L6

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Step2/3:

$$\begin{aligned} x &= \alpha_1(\tau, s) \\ y &= \alpha_2(\tau, s) \\ (u)_z &= \alpha_3(\tau, s) \end{aligned}$$

We invert

$$\begin{aligned} \tau &= T(x, y) \\ s &= S(x, y) \\ u(x, y) &= \alpha_3(T(x, y), S(x, y)) \end{aligned}$$

Call $u(x, y) = Z(T(x, y), S(x, y))$

Lecture 6:

We have $a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u)$

$$\begin{cases} \alpha_1(d) = a(\alpha(s)), \alpha_1(0) = \gamma_1(\tau) \\ \alpha_2'(s) = b(\text{`}a(s)), \alpha_2(0) = \gamma_2(\tau) \\ \alpha_3'(s) = c(\alpha(s)), \alpha_3(0) = f(\tau) \end{cases} \tag{1}$$

Step 2/3: 'Inverting' $x = \alpha_1(s, \tau)$, $y = \alpha_2(s, \tau)$, $z - \alpha_3(s, \tau)$. We have $s = S(x, y)$, $\tau = T(x, y)$, $u(x, y) = \alpha_3(S(x, y), T(x, y))$.

Notation: $U(x, y) = Z(S(x, y), T(x, y))$

Check characteristic system: $\frac{\partial U}{\partial x} = \frac{\partial Z}{\partial S}\frac{\partial S}{\partial x} + \frac{\partial Z}{\partial \tau}\frac{\partial T}{\partial x}$, $\frac{\partial U}{\partial y} = \frac{\partial Z}{\partial s}\frac{\partial S}{\partial y} + \frac{\partial Z}{\partial \tau}\frac{\partial T}{\partial y}$

then we have

$$au_x + bu_y = \frac{\partial Z}{\partial S}\left(a\frac{\partial S}{\partial x} + b\frac{\partial S}{\partial y}\right) + \frac{\partial Z}{\partial \tau}\left(a\frac{\partial T}{\partial x} + b\frac{\partial T}{\partial y}\right) \tag{2}$$

since $x = \alpha_1(s, t)$ $y = \alpha_2(s, \tau)$

$\frac{\partial x}{\partial s} = a(\alpha(s))$, $\frac{\partial y}{\partial s} = b(\alpha(s))$

Plug in equation 2: $a\frac{\partial S}{\partial x} + b\frac{\partial S}{\partial y} = \frac{\partial X}{\partial S}\frac{\partial S}{\partial x} + \frac{\partial Y}{\partial S}\frac{\partial S}{\partial y}$

Equation 2 gives $au_x + bu_y = \frac{\partial Z}{\partial s} = c \implies u$ satisfies the PDE

Theorm 0.1. given a, b, c C^1 function, γ C^1 curve: $|\gamma'(\tau) \neq 0|$ and $f \in C^1$, $a^2 + bb^2 \neq 0$ and let $\det\begin{pmatrix} a(\gamma(\tau), f(\tau)) & b(\gamma(\tau) f(\tau)) \\ \gamma_1'(\tau) & \gamma_2'(\tau) \end{pmatrix} \neq 0 \forall \tau$,

Then there exists a unique *local* C^1 solution defined in a neighbourhood of $\gamma(z)$.

Recall: $u_t + au_x = 0$ is the linear transport equation. In multidimension: let $u = u(t,x) : (0,\infty) \times \Omega \mapsto \mathbb{R}^n, \Omega s \subset \mathbb{R}^n$

Condiser : for a concentration $\rho = \rho(t,x) : (0,\infty) \times \Omega \mapsto \mathbb{R}$

$$PDE\,(IVT) \begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \rho(0,x) = g(x) \end{cases} \tag{3}$$

Case1: $\nabla u = 0$ (Incompressible flow)

pde $\implies \partial_t \rho + u \cdot \nabla \rho = 0$

Figure missing

Incompressible: $|\{g = 1\}| = |\{\rho(t = 10) = 1\}|$ Where $\{\}$ denote the measure, the equality implies they have the same measure.

Other point of view: look at particle

Figure missing

$$\begin{cases} \frac{dx}{dt} = u(t,x) \\ x(0,x_0) = x_0 \end{cases} \tag{4}$$

$$\frac{d}{dt}\rho(t, x(t,x_0)) = [\partial_t \rho + \dot{x} \cdot \nabla \rho](r, x(t,x_0)) \tag{5}$$

$$\text{use CS} = [\partial_t \rho + u \cdot \nabla \rho](t, X(t,x_0)) \overbrace{=}^{\text{by PDE}} 0 \tag{6}$$

ρ is constant on characteristics $\implies \rho(t, X(t,x_0)) = \rho\left(0, \overbrace{X(0,x_0)}^{=x_0}\right) = g(x_0) \implies \rho(t,x_0) = g\left(X\left(\overbrace{0}^{final}; \overbrace{t}^{\text{initial time}}, x\right)\right)$ $X(0;t,x)$ is the inverse map of $X(t,X_0) = X(t;0,x_0)$

Case 2: $\nabla \cdot u \neq 0$

$$\partial_t \rho + u \cdot \nabla \rho = -\rho \cdot \nabla u \tag{7}$$

Then $\frac{d}{dt}\rho(t, x(t,x_0)) = -\rho(t, X(t,x_0)) \nabla \cdot u(t, X(t,x_0))$

It is like $\dot{y} = -ay$

$\rho(t, X(t,x_0)) = \rho(0,x_0) \exp\left(-\int_0^t \nabla \cdot u(\tau, x_0)\, d\tau\right)$.