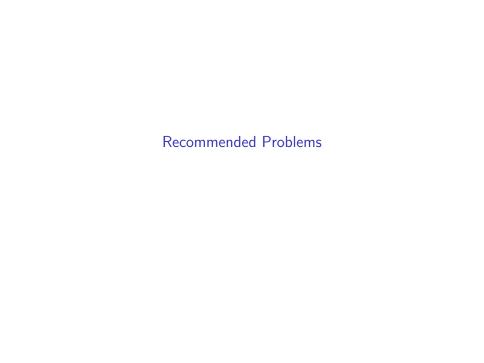
## Recommended Problems



#### Standard Form

Problems 16-19 of Section 12.1 work through the reduction algorithm to standard form in the slightly simpler, but practically useful case of R being a Euclidean Ring.

#### Torsion and Rank

Remember that  $Tor(M) = \{m \in M : \exists r \in R, rm = 0\}$  and that the rank of a module is the maximal size of a linearly independent set of elements of M.

1. (DF Section 12.1 problem 1) Let M be a module over an integral domain R. Show that the rank of M (the maximal number of linearly independent elements of M) is zero. Then show that the rank of M is the same as the rank of  $M/\mathrm{Tor}(M)$ .

#### More on Torsion and Rank

- 2. (DF Section 12.1 problem 2) Let M be a module over an integral domain R.
  - a. Let  $x_1, \ldots, x_n$  be a maximal linearly independent set of elements of M. Let N be the submodule of M generated by the  $x_i$ . Prove that N is isomorphic to  $R^n$  and that M/N is a torsion R-module.
  - b. Conversely, suppose M contains a free R-module N of rank n such that M/N is torsion. Prove that M has rank n.
- 3. (DF Section 12.1 Problem 4) So that if M has rank n,  $N \subset M$  has rank r, and M/N has rank s, then n = r + s.

### **Primary Decomposition**

4. (DF Section 12.1 problem 11) Let R be a PID and let  $a \in R$  be a nonzero element. Let M = R/aR. For any prime p of R, prove that, if n is the power of p dividing a in R, then

$$p^{k-1}M/p^kM = \begin{cases} R/pR & k \le n \\ 0 & k > n \end{cases}$$

# More on primary decomposition (uniqueness part of main theorem)

- 5. (DF Section 12.1 problem 12) Let R be a PID and let p be a prime in R. This problem gives an approach to proving uniqueness of the decomposition into cyclic modules of the form  $R/p^aR$ .
  - a. Let M be a finitely generated torsion R-module. Prove that  $p^{k-1}M/p^kM$  is isomorphic to  $F^{n_k}$  where F is the field R/pR and  $n_k$  is the number of elementary divisors of M which are powers of  $p^{\alpha}$  with  $\alpha \geq k$ .
  - b. Suppose  $M_1$  and  $M_2$  are isomorphic finitely generated torsion R-modules. Use part (a) to prove that, for all  $k \geq 0$ ,  $M_1$  and  $M_2$  have the same number of elementary divisors  $p^{\alpha}$  with  $\alpha \geq k$ . Show that this implies that  $M_1$  and  $M_2$  have the same set of elementary divisors.