L5

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$$\begin{pmatrix} \frac{\partial \alpha_1}{\partial \tau}(\gamma(\tau)) & \frac{\partial a_1}{\partial s}(\psi(\tau)) \\ \frac{\partial \alpha_2}{\partial \tau}(\gamma(\tau)) \end{pmatrix} \neq 0 \text{ forall } \tau \tag{1}$$

equation 1 implies that $\gamma(\tau)$ crosses transversely the characteristics.

Theorm 0.1. $\Omega s \subset \mathbb{R}^2$ domain, $\gamma : I \mapsto \Gamma$, with $a, b, c \in C^1$, $\gamma \in C^1$, $|\gamma'(\tau) \neq 0|, \tau \in I$, $a^2 + b^2 \neq 0$, given $f \in C^1(I)$, then

$$\begin{cases} a(x, y) u_x + b(x, y) u_y = c(x, y) + (d(x, y) u(x, y))^* \\ u(\gamma(\tau)) = f(\tau) \end{cases} \tag{2}$$

has a solution if equation 1 was satisfied.

How do we solve $\frac{d}{ds} u(\alpha(s)) = c(\alpha(s)) + dd(\alpha(s)) u(\alpha(s))$? Use $y' = c + dy$.

Example 0.1. Consider system

$$\begin{cases} -yu_x + xu_y = 4xy, & x > 0, y \in \mathbb{R} \\ u(x, 0) = f(x), & x > 0 \end{cases} \tag{3}$$

$\alpha_1'(\tau, s) = -y = -\alpha_2(\tau, s)$, $\alpha_1(\tau, 0) = \tau$,
$\alpha_2'(\tau, s) = \alpha_1(\psi, s)$, $\alpha_2(\tau, 0) = 0$, then we have

$$\alpha' = \frac{d\alpha}{ds} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \alpha \tag{4}$$

Then we have

$$\alpha(s, \tau) = \begin{pmatrix} \cos(s) & -\sin(s) \\ sin(s) & \cos(s) \end{pmatrix} (\tau, 0) \tag{5}$$

We differentiate the equation above,

$$\alpha'(s, \tau) = \begin{pmatrix} -\sin(s) & -\cos(s) \\ cos(s) & -\sin(s) \end{pmatrix} \begin{pmatrix} \tau & 0 \end{pmatrix} \tag{6}$$

We find $\alpha_1(s, \tau) = \tau \cos(s)$, $\alpha_2 = \tau \sin(s)$

$$\frac{d}{ds} u(\alpha(s)) = 4\tau^2 \cos(s) \sin(s) \tag{7}$$

$$u(\alpha(\tau, s)) = u(\alpha(\tau, 0)) + 4\tau^2 \int_0^s \cos(\sigma) \sin(\sigma)\, d\sigma \tag{8}$$

$$= f(\tau) - 2\tau^2 \cos^2(\sigma)\Big|_0^s \tag{9}$$

Step2:

$$u(\alpha(\tau, s)) = f(\tau) - 2\tau^2 \cos^2(s) + 2\tau^2 \tag{10}$$

Step3: $x = \tau \cos(s)$, $y = \tau \sin(s)$ implies $\tau = x^2 + y^2$ and $s = \arctan\left(\frac{y}{x}\right)$ for $x > 0$.

$$u(x, y) = f\left(\sqrt{x^2 + y^2} - 2x^2 + 2x^2 + y^2\right) \tag{11}$$

Example 0.2. Transport, $a > 0$.

$$\begin{cases} u_t + au_x = du, x \in \mathbb{R}, t \in \mathbb{R} \\ u(0, x) = f(x)\, x \in \mathbb{R}, \gamma(\tau) = (0, \tau)^T \end{cases} \tag{12}$$

Step 1: we know the characteristics are :$at - x = k$, $\alpha_1 = t$, $\alpha_2 = x$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 \\ \frac{dx}{ds} = a & x(0) = \tau \end{cases} \implies \begin{cases} t = s \\ x = as + \tau \\ x = at + \tau \end{cases} \tag{13}$$

Step2: $\frac{d}{ds} u(s, a\tau + s) = du(s, as + \tau)$ This is an ODE, if $g(s) = u(s, as + \tau)$.

$$\frac{dg}{ds} = dg \implies e^{ds} g(0) \tag{14}$$

$$u(s, as + \tau) = e^{ds} u(0, \tau) \tag{15}$$

$$u(t, x) = e^{dt} u(0, x - at) = e^{dt} f(x - at) \tag{16}$$

Note: d<0 and f is bounded function.

$$|u(t, x)| \le e^{dt} \max_\tau |f(\tau)| \overbrace{\rightarrow}^{t \to \infty} = 0 \tag{17}$$

d=0: $u(t, x) = f(x - at)\,(a > 0)$. Figure missing

Generilisation

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u) \tag{18}$$

This is a quasilinear equation with $u(\gamma(\tau)) = f(\tau)$.

Step1: 3D characteristic system

$$\begin{cases} \alpha_1'(s) = a(\alpha(s)) \\ \alpha_2'(s) = b(\alpha(s)) \\ \alpha_3'(s) = c(\alpha(s)) \end{cases} \implies \begin{cases} \alpha_1(0) = \gamma_1(\tau) \\ \alpha_2(0) = \gamma_2(\tau) \\ \alpha_3(0) = \gamma_3(\tau) \end{cases} \tag{19}$$

If $a, b, c \in C^1(\Omega)$, then $\exists!$ solution $a_1(s, \tau)$, $\alpha_2(s, \tau)$, $\alpha_3(s, \tau)$ local solution.