Recommended Problems

Recommended Problems

From DF, Section 10.3, p. 356ff. Note that R is assumed to have a unit element.

Problem 1

Prove that if A and B are sets of the same cardinality then $F(A)$ and $F(B)$ are isomorphic.

Problem 2

Prove that, if R is commutative, then R^n and R^m are isomorphic if and only if $n = m$.

Note: This is false if R is not commutative. Problem 27 constructs an example.

Problem 6

Prove that if M is a finitely generated R-module generated by n elements, then any quotient of M can be generated by at most n elements. In particular, quotients of cyclic modules are cyclic.

Problems 16 and 17

These two problems establish the decomposition theorem for modules. Let M be a module and let I and J be ideals of R.

- Prove that the map $M \to M/IM \times M/JM$ defined by $m \mapsto (m + IM, m + JM)$ is an R module homomorphism with kernel $IM \cap JM$.
- Suppose that $I + J = R$. Prove thatthe map in the previous part is surjective and its kernel is IJM so that

  $$M/IJM = M/IM \times M/JM$$

  in this case.

This can be extended by induction to families of ideals $I_1, \ldots, I_k$ that are pairwise relatively prime ($I_i + I_j = R$ for any pair.)

Problem 24 (optional)

This problem constructs an infinite direct product of free modules that is not free; in fact it shows that the countable direct product of copies of $\mathbb{Z}$ is not free.