MTH 9875 The Volatility Surface: Fall 2019

Lecture 9: Stochastic volatility with jumps, time-changed Lévy models

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Outline of lecture 9

- Stochastic volatility plus stock price jumps (SVJ)
- Stochastic volatility with stock and volatility jumps (SVJJ)
- Comparison of model fits
- The Schoutens et al. paper
- Stochastic time change
- Valuation of the guaranteed stop-loss contract

Stochastic volatility plus stock price jumps (SVJ)

Since jumps generate a steep short-dated skew that dies quickly with time to expiration and stochastic volatility models don't generate enough skew for very short expirations but more or less fit for longer expirations, it is natural to try to combine stock price jumps and stochastic volatility in one model.

- In passing, we give explicit characteristic functions for the popular SVJ and SVJJ models:
  - The SVJ model succeeds in generating a volatility surface that has most of the observed features of the empirical surface with fewer parameters than the SVJJ model.

SVJ process

Suppose we add a simple Merton-style lognormally distributed jump process to the Heston process so that

$$\begin{aligned} dS_t &= \mu\, S_t\, dt + \sqrt{v_t}\, S_t\, dZ_t + \left(e^{\alpha+\delta\,\epsilon} - 1\right)\, S_t\, dq_t \\ dv_t &= -\lambda(v_t - \bar{v})\, dt + \eta\, \sqrt{v_t}\, dW_t \end{aligned}$$

with $\langle dW_t\, dZ_t \rangle = \rho\, dt$, $\epsilon \sim N(0,1)$ and as in the jump-diffusion case, the Poisson process

$$dq_t = \begin{cases} 0 & \text{with probability } 1 - \lambda_J\, dt \\ 1 & \text{with probability } \lambda_J\, dt \end{cases}$$

where λ_J is the jump intensity (or hazard rate).

Reminder: Computing the ATM volatility level and skew

As usual with $X_t = 0$ and $\tau - T - t$, we recall the following formulae from Lecture 8:

(1)

$$\int_0^\infty \frac{du}{u^2+\frac{1}{4}} \operatorname{Re}\left[e^{-iuk}\left(\phi_t^T(u-i/2) - e^{-\frac{1}{2}\left(u^2+\frac{1}{4}\right)\sigma_{BS}(k,\tau)^2\tau}\right)\right] = 0$$

and

(2)

$$\left.\frac{\partial\sigma_{BS}}{\partial k}\right|_{k=0} = -e^{\frac{\sigma_{BS}^2\tau}{8}}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{\tau}}\int_0^\infty du\,\frac{u\operatorname{Im}\left[\phi_t^T(u-i/2)\right]}{u^2+\frac{1}{4}}$$

- Given a characteristic function in closed-form, we may easily compute the ATM volatility and the ATM volatility skew.

SVJ characteristic function

From the definition of the SVJ process, it is easy to see that the SVJ characteristic function is just the product of Heston and jump characteristic functions. Specifically,

(3)

$$\phi_t^T(u) = e^{C(u,\tau)\,\bar{v}+D(u,\tau)\,v}\;e^{\psi(u)\tau}$$

with $\psi(u) = -\lambda_J iu\left(e^{\alpha+\delta^2/2}-1\right)+\lambda_J\left(e^{iu\alpha-u^2\delta^2/2}-1\right)$ and $C(u,\tau)$, $D(u,\tau)$ are as before.

We may substitute this functional form into equations (1) and (2) to get the implied volatilities and at-the-money volatility skew respectively for any given expiration.

Term structure of SVJ skew

- [Figure 1](#fig:combskew) plots the at-the-money variance skew corresponding to the Bakshi-Cao-Chen SVJ model fit together with the sum of the Heston and jump-diffusion at-the-money variance skews with the same parameters (see [Table 1](#table:jumpfits)).
- We see that (at least with this choice of parameters), not only does the characteristic function factorize but the at-the-money variance skew is almost additive.

- One practical consequence of this is that the Heston parameters can be fitted fairly robustly using longer dated options and then jump parameters can be found to generate the required extra skew for short-dated options.
- [Figure 2](#fig:shortdatedskew) plots the at-the-money variance skew corresponding to the SVJ model vs the Heston model skew for short-dated options, highlighting the small difference.

Download some R code

In [1]:
```
download.file(url="http://mfe.baruch.cuny.edu/wp-content/uploads/2019/10/9875-9.zip", des
tfile="9875-9.zip")
unzip(zipfile="9875-9.zip")

library(stinepack)
library(repr)

source("BlackScholes.R")
source("Lewis.R")
source("svi.R")
source("sviVolSurface.R")
source("Heston.R")
```

SVJ ATM skew

Here are BCC parameters for SVJ:

In [2]:
```
subSVJJ <- list(lambda = 2.03,eta = 0.38, rho = -0.57, vbar = 0.04,v = .04,
                lambdaJ = 0.59, alpha = -0.05, delta = 0.07, gammaV = 0.1)
```

Now we code the SVJ characteristic function from equation (3) above:

In [3]:
```
phiSVJ <- function(params){

    lambdaJ <- params$lambdaJ
    alpha <- params$alpha
    delta <- params$delta

    function(u, t){
        psiu <- lambdaJ*(-1i*u*(exp(alpha+delta^2/2)-1) + (exp(1i*u*alpha-u^2*delta^2/2)-
1))
        return(phiHeston(params)(u,t)*exp(psiu*t))
        }
}
```

And code formula (2) to get the ATM volatility skew from the characteristic function (Equation (5.8) of The Volatility Surface, equation (8) of Lecture 8):

In [4]:
```
atmSkew.raw <- function(phi, t){
    atmVol <- impvol.phi(phi)(0, t)
    integrand <-  function(u){Im(u*phi(u - 1i/2, t)/(u^2 + 1/4))}
    res <- -integrate(integrand,lower=0,upper=Inf)$value/sqrt(t)*sqrt(2/pi)*exp(atmVol^2*
t/8)
    return(res)
}

# Vectorize the function
atmSkew <- function(phi,tVec){sapply(tVec,function(t){atmSkew.raw(phi,t)})}

# Now variance skew
atmVarSkew.raw <- function(phi, t){
    volSkew <- atmSkew.raw(phi,t)
    atmVol <- impvol.phi(phi)(0, t)
    return(2*volSkew*atmVol)
}

# Vectorize the function
atmVarSkew <- function(phi,tVec){sapply(tVec,function(t){atmVarSkew.raw(phi,t)})}
```

To construct the SVJ characteristic function, we need to code the Merton jump-diffusion characteristic function (see Lecture 8 again):

In [5]:
```
phiMJD <- function(params){

    sigma <- params$sigma
    lambdaJ <- params$lambdaJ
    alpha <- params$alpha
    delta <- params$delta

    function(u, t){

    psiBS <- -u/2*(u+1i)*sigma^2
    psiJump <- lambdaJ*(-1i*u*(exp(alpha+delta^2/2)-1) + (exp(1i*u*alpha-u^2*delta^2/2)-1
))

    return(exp((psiBS+psiJump)*t))
}
}
```

We now verify that the SVJ skew almost perfectly decomposes as sum of Heston skew and jump skew:

In [6]:
```
parHestJD <- list(lambda = 2.03,eta = 0.38, rho = -0.57, vbar = 0.04,v = .04,
                lambdaJ = 0.59, alpha = -0.05, delta = 0.07, gammaV = 0.1,
                  sigma=0.2)

skewSum <- function(x){atmVarSkew(phiHeston(parHestJD),x)+atmVarSkew(phiMJD(parHestJD),x
)}
```

In [7]:
```
options(repr.plot.height=5)
```

In [8]:
```
curve(atmVarSkew(phiSVJ(parHestJD),x),from=.01,to=10,col="blue",ylab="Variance skew", xla
b="Time to expiration")
curve(skewSum(x),from=.01,to=10,col="red",add=T,lty=2)
```

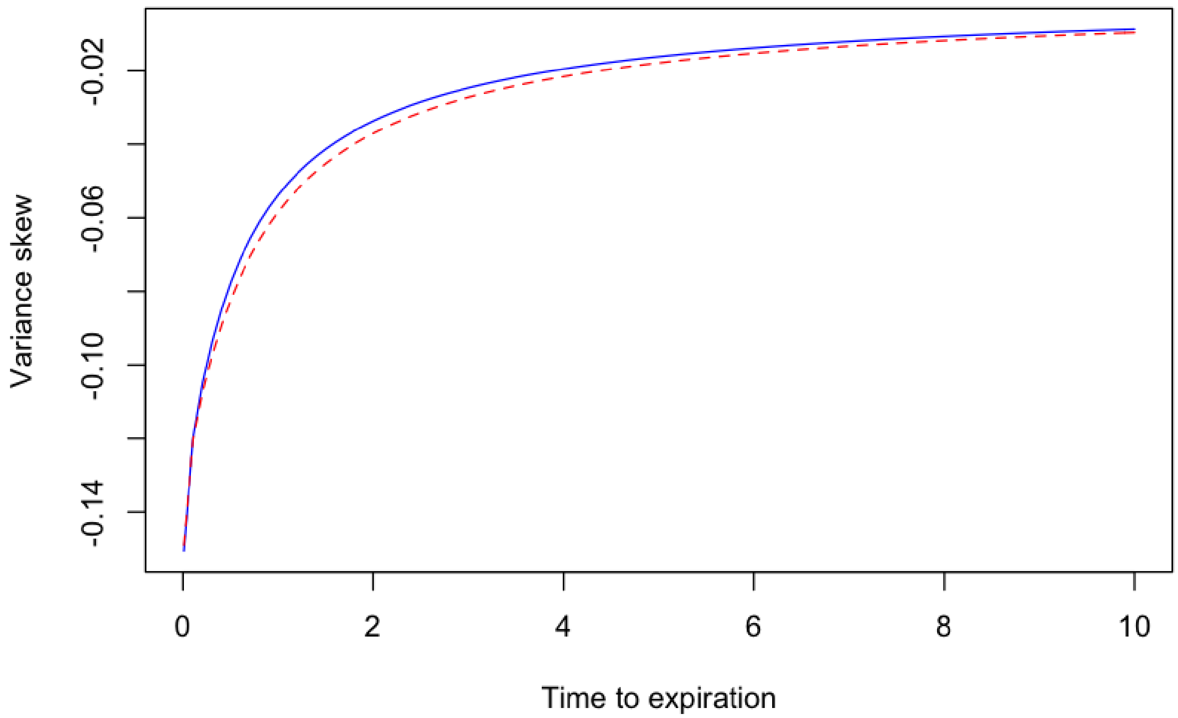


Figure 1: The solid line is a graph of the at-the-money variance skew in the SVJ model with BCC parameters vs time to expiration. The dashed line represents the sum of at-the-money Heston and jump-diffusion skews with the same parameters.

SVJ skew vs Heston skew

In [9]:
```
curve(atmVarSkew(phiSVJ(subSVJJ),x),from=.01,to=1,col="blue",ylab="Variance skew", xlab=
"Time to expiration")
curve(atmVarSkew(phiHeston(subSVJJ),x),from=.01,to=1,col="red",add=T,lty=2)
```

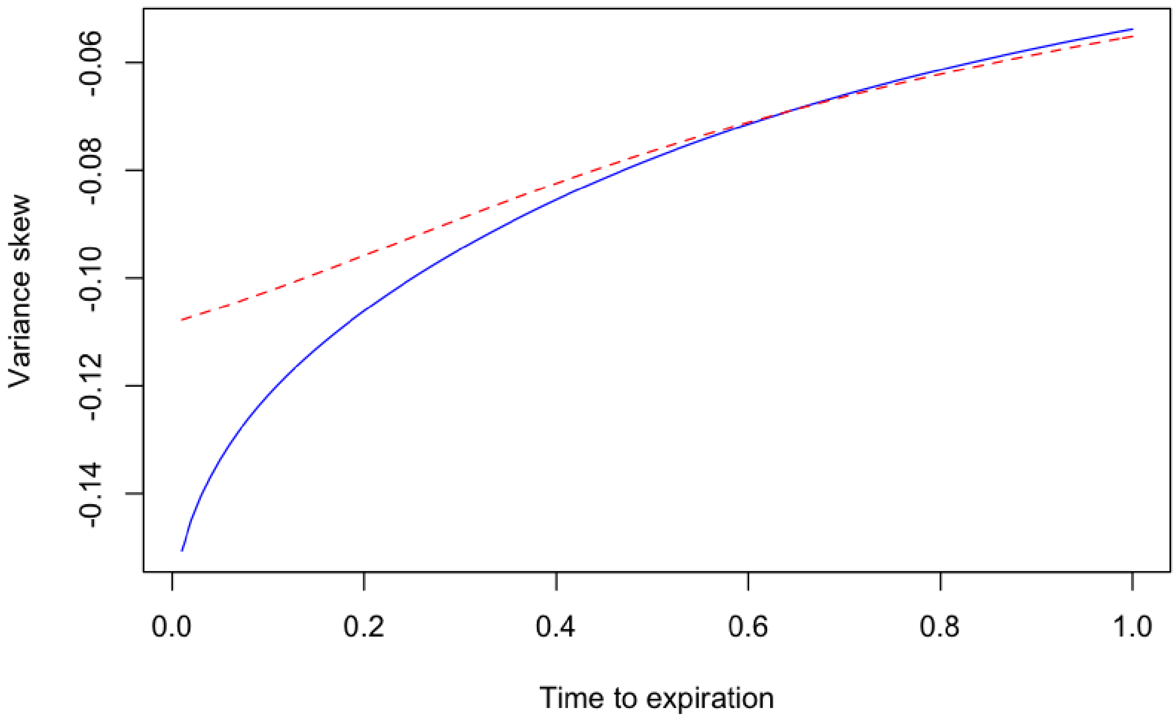


Figure 2: The solid line is a graph of the at-the-money variance skew in the SVJ model with BCC parameters vs time to expiration. The dashed line represents the at-the-money Heston skew with the same parameters.

SVJ model assumptions are unrealistic

- In the SVJ model, after the stock price has jumped, the volatility will stay unchanged because the jump process is uncorrelated with the volatility process.
  - This is inconsistent with both intuition and empirically observed properties of the time series of asset returns.
  - In practice, after a large move in the underlying, implied volatilities always increase substantially (*i.e.* they jump).

Remarks on fitting SVJ

There are only 4 parameters in the jump-diffusion model: the volatility σ, λ_J, α and δ so it's not in principle difficult to perform a fit to option price data. The SVJ model obviously fits the data better because it has more parameters and it's not technically that much harder to perform the fit.

Some empirical fits to the SPX volatility surface

Author(s)	Model	λ	η	ρ	$\bar{v}$	$\lambda_{\mathbf{J}}$	α	δ
AA	JD	NA	NA	NA	0.032	0.089	-0.8898	0.4505
BCC	SVJ	2.03	0.38	-0.57	0.04	0.59	-0.05	0.07
M	SVJ	1.0	0.8	-0.7	0.04	0.5	-0.15	0
DPS	SVJ	3.99	0.27	-0.79	0.014	0.11	-0.12	0.15
JG	SVJ	0.54	0.3	-0.7	0.044	0.13	-0.12	0.10

Author(s)	Reference	Data from
AA	Andersen & Andreasen (2000)	April 1999
BCC	Bakshi, Cao & Chen (1997)	June 1988 – May 1991
M	Matytsin (1999)	1999
DPS	Duffie, Pan & Singleton (2000)	November 1993
JG	Jim G.	September 2005

Table 1: Various fits of jump-diffusion style models to SPX data. JD means Jump Diffusion and SVJ means Stochastic Volatility plus Jumps.

Remarks on the fits

- These estimates all relate to different dates.
  - We can't expect the volatility surfaces they generate to be the same shape.
- Nevertheless, the shape of the SPX volatility surface doesn't really change much over time so it does make some sense to compare them.

A remark on the AA estimate

The one estimate that sticks out is obviously the AA JD fit with a huge expected jump size of -0.8898. At first sight it might seem disconcerting that imputing jump parameters from implied volatility surfaces could give rise to such wildly different parameter estimates.

- The AA jump size estimate turns out to have been driven by requiring the fit to match the 10-year volatility skew
  - As pointed out earlier in Lecture 6, for the characteristic time T^* to be of the order of ten years, we need a huge and infrequent jumps.
- The overall AA fit of JD to the implied volatility surface is very poor.
  - JD is completely misspecified and we can confidently reject JD with AA parameters.

Stochastic volatility with simultaneous jumps in stock price and volatility (SVJJ)

- As we noted earlier in our discussion of the SVJ model, it is unrealistic to suppose that the instantaneous volatility wouldn't jump if the stock price were to jump.

- Conversely, adding a simultaneous upward jump in volatility to jumps in the stock price allows us to maintain the clustering property of stochastic volatility models: recall that "large moves follow large moves and small moves follow small moves".

The SVJJ model

- In a couple of presentations, Andrew Matytsin describes a model that is effectively SVJ with a jump in volatility:
  - Jumps in the stock price are accompanied by a jump $v \mapsto v + \gamma_v$ in the instantaneous volatility.

The SVJJ characteristic function

The SVJJ characteristic function is given by

(4)

$$\phi_t^T(u) = \exp\left\{\hat{C}(u,\tau)\,\bar{v} + \hat{D}(u,\tau)\,v\right\}$$

with $C(u,\tau)$ and $D(u,\tau)$ given by

$$\begin{aligned}\hat{C}(u,\tau) &= C(u,\tau)\\ &\quad + \lambda_J\,\tau\,\left[e^{iu\alpha - u^2\delta^2/2} I(u,\tau) - 1 - i\,u\,\left(e^{\alpha+\delta^2/2} - 1\right)\right]\\ \hat{D}(u,\tau) &= D(u,\tau)\end{aligned}$$

where

$$\begin{aligned}I(u,\tau) &= \frac{1}{\tau}\int_0^\tau e^{\gamma_v D(u,t)}\,dt\\ &= -\frac{1}{\tau}\,\frac{2\gamma_v}{p_+ p_-}\int_0^{-\gamma_v D(u,\tau)} \frac{e^{-z}dz}{(1+z/p_+)(1+z/p_-)}\end{aligned}$$

and

$$p_\pm = \frac{\gamma_v}{\eta^2}(\beta - \rho\eta u i \pm d)$$

Asymptotics of the SVJJ model

- In the limit $\gamma_v \to 0$, we have $I(u,\tau) \to 1$ and by inspection, we retrieve the SVJ model.

- Also, in the limit $\tau \to 0$, $I(u,\tau) \to 1$ and in that limit, the SVJJ characteristic function is identical to the SVJ characteristic function.

- Alternatively, following our earlier heuristic argument, the short-dated volatility skew is a function of the jump compensator only and this compensator is identical in the SVJ and SVJJ cases.
  - Intuitively, when the stock price jumps, the volatility jumps but this has no effect in the $\tau \to 0$ limit because by assumption, an at-the-money option is always out-of-the-money after the jump and its time value is zero no matter what the volatility is.

- On the other hand, in the $\tau \to \infty$ limit, the skew should increase because the effective volatility of volatility increases due to (random) jumps in volatility.

SVJJ ATM skew

Let's now code the SVJJ characteristic function.

In [10]:
```
phiSVJJ <- function(params){

    lambda <- params$lambda
    rho <- params$rho
    eta <- params$eta
    vbar <- params$vbar
    v <- params$v
    lambdaJ <- params$lambdaJ
    alpha <- params$alpha
    delta <- params$delta
    gammaV <- params$gammaV

    function(u, t){

    al <- -u*u/2 - 1i*u/2
    bet <- lambda - rho*eta*1i*u
    gam <- eta^2/2
    d <- sqrt(bet*bet - 4*al*gam)
    rp <- (bet + d)/(2*gam)
    rm <- (bet - d)/(2*gam)
    g <- rm / rp
    D <- rm * (1 - exp(-d*t))/ (1 - g*exp(-d*t))
    Dfunc <- function(t){rm * (1 - exp(-d*t))/ (1 - g*exp(-d*t))}
    C <- lambda * (rm * t - 2/eta^2 * log( (1 - g*exp(-(d*t)))/(1 - g) ) )

    pplus <- gammaV/eta^2*(bet- rho*eta*u*1i + d)
    pminus <- gammaV/eta^2*(bet- rho*eta*u*1i - d)

    integrandR <- function(t){Re(exp(gammaV*Dfunc(t)))}
    integrandI <- function(t){Im(exp(gammaV*Dfunc(t)))}

    bigI <- 1/t*(integrate(integrandR,lower=0,upper=t,rel.tol=0.0000000001,subdivisions=1
000)$value+
                 1i*integrate(integrandI,lower=0,upper=t,rel.tol=0.0000000001,subdivision
s=1000)$value)
    psi <- lambdaJ*t*( (exp(1i*u*alpha-u^2*delta^2/2)*bigI -1)-1i*u*(exp(alpha+delta^2/2)
-1))

    return(exp(C*vbar + D*v + psi))
    }
}
```

Plot the term structure of at-the-money volatility skew with BCC parameters for the various models

Let's compare the ATM vol skew term structure for Heston, SVJ and SVJJ models. To add the SVJJ curve, we need to explicitly vectorize the function:

In [11]:
```
f <- function(u,t){sapply(u,function(u){phiSVJJ(subSVJJ)(u,t)})}
sk <- function(t){sapply(t,function(x){atmSkew(f,x)})}
```

Note that SVJJ comptutations are very slow!!

In [12]:
```
options(repr.plot.width=10,repr.plot.height=7)
```

In [13]:
```
curve(sk(x),from=0.003,to=10,col="dark green",type="l",lwd=2,lty=4,ylab="ATM vol skew",xl
ab="Expiration (years)")
curve(atmSkew(phiHeston(subSVJJ),x),from=.001,to=10,col="red",lty=3,lwd=2,add=T)
curve(atmSkew(phiSVJ(subSVJJ),x),from=.001,to=10,col="blue",lty=2,lwd=2,add=T)

# Compare with sqrt(t) scaling
skew2 <- atmSkew(phiHeston(subSVJJ),2)
curve(skew2/sqrt(x/2),from=.001,to=10,col="orange",ylim=c(-.4,0),add=T)

leg.txt <- c("SVJJ","Heston", "SVJ", "Sqrt scaling")
legend(x=2.8,y=-.2,leg.txt,lty=c(4,3,2,1),col=c("dark green","red","blue","orange"),lwd=c
(2,2,2,1))
```

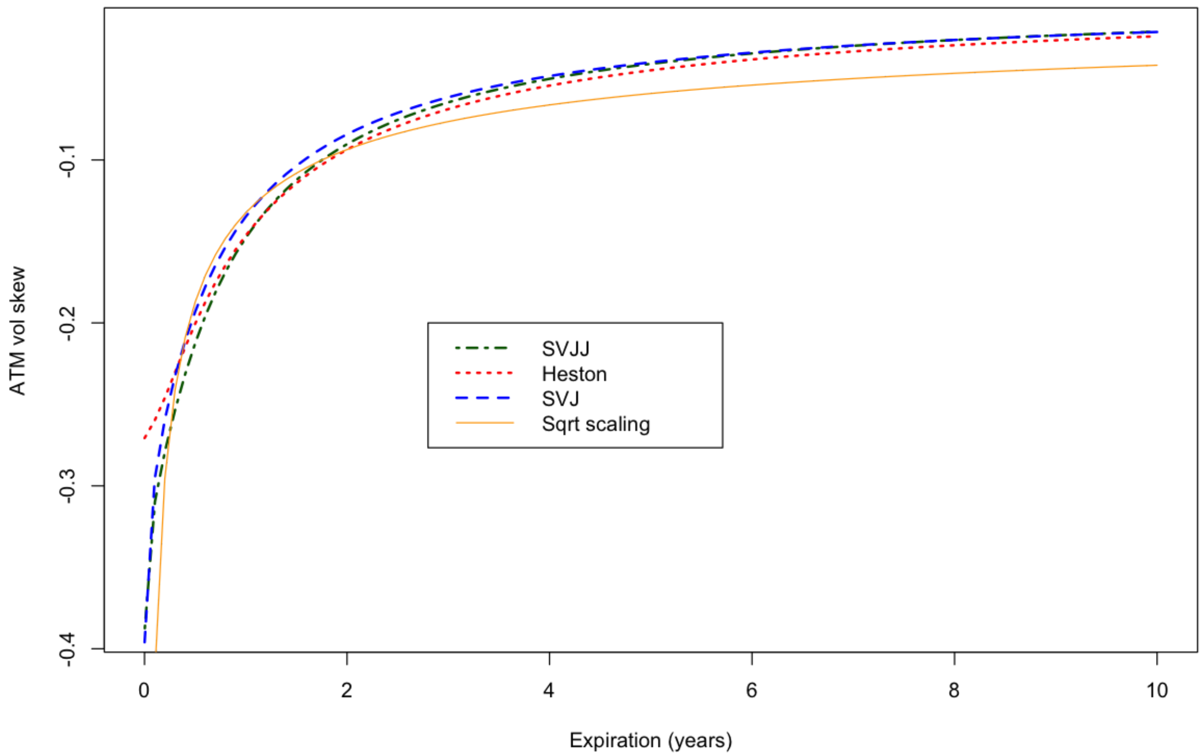


Figure 3: Term structure of at-the-money volatility skew with BCC parameters for various models

SVJJ ATM skew for short dates

Now a zoomed version of Figure 3:

In [14]:
```
curve(sk(x),from=0.0025,to=.25,col="dark green",type="l",lwd=2,lty=4,ylab="ATM vol skew",
xlab="Expiration (years)",ylim=c(-.4,-.2))
curve(atmSkew(phiHeston(subSVJJ),x),from=.001,to=.25,col="red",lty=3,lwd=2,add=T)
curve(atmSkew(phiSVJ(subSVJJ),x),from=.001,to=.25,col="blue",lty=2,lwd=2,add=T)

# Compare with sqrt(t) scaling
skew25 <- atmSkew(phiHeston(subSVJJ),.25)
curve(skew25/(x/.25)^.5,from=.001,to=.25,col="orange",add=T)

leg.txt <- c("SVJJ","Heston", "SVJ", "Sqrt scaling")
legend(x=.15,y=-.32,leg.txt,lty=c(4,3,2,1),col=c("dark green","red","blue","orange"),lwd=
c(2,2,2,1))
```

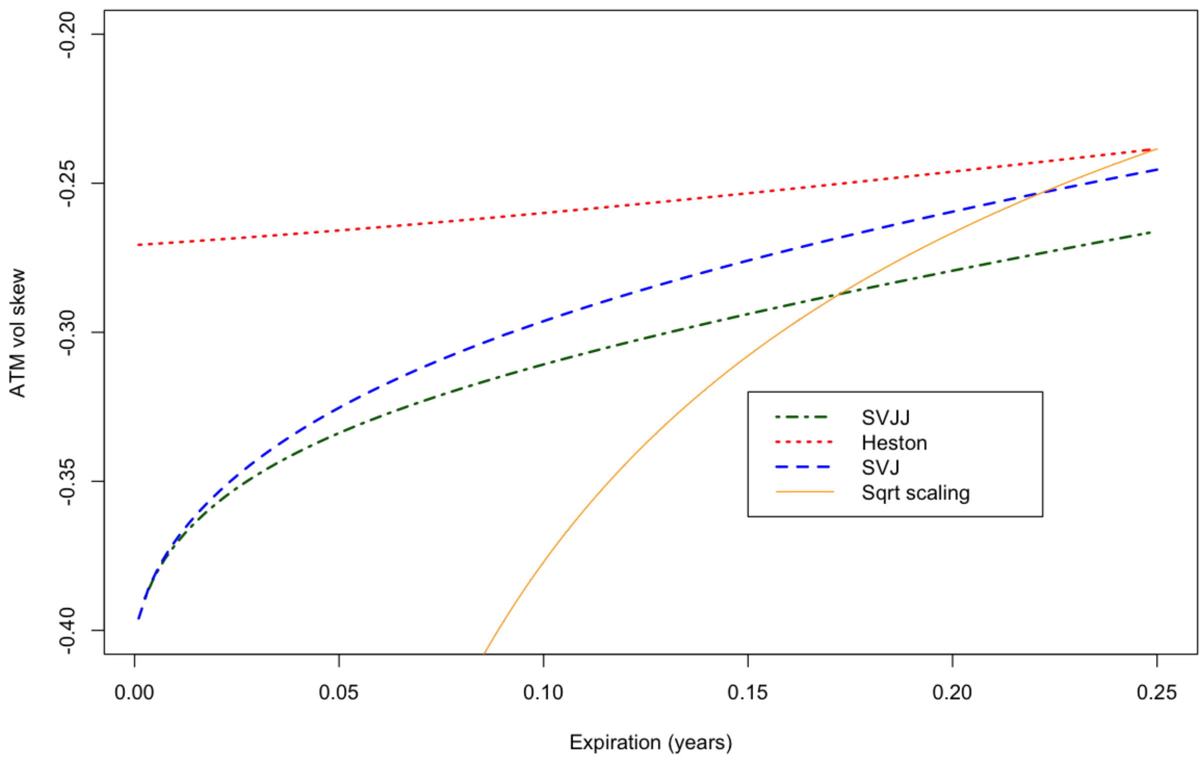


Figure 4: This graph is a short-expiration detailed view of the graph shown in Figure 3.

Remarks on the SVJJ ATM skew

- By substituting the SVJJ characteristic function [(4)](#eq:SVJJcharfn) into equation [(2)](#eq:skew) for the implied volatility skew with the BCC parameters plus a variance jump of $\gamma_v = 0.1$, we obtain the graphs shown in Figures [3](#fig:combskewSVJJ) and [4](#fig:shortdatedskewSVJJ).
  - We note that the term structure of volatility skew is in accordance with our intuition.
  - In particular, adding a jump in volatility doesn't help explain extreme short-dated volatility skews.

- However relative to stochastic volatility and SVJ models, adding a jump in volatility does reduce the volatility of volatility required to fit longer-dated volatility skews even if that comes at the expense of a seemingly even more unreasonable estimate for the average stock price jump.

Heston and SVJ fits to 15-Sep-2005 SPX option data

	Heston	SVJ
v	0.0174	0.0158
$\bar{v}$	0.0354	0.0439
η	0.3877	0.3038
ρ	-0.7165	-0.6974
λ	1.3253	0.5394
λ_J		0.1308
δ		0.0967
α		-0.1151

Table 2: Heston and SVJ fit to the SPX surface as of the close on 15-Sep-2005.

Comparing model and empirical volatility surfaces

- Again we assess SVJ fit quality by comparing the SVJ and empirical volatility surfaces graphically.
- We see from [Figure 5](#fig:spxsvjcompare) that in contrast to the Heston case, the SVJ model succeeds in generating a volatility surface that has the main features of the empirical surface although the fit is not perfect.
  - Again, for longer expirations, the fit is pretty good.

3D plots of SPX and fitted SVJ surfaces

First, here are some SVJ parameters that generate a more or less decent fit to the volatility surface as of September 15, 2005:

In [15]:
```
paramsSVJ050915 <- list(
    v= 0.0158,
    vbar=0.0439,
    eta=0.3038,
    rho=-0.6974,
    lambda=0.5394,
    lambdaJ = 0.1308,
    delta = 0.0967,
    alpha = -0.1151
    )
```

Once again, here are the SVI parameters corresponding to the SVI fit shown in Figures 3.2 and 3.3 of The Volatility Surface:

In [16]:
```
texp <- c(0.003832991, 0.098562628, 0.175336527, 0.251996350, 0.501140771, 0.750171116,
1.248574036, 1.746748802)

svidata <- c(
 c(-0.0001449630, 0.0092965440, 0.0196713280, -0.2941176470,-0.0054273230),
 c(-0.000832134 , 0.024439766 , 0.069869455 , -0.299975308 , 0.02648364  ),
 c(-0.0008676750, 0.0282906450, 0.0873835580, -0.2892204290, 0.0592703000),
 c(-0.0000591593, 0.0331790820, 0.0812872370, -0.3014043240, 0.0652549210),
 c(0.0011431940 , 0.0462796440, 0.1040682980, -0.3530782140, 0.0942000770),
 c(0.0022640980 , 0.0562604150, 0.1305339330, -0.4387409470, 0.1111230690),
 c(0.0040335530 , 0.0733707550, 0.1707947600, -0.4968970370, 0.1496609160),
 c(0.0034526910 , 0.0917230540, 0.2236814130, -0.4942213210, 0.1854128490));

sviMatrix <- as.data.frame(t(array(svidata,dim=c(5,8))));
colnames(sviMatrix)<-c("a","b","sig","rho","m")
```

In [17]:
```
# Inspect the matrix
sviMatrix
```

a	b	sig	rho	m
-0.0001449630	0.009296544	0.01967133	-0.2941176	-0.005427323
-0.0008321340	0.024439766	0.06986945	-0.2999753	0.026483640
-0.0008676750	0.028290645	0.08738356	-0.2892204	0.059270300
-0.0000591593	0.033179082	0.08128724	-0.3014043	0.065254921
0.0011431940	0.046279644	0.10406830	-0.3530782	0.094200077
0.0022640980	0.056260415	0.13053393	-0.4387409	0.111123069
0.0040335530	0.073370755	0.17079476	-0.4968970	0.149660916
0.0034526910	0.091723054	0.22368141	-0.4942213	0.185412849

Now generate a 3D plot of the SVJ surface:

In [18]:
```
k <- seq(-.5,.5,0.01) # Vector of log-strikes
t <- seq(0.04,1.74,0.02) # Vector of times

ivol.SVJ <- function(k,t){
    tryCatch(
    impvol.phi(phiSVJ(subSVJJ))(k,t),error=function(e){return(NA)})
    }
iv <- Vectorize(ivol.SVJ)

z.SVJ <- outer(k,t, FUN = iv)
```

In [19]:
```
options(repr.plot.height=10,repr.plot.width=10)
```

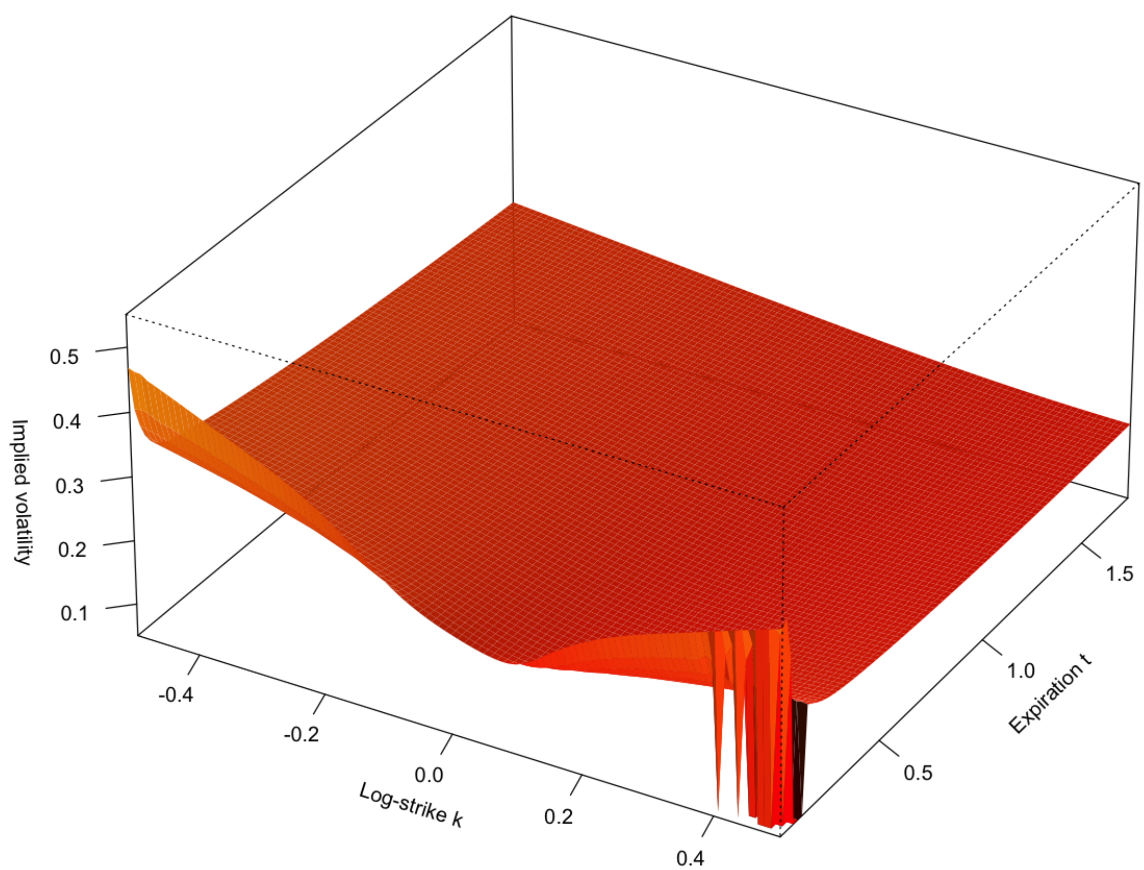
In [20]:
```
plotSurface <- function(k,t,z,color,box=TRUE,phi=30,theta=30){

    z <- (z>.05)*z+(z <= 0.05)*.05 # Clean up vols that are too small

    # Add colors
    nbcol <- 100
    nrz <- nrow(z)
    ncz <- ncol(z)
    # Compute the z-value at the facet centres
    zfacet <- z[-1, -1] + z[-1, -ncz] + z[-nrz, -1] + z[-nrz, -ncz]
    # Recode facet z-values into color indices
    facetcol <- cut(zfacet, nbcol)

    persp(k, t, z, col=color[facetcol], phi=phi, theta=theta,
    r=1/sqrt(3)*20,d=5,expand=.5,ltheta=-135,lphi=20,ticktype="detailed",
    shade=.5,border=NA,
          xlab="Log-strike k",ylab="Expiration t",zlab="Implied volatility",
          zlim=c(.05,.55),box=box)
    }
```

In [21]:
```
plotSurface(k,t,z.SVJ,rainbow(100,start=.0,end=.1))
```

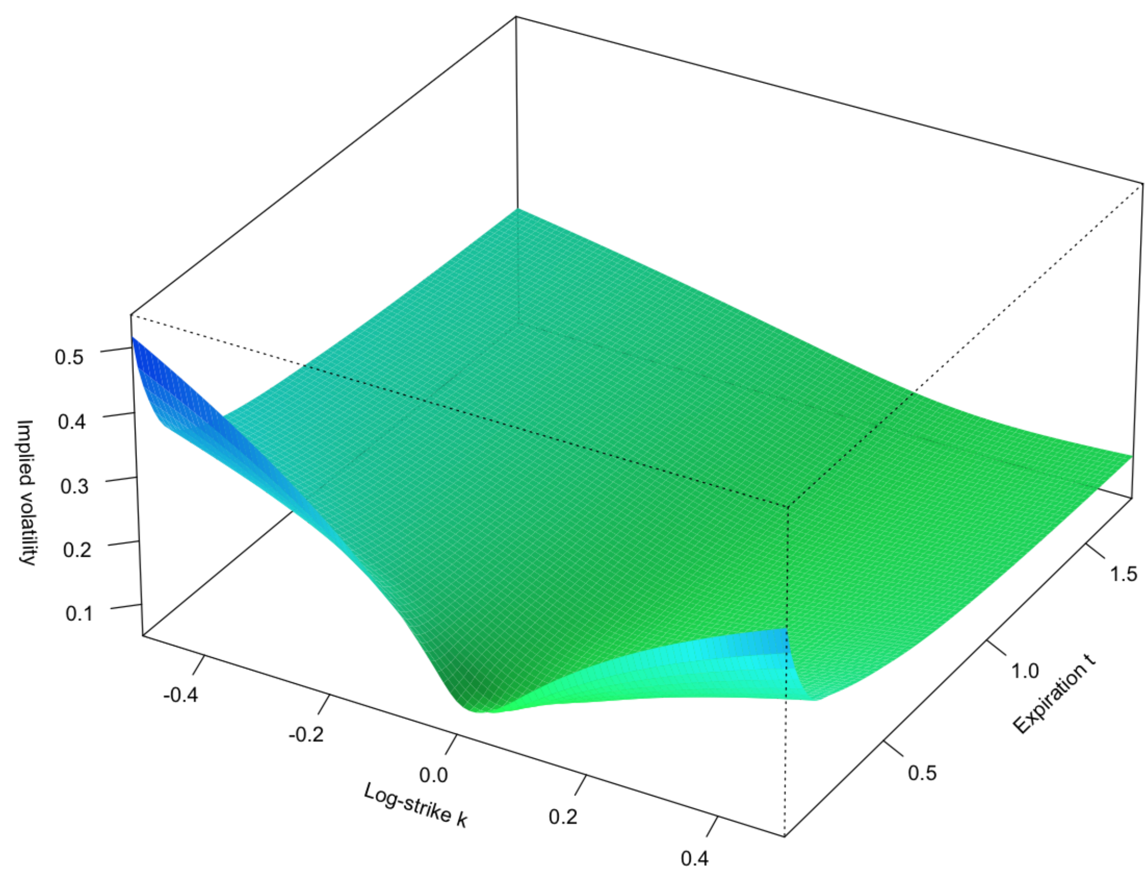


Next plot the empirical surface:

In [22]:
```
volTVS <- function(k,t){sqrt(sviW(sviMatrix,texp,k,t)/t)}

# Setup the plot
k <- seq(-.5,.5,0.01) # Vector of log-strikes
t <- seq(0.04,1.74,0.02) # Vector of times
z.empirical <- t(volTVS(k,t)) # Array of volatilities
```

In [23]:
```
plotSurface(k,t,z.empirical,rainbow(100,start=.4,end=.6))
```



Now plot them together with the empirical surface shifted up (to make the comparison easier):

In [24]:
```
view <- function(phi,theta,z1=z.SVJ,z2=z.empirical,shift=0.1){
    plotSurface(k,t,z1,rainbow(100,start=.0,end=.1),box=FALSE,phi=phi,theta=theta)
    par(new=T)
    plotSurface(k,t,z2+shift,rainbow(100,start=.4,end=.8),box=FALSE,phi=phi,theta=theta)
    }
```

In [25]:
```
par(mfrow=c(2,2))
view(30,30)
view(30,150)
view(30,210)
view(30,300)
par(mfrow=c(1,1))
```

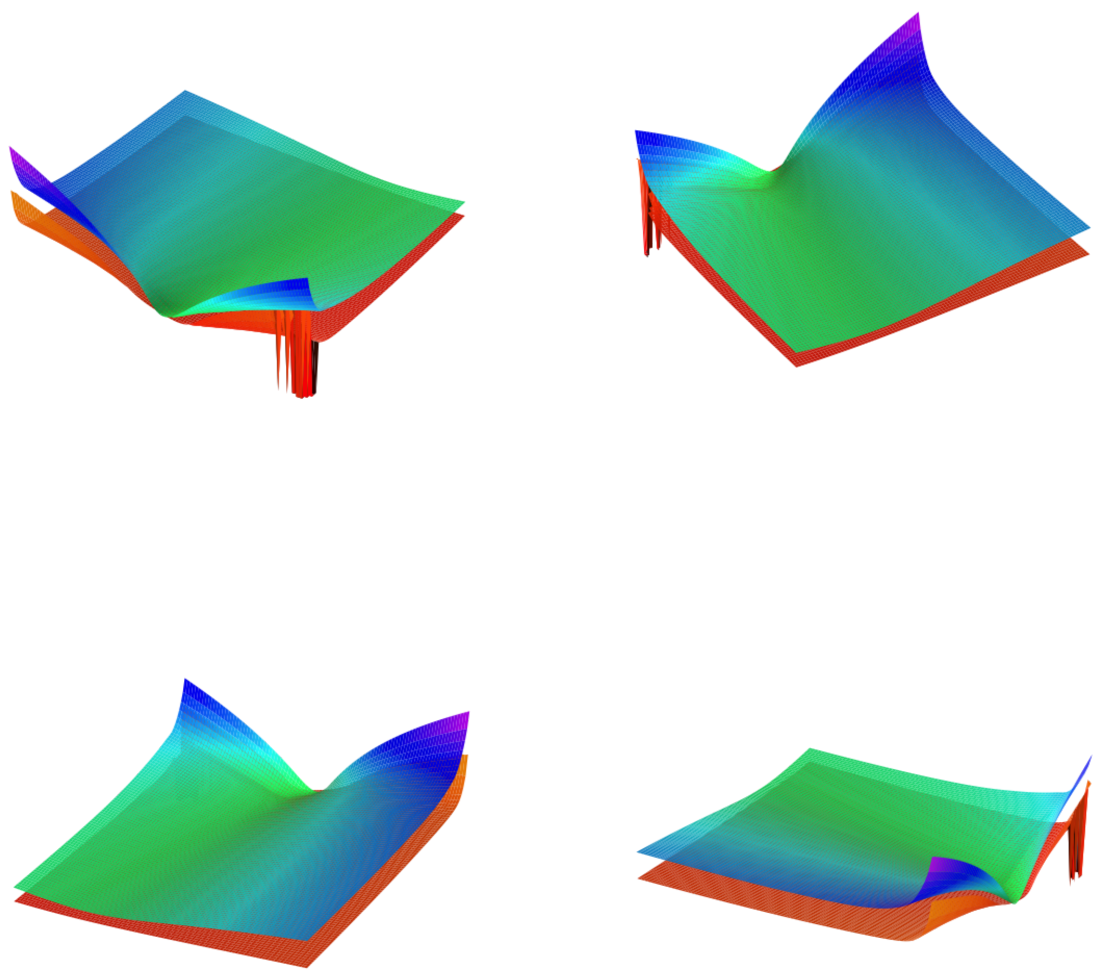


Figure 5: Comparison of the empirical SPX implied volatility surface with the SVJ fit as of September 15, 2005. In contrast to the Heston case, the major features of the empirical surface are replicated by the SVJ model. The upper surface is the empirical SPX volatility surface and the reddish lower one the SVJ fit. The SVJ fit surface has been shifted down by 5 volatility points for ease of visual comparison.

Why the SVJ beats SVJJ

- SVJ fits the observed implied volatility surface reasonably well in contrast to the Heston model.

- You might think that making dynamics more reasonable by including jumps in volatility as in the SVJJ model might generate surfaces that fit even better.
  - Sadly, we can see from Figures [3](#fig:combskewSVJJ) and [4](#fig:shortdatedskewSVJJ) that not only does the SVJJ model have more parameters than the SVJ model, but it's harder to fit to observed option prices.

- The SVJ model thus emerges as a clear winner in the comparison between Heston, SVJ and SVJJ models.
  - However, rough Heston beats all three!

Stochastic time-changed Lévy processes

- So far we have focussed on diffusion models or diffusion models with jumps.
- There has been much academic interest in modeling underlyings using Lévy processes.

- In the rest of this lecture, we will work through a paper by Wim Schoutens and collaborators, that calibrates a number of different models to the EuroStoxx volatility surface.
  - The numerical results presented seem to directly contradict claims made in The Volatility Surface concerning the model-dependence of the value of certain exotic options.
  - Who is right?
- In the process of finding the answer, we will learn about stochastic time changes and the simulation of Lévy processes.

Intuition for down-and-out options

- Suppose we sell a knock-out call option with barrier B equal to the strike price K below the current stock price S_0.
- Suppose further that we hedge this position by buying one stock per option and we charge $S_0 - K$ as the premium.
- With zero rates and dividends, this hedge is perfect.
- In this special case, a knock-out option has no optionality whatsoever. Delta is one, gamma is zero and vega is zero.

Similarity to stop-loss orders

- So long as the stock price remains above the barrier level, we are net flat.
- When the barrier is hit, the option knocks out and we are left long of the stock.

- This is exactly the portfolio we would have if the option buyer had left us a stop-loss order to sell stock if the price ever reached the barrier level B – with one big difference:
  - A barrier option like this guarantees execution at the barrier level but a conventional stop-loss order would get filled at the earliest opportunity after the barrier is hit (usually a bit below the barrier).
  - If we could really trade continuously, there would be no difference between the two contracts.

- In the real world, knock-out options are priced more highly than the diffusion model price to compensate for the risk of the stock price gapping through the barrier level.

Hand-waving extension

- If the strike price K and the barrier level B are not equal but not so far apart with $B \leq K \leq S_0$, it is natural to expect that neither gamma nor vega would be very high relative to the European option with the same strike K.
- Nor would we expect the price of such a knock-out option to be very sensitive to the model used to value it (assuming of course that this model is calibrated to vanilla options).

Overview of the Schoutens et al. paper (SST)

The authors calibrate the following seven models to the EuroStoxx volatility surface as of October 7th, 2003:

- HEST : The Heston model
- HESJ : The Heston model with jumps
- BNS : The Barndorff-Nielsen Shephard model
- VGCIR : The Variance Gamma model with CIR time change
- VGOU : The Variance Gamma model with OU-Gamma time change
- NIGCIR : The Normal Inverse Gaussian model with CIR time change
- NIGOU : The Normal Inverse Gaussian model with OU time change

The Schoutens barrier option challenge

- The authors then price various exotic options using the different models, finding that results vary wildly between models *even though all the models are calibrated to the same European options*.
- In particular, the first row of Table 5 of the SST paper has the following prices for 3-year down-and-out barrier options struck at-the-money with a 95% barrier:

	NIGOU	VGCIR	VGOU	HEST	HESJ	BNS	NIGCIR
Euro	509.76	511.80	509.33	509.39	510.89	509.89	512.21
B=0.95	300.25	293.28	318.35	173.03	174.64	230.25	284.10

- For comparison, with $S_0 = 2461.44$ as of 07-Oct-2003, the difference $S_0 - B$ is $0.05\,S_0 = 123.07$
- Only the HEST and HESJ are close to being consistent with the intuition we developed earlier!
- The VGOU price is more than double our rough estimate!

Intuition for lookback options

- Consider a portfolio long a lookback call (paying the $(S_{max} - K)^+$ struck at K and short two European call options struck at K.
- If the stock price never reaches K, both the lookback and the European option expire worthless.
- If and when the stock price does reach K and increases by some small increment ΔK, the value of the lookback option must increase by ΔK (since $K + \Delta K$ is now the new maximum).

- We must rebalance our hedge portfolio by selling two calls struck at K and buying two calls struck at $K + \Delta K$. The profit generated by rebalancing is just ΔK.
- We conclude that we have a replicating hedge and so the fair value of a lookback call (with zero rates and dividends) is twice the fair value of the corresponding European call.

The Schoutens lookback option challenge

- A similar argument shows that a lookback option that pays the final stock price minus the minimum should be priced as a European straddle.
- Table 3 of the SST paper has the following prices for ATM lookback options:

	NIGOU	VGCIR	VGOU	HEST	HESJ	BNS	NIGCIR
Euro	509.76	511.80	509.33	509.39	510.89	509.89	512.21
Lookback	722.34	724.80	713.49	838.48	845.18	771.28	730.84

- We note that HEST and HESJ are the models that return values closest to our rough guess of $1,000$.

Checking the computations

- Our first assumption might be that the authors' computations were wrong.
- To check their computations, we need to simulate the various models.
  - We don't know what interest rate or dividend assumptions the authors used so for simplicity, we take them to be zero.
- We get to practice generating option prices from characteristic functions.

The Heston characteristic function

Recall the Heston process

$$\begin{aligned} dS_t &= \sqrt{v_t}\, S_t\, dZ_t \\ dv_t &= -\lambda\,(v_t - \bar{v})\, dt + \eta\, \sqrt{v_t}\, dW_t \end{aligned}$$

with $\langle dZ_t\, dW_t \rangle = \rho\, dt$.

The Heston characteristic function is:

$$\phi_{HEST}(u) = \exp\{C(u,\tau)\,\bar{v} + D(u,\tau)\,v\}$$

with $C(u,\tau)$ and $D(u,\tau)$ the usual complicated functions of the model parameters.

The HESJ characteristic function

The Heston model with jumps process is:

$$\begin{aligned} \frac{dS_t}{S_t} &= -\mu_J\, dt + \sqrt{v_t}\, dZ_t + J_t\, dN_t \\ dv_t &= -\lambda\,(v_t - \bar{v})\, dt + \eta\, \sqrt{v_t}\, dW_t \end{aligned}$$

where N_t is a Poisson process with rate λ_J and the jumps J_t are lognormally distributed with mean α and standard deviation δ. As before $\langle dZ_t\, dW_t \rangle = \rho\, dt$.

The HESJ characteristic function is:

$$\phi_{HESJ}(u) = \phi_{HEST}(u)\, \phi_{JUMP}(u)$$

with

$$\begin{aligned}&\phi_{JUMP}(u)\\ &= \exp\left\{-i\, u\, \lambda_J\, \tau\, \left(e^{\alpha+\delta^2/2} - 1\right) + \lambda_J\, \tau\, \left(e^{i\, u\, \alpha - u^2/2\, \delta^2} - 1\right)\right\}.\end{aligned}$$

The BNS process

The version of the Barndorff-Nielsen-Shephard model that is implemented in SST has the following dynamics:

$$\begin{aligned} d\log S_t &= -\left[\lambda\, k(-\rho) + v_t/2\right]\, dt + \sqrt{v_t}\, dW_t + \rho\, dz_{\lambda\, t} \\ dv_t &= -\lambda v_t\, dt + dz_{\lambda\, t}\end{aligned}$$

Here z_t is the compound Poisson process

$$z_t = \sum_{i=1}^{N} x_i$$

where N is a Poisson process with rate a and each x_i follows an exponential law with rate b.

If the process for v is started with an initial value from its stationary Gamma distribution, we obtain

$$k(u) := \mathbb{E}\left[e^{i\, u\, z_1}\right] = \frac{a\, i\, u}{b - i\, u}$$

as the characteristic function of z_1.

The BNS characteristic function

The BNS characteristic function is given by

$$\begin{aligned}\phi_{BNS}(u) = \exp&\left\{-i\, u\, \frac{a\, \lambda\, \rho}{b - \rho}\, \tau\right\}\, \exp\left\{-\frac{1 - e^{-\lambda\, \tau}}{\lambda}\, (u^2 + i\, u)\, v_0/2\right\}\\ \times \exp&\left\{\frac{a}{b - f_2(u)}\left[b\, \log\left(\frac{b - f_1(u)}{b - i\, u\, \rho}\right) + f_2(u)\, \lambda\, \tau\right]\right\}\end{aligned}$$

with

$$\begin{aligned} f_1(u) &= i\, u\, \rho - \frac{1 - e^{-\lambda\, \tau}}{2\, \lambda}\, (u^2 + i\, u) \\ f_2(u) &= i\, u\, \rho - \frac{1}{2\, \lambda}\, (u^2 + i\, u)\end{aligned}$$

The AVG characteristic function

- The Asymmetric Variance Gamma characteristic function is given by

$$\phi_{AVG}(u) = \left(\frac{G\, M}{G\, M + (M - G)\, i\, u + u^2}\right)^C$$

  with $C > 0,\ G > 0,\ M > 0.$

- G and M are parameters controlling the size of up and down jumps respectively. If $M > G$, the density will be left-skewed.

- C controls the rate of jump arrivals.

The NIG characteristic function

- The Normal Inverse Gaussian characteristic function is given by
$$\phi_{NIG}(u) = \exp\left\{-\delta\,\left(\sqrt{\alpha^2-(\beta+i\,u)^2}-\sqrt{\alpha^2-\beta^2}\right)\right\}$$
with $\alpha > 0,\ \beta^2 < \alpha^2,\ \delta > 0.$

- α controls the sizes of jumps and β is an asymmetry parameter.

- δ controls the rate of jump arrivals.

Stochastic time change

- One way to model stochastic volatility is by making time stochastic. The idea is that parameters are constant in business time but vary in calendar (or wall clock) time.

- For example, the Heston model can be viewed as Brownian Motion with a CIR (square-root process) time change.

- Obviously, stochastic time has to be monotonically increasing with respect to calendar time so the process modeling the time change has to be positive.

- For a given positive time-change process $y_s, 0
$$\tau_T = \int_0^T y_s\,ds.$$

Characteristic function of a time-changed Lévy process

Consider the time-changed process $Y_T = X_{\tau_T}$ with X a Lévy process, and X and τ independent. The characteristic function of Y_T may be computed as follows:

(5) \begin{eqnarray} *\phi_{Y_T}(u)&=&\mathbb{E}\left[e^{i\,u\,YT}\right]\nonumber\ &=&\mathbb{E}\left[e^{i\,u\,X{\tau_T}}\right]\nonumber\ &=&\mathbb{E}\left[e^{\psi_X(u)\,\tauT}\right]\nonumber\ &=&\phi{\tau_T}\left(-i\,\psi_X(u)\right) \end{eqnarray}*

where $\psi_X(u) := \log\mathbb{E}\left[e^{i\,u\,X_1}\right]$ is the characteristic exponent of X.

Special case: Hull and White

Hull and White consider the process \begin{eqnarray}*\frac {dS_t} {S_t}&=&\phi\,dt+\sigma_t\,dZ_t\ \frac{dv_t} {v_t}&=&\mu\,dt+\xi\,dW_t \end{eqnarray}*

with $v_t = \sigma_t^2$ and $\mathbb{E}[dW_t\,dZ_t] = 0.$

The option price in this model is given by

(6)

$$C(k, v, T) = \int C_{BS}(k, w_T)\, p(w_T)\, dw_T$$

where as usual $w_T = \int_0^T v_t\, dt$.

- Remark that $w_T = \int_0^T v_t\, dt$ is a stochastic time-change.
- The value of the option is obtained by integrating the Black-Scholes formula over all possible values of the total variance w_T.

- Formula (6) may be rewritten as

  \begin{eqnarray} *C(k,v,T)&=&\mathbb{E}_{wT}\left[C{BS}(k,w)\right]\ &=&\mathbb{E}_{wT}\left[\mathbb{E}{S_w}\left[(S_w-K)^+\right]\right] \end{eqnarray}*

- It's very important to note that this only works if the stock process dW and the volatility process dZ are independent.
  - Otherwise, it's more complicated with the result depending on the details of the assumed stock and volatility processes.

The CIR stochastic clock

In this case, the time change y_t satisfies

$$dy_t = -\kappa(\eta - y_t)\, dt + \lambda\, \sqrt{y_t}\, dZ_t.$$

The characteristic function of the time change $\tau_t = \int_0^t y_s\, ds$ is given by the well-known CIR formula:

$$\phi_{CIR}(u) = \frac{e^{\kappa^2\, \eta\, t/\lambda^2}\, \exp\left\{ \frac{2\, y_0\, i\, u}{\kappa + \gamma/\tanh(\gamma\, t/2)} \right\}}{\left(\cosh(\frac{\gamma\, t}{2}) + \frac{\kappa}{\gamma}\, \sinh\left(\frac{\gamma\, t}{2}\right)\right)^{2\, \kappa\, \eta/\lambda^2}}$$

where

$$\gamma = \sqrt{\kappa^2 - 2\, \lambda^2\, i\, u}$$

The OU-Γ stochastic clock

The time change y_t satisfies

$$dy_t = -\lambda\, y_t\, dt + dz_{\lambda\, t}$$

where the process z_t is the compound Poisson process

$$z_t = \sum_{i=1}^{N} x_i$$

where N is a Poisson process with rate a and each x_i follows an exponential law with rate b.

The characteristic function of the time change $\tau_T = \int_0^T y_s\, ds$ is then given by:

$$\begin{aligned}\phi_{OU}(u) = \exp\Bigg\{ & i\, u\, y_0\, \frac{1 - e^{-\lambda\, T}}{\lambda} \\ & + \frac{\lambda\, a}{i\, u - \lambda\, b} \left[b\, \log\left(\frac{b}{b - i\, u\, (1 - e^{-\lambda\, T})/\lambda} \right) - i\, u\, T \right] \Bigg\}.\end{aligned}$$

Computing option prices from the characteristic function

- Now that we can compute all the desired characteristic functions in closed form, we can compute European option prices.
- Recall equation (5.6) of *The Volatility Surface*:

$$C_t(S,K,T) = S - \sqrt{SK}\,\frac{1}{\pi}\int_0^\infty \frac{du}{u^2+\frac{1}{4}}\,\mathrm{Re}\left[e^{-iuk}\,\phi_t^T\,(u-i/2)\right]$$

  with $k := \log(K/S)$.

Checking SST numerics

- We don't know from the SST paper what dividend and interest rate assumptions were used to calibrate the models and generate option prices.
- All we need to do is to ensure that with zero rates and dividends, all of the models generate the same European option prices

- Equivalently, we can check that these models return:
  - the same densities.
  - the same implied volatility smiles.
- We choose to compare implied volatility smiles (SST choose to compare option values but these are almost indistinguishable between models).

Comparison of volatility smiles: 1 year

First we code BNS, Heston, and SVJ characteristic functions with SST parameters:

In [26]:
```
subBNS <- list(lambda =.5474,a = .6069,b = 18.6075,rho = -4.675,v0 = 0.0433)

# The Barndorf-Nielsen-Shephard model (BNS)
phiBNS <- function(params){

    lambda <- params$lambda
    a <- params$a
    b <- params$b
    rho <- params$rho
    v0 <- params$v0

    function(u,t){
    f1 <- 1i*u*rho - (1-exp(-lambda*t))/(2*lambda)*(u^2+1i*u)
    f2 <- 1i*u*rho - (u^2+1i*u)/(2*lambda)
    term1 <- exp(-1i*u*(a*lambda*rho)/(b - rho)*t)
    term2 <- exp(-(1 - exp(-lambda*t))/(2*lambda)*(u^2 + 1i*u)*v0)
    term3 <- exp(a/(b-f2)* (b*log((b-f1)/(b-1i*u*rho)) + f2*lambda*t))
    return(term1*term2*term3)
    }
}
```

In [27]:
```
subHeston <- list(lambda = 0.6067,rho = -0.7571,eta = 0.2928,vbar = 0.0707,v = .0654)

# The Heston model
phiHeston <- function(params){

    lambda <- params$lambda
    rho <- params$rho
    eta <- params$eta
    vbar <- params$vbar
    v <- params$v

    function(u, t){

    al <- -u*u/2 - 1i*u/2
    bet <- lambda - rho*eta*1i*u
    gam <- eta^2/2
    d <- sqrt(bet*bet - 4*al*gam)
    rp <- (bet + d)/(2*gam)
    rm <- (bet - d)/(2*gam)
    g <- rm / rp
    D <- rm * (1 - exp(-d*t))/ (1 - g*exp(-d*t))
    C <- lambda * (rm * t - 2/eta^2 * log( (1 - g*exp(-(d*t)))/(1 - g) ) )
    return(exp(C*vbar + D*v))
    }
}
```

In [28]:
```
subSVJ = list(lambda = 0.4963,rho = -.99,eta = .2286,vbar = 0.065,v = 0.0576,lambdaJ = .1
382,alpha = -.1791,delta = .1346)

# phiJump
phiJump <- function(params){

    lambdaJ <- params$lambdaJ
    alpha <- params$alpha
    delta <- params$delta

    function(u,t){
    tmp <- exp(-1i*u*lambdaJ*(exp(alpha + delta^2/2) - 1)*t +
    lambdaJ*t*(exp(1i*u*alpha - u^2/2*delta^2) - 1))
    return(tmp)
    }
}

# The HESJ or SVJ model
phiSVJ <- function(params){
    function(u, t){
    return(phiHeston(params)(u,t)*phiJump(params)(u,t))
    }
}
```

Next we code the CIR and OU stochastic clocks:

In [29]:
```
# CIR
phiCIR <- function(params){

    kappa <- params$kappa
    eta <- params$eta
    y0 <- params$y0
    lambda <- params$lambda

    function(u,t){
    gamma <- sqrt(kappa^2 - 2*lambda^2*1i*u)
    numer <- exp(kappa^2*eta*t/lambda^2)*exp(2*y0*1i*u/(kappa+gamma/tanh(gamma*t/2)))
    denom <- (cosh(gamma*t/2) + kappa/gamma*sinh(gamma*t/2))^(2*kappa*eta/lambda^2)
    return(numer/denom)
    }
}
```

In [30]:
```
# OU
phiOU <- function(params){

    y0 <- params$y0
    lambda <- params$lambda
    a <- params$a
    b <- params$b

    function(u,t){
    tmp1 <- 1i*u*y0*(1 - exp(-lambda*t))/lambda
    tmp2 <- lambda*a/(1i*u - lambda*b)
    tmp3 <- b*log(b/(b - 1i*u*(1 - exp(-lambda*t))/lambda))-1i*u*t

    return(exp(tmp1+tmp2*tmp3))
    }
}
```

Now use equation (5) to get time-changed Normal Inverse Gaussian (NIG) with CIR and OU subordination:

Normal Inverse Gaussian (NIG)

In [31]:
```
psiNIG <- function(params){

    alpha <- params$alpha
    beta <- params$beta
    delta <- params$delta

    function(u){
    return(-(sqrt(alpha^2 - (beta + 1i*u)^2) -
    sqrt(alpha^2 - beta^2))*delta)
    }
}
```

The Normal Inverse Gaussian (NIG) model with CIR subordination

In [32]:
```
subNIGCIR <- list(alpha = 16.1975,beta = -3.1804,delta = 1.0867,
                kappa = 1.2101, eta = 0.5507, lambda = 1.7864, y0 = 1)

phiNIGCIR <- function(params){

    function(u, t){
    numer <- phiCIR(params)(-1i*psiNIG(params)(u),t)
    denom <- phiCIR(params)(-1i*psiNIG(params)(-1i),t)^(1i*u)
    return(numer/denom)
    }
}
```

The Normal Inverse Gaussian (NIG) model with OU subordination

In [33]:
```
subNIGOU <-list( alpha = 8.8914, beta = -3.1634, delta = 0.6728, lambda = 1.7478, a = 0.3
442, b = 0.7628, y0 = 1)

phiNIGOU <- function(params){

    function(u, t){
        numer <- phiOU(params)(-1i*psiNIG(params)(u),t)
        denom <- phiOU(params)(-1i*psiNIG(params)(-1i),t)^(1i*u)
        return(numer/denom)
    }
}
```

And repeat this for the Asymmetric Variance Gamma (AVG) model:

Asymmetric Variance Gamma (AVG)

In [34]:
```
psiAVG <- function(params){

    c <- params$c
    g <- params$g
    m <- params$m

    function(u){
    return(c*log((g*m/(g*m + (m - g)*1i*u + u^2))))
    }
}
```

The AVG model with CIR subordination

In [35]:
```
subAVGCIR <- list(c = 18.0968, g = 20.0276, m = 26.3971, kappa= 1.2145,
eta = 0.5501, lambda =  1.7913, y0 = 1)

phiAVGCIR <- function(params){

    function(u, t){
    numer <- phiCIR(params)(-1i*psiAVG(params)(u),t)
    denom <- phiCIR(params)(-1i*psiAVG(params)(-1i),t)^(1i*u)
    return(numer/denom)
    }
}
```

The AVG model with OU subordination

In [36]:
```
subAVGOU <- list( c = 6.1610, g = 9.6443, m = 16.026, lambda = 1.679,
    a = 0.3484, b = 0.7664, y0 = 1)

phiAVGOU <- function(params){

    function(u, t){
    numer <- phiOU(params)(-1i*psiAVG(params)(u),t)
    denom <- phiOU(params)(-1i*psiAVG(params)(-1i),t)^(1i*u)
    return(numer/denom)
    }
}
```

Now we can compare the smiles generated by all of these models:

In [37]:
```
options(repr.plot.height=7,repr.plot.width=10)

leg.txt <- c("Heston", "HESJ", "BNS", "NIGCIR", "NIGOU", "AVGCIR", "AVGOU")

# First compare with 1 year expiration
clr <- rainbow(7)
xrange <- c(-.5,.5);  yrange <- c(.1,.4)
plot(NA,xlim=xrange,ylim=yrange,ylab="Implied vol.",xlab="Log-strike k")

vol <- function(k){sapply(k,function(x){impvol.phi(phiHeston(subHeston))(x,1)})}
curve(vol(x),from=-.5,to=.5,col=clr[1],add=T,lwd=2)

vol <- function(k){sapply(k,function(x){impvol.phi(phiSVJ(subSVJ))(x,1)})}
curve(vol(x),from=-.5,to=.5,col=clr[2],add=T,lwd=2)

vol <- function(k){sapply(k,function(x){impvol.phi(phiBNS(subBNS))(x,1)})}
curve(vol(x),from=-.5,to=.5,col=clr[3],add=T,lwd=2)

vol <- function(k){sapply(k,function(x){impvol.phi(phiNIGCIR(subNIGCIR))(x,1)})}
curve(vol(x),from=-.5,to=.5,col=clr[4],add=T,lwd=2)

vol <- function(k){sapply(k,function(x){impvol.phi(phiNIGOU(subNIGOU))(x,1)})}
curve(vol(x),from=-.5,to=.5,col=clr[5],add=T,lwd=2)

vol <- function(k){sapply(k,function(x){impvol.phi(phiAVGCIR(subAVGCIR))(x,1)})}
curve(vol(x),from=-.5,to=.5,col=clr[6],add=T,lwd=2)

vol <- function(k){sapply(k,function(x){impvol.phi(phiAVGOU(subAVGOU))(x,1)})}
curve(vol(x),from=-.5,to=.5,col=clr[7],add=T,lwd=2)

legend(x=.25,y=.39,leg.txt,lty=1,col=clr,lwd=2)
```

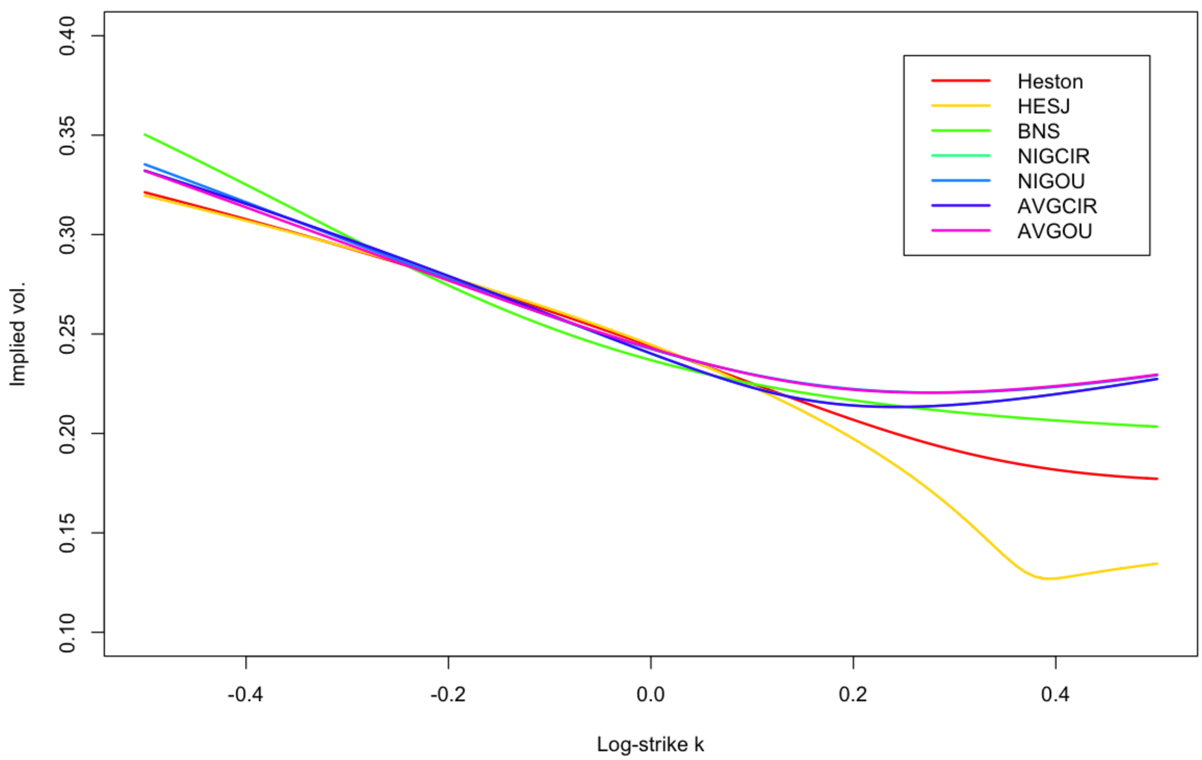


Figure 6: One year volatility smiles generated by various models with SST parameters.

Comparison of volatility smiles: All expiries

Repeating this exercise for all expiries gives the following plot:

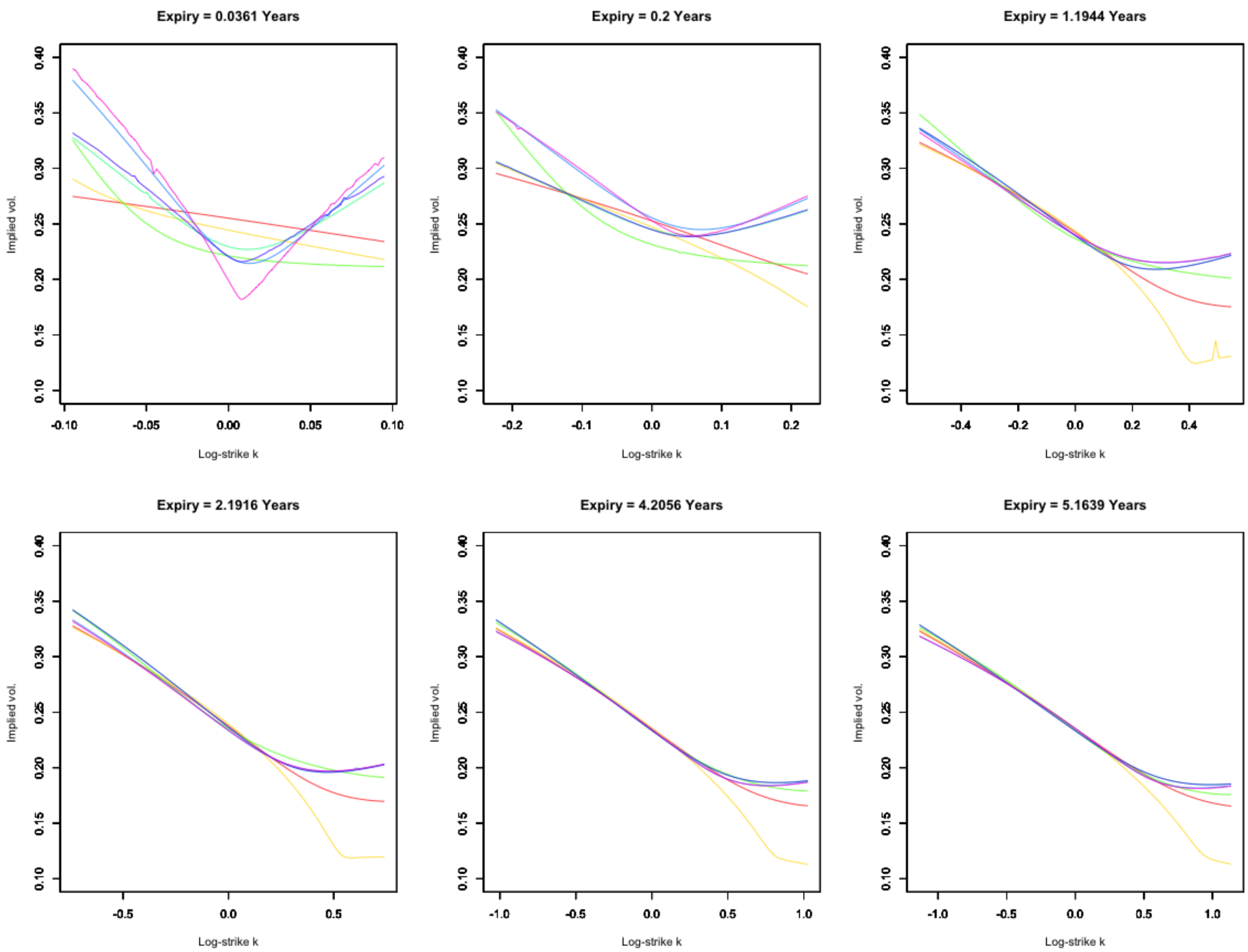


Figure 7: All volatility smiles for the same models as in Figure 6.

Interim conclusion

- We conclude that the SST numerics are correct.
- All models generate more or less the same implied volatilities except for very short expirations.

The guaranteed stop loss contract

- Consider a hypothetical contract that pays the difference between a pre-agreed level (the barrier B) and the first stock price reached after the barrier is breached.
  - If price paths are continuous, the fair value of this contract is zero.
  - The fair value of this contract is sensitive only to the magnitude and frequency of jumps.

- For indices at least, informal canvassing of traders suggests that this contract is worth very little.

- We can approximate the value of this contract in the various models by simulation.
  - As an example, we value this option under NIGOU.

Path generation for time-changed Lévy processes

1. Simulate the time change y_t and compute the new clock τ_t.
2. Simulate the Lévy process $X = \{X_t,\ 0 \leq t \leq \tau_T\}$.
3. Compute $Y_t = X(\tau_t)$.
4. Compute stock prices S_t as

$$S_t = S_0\, \frac{\exp\{X(\tau_t)\}}{\mathbb{E}[\exp\{X(\tau_t)\}]}$$

Simulating the OU-Γ time-change

$$y_i = e^{-\lambda\, \Delta t}\, y_{i-1} + \sum_{k=N_{i-1}+1}^{N_i} x_k\, e^{-\lambda\, \Delta t\, u_k}$$

with $u_k \sim U(0,1)$ and x_k exponentially distributed with rate b.

N_i is the Poisson random variable giving the number of jumps up to time t_i.

Simulating the OU-Γ time-change: R-code

In [38]:
```
simOU <- function(T,nTimes,params){

    dt <- T/nTimes
    vOut <- rep(1,nTimes)  #v0=1
    a <- params$a;  b <- params$b;  lambda <- params$lambda
    elt <- exp(-lambda*dt)

    #Generate Poisson path
    nJumps <- rpois(nTimes,lambda=a*lambda*dt)

    for (i in 1:(nTimes-1))
    {
    xx <- 0
    if (nJumps[i] >0){
    for (k in 1:nJumps[i]){
        u <- runif(1)
        x <- rexp(1,rate=b)*elt^u
        xx <- xx+x
    }}
    vOut[i+1] <- vOut[i]*elt + xx
  }
  return(c(0,cumsum(vOut)*dt))
}
```

A typical simulated OU-Γ time-change

In [39]:
```
options(repr.plot.height=5)
s <- simOU(1,100,subNIGOU)
par(mfrow=c(1,2))
plot(diff(s),type="l",col="red",xlab=NA,ylab=expression(y[t]))
plot(s,type="l",col="red",xlab=NA,ylab=expression(tau))
par(mfrow=c(1,1))
```

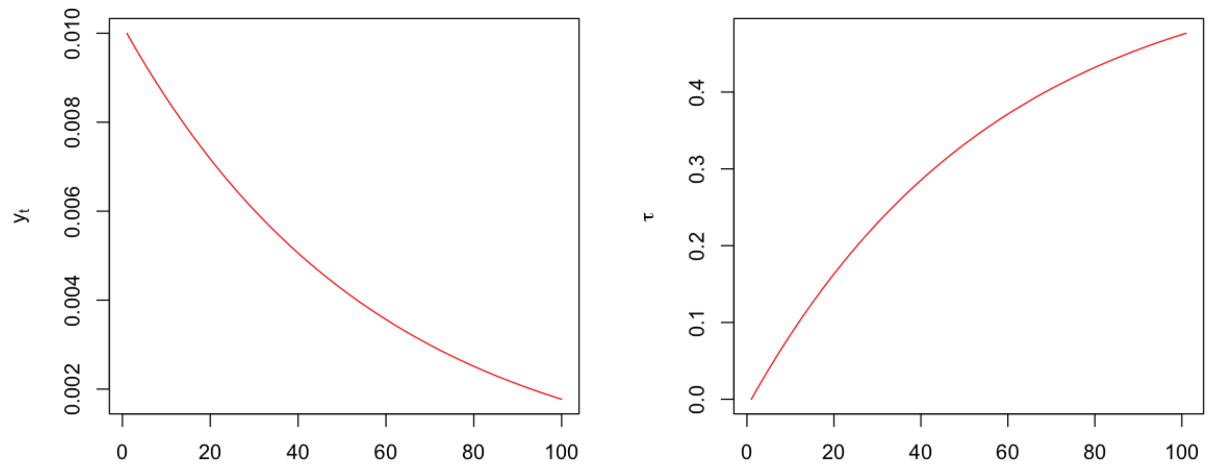


Figure 8: The time-change $\tau_t = \int^t y_s \, ds$

Simulating time-changed NIG: R-code

The R library fBasics has a function to generate NIG random variables. We simulate the time-changed NIG process by generating $NIG(\alpha, \beta, \delta \, \Delta\tau)$ random variables at each step where $\Delta\tau$ is the time-difference according to the chosen stochastic clock.

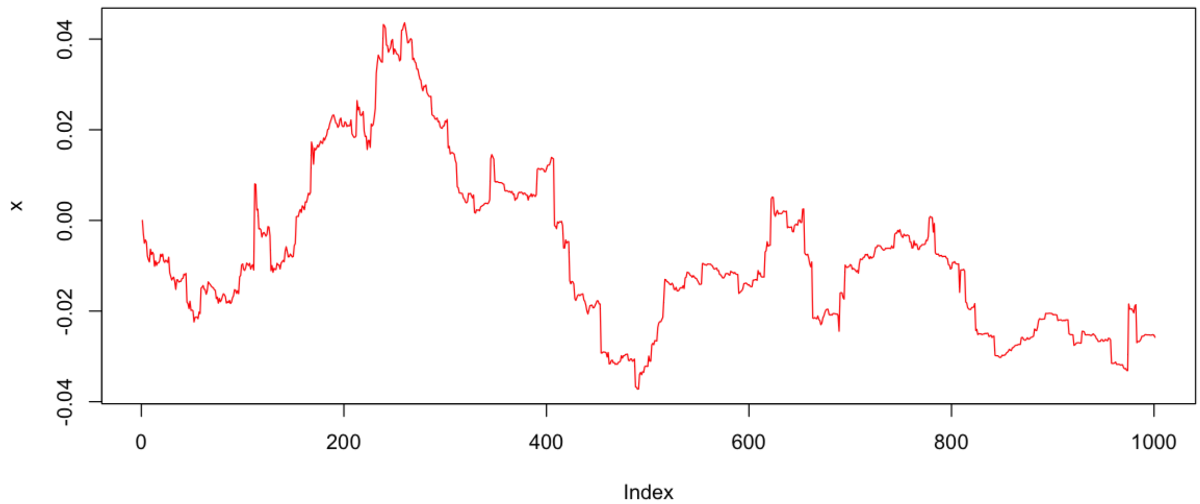
In [40]:
```
library(fBasics)

simNIG <- function(t,nTimes,subordinator,params){

    st <- subordinator(t,nTimes,params)  # Stochastic clock times
    delst <- diff(st)
    x <- sapply(1:nTimes,function(i){rnig(1,alpha=params$alpha,beta=params$beta,delta=par
ams$delta*delst[i]})})
    return(cumsum(c(0,x)))
}
```

```
Loading required package: timeDate
Loading required package: timeSeries
```

In [41]:
```
x <- simNIG(t=1,1000,simOU,subNIGOU)
plot(x,type="l",col="red")
```



NIGOU Monte Carlo vs Analytic

The following code, when uncommented, will generate 10,000 NIGOU paths. This code takes about 8 minutes to run on my machine.

In [42]:
```
#system.time(xNIGOU <- t(sapply(1:10000,function(i){simNIG(t=1,1000,simOU,subNIGOU)})))
```

Alternatively, you can use the following presimulated time-changed NIGOU paths for exotic option valuation. You can download the datafile from here: https://drive.google.com/file/d/0B7j7JiZG5FMvajFjNWs4MHlTcG8/view?usp=sharing (https://drive.google.com/file/d/0B7j7JiZG5FMvajFjNWs4MHlTcG8/view?usp=sharing). Then move the file to your working directory.

In [43]:
```
load("pathsNIGOU.rData")
xNO <- xNIGOU[,1001]
```

In [44]:
```
#Generate final stock prices
sNO <- exp(xNO);  sNO <- sNO/mean(sNO)

# Compute implied vols. from final stock prices
kk <- seq(-.5,.5,.05)
AK <- exp(kk)
res <- bsOut(sNO,1,AK)
```

Plot results and compare with analytic computation

In [45]:
```
options(repr.plot.height=7)
xrange <- c(-.5,.5);  yrange <- c(.2,.4)
x1=.5
plot(kk,res$BSV,xlim=xrange,ylim=yrange,xlab=NA,ylab="Implied Vol.",col="dark green")
arrows(kk,res$BSVH,kk,res$BSVL,angle=90,code=3,length=.1,col="dark green")
par(new=T)
vol <- function(k){sapply(k,function(x){impvol.phi(phiNIGOU(subNIGOU))(x,1)})}
curve(vol(x),from=-x1,to=x1,col="red",xlim=xrange,ylim=yrange,xlab=NA,ylab=NA)
```

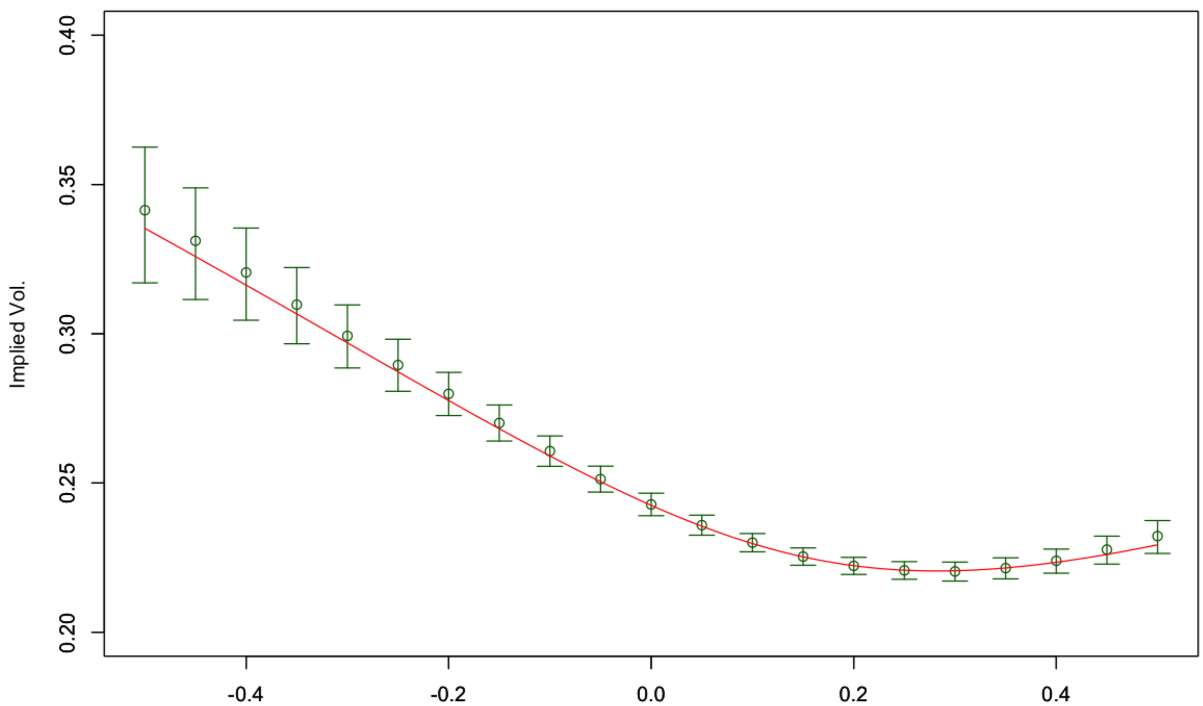


Figure 9: Here, we compare the Monte Carlo (10,000 paths) and analytic (using characteristic function) valuations of 1 year European options

Valuation of the guaranteed stop-loss contract

Having determined that the simulation appears to be correct, we can value the guaranteed stop-loss for different barrier levels.

In [46]:
```
gsl <- function(path,b){
if(min(path)<= b){
    tau <- min(which(path<=b))  # Barrier is hit at this point
    tmp <- b-path[tau]  # Jump in price
}
else tmp <- 0
return(tmp)
}
```

Generate NIGOU sample paths:

In [47]:
```
sNIGOU <- exp(xNIGOU)
```

Now plot value of guaranteed stop-loss as a function of barrier level:

In [48]:
```
gslVal <- function(k){
 tmp <- apply(sNIGOU,1,function(x){gsl(x,k)})
 return(mean(tmp))
}
bb <- c(seq(0.8,.99,0.01),.995,.999,.9999)
y <- sapply(bb,gslVal)
```

In [49]:
```
plot(bb,y,type="b",col="red",xlab="Barrier level", ylab="Value of guaranteed stop-loss")
```

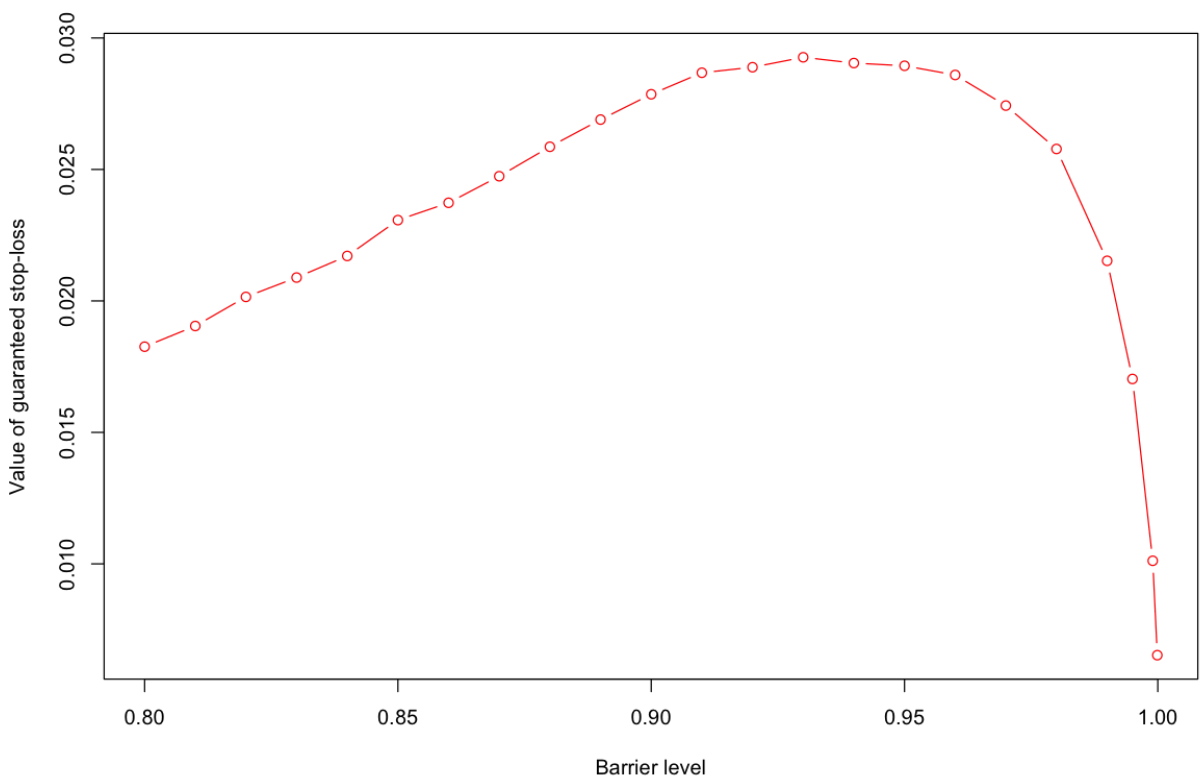


Figure 10: Value of the guaranteed stop-loss contract for different barrier levels

- Note the unreasonably large value under NIGOU!

The NIGOU sample path with the biggest jump

In [50]:
```
options(repr.plot.height=6)
dNO <- apply(sNIGOU,1,diff)  # Compute differences
tmp <- apply(t(dNO),1,function(x){max(x)})
worst <- which(tmp==max(tmp))
plot(sNIGOU[worst,],col="red",type="l",ylab="Stock price",xlab=NA)
```

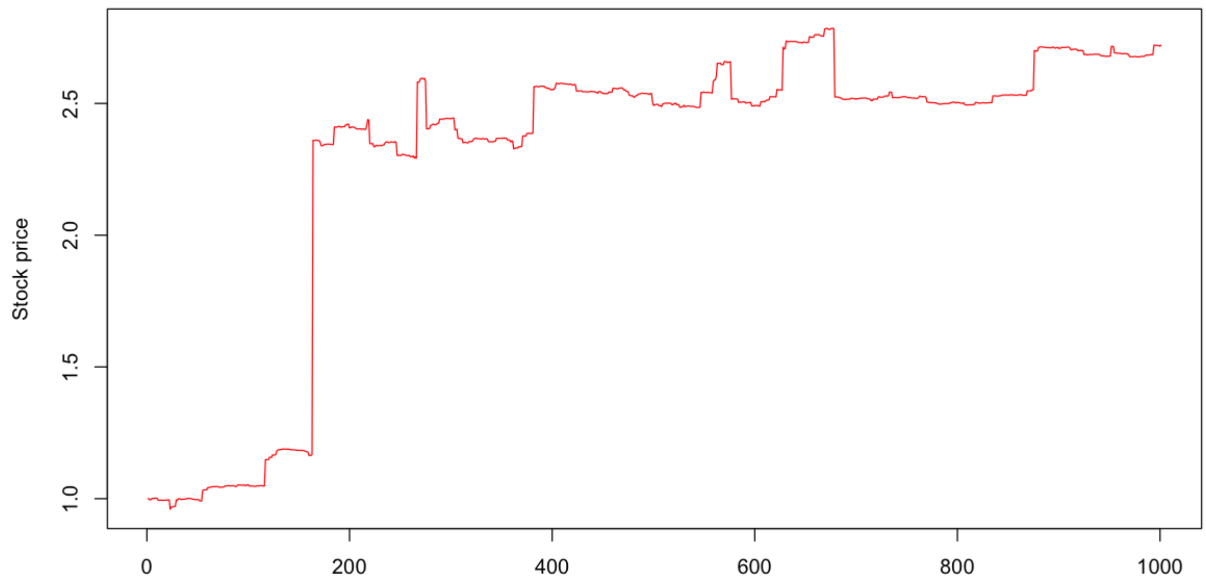


Figure 11: The sample path with the biggest jump

- That's a big jump!

The NIGOU sample path with the biggest negative jump

In [51]:
```
tmp <- apply(t(dNO),1,function(x){min(x)})
worst <- which(tmp==min(tmp))
plot(sNIGOU[worst,],col="red",type="l",ylab="Stock price",xlab=NA)
```

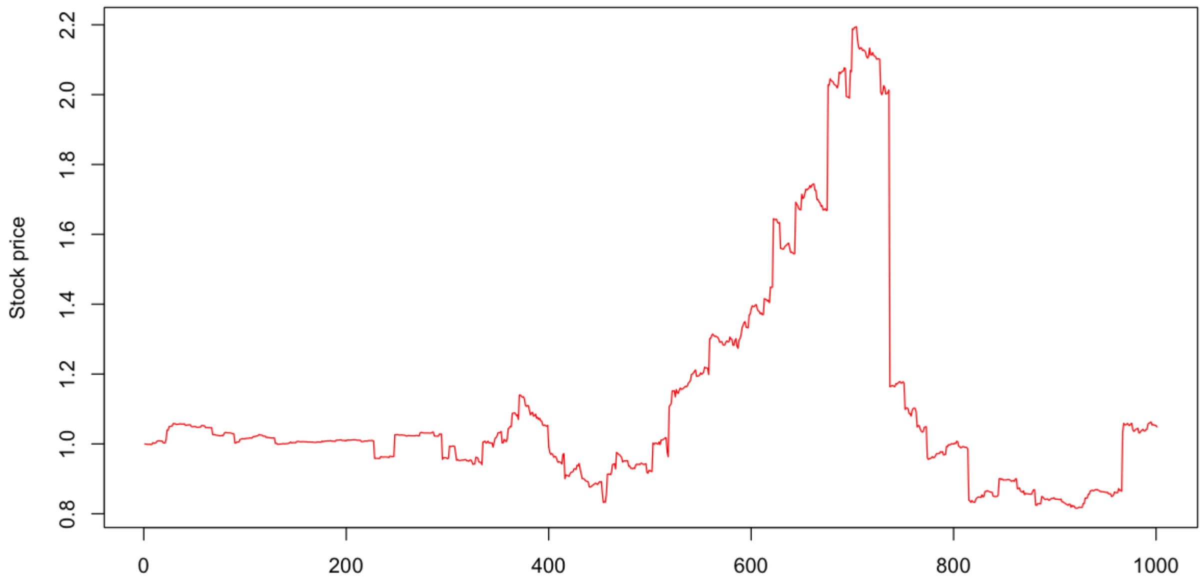


Figure 12: The sample path with the biggest negative jump

- Ouch!

The autocorrelation of squared returns

There's an even easier way to see that time-changed Lévy models of this sort are completely unreallistic.

- Plot the autocorrelation function.

In [52]:
```
dx.worst <- diff(log(sNIGOU[worst,]))
```

In [53]:
```
options(repr.plot.height=5)
acf(dx.worst^2,main="ACF of NIGOU worst path squared returns")
```

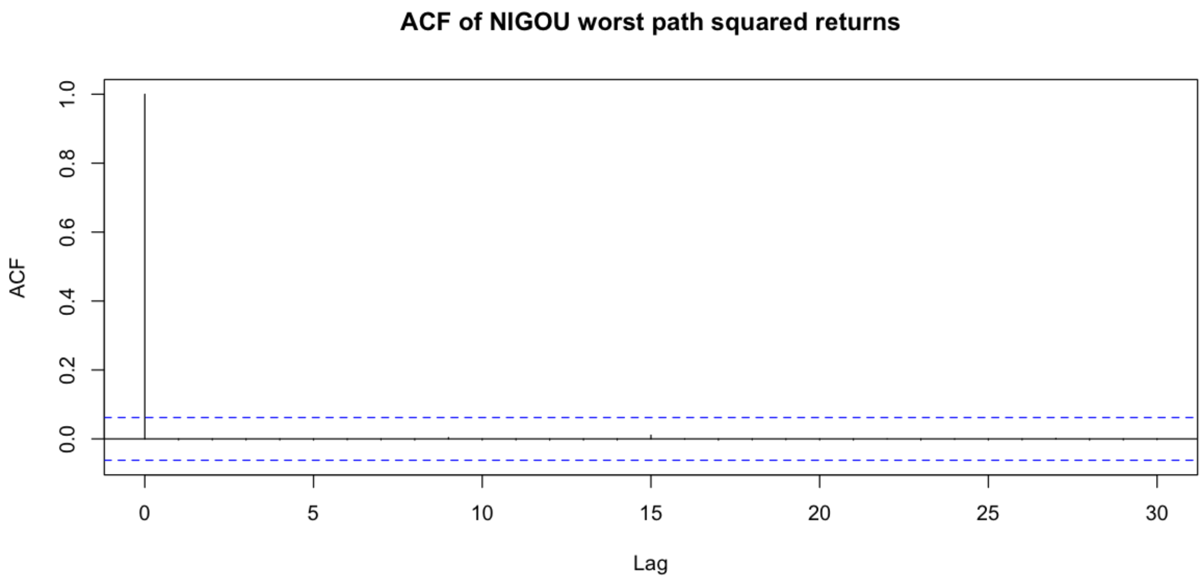


Figure 13: ACF of squared returns of the NIGOU sample path.

ACF of SPX squared returns

In [54]:
```
library(quantmod)
```

```
Loading required package: xts
Loading required package: zoo

Attaching package: ‘zoo’

The following object is masked from ‘package:timeSeries’:

    time<-

The following objects are masked from ‘package:base’:

    as.Date, as.Date.numeric

Loading required package: TTR

Attaching package: ‘TTR’

The following object is masked from ‘package:fBasics’:

    volatility

Version 0.4-0 included new data defaults. See ?getSymbols.
```

In [55]:
```
getSymbols("^GSPC")
```

```
'getSymbols' currently uses auto.assign=TRUE by default, but will
use auto.assign=FALSE in 0.5-0. You will still be able to use
'loadSymbols' to automatically load data. getOption("getSymbols.env")
and getOption("getSymbols.auto.assign") will still be checked for
alternate defaults.

This message is shown once per session and may be disabled by setting
options("getSymbols.warning4.0"=FALSE). See ?getSymbols for details.


WARNING: There have been significant changes to Yahoo Finance data.
Please see the Warning section of '?getSymbols.yahoo' for details.

This message is shown once per session and may be disabled by setting
options("getSymbols.yahoo.warning"=FALSE).
```

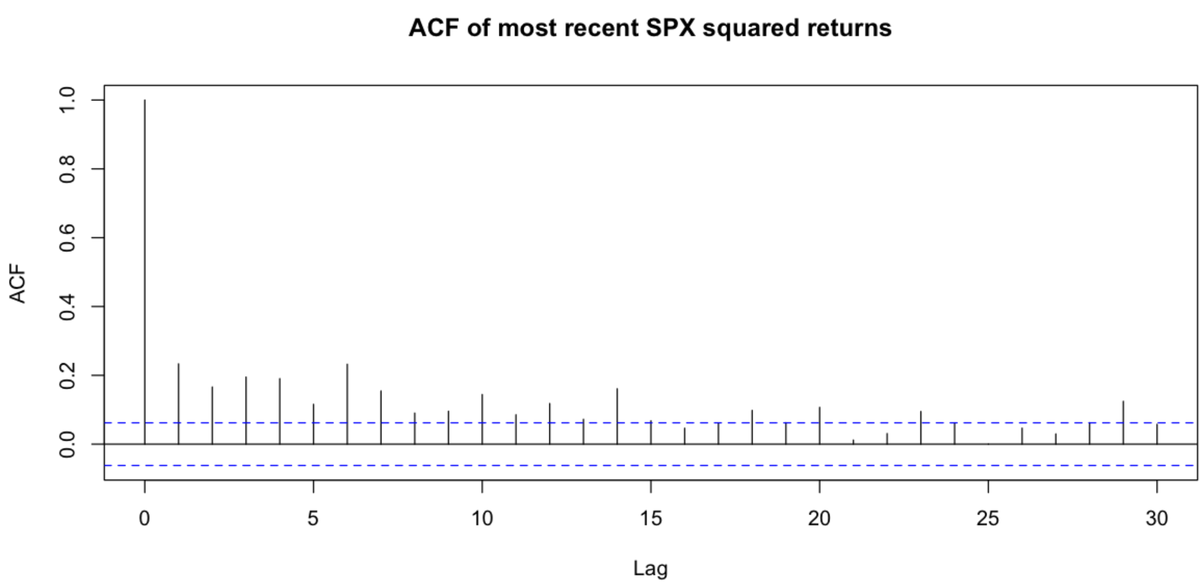
'GSPC'

In [56]:
```
tail(GSPC)
```

	GSPC.Open	GSPC.High	GSPC.Low	GSPC.Close	GSPC.Volume	GSPC.Adjusted
2019-10-18	2996.84	3000.00	2976.31	2986.20	3264290000	2986.20
2019-10-21	2996.48	3007.33	2995.35	3006.72	3271620000	3006.72
2019-10-22	3010.73	3014.57	2995.04	2995.99	3523890000	2995.99
2019-10-23	2994.01	3004.78	2991.21	3004.52	3392870000	3004.52
2019-10-24	3014.78	3016.07	3000.42	3010.29	3692600000	3010.29
2019-10-25	3003.32	3027.39	3001.94	3022.55	3370370000	3022.55

In [57]:
```
p.SPX <- Cl(GSPC)
dx.SPX.1000 <- tail(diff(p.SPX),1000)
```

In [58]:
```
acf(dx.SPX.1000^2,main="ACF of most recent SPX squared returns")
```



Relationship between pricing measure and physical measure

- If price paths are continuous, Girsanov links the physical measure $\mathbb{P}$ and pricing measure $\mathbb{Q}$ through a change of drift.
  - In particular, $\mathbb{P}$ and $\mathbb{Q}$ price paths are qualitatively similar.
- If there are jumps, the relationship between $\mathbb{P}$ and $\mathbb{Q}$ is not so constrained.
  - Afficionados of jumps see this as an advantage: Even if there are no huge jumps in the physical measure, there can be huge jumps in the pricing measure.

- I argue that because dealers need to delta-hedge large portfolios of exotic derivatives at the margin, there should be no significant qualitative difference between price paths under the statistical and risk-neutral measures.
  - Pricing of hypothetical contracts such as the guaranteed stop-loss impose this.
- This excludes processes with crazy jumpy paths such as NIGOU.
  - For indices at least, price paths should be mostly diffusion-like.

Notable quotes

A quote from [Schoutens et al.][5]:

> The conclusion is that great care should be taken when employing attractive fancy-dancy models to price (or even more important, to evaluate hedge parameters for) exotics.

and a quote from page 495 of [Cont and Tankov][2]:

> ... the Bates model appears to be at the same time the simplest and the most flexible of the models.

Note that the Bates model is what we call SVJ.

References

1. ^ Peter Carr and Dilip Madan, Option valuation using the Fast Fourier Transform, *Journal of Computational Finance* **2**(4), 61–73 (1999).
2. ^ Rama Cont and Peter Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC Financial Mathematics Series (2003).
3. ^ Jim Gatheral, *The Volatility Surface: A Practitioner's Guide*, John Wiley and Sons, Hoboken, NJ (2006).
4. ^ Alan L. Lewis, *Option Valuation under Stochastic Volatility with Mathematica Code*, Finance Press: Newport Beach, CA (2000).
5. ^ Wim Schoutens, Erwin Simons, and Jurgen Tistaert, A perfect calibration! Now what?, *Wilmott Magazine*, **2** 66–78 (March 2004).