2. Modules (continued)

More on modules

Sums of modules

Suppose that R is a ring and M is an R-module. Let $N_1, \ldots, N_k$ be submodules of M. Then the sum $N_1 + \ldots + N_k$ is the collection

$$N_1 + \ldots + N_k = \{n_1 + \cdots + n_k : n_i \in N_i\}$$

It is a submodule of M and the smallest submodule containing all the N_i.

One can also consider infinite collections of submodules:

$$\sum_{i \in I} N_i = \{\sum_{j \in J} n_j : n_j \in N_j,\ J \subset I \text{ finite } \}$$

Generating submodules (compare vector spaces)

Suppose $A \subset M$. Then the submodule RA of M *generated by* A is the smallest submodule of M containing A. In practice it is the collection

$$RA = \{r_1 a_1 + \cdots + r_k a_k : r_1, \ldots, r_k \in R, a_1, \ldots, a_k \in A, k \in \mathbb{Z}, k \geq 0\}$$

In linear algebra, we would say that RA is the *submodule of M that is spanned by A* and this terminology can be used here as well.

We can also say that RA is the set of (finite) R-linear combinations of elements of A.

Generating sets - an example

Suppose that V is a $\mathbb{Q}$-vector space of dimension n and $w_1, \ldots, w_k$ are a set of vectors in V.

Since V is also a $\mathbb{Z}$ module (by "restriction of scalars") we can consider the sub-$\mathbb{Z}$-module of V generated by the w_i. This is all $\mathbb{Z}$-linear combinations of the w_i.

For example if $V = \mathbb{Q}^2$ and $A = \{w_1, w_2\}$ are the standard basis elements then $\mathbb{Z}A$ is the subset of V of vectors with integer coefficients in the standard basis.

Finite generation

Definition: An R-module M is finitely generated if there is a finite subset $A \subset M$ such that $RA = M$.

Note that $\mathbb{Q}$ is finitely generated as a $\mathbb{Q}$-module (in fact it's generated by one element) but not as a $\mathbb{Z}$-module.

For vector spaces, finitely generated means finite dimensional. A generating set is the same as a spanning set.

Comparison with vector spaces

A set $m_1, \ldots, m_k$ in an R-module M is *linearly independent* if, whenever $\sum r_i m_i = 0$, all $r_i = 0$.

For vector spaces, a maximal linearly independent set (meaning a linearly independent set which becomes dependent when any nonzero element is added to it) automatically spans the vector space, and we call this a basis.

For modules, this fails. Consider $\mathbb{Z}^2$ and let $e_1 = [2, 0]$ and $e_2 = [0, 2]$. If $e = [a, b]$ then

$$2e - ae_1 - be_2 = 0$$

so e_1, e_2 is a maximal linearly independent set. But they don't generate all of $\mathbb{Z}^2$.

Cyclic modules

Definition: An R module M is cyclic if it is generated by one element: $M = Ra$ for some $a \in M$.

- Cyclic groups are cyclic $\mathbb{Z}$-modules.
- If R is a ring with unity and I is a left ideal, then R/I is a cyclic R-module generated by $1 + I$.
- If R is a ring with unity, an ideal I is a cyclic module if and only if it is a principal ideal.
- If $R = M_n(F)$ for a field F and $M = F^n$ is the space of column vectors viewed as an R-module, then M is cyclic.

If $R = \mathbb{Z}[i]$, then $(1 + i)R$ is a cyclic module for R generated by $(1 + i)$. *But* if we view $(1 + i)R$ as a $\mathbb{Z}$-module inside the $\mathbb{Z}$-module $R = \mathbb{Z} + \mathbb{Z}i$ then $(1 + i)R$ is generated *over* $\mathbb{Z}$ by $1 + i$ and $(1 + i)i = i - 1$; it is not cyclic as a $\mathbb{Z}$-module.

Characterization of cyclic modules

Proposition: Let M be a cyclic R-module. Then M is isomorphic to R/I where I is a left ideal of R.

Proof: Let $m \in M$ generate M. Consider the map $f : R \to M$ defined by $f(r) = rm$. This is a module homomorphism since

$$f(r_1 r_2) = r_1 r_2 m = r_1(r_2 m) = r_1 f(r_2 m).$$

(Remember that we are thinking of R here as an R-module, not a ring.)

Characterization of cyclic modules cont'd

The kernel of the map $f(r) = rm$ is the set $I = \{r \in R : rm = 0\}$.

This is a left ideal since if $rm = 0$ then $srm = 0$ for all $s \in R$.

Since M is cyclic, the map f is surjective.

Therefore by the isomorphism theorem M is isomorphic to R/I.

More on cyclic modules

Recall that a module M for $F[x]$ is the same as an F-vector space V together with a linear map $T : V \to V$.

If M is cyclic then there is an $m \in M$ so that every $m' \in M$ is given by $p(x)m$ for some $p(x) \in F[x]$.

This means that that there is a vector $v \in V$ so that every vector $v' \in V$ is of the form $p(T)v$. In other words, the set $v, Tv, T^2 v, \ldots, T^n v, \ldots$ spans V.

If $V = F^2$ and T satisfies $Te_1 = 0$ and $Te_2 = e_2$ then V is *not* cyclic.

If $Te_1 = 0$ and $Te_2 = e_1$ then V is cyclic and generated by e_2. Also $T^2 e_2 = 0$ and so as an R-module V is isomorphic to $F[x]/(x^2)$.

Direct Sums and Direct Products

Direct Products (definition)

Suppose that $M_1, \ldots, M_k$ are R modules. The direct product $M_1 \times \cdots \times M_k$ of the M_i is the set of "vectors" $(m_1, \ldots, m_k)$ with $m_i \in M_i$. Addition and multiplication by R are done componentwise.

Internal direct sums

Suppose that M is an R-module and $N_1, \ldots, N_k$ are submodules of M. There is a module homomorphism

$$N_1 \times \cdots \times N_k \to N_1 + \cdots N_k \subset M$$

defined by sending $(n_1, \ldots, n_k) \to n_1 + \cdots n_k$.

Internal direct sums (continued)

Definition: The sum map above is an isomorphism if and only if either of the following two conditions are satisfied:

- $N_j \cap (N_1 + \cdots N_{j-1} + N_{j+1} + \cdots N_k) = 0$ for all $j = 1, 2, \ldots, k$
- Any $x \in N_1 + N_2 + \ldots + N_k$ can be written *uniquely* as a sum $x = n_1 + n_2 + \ldots + n_k$ with $n_i \in N_i$.

If M is isomorphic to $N_1 \times \cdots \times N_k$ via the sum map, we say that

$$M = N_1 \oplus N_2 \oplus \cdots \oplus N_k$$

and say that M is the *internal direct sum* of the N_i.

Direct Sums vs Direct Products

Definitions

Suppose that I is a set and M_i is an R-module for each $i \in I$.

The *direct product* $\prod_I M_i$ is the collection of all functions $f : I \to \cup_{i \in I} M_i$ such that $f(i) \in M_i$. It is an R-module: $(f+g)(i) = f(i) + g(i)$ and $(rf)(i) = r(f(i))$.

The *direct sum* $\oplus_I M_i$ is the submodule of $\prod_I M_i$ consistsing of functions f with the additional property that there is a finite subset $J \subset I$ such that $f(i) = 0$ unless $i \in J$.

Notice that if I is finite then these two things are the same.

Countable sums and products

Suppose that $I = \mathbb{N}$, the natural numbers, and M_i is a family of R-modules indexed by I. Then:

- $\prod_{i \in I} M_i$ consists of sequences $(m_1, m_2, \ldots, m_k, \ldots)$ where $m_i \in M_i$.
- $\oplus_{i \in I} M_i$ consists of sequences $(m_1, m_2, \ldots, m_k, \ldots)$ where $m_i \in M_i$ and there is an N such that $m_i = 0$ for all $i \geq N$.

Notice that, if each M_i is countable, then so is $\oplus_{i \in I} M_i$, but $\prod_{i \in I} M_i$ is not.

Free Modules

Definition

Definition: A module M is *free* on a set A of generators if, for every element m of M, there are *unique* $r_1, \dots, r_k$ in R and $a_1, \dots, a_k$ in A such that

$$m = r_1 a_1 + \cdots + r_k a_k.$$

Such a set A is called a *basis* of M, so a module M is free if it has a basis.

Examples and non-examples

If $A = \{a_1, \dots, a_n\}$ is finite, then M is free on A if the map

$$\oplus_{i=1}^{n} R \to M$$

defined by $(r_1, \dots, r_n) \mapsto r_1 a_1 + \cdots + r_n a_n$ is an isomorphism. So basically M is free on a set A with n elements if and only if it is isomorphic to R^n.

If $R = \mathbb{Z}$, then $M = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ is not free on $(1,0)$ and $(0,1)$. Every $m \in M$ is a linear combination $r_1(1,0) + r_2(0,1)$ for $r_1, r_2 \in \mathbb{Z}$, but r_1 and r_2 are not uniquely determined. In fact M is not free on any set of generators.

Any vector space over F is a free F-module.

A principal ideal in a (commutative) ring is a free module, but a non-principal ideal is not. Consider $(2, 1+\sqrt{-5}) \subset \mathbb{Z}[\sqrt{-5}]$. We have $(3)(2) - (1-\sqrt{-5})(1+\sqrt{-5}) = 0$ so 0 doesn't have a unique representation.

Mapping property

Let A be a set. There exists a module $F(A)$, called the *free module on A*, which contains A as a subset.
It satisfies the following property.

Let M be any module and let $f : A \to M$ be any *map of sets*. Then there is a unique module homomorphism $\Phi : F(A) \to M$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\subset"] \arrow[dr, "f"'] & F(A) \arrow[d, "\Phi"] \\ & M \end{tikzcd}$$

Examples of mapping property

- If V is a vector space and B is a basis, then V is free on B. A linear map from $V \to W$ is determined by where you send B. In this situation, $f : B \to W$ is the map of sets sending the basis of V to a subset of W, and Φ is the resulting linear map.
- If A is any set, then $F(A)$ is the R-module of "formal linear combinations of elements of A": the set of sums $\sum r_i a_i$ over finite collections $\{a_1, \ldots, a_n\}$ of elements of A.
- Alternatively it is the set of functions $f : A \to R$ that are zero for all but a finite subset of A with pointwise addition and scalar multiplication.

Uniqueness

Any two free modules on the same set are isomorphic via the module map induced by the identity map on A.

Rank

Let R be an integral domain.

Definition: The rank of an R-module is the maximum number of R-linear independent elements of M.

Proposition: Let M be a free R module of rank n. Then any $n+1$ elements of M are linearly dependent. Thus any submodule of M has rank at most n.

Proof: Let $m_1, \ldots, m_{n+1}$ be elements of M and let $e_1, \ldots, e_n$ be a basis of M. Each m_i is an R-linear combination of the e_i. We can view the m_i as vectors in F^n where F is the fraction field of R. They are linearly dependent in F^n, meaning there is a relation

$$\sum f_i m_i = 0$$

where the f_i are in F. Clearing denominators gives a relation over R. ## Torsion

Torsion Definition

Suppose that R is a ring with unity.

Definition: Let M be an R-module. An element $m \in M$ is a torsion element if $rm = 0$ for some nonzero $r \in R$. The set of torsion elements in M is called $\mathrm{Tor}(M)$.

- Any finite abelian group is a torsion $\mathbb{Z}$-module.
- Any cyclic R-module is torsion.
- Any finite dimensional vector space V over a field F with a linear map $T : V \to V$ is a torsion $F[x]$-module.

Lemma: If R is an integral domain and M is an R-module, then the set of torsion elements is a submodule.

Proof: If m_1 and m_2 are torsion, $r_1 m_1 = 0$ and $r_2 m_2 = 0$, with both r_1 and r_2 nonzero, then $r_1 r_2(m_1 + m2) = 0$ and $r_1 r_2(m_1 m_2) = 0$, and $r_1 r_2$ is nonzero since R is an integral domain.

Torsion-free modules

If R is an integral domain, an R-module M is called torsion-free if $\mathrm{Tor}(M) = 0$.

Any free module is torsion-free, but the converse is false. For example, non-principal ideals in integral domains are not free. This follows from the following lemma.

Lemma: An ideal of R is free if and only if it is principal.

Proof: R is a free module of rank 1, so a submodule has rank at most 1; if it has rank 1, it is a principal ideal.

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