7. Galois Extensions

Galois Extensions

Automorphisms

Let K/F be an extension fields and let $\mathrm{Aut}(K/F)$ be the group of field homomorphisms $\sigma : K \to K$ which are the identity when restricted to F.

Some basics:

- If $\alpha \in K$ is algebraic over F with minimal polynomial $p(x)$ over F, then $\sigma(\alpha)$ is also a root of $p(x)$.
- If $H \subset \mathrm{Aut}(K/F)$ is a subgroup, then the set $K^H = \{x \in K : \sigma(x) = x \forall \sigma \in H\}$ is a subfield of K.
- If $H_1 \subset H_2$ are subgroups of $\mathrm{Aut}(K/F)$, then $K^{H_2} \subset K^{H_1}$.

Galois Extensions

Proposition: Let E/F be the splitting field over F of some polynomial $f(x) \in F[x]$. Then

$$|\mathrm{Aut}(E/F)| \leq [E : F].$$

If $f(x)$ is separable, then this is an equality.

Definition: E/F is called a Galois extension if $|\mathrm{Aut}(E/F)| = [E : F]$. In this case the automorphism group is called the *Galois group* of the extension.

Proposition 5 says that separable splitting fields are Galois extensions.

The Galois correspondence

Let E/F be a Galois extension with Galois group G. Then there is a bijective (inclusion reversing) correspondence between:

- subfields L of E containing F
- subgroups of G

The correspondence is given by $H \to E^H$ for $H \subset G$ in one direction, and $L \to \mathrm{Aut}(E/L) \subset G$ in the other direction.

Further:

- If $L = E^H$ then E/L is Galois with group H.
- If $L = E^H$ then $[E : L] = |H|$ so E is Galois over L.
- If L is a subfield of E containing F, then $|\mathrm{Aut}(E/L)| = [E : L]$ so E is Galois over L.
- The fixed field $L = E^H$ is Galois over F if and only if H is a normal subgroup of G, and in that case $\mathrm{Aut}(L/F) = G/H$.

Some examples

- Quadratic extensions
- $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.
- The splitting field of $x^3 - 2$ over $\mathbb{Q}$.
- The splitting field of $x^4 - 2$ over $\mathbb{Q}$.
- The field $\mathbb{Q}(\zeta_p)$ where ζ_p is a p^{th} root of unity.
- The fields of eigth and ninth roots of unity.
- Finite fields.

Overview of the proof

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