Recommended Problems

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Standard Form

Problems 16-19 of Section 12.1 work through the reduction algorithm to standard form in the slightly simpler, but practically useful case of R being a Euclidean Ring.

Torsion and Rank

Remember that $\mathrm{Tor}(M) = \{m \in M : \exists r \in R, rm = 0\}$ and that the rank of a module is the maximal size of a linearly independent set of elements of M.

1. (DF Section 12.1 problem 1) Let M be a module over an integral domain R. Show that the rank of M (the maximal number of linearly independent elements of M) is zero. Then show that the rank of M is the same as the rank of $M/\mathrm{Tor}(M)$.

More on Torsion and Rank

2. (DF Section 12.1 problem 2) Let M be a module over an integral domain R.
   a. Let $x_1, \dots, x_n$ be a maximal linearly independent set of elements of M. Let N be the submodule of M generated by the x_i. Prove that N is isomorphic to R^n and that M/N is a torsion R-module.
   b. Conversely, suppose M contains a free R-module N of rank n such that M/N is torsion. Prove that M has rank n.
3. (DF Section 12.1 Problem 4) So that if M has rank n, $N \subset M$ has rank r, and M/N has rank s, then $n = r + s$.

Primary Decomposition

4. (DF Section 12.1 problem 11) Let R be a PID and let $a \in R$ be a nonzero element. Let $M = R/aR$. For any prime p of R, prove that, if n is the power of p dividing a in R, then

$$p^{k-1}M/p^kM = \begin{cases} R/pR & k \leq n \\ 0 & k > n \end{cases}$$

More on primary decomposition (uniqueness part of main theorem)

5. (DF Section 12.1 problem 12) Let R be a PID and let p be a prime in R. This problem gives an approach to proving uniqueness of the decomposition into cyclic modules of the form $R/p^a R$.

   a. Let M be a finitely generated torsion R-module. Prove that $p^{k-1}M/p^k M$ is isomorphic to F^{n_k} where F is the field R/pR and n_k is the number of elementary divisors of M which are powers of p^α with $\alpha \geq k$.

   b. Suppose M_1 and M_2 are isomorphic finitely generated torsion R-modules. Use part (a) to prove that, for all $k \geq 0$, M_1 and M_2 have the same number of elementary divisors p^α with $\alpha \geq k$. Show that this implies that M_1 and M_2 have the same set of elementary divisors.