Recommended Problems

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Problem 1

Prove that if A and B are sets of the same cardinality then $F(A)$ and $F(B)$ are isomorphic.

Problem 2

Prove that, if R is commutative, then R^n and R^k are isomorphic if and only if $n = k$.

Strategy:

Assume $f : R^n \to R^k$ is an isomorphism.

1. If R is a field, this follows from the fact that the dimension of a vector space is well-defined.
2. If R is not a field, it has a proper maximal ideal m so that R/m is a field.
3. $R^n/mR^n = (R/m)^n$. Any elememt $(mr_1, mr_2, \ldots, mr_n)$ belongs to $m^n \subset R^n$. Conversely, if $x = (m_1, \ldots, m_n) \in m^n$, then $x = m_1e_1 + m_2e_2 + \cdots + m_ne_n$ where the e_i are the elements of R^n with 1 in position i and 0 elsewhere. Therefore $mR^n = m^n$. Define a map $\pi_n : R^n \to (R/m)^n$ that reduces each component mod m. This is an R-module homomorphism whose kernel is m^n.

Problem 2 continued

4. The composite map $\pi_k \circ f : R^n \to R^k/mR^k$ has mR^n in its kernel.
5. Suppose that $\pi_k \circ f(x) = 0$. Then $f(x)$ belongs to mR^k, so write $f(x) = m_1 y_1 + \cdot + m_j y_j$ with $y_j \in R^k$. Since f is surjective, each $y_j = f(x_j)$ and therefore $f(x) = m_1 f(x_1) + \cdots + m_j f(x_j) = f(b)$ where $b \in mR^n$. Since f is injective, we see that $x \in mR^n$. Thus $\pi_k \circ f$ is surjective and the kernel of $\pi_k \circ f$ is mR^n.
6. By the isomorphism theorem, there is a quotient map $\overline{\pi_k \circ f} : R^n/mR^n \to R^k/mR^k$ that is an isomorphism. It is also R/m linear – it's R-linear because it's an R-module map, and m acts like zero. Therefore it is an isomorphism of vector spaces between $(R/m)^n$ and $(R/m)^k$ and therefore $n = k$.

Note: This is false if R is not commutative. Problem 27 constructs an example.

Problem 6

Prove that if M is a finitely generated R-module generated by n elements, then any quotient of M can be generated by at most n elements. In particular, quotients of cyclic modules are cyclic.

Problems 16 and 17: Decomposition Theorem

These two problems establish the decomposition theorem for modules. We assume that R is commutative with 1. Let M be a module and let I and J be ideals of R.

- Prove that the map $M \to M/IM \times M/JM$ defined by $m \mapsto (m + IM, m + JM)$ is an R module homomorphism with kernel $IM \cap JM$.
- Suppose that $I + J = R$. Prove that the map in the previous part is surjective and its kernel is IJM so that

$$M/IJM = M/IM \times M/JM$$

in this case.

Decomposition theorem continued

This problem is proved in a manner very similar to the situation for rings. The second part is the interesting one. For surjectivity, given $(m_1 + IM, m_2 + JM) \in M/IM \times M/JM$, we want to find m so that $m - m_1 \in IM$ and $m - m_2 \in JM$. Write $1 = i + j$ with $i, j \in R$. Then $m = im + jm$. Let $m_1 = jm$ and $m_2 = im$. Then $m - m_1 = m_2 \in iM$ and $m - m_2 = m_1 \in JM$.

For the second part, suppose $m \in IM \cap JM$. Write $m = im + jm$. Since $m \in JM$, $im \in IJM$, and since $m \in IM$ and R is commutative, $jm \in IJM$. Therefore $m \in IJM$.

This can be extended by induction to families of ideals $I_1, \ldots, I_k$ that are pairwise relatively prime ($I_i + I_j = R$ for any pair.)

Problem 24 (optional)

This problem constructs an infinite direct product of free modules that is not free; in fact it shows that the countable direct product of copies of $\mathbb{Z}$ is not free.