1. Modules

Modules

- Modules are to rings as vector spaces are to fields.
- Modules are to rings as sets with group actions are to groups.

Definition of (left) modules

Definition: Let R be a ring (for now, not necessarily commutative and not necessarily having a unit). A *left R-module* is an abelian group M together with a map $R \times M \to M$ (written $(r, m) \mapsto rm$) such that:

- $r(m_1 + m_2) = rm_1 + rm_2$
- $(r_1 + r_2)m = r_1 m + r_2 m$
- $r_1(r_2 m) = (r_1 r_2)m$

If R has a unit element 1, we also require $1m = m$ for all $m \in M$.

Right modules

A right module is defined by a map $M \times R \to M$ and written $(m, r) \mapsto mr$ and satisfying the property

$$(mr_1)r_2 = m(r_1 r_2).$$

If R is not commutative, these really are different, since for a left module:

- $r_1 r_2$ acts by "first r_2, then r_1

while for a right module

- $r_1 r_2$ acts by "first r_1, then r_2."

Left and Right modules

If R is commutative, and M is a left R-module, then we can define a right R module M' with the same underlying abelian group M and by defining $m'r = (rm)'$. This works because

$$(m'r_1)r_2 = (r_1 m)'r_2 = (r_2(r_1 m))' = ((r_2 r_1)m)' = ((r_1 r_2)m)' = m'(r_1 r_2)$$

Remarks

Vector spaces If R is a field, then a left (or right) R-module is the same as a vector space.

Another definition If M is an abelian group, and R is a ring, then a left R-module structure on M is the same as a ring map

$$R \to \mathrm{End}(M)$$

.

If ϕ_r is the endomorphism associated to $r \in R$, then $rm = \phi_r(m)$. The associativity comes from defining the ring structure on

$$\mathrm{End}(M)$$

as the usual composition of functions:

$$\phi_{r_1 r_2} = \phi_{r_1} \circ \phi_{r_2}.$$

Submodules

Definition: If M is a left R-module, then a submodule N of M is a subgroup with the property that, if $n \in N$, then $rn \in N$ for all $r \in R$.

Observation: A ring R is a left module over itself by ring multiplication. The (left) ideals of R are *exactly the left submodules of R.*

`<a href="beamer/01-modules.pdf"> View as slides </a>`