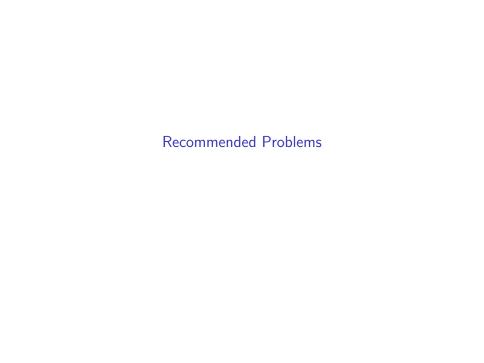
# Recommended Problems



## Problem 1

Prove that if A and B are sets of the same cardinality then F(A) and F(B) are isomorphic.

#### Problem 2

Prove that, if R is commutative, then  $R^n$  and  $R^k$  are isomorphic if and only if n = k.

### Strategy:

Assume  $f: \mathbb{R}^n \to \mathbb{R}^k$  is an isomorphism.

- 1. If *R* is a field, this follows from the fact that the dimension of a vector space is well-defined.
- 2. If R is not a field, it has a proper maximal ideal m so that R/m is a field.
- 3.  $R^n/mR^n=(R/m)^n$ . Any elememt  $(mr_1, mr_2, \ldots, mr_n)$  belongs to  $m^n \subset R^n$ . Conversely, if  $x=(m_1,\ldots,m_n) \in m^n$ , then  $x=m_1e_1+m_2e_2+\cdots+m_ne_n$  where the  $e_i$  are the elements of  $R^n$  with 1 in position i and 0 elsewhere. Therefore  $mR^n=m^n$ . Define a map  $\pi_n:R^n\to (R/m)^n$  that reduces each component mod m. This is an R-module homomorphism whose kernel is  $m^n$ .

#### Problem 2 continued

- 4. The composite map  $\pi_k \circ f: R^n \to R^k/mR^k$  has  $mR^n$  in its kernel.
- 5. Suppose that  $\pi_k \circ f(x) = 0$ . Then f(x) belongs to  $mR^k$ , so write  $f(x) = m_1 y_1 + \cdots + m_j y_j$  with  $y_j \in R^k$ . Since f is surjective, each  $y_j = f(x_j)$  and therefore  $f(x) = m_1 f(x_1) + \cdots + m_j f(x_j) = f(b)$  where  $b \in mR^n$ . Since f is injective, we see that  $x \in mR^n$ . Thus  $\pi_k \circ f$  is surjective and the kernel of  $\pi_k \circ f$  is  $mR^n$ .
- 6. By the isomorphism theorem, there is a quotient map  $\overline{\pi_k \circ f}: R^n/mR^n \to R^k/mR^k$  that is an isomorphism. It is also R/m linear it's R-linear because it's an R-module map, and m acts like zero. Therefore it is an isomorphism of vector spaces between  $(R/m)^n$  and  $(R/m)^k$  and therefore n=k.

**Note:** This is false if R is not commutative. Problem 27 constructs an example.

## Problem 6

Prove that if M is a finitely generated R-module generated by n elements, then any quotient of M can be generated by at most n elements. In particular, quotients of cyclic modules are cyclic.

# Problems 16 and 17: Decomposition Theorem

These two problems establish the decomposition theorem for modules. We assume that R is commutative with 1. Let M be a module and let I and J be ideals of R.

- ▶ Prove that the map  $M \to M/IM \times M/JM$  defined by  $m \mapsto (m + IM, m + JM)$  is an R module homomorphism with kernel  $IM \cap JM$ .
- Suppose that I + J = R. Prove that the map in the previous part is surjective and its kernel is IJM so that

$$M/IJM = M/IM \times M/JM$$

in this case.

## Decomposition theorem continued

This problem is proved in a manner very similar to the situation for rings. The second part is the interesting one. For surjectivity, given  $(m_1+IM,m_2+JM)\in M/IM\times M/JM$ , we want to find m so that  $m-m_1\in IM$  and  $m-m_2\in JM$ . Write 1=i+j with  $i,j\in R$ . Then m=im+jm. Let  $m_1=jm$  and  $m_2=im$ . Then  $m-m_1=m_2\in iM$  and  $m-m_2=m_1\in JM$ .

For the second part, suppose  $m \in IM \cap JM$ . Write m = im + jm. Since  $m \in JM$ ,  $im \in IJM$ , and since  $m \in IM$  and R is commutative,  $jm \in IJM$ . Therefore  $m \in IJM$ .

This can be extended by induction to families of ideals  $I_1, \ldots, I_k$  that are pairwise relatively prime  $(I_i + I_j = R \text{ for any pair.})$ 

# Problem 24 (optional)

This problem constructs an infinite direct product of free modules that is not free; in fact it shows that the countable direct product of copies of  $\mathbb Z$  is not free.