MTH 9875 The Volatility Surface: Fall 2019

Lecture 3: Affine models

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Outline of lecture 3

- Forward variance models
- Affine forward variance models
- The classical Heston model
- The rough Heston model
- The characteristic function of an affine forward variance model
  - Classical Heston and rough Heston characteristic functions
- Implementation of the classical and rough Heston models in R
- Numerical experiments

Forward variance models

Following [Bergomi and Guyon][3], forward variance models may be written in the form

$$\begin{aligned} dS_t &= S_t\,\sqrt{v_t}\,\left(\rho dW_t + \sqrt{1-\rho^2}\,dW_t^\perp\right) \\ d\xi_t(u) &= \eta_t(u;\omega)\,dW_t, \end{aligned}$$

where $W, W^\perp$ are independent Brownian motions, the $\mathbb{R}_{\geqslant 0}$-valued stochastic process $\eta_t(u;\omega)$ is progressively measurable for all $u > 0$ and ξ is linked to the instantaneous variance v by

$$\xi_t(T) = \mathbb{E}\left[v_T|\,\mathcal{F}_t\right].$$

- Models of this form were also studied by Hans Bühler as *variance curve models*.

- If v is continuous and uniformly integrable, we can recover v_t from $\xi_t(u)$ as $v_t = \lim_{u\downarrow t}\,\xi_t(u)$. For our purposes, $v_t = \xi_t(t)$.

- The initial conditions of a forward variance model are the initial stock price S_t and the initial forward variance curve $\xi_t(u)_{u\succ t}$.

Important remark

As noted by [Bergomi and Guyon][3], all conventional finite-dimensional Markovian stochastic volatility models may be cast as forward variance models.

The classical Heston model

The classical Heston stochastic volatility model may be written as

$$\begin{aligned}\frac{dS_t}{S_t} &= \sqrt{v_t}\, dZ_t \\ dv_t &= -\lambda\,(v_t - \bar v)\, dt + \eta\, \sqrt{v_t}\, dW_t\end{aligned}$$

with $\mathbb{E}\left[dZ_t\, dW_t\right] = \rho\, dt$ and where λ is the speed of reversion of v_t to its long term mean $\bar v$.

- The process followed by the instantaneous variance v_t may be recognized as a version of the square root process or CIR process of [Cox, Ingersoll, Ross][5].
- It is a (jump-free) special case of a so-called *affine jump diffusion (AJD)*.
  - Roughly speaking a jump-diffusion process for which the drifts and covariances and jump intensities are linear in the state vector (which is $\{x, v\}$ in this case with $x = \log(S)$).

$\mathbb{P}$ and $\mathbb{Q}$ measures

- In Heston's original paper, the price of risk is assumed to be linear in the instantaneous variance v in order to be able to retain the form of the dynamics under the transformation from the physical measure $\mathbb{P}$ to the pricing measure $\mathbb{Q}$.
- In contrast, in the following, we assume that model dynamics with parameters fitted to option prices generates the pricing measure.
  - All expectations in the following are under $\mathbb{Q}$.

Forward variance in the Heston model

Recall the Heston variance SDE

$$dv_t = -\lambda(v_t - \bar v)\, dt + \eta\, \sqrt{v_t}\, dW_t.$$

With $\xi_t(u) = \mathbb{E}\left[v_u \middle| \mathcal{F}_t\right]$, take expectations to get

$$d\xi_t(u) = -\lambda\,(\xi_t(u) - \bar v)\, du.$$

This ODE has the solution

$$\xi_t(u) = (\xi_t(t) - \bar v)\, e^{-\lambda\,(u-t)} + \bar v = (v_t - \bar v)\, e^{-\lambda\,(u-t)} + \bar v.$$

The Heston model in forward variance form

For each u, $\xi_t(u)$ is a conditional expectation and so a martingale in t. It is then immediate from the last equation that

$$d\xi_t(u) = e^{-\lambda\,(u-t)}\;\widehat{dv_t}\; = \eta\, e^{-\lambda\,(u-t)}\;\sqrt{v_t}\,dW_t$$

where $\widehat{dv_t}$ denotes the martingale part of dv_t.

- It is easy to check explicitly that all drift (*i.e.* dt) terms cancel.

The rough Heston model

By considering the limit of a simple Hawkes process-based model of order flow, [El Euch and Rosenbaum][6] derive a rough Heston model. The equation for variance in this model takes the form

$$v_u\; = \theta(u) - \frac{1}{\Gamma(\alpha)}\int_t^u (u-s)^{\alpha-1}\lambda\, v_s\,ds + \frac{1}{\Gamma(\alpha)}\int_t^u (u-s)^{\alpha-1}\eta\sqrt{v_s}\,dW_s.$$

where $\alpha = H + \frac{1}{2}$.

- $H \in (0, \frac{1}{2}]$ is the Hurst exponent of the volatility, $\lambda > 0$ is the mean reversion parameter, $\eta > 0$ is the volatility of volatility parameter.
- The function θ is assumed to be continuous and represents a time-dependent mean reversion level.
- The rough Heston model generalizes the classical Heston model which is recovered when $H = 1/2$.

Forward variance in the rough Heston model

- We will consider only the special case $\lambda = 0$. In this case, $\xi_t(u) = \mathbb{E}\left[v_u \middle| \mathcal{F}_t\right] = \theta(u)$.
- It follows that
$$v_u\; = \xi_t(u) + \frac{1}{\Gamma(H+1/2)}\int_t^u (u-s)^{H-1/2}\eta\sqrt{v_s}\,dW_s.$$
- Also
$$v_u\; = \xi_{t+h}(u) + \frac{1}{\Gamma(H+1/2)}\int_{t+h}^u (u-s)^{H-1/2}\eta\sqrt{v_s}\,dW_s.$$

The rough Heston model in forward variance form

Subtracting these two equations gives

$$\xi_{t+h}(u) - \xi_t(u) = \frac{1}{\Gamma(H+1/2)} \int_t^{t+h} (u-s)^{H-1/2} \eta \sqrt{v_s}\, dW_s.$$

Taking the limit $h \to 0$, we obtain

$$d\xi_t(u) = \frac{\eta}{\Gamma(H+1/2)} (u-t)^{H-1/2} \sqrt{v_t}\, dW_t,$$

the rough Heston model in forward variance form.

Non-Markovianity of the rough Heston model

- Note that the limit $u \to t$ of the rough Heston model makes no sense.
  - This reflects the fact that the rough Heston model is not Markovian.
    - There is no SDE for v_t and no corresponding PDE.
  - On the other hand, we can write an SDE for each $\xi_t(u),\ u > t$.
    - We can even apply Itô's Lemma!
- The rough Heston model is Markovian in the infinite-dimensional forward variance curve $\xi_t(u),\ u > t$.

Affine processes

The following explanation is due to Martin Keller-Ressel:

> An *affine process* can be described as a Markov process whose log-characteristic function is an affine function of its initial state vector.

And here's a definition of the word *affine* from Wikipedia:

> In geometry, an affine transformation or affine map or an affinity (from the Latin, *affinis*, "connected with") between two vector spaces (strictly speaking, two affine spaces) consists of a linear transformation followed by a translation:
> $$x \mapsto A\,x + b$$

Affine CGF

Let $X_t = \log S_t$. According to Definition 2.2 of [Gatheral and Keller-Ressel][9], we say that a forward variance model has an *affine cumulant generating function* determined by $g(t; u)$, if its conditional cumulant generating function is of the form

(1)

$$\log \mathbb{E}\left[e^{u(X_T - X_t)} \middle| \mathcal{F}_t \right] = \int_t^T g(T-s; u)\, \xi_t(s) ds.$$

for all $u \in [0, 1]$, $0 \leq t \leq T$ and $g(.\,; u)$ is $\mathbb{R}_{\leqslant 0}$-valued and continuous on $[0, T]$ for all $T > 0$ and $u \in [0, 1]$.

- The restriction $u \in [0, 1]$ is for mathematical convenience. We will later allow complex u.

When is a forward variance model affine?

Theorem 2.4 of [Gatheral and Keller-Ressel][9] states that a forward variance model has an affine CGF if and only if it takes the form

$$\begin{aligned}\frac{dS_t}{S_t} &= \sqrt{v_t}\,dZ_t \\ d\xi_t(u) &= \sqrt{v_t}\,\kappa(u-t)\,dW_t\end{aligned}$$

for some deterministic, non-negative decreasing kernel κ, which satisfies $\int_0^T \kappa(r)dr < \infty$ for all $T > 0$.

Moreover, $g(.\,, u) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\leqslant 0}$ in the definition (1) of the CGF is the unique global continuous solution of the convolution Riccati equation

$$g(t,u) = R_V\left(u, \int_0^t \kappa(t-s)g(s,u)ds\right) = R_V\Big(u, (\kappa \star g)(t,u)\Big), \quad t \geq 0$$

where

$$R_V(u,w) = \frac{1}{2}(u^2-u) + \rho\,u\,w + \frac{1}{2}\,w^2.$$

Alternatively, $g(t, u)$ can be written as

$$g(t,u) = R_V(u, f(t,u)),$$

where $f(t, u)$ is the unique global solution of the non-linear Volterra equation

$$f(t,u) = \int_0^t \kappa(t-s)R_V(u, f(t,s))ds.$$

Derivation of the Riccati equation

From the definition definition (1) of the CGF,

$$M_t = \mathbb{E}\left[e^{u\,X_T}\middle|\mathcal{F}_t\right] = \exp\left\{u\,X_t + \int_t^T \xi_t(s)\,g(T-s;u)\,ds\right\} =: \exp\{u\,X_t + G_t\}$$

is a conditional expectation and thus a martingale in t.

Applying Itô's Lemma to M gives

$$\frac{dM_t}{M_t} = u\,dX_t + dG_t + \frac{u^2}{2}\,d\langle X\rangle_t + \frac{1}{2}\,d\langle G\rangle_t + u\,d\langle X,G\rangle_t.$$

Now

$$\begin{aligned} dX_t &= -\frac{1}{2}\,v_t\,dt + \sqrt{v_t}\,dZ_t \\ dG_t &= -\xi_t(t)\,g(T-t;u)\,dt + \int_t^T d\xi_t(s)\,g(T-s;u)\,ds \\ &= -v_t\,g(T-t;u)\,dt + \int_t^T \kappa(s-t)\,\sqrt{v_t}\,dW_t\,g(T-s;u)\,ds. \end{aligned}$$

We compute

$$\begin{aligned} d\langle X\rangle_t &= v_t\,dt \\ d\langle G\rangle_t &= v_t\,dt\left(\int_t^T \kappa(s-t)\,g(T-s;u)\,ds\right)^2 \\ d\langle X,G\rangle_t &= \rho\,v_t\,dt\int_t^T \kappa(s-t)\,g(T-s;u)\,ds. \end{aligned}$$

Imposing $\mathbb{E}\left[dM_t\right] = 0$ and letting $\tau = T - t$ gives

$$0 = v_t\,dt\,\left\{-\frac{1}{2}\,u + \frac{1}{2}\,u^2 - g(\tau;u) + \rho\,u\,(\kappa\star g)(\tau,u) + \frac{1}{2}\,(\kappa\star g)(\tau,u)^2\right\}$$

where the convolution integral is given by

$$(\kappa\star g)(\tau,u) = \int_0^\tau \kappa(\tau-s)\,g(s;u)\,ds.$$

- It is almost obvious why the CGF is affine if an only if the forward variance process is of the form $d\xi_t(u) = \sqrt{v_t}\,\kappa(u-t)\,dW_t$.

The convolution Riccati equation

Rearranging gives

$$g(\tau;u) = \frac{1}{2}\,u(u-1) + \rho\,u\,(\kappa\star g)(\tau;u) + \frac{1}{2}\,(\kappa\star g)(\tau;u)^2 = R_V(u,(\kappa\star g)),$$

as required.

Example: The rough Heston model (with $\lambda = 0$)

In this case, with $\alpha = H + \frac{1}{2}$, $\kappa(\tau) = \frac{\eta}{\Gamma(\alpha)}\,\tau^{\alpha-1}$ and

$$
\begin{aligned}
\eta\, h(\tau;u) := (\kappa \star g)(\tau;u) &= \frac{\eta}{\Gamma(\alpha)} \int_0^\tau (\tau-s)^{\alpha-1}\, g(s;u)\, ds \\
&= \eta\, I^\alpha g(\tau;u).
\end{aligned}
$$

Inverting this gives $g(\tau;u) = D^\alpha h(\tau;u)$.

The convolution integral Riccati equation then reads

$$
D^\alpha h(\tau;u) = \frac{1}{2}\, u\,(u-1) + \rho\,\eta\,u\,h(\tau;u) + \frac{1}{2}\,\eta^2\, h(\tau;u)^2,
$$

consistent with [El Euch and Rosenbaum][6].

An aside: Fractional calculus

Define the fractional integral and differential operators:

$$
I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds; \quad D^\alpha f(t) = \frac{d}{dt}\, I^{1-\alpha} f(t).
$$

The fractional integral is a natural generalization of the ordinary integral using the Cauchy formula for repeated integration:

$$
\begin{aligned}
I^n f(t) &:= \int_0^t dt_1 \int_0^{t_1} \ldots dt_{n-1} \int_0^{t_{n-1}} f(t_n)\, dt_n \\
&= \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s)\, ds.
\end{aligned}
$$

The induction step

$$
\begin{aligned}
I^{n+1} f(t) &:= \int_0^t I^n f(s)\, ds \\
&= \frac{1}{(n-1)!} \int_0^t ds \int_0^s (s-r)^{n-1} f(r)\, dr \\
&= \frac{1}{(n-1)!} \int_0^t f(r)\, dr \int_r^t (s-r)^{n-1}\, ds \\
&= \frac{1}{n!} \int_0^t (t-r)^n f(r)\, dr.
\end{aligned}
$$

A microstructural foundation of the rough Heston model

- The rough Heston model emerges as the continuous time limit of a Hawkes process-based model of order flow.
  - Buys(sells) make the price go up(down)
  - Buys(sells) induce more buys(sells). That is, the processes are self-exciting.
  - Excitations decay as a power law.

- This gives us a clue as to why rough volatility appears to be universal.

Example: The classical Heston model

In this case, $\kappa(\tau) = \eta\, e^{-\lambda\, \tau}$.

Then

$$\eta\, h(\tau; u) := (\kappa \star g)(\tau; u) = \eta \int_0^\tau e^{-\lambda\,(\tau-s)}\, g(s; u)\, ds.$$

Also, $\partial_\tau h(\tau; u) = -\lambda\, h(\tau; u) + g(\tau; u)$. The convolution Riccati equation then becomes

(2)

$$\partial_\tau h(\tau; u) = \frac{1}{2}\, u\,(u-1) - (\lambda - \rho\,\eta\, u)\, h(\tau; u) + \frac{1}{2}\,\eta^2\, h(\tau; u)^2$$

consistent with the classical derivation in (for example) Chapter 2 of [The Volatility Surface][7].

Solution of the classical Heston Riccati equation

We rewrite equation (2) in the form

(3)

$$\partial_\tau h = \frac{1}{2}\,\eta^2\,(h - r_+)(h - r_-)$$

where

$$\begin{aligned} r_\pm &= \frac{\lambda - \rho\,\eta\, u \pm \sqrt{\lambda^2 + u\,(\eta^2 - 2\rho\,\eta\,\lambda) - u^2\,\eta^2\,\left(1-\rho^2\right)}}{\eta^2} \\ &=: \frac{\lambda - \rho\,\eta\, u \pm d}{\eta^2}. \end{aligned}$$

Integrating (3) with the terminal condition $h(0; u) = (\kappa \star g)(0, u) = 0$ gives

$$h(\tau; u) = r_-\,\frac{1 - e^{-d\,\tau}}{1 - \frac{r_-}{r_+}\, e^{-d\,\tau}}.$$

The Heston characteristic function

The characteristic function is given by

$$\varphi_t^T(a) = \mathbb{E}\left[e^{\mathrm{i}aX_T}\middle|\mathcal{F}_t\right] = \exp\left\{\mathrm{i}aX_t + \int_t^T \xi_t(s)\, g(T-s; \mathrm{i}a)\, ds\right\}$$

Recall that in the Heston model,

$$\xi_t(s) = (v_t - \bar{v})\, e^{-\lambda(s-t)} + \bar{v}.$$

First note that

$$\int_t^T e^{-\lambda\,(s-t)}\, g(T-s; \mathrm{i}a)\, ds = h(T-t; \mathrm{i}a).$$

Also,

$$g(\tau; u) = \partial_\tau h(\tau; u) + \lambda\, h(\tau; u)$$

so

$$\int_t^T g(T-s; \mathrm{i}a)\, ds = h(T-t; \mathrm{i}a) + \lambda \int_t^T h(T-s; \mathrm{i}a)\, ds.$$

Identifying $h(\tau; \mathrm{i}a)$ with $D(a,\tau)$ and with

$$C(a,\tau) = \lambda \int_t^T h(T-s; \mathrm{i}a)\, ds,$$

we obtain

(4)

$$\begin{aligned} D(a,\tau) &= r_- \frac{1 - e^{-d\,\tau}}{1 - \frac{r_-}{r_+}\, e^{-d\,\tau}} \\ C(a,\tau) &= \lambda \left\{ r_- \tau - \frac{2}{\eta^2} \log\left(\frac{1 - \frac{r_-}{r_+}\, e^{-d\,\tau}}{1 - \frac{r_-}{r_+}} \right) \right\} \end{aligned}$$

as in equation (2.12) of [The Volatility Surface][7].

The Heston characteristic function

For emphasis, the Heston characteristic function is given by

The Heston characteristic function

$$\begin{aligned} \varphi_t^T(a) &= \mathbb{E}\left[e^{\mathrm{i}aX_T} \middle| \mathcal{F}_t \right] \\ &= \exp\{ \mathrm{i}aX_t + C(a, T-t)\, \bar{v} + D(a, T-t)\, v_t \} \end{aligned}$$

The Heston density

The Heston probability density function may be computed by Fourier inversion as

(5)

$$
\begin{aligned}
p(y,\tau) &= \int_{-\infty}^{\infty} \frac{du}{2\pi}\, e^{-i\,u\,y}\, \varphi(u) \\
&= \frac{1}{\pi} \int_0^{\infty} du\, Re\left[\exp\{C(u,\tau)\,\bar{v} + D(u,\tau)\,v - i\,u\,y\}\right]
\end{aligned}
$$

- Note that $y = \log S_T/S_t$ is the forward variable.

Information content of the characteristic function

The following are equivalent in the sense that given one of these, all of the others may be derived:

- All T expiration European option prices.
- The T expiration risk-neutral density.
- The T expiration implied volatilities.
- The T expiration characteristic function.

R-implementation of the Heston characteristic function

In [1]:
```
download.file(
    url="http://mfe.baruch.cuny.edu/wp-content/uploads/2019/09/9875-3.zip",
    destfile="9875-3.zip")
unzip(zipfile="9875-3.zip")
```

In [2]:
```
source("BlackScholes.R")
source("Lewis.R")
source("Heston.R")
source("roughHestonPade.R")
```

In [3]:
```
phiHeston
```

```
function (params)
{
    lambda <- params$lambda
    rho <- params$rho
    eta <- params$eta
    vbar <- params$vbar
    v <- params$v
    function(u, t) {
        al <- -u * u/2 - (0+1i) * u/2
        bet <- lambda - rho * eta * (0+1i) * u
        gam <- eta^2/2
        d <- sqrt(bet * bet - 4 * al * gam)
        rp <- (bet + d)/(2 * gam)
        rm <- (bet - d)/(2 * gam)
        g <- rm/rp
        D <- rm * (1 - exp(-d * t))/(1 - g * exp(-d * t))
        C <- lambda * (rm * t - 2/eta^2 * log((1 - g * exp(-(d *
            t)))/(1 - g)))
        return(exp(C * vbar + D * v))
    }
}
```

A digression: the complex logarithm in (4)

In Heston's original paper and in most other papers on the subject, $C(u, \tau)$ is written (almost) equivalently as

(6)

$$C(u, \tau) = \lambda \left\{ r_+ \tau - \frac{2}{\eta^2} \log\left(\frac{e^{+d\,\tau} - g}{1 - g} \right) \right\}$$

with $g = \frac{r_-}{r_+}$.

The reason for the qualification "almost" is that this definition coincides with our previous one only if the imaginary part of the complex logarithm is chosen so that $C(u, \tau)$ is continuous with respect to u.

It turns out that taking the principal value of the logarithm in (6) causes $C(u, \tau)$ to jump discontinuously each time the imaginary part of the argument of the logarithm crosses the negative real axis.

BCC parameters

[Bakshi, Cao and Chen][2] found (more or less) the following Heston (SV) parameters in a fit of the Heston model to historical volatility surfaces:

Table 1: BCC parameters

v	0.04
$\bar{v}$	0.04
λ	1.15
η	0.39
ρ	-0.64

A numerical example with BCC parameters

Recall the two versions of $C(u, \tau)$:

$$C^{JG}(u, \tau) = \lambda \left\{ r_{-}\tau - \frac{2}{\eta^2} \log\left(\frac{1 - g\, e^{-d\,\tau}}{1 - g} \right) \right\}$$

and

$$C^{Heston}(u, \tau) = \lambda \left\{ r_{+}\tau - \frac{2}{\eta^2} \log\left(\frac{e^{+d\,\tau} - g}{1 - g} \right) \right\}.$$

We code these below.

In [4]:
```
bigC <- function(params){

    lambda <- params$lambda
    rho <- params$rho
    eta <- params$eta
    vbar <- params$vbar
    v <- params$v

    fj <- function(u, t, j){

            al <- -u*u/2 - 1i*u/2 + 1i*j*u
            bet <- lambda - rho*eta*1i*u-rho*eta*j
            gam <- eta^2/2
            d <- sqrt(bet*bet - 4*al*gam)
            rp <- (bet + d)/(2*gam)
            rm <- (bet - d)/(2*gam)
            g <- rm / rp
            C.JG <- lambda * (rm * t - 2/eta^2 * log( (1 - g*exp(-(d*t)))/(1
            C.Heston <- lambda * (rp * t - 2/eta^2 * log( (exp(d*t) - g)/(1
            return(list(C.JG=C.JG,C.Heston=C.Heston))
    }
}
```

Check the two versions with $\tau = 3$, and with BCC parameters:

In [5]:
```
paramsBCC <- list(lambda = 1.15,rho = -0.64,eta = 0.39,vbar = 0.04,v = .04)
```

In [6]:
```
bigC(paramsBCC)(u=10,t=3,j=0)
```

$C.JG

-47.5424985325402+26.838817444334i

$C.Heston

-47.54249853254+121.850824061777i

The imaginary parts are very different. Which function works better?

Comparison plot

In [7]:
```
library(repr)
options(repr.plot.height=5)
```

In [8]:
```
u.vec <- seq(0,10,.01)
x <- bigC(paramsBCC)(u=u.vec,t=3,j=0)
```

In [9]:
```
plot(Re(x$C.Heston),Im(x$C.Heston),type="l",col="red",lty=2,lwd=2,xlab="Re(C
lines(Re(x$C.JG),Im(x$C.JG),type="l",col="green4",lwd=1.5)
```

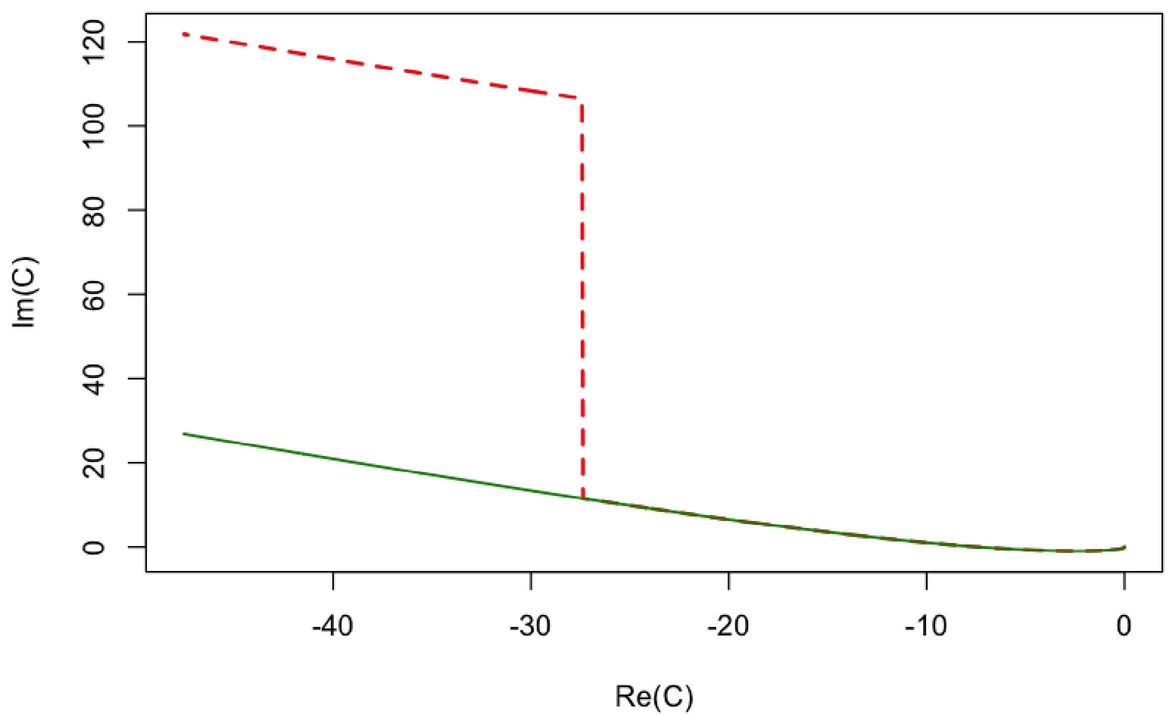


Figure 1: $C^{Heston}(\cdot)$ in red, $C^{JG}(\cdot)$ in green.

Solutions in the literature

One conventional resolution is to keep careful track of the winding number in the integration (14) so as to remain on the same Riemann sheet.

This leads to practical implementation problems because standard numerical integration routines cannot be used. The paper of [Kahl and Jäckel][10] concerns itself with this problem and provides an ingenious resolution.

A better solution

- With *our* definition (4) of $C(u, \tau)$, whenever the imaginary part of the argument of the logarithm is zero, the real part is positive.
  - Plotted in the complex plane, the argument of the logarithm never cuts the negative real axis.
  - This result was proved by [Albrecher, Mayer, Schoutens and Tistaert][1].
- It follows that with our definition of $C(u, \tau)$, taking the principal value of the logarithm leads to a continuous integrand over the full range of integration.

The rough Heston characteristic function

- There exist a number of standard numerical techniques, such as the Adams scheme, for solving fractional differential equations such as the rough Heston fractional Riccati equation.
  - These techniques are all slow!
- Recently, [Gatheral and Radoičić][10] showed how to approximate the solution of the rough Heston fractional Riccati equation by a rational function.
  - This approximation solution is just as fast as the classical Heston solution and appears to be more accurate than the Adams scheme for any reasonable number of time steps!

Computing option prices from the characteristic function

It turns out (see [Carr and Madan][4] and [Lewis][11]) that it is quite straightforward to get option prices by inverting the characteristic function of a given stochastic process (if it is known in closed-form).

The formula we will use is a special case of formula (2.10) of Lewis (as usual we assume zero interest rates and dividends):

Formula (2.10) of Lewis

(7)
$$C(S,K,t,T) = S - \sqrt{SK}\frac{1}{\pi}\int_0^\infty \frac{du}{u^2+\frac{1}{4}}\,\mathrm{Re}\left[e^{-iuk}\,\varphi_t^T\,(u-i/2)\right]$$

with $k = \log\left(\frac{K}{S}\right)$. We now proceed to prove this formula.

Heston greeks

It is worth noting that taking derivatives of the Heston option value with respect to x or v in order to compute delta, gamma, vega and so on is extremely straightforward because the functions $C(u,\tau)$ and $D(u,\tau)$ are independent of x and v.

Proof of (7)

A covered call position has the payoff $\min[S_T, K]$ where S_T is the stock price at time T and K is the strike price of the call.

Consider the Fourier transform of this covered call position $G(k,\tau)$ with respect to the log-strike $k := \log\,(K/F)$ defined by

$$\hat{G}(u,\tau) = \int_{-\infty}^{\infty} e^{i\,u\,k}\,G(k,\tau)\,dk$$

Denoting time-to-expiration by $\tau = T - t$, and setting interest rates and dividends to zero as usual, we have that

$$
\begin{aligned}
\frac{1}{S}\hat{G}(u,\tau) &= \int\limits_{-\infty}^{\infty} e^{i\,u\,k}\,\mathbb{E}\left[\min[e^{x_\tau},e^{k})^+]\right]\,dk \\
&= \mathbb{E}\left[\int\limits_{-\infty}^{\infty} e^{i\,u\,k}\,\min[e^{x_\tau},e^{k})^+]\,dk\right] \\
&= \mathbb{E}\left[\int\limits_{-\infty}^{x_\tau} e^{i\,u\,k}\,e^{k}\,dk + \int\limits_{x_\tau}^{\infty} e^{i\,u\,k}\,e^{x_\tau}\,dk\right] \\
&= \mathbb{E}\left[\frac{e^{(1+iu)x_\tau}}{1+iu} - \frac{e^{(1+iu)x_\tau}}{iu}\right] \text{ only if } 0 < \mathrm{Im}[u] < 1! \\
&= \frac{1}{u(u-i)}\mathbb{E}\left[e^{(1+iu)x_\tau}\right] \\
&= \frac{1}{u(u-i)}\,\phi_\tau(u-i)
\end{aligned}
$$

by definition of the characteristic function $\phi_\tau(u)$.

- Note that the transform of the covered call value exists only if $0 < \mathrm{Im}[u] < 1$.
  - It is easy to see that this derivation would go through pretty much as above with other payoffs.
  - The region where the transform exists depends on the payoff.

To get the call price in terms of the characteristic function, we express it in terms of the covered call and invert the Fourier transform, integrating along the line $\mathrm{Im}[u] = 1/2$. Then

$$
\begin{aligned}
C(S,K,T) &= S - S\frac{1}{2\pi}\int_{-\infty+i/2}^{\infty+i/2}\frac{du}{u(u-i)}\,\phi_T(u-i)\,e^{-iku} \\
&= S - S\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{du}{(u+i/2)(u-i/2)}\,\phi_T(u-i/2)\,e^{-ik(u+i/2)} \\
&= S - \sqrt{SK}\frac{1}{\pi}\int_0^{\infty}\frac{du}{u^2+\frac{1}{4}}\,\mathrm{Re}\left[e^{-iuk}\,\phi_T\,(u-i/2)\right]
\end{aligned}
$$

with $k = \log\left(\frac{K}{S}\right)$.

R implementation of the Lewis formula

In [10]:
```
option.OTM.raw
```

```
function (phi, k, t)
{
    integrand <- function(u) {
        Re(exp(-(0+1i) * u * k) * phi(u - (0+1i)/2, t)/(u^2 +
            1/4))
    }
    k.minus <- (k < 0) * k
    res <- exp(k.minus) - exp(k/2)/pi * integrate(integrand,
        lower = 0, upper = Inf, rel.tol = 1e-08)$value
    return(res)
}
```

- Note that there is only one numerical integration here.
- In the version presented in Chapter 2 of [The Volatility Surface][7], two integrations are required.
    - The option price is then given by the difference between two large numbers, introducing numerical error.

The Heston model in R

- In the next few slides, we will exhibit an implementation of the Heston model in R.
- We can then develop some intuition for the behavior of the model through experiment.

Black-Scholes

First, we need an implementation of the Black-Scholes formula (this is in `BlackScholes.R`).

In [11]:
```
BSFormula
```

```
function (S0, K, T, r, sigma)
{
    x <- log(S0/K) + r * T
    sig <- sigma * sqrt(T)
    d1 <- x/sig + sig/2
    d2 <- d1 - sig
    pv <- exp(-r * T)
    return(S0 * pnorm(d1) - pv * K * pnorm(d2))
}
```

Implied volatility computation

Then, we need an implied volatility computation (this is also in `BlackScholes.R`).

In [12]:
```
BSImpliedVolCall
```

```
function (S0, K, T, r, C)
{
    nK <- length(K)
    sigmaL <- rep(1e-10, nK)
    CL <- BSFormula(S0, K, T, r, sigmaL)
    sigmaH <- rep(10, nK)
    CH <- BSFormula(S0, K, T, r, sigmaH)
    while (mean(sigmaH - sigmaL) > 1e-10) {
        sigma <- (sigmaL + sigmaH)/2
        CM <- BSFormula(S0, K, T, r, sigma)
        CL <- CL + (CM < C) * (CM - CL)
        sigmaL <- sigmaL + (CM < C) * (sigma - sigmaL)
        CH <- CH + (CM >= C) * (CM - CH)
        sigmaH <- sigmaH + (CM >= C) * (sigma - sigmaH)
    }
    return(sigma)
}
```

A numerical example

First generate some option prices.

In [13]:
```
vols <- c(0.23,0.20,0.18)
K <- c(0.9,1.0,1.1)
(optVals <- BSFormula(S0=1,K,T=1,r=0,sigma=vols))
```

0.145896960390544 0.079655674554058 0.0355767789605738

Then compute implied volatilities.

In [14]:
```
(impVols <- BSImpliedVolCall(S0=1, K, T=1, r=0, C=optVals))
```

0.230000000035418 0.200000000065986 0.179999999989352

Pick some Heston parameters

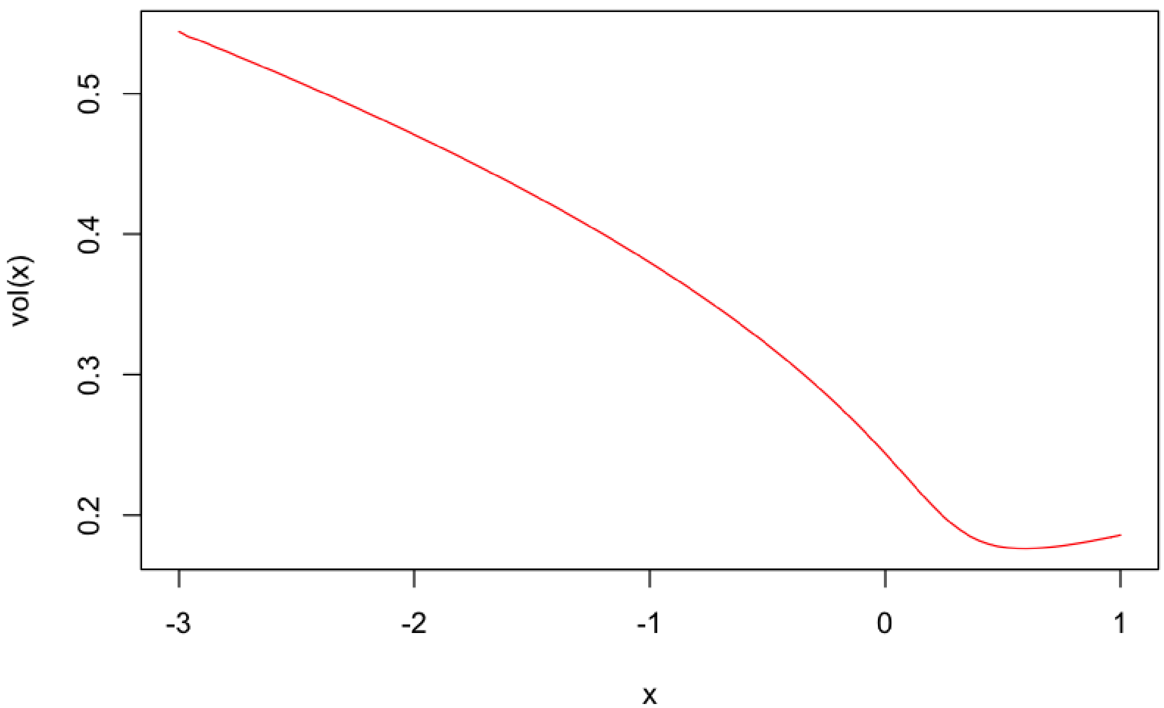
In [15]:
```
subHeston <- list(lambda = 0.6067,rho = -0.7571,eta = 0.2928,vbar = 0.0707,v

option.OTM(phiHeston(subHeston), 0, 1)

impvol.phi(phiHeston(subHeston))(0,1)
```

0.0970140580426301

0.243780488551484

Now draw a Heston 1-year smile

In [16]:
```
vol <- function(k){sapply(k,function(x){impvol.phi(phiHeston(subHeston))(x,1
curve(vol(x),from=-3,to=1,col="red")
```



Some notable features of R

- Complex arithmetic with $1i$.
- Functional programming:
  - This is what allows us to code a function which is called as:

    ```
    impvol.phi(phiHeston(paramsBCC))(0,1)
    ```

  - We can define a function that returns a function (and so on indefinitely).
  - We could even conveniently define a new function:

    ```
    impvolBCC <-
        impvol.phi(phiHeston(paramsBCC))
    ```

- We can conceptually separate parameters and variables rather than having to carry all the parameters around with each function call.

The Heston smile with BCC parameters

First we list the BCC parameters.

In [17]:
```
paramsBCC <- list(lambda = 1.15,rho = -0.64,eta = 0.39,vbar = 0.04,v = .04)
```

Then, generate the smile as before.

In [18]:
```
vol <- function(k){
    sapply(k,function(x){impvol.phi(phiHeston(paramsBCC))(x,1)})}
system.time(curve(vol(x),from=-.4,to=.4,col="red"))
```

```
   user  system elapsed 
  0.071   0.005   0.077 
```

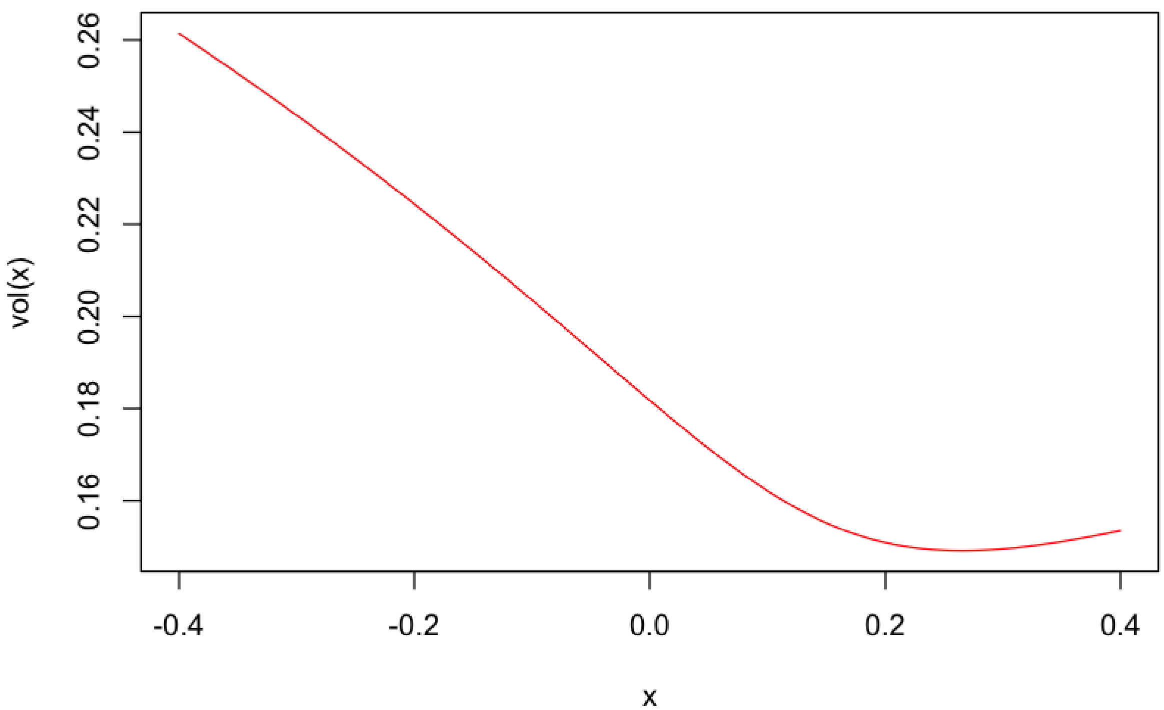


Figure 2: The $\tau = 1$ Heston smile with BCC parameters.

Sensitivity of the Heston smile to ρ

In [19]:
```
subH <- function(incr){
    tmp <- paramsBCC
    tmp$rho <- tmp$rho+incr
    return(tmp)
}

vol <- function(phi)function(k){sapply(k,function(x){impvol.phi(phi)(x,1)})}
```

In [20]:
```
yrange <- c(0.15,.3)
curve(vol(phiHeston(paramsBCC))(x),from=-.5,to=.5,col="red",ylim=yrange,lwd=
for (incr in seq(0.1,0.5,0.1))
    {
    curve(vol(phiHeston(subH(incr)))(x),from=-.5,to=.5,col="red",lty=2,add=T
    }
```

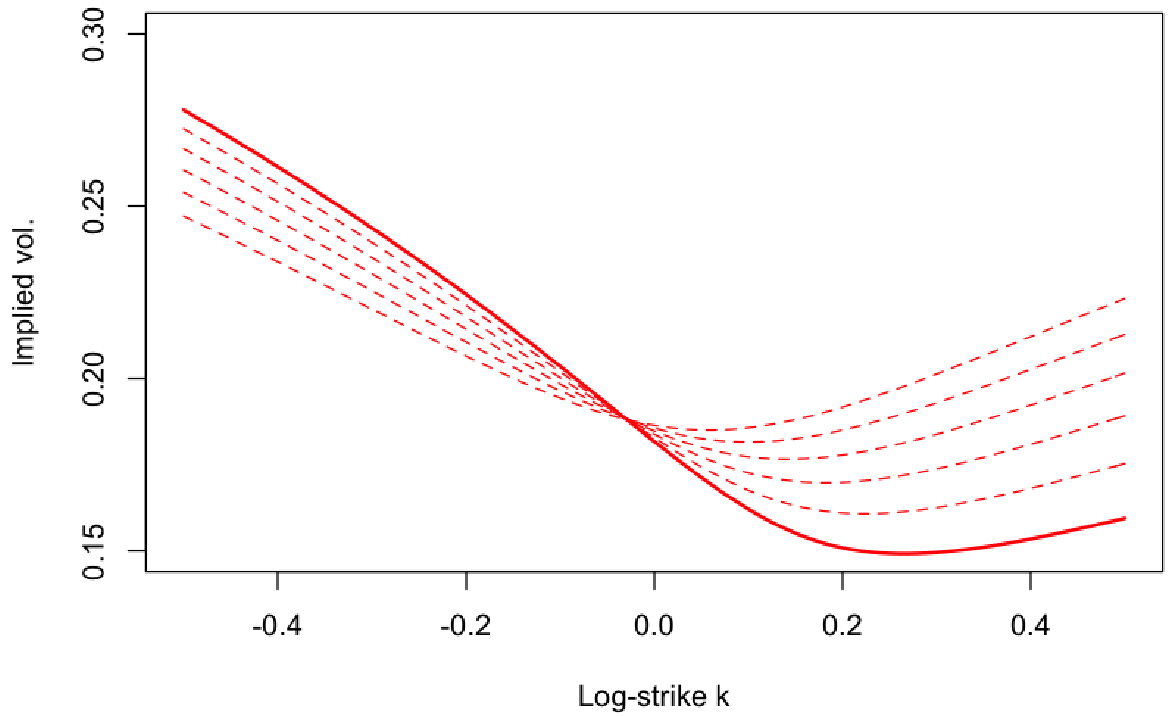


Figure 3: The dotted lines are smiles with $\rho = \rho_{BCC} + \{0.1, 0.2, 0.3, 0.4, 0.5\}$.

Sensitivity of the Heston smile to η

In [21]:
```
subH <- function(incr){
    tmp <- paramsBCC
    tmp$eta <- tmp$eta+incr
    return(tmp)
}
```

In [22]:
```
yrange <- c(0.1,.35)
curve(vol(phiHeston(paramsBCC))(x),from=-.5,to=.5,col="red",ylim=yrange,lwd=
for (incr in seq(0.1,0.5,0.1))
    {
    curve(vol(phiHeston(subH(incr)))(x),from=-.5,to=.5,col="red",lty=2,add=T
    }
```

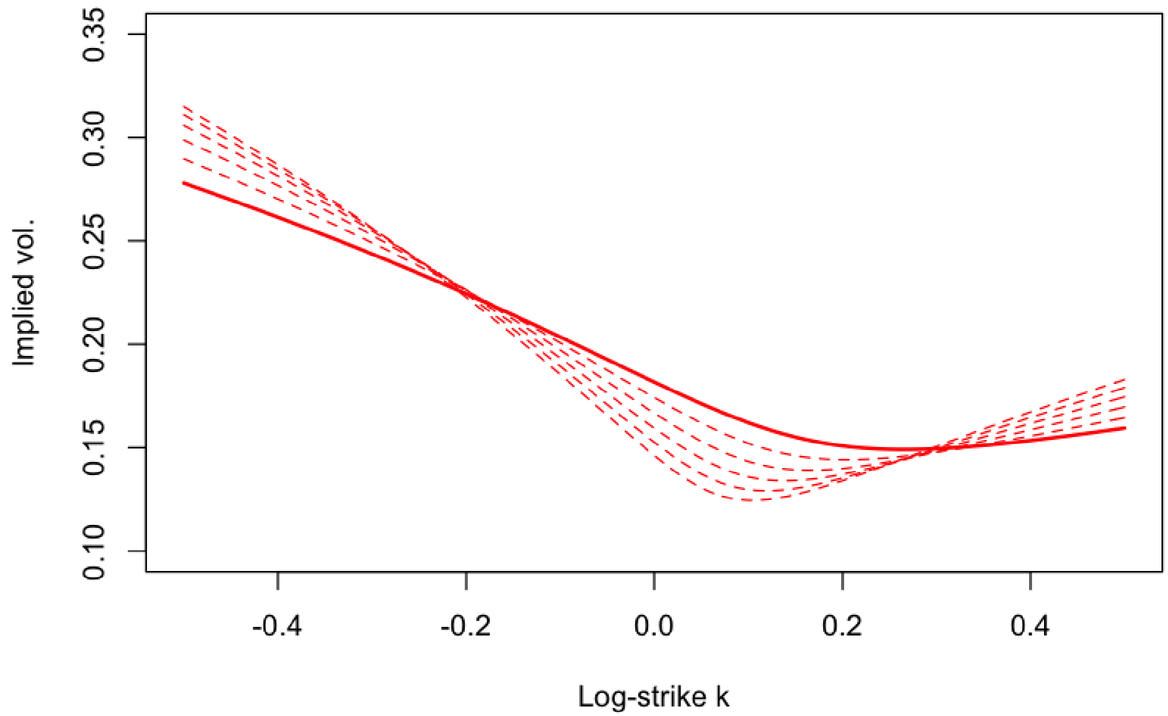


Figure 4: The dotted lines are smiles with $\eta = \eta_{BCC} + \{0.1, 0.2, 0.3, 0.4, 0.5\}$.

Sensitivity of the Heston smile to λ

In [23]:
```
subH <- function(incr){
    tmp <- paramsBCC
    tmp$lambda <- tmp$lambda+incr
    return(tmp)
}
```

In [24]:
```
yrange <- c(0.15,.3)
curve(vol(phiHeston(paramsBCC))(x),from=-.5,to=.5,col="red",ylim=yrange,lwd=
for (incr in seq(0.5,2.5,0.5))
    {
    curve(vol(phiHeston(subH(incr)))(x),from=-.5,to=.5,col="red",lty=2,add=T
    }
```

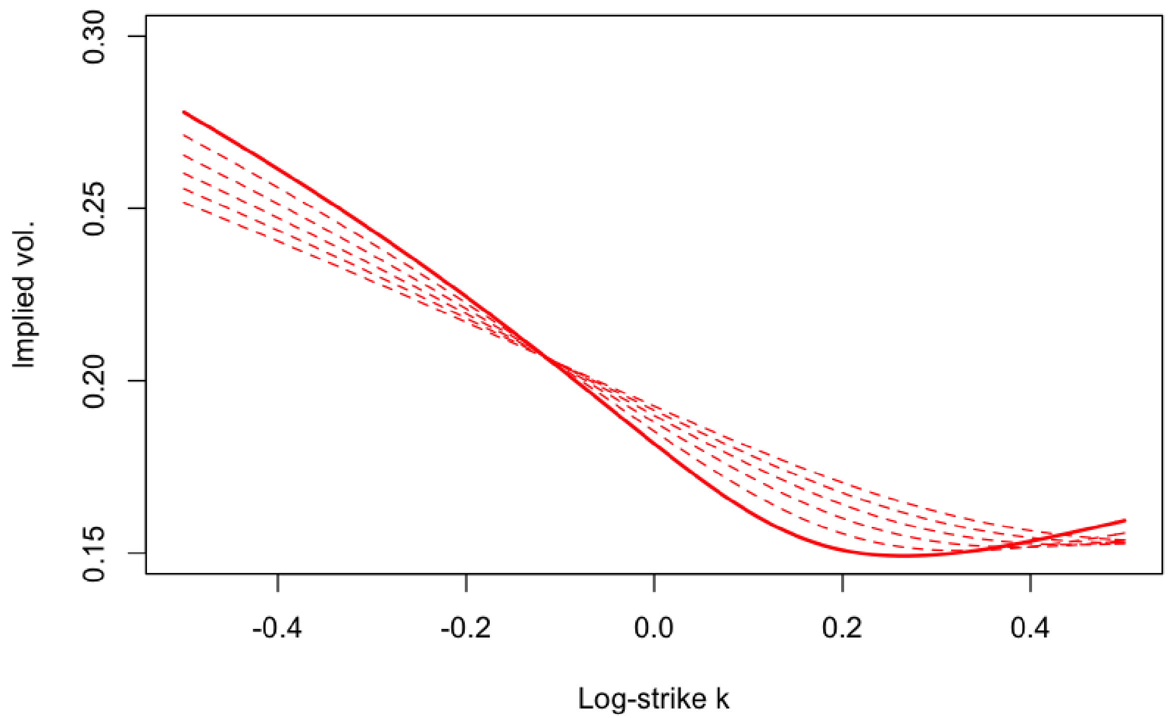


Figure 5: The dotted lines are smiles with $\lambda = \lambda_{BCC} + \{0.5, 1.0, 1.5, 2.0, 2.5\}$.

Another sensitivity plot

Now increase both λ and η keeping the ratio η^2/λ constant.

In [25]:
```
subH <- function(incr){
    tmp <- paramsBCC
    tmp$eta <- tmp$eta+incr
    tmp$lambda <- tmp$lambda*(tmp$eta/paramsBCC$eta)^2
    return(tmp)
}
```

In [26]:
```
yrange <- c(0.1,.35)
curve(vol(phiHeston(paramsBCC))(x),from=-.5,to=.5,col="red",ylim=yrange,lwd=
for (incr in seq(0.1,0.5,0.1))
    {
    curve(vol(phiHeston(subH(incr)))(x),from=-.5,to=.5,col="red",lty=2,add=T
    }
```

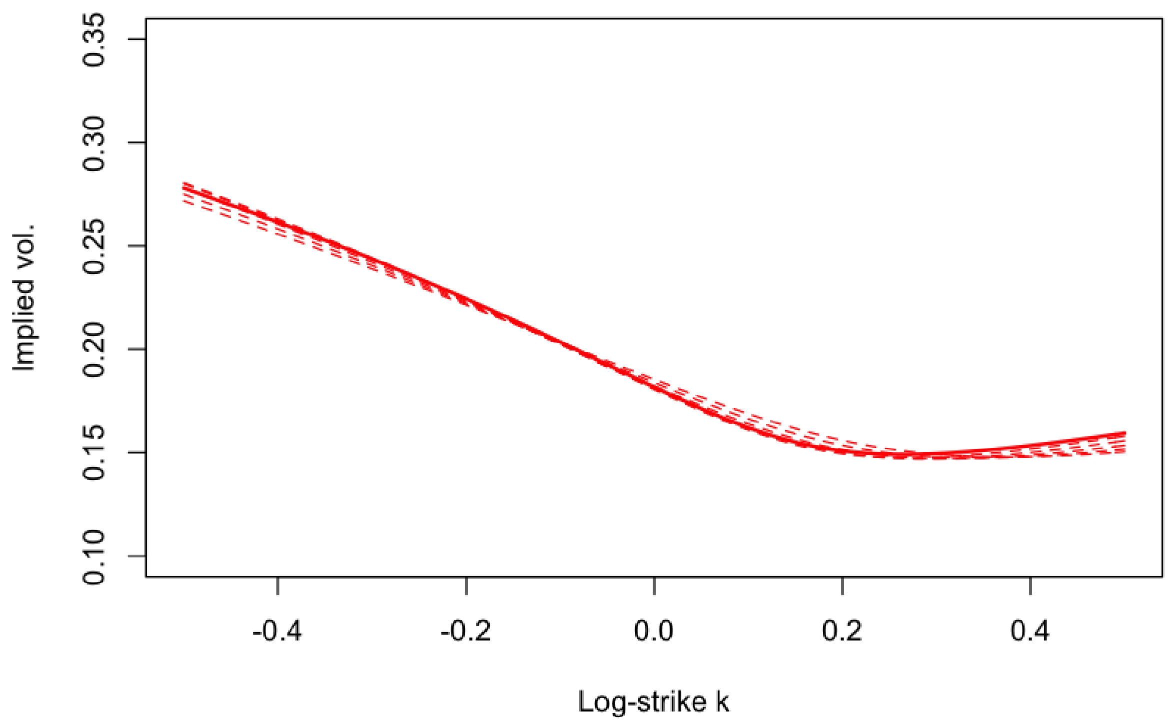


Figure 6: The dotted lines are smiles with $\eta = \eta_{BCC} + \{0.1, 0.2, 0.3, 0.4, 0.5\}$ and λ incremented so as to keep the ratio λ/η^2 constant.

Variance of v_T in the Heston model (old version)

Applying Itô's Lemma, we get

$$dv^2 = 2\, v\, dv + (dv)^2$$

With $\hat{v} = \mathbb{E}[v]$, this gives

$$\mathbb{E}[dv^2] = -2\,\lambda\, \left(\mathbb{E}[v^2] - \bar{v}\,\hat{v}\right)\, dt + \eta^2\, \hat{v}\, dt$$

Also

$$d\hat{v}^2 = -2\,\lambda\,\left(\hat{v}^2 - \bar{v}\,\hat{v}\right)\,dt$$

So

$$d\mathrm{Var}(v) = -2\,\lambda\,\mathrm{Var}(v)\,dt + \eta^2\,\hat{v}\,dt$$

Integrating, we get

$$\mathrm{Var}(v_T) = \int_0^T e^{-2\,\lambda\,(T-t)}\,\eta^2\,\hat{v}_t\,dt$$

Putting $\hat{v}_t = \bar{v}$ for simplicity, we obtain

$$\begin{aligned}\mathrm{Var}(v_T) &= \eta^2\,\bar{v}\int_0^T e^{-2\,\lambda\,(T-t)}\,dt\\ &= \eta^2\,\bar{v}\,\frac{1-e^{-2\,\lambda\,T}}{2\,\lambda}\\ &\to \frac{\eta^2\,\bar{v}}{2\,\lambda}\ \text{as } T\to\infty\end{aligned}$$

- The smile depends only on the distribution of integrated variance and its correlation with the stock price, so for longer expirations, the smile should depend (roughly) only on ρ and the ratio $\eta^2\,\bar{v}/\lambda$.

Covariance of v_T and $v_{T+\Delta}$ in affine forward variance models

Recall that $\xi_t(s) = \mathbb{E}\left[v_s|\,\mathcal{F}_t\right]$.

Then

$$\begin{aligned}\mathrm{cov}\left[v_T, v_{T+\Delta}|\,\mathcal{F}_t\right] &= \mathbb{E}\left[v_T\,v_{T+\Delta}|\,\mathcal{F}_t\right] - \mathbb{E}\left[v_T|\,\mathcal{F}_t\right]\mathbb{E}\left[v_{T+\Delta}|\,\mathcal{F}_t\right]\\ &= \mathbb{E}\left[\xi_T(T)\,\xi_T(T+\Delta)|\,\mathcal{F}_t\right] - \xi_T(T)\,\xi_t(T+\Delta) \quad \text{using the tower law}\\ &= \mathbb{E}\left[(\xi_T(T)-\xi_t(T))\,(\xi_T(T+\Delta)-\xi_t(T+\Delta))|\,\mathcal{F}_t\right] \quad \text{using that } \xi_t(u) \text{ is a ma}\\ &= \mathbb{E}\left[\int_t^T d\langle\xi(T),\xi(T+\Delta)\rangle_s\middle|\,\mathcal{F}_t\right]\\ &= \int_t^T \mathbb{E}\left[v_s|\,\mathcal{F}_t\right]\,\kappa(T-s)\,\kappa(T+\Delta-s)\,ds\\ &= \int_t^T \xi_t(s)\,\kappa(T-s)\,\kappa(T+\Delta-s)\,ds\end{aligned}$$

Variance of v_T in affine forward variance models

An immediate corollary is that

$$\mathrm{var}\left[v_T|\,\mathcal{F}_t\right] = \int_t^T \xi_t(s)\,\kappa(s-t)^2\,ds.$$

Application to the Heston model

With $\kappa(\tau) = \eta\, e^{-\lambda\,\tau}$ and putting $\xi_t(s) = \bar{v}$ (a flat curve), we obtain

$$\begin{aligned}\mathrm{var}\left[v_T | \mathcal{F}_t\right] &= \eta^2\,\bar{v}\,\int_t^T e^{-2\lambda\,(s-t)}\,ds \\ &= \frac{\eta^2\,\bar{v}}{2\,\lambda}\left\{1 - e^{-2\lambda\,(T-t)}\right\}.\end{aligned}$$

- The same results as before.

Application to the rough Heston model

With $\kappa(\tau) = \frac{1}{\Gamma(\alpha)}\,\eta\,\tau^{\alpha-1}$ and putting $\xi_t(s) = \bar{v}$ (a flat curve), we obtain

$$\begin{aligned}\mathrm{var}\left[v_T | \mathcal{F}_t\right] &= \frac{\eta^2\,\bar{v}}{\Gamma(\alpha)^2}\int_t^T (s-t)^{2\alpha-2}\,ds \\ &= \frac{\eta^2\,\bar{v}}{(2\alpha-1)\,\Gamma(\alpha)^2}\,(T-t)^{2\,\alpha-1} \\ &= \frac{\eta^2\,\bar{v}}{2H\,\Gamma(\alpha)^2}\,(T-t)^{2H}\end{aligned}$$

- Variance of v_T is power-law in $\tau = T - t$.

Autocorrelation of variance in the Heston model

Applying the above formula,

$$\begin{aligned}\mathrm{cov}\left[v_T, v_{T+\Delta} | \mathcal{F}_t\right] &= \int_t^T \xi_t(s)\,\kappa(T-s)\,\kappa(T+\Delta-s)\,ds \\ &= \bar{v}\int_t^T e^{-\lambda\,(T-s)}\,e^{-\lambda\,(T+\Delta-s)}\,ds \\ &= e^{-\lambda\,\Delta}\,\bar{v}\int_t^T e^{-2\,\lambda\,(T-s)}\,ds.\end{aligned}$$

Thus

$$\rho(\Delta) \sim \frac{\mathrm{cov}\left[v_T, v_{T+\Delta} | \mathcal{F}_t\right]}{\mathrm{var}\left[v_T | \mathcal{F}_t\right]} = e^{-\lambda\,\Delta}.$$

Autocorrelation of variance in the rough Heston model

Similarly,

$$
\begin{aligned}
\operatorname{cov}\left[v_T, v_{T+\Delta} \middle| \mathcal{F}_t\right] &= \int_t^T \xi_t(s)\, \kappa(T-s)\, \kappa(T+\Delta-s)\, ds \\
&= \frac{\eta^2}{\Gamma(\alpha)^2}\, \bar{v} \int_t^T (T-s)^{\alpha-1}\, (T+\Delta-s)^{\alpha-1}\, ds \\
&= \frac{\eta^2}{\Gamma(\alpha)^2}\, \bar{v}\, (T-t)^{2\,\alpha-1}\, G\left(\frac{\Delta}{T-t}\right).
\end{aligned}
$$

where (not very usefully),

$$
G(\Delta) = \frac{\Delta^{\alpha-1}}{\alpha\,(2\,\alpha-1)}\, {}_2F_1\left(1-\alpha, \alpha; \alpha+1; -\frac{1}{\Delta}\right).
$$

Thus

$$
\rho(\Delta) \sim \frac{\operatorname{cov}\left[v_T, v_{T+\Delta} \middle| \mathcal{F}_t\right]}{\operatorname{var}\left[v_T \middle| \mathcal{F}_t\right]} = G\left(\frac{\Delta}{T-t}\right).
$$

- What T to choose?
- There's only one timescale in this problem, namely 1 day.
- We end up with
$$
\rho(\Delta) \sim G\,(\Delta)\,.
$$

Thus

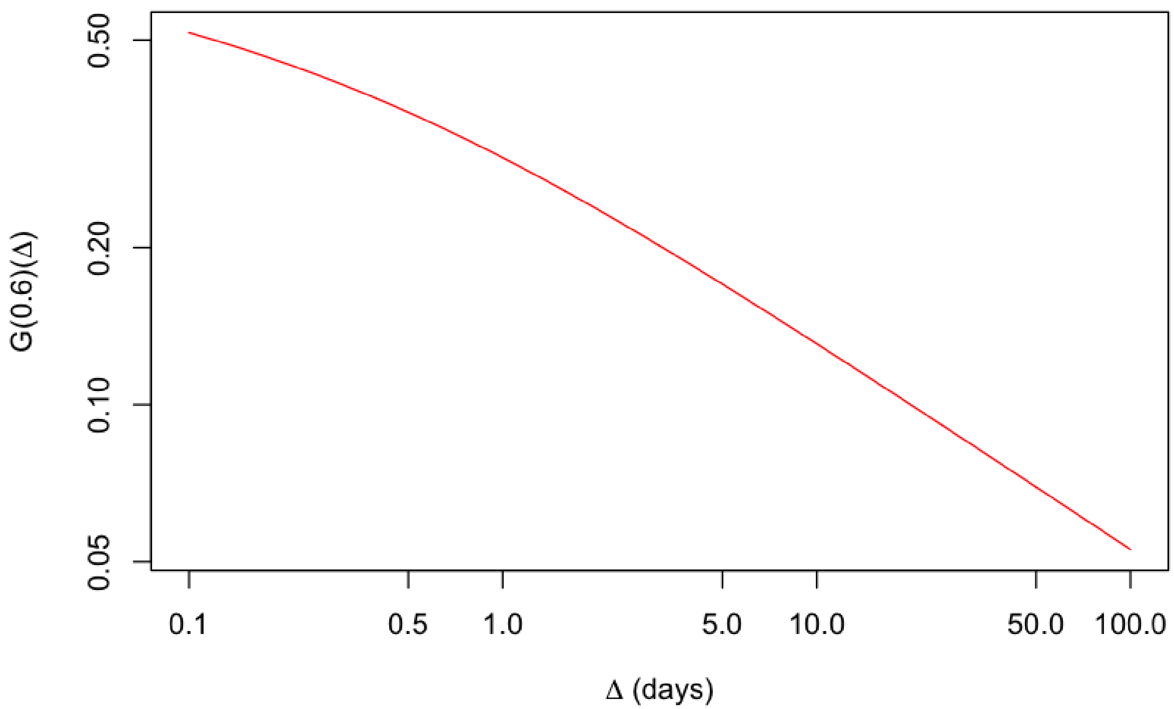
$$
\rho(\Delta) \sim \frac{\operatorname{cov}\left[v_T, v_{T+\Delta} \middle| \mathcal{F}_t\right]}{\operatorname{var}\left[v_T \middle| \mathcal{F}_t\right]} = G\left(\frac{\Delta}{T-t}\right).
$$

- What $T-t$ to choose?
  - We leave the timescale free as a parameter θ to be fitted.
- We end up with
$$
\rho(\Delta) \sim G\left(\frac{\Delta}{\theta}\right).
$$

Function to compute $G(\Delta)$

In [99]:
```
bigG.raw <- function(alpha)function(Delta){

    H <- alpha-1/2
    integr <- function(s){s^(alpha-1)*(s+Delta)^(alpha-1)}*(2*H)
    res <- integrate(integr,lower=0,upper=1)$value
    return(res)
}


bigG <- function(alpha){Vectorize(bigG.raw(alpha))}
```

In [100]:
```
curve(bigG(alpha=.6)(x),from=.1,to=100,n=1000, col="red",
      xlab=expression(paste(Delta," (days)")),ylab=expression(paste("G(0.6)(
```



Empirical autocorrelation of variance

In [27]:
```
load("oxfordRVnew.rData")
```

In [28]:
```
names(rv.list)
```

'.AEX' '.AORD' '.BFX' '.BSESN' '.BVLG' '.BVSP' '.DJI' '.FCHI' '.FTMIB' '.FTSE' '.GDAXI' '.GSPTSE' '.HSI' '.IBEX' '.IXIC' '.KS11' '.KSE' '.MXX' '.N225' '.NSEI' '.OMXC20' '.OMXHPI' '.OMXSPI' '.OSEAX' '.RUT' '.SMSI' '.SPX' '.SSEC' '.SSMI' '.STI' '.STOXX50E'

In [29]:
```
library(xts)
tail(rv.list[[".SPX"]])
```

```
Loading required package: zoo

Attaching package: 'zoo'

The following objects are masked from 'package:base':

    as.Date, as.Date.numeric


                     rk   Close    Open
2019-09-06 1.321994e-05 2978.73 2980.33
2019-09-09 1.372631e-05 2978.58 2988.43
2019-09-10 2.370667e-05 2979.01 2971.01
2019-09-11 1.219658e-05 3000.55 2981.41
2019-09-12 5.074504e-05 3009.78 3009.08
2019-09-13 1.330606e-05 3007.07 3012.21
```

In [51]:
```
v <- as.numeric(rv.list[[".SPX"]][,1])  # Pick spx.rk
acv <-acf(v,lag=100,plot=F)
```

Empirical autocorrelation of variance

In [52]:
```
plot(acv$lag[-1],acv$acf[-1],pch=20,ylab=expression(rho(Delta)),xlab=express
```

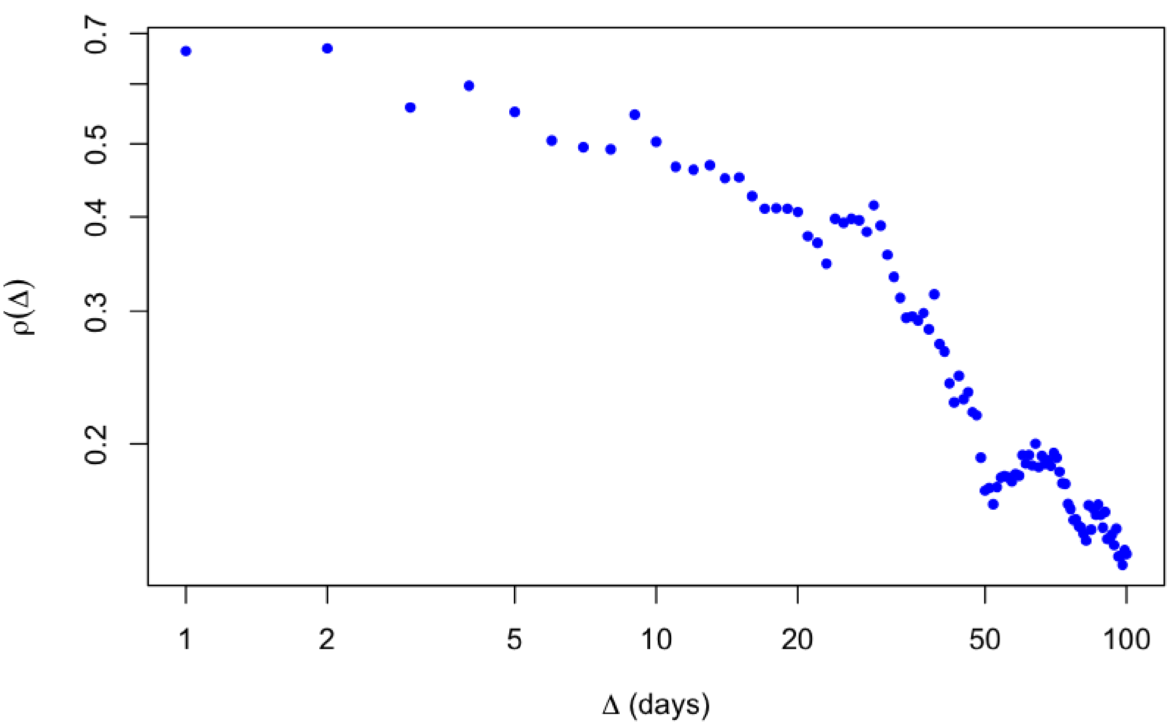


Figure 7: The empirical autocovariance function of SPX realized kernal estimates of integrated

variance.

Fit the Heston autocorrelation function

In [148]:
```
obj.Heston <- function(paramVec){

    rho0 <- paramVec[1]
    lambda <- paramVec[2]
    delta <- 1:100

    return(sum((acv$acf[-1]-rho0*exp(-lambda*delta/252))^2)*1000)
}
```

In [151]:
```
obj.Heston(c(.6, 5))
```

131.343742123962

In [152]:
```
(res <- optim(c(.6,1),obj.Heston, method="L-BFGS-B",lower=c(.4,1),upper=c(1,
```

$par

0.590910792654928 4.49473070865386

$value

105.351901415746

$counts

function	15
gradient	15

$convergence

0

$message

'CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH'

In [153]:
```
rho0 <- res$par[1]
lambda <- res$par[2]
```

Plot the fit

In [154]:
```
plot(acv$lag[-1],acv$acf[-1],pch=20,ylab=expression(rho(Delta)),ylim=c(0.1,1
     xlab=expression(paste(Delta," (days)")),log="xy",col="blue")
curve(rho0*exp(-lambda*x/252),from=0.0001,to=100,col="green4",lwd=2,add=T)
```

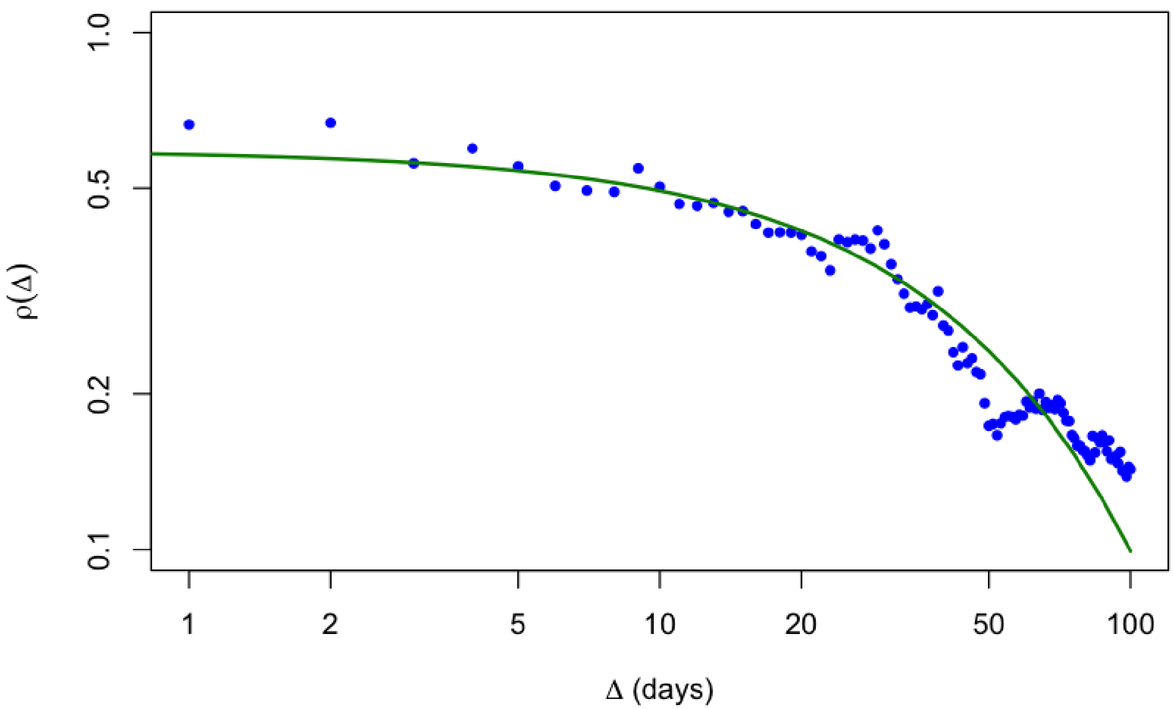


Figure 8: The empirical autocovariance function of SPX realized kernel estimates of integrated variance is reasonably consistent with classical Heston.

Fit the rough Heston autocorrelation function

In [136]:
```
obj.rHeston <- function(paramVec){

    alpha <- paramVec[1]
    scale <- paramVec[2]
    delta <- 1:100

    return(sum((acv$acf[-1]-bigG(alpha)(delta/scale))^2)*1000)
}
```

In [137]:
```
obj.rHeston(c(.6,20))
```

479.893870983545

In [155]:
```
(res.rough <- optim(c(.6,20),obj.rHeston, method="L-BFGS-B",lower=c(.4,1),up
```

$par

0.633315739271412 13.659159366761

$value

327.261726151837

$counts

function	19
gradient	19

$convergence

0

$message

'CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH'

In [145]:
```
alpha <- res.rough$par[1]
scale <- res.rough$par[2]
```

Plot the fit

In [146]:
```
plot(acv$lag[-1],acv$acf[-1],pch=20,ylab=expression(rho(Delta)),ylim=c(0.1,1
     xlab=expression(paste(Delta," (days)")),log="xy",col="blue")
curve(bigG(alpha)(x/scale),from=0.0001,to=100,col="green4",lwd=2,add=T)
```

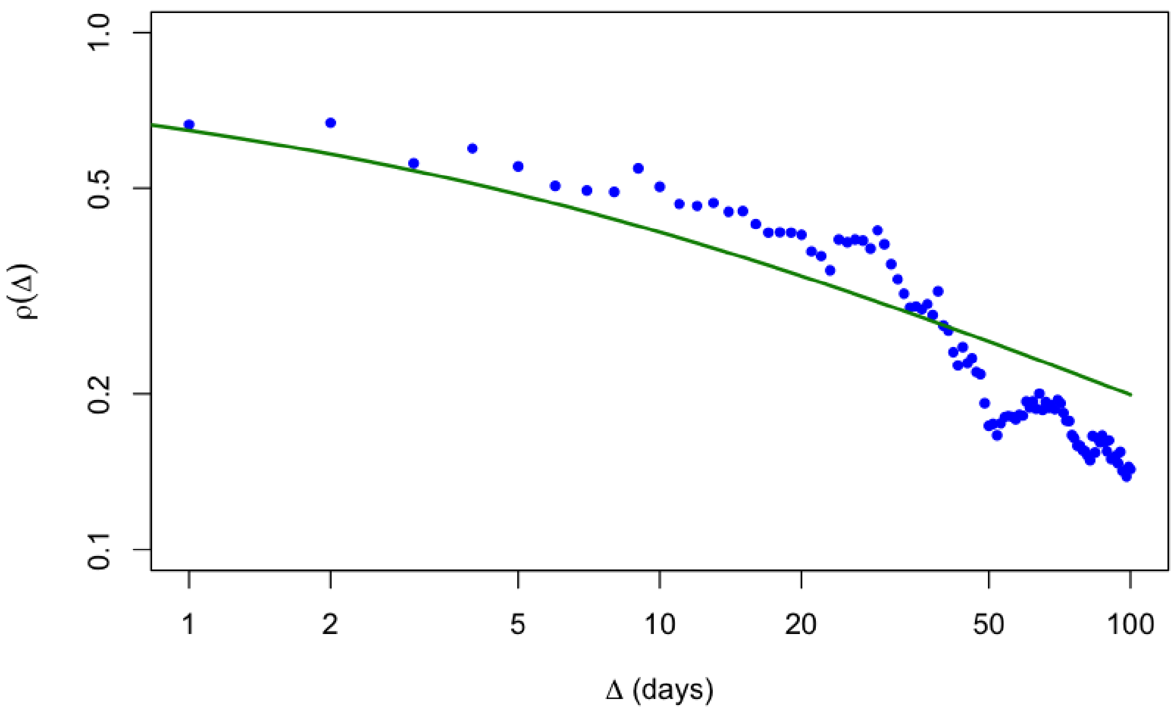


Figure 9: The empirical autocovariance function of SPX realized kernel estimates of integrated variance is not too inconsistent with rough Heston.

Empirical autocorrelation of log-variance

In [54]:
```
aclog <-acf(log(v),lag=100,plot=F)
```

In [55]:
```
plot(aclog$lag[-1],aclog$acf[-1],pch=20,ylab=expression(rho(Delta)),xlab=exp
```

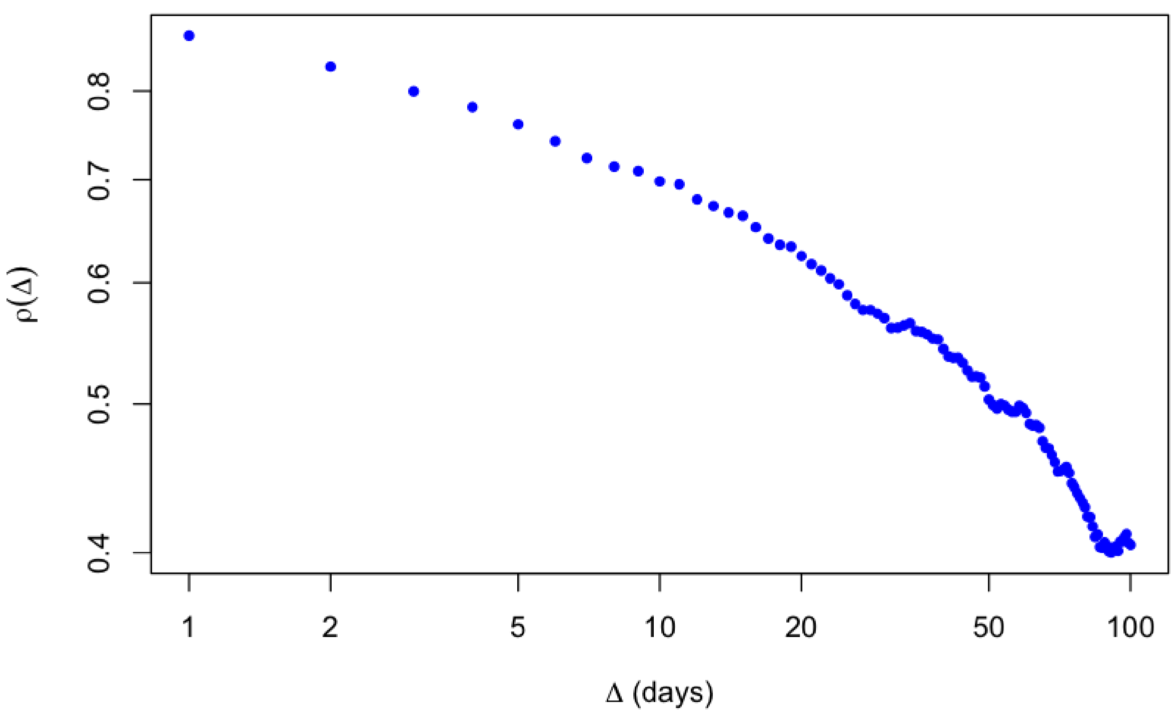


Figure 10: The empirical autocovariance function of SPX realized kernal estimates of log-integrated variance.

Remarks

- See how much nicer Figure 10 looks compared to Figure 7!
  - This is likely because variance is close to lognormally distributed, rather than normally distributed.

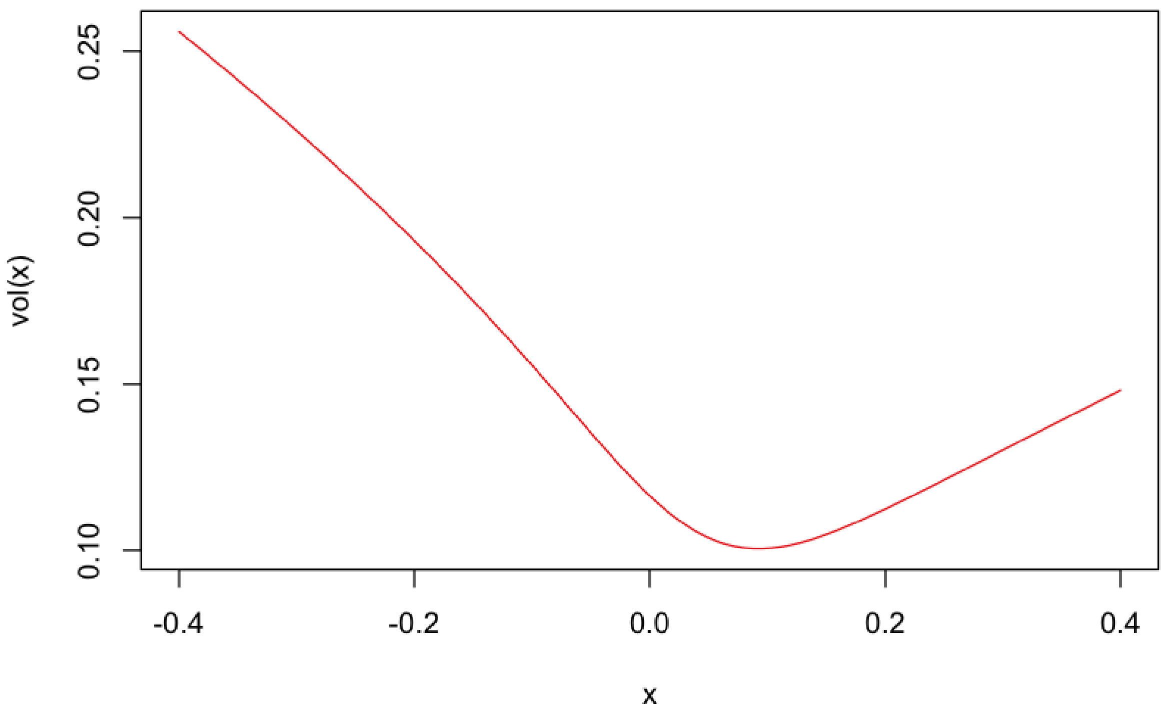
The rough Heston smile

In [33]:
```
params.rHeston <- list(H=0.05,nu=0.4,rho=-.65,eta=0.4)
xiCurve <- function(t){0.025+0*t}
```

In [34]:
```
phi <- phiRoughHestonDhApprox(params.rHeston, xiCurve, dh.approx= d.h.Pade3:
```

In [35]:
```
vol <- function(k){
    sapply(k,function(x){impvol.phi(phi)(x,1)})}
system.time(curve(vol(x),from=-.4,to=.4,col="red"))
```

```
   user  system elapsed 
  1.074   0.038   1.112 
```



Sensitivity of the rough Heston smile to η

In [36]:
```
subH <- function(incr){
    tmp <- params.rHeston
    tmp$eta <- tmp$eta+incr
    return(tmp)
}

vol <- function(params)function(k){
    phi <- phiRoughHestonDhApprox(params, xiCurve, dh.approx= d.h.Pade33, n=
    sapply(k,function(x){impvol.phi(phi)(x,1)})}
```

In [37]:
```
yrange <- c(0.05,.35)
curve(vol(subH(0))(x),from=-.5,to=.5,col="red",ylim=yrange,lwd=2,ylab="Impli
for (incr in seq(0.1,0.5,0.1))
    {
    curve(vol(subH(incr))(x),from=-.5,to=.5,col="red",lty=2,add=T)
    }
```

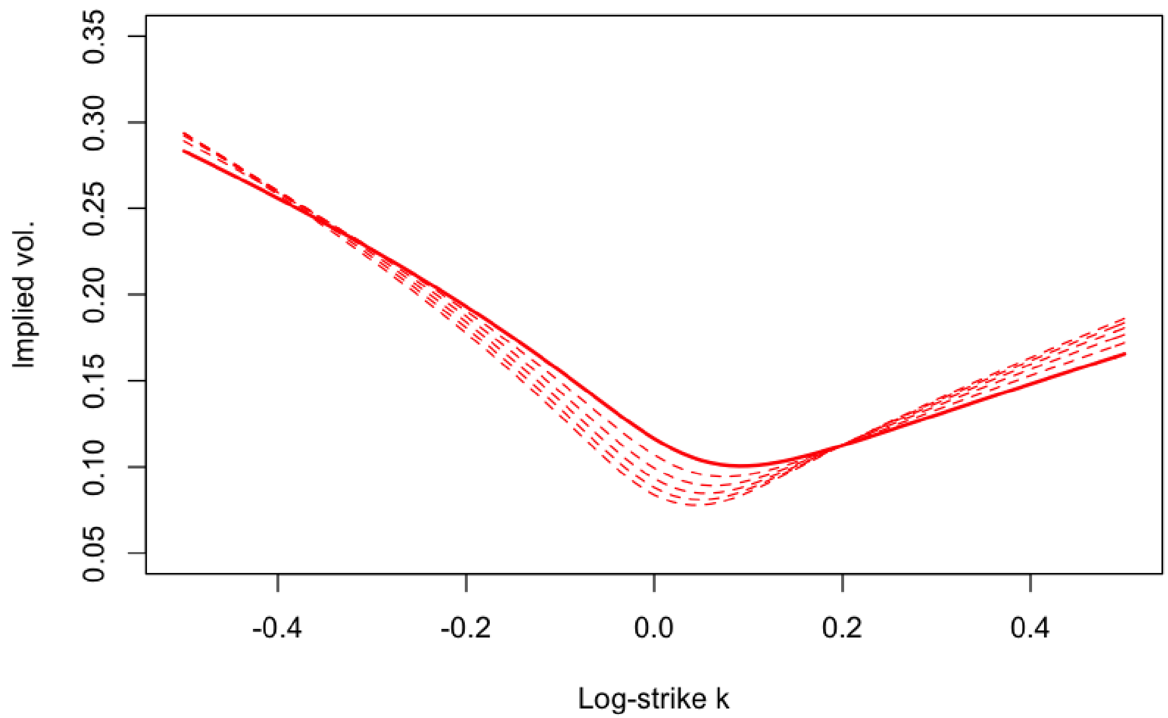


Figure 11: The dotted lines are smiles with $\eta = \eta_{BCC} + \{0.1, 0.2, 0.3, 0.4, 0.5\}$.

Sensitivity of the rough Heston 1 year smile to H

In [38]:
```
subH <- function(incr){
    tmp <- params.rHeston
    tmp$H <- tmp$H+incr
    return(tmp)
}

vol <- function(params)function(k){
    phi <- phiRoughHestonDhApprox(params, xiCurve, dh.approx= d.h.Pade33, n=
    sapply(k,function(x){impvol.phi(phi)(x,1)})}
```

In [39]:
```
yrange <- c(0.05,.35)
curve(vol(subH(0))(x),from=-.5,to=.5,col="red",ylim=yrange,lwd=2,ylab="Impli
for (incr in seq(0.1,0.4,0.1))
    {
    curve(vol(subH(incr))(x),from=-.5,to=.5,col="red",lty=2,add=T)
    }
```

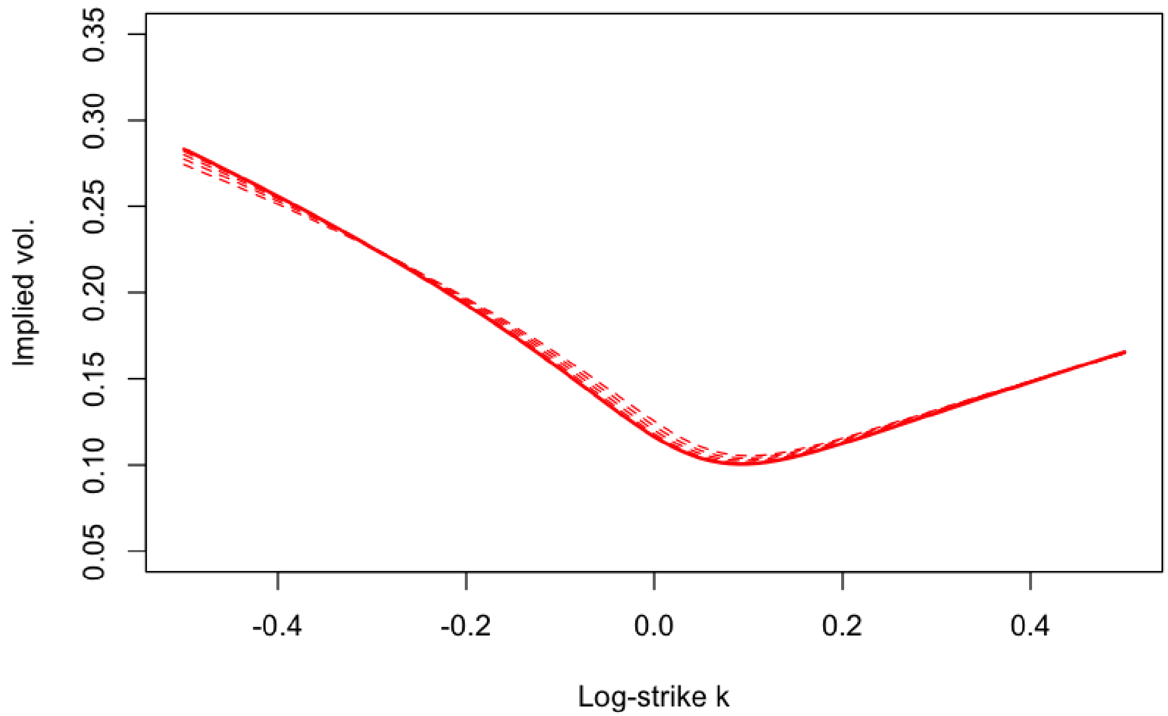


Figure 12: The dotted lines are 1 year smiles with $H \mapsto H + \{0.1, 0.2, 0.3, 0.4\}$.

Sensitivity of the rough Heston 1 week smile to H

In [40]:
```
subH <- function(incr){
    tmp <- params.rHeston
    tmp$H <- tmp$H+incr
    return(tmp)
}

vol <- function(params)function(k){
    phi <- phiRoughHestonDhApprox(params, xiCurve, dh.approx= d.h.Pade33, n=
    sapply(k,function(x){impvol.phi(phi)(x,1/52)})}
```

In [41]:
```
yrange <- c(0.05,.4)
curve(vol(subH(0))(x),from=-.15,to=.15,col="red",ylim=yrange,lwd=2,ylab="Imp
for (incr in seq(0.1,0.4,0.1))
    {
    curve(vol(subH(incr))(x),from=-.15,to=.15,col="red",lty=2,add=T)
    }
```

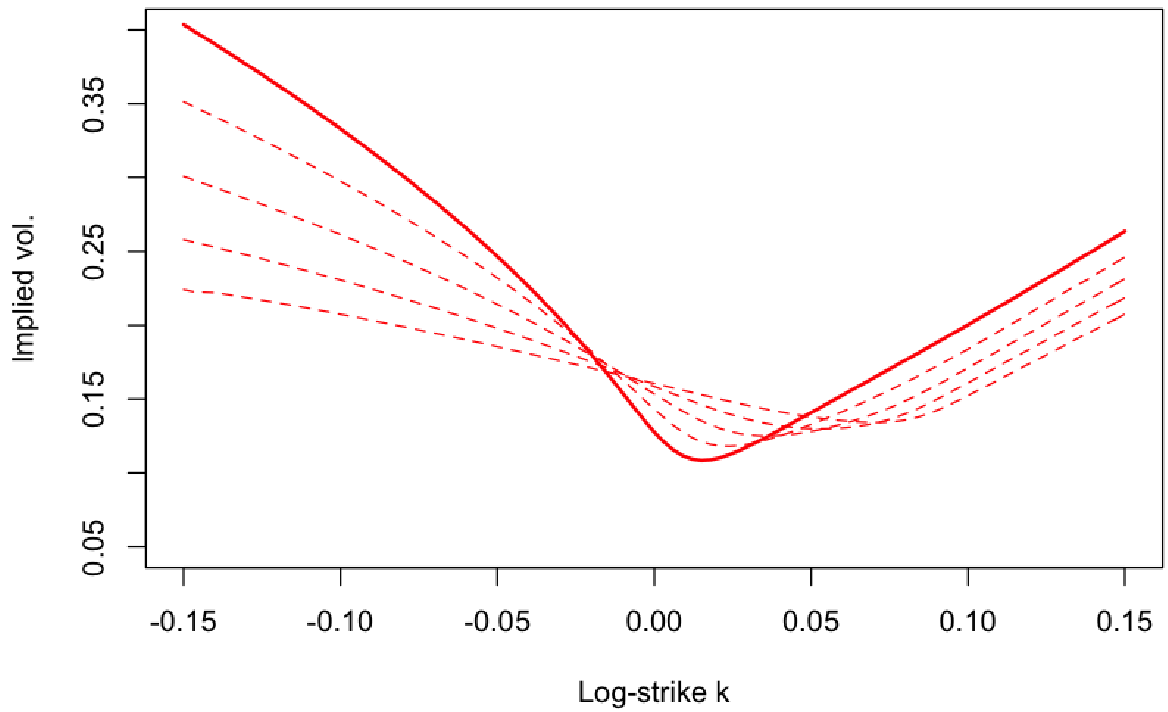


Figure 13: The dotted lines are 1 week smiles with $H \mapsto H + \{0.1, 0.2, 0.3, 0.4\}$. The smile flattens as we increase H.

Summary of Lecture 3

- We introduce forward variance and affine models.
- We showed how to compute the characteristic function for any affine forward variance model.
- We applied these general results to the classical Heston and rough Heston models.

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