4. Canonical Forms

Canonical Forms: Preliminaries

Set-up

Let F be a field, let V be a nontrivial finite dimensional vector space over F, and let $T : V \to V$ be a linear transformation.

Then V is an $F[x]$-module.

Lemma: V is a finitely generated torsion module over $F[x]$.

Proof: V is generated by a basis, which is finite by assumption. If V were not torsion, there would be a $v \in V$ so that the map $F[x] \to V$ given by $f(x) \mapsto f(T)v$ is injective; but that contradicts the fact that V is finite dimensional.

Eigenvalues and the Characteristic Polynomial.

Definition: An element $\lambda \in F$ is an eigevanlue of T if there is a nonzero $v \in V$ with $Tv = \lambda v$. The vector v is an eigenvector for this eigenvalue.

Definition: Let $c(x) = \det(xI - T)$. Then $c(x)$ is a monic polynomial of degree n called the *characteristic polynomial* of T.

Lemma: λ is an eigenvalue of T if and only if $c(\lambda) = 0$.

Proof: If $c(\lambda) = 0$, then $\det(\lambda I - T) = 0$ so $\lambda I - T$ has a nontrivial kernel; an element of the kernel is an eigenvector with eigenvalue λ. Conversely, if λ is an eigenvalue, then $\lambda I - T$ has nontrivial kernel, so its determinant is zero.

The Minimal polynomial

Since V is a torsion $F[x]$ module, its annihilator

$$\mathrm{Ann}(V) = \{f(x) \in F[x] : f(T)v = 0 \forall v \in V\}$$

is a nonzero ideal in $F[x]$, hence a principal ideal generated by a unique monic polynomial $m(x)$.

Definition: The unique monic generator $m(x)$ of $\mathrm{Ann}(V)$ is called the minimal polynomial of T.

Lemma: The minimal polynomial $m(x)$ is the monic polynomial of minimal degree such that $m(T)v = 0$ for all $v \in V$.

Proof: By definition $m(T)v = 0$ for all $v \in V$. The collection of polynomials $h(x)$ such that $h(T)v = 0$ is exactly $\mathrm{Ann}(V)$, and the generator of the ideal $\mathrm{Ann}(V)$ in the Euclidean ring $F[x]$ is its monic polynomial element of minimal degree.

The structure of V

By the fundamental theorem of finitely generated modules over PID's, we have two different ways to represent V as an $F[x]$ module.

Invariant Factors: There are monic polynomials $f_1(x)|f_2(x)|\cdots|f_k(x)$ such that

$$V = F[x]/(f_1(x)) \oplus F[x]/(f_2(x)) \oplus \cdots \oplus F[x]/(f_k(x)).$$
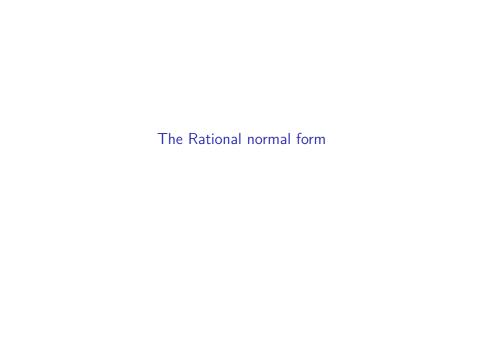
Elementary Divisors: There are irreducible polynomials $f_1(x), \ldots, f_k(x)$ and nonnegative integers $e_1, \ldots, e_k$ such that

$$V = F[x]/(f_1(x)^{e_1}) \oplus F[x]/(f_2(x)^{e_2}) \oplus \cdots \oplus F[x]/(f_k(x)^{e_k})$$

The invariant factors are uniquely determined; and the elementary divisors are uniquely determined up to order.

Invariant factors and the minimal polynomial

Lemma: The minimal polynomial $m(x)$ of T is the last (the "largest") invariant factor of T acting on V; all the invariant factors divide $m(x)$.

The Rational normal form

The cyclic case

Let's focus our attention for the moment on a cyclic $F[x]$ module of the form $M = F[x]/(f(x))$ where

$$f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0$$

- M is a finite dimensional vector space of degree equal to the degree d of $f(x)$.
- $1, x, x^2, \ldots, x^{d-1}$ is a basis for this module.
- x is an F-linear transformation $M \to M$.

Matrix of x

The linear map given by multiplication by x has the following matrix form in the basis $1, x, x^2, \ldots, x^{d-1}$:

$$[x] = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{d-1} \end{pmatrix}$$

Definition: The matrix for $[x]$ above is called the companion matrix for the polynomial $f(x)$.

Characteristic and minimal polynomials of the companion matrix

Lemma: The characteristic and minimal polynomials of this linear transformation are both $f(x)$.

Proof: The fact that the characteristic polynomial of $[x]$ is $f(x)$ is a computation. The fact that $f(x)$ is the minimal polynomial follows from the fact that x clearly satisfies $f(x)$, and, since $1, x, \ldots, x^{d-1}$ are linearly dependent, there is no relation $\sum a_i x^i = 0$ of degree less than d.