L7

Hanwen Jin

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$\partial_t + \nabla (\rho u) = 0$, $\rho (0, x) = g (x)$
$u : (-\infty, \infty) \times \Omega \mapsto \mathbb{R}^n$ given , $\rho : (0, \infty) \times \Omega \mapsto \mathbb{R}$

$$\text{CS} : \begin{cases} X (t; x) = U (t, X (t; x)) \\ X (0; x) = x \in \Omega \end{cases} \tag{1}$$

1. $\nabla \cdot u = 0 : \rho (X (t; x)) = g (x) \mapsto \rho (t, x) = g (X (0; t, x))$
2. $\nabla u \neq 0 \quad \rho (t, X (t; x)) = g (x) \exp \left(- \int_0^t \nabla \cdot u (\tau, X (\tau, x)) \, d\tau \right)$. Where the $\exp (\ldots)$ is the Jacobian.

$$J (x, t) = \det (\nabla X) (x, t) \tag{2}$$

$$\partial_t J (x, t) = J (x, t) \nabla \cdot u (t, X (t; x)) \tag{3}$$

$$J (x, 0) = 1 \tag{4}$$

If $\nabla u = 0 \implies J (x, t) = 1 \forall t$.

0.1 Numerical approximation of PDEs

Consider characteristic system

$$\begin{cases} u_t + c u_x = 0 \\ u (0, x) = f (x) \end{cases} \tag{5}$$

We know $g (\overline{x})$, want to guess $g (\overline{x} + 1)$
Know $g' (x) \implies g (\overline{x} + 1) = g (\overline{x}) + g' (\overline{x}) \times 1$
Recall: given $u (x)$, $u' (x) = \lim_{n \to 0} \frac{u(x+h) - u(x)}{h}$.
Can take $u' (x_0) \approx \frac{u(x+h) - u(x_0)}{h}$.
if $u \in C^2$, $\frac{u(x_0+h) - u(x_0)}{h} = u' (x_0) + \frac{1}{2} u\prime\prime (\xi) h = u' (x_0) + \mathcal{O} (h)$
Where $\xi \in (x_0, x_0 + h)$
$\implies \left| u' \left(x_0 - \frac{u(x_0+h) - u(x_0)}{h} \right) \right| \leq Mh$, this is the order 1 approximation. n is to power.

How to improve? $u (x + h) = u (x) + u' (x) h + \frac{u''(x)}{2} h^2 + \frac{u\prime\prime\prime \xi_1}{6} h^3 \xi_1 \in (x, x + h)$

$u (x - h) = u (x) - u' (x) h + \frac{u\prime\prime(x)}{2} h^2 - \frac{u\prime\prime\prime(\xi_2)}{6} h^3 \xi_2 \in (x - h, x)$, we add them up, and have

$$\frac{u (x + h) - u (x - h)}{2h} = u' (x) + \frac{1}{12} (u\prime\prime\prime (\xi_1) + u\prime\prime\prime (\xi_2)) h^2 \tag{6}$$

if $u \in C^3$, $\left|\frac{u(x+h)-u(x-h)}{2h}\right| \leq Mh^2$, which is 2nd order.

Transport equation :

$$\begin{cases} u_t + cu_x = 0, x \in \mathbb{R}, t > 0 \\ u\left(0, x\right) = f\left(x\right), x \in \mathbb{R} \end{cases} \tag{7}$$

Recall: $u\left(t, x\right) = f\left(x - ct\right)$, u is continuous on line $x - ct = k$, $k \in \mathbb{R}$.

Note: u is transportation of f by $\frac{k}{t}$ to right if $c > 0$, left if $c < 0$

Figure missing

We choose a mesh(t_j, x_i), $j \in N \cup \{0\}, i \in \mathbb{Z}$.

$$\left.\begin{cases} \Delta x = x_{i+1} - x_i \forall i \\ \Delta t = t_{j+1} - t_j \forall j \end{cases}\right\} \text{mesh information} \tag{8}$$

How to compute $U_{ij} = U\left(x_i, t_j\right)$

$$\begin{aligned} U_t\left(t_j, x_i\right) &= \frac{U_{i,j+1} - U_{ij}}{\Delta t} \\ U_x\left(t_j, x_i\right) &= \frac{U_{i+1,j} - U_{ij}}{\Delta x} \end{aligned}$$

u_{ij} depend on $u\left(k, 0\right)$ for $k \in \{i, \ldots, i+j\}$

$$\begin{aligned} \frac{U_{i,j+1} - U_{ij}}{\Delta t} + c\frac{U_{i+1,j} - U_{ij}}{\Delta x} &= 0 \\ \iff u_{i,j+1} = -\sigma u_{i+1,j} + \left(\sigma + 1\right) u_{ij} \end{aligned}$$

Where $\sigma = \frac{c\Delta t}{\Delta i}$ note: the sign of c matters.

Simulations:

1. $c > 0$: very bad 'oscillation' not good approximation.

2. $c < 0$: not too negative: looks ok

3. $c < 0$, very negative, bad approximation.

Case 1: $x - ct > k \implies t = \frac{q}{c}x + \tilde{k}$. This means that u_{ij} has nothing to do with $u()t_{j,x_i}$. Because we are not using characteritics.

Case 2: $c < 0$ Not too negative: $t = \frac{1}{c}x + k$, some inforamtion of characteristics will be used.

Case 3: $c < 0$ and very negative, same problem as $c < 0$.

In case 3, $x_i - xt_j \notin [x_i, x_{i+1}]$, this implies the foot of the characters doesn't belong to the interval $[x_i, x_{i+j}]$.

Case 3 precise c value:

$$\begin{aligned} x_i \leq x_i - ct_j \leq x_{i+j} &\iff 0 \leq -ct_j \leq x_{i+j} - x_i \\ &\iff 0 \leq -cj\Delta t \leq j\Delta x \\ &\iff 0 \leq -c\sigma \leq 1 \\ &- 1 \leq \sigma \leq 0 \end{aligned}$$

sicne $c < 0$, $\implies |\sigma| \leq 1$.

given c, $\frac{\Delta t}{\Delta x} \leq \frac{1}{|c|}$ this means $\frac{\Delta x}{\Delta t} \geq c$

'CFL condition' , -Courant-Friedrichs-Lewy' consition. It is a stability consition.

If $c > 0$, $\frac{u_{ij+1} - u_{ij}}{\Delta t} = c\frac{u_{ij} - u_{i-1j}}{\Delta x}$

Convergence order? 1st order convergence. Improve?

$$\frac{u_{ij} + 1 - u_{ij}}{\Delta t} = c\frac{u_{i+1j} - u_{i-1j}}{2\Delta x} \tag{9}$$

This is always unstable, use inforamtion about initial domain is useless.

The two 1st order shcemes are the good ones. They are called upwind.

Conservation laws

$$u_t + (a(u))_x = 0 \iff u_t + a'(u)\, u_x = 0 \tag{10}$$

For Burgers equaiton, $a(u) = \frac{1}{2}u^2$. We get

$$\begin{cases} u_t + uu_x = 0, t > 0 \\ u(0, x) = f(x), x \in \mathbb{R} \end{cases} \tag{11}$$

Using the method of characteristics

$$\frac{dx}{ds} = u, \frac{dt}{ds} = 1$$
$$x(0) = \tau, t(0) = 0, u(0) = f(\tau)$$
$$x(s) = su + \tau, t(s) = s, u(s) = f(\tau)$$

$$\frac{d}{dt}(u(t, tf(\tau) + \tau)) = u_t + f(\tau)\, u_x \tag{12}$$

$$u(t, f(\tau) + \tau) = u(0, \tau) = f(\tau) \tag{13}$$

$\implies$ f is transported through the lines $x - tf(\tau) = \tau$.

How to invert and solve for τ? Implicit function theorem tells me that If

$$F(x, t, \tau) = 0 \tag{14}$$

Suppose $F \in C^1$, $\frac{\partial F}{\partial \tau} \neq 0$ at (t_0, x_0, τ_0), then one can invert in a small neighbourhood of t_0, x_0, τ_0.

Example 0.1. $F(x, y) = x^2 - y = 0$, solve for x, $\partial_x F = 2x$ this means can solve everwhere except near $(0, 0) : x = \pm\sqrt{y}$ not a function.