# Lecture 1

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The personal webpage of the lecture is `wwwf.imperial.ac.uk/~mcptozel/PDE.thml` His website is `m.coti-zelati@imperial.ac.uk`

There are four referece in the syalibus. And three coursework. Office hour is wednesday 11-12.

**Definition 1** (PDE)**.** PDE is an equation of the form

$$f\left(x_1, \ldots x_m, \ldots, u_{x_n}, u_{x_1 x_1}, \ldots u_{x_n x_n} \ldots u_{x_1 x_2}, \ldots\right) \tag{1}$$

The unknpown is a function $u = u\left(x_1, \ldots, x_n\right)$ $u_{x_1}, u_{x_2} \ldots$ are partial derivatives.

**Definition 2** (Order of PDE)**.** It is the highest order of differentiation.

**Definition 3** (Linearity)**.** $F$ is linear in $u$ and its partial derivatives. e.g$u_{x_1} + u = x_2^2$. is linear first order . $u_{x_1 x_1} + u u_{x_2} = 0$ has a quadratic piece, so it is non linear. And it is second order.

Nonlinear PDEs can also have

- Semilinear: $F$ is non-linear wrt u, but linear wrt partial derivatives. e.g. $u_{xx} = u^3$.
- Quasilinear: F is linear in the highest differntiation order, but non-linear in other derivatives. e.g. $u_{xx} + \left|u_x\right|^2 = u^3$
- Fully nonlinear: F is non-linear in the highest order. $\left|u_{x_1}^2\right| + \left|u_{x_2}\right|^2 = x_1^2 + x_2^2$.

Linear PDEs/Homogeneous PDES will be biggest part of this course.

**Definition 4** (Linear operator)**.** A linear differntial operator $L$ is a sum of basic derivatives. $L$ acts on functions in a linear way, such that $L\left(u + v\right) = L\left(u\right) + L\left(v\right) \forall u, v$. $L\left(cu\right) = cL\left(u\right) \forall c \in \mathbb{R}$.

**Definition 5** (Homogenoeous)**.** Each term involves $u$ and or its partial derivatives.

**Example 1.** $u_{xx} + \frac{u}{1+x^2} = 0$ is linear and homoge

**Example 2.** $L = \partial_{xx}$for any$u, v \in C^2\mathbb{R}, c \in \mathbb{R}$. something is missing here
$L\left(u + v\right) =$
$frac\partial\partial x$

**Theorm 1** (Superposition principle)**.** If $u_1, \ldots, n_n$ is a solution of $L(u) = 0$, which is a linear homogeneous PDE, then $u = \sum_{n=1}^{k} c_n u_n$ is also a solution.

Proof: $L(u) = L(c_1 u_1, \ldots, c_k u_k) = L(c_1 u_1, \ldots, c_{k-1} u_{k-1}) + L(c_k u_k) = 0$, repeat this process, you will get $L(u) = L(c_1 u_1) + \ldots + L(u_n u_n)$ which all of the terms are 0. so this is also a solution.

The existance of unique solution requires

$$x' = x \tag{2}$$

$$x(0) = x_0 \tag{3}$$

The top bit may be wrong.

**Example 3.** Heat equation: $u_t = u_{xx}$, $(t, x) \in D \subset R^2$ Classical solution: $u \in \mathbb{C}^2_{t,x}$ and need to hold forall $t, d \in D$

- $u(x,t) = t + \frac{1}{2}x^2$ in $\mathbb{R}^2$.
- $u_t = 1$
- $u_x = 1$, $u_{xx} = 1$.
- $\tilde{u}(t,x) = \exp\left(-\frac{x^2}{4t}\right) / \left(2\sqrt{\pi t}\right)$ in $(0,\infty) \times \mathbb{R}$.

Different behaviour as $|x| \to \infty$. Also $u + \tilde{u}$ is a solution by superposition principle.

There is no statement like LIpchiz's result, so in this course we will have numberous example.

## 0.1 Boundary condition

- Dirichlet Boundary condition: assign value of $u$ along the boundary. E.g. to find the temperature, assign the temperature in the walls.
- Newmann: assign the normal derivative alone the boundary. ($\nabla$ = outer normal, $\frac{\partial u}{\partial \nu} = \nabla u.\nu$)
- Mixed: D on some part of boundary, N on the other.

**Example 4.** For the heat equation again,

- $u_t = U_{xx}$in $\{t > 0\} \times (0,1)$
- $u(0,x) = f(x)$
- $u(t,0) = 0 \forall t$
- $u(t,1) = 0 \forall t$. This is an example of a homogeneous boundary condition.

We will give a list of PDEs

1. Transport equation $u_t + v.\nabla u = 0$, it is 1st order. $u$ is the concentration in a channel and $v$ is the given velocity of the steam.

2. Heat/Diffusion 2nd order:$u_t - D\Delta u = 0$. Where $\Delta =$ Laplacian $= \frac{\partial}{\partial x_1 x_1} + \frac{\partial^2}{\partial x_2 x_2}$, $u$ is the temperature in the medium.
3. Wave equation 2nd order , $u_{tt} - c^2 \Delta u = 0$. $u$ in 1D is the vertical displacement of the string. In 2D it is like the vibration of the drum.

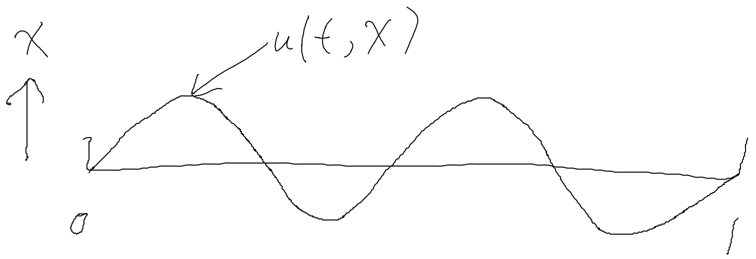



Figure 1: L2F1

Note this is 2nd order we need the velocity of displacement and also the initial displacement to solve this equation

Also note that the heat equation will become smooth over time, whereas the wave equation will not become smooth.

4. Laplace equarion: $\Delta u = 0$
5. Possion equation : $\Delta u = f$ Laplace and Possion equation are the steady state solutuion of wave and heat equation.
6. Burger's equation: $u_t + uu_x = 0$ or $u_{xx}$ this is similar to transport, except the velocity is given by $u$ itself. it is 1st order nonlinear.(shocks).
7. Navier-Stokes Equations$\frac{\partial}{\partial t} u + (u \cdot \nabla) u = -\nabla p + \Delta u$. div$u = 0$. $u = (u_1, u_2, u_3) : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3 =$ velocity of fluid, $P = P(t, x) : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R} =$ Pressure.

Differential Operators

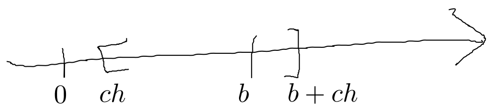
- $\nabla u = \begin{pmatrix} d_{x_1} u \\ \vdots \\ d_{x_n} u \end{pmatrix}$
- div$\mathbf{F} = \partial_{x_1} F_1 + \ldots + \partial_{x_n} F_n$
- $\Delta u = \partial_{x_1 x_1} u + \ldots + \partial_{x_n x_n}$

- for n=3 curl$\mathbf{F} = \begin{pmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{pmatrix}$.

PDE from modelling

1. Simple transport: consider fluid flowing at a constant rate $c > 0$, in a horizontal pipe in the x-direction. $u(t,x)$ = concentration of a substance

$M = \int_0^b u(t,x)\mathrm{dx}$=amount of substance in $[0,b]$ at time t




at time $t+h$, the same molecule moved to the interval [ch,b+ch]

Figure 2: L2F2

From figure 2, we can see that

$$\int_b^0 u(t,x)\,\mathrm{dx} = \int_{b+ch}^{ch} u(t+h,x)\,\mathrm{dx} \tag{4}$$

Differntiate it wrt $b$,

$$u(t,b) = u(t+h, b+ch) \tag{5}$$

Take derivative in $h$,

$$0 = u_t + cu_x \tag{6}$$