Solutions of Assignment 11

> Theory of Computation, Fall 2022

Keys

Q1

Proof:

I)

- L_1 is not recursive:

  We can construct a reduction from H to L_1 to show that L_1 is not recursive.

  There is a recursive function $f(''M''''w'') = M_{''M''''w''}$, in which $M_{''M''''w''}$ is defined as follows.

$$M_{''M''''w''} = \begin{array}{rl} & \text{on input } u: \\ 1. & \text{Run } M \text{ on } w \end{array} \tag{1}$$

  For any input $''M''''w''$ of H,

  - $$\begin{gathered} ''M''''w'' \in H \implies M \text{ halts on } w \implies \\ M_{''M''''w''} \text{ halts on every input } \implies \\ M_{''M''''w''} \text{ halts on at least 2023 strings } \implies f(''M''''w'') \in L_1 \end{gathered} \tag{2}$$

  - $$\begin{gathered} ''M''''w'' \in \overline{H} \implies M \text{ does not halt on } w \implies \\ M_{''M''''w''} \text{ halts on no input } \implies \\ M_{''M''''w''} \text{ halts on less than 2023 strings } \implies f(''M''''w'') \in \overline{L_1} \end{gathered} \tag{3}$$

  Therefore, $f(''M''''w'') = M_{''M''''w''}$ is a reduction from H to L_1. Thus L_1 is not recursive.

II)

- L_1 is recursively enumerable:

  Since Σ^* is recursive, it is Turing enumerable, so we can construct a permutation $s_1, s_2, \ldots$ of strings in Σ^*.

  Construct a Turing machine M_1 as follows:

$$\begin{aligned} M_1 = & \quad && \text{on input } ''M'' : \\ & 1. && \text{Set } c = 0 \\ & 2. && \text{For } i = 1, 2, \ldots \text{ loop} \\ & 3. && \quad \text{For } s = s_1, s_2, \ldots, s_i \text{ without mark} \\ & 4. && \quad\quad \text{Run } M \text{ on } s \text{ for } i \text{ steps} \\ & 5. && \quad\quad \text{If } M \text{ halts on } s \text{ in } i \text{ steps} \\ & 6. && \quad\quad\quad \text{Mark } s_i \\ & 7. && \quad\quad\quad \text{Set } c = c + 1 \\ & 8. && \quad\quad\quad \text{If } c \geq 2023 \\ & 9. && \quad\quad\quad\quad \text{Halt} \end{aligned} \tag{4}$$

  We have that M_1 semidecides L_1. Thus L_1 is recursively enumerable.

Q2

Proof:

We can construct a reduction from $\overline{H}$ to L_2 to show that L_2 is not recursively enumerable.

There is a recursive function $f(''M''''w'') = M_{''M''''w''}$, in which $M_{''M''''w''}$ is defined as follows.

$$\begin{aligned} M_{''M''''w''} = & \quad && \text{on input } u : \\ & 1. && \text{Run } M \text{ on } w \end{aligned} \tag{5}$$

For any input $''M''''w''$ of H,

- $$\begin{gathered} ''M''''w'' \in \overline{H} \implies M \text{ does not halt on } w \implies \\ M_{''M''''w''} \text{ halts on no input } \implies \\ M_{''M''''w''} \text{ halts on at most 2022 strings } \implies f(''M''''w'') \in L_2 \end{gathered} \tag{6}$$

- $$\begin{gathered} ''M''''w'' \in H \implies M \text{ halts on } w \implies \\ M_{''M''''w''} \text{ halts on every input } \implies \\ M_{''M''''w''} \text{ halts on less than 2022 strings } \implies f(''M''''w'') \in \overline{L_2} \end{gathered} \tag{7}$$

Therefore, $f(''M''''w'') = M_{''M''''w''}$ is a reduction from $\overline{H}$ to L_2. Thus L_2 is not recursively enumerable.

> **Note:** Ignoring the input that is not an encoded Turing machine, L_2 is the complement of L_1. You can also prove **Q2** using the conclusion of **Q1** and the fact that if a language A is recursively enumerable but not recursive, then if $\overline{A}$ is not recursively enumerable.

Q3

We can construct a reduction from $\overline{H}$ to L_3 to show that L_3 is not recursive.

There is a recursive function $f(''M''''w'') = M_{''M''''w''}$, in which $M_{''M''''w''}$ is defined as follows.

$$M_{''M''''w''} = \begin{array}{rl} & \text{on input } u: \\ 1. & \text{Run } M \text{ on } w \end{array} \tag{8}$$

For any input $''M''''w''$ of H,

- $$\begin{gathered} ''M''''w'' \in \overline{H} \implies M \text{ does not halt on } w \implies \\ M_{''M''''w''} \text{ halts on no input } \implies \\ \text{At least 2023 strings on which } M \text{ does not halt } \implies f(''M''''w'') \in L_3 \end{gathered} \tag{9}$$

- $$\begin{gathered} ''M''''w'' \in H \implies M \text{ halts on } w \implies \\ M_{''M''''w''} \text{ halts on every input } \implies \\ \text{Less than 2023 strings on which } M \text{ does not halt } \implies f(''M''''w'') \in \overline{L_3} \end{gathered} \tag{10}$$

Therefore, $f(''M''''w'') = M_{''M''''w''}$ is a reduction from $\overline{H}$ to L_3. Thus L_3 is not recursively enumerable.

Q4

We can construct a reduction from $\overline{H}$ to L_4 to show that L_4 is not recursively enumerable.

There is a recursive function $f(''M''''w'') = M_{''M''''w''}$, in which $M_{''M''''w''}$ is defined as follows.

$$M_{''M''''w''} = \begin{array}{rl} & \text{on input } u: \\ 1. & \text{Run } M \text{ on } w \text{ for } |u| \text{ steps} \\ 2. & \text{If } M \text{ halts on } w \text{ in } |u| \text{ steps} \\ 3. & \quad \text{Loop forever} \\ 4. & \text{Else } (M \text{ does not halt on } w \text{ in } |u| \text{ steps}) \\ 5. & \quad \text{Halt} \end{array} \tag{11}$$

For any input $"M""w"$ of H,

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$$
\begin{aligned}
"M""w" \in \overline{H} \implies M \text{ does not halt on } w \implies \\
M_{"M""w"} \text{ halts on every input } \implies \\
\text{At most 2022 strings on which } M \text{ does not halt } \implies f("M""w") \in L_4
\end{aligned}
\tag{12}
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\begin{aligned}
"M""w" \in H \implies M \text{ halts on } w \implies \\
\exists n > 0, \forall u \text{ with } |u| \geq n, M_{"M""w"} \text{ does not halt on } u \implies \\
\text{More than 2022 strings on which } M \text{ does not halt } \implies f("M""w") \in \overline{L_4}
\end{aligned}
\tag{13}
$$

Therefore, $f("M""w") = M_{"M""w"}$ is a reduction from $\overline{H}$ to L_4. Thus L_4 is not recursively enumerable.

Q5

The grammar $G = (V, \Sigma, S, R)$ generates $\{ww : w \in \{a, b\}^*\}$ where

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$$V = \{T, A, B, S_1, S, a, b\} \tag{14}$$

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$$\Sigma = \{a, b\} \tag{15}$$

- R is the set of the following rules

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\begin{aligned}
S &\rightarrow TS_1 \\
S_1 &\rightarrow AaS_1 \\
S_1 &\rightarrow BbS_1 \\
S_1 &\rightarrow e \\
aA &\rightarrow Aa \\
bA &\rightarrow Ab \\
aB &\rightarrow Ba \\
bB &\rightarrow Bb \\
TA &\rightarrow aT \\
TB &\rightarrow bT \\
T &\rightarrow e
\end{aligned}
\tag{16}
$$

Grading

100 pts. in total

- Q1. 32 pts. = 12 pts. + 20 pts.
- Q2. 12 pts.
- Q3. 12 pts.
- Q4. 20 pts.
- Q5. 24 pts.