Coursework 3

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Question 3 Since $f = 0$,we insert $g(x)$ into $u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$.

$$
\begin{aligned}
u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \\
&= \frac{1}{4}\int_{x-ct}^{x+ct} s\exp\left(-s^2\right) ds \\
&= \frac{1}{4}\left[-\frac{1}{2}\exp\left(-s^2\right)\right]_{x-ct}^{x+ct} \\
&= \frac{1}{8}\left[\exp\left[-(x-ct)^2\right] - \exp\left[-(x+ct)^2\right]\right]
\end{aligned}
$$

If we change the g in interval $(-1,1)$, note that the u value only depends on the g value in $(x-2t, x+2t)$. Also note that $t > 0$, so if $x - 2t > 1 \implies x > 1+2t$ or $x + 2t < -1 \implies x < -1-2t$, then the solution is not affected. So the region affected is $(-1-2t, 1+2t)$.

Question 7 Since Ω is an open set, for x_0 and y_0 there exists r_1, r_2 such that $B(x_0;r_1) \subset \Omega$ and $B(y_0,r_2) \subset \Omega$. By max/min value principle,

$$
\begin{aligned}
\underbrace{\min_{z\in\partial B(x_0,r_1)} u(z)}_{\text{denote A}} \le u(x_0) \le \underbrace{\max_{z\in\partial B(x_0,r_1)} u(z)}_{B} \\
\underbrace{\min_{z\in\partial B(y_0,r_2)} u(z)}_{C} \le u(y_0) \le \underbrace{\max_{z\in\partial B(y_0,r_2)} u(z)}_{D}
\end{aligned}
$$

And if there is a $=$ rather than $<$, then u is constant within the ball, and one can show that u is constant throughout Ω, since $\Delta u = 0$ in Ω, and $\nabla u = \vec{0}$ in the ball, then $u(x_0) + u(y_0) = M$, and any x, y satisfy $u(x) + u(y) = M$. Now let's prove the case when where isn's a equality.

Theorm 1 (Higher dimensional Intermediate Value theorem)**.** If S is a path-connected subset of $\mathbb{R}^n$, and $u : S \to \mathbb{R}$ is continuous. If $a, b \in S$ and

$$
u(a) < t < u(b) \tag{1}
$$

Then there exist a point $c \in S$ such that $f(c) = t$

This theorem was proofed in `http://www.math.toronto.edu/courses/mat237y1/20189/notes/Chapter1/S1.5.html`. Also A ball is path-connected.

Now we denote $\varepsilon = \min\{u(y_0) - C, D - u(y_0), u(x_0) - A, B - u(x_0)\}$. And choose $E \in (0, \varepsilon)$.

Clearly $u(x_0) - E \in (A, B)$, $\exists x$ such that $u(x) = u(x_0) - E$.

Similarly $u(y_0) + E \in (C, D)$, $\exists y$ such that $u(y) = u(y_0) + E$.

We have $u(x) + u(y) = u(x_0) - E + u(y_0) + E = M$. And there are uncountably infinity number in $(0, \varepsilon)$. And each number correspond to a different set of x, y so there is infinitely many pairs $(x, y) \in \Omega \times \Omega$ such that $u(x) + u(y) = M$. QED

Question 9: (a):we define the function $\phi(r) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u d\sigma$. Where ω_n is the volume of an n dimensional unit sphere. Clearly the $n\omega_n r^{n-1}$ is the surface area of $\overline{B(x,r)}$. Clearly $\lim_{r\to 0} \phi(r) = u(x)$, and $\phi(r) \equiv \fint_{\partial B(x,r)} u d\sigma$. Now we compute $\phi'(r)$.

$$\begin{aligned}
\phi(r) &= \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u(\sigma)\, d\sigma \\
\implies \phi(r) &= \frac{1}{n\omega_n} \int_{\partial B(0,1)} u(x + r\omega)\, d\omega \\
\implies \phi'_r &= \frac{1}{n\omega_n} \int_{\partial B(0,1)} \nabla u \cdot \omega d\omega \\
&= \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \eta} d\sigma \\
&= \frac{1}{n\omega_n r^{n-1}} \int_{B(x,r)} \Delta u dy
\end{aligned}$$

But we know that $\Delta u \le 0$, so

$$\begin{aligned}
\phi'(r) &= \frac{1}{n\omega_n r^{n-1}} \int_{B(x,r)} \Delta u dy \\
&\le 0 \\
&\implies \phi(0) \ge \phi(r)\, \forall r > 0 \\
&\implies u(x) \ge \fint_{\partial B(x,t)} u d\sigma
\end{aligned}$$

(b): We will show that if $x_0 \in \Omega$ is a minimum, then u is a constant in Ω and $\partial\Omega$.

First, we prove that $u(x) \ge \fint_{B(x,r)} u dy$.

$$\begin{aligned}
\frac{1}{\omega_n r^n} \int_{B(x,r)} u dy &= \int_0^r \int_{\partial B(x,s)} u d\sigma ds \\
&\le \frac{1}{n\omega_n} \int_0^r n\omega_n s^{n-1} u(x)\, ds \\
&= \frac{1}{\omega_n r^n} \omega_n r^n u(x) \\
&= u(x)
\end{aligned}$$

Then suppose $x_0 \in \Omega$ the point where u is minimum. Then $u(x) \ge u(x_0)\, \forall x \in \Omega$. We first take $B(x_0, r_0) \subset \Omega$. Suppose there exist point z such that

$u(z) > u(x_0)$, then $u(z) - u(x) = \epsilon > 0$. Since $u \in C^2(\Omega)$, there exist δ such that if $x \in B(z,\delta) \subset \Omega$, then $|u(x) - u(z)| < \frac{\epsilon}{2}$. This implies $u(x) - u(x_0) > \frac{\epsilon}{2} \forall x \in B(z,\delta)$. We now choose $\delta' < \delta$ such that $B(z,\delta') \subset B(x_0,r_0)$. And we compute $\frac{1}{|B(x_0,r_0)|}\int_{B(x_0,r_0)} u(y)\,dy$.

$$\begin{aligned}
\frac{1}{|B(x_0,r_0)|}\int_{B(x_0,r_0)} udy &= \frac{1}{|B(x_0,r_0)|}\left[\int_{B(x_0,r_0)/B(z,\delta')} \underbrace{u(y)}_{\geq u(x_0)}\, dy + \int_{B(z,\delta')} \underbrace{u(y)}_{> u(x_0)+\frac{\epsilon}{2}}\, dy\right] \\
&> \frac{1}{|B(x_0,r_0)|}\int_{B(x_0,r_0)} udy
\end{aligned}$$

This contradicts the proposition we proved in part (a). So there is no point z in the ball such that $u(z) > u(x_0)$, and we know $u(z) \geq u(x_0) \forall z \in B(x_0,r_0)$, so $u(z) = u(z_0)$

Now we consider an arbirary point $q \in \Omega$, we can find $y_0, \ldots, y_n$ and balls such that $y_j \in \overline{B(y_{j-1})} \subset \Omega$, $j = 1, \ldots N$. (Note that x_0 is the minimum point we have defined above). And also $y_n = q$, $y_0 = x_0$. Using the same argument as above, we can show that $u = u(x_0)$ on all of $B(y_j)$.

So if we find a minimum x_0 in Ω, then u is constant in Ω. But $u \in C\left(\overline{\Omega}\right)$, so $u(x) = u(x_0) \forall x \in \overline{\Omega}$

(c): we consider the function $u - v$, clearly $u - v \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$. and $\Delta(u - v) \leq 0$ in Ω. We also know that $v \leq u$ on $\overline{\Omega}$, so $u - v \geq 0$ on $\overline{\Omega}$. but $\min_{\overline{\Omega}}(u - v) = \min_{\partial\Omega}(u - v) \geq 0$. So $u - v \geq 0 \implies u \geq v$ in $\overline{\Omega}$.