# 4. Canonical Forms

## Canonical Forms: Preliminaries

### Set-up

Let $F$ be a field, let $V$ be a nontrivial finite dimensional vector space over $F$, and let $T : V \to V$ be a linear transformation.

Then $V$ is an $F[x]$-module.

**Lemma:** $V$ is a finitely generated torsion module over $F[x]$.

**Proof:** $V$ is generated by a basis, which is finite by assumption. If $V$ were not torsion, there would be a $v \in V$ so that the map $F[x] \to V$ given by $f(x) \mapsto f(T)v$ is injective; but that contradicts the fact that $V$ is finite dimensional.

### Eigenvalues and the Characteristic Polynomial.

**Definition:** An element $\lambda \in F$ is an eigevanlue of $T$ if there is a nonzero $v \in V$ with $Tv = \lambda v$. The vector $v$ is an eigenvector for this eigenvalue.

**Definition:** Let $c(x) = \det(xI - T)$. Then $c(x)$ is a monic polynomial of degree $n$ called the *characteristic polynomial* of $T$.

**Lemma:** $\lambda$ is an eigenvalue of $T$ if and only if $c(\lambda) = 0$.

**Proof:** If $c(\lambda) = 0$, then $\det(\lambda I - T) = 0$ so $\lambda I - T$ has a nontrivial kernel; an element of the kernel is an eigenvector with eigenvalue $\lambda$. Conversely, if $\lambda$ is an eigenvalue, then $\lambda I - T$ has nontrivial kernel, so its determinant is zero.

### The Minimal polynomial

Since $V$ is a torsion $F[x]$ module, its annihilator

$$\mathrm{Ann}(V) = \{f(x) \in F[x] : f(T)v = 0 \forall v \in V$$

is a nonzero ideal in $F[x]$, hence a principal ideal generated by a unique monic polynomial $m(x)$.

**Definition:** The unique monic generator $m(x)$ of $\mathrm{Ann}(V)$ is called the minimal polynomial of $T$.

**Lemma:** The minimal polynomial $m(x)$ is the monic polynomial of minimal degree such that $m(T)v = 0$ for all $v \in V$.

**Proof:** By definition $m(T)v = 0$ for all $v \in V$. The collection of polynomials $h(x)$ such that $h(T)v = 0$ is exactly $\mathrm{Ann}(V)$, and the generator of the ideal $\mathrm{Ann}(V)$ in the Euclidean ring $F[x]$ is its monic polynomial element of minimal degree.

#### The structure of $V$

By the fundamental theorem of finitely generated modules over PID's, we have two different ways to represent $V$ as an $F[x]$ module.

**Invariant Factors:** There are monic polynomials $f_1(x)|f_2(x)|\cdots|f_k(x)$ such that

$$V = F[x]/(f_1(x)) \oplus F[x]/(f_2(x)) \oplus \cdots \oplus F[x]/(f_k(x)).$$

**Elementary Divisors:** There are irreducible polynomials $f_1(x), \ldots, f_k(x)$ and nonnegative integers $e_1, \ldots, e_k$ such that

$$V = F[x]/(f_1(x)^{e_1}) \oplus F[x]/(f_2(x)^{e_2}) \oplus \cdots \oplus F[x]/(f_k(x)^{e_k})$$

The invariant factors are uniquely determined; and the elementary divisors are uniquely determined up to order.

#### Invariant factors and the minimal polynomial

**Lemma:** The minimal polynomial $m(x)$ of $T$ is the last (the "largest") invariant factor of $T$ acting on $V$; all the invariant factors divide $m(x)$.

## The Rational normal form

#### The cyclic case

Let's focus our attention for the moment on a cyclic $F[x]$ module of the form $M = F[x]/(f(x))$ where

$$f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0$$

- $M$ is a finite dimensional vector space of degree equal to the degree $d$ of $f(x)$.
- $1, x, x^2, \ldots, x^{d-1}$ is a basis for this module.
- $x$ is an $F$-linear transformation $M \to M$.

#### Matrix of x

The linear map given by multiplication by $x$ has the following matrix form in the basis $1, x, x^2, \ldots, x^{d-1}$:

$$[x] = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{d-1} \end{pmatrix}$$

**Definition:** The matrix for $[x]$ above is called the companion matrix for the polynomial $f(x)$.

### Characteristic and minimal polynomials of the companion matrix

**Lemma:** The characteristic and minimal polynomials of this linear transformation are both $f(x)$.

**Proof:** The fact that the characteristic polynomial of $[x]$ is $f(x)$ is a computation. The fact that $f(x)$ is the minimal polynomial follows from the fact that $x$ clearly satisfies $f(x)$, and, since $1, x, \ldots, x^{d-1}$ are linearly dependent, there is no relation $\sum a_i x^i = 0$ of degree less than $d$.

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