# L3

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Tools from calculus: Laplace equation:

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \subset \mathbb{R}^n \\ u(x) = 0 \text{for} x \in \partial\Omega \end{cases} \tag{1}$$

In general $\Omega$ is an open set, and $\partial\Omega$ is boundary set, it has some regularity.

**Definition 0.1.** $\Omega$ is a $C^1$ domain, if $\forall x \in \partial\Omega$, there exists a system of coordinates $(y_1, \ldots, y_{n-1}, y_n) \equiv (y', y_n)$, where$y'$ is the vector contianing all $y$s. with origin at $x$, a ball $B(x)$ around x and a function $\varphi$ in a neighbourhood $N \subset \mathbb{R}^{n-1}$ of $y' = 0'$ such that $\varphi$ is $C^1$ in the neighbourhood , $\varphi(0') = 0$ and two things happened.

1. $\partial\Omega \cap B(x) = \{(y', y_n) : y_n = \varphi(y'), y' \in \mathcal{N}\}$
2. $\Omega \cap B(x) = \{y', y_n : y_n > \varphi(y')\, y' \in \mathcal{N}\}$

**Remark.** 1 says that locally, $\partial\Omega$ is the graph of the $C^1$ function . 2 says that locally $\Omega$ lies on one side of the graph of $\varphi$.

**Remark.** A $C^1$ domain does not have corners, and the tangent line (n=2), the tangent plane (n=3) is always well defined.

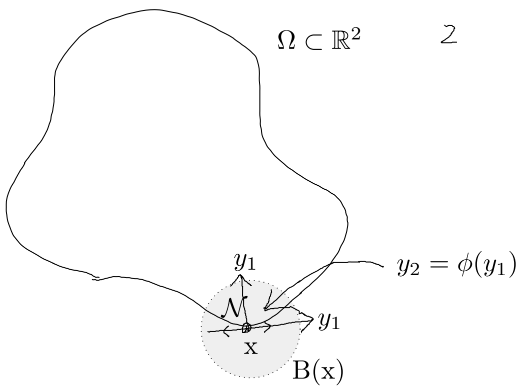


Figure 1: L3F1

If $\varphi \in \mathbb{C}^k \implies \mathbb{C}^k - domain\,(c^{\infty}\text{smooth domain})$
If $\varphi \in \mathrm{Lip} \implies$ Lipschitz domain

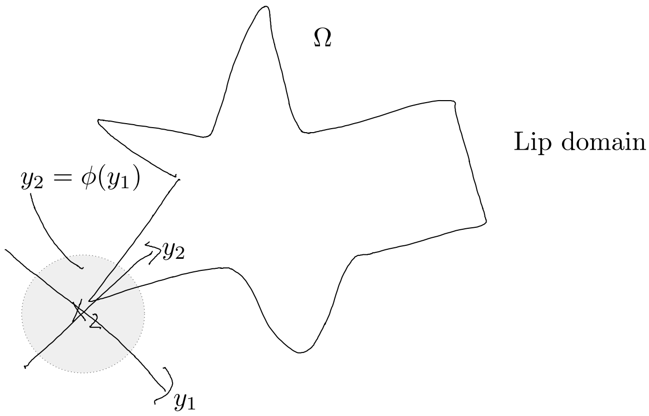


Figure 2: L3F2

Integration by parts

$\Omega \in \mathbb{R}^n$ is $\mathbb{C}^1$, take a vector fields $F = (F_1, \ldots F_n) : \Omega \mapsto \mathbb{R}^n, F \in C^1(\Omega)$, we have Gauss divergence theorem.

$$\int_\Omega \mathrm{div} F dx = \int_{\partial\Omega} F \cdot v d\sigma \tag{2}$$

Where $\nu$ is the outer normal vector. and $d\sigma = \sqrt{1 + |\nabla\varphi(y')|}dy'$ it is the surface measure locally defined.

The consequences are : Take $v \cdot F$ where $v \in C^1(\Omega)$ (scalar).

$$\int_\Omega \mathrm{div}\,(vE) = \int_{\partial\Omega} vF \cdot \nu \tag{3}$$

$$\int_\Omega \mathrm{div}\,(vF) = \int_\Omega v \mathrm{div} F + \int_\Omega \nabla v \cdot F = \int_{\partial\Omega} vF \cdot \nu \tag{4}$$

Special case $F = \nabla u$

$$\mathrm{div} \nabla u = \Delta u \tag{5}$$

$$\int_\Omega v\Delta u = -\int_\Omega \nabla v \cdot \nabla u + \int_{\partial\Omega} v\partial\nu u \tag{6}$$

1. $\nu = 1$: this is the Newmann boundary condition.

$$\int_\Omega \Delta u = \int_{\partial\Omega} \partial_\nu u \tag{7}$$

2. $v = u$

$$\int_\Omega u\Delta u = -\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u\partial_\nu u \tag{8}$$

Two useful theorems

**Theorm 0.1** (ODES)**.** Fix $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^n$ $a, b > 0$, and define

$$R = \{(t, y) : t_0 \leq t \leq t_0 + a, |y - y_0| \leq b\} \tag{9}$$

Consider the ODE

$$y'(t) = f(t, y(t)), y(t_0) = y_0 \tag{10}$$

Where f is a ctn on R and uniformly Lipschitz in y ($\exists L > 0 : |f(t, y_1) - f(t, y_2)| \leq L(y_1 - y_2) \forall t_1, y_1, y_2$ (with maximum equal to $M \geq 0$ in $\mathbb{R}$) Then (ODE) has a unique solution $y(t)$ defined on $[r_0, t_0 + T]$ where $T = \min\left\{a, \frac{b}{M}\right\}$

**Theorm 0.2** (Inverse function theorem)**.** Let $F : R^n \mapsto \mathbb{R}^n$ be $C^1$ assume $DF(a)$ is invertable for some $a \in \mathbb{R}^n$, Where $DF$ is a matrix of $[\partial_{x_i} F_j]$. for some $a \in \mathbb{R}^n$ let $b = F(a)$, then

1. $\exists\, U, V$ open in $\mathbb{R}^n$ such that $a \in U, b \in V$ F is bijecytion on $U$, $F(u) = V$

2. If G is the inverse of F in V(it exists by 1) defined by $G(F(x)) = x$ then $G \in C^1(v)$ Roughly: a $C^1$ mapping F is invertable and in a neighbourhood of a point $a \in \mathbb{R}^n$ at which the matrix $DF(a)$ is invertable.

Consequence: write the equation $Y = F(x)$ componentwise $Y_i = F_i(x_1, \ldots x_n)$, the system can be solved for $x_1, \ldots, xn$ in terms of $y_1, \ldots, y_n$ if we restrict $x$ and $y$ to a small neighbourhood of $a$ and $b$ .

The solutions are unique and $C^1$.