L4

Hanwen Jin

November 24, 2019

1 Method of characteristics

$$a\left(x,y\right)u_x+b\left(x,y\right)u_y=c(x,y) \tag{1}$$

$x,y\in\Omega s\subset\mathbb{R}^2$

Transport $=u_t+au_x=0$. look at $\mathbf{v}=\left(a,1\right)^t$

$au_x+u_t=\mathbf{v}\cdot\nabla_{u,t}u=0$

∇u perpendicular to $\mathbf{v}$, also, ∇u is perpendicular to level lines of u on which u is constant $\implies$ u is a constant on lines parallel to v.

Figure missing

consider $u(t,k+at)$, $\frac{d}{dt}u(t,k+at)=(u_t+au_x)(t,k+at)=0$.

Step2: $u(t,k+at)=u(0,k)$ which is usually assigned

Step3: $u(t,x)=u(0,at-x)$ if $u\left(0,x\right)=f\left(x\right)$ given , then $u\left(t,x\right)=f\left(at-x\right)$.

Figure missing

How to assign an initial daturn? Assign it on a curve $\gamma\left(\tau\right)$, $\tau\in\mathbb{R}$ such that $\gamma\left(\tau\right)$intercept the characteristics curves only once for each k.

In general, $a(x,y)u_x+b(x,y)u_y=c(x,y)$, $x,y\in\Omega$, $\gamma\left(\tau\right)=\left(\gamma_1\left(\tau\right),\gamma_2\left(\tau\right)\right)^t$.

Figure missing

Assumption:

1. $a,b,c\in C^1\left(\Omega\right)a^2+b^2\neq0\forall x,y$.
2. $fs\subset C^1$
3. $\gamma\in C^1$, $|\gamma'|\neq0$.

Procedure

1. Compute characteristic curves
2. Solve along characteristic curves
3. Reconstruct solutions (if possible)

Step 1: they are curves $\alpha_1\left(s\right)=\left(a_1\left(s\right),\alpha_2\left(s\right)\right)^T$. Such that $u\left(\alpha_1\left(s\right),\alpha_2\left(s\right)\right)$ satiefies

$$\frac{du\left(\alpha\left(s\right)\right)}{ds}=\alpha_1\left(s\right)u_x+\alpha_2'\left(s\right)u_y:=a\left(\alpha\left(s\right)\right)u_x+b\left(\alpha\left(s\right)\right)u_y=c\left(\alpha\left(s\right)\right) \tag{2}$$

Definition 1.1 (Characteristic system)**.** Characteristic system was defined as

$$\begin{cases} \alpha_1'(s) = a(\alpha(s)) \\ \alpha_2'(s) = b(\alpha(s)) \end{cases} \tag{3}$$

Initial condition wwas defined as

$$\begin{cases} \alpha_1'(s) = \gamma_1(\tau) \\ \alpha_2'(0) = \gamma_2(\tau) \end{cases} \tag{4}$$

By ODE, $\exists! a, b$ are C^1, $\gamma \in C^1$, $\exists!$ local solution for characteristic systems.
Step2: Solve on characteristics

$$\begin{aligned} \frac{d}{ds} u(\alpha(s)) = c(\alpha(s)) \, &\text{integtrate in s} \\ u(\alpha(s)) - u(\alpha(0)) = &\int_0^s c(\alpha(\sigma)) \, d\sigma \\ u(\alpha(s)) - f(\tau) = &\int_0^s c(\alpha(\sigma)) \, d\sigma \end{aligned}$$

Precisely

$$u(\alpha(\tau, s)) = f(\tau) + \int_0^s c(\alpha(\tau, \sigma)) \, d\sigma \tag{5}$$

Step3: Reconstruct solution

$$x = \alpha_1(\tau, s), y = \alpha_2(\tau, s) \tag{6}$$

I want this to be invertable near $\gamma(\tau)$, or $s = 0$.
Implicit function theorem tells me that the determinant of Jacobian

$$\det \frac{\partial \alpha}{\partial(\tau, s)}\bigg|_{s=0} \neq 0 \tag{7}$$

, then τ is invertable.