L8

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Conservation law

$$\begin{cases} u_t + uu_x = 0, t \in (0, \infty), x \in \mathbb{R} \\ u(0, x) = f(x), x \in \mathbb{R} \end{cases} \tag{1}$$

Characteristic system: $x - tf(\tau) = \tau$, $\frac{d}{dt}(u(t, x - tf(\tau))) = 0$

$u(t, x - tf(\tau)) = f(\tau)$. Can we invert $x - tf(\tau) = \tau$? If we can then $\implies \tau = f(x, t)$

Implicit function theorem : Assume $F \in C^1(\mathbb{R}^3)$, $F = F(t, x, \tau) = 0$, suppose at some point (t_0, x_0, τ_0) , we know that $\frac{\partial F}{\partial \tau} \neq 0$, $\implies \exists g$, such that $\tau = g(x, t), g \in C^1$. In a neighbourhood of t_0, x_0.

Inversion is possible in the union of all neighbourhood of point where above is statified.

In our case, $F(t, x, \tau) = x - tf(\tau) = \tau$. Inversion set$=\left\{(x, t, \tau) : \frac{\partial F}{\partial \tau} \neq 0\right\}$.

$$\frac{\partial F}{\partial \tau} \neq 0 \iff 1 + tf'(\tau) \neq 0 \tag{2}$$

Assume $f \in C^1$, $|f'(x) \leq M| \forall x$ for some $M > 0$. if $t < \frac{1}{M}$, then $1 + t(f'(\tau)) \geq 1 - t|f'(\tau)| \geq 1 - Mt \geq 0$.

This is loval existance theorem: if $f \in C^1(\mathbb{R}) : |f'(x)| \leq M \forall x \in \mathbb{R}$, then there exist unique classical solution of Burger's equation.

If $f'(\tau) \geq 0$, $\forall \tau \implies {}_1 + tf'(\tau \geq 1 > 0) \forall t > 0$, the condition of classical solution always fullfill, solution is global in time. In this case,

Theorm 0.1. If $f \in C^1(\mathbb{R})$, then the solution to Burger equation is global iff f is non decreasing. We have proved $\impliedby$, what about $\implies$?.

If f decreases, then there are τ_1, τ_2 such that $f(\tau_1) < f(\tau_2)$. Follow characteristics.

$x - tf(\tau_1) = \tau_1$, u is constant and $= f(\tau_1)$ .. . (1)

$x - tf(\tau_2) = \tau_2$, u is constant and $= f(\tau_2)$ .. . (2)

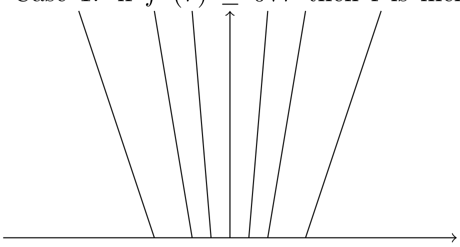
We will hsow that they intercept. so $u = f(\tau_1)$ and $u = f(\tau_2)$, so u is not continuous.

(2)-(1): $t(f(\tau_1) - f(\tau_2)) = \tau_2 - \tau_1$

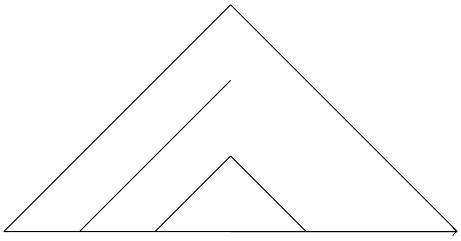
$$t = \frac{\tau_2 - \tau_1}{f(\tau_1) - f(\tau_2)} \implies \text{characteristic intercepts} \tag{3}$$

at this t, which is positive, $\implies$ solution become discontinuous, hencen ot global.

Graphical interpreteationo Case 1: if $f'(\tau) \geq 0 \forall \tau$ then f is increasing.



$t = \frac{1}{f(\tau)}x - \frac{\tau}{f(\tau)}$. Then we have



Case 2: $f'(\tau) < 0$, f is decreasing.
There are points of discontinuity.

What happens to solution in case 2? We have local solution up to $t < \frac{1}{M}$, for f decreasing, $f'(\tau) \leq M$. $Q : t \to \frac{1}{M}$? let τ_* be such that $f'(\tau_*) = -M$.

$x - f(\tau_*) = \tau_*$,

$$u(t,x) = f(\tau) f(x - tf(\tau))$$
$$u(t,x) = f(x - tu(t,x))$$
$$u_x = f'(x - tu) \cdot (1 - tu_x)$$
$$\Longrightarrow u_x = \frac{f'(x-tu)}{1 + tf'(x-tu)}$$

Take $t \to \frac{1}{M}$ on the characteristic, $x - tf(\tau) = \tau_*$.

$$\frac{f'(x-tu)}{1+tf'(x-tu)} \to \frac{-M}{1-tM} \to \infty \text{as } t \to \frac{1}{M} \tag{4}$$

u stops being C.