2A. Finite generation and the Noetherian property



Finite generation and the Noetherian Property.

Let R be a ring and M a left R module. M is finitely generated if there are finitely many elements $m_1, \ldots, m_k \in M$ so that, for any $m \in M$, there are $r_i \in R$, such that

$$m=\sum_{i=1}^k r_i m_i.$$

Equivalently, M is finitely generated if, for some $n \ge 0$ in $\mathbb Z$ there is a surjective R-module homomorphism

$$\pi: \bigoplus_{i=1}^n R = R^n \to M.$$

The images of the basis elements of R^n give the generating set m_i .

Definition: A module M satisfies the ascending chain condition if any increasing sequence of submodules

$$M_1 \subset M_2 \subset \cdots \subset M_k \subset$$

Non-noetherian rings and modules

The polynomial ring in countably many variables is not Noetherian. The ring of continuous functions on \mathbb{R} (or on [-1,1]) is not Noetherian because you can let M_i be the space of continuous functions which vanish on [-1/i,1/i] for $i=1,\ldots$ These are ideals in the ring and we have $M_i\subset M_{i+1}$ but the sequence doesn't stabilize.