Some Topics in Algebraic Combinatorics

Hanzhang Yin

May 2 2022

Outline

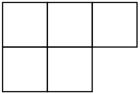
1. Examples of Combinatorics Problems

Outline

1. Examples of Combinatorics Problems
2. Definition of Posets

Outline

1. Examples of Combinatorics Problems
2. Definition of Posets
3. Young tableaux



Example Combinatorial Problems

- n-colorings of 1×5 boards.

Example Combinatorial Problems

- n-colorings of 1×5 boards.

- Squares are colored with letters.

A	B	C	D	E

Example Combinatorial Problems

- n-colorings of 1×5 boards.

- Squares are colored with letters.

A	B	C	D	E

- Rotating the board 180°, gives a new coloring.

Example Combinatorial Problems

- n-colorings of 1×5 boards.

$$\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline \end{array}$$

- Squares are colored with letters.

$$\begin{array}{|c|c|c|c|c|}\hline A & B & C & D & E \\ \hline \end{array}$$

- Rotating the board 180°, gives a new coloring.

$$\begin{array}{|c|c|c|c|c|}\hline A & B & C & D & E \\ \hline \end{array} \xrightarrow{\quad 180^\circ \quad} \begin{array}{|c|c|c|c|c|}\hline E & D & C & B & A \\ \hline \end{array}$$

Example Combinatorial Problems

- n-colorings of 1×5 boards.

$$\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline \end{array}$$

- Squares are colored with letters.

$$\begin{array}{|c|c|c|c|c|}\hline A & B & C & D & E \\ \hline \end{array}$$

- Rotating the board 180°, gives a new coloring.

$$\begin{array}{|c|c|c|c|c|}\hline A & B & C & D & E \\ \hline \end{array} \xrightarrow{\quad 180^\circ \quad} \begin{array}{|c|c|c|c|c|}\hline E & D & C & B & A \\ \hline \end{array}$$

- We define two colorings are "the same" if rotating one results in the other.

Example Combinatorial Problems

- n-colorings of 1×5 boards.

$$\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline \end{array}$$

- Squares are colored with letters.

$$\begin{array}{|c|c|c|c|c|}\hline A & B & C & D & E \\ \hline \end{array}$$

- Rotating the board 180°, gives a new coloring.

$$\begin{array}{|c|c|c|c|c|}\hline A & B & C & D & E \\ \hline \end{array} \xrightarrow{180^\circ} \begin{array}{|c|c|c|c|c|}\hline E & D & C & B & A \\ \hline \end{array}$$

- We define two colorings are "the same" if rotating one results in the other.
- Special Case

$$\begin{array}{|c|c|c|c|c|}\hline A & B & C & B & A \\ \hline \end{array} \xrightarrow{180^\circ} \begin{array}{|c|c|c|c|c|}\hline A & B & C & B & A \\ \hline \end{array}$$

n-Colorings of 1×5 Board

- Goal: Count the number of unique colorings with 180° flips.

n-Colorings of 1×5 Board

- Goal: Count the number of unique colorings with 180° flips.
- Case 1.

$$\boxed{n}\boxed{n}\boxed{n}\boxed{n}\boxed{n} \longrightarrow n^5$$

n-Colorings of 1×5 Board

- Goal: Count the number of unique colorings with 180° flips.
- Case 1.

$$\boxed{n}\boxed{n}\boxed{n}\boxed{n}\boxed{n} \longrightarrow n^5$$

- Case 2.

$$\boxed{n}\boxed{n}\boxed{n}\boxed{1}\boxed{1} \longrightarrow n^3$$

n-Colorings of 1×5 Board

- Goal: Count the number of unique colorings with 180° flips.
- Case 1.

$$\begin{array}{|c|c|c|c|c|}\hline n & n & n & n & n \\ \hline\end{array} \longrightarrow n^5$$

- Case 2.

$$\begin{array}{|c|c|c|c|c|}\hline n & n & n & 1 & 1 \\ \hline\end{array} \longrightarrow n^3$$

- The number of colorings which don't equal their 180° rotation is

$$n^5 - n^3$$

n-Colorings of 1×5 Board

- Goal: Count the number of unique colorings with 180° flips.
- Case 1.

$$\boxed{n}\boxed{n}\boxed{n}\boxed{n}\boxed{n} \longrightarrow n^5$$

- Case 2.

$$\boxed{n}\boxed{n}\boxed{n}\boxed{1}\boxed{1} \longrightarrow n^3$$

- The number of colorings which don't equal their 180° rotation is

$$n^5 - n^3$$

- The number of equivalence class of colorings which don't equal their 180° rotation is

$$\frac{1}{2}(n^5 - n^3)$$

Burnside Lemma:

- Let Y be a finite set and G a subgroup of a symmetric group. For each $\pi \in G$, let

$$\mathrm{Fix}(\pi) = \{y \in Y : \pi(y) = y\},$$

so $\#\mathrm{Fix}(\pi)$ is the number of cycles of length one in the permutation π. Let Y/G be the set of orbits of G. Then

$$|Y/G| = \frac{1}{\#G} \sum_{\pi \in G} \#\mathrm{Fix}(\pi)$$

Burnside Lemma: Example

- $Y = \{\text{colorings of } 1 \times 5 \text{ Board}\} = \{ABCDE, \cdots, ABCBA, \cdots\}$ and $G = \{e, r\}$ where r represents rotation.

Burnside Lemma: Example

- $Y = \{\text{colorings of } 1 \times 5 \text{ Board}\} = \{ABCDE, \cdots, ABCBA, \cdots\}$ and $G = \{e, r\}$ where r represents rotation.
- For any $y \in Y$, $e \cdot y = y$.

$$\mathrm{Fix}(e) = Y$$

$$\#\mathrm{Fix}(e) = n^5$$

Burnside Lemma: Example

- $Y = \{\text{colorings of } 1 \times 5 \text{ Board}\} = \{ABCDE, \cdots, ABCBA, \cdots\}$ and $G = \{e, r\}$ where r represents rotation.
- For any $y \in Y$, $e \cdot y = y$.

$$\text{Fix}(e) = Y$$

$$\#\text{Fix}(e) = n^5$$

- For r, $r \cdot ABCDE = EDCBA$

Burnside Lemma: Example

- $Y = \{\text{colorings of } 1 \times 5 \text{ Board}\} = \{ABCDE, \cdots, ABCBA, \cdots\}$ and $G = \{e, r\}$ where r represents rotation.
- For any $y \in Y$, $e \cdot y = y$.

$$\mathrm{Fix}(e) = Y$$

$$\#\mathrm{Fix}(e) = n^5$$

- For r, $r \cdot ABCDE = EDCBA$

$$\mathrm{Fix}(r) = \{ABCBA, \ldots\}$$

Burnside Lemma: Example

- $Y = \{\text{colorings of } 1 \times 5 \text{ Board}\} = \{ABCDE, \cdots, ABCBA, \cdots\}$ and $G = \{e, r\}$ where r represents rotation.
- For any $y \in Y$, $e \cdot y = y$.

$$\text{Fix}(e) = Y$$

$$\#\text{Fix}(e) = n^5$$

- For r, $r \cdot ABCDE = EDCBA$

$$\text{Fix}(r) = \{ABCBA, \dots\}$$

$$\#\text{Fix}(r) = n^3$$

Burnside Lemma: Example

$$|Y/G| = \frac{1}{\#G} \sum_{\pi \in G} \#\mathrm{Fix}(\pi) = \frac{1}{2}(n^3 + n^5)$$

Burnside Lemma: Example

$$|Y/G| = \frac{1}{\#G}\sum_{\pi\in G}\#\mathrm{Fix}(\pi) = \frac{1}{2}(n^3 + n^5)$$

The number of equivalence class of colorings which don't equal their 180° rotation.

$$\frac{1}{2}(n^3 + n^5) - n^3 = \frac{1}{2}n^5 - \frac{1}{2}n^3$$

Burnside Lemma: Example

$$|Y/G| = \frac{1}{\#G}\sum_{\pi\in G}\#\mathrm{Fix}(\pi) = \frac{1}{2}(n^3+n^5)$$

The number of equivalence class of colorings which don't equal their 180° rotation.

$$\frac{1}{2}(n^3+n^5) - n^3 = \frac{1}{2}n^5 - \frac{1}{2}n^3$$

- The total number of equivalence class is $\frac{1}{2}(n^3+n^5)$
- The total number of rectangle colorings which equal their 180° is n^3
- The number of equivalence class of colorings which don't equal their 180° rotation is $\frac{1}{2}n^5 - \frac{1}{2}n^3$

Burnside Lemma: Example

Let X be the 2×2 chessboard and let it be labeled as

1	2
3	4

$C = \{r, b, y\}$. A typical coloring can be

r	b
y	r

How many ways can we color a 2×2 chessboard with two red squares, one blue square and one yellow square?

Burnside Lemma: Example

Let X be the 2×2 chessboard and let it be labeled as

1	2
3	4

$C = \{r, b, y\}$. A typical coloring can be

r	b
y	r

How many ways can we color a 2×2 chessboard with two red squares, one blue square and one yellow square? 12

we denote Y is the set of all 12 colorings.

Burnside Lemma: Example

There are many possible choices of a symmetry group G, and this will affect when two colorings are equivalent.

- G_1 consists of only the identity permutation (1).

Burnside Lemma: Example

There are many possible choices of a symmetry group G, and this will affect when two colorings are equivalent.

- G_1 consists of only the identity permutation (1).
- $G_2 = \{(1), (12)(34)\}$ is the group generated by a vertical reflection.

Burnside Lemma: Example

There are many possible choices of a symmetry group G, and this will affect when two colorings are equivalent.

- G_1 consists of only the identity permutation (1).
- $G_2 = \{(1), (12)(34)\}$ is the group generated by a vertical reflection.
- $G_3 = \{(1), (23)\}$ is the group generated by a reflection in the main diagonal.

Burnside Lemma: Example

There are many possible choices of a symmetry group G, and this will affect when two colorings are equivalent.

- G_1 consists of only the identity permutation (1).
- $G_2 = \{(1), (12)(34)\}$ is the group generated by a vertical reflection.
- $G_3 = \{(1), (23)\}$ is the group generated by a reflection in the main diagonal.
- $G_4 = \{(1), (1243), (14)(23), (1342)\}$ is the group of all rotations.

Burnside Lemma: Example

There are many possible choices of a symmetry group G, and this will affect when two colorings are equivalent.

- G_1 consists of only the identity permutation (1).
- $G_2 = \{(1), (12)(34)\}$ is the group generated by a vertical reflection.
- $G_3 = \{(1), (23)\}$ is the group generated by a reflection in the main diagonal.
- $G_4 = \{(1), (1243), (14)(23), (1342)\}$ is the group of all rotations.
- G_5 is the group of all rotations and reflections.

Burnside Lemma: Example

There are many possible choices of a symmetry group G, and this will affect when two colorings are equivalent.

- G_1 consists of only the identity permutation (1).
- $G_2 = \{(1), (12)(34)\}$ is the group generated by a vertical reflection.
- $G_3 = \{(1), (23)\}$ is the group generated by a reflection in the main diagonal.
- $G_4 = \{(1), (1243), (14)(23), (1342)\}$ is the group of all rotations.
- G_5 is the group of all rotations and reflections.
- G_6 is the symmetric group of all 24 permutations of Y.

For each of these groups, how many inequivalence classes do we get? For this we can use Burnside Theorem.

Burnside Lemma: Example(G_1)

- G_1 consists of only the identity permutation (1).

Burnside Lemma: Example(G_1)

- G_1 consists of only the identity permutation (1).
- There are 12 colorings in all with two red squares, one blue square, and one yellow square, and all are inequivalent under the trivial group.

Burnside Lemma: Example(G_2)

- $G_2 = \{(1), (12)(34)\}$ is the group generated by a vertical reflection.

Burnside Lemma: Example(G_2)

- $G_2 = \{(1), (12)(34)\}$ is the group generated by a vertical reflection.
- $\#\mathrm{Fix}(1) = 12$, $\#\mathrm{Fix}((12)(34)) = 0$

Burnside Lemma: Example(G_2)

- $G_2 = \{(1), (12)(34)\}$ is the group generated by a vertical reflection.
- $\#\mathrm{Fix}(1) = 12$, $\#\mathrm{Fix}((12)(34)) = 0$

$$|Y/G_2| = \frac{1}{2}(12 + 0) = 6$$

Burnside Lemma: Example(G_2)

- $G_2 = \{(1), (12)(34)\}$ is the group generated by a vertical reflection.
- $\#\text{Fix}(1) = 12$, $\#\text{Fix}((12)(34)) = 0$

$$|Y/G_2| = \frac{1}{2}(12 + 0) = 6$$

- There are 6 inequivalent colorings.

$$\begin{array}{|c|c|}\hline r & r \\ \hline b & y \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline r & b \\ \hline r & y \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline r & y \\ \hline r & b \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline b & y \\ \hline r & r \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline r & b \\ \hline y & r \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline r & y \\ \hline b & r \\ \hline\end{array}$$

Burnside Lemma: Example(G_3)

- $G_3 = \{(1), (23)\}$ is the group generated by a reflection in the main diagonal.

Burnside Lemma: Example(G_3)

- $G_3 = \{(1), (23)\}$ is the group generated by a reflection in the main diagonal.
- $\#\mathrm{Fix}(1) = 12$, $\#\mathrm{Fix}((23)) = 2$

Burnside Lemma: Example(G_3)

- $G_3 = \{(1), (23)\}$ is the group generated by a reflection in the main diagonal.
- $\#\mathrm{Fix}(1) = 12$, $\#\mathrm{Fix}((23)) = 2$

$$|Y/G_3| = \frac{1}{2}(12 + 2) = 7$$

Burnside Lemma: Example(G_3)

- $G_3 = \{(1), (23)\}$ is the group generated by a reflection in the main diagonal.
- $\#\mathrm{Fix}(1) = 12$, $\#\mathrm{Fix}((23)) = 2$

$$|Y/G_3| = \frac{1}{2}(12 + 2) = 7$$

- There are 7 inequivalent colorings.

$$\begin{array}{|c|c|}\hline r & r \\ \hline b & y \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline r & r \\ \hline y & b \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline b & y \\ \hline r & r \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline y & b \\ \hline r & r \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline r & b \\ \hline y & r \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline b & r \\ \hline r & y \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline y & r \\ \hline r & b \\ \hline\end{array}$$

Burnside Lemma: Example(G_4)

- $G_4 = \{(1), (1243), (14)(23), (1342)\}$ is the group of all rotations.
- $\#\mathrm{Fix}(1) = 12$, $\#\mathrm{Fix}((1243)) = 0$, $\#\mathrm{Fix}((14)(23)) = 0$, $\#\mathrm{Fix}((1342)) = 0$

Burnside Lemma: Example(G_4)

- $G_4 = \{(1), (1243), (14)(23), (1342)\}$ is the group of all rotations.
- $\#\mathrm{Fix}(1) = 12$, $\#\mathrm{Fix}((1243)) = 0$, $\#\mathrm{Fix}((14)(23)) = 0$, $\#\mathrm{Fix}((1342)) = 0$

$$|Y/G_4| = \frac{1}{4}(12 + 0 + 0 + 0) = 3$$

Burnside Lemma: Example(G_4)

- $G_4 = \{(1), (1243), (14)(23), (1342)\}$ is the group of all rotations.
- $\#\text{Fix}(1) = 12$, $\#\text{Fix}((1243)) = 0$, $\#\text{Fix}((14)(23)) = 0$, $\#\text{Fix}((1342)) = 0$

$$|Y/G_4| = \frac{1}{4}(12 + 0 + 0 + 0) = 3$$

Burnside Lemma: Example(G_4)

- $G_4 = \{(1), (1243), (14)(23), (1342)\}$ is the group of all rotations.
- $\#\text{Fix}(1) = 12$, $\#\text{Fix}((1243)) = 0$, $\#\text{Fix}((14)(23)) = 0$, $\#\text{Fix}((1342)) = 0$

$$|Y/G_4| = \frac{1}{4}(12 + 0 + 0 + 0) = 3$$

- There are 3 inequivalent colorings.

r	r
y	b

r	r
b	y

r	b
y	r

Burnside Lemma: Example(G_5)

- G_5 is the group of all rotations and reflections. Therefore,

$$G_5 = \{(1), (1243), (14)(23), (1342), (12)(34), (13)(24), (23), (14)\}$$

Burnside Lemma: Example(G_5)

- G_5 is the group of all rotations and reflections. Therefore,

$$G_5 = \{(1), (1243), (14)(23), (1342), (12)(34), (13)(24), (23), (14)\}$$

$$\#\text{Fix}(1) = 12, \#\text{Fix}(14) = 2, \#\text{Fix}(23) = 2$$

- $$|Y/G_5| = \frac{1}{8}(12 + 2 + 2) = 2$$

- There are 2 inequivalent colorings.

r	r
b	y

r	b
y	r

Burnside Lemma: Example(G_6)

- G_6 is the symmetric group of all 24 permutations of X.

Burnside Lemma: Example(G_6)

- G_6 is the symmetric group of all 24 permutations of X.
- There is only one equivalence class.

Poset

- A poset P is a set, also denoted P, together with a binary relation denoted $\leq$ satisfying the following axioms:

Poset

- A poset P is a set, also denoted P, together with a binary relation denoted $\leq$ satisfying the following axioms:
- (Reflexivity) $x \leq x$ for all $x \in P$.

Poset

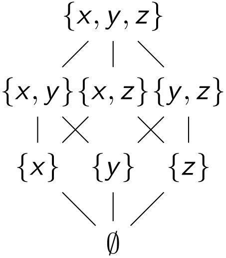
- A poset P is a set, also denoted P, together with a binary relation denoted $\leq$ satisfying the following axioms:
- (Reflexivity) $x \leq x$ for all $x \in P$.
- (Antisymmetry) If $x \leq y$ and $y \leq x$,then $x = y$.

Poset

- A poset P is a set, also denoted P, together with a binary relation denoted $\leq$ satisfying the following axioms:
- (Reflexivity) $x \leq x$ for all $x \in P$.
- (Antisymmetry) If $x \leq y$ and $y \leq x$,then $x = y$.
- (Transitivity) If $x \leq y$ and $y \leq z$,then $x \leq z$.

Poset

- A poset P is a set, also denoted P, together with a binary relation denoted $\leq$ satisfying the following axioms:
- (Reflexivity) $x \leq x$ for all $x \in P$.
- (Antisymmetry) If $x \leq y$ and $y \leq x$,then $x = y$.
- (Transitivity) If $x \leq y$ and $y \leq z$,then $x \leq z$.



Examples of Poset

- E.g. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ with usual ordering.

Examples of Poset

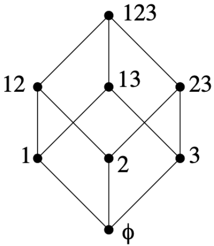
- E.g. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ with usual ordering.
- Assume that $S = \{1, 2, 3\}$ is a 3-element set and P contains all its subsets.

Examples of Poset

- E.g. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ with usual ordering.
- Assume that $S = \{1, 2, 3\}$ is a 3-element set and P contains all its subsets.
- P in this case called a finite boolean algebra of rank 3 and is denoted by $B_{\{1,2,3\}}$.
- If $x, y \in P$,then define $x \leq y$ in P if $x \subseteq y$ as sets.

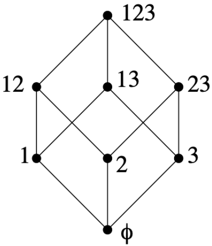
Hasse Diagrams

- Hasse diagram can be a simple way to represent small posets.



Hasse Diagrams

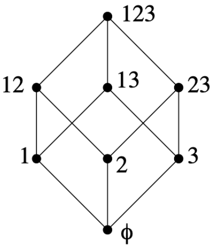
- Hasse diagram can be a simple way to represent small posets.



- If $x < y$ in P (i.e.,$x \leq y$ and $x \neq y$),then y is drawn "above" x.

Hasse Diagrams

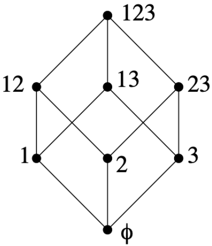
- Hasse diagram can be a simple way to represent small posets.



- If $x < y$ in P (i.e.,$x \leq y$ and $x \neq y$),then y is drawn "above" x.
- An edge is drawn between x and y if y covers x,i.e.,$x < y$ and no element z satisfies $x < z < y$. We then write $x \lessdot y$ or $y \gtrdot x$.

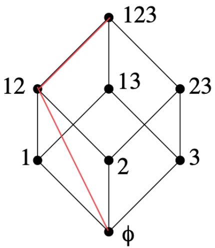
Hasse Diagrams

- Hasse diagram can be a simple way to represent small posets.



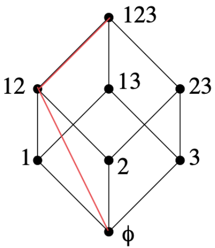
- If $x < y$ in P (i.e.,$x \leq y$ and $x \neq y$),then y is drawn "above" x.
- An edge is drawn between x and y if y covers x,i.e.,$x < y$ and no element z satisfies $x < z < y$. We then write $x \lessdot y$ or $y \gtrdot x$.
- By the transitivity property (P3), all the relations of a finite poset are determined by the cover relations, so the Hasse diagram determines P.

Chain



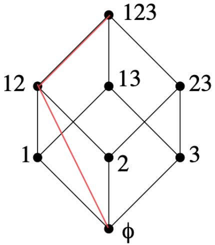
- A chain C in a poset is a totally ordered subset of P, i.e., if $x, y \in C$ then either $x \leq y$ or $y \leq x$ in P.

Chain



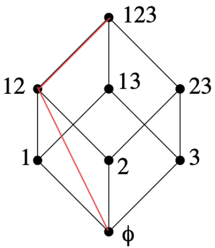
- A chain C in a poset is a totally ordered subset of P, i.e., if $x, y \in C$ then either $x \leq y$ or $y \leq x$ in P.
- E.g. $\{\emptyset, \{1,2\}, \{1,2,3\}\}$ is a chain.

Chain



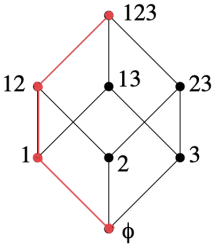
- A chain C in a poset is a totally ordered subset of P, i.e., if $x, y \in C$ then either $x \le y$ or $y \le x$ in P.
- E.g. $\{\emptyset, \{1,2\}, \{1,2,3\}\}$ is a chain.
- A finite chain is said to have length n if it has $n+1$ elements.

Chain



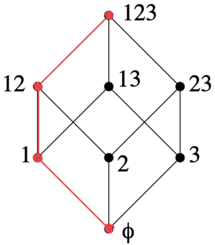
- A chain C in a poset is a totally ordered subset of P, i.e., if $x, y \in C$ then either $x \leq y$ or $y \leq x$ in P.
- E.g. $\{\emptyset, \{1,2\}, \{1,2,3\}\}$ is a chain.
- A finite chain is said to have length n if it has $n+1$ elements.
- $\{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\}$ has length 3.
  $\{\emptyset, \{1\}, \{1,2\}\}$ has length 2.

Chain



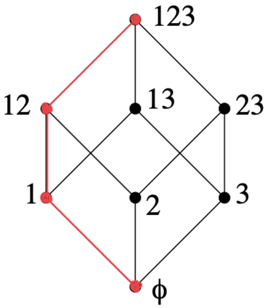
- A chain is maximal if it's contained in no larger chain.

Chain



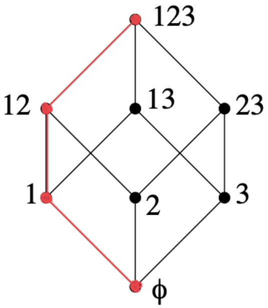
- A chain is maximal if it's contained in no larger chain.
- E.g. $\{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$,
  $\{\emptyset, \{1\}, \{1, 3\}, \{1, 2, 3\}\}$,
  $\{\emptyset, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$ are all maximal and many more.

Chain



- A finite poset is *graded* of rank n if every maximal chain has length n.

Chain



- A finite poset is *graded* of rank n if every maximal chain has length n.
- $P = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ is finite and has rank 3

Partition

- A partition λ of an integer $n \geq 0$ is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of integers $\lambda_i \geq 0$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_{i \geq 1} \lambda_i = n$.

Partition

- A partition λ of an integer $n \geq 0$ is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of integers $\lambda_i \geq 0$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_{i \geq 1} \lambda_i = n$.
- All but finitely many λ_i are equal to 0.

Partition

- A partition λ of an integer $n \geq 0$ is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of integers $\lambda_i \geq 0$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_{i \geq 1} \lambda_i = n$.
- All but finitely many λ_i are equal to 0.
- Only the $\lambda_i > 0$ will be called a part of λ.

Partition

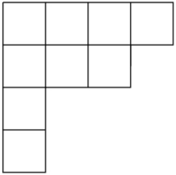
- A partition λ of an integer $n \geq 0$ is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of integers $\lambda_i \geq 0$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_{i \geq 1} \lambda_i = n$.
- All but finitely many λ_i are equal to 0.
- Only the $\lambda_i > 0$ will be called a part of λ.
- If λ is a partition of n, then we denote this by $|\lambda| = n$. Written $\lambda \vdash n$.

Partition

- A partition λ of an integer $n \geq 0$ is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of integers $\lambda_i \geq 0$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_{i \geq 1} \lambda_i = n$.
- All but finitely many λ_i are equal to 0.
- Only the $\lambda_i > 0$ will be called a part of λ.
- If λ is a partition of n, then we denote this by $|\lambda| = n$. Written $\lambda \vdash n$.
- The seven partitions of 5 are (5), $(4, 1)$, $(3, 2)$, $(3, 1, 1)$, $(2, 2, 1)$, $(2, 1, 1, 1)$, and $(1, 1, 1, 1, 1)$.

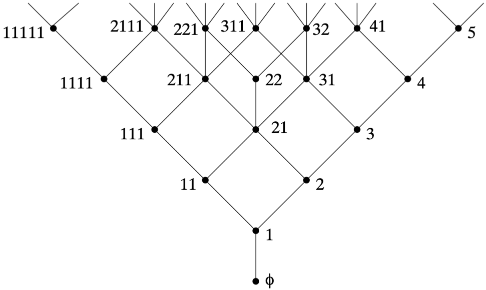
Young Diagram

- The Young Diagram (Diagram) of a partiton λ is a left-justified array of squares, with λ_i squares in the ith row.



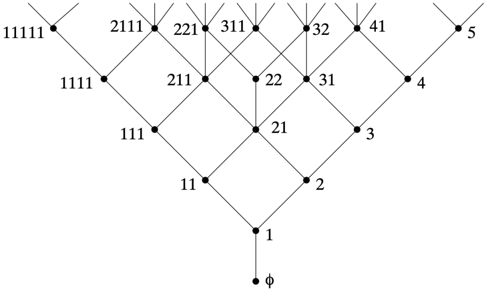
- The example above is the Young diagram of $(4, 3, 1, 1)$.

Hasse Walk



- Hasse diagram of Y has no loops or multiple edges, a walk of length n is specified by a sequence $\lambda_0, \lambda_1, \cdots, \lambda_n$ of vertices of Y .

Hasse Walk



- Hasse diagram of Y has no loops or multiple edges, a walk of length n is specified by a sequence $\lambda_0, \lambda_1, \cdots, \lambda_n$ of vertices of Y .
- A walk in the Hasse diagram of a poset is a *Hasse walk* (or just walk for this section.)

Young tableaux

- There are two types of walk.

Young tableaux

- There are two types of walk.
- Type U (Up):
- $\lambda^i < \lambda^{i+1}$ and $|\lambda^i| = |\lambda^{i+1}| - 1$

Young tableaux

- There are two types of walk.
- Type U (Up):
- $\lambda^i < \lambda^{i+1}$ and $|\lambda^i| = |\lambda^{i+1}| - 1$
- Type D (Down):
- $\lambda^i > \lambda^{i+1}$ and $|\lambda^i| = |\lambda^{i+1}| + 1$

Young tableaux

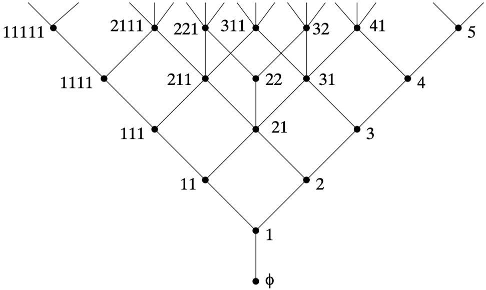
- There are two types of walk.
- Type U (Up):
- $\lambda^i < \lambda^{i+1}$ and $|\lambda^i| = |\lambda^{i+1}| - 1$
- Type D (Down):
- $\lambda^i > \lambda^{i+1}$ and $|\lambda^i| = |\lambda^{i+1}| + 1$
- If the walk W has steps of types $A_1, A_2, \cdots, A_{n-1}, A_n$, respectively, where each A_i is either U or D, then we say that W is of type

$$A_n A_{n-1} \cdots A_2 A_1$$

Young tableaux

- There are two types of walk.
- Type U (Up):
- $\lambda^i < \lambda^{i+1}$ and $|\lambda^i| = |\lambda^{i+1}| - 1$
- Type D (Down):
- $\lambda^i > \lambda^{i+1}$ and $|\lambda^i| = |\lambda^{i+1}| + 1$
- If the walk W has steps of types $A_1, A_2, \cdots, A_{n-1}, A_n$, respectively, where each A_i is either U or D, then we say that W is of type
$$A_n A_{n-1} \cdots A_2 A_1$$
- The reason that the type of a walk is written in the opposite order to that of the walk is because we regard U and D as linear transformations.

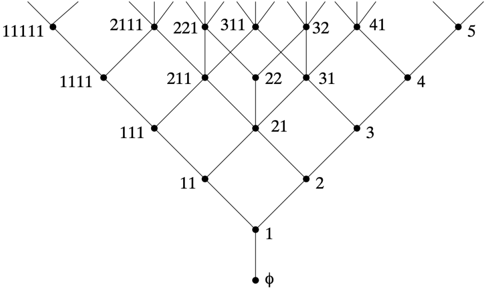
Example



- The walk $\emptyset, 1, 2, 1, 11, 111, 211, 21, 22, 21, 31, 41$ is of type $UUDDUUUUDUU = U^2D^2U^4DU^2$.

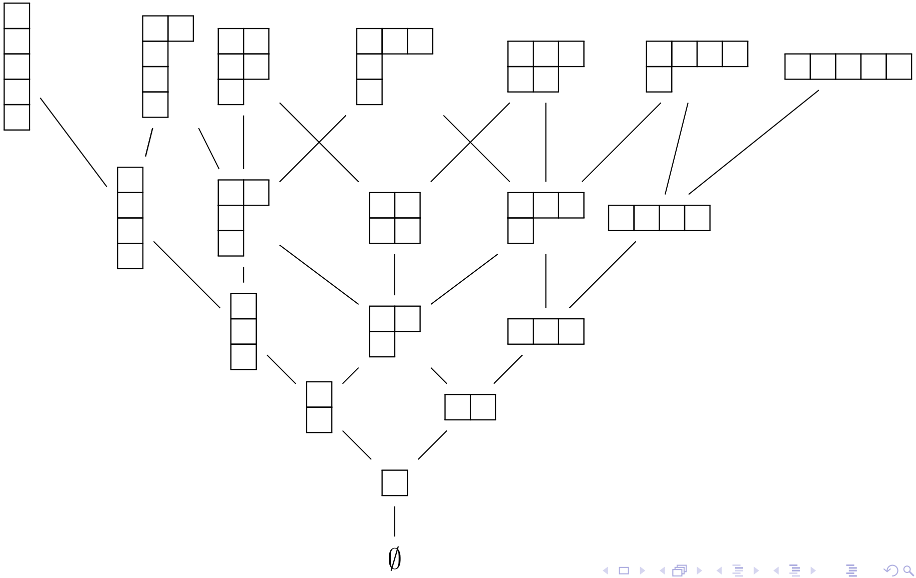
The Walks of Type U^n which begin at $\emptyset$

- The Walks of Type U^n which begin at $\emptyset$ are just saturated chain $\emptyset = \lambda^0 \lessdot \lambda^1 \lessdot \cdots \lessdot \lambda^n$.

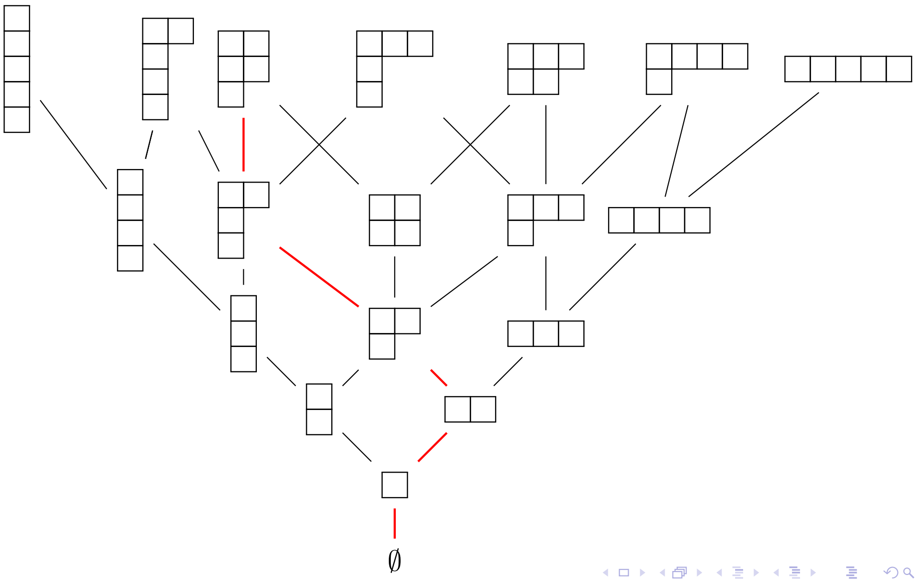


- Goal: Count the number of walks of type U^n.

Young Lattice



An example of U^5



An example of U^5

- Suppose a walk of type U^5 is ended at $\lambda = (221)$.

An example of U^5

- Suppose a walk of type U^5 is ended at $\lambda = (221)$.
- Then this walk can be a sequences of young diagram.

An example of U^5

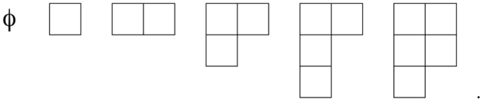
- Suppose a walk of type U^5 is ended at $\lambda = (221)$.
- Then this walk can be a sequences of young diagram.

$$\lambda^0 = \emptyset \lessdot \lambda_1 = (1) \lessdot \lambda^2 = (2) \lessdot \lambda^3 = (21) \lessdot \lambda^4 = (211) \lessdot \lambda^5 = (221)$$

An example of U^5

- Suppose a walk of type U^5 is ended at $\lambda = (221)$.
- Then this walk can be a sequences of young diagram.

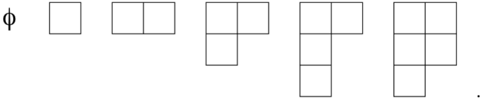
$$\lambda^0 = \emptyset \lessdot \lambda_1 = (1) \lessdot \lambda^2 = (2) \lessdot \lambda^3 = (21) \lessdot \lambda^4 = (211) \lessdot \lambda^5 = (221)$$



An example of U^5

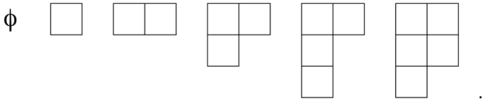
- Suppose a walk of type U^5 is ended at $\lambda = (221)$.
- Then this walk can be a sequences of young diagram.

$$\lambda^0 = \emptyset \lessdot \lambda_1 = (1) \lessdot \lambda^2 = (2) \lessdot \lambda^3 = (21) \lessdot \lambda^4 = (211) \lessdot \lambda^5 = (221)$$



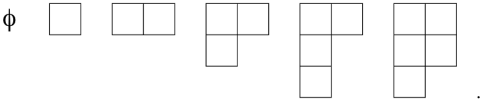
- We can visualize it on the Hasse diagram.

An example of U^5



- We can specify the walk by taking the final diagram and inserting an i into square s if s was added at the ith step.

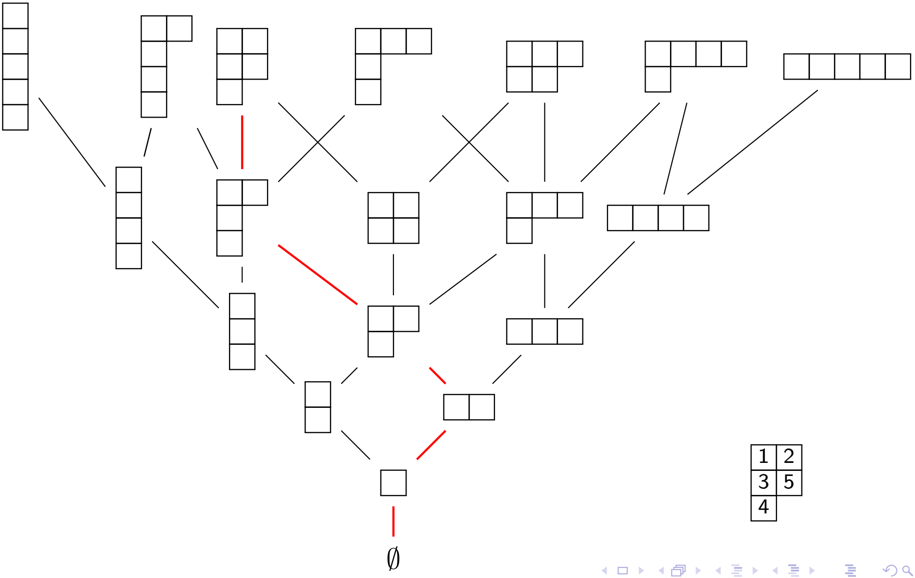
An example of U^5



- We can specify the walk by taking the final diagram and inserting an i into square s if s was added at the ith step.
- Thus the above walk is encoded by the "tableau"

1	2
3	5
4	

An example of U^5



1	2
3	5
4	

Standard Young Tableaux (SYT)

- The standard Young Tableaux consists of the Young diagram D of some partition λ of an integer n, together with number number $1, 2, \ldots, n$ inserted into the squares of D.

Standard Young Tableaux (SYT)

- The standard Young Tableaux consists of the Young diagram D of some partition λ of an integer n, together with number number $1, 2, \ldots, n$ inserted into the squares of D.
- Each number appears once.

Standard Young Tableaux (SYT)

- The standard Young Tableaux consists of the Young diagram D of some partition λ of an integer n, together with number number $1, 2, \ldots, n$ inserted into the squares of D.
- Each number appears once.
- Every row and column is increasing.

1	2
3	4
5	

Standard Young Tableaux (SYT)

- The standard Young Tableaux consists of the Young diagram D of some partition λ of an integer n, together with number number $1, 2, \ldots, n$ inserted into the squares of D.
- Each number appears once.
- Every row and column is increasing.

1	2
3	4
5	

- We call λ the shape of the SYT τ, denoted $\lambda = \mathrm{sh}(\tau)$.

Standard Young Tableaux (SYT)

- The standard Young Tableaux consists of the Young diagram D of some partition λ of an integer n, together with number number $1, 2, \ldots, n$ inserted into the squares of D.
- Each number appears once.
- Every row and column is increasing.

1	2
3	4
5	

- We call λ the shape of the SYT τ, denoted $\lambda = \mathrm{sh}(\tau)$.
- Define f^{λ} the number of SYT of shape λ.

The example of SYT of $\lambda = (2, 2, 1)$

- There are 5 SYT of shape $(2, 2, 1)$, given by

The example of SYT of $\lambda = (2, 2, 1)$

- There are 5 SYT of shape $(2, 2, 1)$, given by

$$
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}
$$

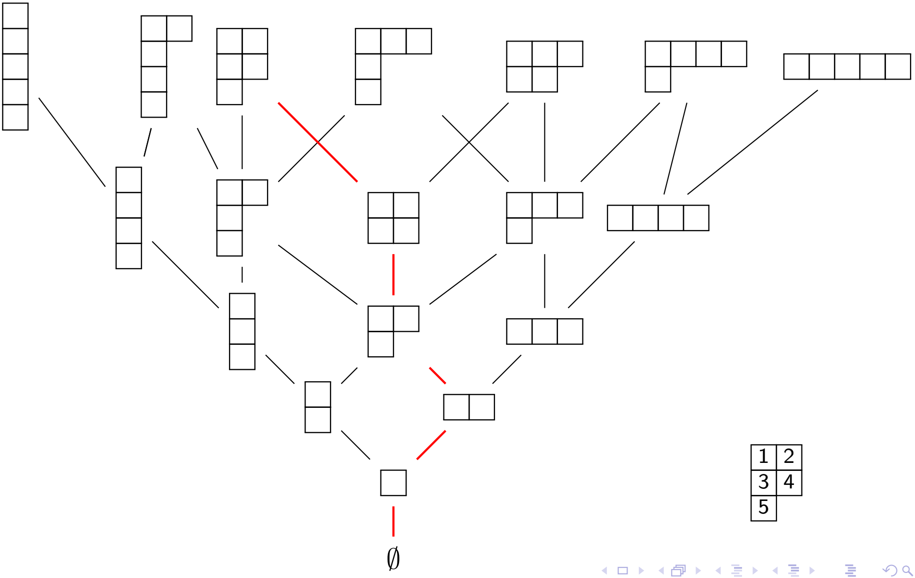
The example of SYT of $\lambda = (2, 2, 1)$

- There are 5 SYT of shape $(2, 2, 1)$, given by

$$
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}
$$

- Thus $f^{(2,2,1)} = 5$

An example of U^5



1	2
3	4
5	

Hook Length Formula

- Subgoal: How many SYT are there for a single shape λ?
- Let u be a square of the Young diagram of the partition λ. Define the hook $H(u)$ of u to be the set of all squares directly to the right of u or directly below u, including u itself and $h(u) = |H(u)|$.

Hook Length Formula

- Subgoal: How many SYT are there for a single shape λ?
- Let u be a square of the Young diagram of the partition λ. Define the hook $H(u)$ of u to be the set of all squares directly to the right of u or directly below u, including u itself and $h(u) = |H(u)|$.
- Below is the diagram of the partition $(4, 2, 2)$.

6	5	2	1
3	2		
2	1		

Theorem: Hook Length Formula

- **Theorem:** Let $\lambda \vdash n$. Then
$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)}.$$

Example: $\lambda = (4, 2, 2)$

- **Theorem:** Let $\lambda \vdash n$. Then

$$f^{\lambda} = \frac{n!}{\prod_{u \in \lambda} h(u)}.$$

6	5	2	1
3	2		
2	1		

Example: $\lambda = (4, 2, 2)$

- **Theorem:** Let $\lambda \vdash n$. Then

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)}.$$

6	5	2	1
3	2		
2	1		

$$f^{(4,2,2)} = \frac{(4+2+2)!}{6 \cdot 5 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 1} = 56.$$

Example: $\lambda = (4, 2, 2)$

- **Theorem:** Let $\lambda \vdash n$. Then

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)}.$$

6	5	2	1
3	2		
2	1		

$$f^{(4,2,2)} = \frac{(4+2+2)!}{6 \cdot 5 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 1} = 56.$$

- Hence, there are 56 SYT of shape $(4, 2, 2)$

Example: $\lambda = (2, 2, 1)$

4	2
3	1
1	

Example: $\lambda = (2, 2, 1)$

$$\begin{array}{|c|c|}
\hline
4 & 2 \\
\hline
3 & 1 \\
\hline
1 \\
\cline{1-1}
\end{array}$$

$$f^{(2,2,1)} = \frac{(2+2+1)!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5.$$

Example: $\lambda = (2, 2, 1)$

4	2
3	1
1	

$$f^{(2,2,1)} = \frac{(2+2+1)!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5.$$

- Hence, there are 5 SYT of shape $(2, 2, 1)$

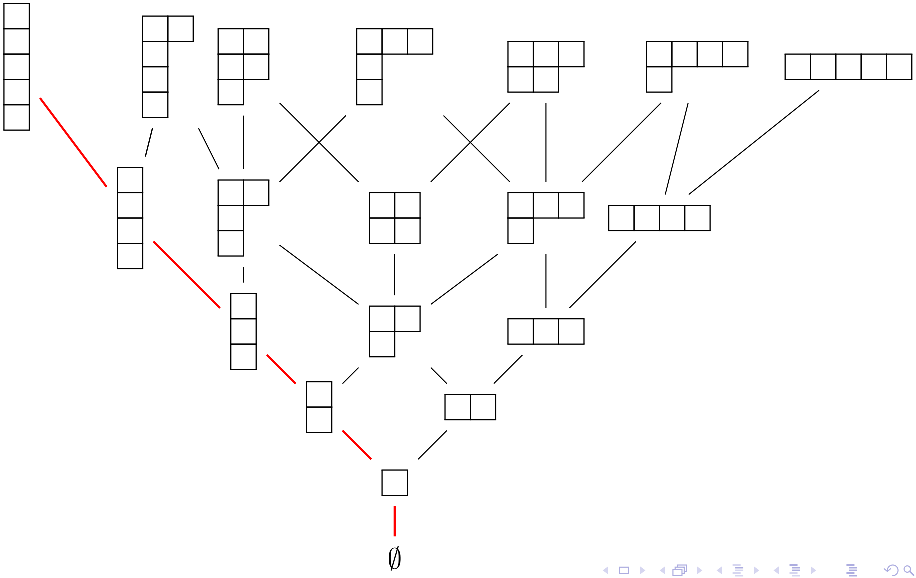
Goal

- Goal: Count the number of walks of type U^n. Call this number T_n.
- Let's do the case where $n = 5$.

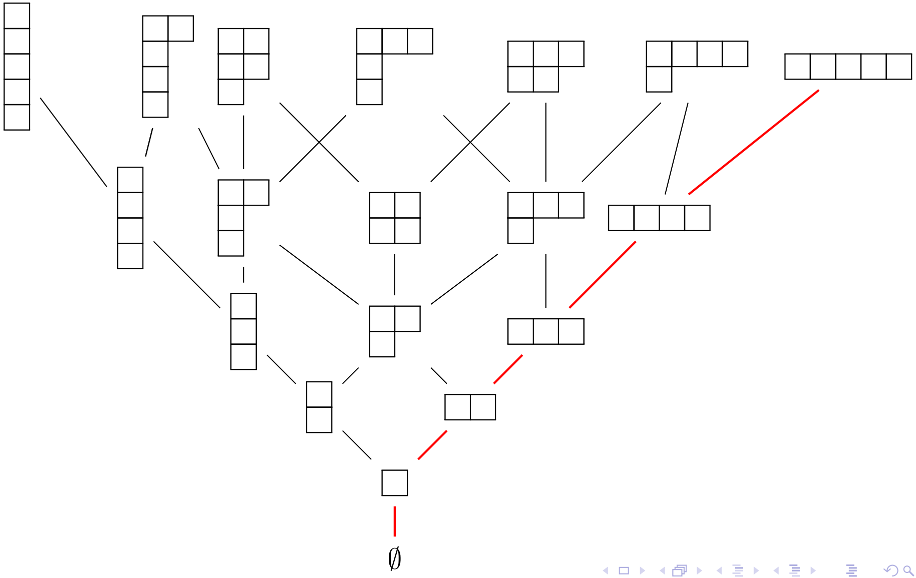
$$T_n = \sum_{\lambda \vdash 5} f^\lambda = \sum_{\lambda \vdash 5} \frac{n!}{\prod_{u \in \lambda} h(u)}$$

Now we calculate T_5

$\lambda = (11111)$



$\lambda = (5)$



U^5

$$\begin{array}{|c|c|}\hline 5 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

$$f^{(2,1,1,1)} = \frac{5!}{5 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 4.$$

$$\begin{array}{|c|c|c|}\hline 5 & 2 & 1 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

$$f^{(3,1,1)} = \frac{5!}{5 \cdot 2 \cdot 2 \cdot 1 \cdot 1} = 3.$$

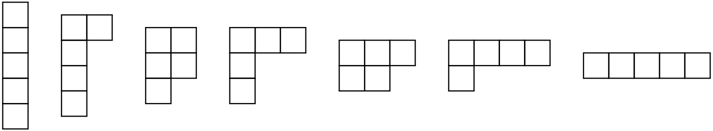
$$\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

$$f^{(3,1,1)} = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5.$$

$$\begin{array}{|c|c|c|c|}\hline 5 & 3 & 2 & 1 \\ \hline 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

$$f^{(4,1)} = \frac{5!}{5 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 4.$$

Goal



$$T_5 = 1 + 4 + 5 + 3 + 5 + 4 + 1 = 23.$$

Thank you

Citation:
Stanley, Richard, "Topics in Algebraic Combinatorics"