

1. Modules

Modules

Modules

- ▶ Modules are to rings as vector spaces are to fields.
- ▶ Modules are to rings as sets with group actions are to groups.

Definition of (left) modules

Definition: Let R be a ring (for now, not necessarily commutative and not necessarily having a unit). A *left R -module* is an abelian group M together with a map $R \times M \rightarrow M$ (written $(r, m) \mapsto rm$) such that:

- ▶ $r(m_1 + m_2) = rm_1 + rm_2$
- ▶ $(r_1 + r_2)m = r_1m + r_2m$
- ▶ $r_1(r_2m) = (r_1r_2)m$

If R has a unit element 1 , we also require $1m = m$ for all $m \in M$.

Right modules

A right module is defined by a map $M \times R \rightarrow M$ and written $(m, r) \mapsto mr$ and satisfying the property

$$(mr_1)r_2 = m(r_1r_2).$$

If R is not commutative, these really are different, since for a left module:

- ▶ r_1r_2 acts by “first r_2 , then r_1 ”

while for a right module

- ▶ r_1r_2 acts by “first r_1 , then r_2 .”

Left and Right modules

If R is commutative, and M is a left R -module, then we can define a right R module M' with the same underlying abelian group M and by defining $m'r = (rm)'$. This works because

$$(m'r_1)r_2 = (r_1m)'r_2 = (r_2(r_1m))' = ((r_2r_1)m)' = ((r_1r_2)m)' = m'(r_1r_2)$$

Remarks

Vector spaces

If R is a field, then a left (or right) R -module is the same as a vector space.

Another definition

If M is an abelian group, and R is a ring, then a left R -module structure on M is the same as a ring map

$$R \rightarrow \text{End}(M)$$

. If ϕ_r is the endomorphism associated to $r \in R$, then $rm = \phi_r(m)$.
The associativity comes from defining the ring structure on

$$\text{End}(M)$$

as the usual composition of functions:

$$\phi_{r_1 r_2} = \phi_{r_1} \circ \phi_{r_2}.$$

Testing $\text{GL}(\mathbb{R})$