2A. Finite generation and the Noetherian property

Finite generation and the Noetherian Property.

Let R be a ring and M a left R module. M is finitely generated if there are finitely many elements $m_1, \ldots, m_k \in M$ so that, for any $m \in M$, there are $r_i \in R$, such that

$$m = \sum_{i=1}^{k} r_i m_i.$$

Equivalently, M is finitely generated if, for some $n \geq 0$ in $\mathbb{Z}$ there is a surjective R-module homomorphism

$$\pi : \oplus_{i=1}^{n} R = R^n \to M.$$

The images of the basis elements of R^n give the generating set m_i.

Definition: A module M satisfies the *ascending chain condition* if any increasing sequence of submodules

$$M_1 \subset M_2 \subset \cdots \subset M_k \subset$$

eventually stabilizes, meaning that there is an N so that $M_i = M_j$ for all $i, j \geq N$.

Proposition: The following are equivalent:

1. M satisfies the ascending chain condition.
2. Every nonempty set of submodules of M has a maximal element.
3. Every submodule of M is finitely generated.

A module that satisfies these equivalent conditions is called (left) *Noetherian* after Emmy Noether. A ring is Noetherian if it is Noetherian as a left module over itself. Since the submodules of a ring are the ideals, a ring is Noetherian if every ideal is finitely generated.

Proof: Suppose M satisfies the ascending chain condition and let $\mathcal{M}$ be a nonempty collection of submodules of M. Every ascending chain in $\mathcal{M}$ has a maximal element (that's basically what the chain condition says) and therefore by Zorn's lemma there is a maximal element for $\mathcal{M}$. Now suppose N is any submodule of M. Let $\mathcal{N}$ be the collection of finitely generated submodules of N. Since the zero module is in $\mathcal{N}$, it is nonempty, so it has a maximal element $N' \subset N$. Choose $x \in N$. Then $N' + Rx$ is a finitely generated submodule of N, and since N' is maximal, we must have $N' + Rx = N'$. This means $x \in N'$. Therefore $N = N'$ so N is finitely generated. Finally, if

$$M_1 \subset M_2 \subset \cdots$$

is an increasing chain of submodules, their union M_∞ is a submodule which must be finitely generated by, say, $m_1, \ldots, m_n$. Then there is an integer k such that M_k that contains $m_1, \ldots, m_n$ so $M_k = M_\infty$ and the increasing chain stabilizes at k.

Proposition: Any principal ideal domain is Noetherian.

Proof: Any ideal is generated by one element.

Proposition: If M is Noetherian, so is any quotient module of M.

Proof: Suppose $N = M/J$ where J is a submodule of M. If $K \subset N$ is a submodule, then by the isomorphism theorem $K = K'/J$ for some $K' \subset M$ containing J. Since M is Noetherian, K' is finitely generated by, say $k_1, \ldots, k_r$ and then the corresponding $k_i + J$ generate N.

Proposition: If R is Noetherian, so is R^n.

Proof: By induction on n. We know the result for $n = 1$. Suppose it's true for R^{n-1}. Let M be a submodule of $R^n = R^{n-1} \oplus R$. Let $\pi : M \to R$ be the projection of M onto the last component. Then $\pi(M)$ is an ideal of R, hence finitely generated. If $x_1, \ldots, x_k$ generate $\pi(M)$, then we know that each $x_i = \pi(m_i)$ for $m_i \in M$ and also, for any $m \in M$, we have

$$\phi(m) = \sum r_i x_i = \phi(\sum r_i m_i).$$

This means that any $m \in M$ can be written $m = m_0 + \sum_{i=1}^{k} r_i m_i$ with $m_0 \in \ker(\pi) \subset R^{n-1}$. Since $\ker(\pi)$ is finitely generated by induction, this shows that M is finitely generated (by the finite set of generators for $\ker(M)$ together with $m_1, \ldots, m_k$)

Remark 1: The proof shows that if $N \subset M$ is finitely generated, and M/N is finitely generated, then M is finitely generated.

Proposition: If R is Noetherian, and M is a finitely generated R-module, then M is Noetherian.

Proof: In this case M is a quotient of R^n.

Proposition: If R is a PID, and M is a submodule of R^n, then M can be generated by n or fewer elements.

Proof: By induction. If $n = 1$ this is true because M is an ideal of R, hence generated by one element. If $M \subset R^n = R^{n-1} \oplus R$, and $\pi : M \to R$ is the projection on the last component, then $\pi(M)$ is generated by one elememt m_n (which could be zero). Thus any element of M is of the form $m_0 + rm_n$ with $m_0 \in \ker(\pi) \in R^{n-1}$. By induction, $\ker(\pi)$ is generated by at most $n-1$ elements.

Proposition: (Hilbert) If R is Noetherian, so is $R[x]$. This is called the Hilbert Basis Theorem; we won't prove it. Paul Gordan is alleged to have said of this result "This isn't mathematics, it is theology!" (quite possibly this is just a folk tale in mathematics.)

Non-noetherian rings and modules

The polynomial ring in countably many variables is not Noetherian. The ring of continuous functions on $\mathbb{R}$ (or on $[-1, 1]$) is not Noetherian because you can let M_i be the space of continuous functions which vanish on $[-1/i, 1/i]$ for $i = 1, \ldots$. These are ideals in the ring and we have $M_i \subset M_{i+1}$ but the sequence doesn't stabilize.

- There are finitely generated modules (over non-Noetherian rings) whose submodules are not finitely generated.
- There are modules over non-commutative rings that are left-Noetherian, but not right-Noetherian.