Solutions of Assignment 6

> Theory of Computation, Fall 2022

Keys

Q1

Proof Using CFG:

Since L is context-free, there is some CFG $G = (V, \Sigma, S, R)$ that generates L.

Construct a CFG $G' = (V, \Sigma, S, R')$ with R' as follows:

$$R' = \{(A, u^R) \mid (A, u) \in R\} \tag{1}$$

G' generates L^R. Thus L^R is context-free.

Proof Using PDA:

Since L is context-free, there is some PDA $P = (K, \Sigma, \Gamma, \Delta, s, F)$ that accepts L.

Construct a PDA $P' = (K', \Sigma, \Gamma, \Delta', s', F')$ as follows. Here s' is a newly constructed state.

$$\begin{array}{rcl} K' & = & K \cup \{s'\} \text{ where } s' \notin K \\ F' & = & \{s\} \\ \Delta' & = & \{((p, c, \alpha), (q, \beta)) \mid ((q, c, \beta), (p, \alpha)) \in \Delta\} \quad \cup \\ & & \{((s', e, e), (f, e)) \mid f \in F\} \end{array} \tag{2}$$

P' accepts L^R. Thus L^R is context-free.

Q2

Assume that $A = \{wcw \mid w \in \{a,b\}^*\}$ is context-free.

p being the pumping length, we pick $s = wcw = a^p b^p c a^p b^p \in A$ and $|s| \geq p$.

By pumping theorem, s can be **arbitrary** divided into $s = uvxyz$ $s.t.$

1.

$$\forall i \geq 0, uv^i xy^i z \in L \tag{3}$$

2.

$$|v| + |y| > 0 \tag{4}$$

3.

$$|vxy| \leq p \tag{5}$$

Given $s = uvxyz = w_1 c w_2$ where $w_1 = w_2$, we divide all possible cases as follows.

- If v or y contains c, then for $i \neq 1$, the string after pumping $uv^i xy^i z$ would contain not exactly one c.

  Thus $uv^i xy^i z \notin A$, contradicting to condition 1 of pumping theorem.
- If x contains c, then v is in w_1 and y is in w_2, and there must be $v = b^{|v|}$ and $y = a^{|y|}$.

  For $i = 0$, $uv^i xy^i z = a^p b^{p-1} c a^{p-1} b^p \notin A$, contradicting to condition 1 of pumping theorem.
- If x does not contain c, then either v and y both in w_1 or v and y both in w_2 stands.

  For $i = 0$, given $uv^i xy^i z = w_1' c w_2'$, there is $|w_1'| < |w_2'|$ if v and y are both in w_1, and $|w_1'| > |w_2'|$ if v and y are both in w_2.

  Thus $uv^i xy^i z \notin A$ no matter which case above stands, contradicting to condition 1 of pumping theorem.

To sum up, $\forall p, \exists s$ with $|s| \geq p$ that violates the pumping theorem, hence A is not context-free.

Q3

Since A is context-free and B is regular,

- There is some PDA $P_A = (K_A, \Sigma, \Gamma, \Delta_A, s_A, F_A)$ accepting A;
- There is some NFA $M_B = (K_B, \Sigma, \Delta_B, s_B, F_B)$ accepting B.

Construct a new PDA $P_\cap = (K_\cap, \Sigma, \Delta_\cap, s_\cap, F_\cap)$ as follows.

$$
\begin{aligned}
K_\cap &= K_A \times K_B \\
s_\cap &= (s_A, s_B) \\
F_\cap &= F_A \times F_B \\
\Delta_\cap &= \{(((q_A, q_B), c, \alpha), ((p_A, p_B), \beta)) \mid \\
&\quad ((q_A, c, \alpha), (p_A, \beta)) \in \Delta_A \wedge ((q_B, c), p_B) \in \Delta_B\}
\end{aligned}
\tag{6}
$$

$P_\cap$ accepts $A \cap B$, hence $A \cap B$ is context-free.

Q4

(a)

Suppose that A is context-free. By the conclusion of **Q3**, we have that $A \cap a^*b^*c^* = \{a^nb^nc^n \mid n \geq 0\}$ is also context-free. However, we have proved that $\{a^nb^nc^n \mid n \geq 0\}$ is not context-free, which makes a contradiction. Therefore, A is not context-free.

(b)

$\overline{A}$ can be written as a union of the following 4 languages. Here $\#a$ is the number of a's in w, and $\#b$ and $\#c$ is defined similarly.

$$
\begin{gathered}
A_1 = \{w \in \{a, b, c\}^* \mid \#a > \#b\} \\
A_2 = \{w \in \{a, b, c\}^* \mid \#a < \#b\} \\
A_3 = \{w \in \{a, b, c\}^* \mid \#a > \#c\} \\
A_4 = \{w \in \{a, b, c\}^* \mid \#a < \#c\} \\
\overline{A} = \bigcup_{i=1}^{4} A_i
\end{gathered}
\tag{7}
$$

We can a construct CFGs that accept or PDAs that generate A_i $(i = 1, 2, 3, 4)$ to show that they are context-free.

Since the class of context-free languages is closed under union, $\overline{A}$ is context-free.

(c)

The class of context-free languages is not closed under complementation.

Q5

- $L - R$ is necessarily context-free.

  We have that $L - R = L \cap \overline{R}$. The class of regular languages is closed under complementation, so $\overline{R}$ is also regular.

  Therefore, $L - R$ is the intersection of a context-free language and a regular language. Thus it is context-free based on the conclusion of **Q3**.

- $R - L$ is not necessarily context-free.

  We have that Σ^* is regular, and $\Sigma^* - L = \overline{L}$, which is not necessarily context-free based on the conclusion of **Q4**.

Grading

100 pts. in total

- Q1. 15 pts.
- Q2. 25 pts.
- Q3. 15 pts.
- Q4. 25 pts. = (a) 10 pts. + (b) 10 pts. + (c) 5 pts.
- Q5. 20 pts.

Notes

Q1

Idea: Reverse the result of every derivation when parsing, or reverse all the arrows in the PDA.

Note that in the original PDA there may be not exactly 1 final state. We have to construct a new initial state in the "reversed" PDA.

Q2

Note that to prove a language is **not** context-free, we have to prove that all possible division of $s = uvxyz$ violate the pumping theorem. That is to say, the division is **arbitrary**. You are not supposed to specify the division by yourself.

Besides, be careful not to leave out some cases.

Q3

Similar to **Assignment1 Q5**. Useful conclusion for **Q4** and **Q5**.

Q4

- (a) We can construct another 2 languages as follows:

  $$\begin{aligned} A_5 &= \{w \in \{a,b,c\}^* \mid \#b > \#c\} \\ A_6 &= \{w \in \{a,b,c\}^* \mid \#b < \#c\} \end{aligned} \tag{8}$$

  Here $\overline{A} = \bigcup_{i=1}^{6} A_i$. It is also correct, but unnecessary, since $A_5 \cup A_6 \subseteq \bigcup_{i=1}^{4} A_i$.

  Besides, proof using pumping theorem is also acceptable. Let p be the pumping length, we can take string $a^p b^p c^p \in A$ to violate the pumping theorem.

- (b) Direct proof by constructing a PDA accepting or a CFG generating $\overline{A}$ is also acceptable.
- (c) Useful conclusion for **Q5**.

  Note: The statement "the complement of a context-free language is not context-free and vice versa" is incorrect. The complement of a context-free language is not **necessarily** context-free, but it **may** be context-free.

Q5

Make use of the conclusions of the previous questions.

Besides, to disprove a conclusion, just use some ordinary case to construct a counterexample.