Q1.

Q 1.

$$f(x) = \frac{1}{2} x^T Q x + a^T x$$

$$\nabla f(x)^T = Qx + a$$

$$g_k = \nabla f(x_k)^T = Q x_k + a$$

(a) $$f(x_{k+1}) = \frac{1}{2} x_{k+1}^T Q x_{k+1} + a^T x_{k+1}$$

$$= \frac{1}{2} (x_k - \alpha g_k)^T Q (x_k - \alpha g_k) + a^T (x_k - \alpha g_k)$$

$$= \underbrace{\frac{1}{2} x_k^T Q x_k}_{} - \alpha x_k^T Q g_k + \frac{1}{2} \alpha^2 g_k^T Q g_k + \underbrace{a^T x_k}_{} - \alpha \cdot a^T g_k$$

$f(x_k) = \frac{1}{2} x_k^T Q x_k + a^T x_k$

$$= f(x_k) - \underbrace{\alpha x_k^T Q g_k}_{} + \frac{1}{2} \alpha^2 g_k^T Q g_k - \underbrace{\alpha a^T g_k}_{}$$

$$= f(x_k) - \alpha (Q x_k + a)^T g_k + \frac{1}{2} \alpha^2 g_k^T Q g_k$$

$$= f(x_k) - \alpha g_k^T g_k + \frac{1}{2} \alpha^2 g_k^T Q g_k$$

$$= f(x_k) - \alpha\left(g_k^T g_k - \frac{1}{2}\alpha\, g_k^T Q g_k\right)$$

$$= f(x_k) - \alpha\left(g_k^T\left(I - \frac{1}{2}\alpha Q\right) g_k\right)$$

Therefore,

$$f(x_k) - f(x_{k+1}) = \alpha\cdot\left(g_k^T \underbrace{\left(I - \frac{1}{2}\alpha Q\right)} g_k\right)$$

If we want to ensure that

$f(x_k) - f(x_{k+1}) > 0$ whenever $g_k \neq 0$,

then $I - \frac{1}{2}\alpha Q$ needs to be positive definite.

Note that when Q has eigenvalues $\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 > 0$,

$I - \frac{1}{2}\alpha Q$ has the following eigenvalues:

$$1 - \frac{1}{2}\alpha\lambda_n \leq 1 - \frac{1}{2}\alpha\lambda_{n-1} \leq \cdots \leq 1 - \frac{1}{2}\alpha\cdot\lambda_1$$

(This can be seen because when x_i is an eigenvector of Q associated with the eigenvalue λ_i,

$$\left(I - \frac{1}{2}\alpha Q\right)x_i = x_i - \frac{1}{2}\alpha Q x_i = x_i - \frac{1}{2}\alpha\cdot\lambda_i x_i$$
$$= \left(1 - \frac{1}{2}\alpha\cdot\lambda_i\right)x_i \quad)$$

For $I - \frac{1}{2}\alpha Q$ to be positive definite,

its smallest eigenvalue, $1 - \frac{1}{2}\alpha\lambda_n$, has to be positive.

$$1 - \frac{1}{2}\alpha\lambda_n > 0$$

$$\Leftrightarrow \quad \alpha < \frac{2}{\lambda_n}$$

Therefore, the condition is $\alpha \in (0, \frac{2}{\lambda_n})$.

(b) Suppose that α is a constant in $(0, \frac{2}{\lambda_n})$.

$\Gamma := \{x \in \mathbb{R}^n : \nabla f(x) = 0\}$.

[redacted], $A(x) = x - \alpha \cdot \nabla f(x)^T$.

Let y denote $A(x)$.

① When $x \notin \Gamma$,

$\nabla f(x)^T \neq \vec{0}$, and in (a), we showed that

$$f(y) < f(x).$$

③ when $x \in \Gamma$,

$\nabla f(x)^T = \vec{0}$, and $y = x - \alpha \cdot \nabla f(x)^T = x$.

Therefore, $f(y) = f(x)$.

(Hence, " $f(y) \leq f(x)$ " is satisfied).

Therefore, $Z := f$ is a descent function

for Γ and A.

(c) $A(x) = x - \alpha \cdot \nabla f(x)^T$

$= x - \alpha \cdot (Qx + a)$

A is a continuous, point-to-point mapping on $\mathbb{R}^n$.

Therefore, A is closed on $\mathbb{R}^n$.

(d) Because $\{x_k\}$ is bounded, there exists $M > 0$

such that $\|x_k\| < M$ for all k.

$\boxed{\text{① Therefore, } x_k \text{ is included in the following compact subset of } \mathbb{R}^n\text{: } x_k \in \{x \in \mathbb{R}^n : \|x\| \le M\} \text{ for all } k.}$

In addition, in (b), we showed that

② $\boxed{Z := f \text{ is a descent function for } \Gamma \text{ and } A.}$

And, in (c), we showed that

③ $\boxed{A \text{ is closed on } \mathbb{R}^n.}$

① ~ ③ implies that all the conditions of the Global Convergence Theorem hold

$\therefore$ Every limit point of $\{x_k\}$ is in Γ.

(Review optimality conditions of convex minimization)

(e) Note that the Hessian

$$F(x) = Q$$

is <u>positive definite</u> for all $x \in \mathbb{R}^n$.

Therefore, f is convex.

Since f is convex, and the problem is unconstrained,

$\nabla f(x)^T = \vec{0}$ implies that

x is a global minimum point.

In (d), we concluded that

every limit point $\bar{x}$ of $\{x_k\}$ is in Γ,

i.e., $\nabla f(\bar{x})^T = \vec{0}$.

Combining <u>this</u> with <u>the above statement</u>,

we can have the following statement.

"Every limit point $\bar{x}$ of $\{x_k\}$ is

a global minimum point."