9875_final_exame

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1 Final Summary review

Five question, 100

one question that is same. why plot come from stochastic vol or local vol

1. Local Volatility Assume $dS(t) = \sigma_t(S_t, t)dZ_t$

$$C(S,K,T) = \int_K^\infty dS_T \phi(S_t, T, S_0)(S_T - K)$$

$$\frac{1}{2}\frac{\partial^2}{\partial S^2}(\sigma^2(S_T, T)S_T^2\phi) = \frac{\partial \phi}{\partial T}$$

$$\frac{\partial C}{\partial T} = \sigma_l^2(K,T)\frac{K^2}{2} * \frac{\partial^2 C}{\partial K^2}$$

$$C = \int_0^\infty dS_t \frac{1}{2}\frac{\partial^2}{\partial S_T^2}(\sigma_l^2(S_T, T)S_T^2\phi)(S_T - K)^+ = \frac{1}{2}\sigma_l^w K^2 \phi(k) = \frac{1}{2}\sigma_l^2 K^2 \frac{\partial^2 \phi}{\partial K^2}$$

(b)

$$\frac{dS_t}{S_t} = \sqrt{v_t}dZ_t$$

Show $\sigma_l^2(K,T) = E[v_T | S_T = K]$

$$d(S_T - K)^+ = \partial(S_T - K)dS_T + \frac{1}{2}v_t S_T^2 \delta(S_T - K)dT$$

$$\begin{aligned}\frac{\partial C}{\partial T} &= \frac{dE[(S_T - K)^+]}{dT} \\ &= \frac{1}{2}K^2 E[v_T | S_T = K]E[\delta(S_T - K)]\end{aligned}$$

2

$$\frac{dS_t}{S_t} = \sqrt{v_t}dZ_t$$

$$dv_t = -x(v_t - \bar{v})dt + \alpha v_t dW_t$$

Need to know what is the variance swap

(a)

$$\xi_t(n) = E[v_n|F_t]$$
$$\frac{dE[v_t]}{dt} = -x(E[v_t] - \bar{v})$$
$$\xi_t(n) = E_t[v_n] = (v_t - \bar{v})e^{-x(n-t)} + \bar{v}$$
$$d\xi_t(n) = e^{-x(n-t)}\alpha v_t dW_t$$

(b)

$$\sigma_{BC}(K,t) = \hat{\sigma_T} + S_T K + o(\epsilon^2)$$
$$S_T = \sqrt{\frac{w}{T}}\frac{1}{2w^2}C^{x\xi}$$
$$C^{x\xi} = \int_0^T dt \int_t^T dn \frac{E[dn_t d\xi_t(n)]}{dt}$$

$$E_0[dn_t d\xi_t(n)] = E[\sqrt{v_t}dZ_t e^{-x(n-t)}\alpha v_t dW_t]$$
$$= E_0[v_t^{\frac{3}{2}} dt\rho\alpha e^{-x(n-t)}]$$
$$= \xi_0(t)^{\frac{3}{2}}\rho * \alpha e^{-x(n-t)} + o(\epsilon^2)$$

$$C^{n_\xi} = \xi_0(t)^{\frac{3}{2}}\rho\alpha \int_0^T dt \frac{1}{K}(1 - e^{-x(T-t)})$$
$$= \xi_0(t)^{\frac{3}{2}}\rho\frac{\alpha}{K}(T - \frac{1}{K}(1 - e^{-KT}))$$

$$\phi_l(T) = \frac{\partial}{\partial K}\sigma_{BS}(K,T)|_{k=0}$$

(c)

$$\phi_L(T) = S_T = \frac{1}{2T}\frac{1}{\bar{v}^{3/2}}\bar{v}^{-3/2}\frac{\rho\alpha}{K}(T - \frac{1-e^{-KT}}{K}) = \frac{\rho\alpha}{2}\frac{1}{x^T}(1 - \frac{1-e^{-KT}}{K_T})$$

(d)

$$\phi_N(T) = \frac{\rho n}{2\sqrt{\bar{v}}}\frac{1}{K_T}(1 - \frac{1-e^{-KT}}{K_T})$$

3

$$w(k,T) = \sigma_{BS}^2(K,T)T$$
$$w(k,T) = \frac{\theta(T)}{2}(1 + \rho\Phi(T) + \sqrt{(\Phi(T)K + \rho)^2 + 1 - \rho^2})$$

(a)

$$w = \theta(1 - \Phi_k)^+$$

(b) convert

$$w(0, T) = \theta \tag{1}$$
$$\partial_k w(0, T) = -\theta \Phi I_{k<1/\Phi} \tag{2}$$
$$\partial_{k,k} w(0, T) = 0 \tag{3}$$

(c)

$$v_l(K, T) = \frac{\frac{\partial w}{T}}{(1 - \frac{1}{2}\frac{k}{w}\frac{\partial w}{\partial k})^2 - \frac{1}{4}(\frac{1}{4} + \frac{1}{w})(\frac{\partial w}{\partial K})^2 + \frac{1}{2}\frac{\partial^2 w}{\partial K^2}}$$

$$\Phi(T) = \frac{\alpha}{T^\gamma}, \alpha > 0$$

Show $\gamma >= 1$, T large and $\gamma <= \frac{1}{2}$ T small
Large T

$$\begin{aligned} f(T) &\approx 1 - \frac{1}{4}(\frac{1}{4} + \frac{1}{w})(\frac{\partial w}{\partial K})^2 \\ &\approx 1 - \frac{1}{4}(\theta\Phi)^2 \\ &\approx 1 - \frac{1}{16}\alpha^2 T^{2-2\gamma}, \gamma >= 1 \end{aligned}$$

(d) Show T

$$\begin{aligned} f(T) &\approx 1 - \frac{1}{4}(\frac{1}{4} + \frac{1}{w})(\frac{\partial w}{\partial K} \\ &\approx 1 - \frac{1}{4\theta}(\theta\Phi)^2 \\ &\approx 1 - \frac{1}{16}\alpha^2 T^{1-2\alpha}, \gamma <= 1/2 \end{aligned}$$

(e)

$$\Phi(T) = \frac{1}{T^\gamma}\frac{1}{(T_0 + T)^{1-\gamma}}, \gamma <= 1/2$$

4

$$1 + max[0, maxcoupon + \sum_{i=1}^{N} min(0, r_i)]$$
$$r_i = \frac{S_i - S_{i-1}}{S_{i-1}}$$

(c)

$$1 + maxCoupon + \sum_{i=1}^{N} min(0, r_i)$$

Short put

(d) 1

(e) Local vol is less than stochastic vol. Short a bunch of put spread. forward skew is small under local vol

5 Heston model.

$$dS_t = \sqrt{v_t} S_t dZ_t$$
$$dv_t = -\lambda(v_t - \bar{v})dt + n\sqrt{v_t} dW_t$$

(a)

$$\hat{v}(t) = E[v_t]$$
$$\frac{d\hat{v}}{dt} = -\lambda(\hat{v} - \bar{v})$$

(b)

$$v(T) = \frac{1}{T} E[\int_0^T v_t dt] = \frac{1}{T}(v_0 - \bar{v})\frac{1 - e^{-\lambda T}}{\lambda} + \bar{v}T)$$

(c)

$$u(t) = E[S_t v_t]$$
$$d(S_t v_t) = S_t dv_t + v_t dS_t + dS_t dv_t$$

$$\frac{du}{dt} = -\lambda(u - S_t \bar{v}) + \rho n u$$
$$= -\lambda'(u - \bar{u}')dt$$

$$\lambda' = \lambda - \rho n$$
$$\bar{u}' = \frac{\lambda S_0 \bar{v}}{\lambda'}$$

(d)

$$g(T) = \frac{1}{T} E[\int_0^T \frac{S_t}{S_0} v_t dt]$$
$$= \frac{1}{T}((v_0 - \bar{v}')\frac{1 - e^{-\lambda' T}}{\lambda'} + \bar{v}'), \bar{v}' = \frac{\lambda \bar{v}}{\lambda'}$$

(f)

$$\rho = 0, h = g - v = 0$$

$$L(T) = \frac{1}{S_0}\int_0^T Cov[S_t, v_t]dt$$

$$Cov(S_t, v_t) = E[S_t v_t] - E[S_t]E[v_T]$$
$$L(T) = E[\int_0^T \frac{S_t}{S_0} v_t dt] - \int_0^T E[v_t]dt$$
$$G(T) - V(T)$$

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