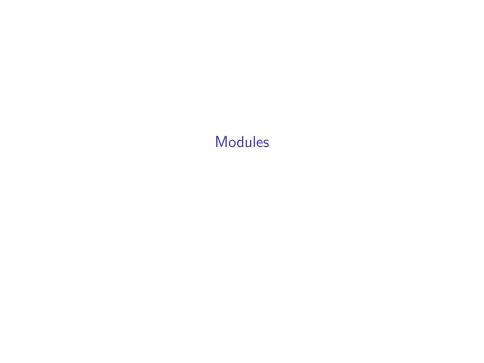
1. Modules



How to think of modules

- Modules are to rings as vector spaces are to fields.
- ▶ Modules are to rings as sets with group actions are to groups.

Definition of (left) modules

Definition: Let R be a ring (for now, not necessarily commutative and not necessarily having a unit). A *left* R-module is an abelian group M together with a map $R \times M \to M$ (written $(r, m) \mapsto rm$) such that:

- $r(m_1 + m_2) = rm_1 + rm_2$
- $(r_1 + r_2)m = r_1m + r_2m$
- $ightharpoonup r_1(r_2m) = (r_1r_2)m$

If R has a unit element 1, we also require 1m = m for all $m \in M$.

Right modules

A right module is defined by a map $M \times R \to M$ and written $(m,r) \mapsto mr$ and satisfying the property

$$(mr_1)r_2=m(r_1r_2).$$

If *R* is not commutative, these really are different, since for a left module:

- $ightharpoonup r_1 r_2$ acts by "first r_2 , then r_1
- while for a right module
 - $ightharpoonup r_1 r_2$ acts by "first r_1 , then r_2 ."

Left and Right modules

If R is commutative, and M is a left R-module, then we can define a right R module M' with the same underlying abelian group M and by defining m'r = (rm)'. This works because

$$(m'r_1)r_2 = (r_1m)'r_2 = (r_2(r_1m))' = ((r_2r_1)m)' = ((r_1r_2)m)' = m'(r_1r_2)$$

Remarks

Vector spaces

If R is a field, then a left (or right) R-module is the same as a vector space.

Another definition

If M is an abelian group, and R is a ring, then a left R-module structure on M is the same as a ring map

$$R \to \operatorname{End}(M)$$
.

If ϕ_r is the endomorphism associated to $r \in R$, then $rm = \phi_r(m)$. The associativity comes from defining the ring structure on

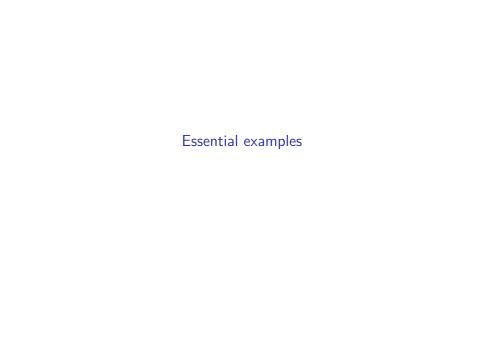
as the usual composition of functions:

$$\phi_{r_1r_2} = \phi_{r_1} \circ \phi_{r_2}.$$

Submodules

Definition: If M is a left R-module, then a submodule N of M is a subgroup with the property that, if $n \in N$, then $rn \in N$ for all $r \in R$.

Observation: A ring R is a left module over itself by ring multiplication. The (left) ideals of R are exactly the left submodules of R.



Rings as modules over themselves

- ► Every ring *R* is a left module over itself. The submodules of *R* are the left ideals.
- ▶ *R* is also a right module over itself, with the right ideals being the right submodules.

If F is a field and n > 1, let $R = M_n(F)$ be the $n \times n$ matrix ring over F. The matrices with arbitrary first column and zeros elsewhere form a left ideal J and therefore a left submodule of R as left R-module. But J is *not* a right R-submodule.

A field F is a one-dimensional vector space over itself, and a commutative ring R is a module (left and right) over itself with the ideals of R being the submodules.

Free modules

Let R be a ring with unity and let $n \ge 1$ be a positive integer. Then

$$R^n = \{(r_1, \ldots, r_n) : r_i \in R \text{ for } i = 1, \ldots, n\}$$

is an R module with componentwise addition and multiplication given by $r(r_1, \ldots, r_n) = (rr_1, \ldots, rr_n)$.

This is called the *free R-module of rank n*.

Free modules and vector spaces

- ▶ If *R* is a field, the free *R*-module of rank *n* is an *n*-dimensional vector space.
- ► The submodules of a finite dimensional vector space are all subspaces which are copies of R^k for $k \le n$.
- For more general R the picture is more complicated. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}^2$. Then:
 - ▶ $\{(n,0): n \in \mathbb{Z}\}$ is a submodule of M which "looks like" a subspace.
 - ▶ $2M = \{(a, b) : a, b \in 2\mathbb{Z}\}$ is a submodule of M which does not.

Change of rings (restriction of scalars)

- ► An abelian group *M* may be an *R* module for different rings *R*. For example:
 - Q is a module over Q, where it is a one dimensional vector space and its only Q-submodules are 0 and itself.
 - $ightharpoonup \mathbb{Q}$ is a module over \mathbb{Z} , and it has many \mathbb{Z} -submodules, such as $\mathbb{Z}[1/2]$.

More generally, if $R \subset S$ is a subring, and M is an S-module, then it is an R-module. This is called *restriction of scalars*.

\mathbb{Z} -modules are the same as abelian groups

Let M be an abelian group. Then it is automatically a \mathbb{Z} -module where we define

$$nx = \overbrace{x + x + \cdots + x}^{n}$$
.

Furthermore, given any \mathbb{Z} -module, it must be the case that

$$nx = (\overbrace{1+1+\cdots+1}^{n})x = \overbrace{x+x+\cdots+x}^{n}.$$

(Note: this is why we require 1x = x when R is a ring with unity in the module axioms).

Further, submodules of M (as \mathbb{Z} -module) are just the subgroups of M (as abelian group).

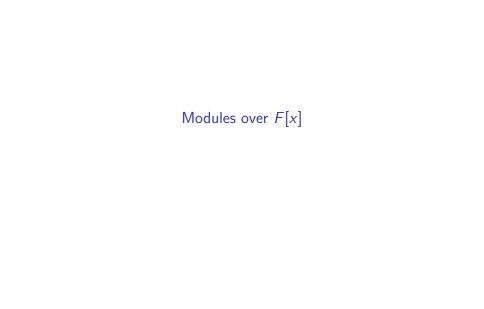
Change of rings (quotients)

Suppose that M is a left R module and $I \subset R$ is a two-sided ideal with the property that, for all $y \in I$, and all $x \in M$, we have yx = 0. In this case we say that I annihilates M or that IM = 0.

With this hypothesis, we may view M as an R/I module by defining (r+I)m=rm for any coset representative $r+I\in R/I$. This is well-defined since two different coset representatives r,r' satisfy r'=r+i for some $i\in I$ and therefore r'm=(r+i)m=rm since im=0.

If M is an abelian group and $m \in Z$ is a positive integer such that mM = 0, then M can be viewed as a module over $\mathbb{Z}/m\mathbb{Z}$ by this process.

This operation is a special case of a general operation called *base* change that we will study in more detail later.



Basic construction

Let F be a field, let V be a vector space over F, and let $T:V\to V$ be an F-linear transformation. Define a homomorphism

$$F[x] \to \operatorname{End}(V)$$

by sending

$$x^k \mapsto T^k = \overbrace{T \circ T \circ \cdots \circ T}^n.$$

This construction makes V into a module for F[x] which depends on the choice of the linear transformation T.

Polynomials and linear transformations

For example let $V={\it F}^2$ and let ${\it T}$ be the linear transformation given by the matrix

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

If e_0 and e_1 are the standard basis elements of F^2 then

$$Te_0 = e_1$$
 $T^2e_0 = Te_1 = e_0 + e_1 = e_0 + Te_0 = (1+T)e_0$

from which we see that $(T^2 - T - 1)e_0 = 0$ and

$$(T^2 - T - 1)e_1 = (T^2 - T - 1)Te_0 = T(T^2 - T - 1)e_0 = 0$$

so the polynomial $x^2 - x - 1$ is in the kernel of the map from $F[x] \to \operatorname{End}(V)$

Characterization of F[x] modules

We saw above that, given an F-vector space V with a linear transformation T, we get an F[x] module where x acts on V through T.

Conversely, suppose that M is an module over F[x]. Then M is an F vector space (via the restriction of scalars from F[x] to F). Furthermore, the element $x \in F[x]$ acts on M as an F-linear transformation because that's what the module axioms amount to.

Therefore there is an equivalence between

 $\{F[x] - \text{modules}\} \Leftrightarrow \{\text{vector spaces } V \text{ over } F \text{ with a given linear map}$

Submodules of F[x] modules

In the correspondence above, a submodule of an F[x] module M corresponds to a subspace $W \subset V$ that is *preserved by* T, meaning $TW \subset W$.

Thus, not all subspaces of V correspond to submodules.

In the example given earlier, the only T-stable proper subspace of V is the zero subspace.

If we consider instead the linear map on F^2 satisfying $Ue_0=0$ and $Ue_1=e_0$, then the one dimensional subspace spanned by e_0 is U-stable and F^2 viewed as an F[x] module via U has a submodule corresponding to that subspace.

Checking the submodule property

Proposition: A subset N of a left R-module M is a submodule if it is nonempty and, for all $x, y \in N$ and $r \in R$, we have $x + ry \in N$. Alternatively, if N is a subgroup of the abelian group M and $rN \subset N$ for all $r \in R$ then N is a submodule.

Algebras

Definition: Let R be a commutative ring with unity. An R-algebra is a (not necessarily commutative) ring S with a ring homomorphism $f: R \to S$ carrying 1_R to 1_S such that f(R) is in the center of S.

The polynomial ring F[x] is an F-algebra, as is the matrix ring $M_n(F)$ where the homomorphism $f: F \to M_n(F)$ embeds F as the diagonal matrices. More generally, any F-algebra A, where F is a field, contains F in its center and the identites of A and F are the same.

The ring $\mathbb{Z}/p\mathbb{Z}$ is a \mathbb{Z} -algebra. In fact any ring S with 1 is a \mathbb{Z} algebra by the map sending $n \in \mathbb{Z}$ to $n1_S$.

The ring $\mathbb{Q}[x]$ is a $\mathbb{Z}[x]$ algebra.

We typically omit the explicit map f and just think of R as "contained in" A; this can be misleading since f doesn't need to be injective, but it works in practice.

Algebra morphisms

Definition: A map of R-algebras $f:A\to B$ is a ring homomorphism that is R-linear in the sense that f(ra)=rf(a) for all $r\in R$ and $a\in A$.

Any homomorphism of rings with unity is a \mathbb{Z} -algebra morphism.