2A. Finite generation and the Noetherian property

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Let R be a ring and M a left R module. M is finitely generated if there are finitely many elements $m_1, \ldots, m_k \in M$ so that, for any $m \in M$, there are $r_i \in R$, such that

$$m = \sum_{i=1}^{k} r_i m_i.$$

Equivalently, M is finitely generated if, for some $n \geq 0$ in $\mathbb{Z}$ there is a surjective R-module homomorphism

$$\pi : \oplus_{i=1}^{n} R = R^n \to M.$$

The images of the basis elements of R^n give the generating set m_i.

Definition: A module M satisfies the *ascending chain condition* if any increasing sequence of submodules

$$M_1 \subset M_2 \subset \cdots \subset M_k \subset$$

Non-noetherian rings and modules

The polynomial ring in countably many variables is not Noetherian. The ring of continuous functions on $\mathbb{R}$ (or on $[-1, 1]$) is not Noetherian because you can let M_i be the space of continuous functions which vanish on $[-1/i, 1/i]$ for $i = 1, \ldots$. These are ideals in the ring and we have $M_i \subset M_{i+1}$ but the sequence doesn't stabilize.