Recall that $\sigma_\ell^2(k,T) = \dfrac{\frac{\partial w}{\partial T}}{\left(1-\frac{1}{2}\frac{k}{w}\frac{\partial w}{\partial k}\right)^2 - \frac{1}{4}\left(\frac{1}{4}+\frac{1}{w}\right)\left(\frac{\partial w}{\partial k}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial k^2}}$

(a) As $T \to 0$, $\sigma_{BS}(K,T) = \sigma_0(K) + \mathcal{O}(T)$ (which is true for conventional SVM), write an equation of $\sigma_\ell(K,T)$ in term of $\sigma_0(K)$ and with order $\mathcal{O}1$.

(b) Show the expression can be re-written to $\mathcal{O}(1)$ as $\frac{\partial}{\partial k}\left(\frac{k}{\sigma_0(K)}\right) = \frac{1}{\sigma_\ell(K,0)}$

Hint: $1-\frac{1}{2}\frac{k}{w}\frac{\partial w}{\partial k} = \sqrt{w}\,\frac{\partial}{\partial k}\left(\frac{k}{\sqrt{w}}\right) = \sigma_0(K)\cdot\frac{\partial}{\partial k}\left(\frac{K}{\sigma_\ell(K)}\right)$

(c) Deduce $\frac{1}{\sigma_0(K)} = \frac{1}{K}\int_0^k \frac{dy}{\sigma_\ell(y,0)}$

(d) Deduce $\frac{1}{\sigma_0(K)} = \int_0^1 \frac{d\alpha}{\sigma_\ell(\alpha k, 0)}$ (BBF) $\longrightarrow$ derivatives

(e) Deduce $\sigma_0(0) = \sigma_\ell(0,0)$

(f) Show $T \to 0$, ATM imp vol skew $= \frac{1}{2}\cdot$ ATM local vol skew. $\sigma_0'(0) = \frac{1}{2}\frac{\partial}{\partial k}\sigma_\ell(k,0)\Big|_{k=0}$

2. (N-V) $dv_t = -\lambda(v_t-\bar{v})\,dt + \xi v_t\, dZ_t$.

(a) $f(v_t,t,T) = \mathbb{E}[g(x_T)\,|\,\mathcal{F}_t]$, Find $\mathcal{L}$ s.t. $\frac{\partial f}{\partial t} + \mathcal{L}f = 0$ $\left(\mathcal{L} = -\lambda(v-\bar{v})\partial_v + \frac{1}{2}\xi^2 v^2\partial_{vv}\right)$.

(b) $\mathcal{L} = \mathcal{L}_0 + \frac{1}{2}\mathcal{L}_1^2$ where $\mathcal{L}_1 = \xi v\partial_v$. Show $\mathcal{L}_0$. $\left(\mathcal{L}_0 = -\lambda(v-\bar{v}) - \frac{1}{2}\xi^2 v\,\partial_v\right)$

(c) Solve $\frac{\partial}{\partial t}v(t) = \mathcal{L}_1 v(t)$ and $\frac{\partial}{\partial t}v(t) = \mathcal{L}_0 v(t)$ with $v(0)=v_0$

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$v(t) = v_0 e^{\xi t}$ $\qquad$ $v(t) = \bar{v}' + (v_0-\bar{v}')e^{-\lambda' t}$ $\quad \lambda' = \lambda + \frac{1}{2}\xi^2 \quad \bar{v}' = \frac{\lambda\bar{v}}{\lambda'}$

(d) $\partial_t f + \{\mathcal{L}_0 + \frac{1}{2}\mathcal{L}_1^2\}f = 0$

$x_{t+\Delta} = e^{\mathcal{L}_0\frac{\Delta}{2}}e^{\mathcal{L}_1\sqrt{\Delta}Z}e^{\mathcal{L}_0\frac{\Delta}{2}}x_t$ Deduce the expression of $x_{t+\Delta}$ in term of x_t.

Hint: $v_1 = e^{\frac{1}{2}\mathcal{L}_0\Delta}v_t = \bar{v}' + (v_t-\bar{v}')e^{-\lambda'\Delta/2}$, $v_2 = e^{\mathcal{L}_1\sqrt{\Delta}Z}v_1 = v_1 e^{\xi\sqrt{\Delta}Z}$, $v_{t+\Delta} = \bar{v}' + (v_2-\bar{v}')e^{-\lambda'\Delta/2}$

$\frac{\partial}{\partial t}\left(e^{\mathcal{L}_i t}\right) = \mathcal{L}_i \underbrace{e^{\mathcal{L}_i t}}_{\text{"}v(t)\text{"}}$

3. $\varphi_T(u) = \mathbb{E}[e^{iux_T}] = \exp\left\{iu\omega T - \frac{1}{2}u^2\sigma^2 T + T\int(e^{iu\xi}-1)\mu(\xi)d\xi\right\}$ $\quad x_T = \ln\left(\frac{S_T}{F_T}\right)$

(a) Note $\mathbb{E}[S_T] = S_0$. Let $u=-i$, show that $\omega = -\frac{1}{2}\sigma^2 - \int[e^{\xi}-1]\mu(\xi)d\xi$

(b) Let $\frac{dS_t}{S_t} = m\,dt + dq_t \leftarrow$ compounded Poisson with $q_t \sim \exp(\nu)$ and arrival rate λ; $q_t < 0$.

show $\mu(\xi) = \lambda\nu e^{+\nu\xi}\mathbb{1}_{\xi<0}$

(c) Write down for this dynamics the characteristic exponent $\Phi_T(u) = \frac{\ln\varphi_T(u)}{T}$

(d) $\mathbb{E}\left[\left(\frac{S_T}{S_0}\right)^\alpha\right] = \varphi_T(-i\alpha) = \cdots$ (power contract)

Answer for (e): We know $\mathbb{E}\left[\left(\frac{S_t}{S_0}\right)^{\alpha}\right] = \exp\left\{\lambda T \frac{\alpha(\alpha-1)}{(\nu+1)(\nu+\alpha)}\right\}$.

Right: Finite Moment when $\alpha > 0 \xrightarrow{\text{Roger Lee}}$ imp vol will not explode when $\alpha > 0 \rightarrow$ it ~~goes~~ converges to

Left: Moment explode when $\alpha = -\nu$. Typical jump size $= \frac{1}{\nu}$.

jump size $\uparrow$, $\frac{1}{\nu}\uparrow$, $\nu\downarrow$, Moment explode at smaller moment $\xrightarrow{\text{Roger Lee}}$ slope at left $\uparrow \rightarrow 2$.

4. $\frac{dS_t}{S_t} = \sqrt{v_t}\, dZ_t$. $d\xi_t(u) = \frac{\eta}{\Gamma(H+\frac{1}{2})} \sqrt{v_t} \cdot \frac{dW_t}{(u-t)^{\frac{1}{2}-H}}$ with correlation ρ.

(a) $v_t = \xi_t(t)$

(b) Why $\xi_t(u)$ is a martingale?

(c) How do H, η, ρ affect the shape?

(d) $X_t = \ln(S_t/S_0)$. Recall Bergomi-Guyon: $C^{x\xi} = (X \diamond M)_t(T) = \int_t^T ds \int_s^T du\, \mathbb{E}\left[\frac{dx_s\, d\xi_s(u)}{ds} \middle| \mathcal{F}_t\right]$

Hint: $= \int_t^T ds \int_s^T du\, \mathbb{E}[v_s | \mathcal{F}_t] \cdot \frac{\rho\eta}{\Gamma(H+\frac{1}{2})} \cdot \frac{1}{(u-s)^{\frac{1}{2}-H}}$

(e) BG expansion: $\sigma_{BS}(K,T) = \hat{\sigma}_T + S_T k + O(\eta^2)$ with $S_T = \sqrt{\frac{w}{T-t}} \cdot \frac{1}{2w^2} C^{x\xi}$

Find the expression of ATM volatility skew. S_T

(f) How to get H from ATM imp skew. (Regression on the log-log plot)

5. [Similar] (Mental Monte Carlo) payoff $= 1 + \max\left\{0, \text{max-cpn} + \sum_{i=1}^{N} \min\{0, r_i\}\right\}$

(a) Which part of the payoff reflect the principle protection (1)

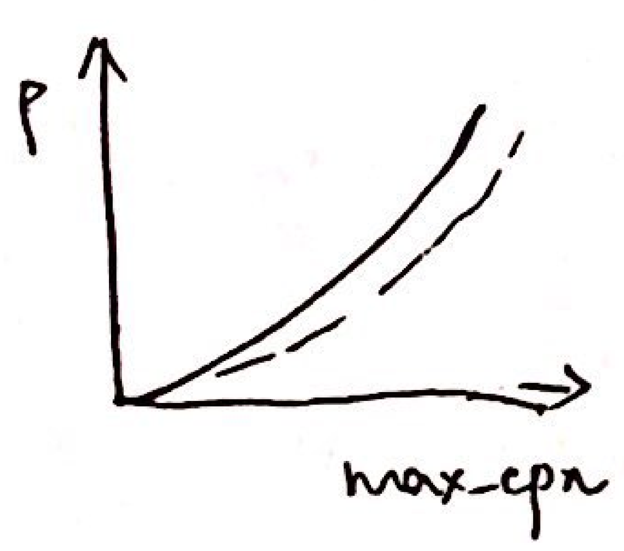
(b) What's the max redemption amount and how does it happen? $(1 + \text{max-cpn})$ $(r_i > 0)$

(c) If the principle guaranteed is taken, what's the derivatives this payoff corresponds to? ~~(puts)~~ ~~(cliquet)~~ ~~of puts~~ ATM

Hint: payoff $= 1 + \text{max-cpn} - \sum_{i=1}^{N} \max(0, -r_i)$ (cliquets / strips of forward ATM puts)

(d) What's the payoff if max-cpn $= 0$?

(e) (repeated)



assume negative ATM volatility skew, which one is SVM and which is LVM?

(1st put spread makes no difference).

In LVM, skew gets flatter $\rightarrow$ the value of ~~ATM~~ put spread $\downarrow \Rightarrow \xrightarrow{\text{Long the spread for hedge}}$ value $P \downarrow$.

(f) Why the two lines coincide at max-cpn $= 0$?