

2. Modules (continued)

More on modules

Sums of modules

Suppose that R is a ring and M is an R -module. Let N_1, \dots, N_k be submodules of M . Then the sum $N_1 + \dots + N_k$ is the collection

$$N_1 + \dots + N_k = \{n_1 + \dots + n_k : n_i \in N_i\}$$

It is a submodule of M and the smallest submodule containing all the N_i .

One can also consider infinite collections of submodules:

$$\sum_{i \in I} N_i = \left\{ \sum_{j \in J} n_j : n_j \in N_j, J \subset I \text{ finite} \right\}$$

Generating submodules (compare vector spaces)

Suppose $A \subset M$. Then the submodule RA of M *generated by* A is the smallest submodule of M containing A . In practice it is the collection

$$RA = \{r_1 a_1 + \cdots + r_k a_k : r_1, \dots, r_k \in R, a_1, \dots, a_k \in A, k \in \mathbb{Z}, k \geq 0\}$$

In linear algebra, we would say that RA is the *submodule of* M *that is spanned by* A and this terminology can be used here as well.

We can also say that RA is the set of (finite) R -linear combinations of elements of A .

Generating sets - an example

Suppose that V is a \mathbb{Q} -vector space of dimension n and w_1, \dots, w_k are a set of vectors in V .

Since V is also a \mathbb{Z} module (by “restriction of scalars”) we can consider the sub- \mathbb{Z} -module of V generated by the w_i . This is all \mathbb{Z} -linear combinations of the w_i .

For example if $V = \mathbb{Q}^2$ and $A = \{w_1, w_2\}$ are the standard basis elements then $\mathbb{Z}A$ is the subset of V of vectors with integer coefficients in the standard basis.

Finite generation

Definition: An R -module M is finitely generated if there is a finite subset $A \subset M$ such that $RA = M$.

Note that \mathbb{Q} is finitely generated as a \mathbb{Q} -module (in fact it's generated by one element) but not as a \mathbb{Z} -module.

For vector spaces, finitely generated means finite dimensional. A generating set is the same as a spanning set.

Comparison with vector spaces

A set m_1, \dots, m_k in an R -module M is *linearly independent* if, whenever $\sum r_i m_i = 0$, all $r_i = 0$.

For vector spaces, a maximal linearly independent set (meaning a linearly independent set which becomes dependent when any nonzero element is added to it) automatically spans the vector space, and we call this a basis.

For modules, this fails. Consider \mathbb{Z}^2 and let $e_1 = [2, 0]$ and $e_2 = [0, 2]$. If $e = [a, b]$ then

$$2e - ae_1 - be_2 = 0$$

so e_1, e_2 is a maximal linearly independent set. But they don't generate all of \mathbb{Z}^2 .

Cyclic modules

Definition: An R module M is cyclic if it is generated by one element: $M = Ra$ for some $a \in M$.

- ▶ Cyclic groups are cyclic \mathbb{Z} -modules.
- ▶ If R is a ring with unity and I is a left ideal, then R/I is a cyclic R -module generated by $1 + I$.
- ▶ If R is a ring with unity, an ideal I is a cyclic module if and only if it is a principal ideal.
- ▶ If $R = M_n(F)$ for a field F and $M = F^n$ is the space of column vectors viewed as an R -module, then M is cyclic.

If $R = \mathbb{Z}[i]$, then $(1 + i)R$ is a cyclic module for R generated by $(1 + i)$. But if we view $(1 + i)R$ as a \mathbb{Z} -module inside the \mathbb{Z} -module $R = \mathbb{Z} + \mathbb{Z}i$ then $(1 + i)R$ is generated over \mathbb{Z} by $1 + i$ and $(1 + i)i = i - 1$; it is not cyclic as a \mathbb{Z} -module.

Characterization of cyclic modules

Proposition: Let M be a cyclic R -module. Then M is isomorphic to R/I where I is a left ideal of R .

Proof: Let $m \in M$ generate M . Consider the map $f : R \rightarrow M$ defined by $f(r) = rm$. This is a module homomorphism since

$$f(r_1 r_2) = r_1 r_2 m = r_1 (r_2 m) = r_1 f(r_2 m).$$

(Remember that we are thinking of R here as an R -module, not a ring.)

The kernel of this map is the set $I = \{r \in R : rm = 0\}$. This is a left ideal since if $rm = 0$ then $sr m = 0$ for all $s \in R$. Since M is cyclic, the map f is surjective. Therefore by the isomorphism theorem M is isomorphic to R/I . ### More on cyclic modules

Recall that a module M for $F[x]$ is the same as an F -vector space V together with a linear map $T : V \rightarrow V$.

If M is cyclic then there is an $m \in M$ so that every $m' \in M$ is given

Direct Sums and Direct Products

Direct Products (definition)

Suppose that M_1, \dots, M_k are R modules. The direct product $M_1 \times \dots \times M_k$ of the M_i is the set of “vectors” (m_1, \dots, m_k) with $m_i \in M_i$. Addition and multiplication by R are done componentwise.

Internal direct sums

Suppose that M is an R -module and N_1, \dots, N_k are submodules of M . There is a module homomorphism

$$N_1 \times \cdots \times N_k \rightarrow N_1 + \cdots + N_k \subset M$$

defined by sending $(n_1, \dots, n_k) \rightarrow n_1 + \cdots + n_k$.

Internal direct sums (continued)

Definition: The sum map above is an isomorphism if and only if either of the following two conditions are satisfied:

- ▶ $N_j \cap (N_1 + \cdots + N_{j-1} + N_{j+1} + \cdots + N_k) = 0$ for all $j = 1, 2, \dots, k$
- ▶ Any $x \in N_1 + N_2 + \cdots + N_k$ can be written *uniquely* as a sum $x = n_1 + n_2 + \cdots + n_k$ with $n_i \in N_i$.

If M is isomorphic to $N_1 \times \cdots \times N_k$ via the sum map, we say that

$$M = N_1 \oplus N_2 \oplus \cdots \oplus N_k$$

and say that M is the *internal direct sum* of the N_i .

Direct Sums vs Direct Products

Definitions

Suppose that I is a set and M_i is an R -module for each $i \in I$.

The *direct product* $\prod_I M_i$ is the collection of all functions $f : I \rightarrow \cup_{i \in I} M_i$ such that $f(i) \in M_i$. It is an R -module:
 $(f + g)(i) = f(i) + g(i)$ and $(rf)(i) = r(f(i))$.

The *direct sum* $\oplus_I M_i$ is the submodule of $\prod_I M_i$ consisting of functions f with the additional property that there is a finite subset $J \subset I$ such that $f(i) = 0$ unless $i \in J$.

Notice that if I is finite then these two things are the same.

Countable sums and products

Suppose that $I = \mathbb{N}$, the natural numbers, and M_i is a family of R -modules indexed by I . Then:

- ▶ $\prod_{i \in I} M_i$ consists of sequences $(m_1, m_2, \dots, m_k, \dots)$ where $m_i \in M_i$.
- ▶ $\bigoplus_{i \in I} M_i$ consists of sequences $(m_1, m_2, \dots, m_k, \dots)$ where $m_i \in M_i$ and there is an N such that $m_i = 0$ for all $i \geq N$.

Notice that, if each M_i is countable, then so is $\bigoplus_{i \in I} M_i$, but $\prod_{i \in I} M_i$ is not.

Free Modules

Definition

Definition: A module M is *free* on a set A of generators if, for every element m of M , there are *unique* r_1, \dots, r_k in R and a_1, \dots, a_k in A such that

$$m = r_1 a_1 + \cdots + r_k a_k.$$

Examples and non-examples

If $A = \{a_1, \dots, a_n\}$ is finite, then M is free on A if the map

$$\bigoplus_{i=1}^n R \rightarrow M$$

defined by $(r_1, \dots, r_n) \mapsto r_1 a_1 + \dots + r_n a_n$ is an isomorphism. So basically M is free on a set A with n elements if and only if it is isomorphic to R^n .

If $R = \mathbb{Z}$, then $M = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ is not free on $(1, 0)$ and $(0, 1)$. Every $m \in M$ is a linear combination $r_1(1, 0) + r_2(0, 1)$ for $r_1, r_2 \in \mathbb{Z}$, but r_1 and r_2 are not uniquely determined. In fact M is not free on any set of generators.

Any vector space over F is a free F -module.

Mapping property

Let A be a set. There exists a module $F(A)$, called the *free module on A* , which contains A as a subset.

It satisfies the following property.

Let M be any module and let $f : A \rightarrow M$ be any *map of sets*. Then there is a unique module homomorphism $\Phi : F(A) \rightarrow M$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\subset} & F(A) \\ & \searrow f & \downarrow \Phi \\ & & M \end{array}$$

Examples of mapping property

- ▶ If V is a vector space and B is a basis, then V is free on B . A linear map from $V \rightarrow W$ is determined by where you send B . In this situation, $f : B \rightarrow W$ is the map of sets sending the basis of V to a subset of W , and Φ is the resulting linear map.
- ▶ If A is any set, then $F(A)$ is the R -module of “formal linear combinations of elements of A ”: the set of sums $\sum r_i a_i$ over finite collections $\{a_1, \dots, a_n\}$ of elements of A .
- ▶ Alternatively it is the set of functions $f : A \rightarrow R$ that are zero for all but a finite subset of A with pointwise addition and scalar multiplication.

Uniqueness

Any two free modules on the same set are isomorphic via the module map induced by the identity map on A .