \epsilon\,\sqrt{v_t}\,dW_t + J\,dN_t.\end{aligned}$$

where N is a Poisson process with intensity γ. Jumps are exponentially distributed with density

$$\frac{1}{\eta}e^{-J/\eta}.$$

- Roughly speaking, Heston with a positive jump in the volatility. $\sigma(t)$ is a piecewise constant function calibrated to match the term structure of VIX futures.

Parameter estimates from July 25, 2007 are:

Parameter	Value
κ	2.26
ϵ	1.66
η	2.54
γ	0.31

- We see that the mean jump size is huge with significant probability!

- Sepp computes the characteristic function with respect to $x = \log S$, v, and the integrated variance $I = \int_t^T \sigma^2(t)\,v_s\,ds$. (That's cool!)
- Using this characteristic function, he can compute the prices of VIX options and SPX options using (at most) 2-dimensional numerical integration.

The Cont-Kokholm model

Forward starting variance swaps are modeled directly. Let

$$V_t^i = \int_{T_i}^{T_{i+1}} \xi_t(u)\,du$$

where $\xi_t(u)$ is the forward variance curve. Then in the model of [Cont and Kokholm][6] ,

$$V_t^i = V_0^i \exp\left\{ \int_0^t \mu_s^i \, ds + \omega \int_0^t e^{-\kappa_1 (T_i - s)} \, dW_s + \sum_{j=0}^{N_t} e^{-\kappa_2 (T_i - \tau_j)} \, Y_j \right\}$$

where N is a Poisson process with intensity λ, the τ_j are its jump times and the Y_j are jumps with distribution F independent of W.

- Jumps Y_j could for example be lognormally or exponentially distributed.
  - with simultaneous jumps in SPX of roughly $b_i x$ where x is the size of the jump in volatility.
- There is an accurate approximation for VIX.
  - VIX options are priced in closed form.
- The positive skew in VIX options comes from positive jumps in volatility.
- The negative skew in SPX comes from simultaneous negative jumps.

- Calibrated parameters correspond to roughly
  - 3 or 4 jumps per year
  - a $+50\%$ mean jump in variance
  - a -10% mean jump in the index
- VIX options give the size and frequency of jumps
- SPX options are used to fit parameters that
  - relate the index jump to the volatility jump
  - determine a piecewise constant diffusion coefficient

Some philosophy

- While all of the above models can more or less fit SPX and VIX jointly, they do this with many parameters.
  - Moreover some models have more reasonable dynamics than others.
- To hedge options such as options on VIX knocking out based on the level of SPX, we need to have reasonable dynamics.
- Rough volatility dynamics seem reasonable!

Rough volatility models

- Though we don't show it here, it's clear that rough Heston will generate the same unreasonable downward sloping VIX smiles as the classical Heston model.
- The rough Bergomi model generates almost flat VIX smiles so cannot fit VIX smiles.
- Bergomi addresses this issue by modeling instantaneous variance as a linear combination of two Ornstein-Uhlenback (OU) processes with different mean reversion rate rather than as lognormal.
- [Horvath, Jacquier and Tankov][11] implement a rough version of Bergomi's idea by modeling instantaneous variance as a sum of Riemann-Liouville processes.

The rough Bergomi model

The rough Bergomi model may be written as

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \sqrt{v_t}\,\left\{\rho\,dW_t + \sqrt{1-\rho^2}\,dW_t^\perp\right\}\\
v_u &= \xi_t(u)\,\mathcal{E}\left(\eta\,\sqrt{2\,H}\int_t^u \frac{dW_s}{(u-s)^\gamma}\right)
\end{aligned}
$$

with $\gamma = \frac{1}{2} - H$.

The forward variances $\xi_t(u) = \mathbb{E}\left[v_u \middle| \mathcal{F}_t\right]$ are (at least in principle) tradable and observed in the market. In forward variance form

$$
\frac{d\xi_t(u)}{\xi_t(u)} = \tilde{\eta}\,\frac{dW_t}{(u-t)^\gamma}
$$

where $\tilde{\eta} = \eta\,\sqrt{2\,H}$.

- In particular, forward variances are lognormal.

VIX options and futures under rough Bergomi

- The rough Bergomi model generates more or less flat VIX smiles.
  - Inconsistent with observed VIX smiles.
- Nevertheless, we can still try to compute the term structure of VIX futures to impute the rough Bergomi parameters H and η.

The distribution of VIX future payoffs

- Denote the terminal value of the VIX futures by $\sqrt{\zeta(T)}$. Then, by definition,

$$
\zeta(T) = \frac{1}{\Delta}\int_T^{T+\Delta} \mathbb{E}[v_u|\mathcal{F}_T]\,du = \frac{1}{\Delta}\int_T^{T+\Delta} \xi_t(u)\,du.
$$

  where Δ is one month.

- Though a sum of lognormal random variables is not itself lognormal, if Δ is sufficiently small, $\zeta(T)$ should be approximately lognormal.

- The VIX payoff $\sqrt{\zeta(T)}$ should also be approximately lognormally distributed.
  - The quality of this approximation was confirmed by [Jacquier, Martini and Muguruza][12].
  - In that case, the terminal distribution of $\zeta(T)$ is completely determined by $\mathbb{E}\left[\zeta(T)\middle|\mathcal{F}_t\right]$ and $\mathrm{var}[\log\zeta(T)|\mathcal{F}_t]$.

- Obviously

$$
\mathbb{E}\left[\zeta(T)\middle|\mathcal{F}_t\right] = \frac{1}{\Delta}\int_T^{T+\Delta} \xi_t(u)\,du.
$$

  - Recall that forward variances $\xi_t(u)$ may be estimated from variance swaps which can themselves be proxied by the log-strip (see Chapter 11 of [The Volatility Surface][8] again).
  - Alternatively they may be estimated from linear strips of VIX options.

Approximating the conditional variance of $\zeta(T)$

- To estimate the conditional variance of $\zeta(T)$, we approximate the arithmetic mean by the geometric mean as follows:

$$\zeta(T) \approx \exp\left\{ \frac{1}{\Delta} \int_T^{T+\Delta} \mathbb{E}[\log v_u|\mathcal{F}_T]\, du \right\}.$$

Let $y_u = \log v_u$ and recall that $\gamma = \frac{1}{2} - H$. Apart from $\mathcal{F}_t$ measurable terms (abbreviated as ``drift''), we have

$$\begin{aligned}\int_T^{T+\Delta} E[y_u|\mathcal{F}_T]\, du &= \eta\sqrt{2H}\int_t^T \frac{dW_s}{(u-s)^\gamma} du + \text{drift}\\ &= \eta\sqrt{2H}\int_t^T \int_T^{T+\Delta} \frac{du}{(u-s)^\gamma} dW_s + \text{drift}\\ &= \eta\frac{\sqrt{2H}}{1-\gamma}\int_t^T \left[(T+\Delta-s)^{1-\gamma} - (T-s)^{1-\gamma}\right] dW_s + \text{drift}.\end{aligned}$$

This gives

$$\begin{aligned}\text{var}[\log\zeta(T)|\mathcal{F}_t] &\approx \frac{\eta^2}{\Delta^2}\,\frac{2H}{(H+1/2)^2}\int_t^T \left[(T+\Delta-s)^{1/2+H} - (T-s)^{1/2+H}\right]^2 ds\\ &= \eta^2\,(T-t)^{2\,H} f^H\left(\frac{\Delta}{T-t}\right)\end{aligned}$$

where

$$f^H(\theta) = \frac{2H}{(H+1/2)^2}\,\frac{1}{\theta^2}\int_0^1 \left[(1+\theta-x)^{1/2+H} - (1-x)^{1/2+H}\right]^2 dx.$$

The approximate fair value of VIX futures

- In [Bayer, Friz and Gatheral][2], we chose to study the term structure of VVIX (the VIX of VIX).
- It is more natural to follow [Jacquier, Martini and Muguruza][9] and approximate the fair value of VIX futures.
- Under the lognormal approximation, the fair value of the T-maturity VIX future is given by

  (5)
  $$\mathbb{E}\left[\sqrt{\zeta(T)}\middle|\mathcal{F}_t\right] = \sqrt{\mathbb{E}\left[\zeta(T)|\mathcal{F}_t\right]}\,\exp\left\{-\frac{1}{8}\,\text{var}[\log\zeta(T)|\mathcal{F}_t]\right\}.$$

Results

- Both [Bayer, Friz and Gatheral][2] and [Jacquier, Martini and Muguruza][9] find that the volatility of volatility parameter η estimated by calibration to the SPX is roughly 20% greater than the estimate from VIX futures and options.
  - Arbitrage or model mis-specification?
- A model with very few parameters can be useful for assessing relative value.
  - Derivative bookrunners need models that fit however!
  - Here we see that rough Bergomi is inconsistent with SPX and VIX option prices jointly.

Calibration of H and η on January 4, 2016.

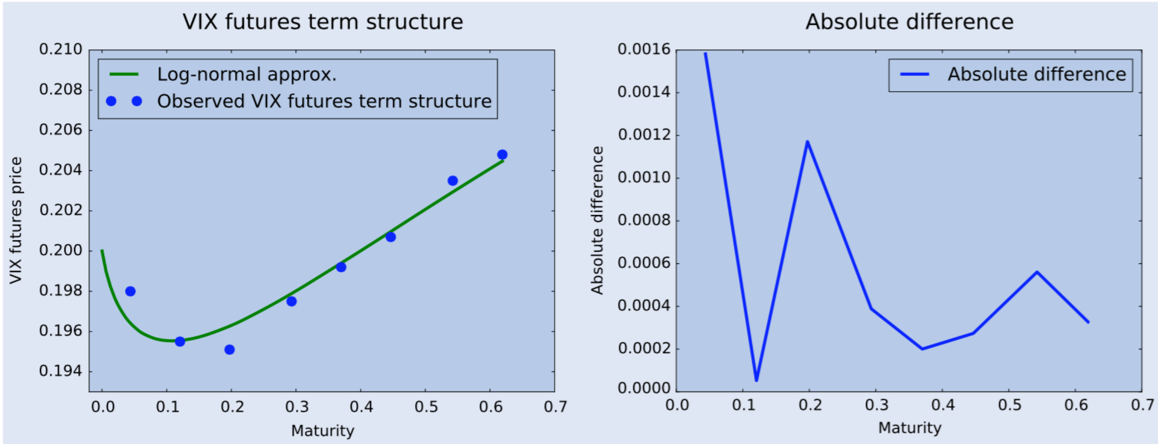


Figure 11. VIX futures calibration on 4 January 2016. Optimal parameters: $(H, \nu) = (0.0509, 1.2937)$.

- We see that rough volatility can generate realistic term structures of VIX futures.
  - This term structure depends on forecast future volatility (see Lecture 13) and the volatility risk premium.
- Note again that the calibrated value of η is significantly lower than that estimated from SPX options.

VIX-based ETNs

- There exist various ETNs that contain positions in Treasuries and VIX futures:
  - VXX: Long short-term VIX futures (Barclays)
  - VXZ: Long medium-term VIX futures (Barclays)
  - VIIX: Long VIX Short Term ETNs (Credit Suisse)
  - ZIV: Inverse VIX Medium Term ETNs (Credit Suisse)
  - TVIX: $2\times$ Long VIX Short Term ETNs (Credit Suisse)
- The following ETNs died (were accelerated) in 2018:
  - XIV: Inverse VIX Short Term ETN (Credit Suisse)
  - VIIZ: Long VIX Medium Term ETNs (Credit Suisse)
  - TVIZ: $2\times$ Long VIX Medium Term ETNs (Credit Suisse)

VXX

From the iPath website:

> The S&P 500 VIX Short-Term Futures™ Index Total Return offers exposure to a daily rolling long position in the first and second month VIX futures contracts. On each business day, a fraction of the first month VIX futures holding is sold and an equal notional amount of the second month future is bought, maintaining a constant weighted average maturity of one month.

VXZ

From the iPath website:

> The S&P 500 VIX Mid-Term Futures™ Index Total Return offers exposure to a daily rolling long position in the fourth, fifth, sixth and seventh month VIX futures contracts. On each business day, a fraction of the fourth month VIX futures holding is sold and an equal notional amount of the seventh month future is bought, maintaining a constant weighted average maturity of five months.

ETFs according to Avellaneda and Zhang

Following [Avellaneda and Zhang][1], denote the spot price of the underlying by S_t, the price of the leveraged ETF (LETF) by L_t, and leverage ratio by β.

Assume that

$$\frac{dS_t}{S_t} = \mu_t\, dt + \sigma_t\, dW_t.$$

Then assuming zero interest rates, dividends and borrowing costs, we have

$$\frac{dL_t}{L_t} = \beta\, \frac{dS_t}{S_t}$$

Leveraged ETF valuation

Applying Itô's Lemma,

$$d \log S_t = \frac{dS_t}{S_t} - \frac{1}{2}\, \sigma_t^2\, dt$$

and

$$d \log L_t = \frac{dL_t}{L_t} - \frac{1}{2}\, \beta^2\, \sigma_t^2\, dt = \beta\, \frac{dS_t}{S_t} - \frac{1}{2}\, \beta^2\, \sigma_t^2\, dt$$

Then

(7)

$$d \log L_t - \beta\, d \log S_t = -\frac{1}{2}\, (\beta^2 - \beta)\, \sigma_t^2\, dt$$

Integrating (7) gives

(8)

$$\frac{L_t}{L_0} = \left(\frac{S_t}{S_0}\right)^{\beta} \exp\left\{ \frac{1}{2}\, (\beta - \beta^2)\, \int_0^t \sigma_s^2\, ds \right\}$$

Volatility exposure of leveraged ETF

- We see from (7) and (8) that long positions in leveraged ETFs are effectively short volatility.
- For example:
  - If $\beta = 2$,

$$d \log L_t - \beta\, d \log S_t = -\sigma_t^2\, dt.$$

  - Likewise, if $\beta = -1$,

$$d \log L_t - \beta\, d \log S_t = -\sigma_t^2\, dt.$$

- This is because the LETF has to buy the underlying when it goes up and sell it when it goes down.

Rebalancing

- Wlog, suppose the starting value of the LETF is $N_0 = \$1$.

- At the close of day 1, the notional is

$$V_1 = 1 + \beta \frac{\Delta S_1}{S_0}$$

- After rebalancing, at the close of day 2, the value is

(9)

$$V_2 = V_1 \left(1 + \beta \frac{\Delta S_2}{S_1}\right) = \left(1 + \beta \frac{\Delta S_2}{S_1}\right) \left(1 + \beta \frac{\Delta S_1}{S_0}\right)$$

- On the other hand if there is no rebalancing the value would be

(10)

$$V_2' = \left(1 + \beta \frac{\Delta S_1 + \Delta S_2}{S_0}\right)$$

dollars worth of underlying.

- Subtracting (10) from (9), we obtain

$$\begin{aligned} V_2 - V_2' &= \beta \left\{ \frac{\Delta S_2}{S_1} - \frac{\Delta S_2}{S_0} + \beta \frac{\Delta S_2}{S_1} \frac{\Delta S_1}{S_0} \right\} \\ &= \beta (\beta - 1) \frac{\Delta S_1}{S_0} \frac{\Delta S_2}{S_1} \end{aligned}$$

from which we conclude that the rebalancing at the end of day 1 involved buying

$$\beta (\beta - 1) \frac{\Delta S_1}{S_0}$$

dollars worth of the underlying.

- Being long a leveraged ETF is therefore like being short gamma: you effectively buy(sell) the underlying if the price rises(falls).
  - In aggregate therefore, leveraged ETFs exarcebate market moves.

Regression of VXX vs VIX

Figure 12: Regression of VXX vs VIX

Regression of XIV vs VXX

Figure 13: Regression of XIV vs VXX

The sad story of XIV

From Reuters on February 6, 2018:

> LONDON (Reuters) - Three U.S.-listed inverse VIX ETFs were halted from trading on Tuesday after suffering heavy losses on Monday when a spike in U.S. stocks volatility caused a violent unwind of trades betting volatility would stay low.
>
> Trading in the VelocityShares Inverse Vix Short-Term ETN (XIV.P), ProShares Short VIX Short-Term Futures ETF (SVXY.K) and VelocityShares Daily Inverse VIX Medium-Term ETN (ZIV.O) was halted and all three products had a short sell restriction placed on them, Thomson Reuters data showed.
>
> "Trading is halted pending the release of material news," Nasdaq said on its website.
>
> Yesterday's spike in the VIX gauge of U.S. stocks volatility (.VIX) sent the VelocityShares Daily Inverse VIX Short-Term ETN down 84 percent in after-hours trading, while the ProShares Short VIX ETF fell nearly 79 percent.
>
> Credit Suisse (.CSGN.S), the issuer of the VelocityShares ETN (XIV.P), in which it has a 32 percent stake, said it had not suffered any loss on the product.

Credit Suisse announces plan to liquidate XIV inverse volatility ETN

From MarketWatch on February 6, 2018:

> Credit Suisse says that a popular exchange-traded note that was designed to bet against rising volatility will be liquidated on Feb. 21 after the VelocityShares Daily Inverse VIX Short Term ETN lost more than 80% of its value in after-hours trade, triggering a technical liquidation, or "acceleration" event. "On the acceleration date, investors will receive a cash payment per ETN in an amount equal to the closing indicative value of XIV on the accelerated valuation date. The last day of trading for XIV is expected to be February 20, 2018. As of the date hereof, Credit Suisse will no longer issue new units of XIV ETNs," the bank wrote in a statement. The tumble for the so-called inverse VIX product which is pegged to moves in the CBOE Volatility Index VIX, +3.83% which jumped 115%, reflecting its sharpest daily surge in history, to 37.32 on Monday, crushing bets that a sustained period of calm would persist. The VIX was trading around 50 early Tuesday. The move for the VIX, known as Wall Street's fear gauge, also comes after the stock market suffered one of its worst declines in recent memory, with the Dow Jones Industrial Average DJIA, -0.77% registering its most severe point drop in history.

VIX trading strategies

- Misha Fomytskyi pointed out to me that a regularly rebalanced portfolio of long 25% XIV and 75% cash actually performed pretty well.
- XIV prices are no longer available so we will make do with SVXY.
- In passing, we will see how easy it is to do rough backtests of portfolio strategies in R.

In [9]:
```
getSymbols(c('SVXY','VXX'))
```

```
'getSymbols' currently uses auto.assign=TRUE by default, but will
use auto.assign=FALSE in 0.5-0. You will still be able to use
'loadSymbols' to automatically load data. getOption("getSymbols.env")
and getOption("getSymbols.auto.assign") will still be checked for
alternate defaults.

This message is shown once per session and may be disabled by setting
options("getSymbols.warning4.0"=FALSE). See ?getSymbols for details.


WARNING: There have been significant changes to Yahoo Finance data.
Please see the Warning section of '?getSymbols.yahoo' for details.

This message is shown once per session and may be disabled by setting
options("getSymbols.yahoo.warning"=FALSE).
```

'SVXY' 'VXX'

Merge price histories into one dataset and label columns

In [10]:
```
portfolio.prices <-as.xts(merge(Cl(SVXY),Cl(VXX),1)) # Add column for cash
portfolio.prices <-portfolio.prices["20140101::"] # Last five years only
names(portfolio.prices) <- c("SVXY","VXX","Cash")

portfolio.returns <-na.omit(ROC(portfolio.prices,1,"discrete"))
```

In [11]:
```
plot(portfolio.prices["20150101::20151231"],main="",col=c("blue","red"))
```

Figure 14: VXX in red; SVXY in blue.

Rebalance the portfolio according to 50% SVXY, 50% cash

In [12]:
```
w1 <- c(.5,0,0.5) # 50% SVXY, 50% cash
portfolio.rebal1 <-Return.portfolio(portfolio.returns,
                                    rebalance_on="weeks",
                                    weights=w1,wealth.index=TRUE,verbose=TRUE)
```

Rebalance the portfolio according to short 25% VXX, 125% cash

In [13]:
```
w2 <- c(1.25,0,-.25)# -25% VXX 125% cash
portfolio.rebal2 <-Return.portfolio(portfolio.returns,
                                        rebalance_on="weeks",
                                        weights=w2,wealth.index=TRUE,verbose=TRUE)
```

Rebalance the portfolio according to 50% SVXY, 25% VXX, 25% cash

In [14]:
```
w3 <- c(.25,.5,.25) # 25% cash, 50% SVXY, long 25% VXX
portfolio.rebal3 <-Return.portfolio(portfolio.returns,
                                        rebalance_on="weeks",
                                        weights=w3,wealth.index=TRUE,verbose=TRUE)
```

Buy and hold SVXY (no rebalancing)

In [15]:
```
w4 <- c(1,0,0) # 100% SVXY
portfolio.bh <-Return.portfolio(portfolio.returns,
                                        weights=w4,wealth.index=TRUE,verbose=TRUE)
```

Merge portfolio returns into one dataset and label columns

In [16]:
```
portfolios <-cbind(portfolio.rebal1$returns,portfolio.rebal2$returns,portfolio.rebal3$returns,p
colnames(portfolios) <-c("50% SVXY, 50% cash","Short 25% VXX, 125% cash","50% SVXY, 25% VXX, 25
```

VIX ETFs just keep wasting away

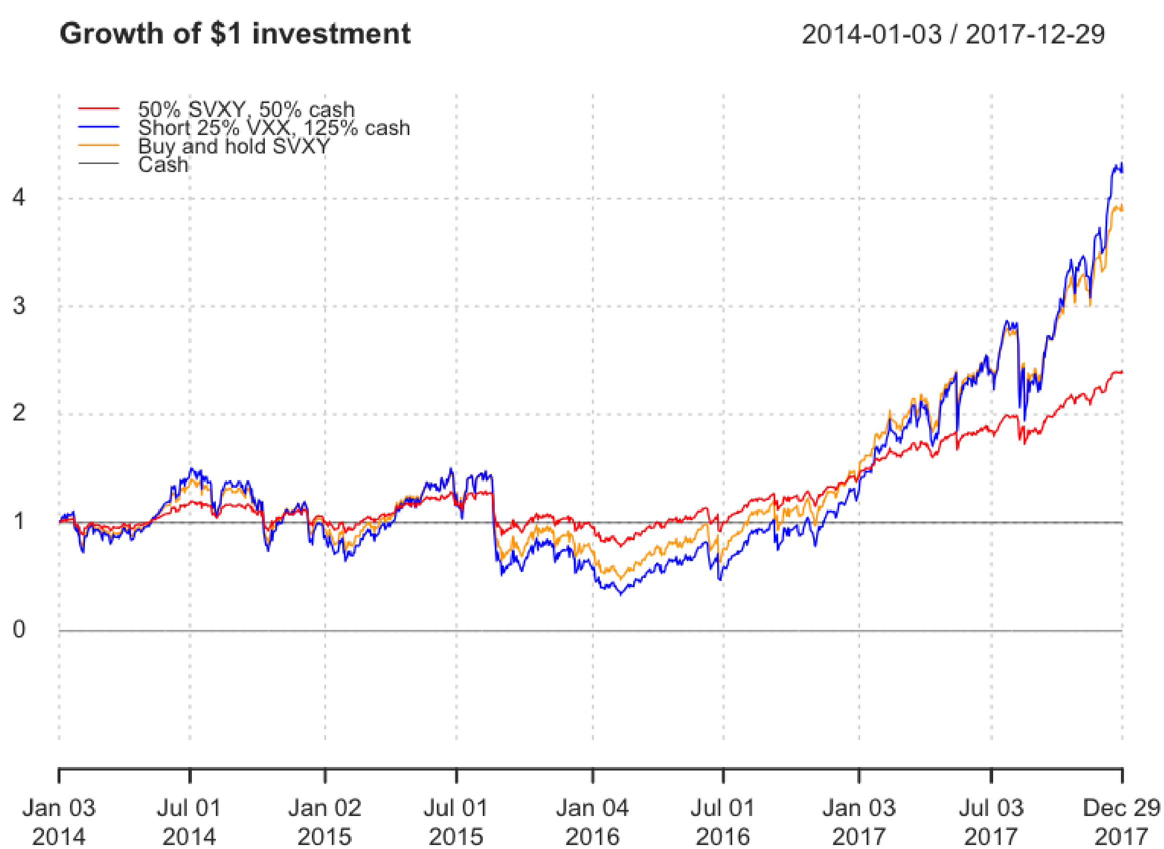
- Here's what happened to $1 invested in 50% SVXY, 25% VXX, 25% cash, rebalance weekly.

In [17]:
```
chart.CumReturns(portfolios[,3],
                 wealth.index=TRUE,
                 legend.loc="bottomleft",
                 main="Growth of $1 investment",
                 ylab="$",col="green4")
```

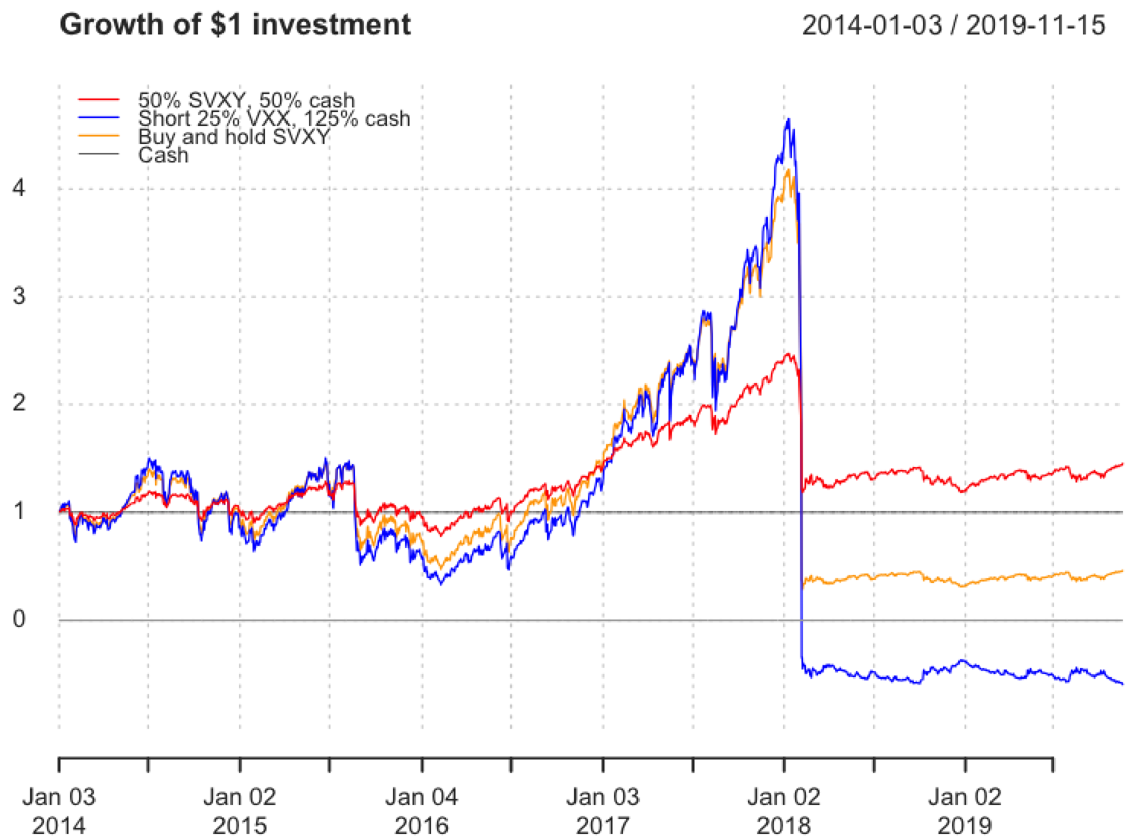
Long SVXY vs short VXX

- These trades looked great until February of 2018.

In [18]:
```
chart.CumReturns(portfolios[,-3]["20140101::20171231"],
                 wealth.index=TRUE,
                 legend.loc="topleft",
                 main="Growth of $1 investment",
                 ylab="$",lwd=c(1,1,1,.5),col=c("red","blue","orange","black"),ylim=c(-1,5))
```



In [19]:
```
chart.CumReturns(portfolios[,-3],
                 wealth.index=TRUE,
                 legend.loc="topleft",
                 main="Growth of $1 investment",
                 ylab="$",lwd=c(1,1,1,.5),col=c("red","blue","orange","black"),ylim=c(-1,5))
```



Compute risk and return

In [20]:
```
table.AnnualizedReturns(portfolios[,1]["20140101::20171231"])
```

	50% SVXY, 50% cash
Annualized Return	0.2429
Annualized Std Dev	0.3085
Annualized Sharpe (Rf=0%)	0.7874

In [21]:
```
table.AnnualizedReturns(portfolios[,1]["20140101::"])
```

	50% SVXY, 50% cash
Annualized Return	0.0666
Annualized Std Dev	0.3145
Annualized Sharpe (Rf=0%)	0.2118

Summary

- Volatility derivatives are now very actively traded, not least because from a practical perspective, valuation is well-understood.
- We saw that although the relationship between the valuation of options on variance and European option prices is currently only partially understood, it's pretty clear from a practical perspective how to value them.
- On the theoretical side, connections between the valuation of volatility derivatives and the dynamics of the underlying process continue to be investigated.

- VIX trading continues to be very popular although volume in VIX-based ETFs and ETNs is obviously down.
- Work remains to be done on potential arbitrage between VIX and SPX.

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