Chapter 8

Frequency Response Methods

- In conventional control system analysis

→2 basic methods for predicting and adjusting the performance of a system without finding the solution of the system’s differential equation

→Root locus and frequency response methods

- Why not solve the system’s differential equation?

→No powerful computers in the past to solve high-order systems

→Even if computers are available, these 2 methods can provide much insight into system’s design and analysis

8.1 Introduction

What

- Frequency response of a system

→steady-state response to a sinusoidal input signal

- 3 parameters of a sinusoid

→ amplitude, frequency, and phase

- Frequency response of LTI

→ A sinusoid that differs from the input only in amplitude and phase angle

$Y(s) = T(s)R(s)$ with a sinusoidal input $r(t) = A \sin \omega t.$ $R(s) = \dfrac{A\omega}{s^2 + \omega^2}$

Assume distinct poles in TF: $T(s) = \dfrac{m(s)}{q(s)} = \dfrac{m(s)}{\prod_{i=1}^{n}(s + p_i)},$

input

Output in partial fraction form: $Y(s) = \dfrac{k_1}{s + p_1} + \cdots + \dfrac{k_n}{s + p_n} + \dfrac{\alpha s + \beta}{s^2 + \omega^2}.$

Inverse LT: $y(t) = k_1 e^{-p_1 t} + \cdots + k_n e^{-p_n t} + \mathscr{L}^{-1}\left\{\dfrac{\alpha s + \beta}{s^2 + \omega^2}\right\},$

For a stable system: $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \mathscr{L}^{-1}\left\{\dfrac{\alpha s + \beta}{s^2 + \omega^2}\right\},$

For the steady state: $y(t) = \mathscr{L}^{-1}\left[\dfrac{\alpha s + \beta}{s^2 + \omega^2}\right] = \dfrac{1}{\omega}\left|A\omega T(j\omega)\right| \sin(\omega t + \phi)$

$= A|T(j\omega)| \sin(\omega t + \phi),$ where $\phi = \underline{/T(j\omega)}.$

$$y(t) = \mathscr{L}^{-1}\left[\frac{\alpha s + \beta}{s^2 + \omega^2}\right] = \frac{1}{\omega}\left|A\omega T(j\omega)\right|\sin(\omega t + \phi)$$

$$= A|T(j\omega)|\sin(\omega t + \phi), \qquad \text{where } \phi = \underline{/T(j\omega)}.$$

- the steady-state output signal depends only on the magnitude and phase of *T*(*j*w) at a specific frequency
- Steady-state response described above is true only for stable systems

3W Questions: what, why, and how

Why

1. The ready availability of sinusoid test signals for various ranges of frequencies and amplitudes

→ experimental determination of the system frequency response is easily accomplished.

→ unknown transfer function of a system can be deduced

2. Design of a system in the frequency domain provides the designer with control of the bandwidth of a system, as well as some measure of the response of the system to undesired noise and disturbances.
3. The magnitude and phase angle of T(jw) are readily represented by graphical plots that provide significant insight into the analysis and design of control systems.

Disadvantage of frequency response analysis and design

- Indirect link between the frequency and the time domain
- Indirect link between the frequency response and the corresponding transient response

→ except for 2nd-order systems

Laplace and Fourier Transforms

- LT

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\,ds,$$

- FT

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega.$$

Questions to be asked:

- T(jw)=T(s) with s=jw, why?
- LT and FT are closely related. Why not use the LT? Why use the FT at all?

Caution!

- $F(j\omega) = F(s)|_{s=j\omega}$ is not always true.

For example, $\mathrm{f(t)} = \sin\omega t, F(s) = \frac{\omega}{s^2+\omega^2}$, but $F(s)|_{s=j\omega}$ does not exist while $F(j\omega)$ exists as a sum of two delta functions

- If $\mathrm{f(t)}$ is absolutely integrable, then $F(j\omega) = F(s)|_{s=j\omega}$ is true.

8.2 Frequency Response Plots

- Transfer function of a system G(s)

→ Sinusoidal steady-state transfer function G(jw)

Cartesian representation

$$G(j\omega) = G(s)|_{s=j\omega} = R(\omega) + jX(\omega),$$

$$R(\omega) = \mathrm{Re}[G(j\omega)] \quad \text{and} \quad X(\omega) = \mathrm{Im}[G(j\omega)].$$

Polar representation

$$G(j\omega) = |G(j\omega)|e^{j\phi(\omega)} = |G(j\omega)|\underline{/\phi(\omega)},$$

$$\phi(\omega) = \tan^{-1}\frac{X(\omega)}{R(\omega)} \quad \text{and} \quad |G(j\omega)|^2 = [R(\omega)]^2 + [X(\omega)]^2.$$

3 types of Frequency Plots

1. Polar plot (Nyquist plot)

→ The locus of real and imaginary parts for various values of w

2. Logarithmic plot (Bode plot)

→ The magnitude and phase plots versus w

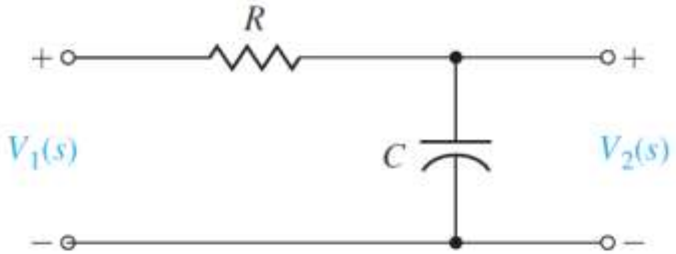
3. Log-magnitude–phase diagram (+M and N circles=Nichols chart): a plot of magnitude versus phase for various values of w

→ These 3 plots can be obtained using the below equations:

Cartesian representation $$G(j\omega) = G(s)|_{s=j\omega} = R(\omega) + jX(\omega),$$

Polar representation $$G(j\omega) = |G(j\omega)|e^{j\phi(\omega)} = |G(j\omega)|\underline{/\phi(\omega)},$$

Example 8.1



- Transfer function

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs + 1},$$

- Sinusoidal steady-state transfer function

$$G(j\omega) = \frac{1}{j\omega(RC) + 1} = \frac{1}{j(\omega/\omega_1) + 1}, \qquad \omega_1 = \frac{1}{RC}.$$

$$G(j\omega) = R(\omega) + jX(\omega) = \frac{1 - j(\omega/\omega_1)}{(\omega/\omega_1)^2 + 1}$$

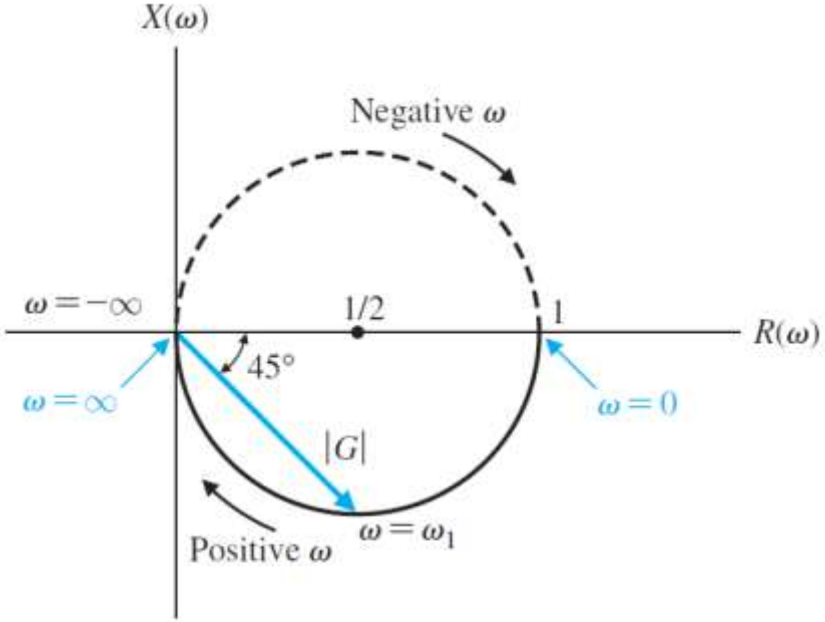
$$= \frac{1}{1 + (\omega/\omega_1)^2} - \frac{j(\omega/\omega_1)}{1 + (\omega/\omega_1)^2}.$$

Find the real and imaginary parts

Cartesian representation

$$G(j\omega) = R(\omega) + jX(\omega) = \frac{1 - j(\omega/\omega_1)}{(\omega/\omega_1)^2 + 1}$$

$$= \frac{1}{1 + (\omega/\omega_1)^2} - \frac{j(\omega/\omega_1)}{1 + (\omega/\omega_1)^2}.$$



Polar representation

$$|G(j\omega)| = \frac{1}{[1 + (\omega/\omega_1)^2]^{1/2}} \quad \text{and} \quad \phi(\omega) = -\tan^{-1}(\omega/\omega_1).$$

Example 8.2

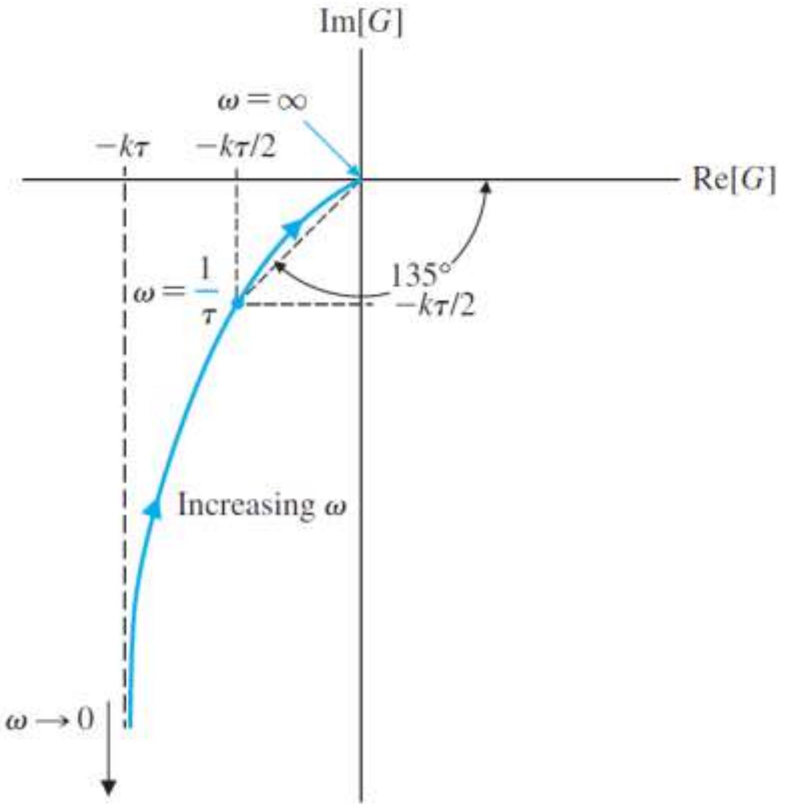
$$G(s)|_{s=j\omega} = G(j\omega) = \frac{K}{j\omega(j\omega\tau + 1)} = \frac{K}{j\omega - \omega^2\tau}.$$

Polar representation

$$|G(j\omega)| = \frac{K}{(\omega^2 + \omega^4\tau^2)^{1/2}} \quad \text{and} \quad \phi(\omega) = -\tan^{-1}\frac{1}{-\omega\tau}.$$

Cartesian representation

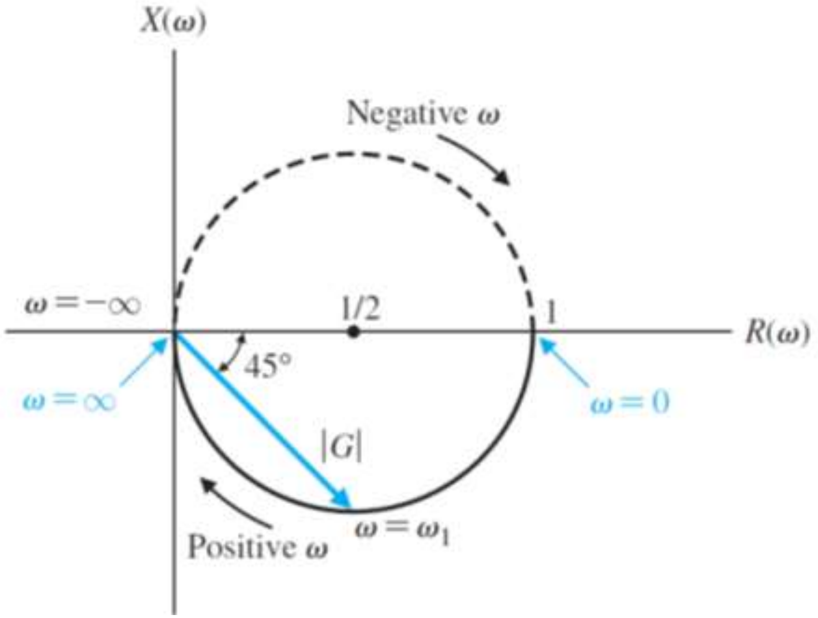
$$G(j\omega) = \frac{K}{j\omega - \omega^2\tau} = \frac{K(-j\omega - \omega^2\tau)}{\omega^2 + \omega^4\tau^2} = R(\omega) + jX(\omega),$$



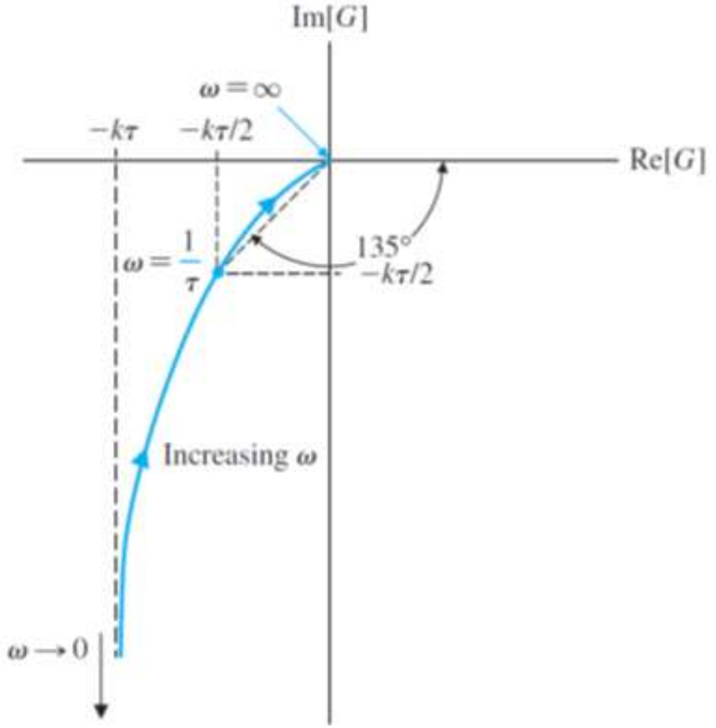
Limitations of Polar Plots

1. Addition of poles or zeros requires the recalculation of the frequency response
2. Calculation is tedious and does not indicate the effect of the individual poles and zeros

Ex 8.1: $G(s) = \frac{1}{\tau s+1}$



Ex 8.2: $G(s) = \frac{1}{s(\tau s+1)}$



Logarithmic Plots or Bode plots

- In honor of H. W. Bode who used them extensively in his studies of feedback amplifiers (This explains why it is called Bode plots)

→Addition of poles or zeros requires no recalculation of the frequency response

→Bode plot can indicate the effect of the individual poles and zeros

- Transfer function in frequency domain

$$G(j\omega) = |G(j\omega)|e^{j\phi(\omega)}.$$

$$\text{Logarithmic gain} = 20\log_{10}|G(j\omega)|, \quad \textbf{decibels (dB)}$$

This explains why it is also called logarithmic plots

Example 8.3

- Transfer function of Ex 8.1 $G(j\omega) = \dfrac{1}{j\omega(RC) + 1} = \dfrac{1}{j\omega\tau + 1}, \quad \tau = RC,$
- Logarithmic gain:

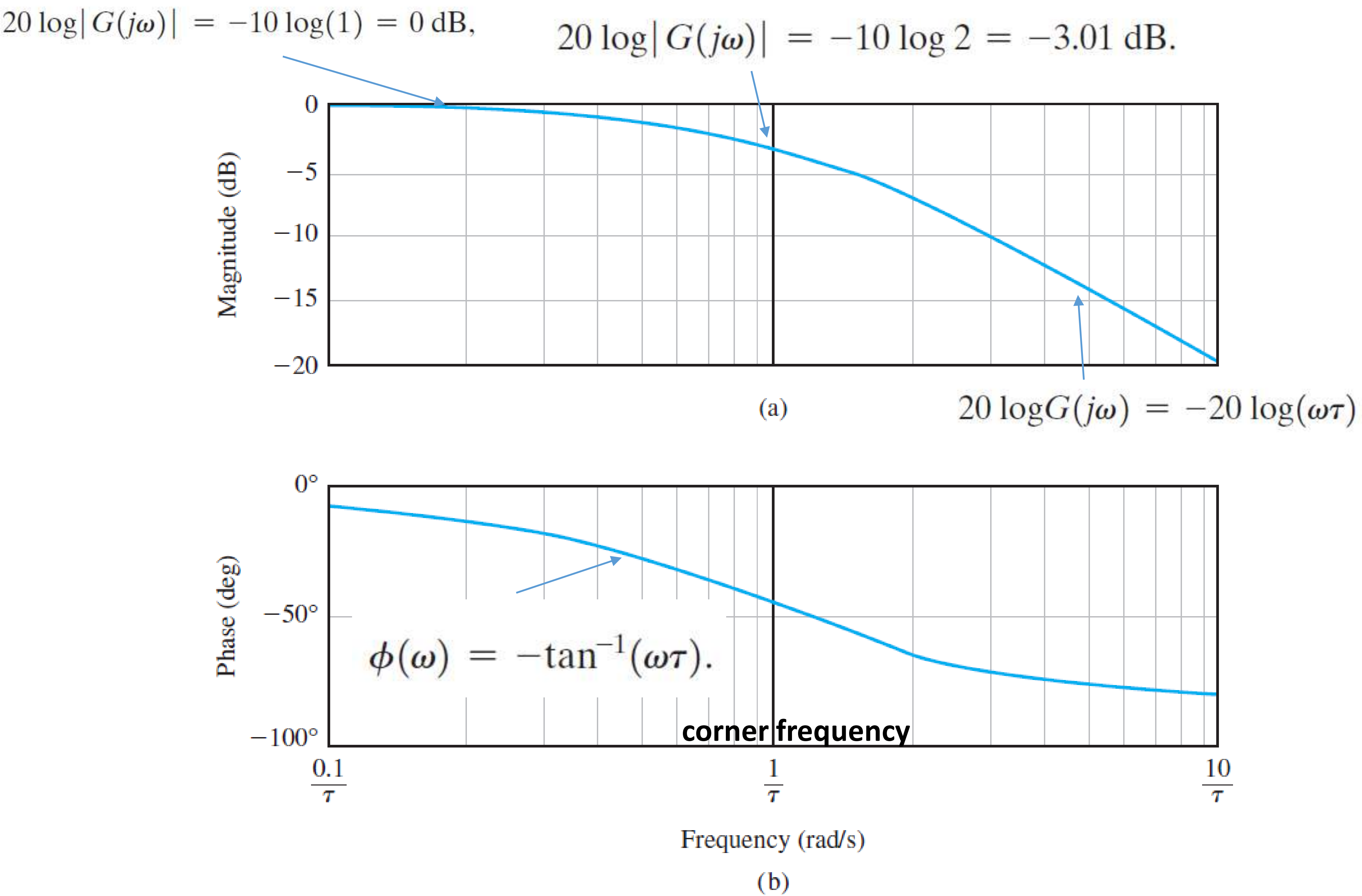
$$20 \log|G(j\omega)| = 20 \log\left(\frac{1}{1 + (\omega\tau)^2}\right)^{1/2} = -10 \log(1 + (\omega\tau)^2).$$

$$\omega \ll 1/\tau. \qquad 20 \log|G(j\omega)| = -10 \log(1) = 0 \text{ dB},$$

$$\omega \gg 1/\tau, \qquad 20 \log G(j\omega) = -20 \log(\omega\tau)$$

$$\omega = 1/\tau, \qquad 20 \log|G(j\omega)| = -10 \log 2 = -3.01 \text{ dB}.$$

break frequency or **corner frequency**



$20 \log|G(j\omega)| = -10 \log(1) = 0$ dB,
$20 \log|G(j\omega)| = -10 \log 2 = -3.01$ dB.
0
−5
−10
−15
−20
Magnitude (dB)
(a)
$20 \log G(j\omega) = -20 \log(\omega\tau)$
0°
−50°
−100°
Phase (deg)
$\phi(\omega) = -\tan^{-1}(\omega\tau)$.
corner frequency
$\frac{0.1}{\tau}$
$\frac{1}{\tau}$
$\frac{10}{\tau}$
Frequency (rad/s)
(b)

- A linear scale of frequency is not the most convenient or judicious choice

→we consider the use of a logarithmic scale of frequency

$$\omega \gg 1/\tau, \qquad 20 \log |G(j\omega)| = -20 \log(\omega\tau) = -20 \log \tau - 20 \log \omega.$$

- A **decade**

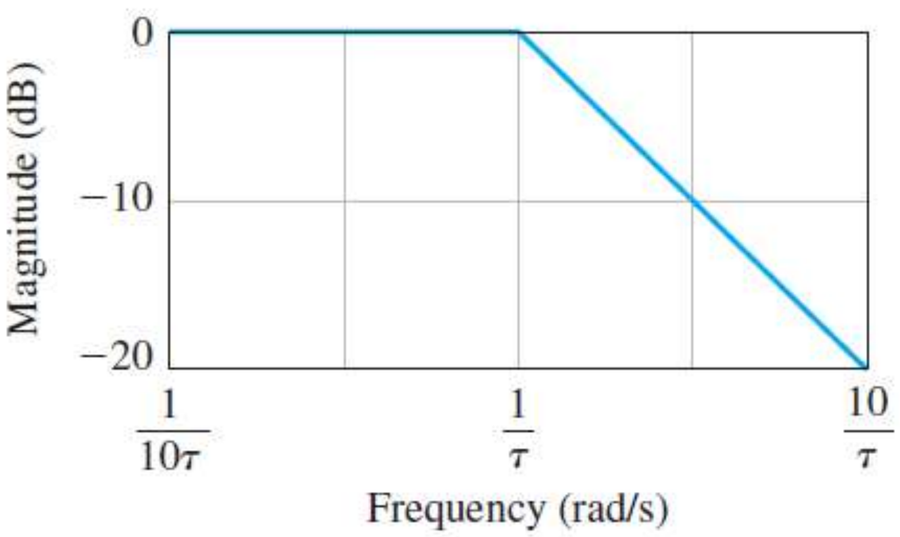
A linear function in log w

→an interval of two frequencies with a ratio equal to 10

- Difference between the logarithmic gains, for $\omega \gg 1/\tau,$ over a decade of frequency is ($\omega_2 = 10\omega_1$)

$$\begin{aligned} 20 \log |G(j\omega_1)| - 20 \log |G(j\omega_2)| &= -20 \log(\omega_1\tau) - (-20 \log(\omega_2\tau)) \\ &= -20 \log \frac{\omega_1\tau}{\omega_2\tau} \\ &= -20 \log \frac{1}{10} = +20 \text{ dB}; \end{aligned}$$

Slope= -20db/decade



- An **octave**

→an interval of two frequencies with a ratio equal to 2

- Show that the slope of the asymptotic line is -6 dB/octave

Advantage of the Bode Plots

- The conversion of multiplicative factors into additive factors

$$G(j\omega) = \frac{K_b \prod_{i=1}^{Q} (1 + j\omega\tau_i) \prod_{l=1}^{P} [(1 + (2\zeta_l/\omega_{n_l})j\omega + (j\omega/\omega_{n_l})^2)]}{(j\omega)^N \prod_{m=1}^{M} (1 + j\omega\tau_m) \prod_{k=1}^{R} [(1 + (2\zeta_k/\omega_{n_k})j\omega + (j\omega/\omega_{n_k})^2)]}.$$

- This transfer function includes *Q* zeros, *N* poles at the origin, *M* poles on the real axis, *P* pairs of complex conjugate zeros, and *R* pairs of complex conjugate poles.

Transfer function

$$G(j\omega) = \frac{K_b \prod_{i=1}^{Q} (1 + j\omega\tau_i) \prod_{l=1}^{P} [(1 + (2\zeta_l/\omega_{n_l})j\omega + (j\omega/\omega_{n_l})^2)]}{(j\omega)^N \prod_{m=1}^{M} (1 + j\omega\tau_m) \prod_{k=1}^{R} [(1 + (2\zeta_k/\omega_{n_k})j\omega + (j\omega/\omega_{n_k})^2)]}.$$

Magnitude plot: adding the contribution of each individual factor.

$$\begin{aligned} 20 \log|G(j\omega)| &= 20 \log K_b + 20 \sum_{i=1}^{Q} \log|1 + j\omega\tau_i| \\ &\quad -20 \log|(j\omega)^N| - 20 \sum_{m=1}^{M} \log|1 + j\omega\tau_m| \\ &\quad + 20 \sum_{l=1}^{P} \log\left|1 + \frac{2\zeta_l}{\omega_{n_l}} j\omega + \left(\frac{j\omega}{\omega_{nl}}\right)^2\right| - 20 \sum_{k=1}^{R} \log\left|1 + \frac{2\zeta_k}{\omega_{n_k}} j\omega + \left(\frac{j\omega}{\omega_{n_k}}\right)^2\right|, \end{aligned}$$

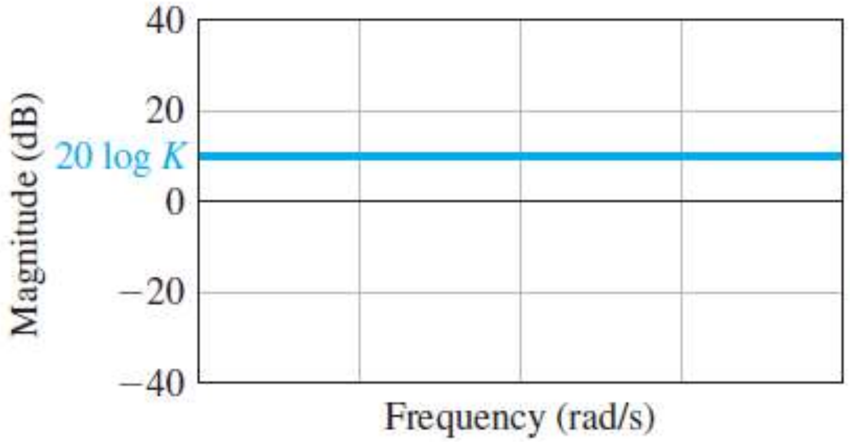
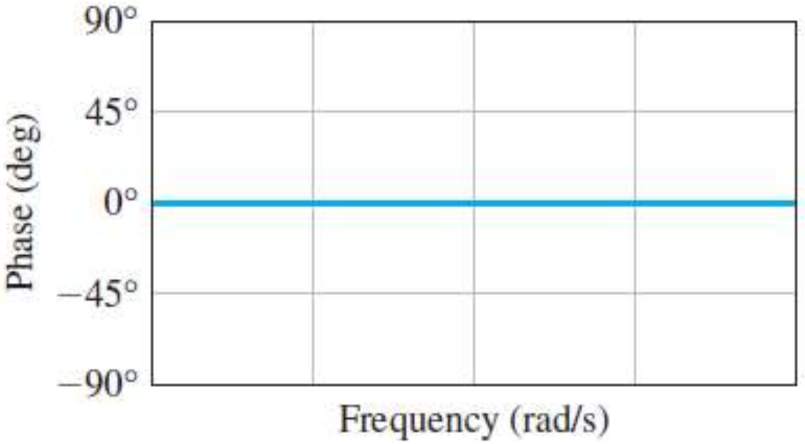
Phase plot: summation of the phase angles due to each individual factor of the transfer function.

$$\begin{aligned} \phi(\omega) &= +\sum_{i=1}^{Q} \tan^{-1}(\omega\tau_i) - N(90^\circ) - \sum_{m=1}^{M} \tan^{-1}(\omega\tau_m) \\ &\quad - \sum_{k=1}^{R} \tan^{-1} \frac{2\zeta_k\omega_{n_k}\omega}{\omega_{n_k}^2 - \omega^2}, + \sum_{l=1}^{P} \tan^{-1} \frac{2\zeta_l\omega_{n_l}\omega}{\omega_{n_l}^2 - \omega^2}, \end{aligned}$$

4 Building Blocks of Bode Plots

1. Constant gain K_b
2. Poles (or zeros) at the origin $(j\omega)$
3. Poles (or zeros) on the real axis $(j\omega\tau + 1)$
4. Complex conjugate poles (or zeros) $[1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2]$

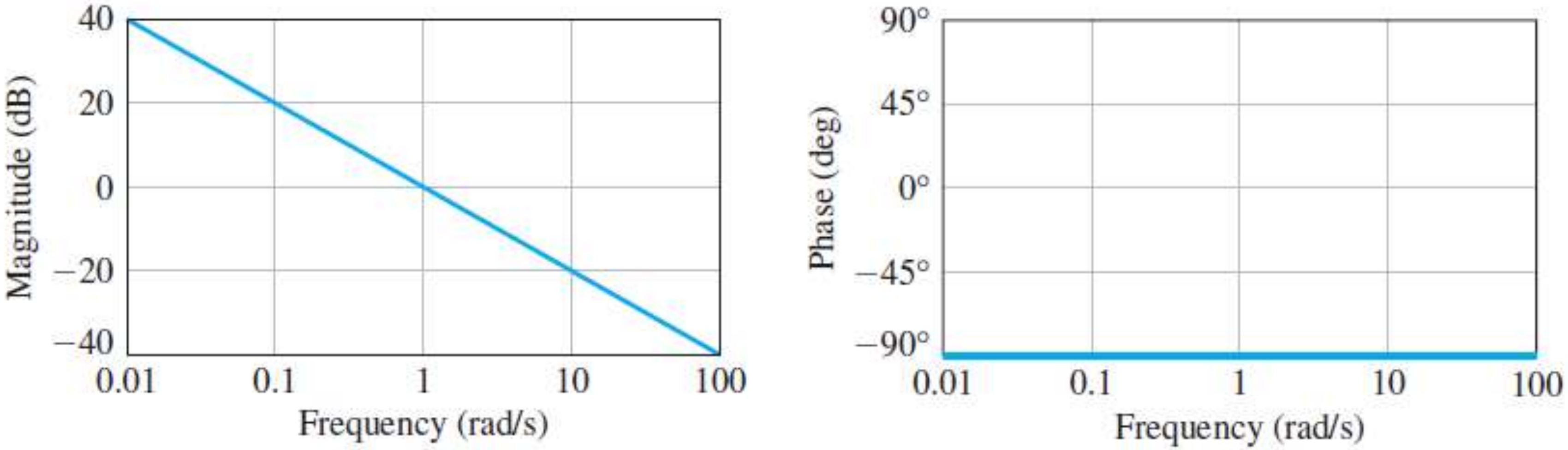
Table 8.1 Asymptotic Curves for Basic Terms of a Transfer Function

Term	Magnitude $20 \log_{10}\lvert G(j\omega)\rvert$	Phase $\phi(\omega)$
1. Gain, $G(j\omega) = K$		

4 Building Blocks of Bode Plots

1. Constant gain K_b
2. Poles (or zeros) at the origin $(j\omega)$
3. Poles (or zeros) on the real axis $(j\omega\tau + 1)$
4. Complex conjugate poles (or zeros) $[1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2]$

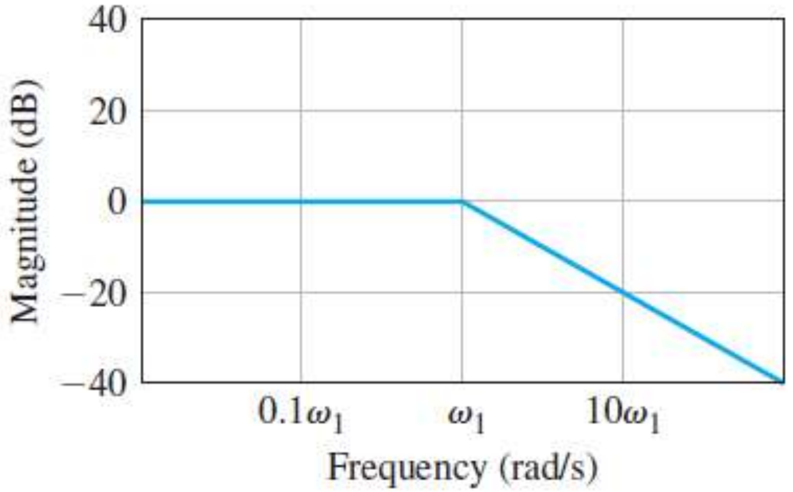
2 Pole at the origin, $G(j\omega) = 1/j\omega$

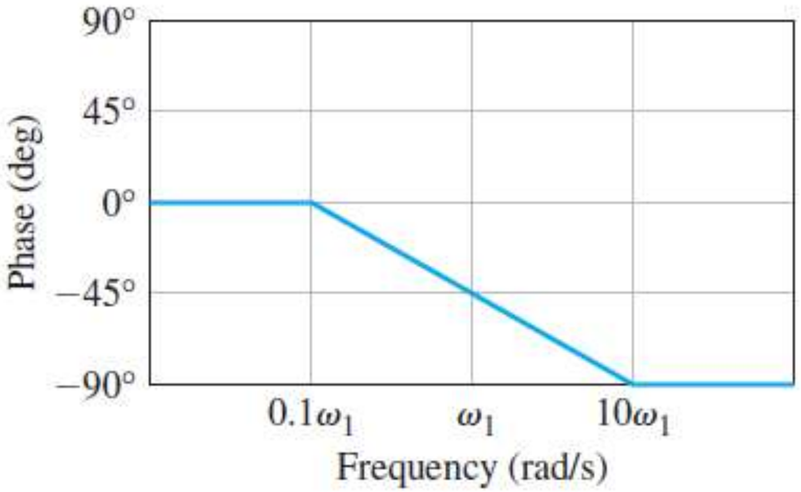


4 Building Blocks of Bode Plots

1. Constant gain K_b
2. Poles (or zeros) at the origin $(j\omega)$
3. Poles (or zeros) on the real axis $(j\omega\tau + 1)$
4. Complex conjugate poles (or zeros) $[1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2]$

3 Pole, $G(j\omega) = (1 + j\omega/\omega_1)^{-1}$



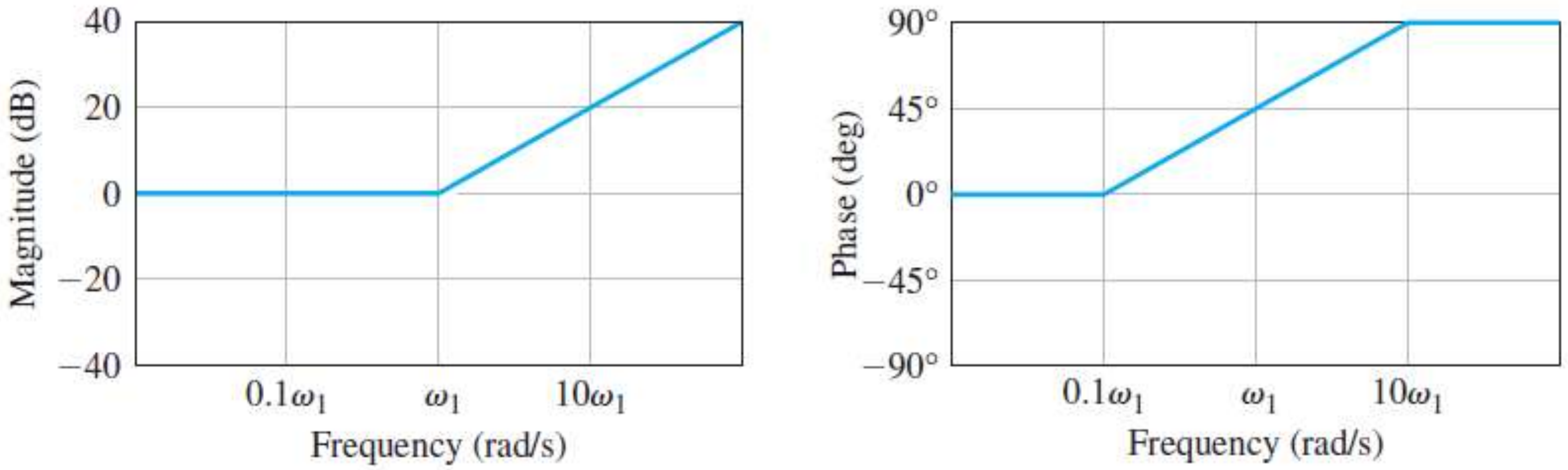


How about the Bode plot of zeros?

4 Building Blocks of Bode Plots

1. Constant gain K_b
2. Poles (or zeros) at the origin $(j\omega)$
3. Poles (or zeros) on the real axis $(j\omega\tau + 1)$
4. Complex conjugate poles (or zeros) $[1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2]$

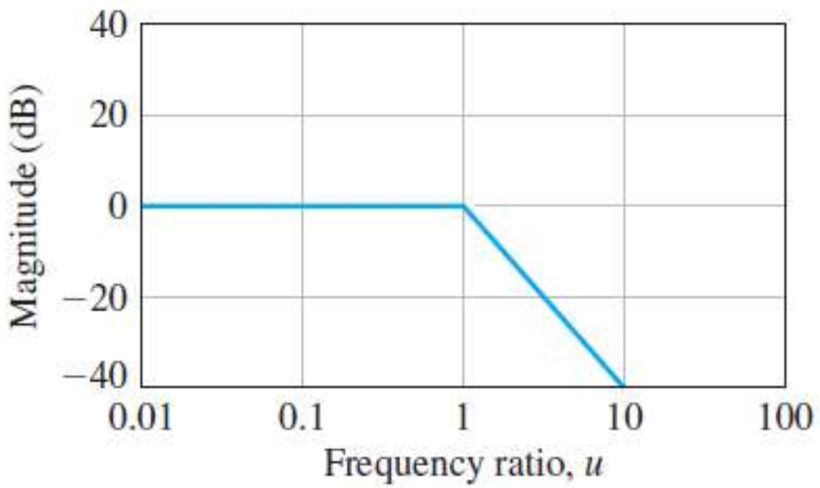
3 Zero, $G(j\omega) = 1 + j\omega/\omega_1$

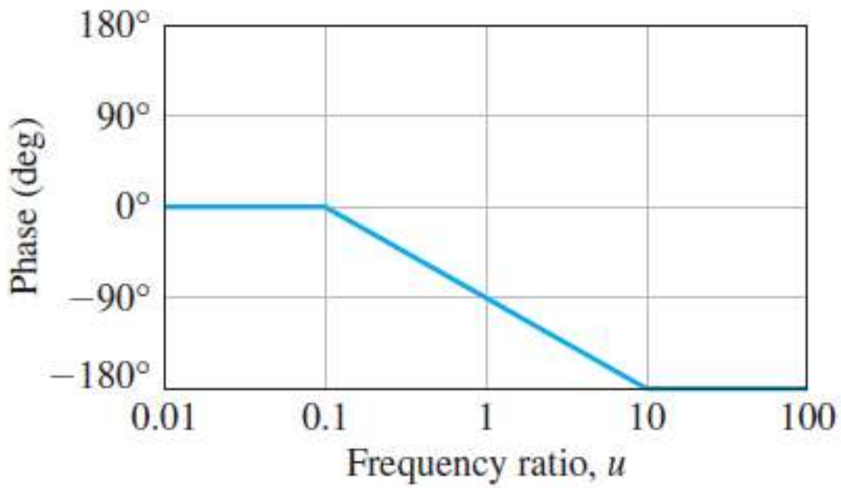


4 Building Blocks of Bode Plots

1. Constant gain K_b
2. Poles (or zeros) at the origin $(j\omega)$
3. Poles (or zeros) on the real axis $(j\omega\tau + 1)$
4. Complex conjugate poles (or zeros) $[1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2]$

4 Two complex poles, $0.1 < \zeta < 1$, $G(j\omega) = (1 + j2\zeta u - u^2)^{-1}$ $u = \omega/\omega_n$





How about the role of the damping ratio?

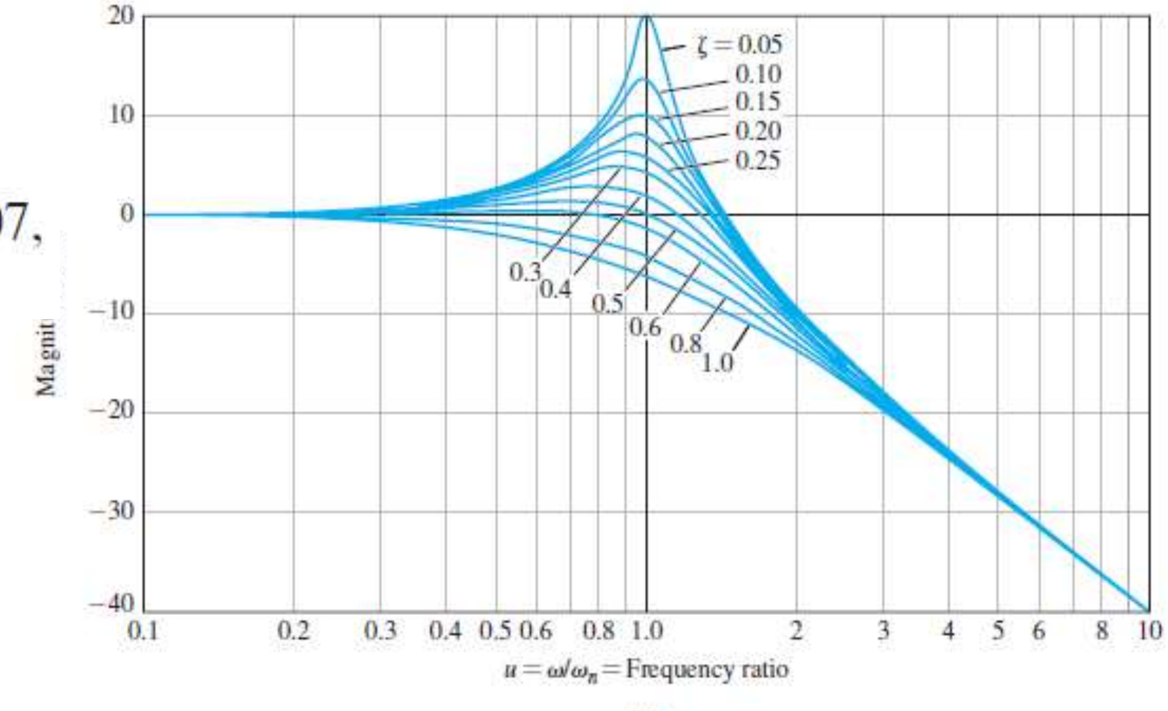
Resonant peak $M_{p\omega}$

$$\boxed{M_{p\omega} = |G(j\omega_r)| = (2\zeta\sqrt{1-\zeta^2})^{-1},} \quad \zeta < 0.707,$$

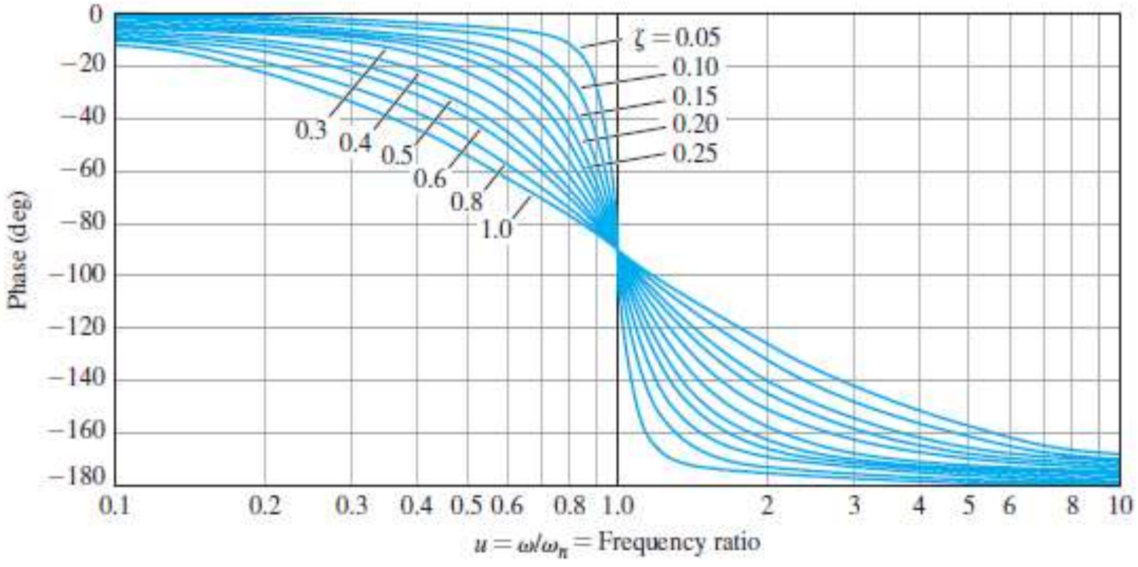
occurs at the **resonant frequency** ω_r

$$\boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2},} \quad \zeta < 0.707,$$

As zeta → 0, resonant frequency becomes natural frequency
As zeta → 1, frequency response is similar to that of two poles located at the resonant frequency



(a)



(b)

Summary of Magnitude Response

- A Pole

→-20db/decade at corner frequency

- A zero

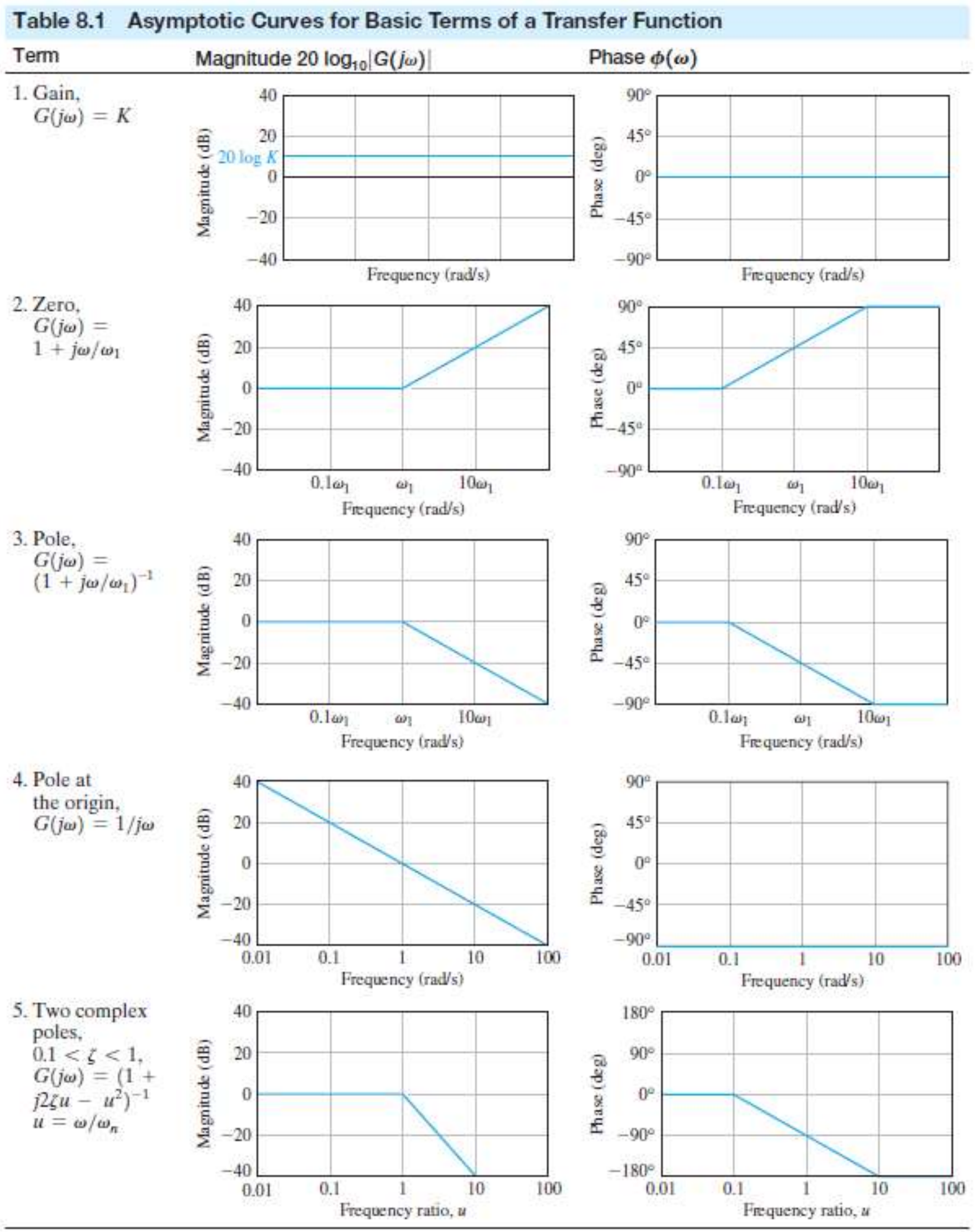
→+20db/decade at corner frequency

- Two complex poles

→-40db/decade at natural frequency

- Two complex zeros

→+40db/decade at natural frequency



Minimum and Nonminimum Phase Systems

- Minimum phase transfer function

→all its zeros lie in the left-hand s-plane.

- Nonminimum phase transfer function

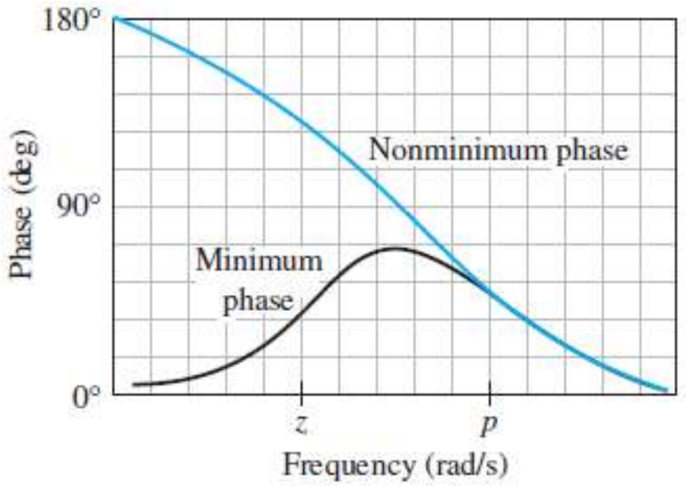
→at least one zero in the right-hand s-plane.

$$G_1(s) = \frac{s+z}{s+p}$$

Minimum phase

$$G_2(s) = \frac{s-z}{s+p}$$

Nonminimum phase



- Physical meaning of a nonminimum phase system

→Go in the wrong direction in order to get it right

G =

$$G = \frac{-0.5\,s + 0.05}{s^3 + 2\,s^2 + s}$$

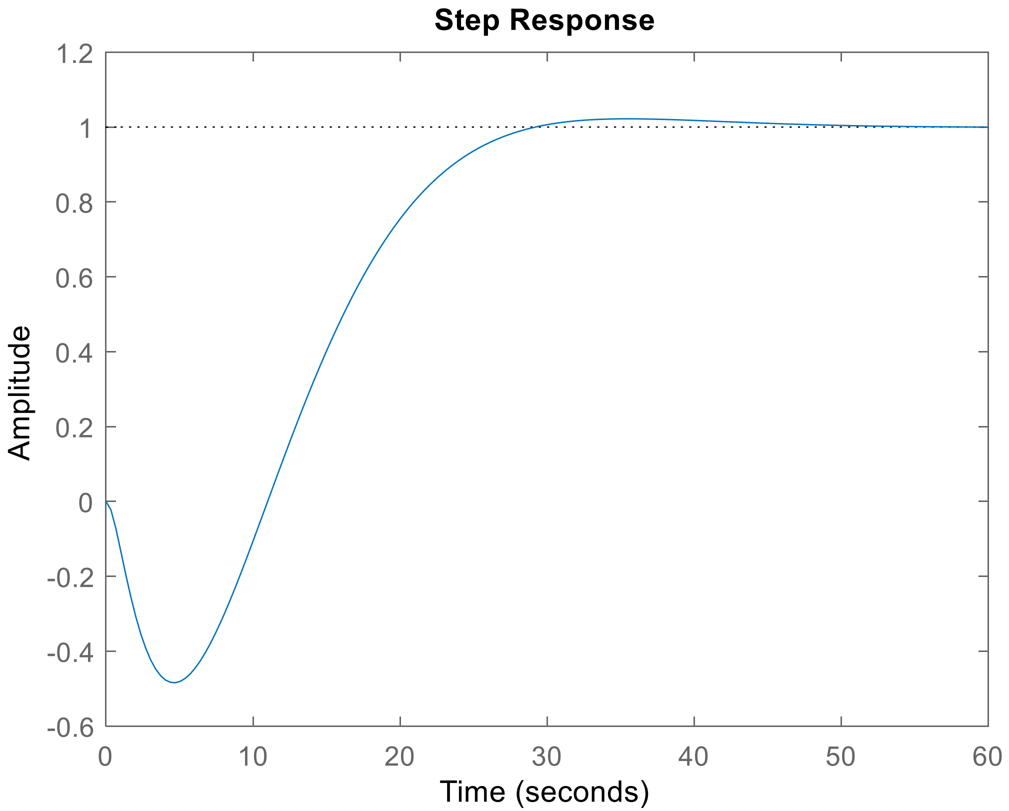
Unity feedback

T =

$$T = \frac{-0.5\,s + 0.05}{s^3 + 2\,s^2 + 0.5\,s + 0.05}$$

Type 1, nonminimum phase

Stable, nonminimum phase




```
num=conv(-0.5,[1 -0.1]);
dec=conv([1 0],[1 1]);
dec=conv(dec,[1 1]);
G=tf(num,dec)
T=feedback(G,1)
step(T)
```

Nonminimum Phase Systems

- Difficult to design and hard to get good performance

Example of nonminimum phase problems

1. Control an aircraft to a higher altitude

→ Kick the elevators (升降舵) to give a downforce

→ Center of mass goes down before going up

2. Parallel park

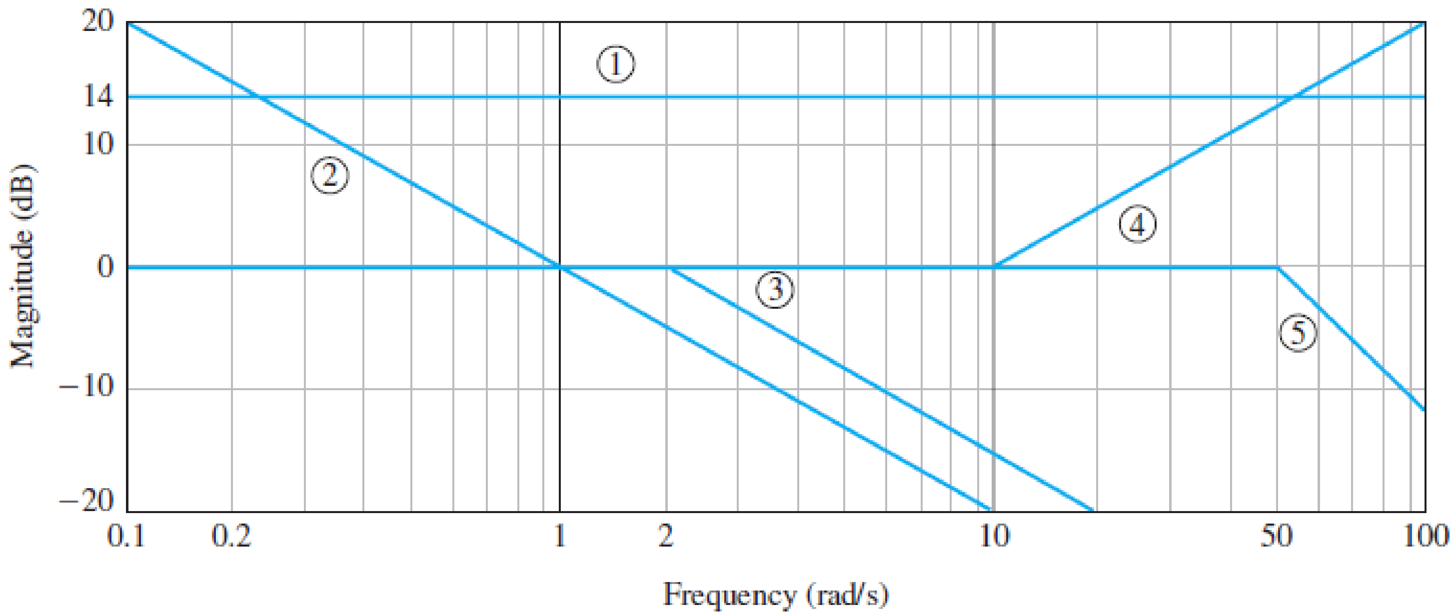
→Move away from a curb before you can move close to it

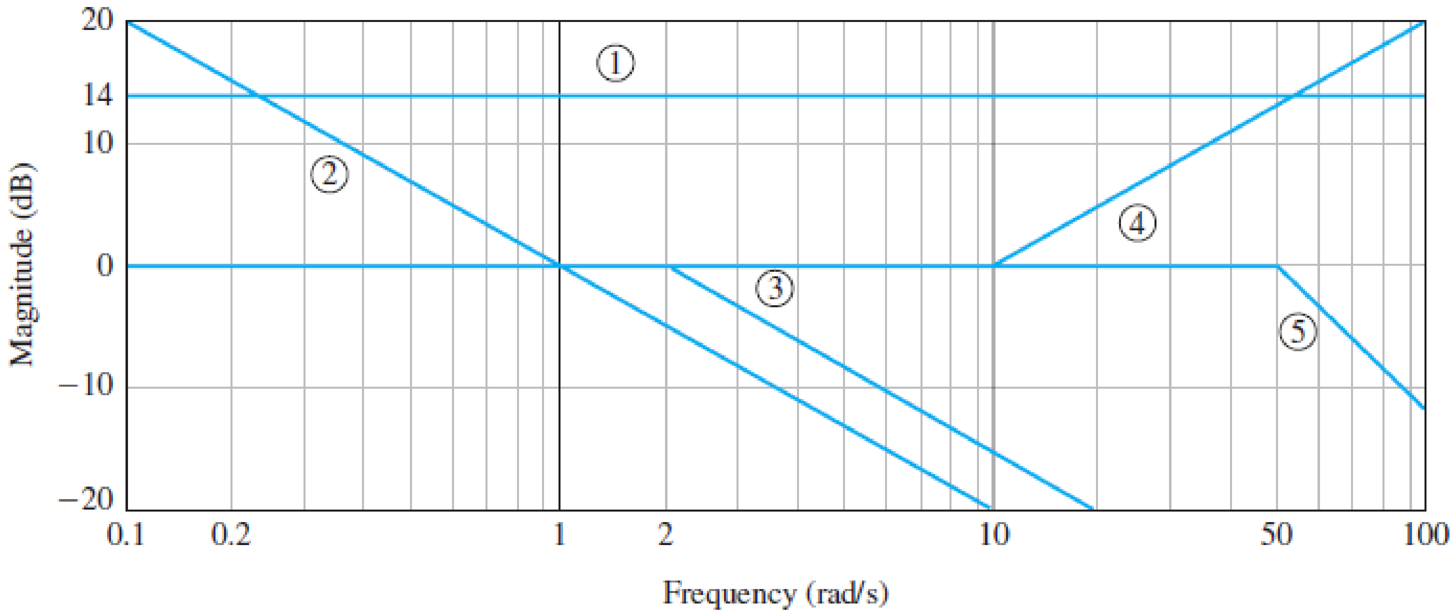
3. Life!

Example 8.5 (Sketching a Bode plot)

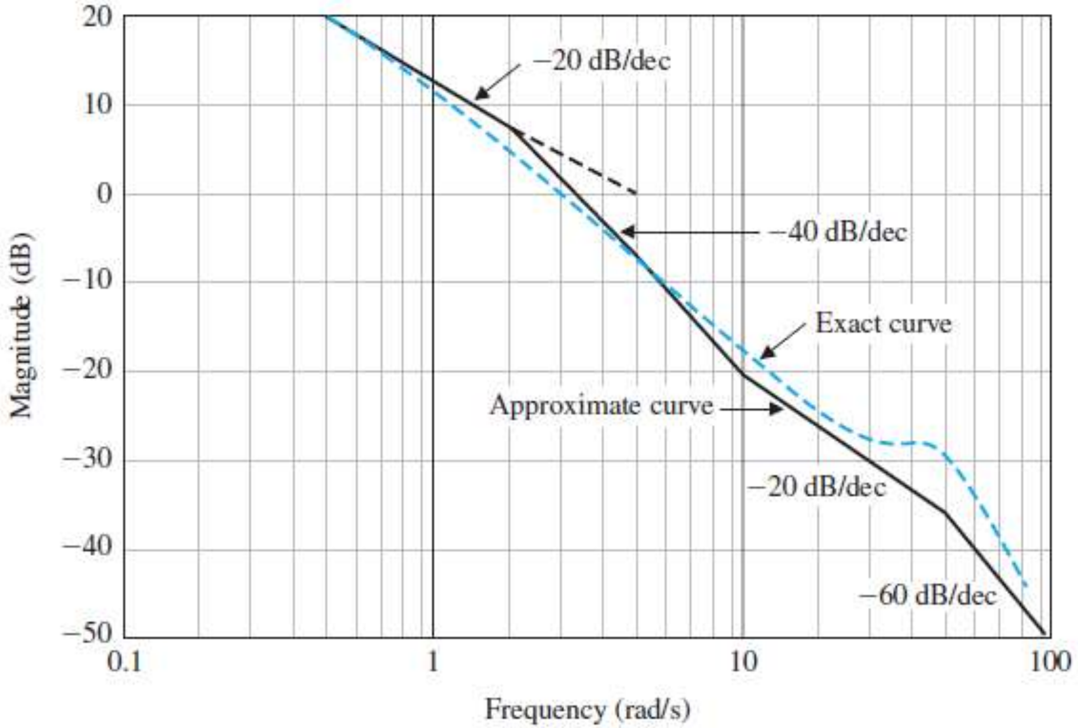
$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)(1 + j0.6(\omega/50) + (j\omega/50)^2)}.$$

1. A constant gain $K = 5$
2. A pole at the origin
3. A pole at $\omega = 2$
4. A zero at $\omega = 10$
5. A pair of complex poles at $\omega = \omega_n = 50$





20
14
10
0
−10
−20
0.1
0.2
1
2
10
50
100
Magnitude (dB)
Frequency (rad/s)
1
2
3
4
5



20
10
0
−10
−20
−30
−40
−50
0.1
1
10
100
Magnitude (dB)
Frequency (rad/s)
−20 dB/dec
−40 dB/dec
Exact curve
Approximate curve
−20 dB/dec
−60 dB/dec

8.3 Frequency Response Measurements

- How to measure the frequency response of a system

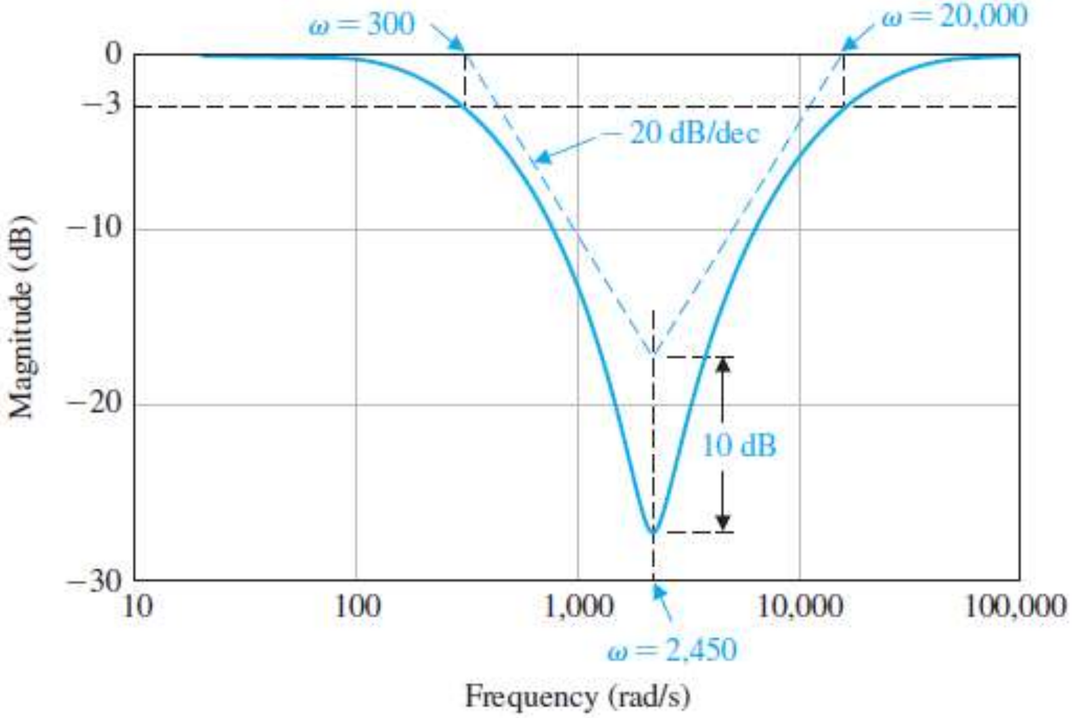
→A sine wave can be used to measure the frequency response of a system (how?)

→A wave analyzer can be used to measure the amplitude and phase variations as the frequency of the input sine wave is altered

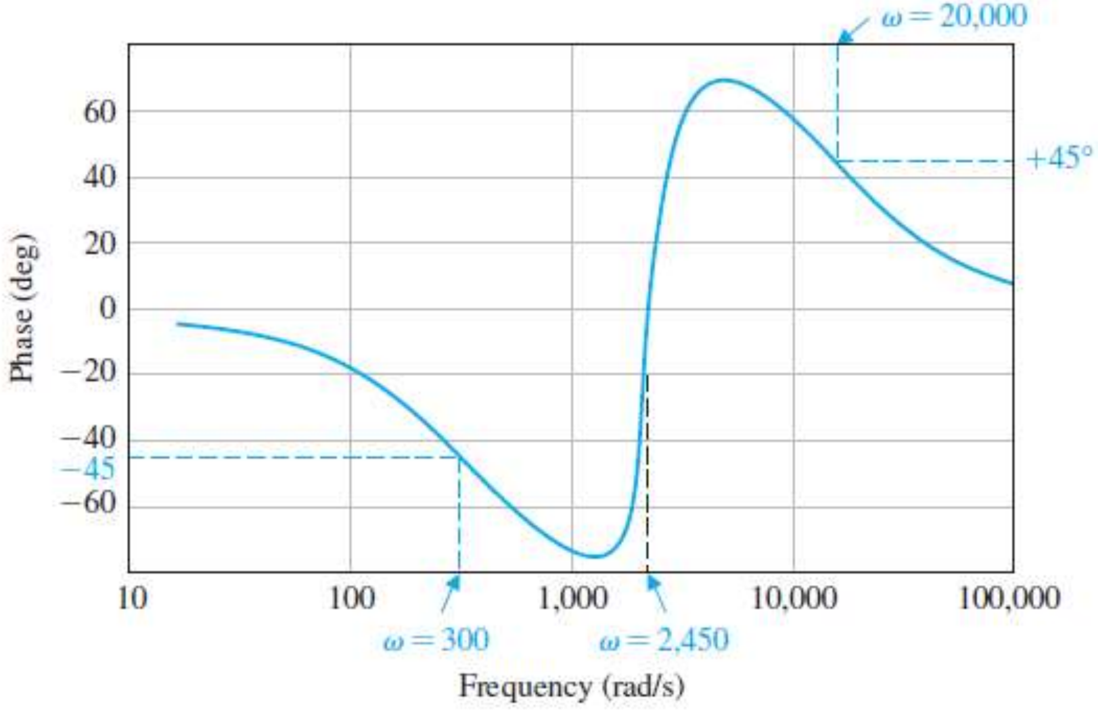
→A transfer function analyzer can be used to measure the loop transfer function and closed-loop transfer functions

→A typical signal analyzer instrument: DC to 100 kHz

Example



(a)



(b)

$$T(s) = \frac{(s/\omega_n)^2 + (2\zeta/\omega_n)s + 1}{(s/p_1 + 1)(s/p_2 + 1)}.$$

- Given transfer function

→Bode plot (section 8.2)

- Given a Bode plot

→ Transfer function (section 8.3)

8.4 Performance Specs in Frequency Domain

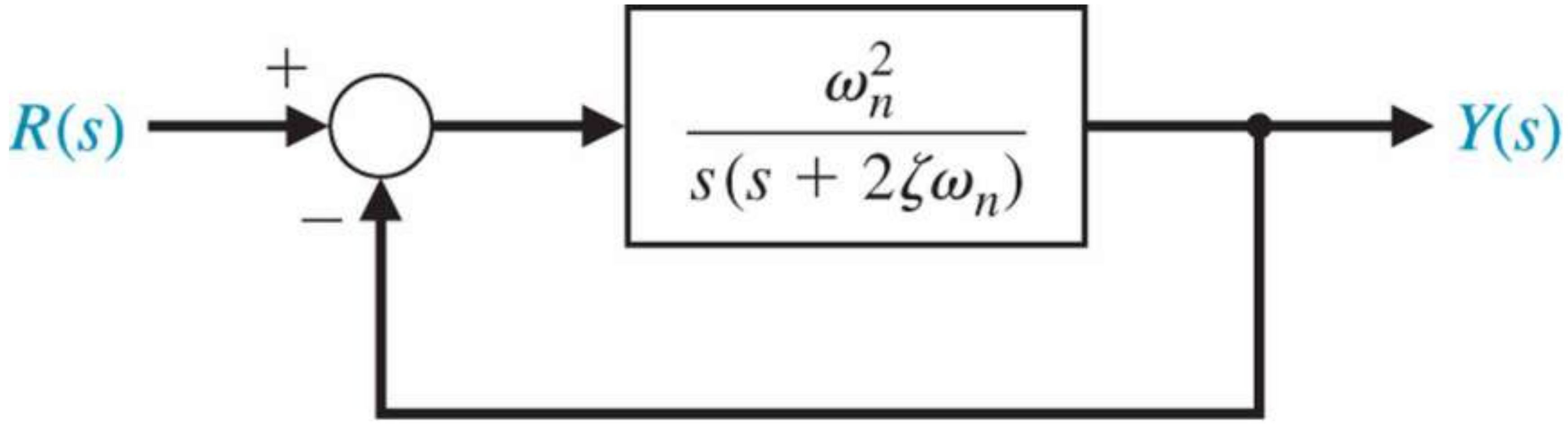


Figure 8.24 A second-order closed-loop system.

Characterization of transient response (2nd system)

- Time domain→ rise time, settling time, PO, …
- Frequency domain→ ?

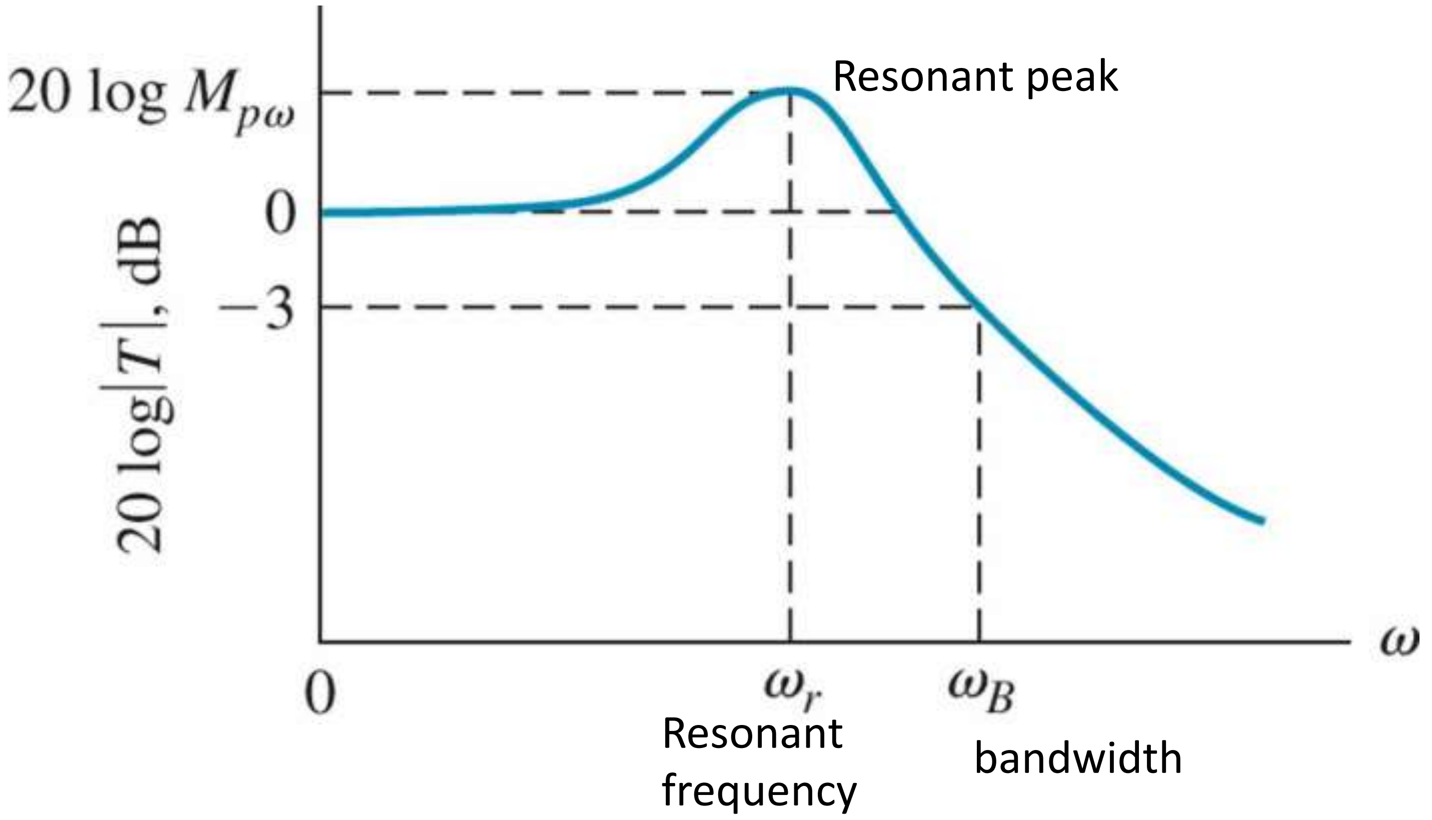


Figure 8.25 Magnitude characteristic of the second-order system.

- **Bandwidth** is the frequency at which the frequency response has declined 3 dB from its low-frequency value
- **Resonant peak** is the maximum frequency response attained at resonant frequency

Transient Response Relationships

- Bandwidth increases

→Rise time decreases

- Resonant peak increases

→Damping ratio decreases

→P.O. increases

- Example of frequency-domain specs

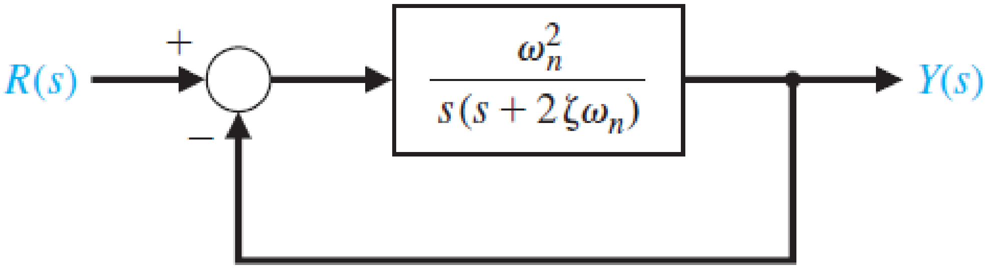
General rules of specs:
* Small resonant peak
(relative stability)
* Large bandwidth
(large natural frequency->small time constant->swift response)

1. Relatively small resonant magnitudes: $M_{p\omega} < 1.5$, for example.
2. Relatively large bandwidths so that the system time constant $\tau = 1/(\zeta\omega_n)$ is sufficiently small.

- Frequency response specs and their relation to the actual transient performance

→ Usually determined by dominant roots

Steady-State Relationship

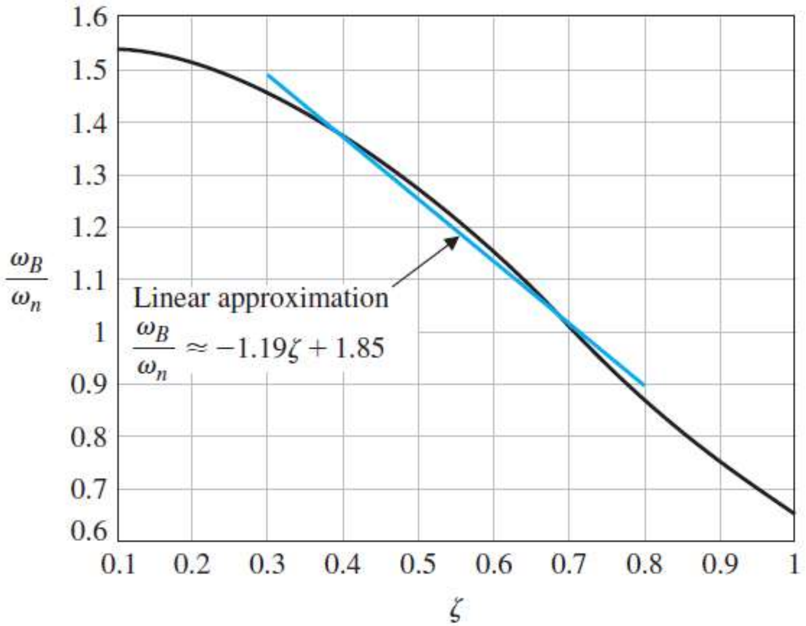


- Steady-state error for a ramp input is specified in terms of Kv, the velocity constant

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\left(\frac{\omega_n^2}{s(s+2\zeta\omega_n)}\right) = \frac{\omega_n}{2\zeta}.$$

related to bandwidth

FIGURE 8.26 Normalized bandwidth, ω_B/ω_n, versus ζ for a second-order system (Equation 8.46). The linear approximation $\omega_B/\omega_n = -1.19\zeta + 1.85$ is accurate for $0.3 \le \zeta \le 0.8$.



- frequency response characteristics represent the performance of a system quite adequately

→ with some experience, they are quite useful for the analysis and design of feedback control systems.

8.5 Log-magnitude–phase Diagram

1. Polar plot (Nyquist plot) OK

→ The locus of real and imaginary parts for various values of w

2. Logarithmic plot (Bode plot) OK

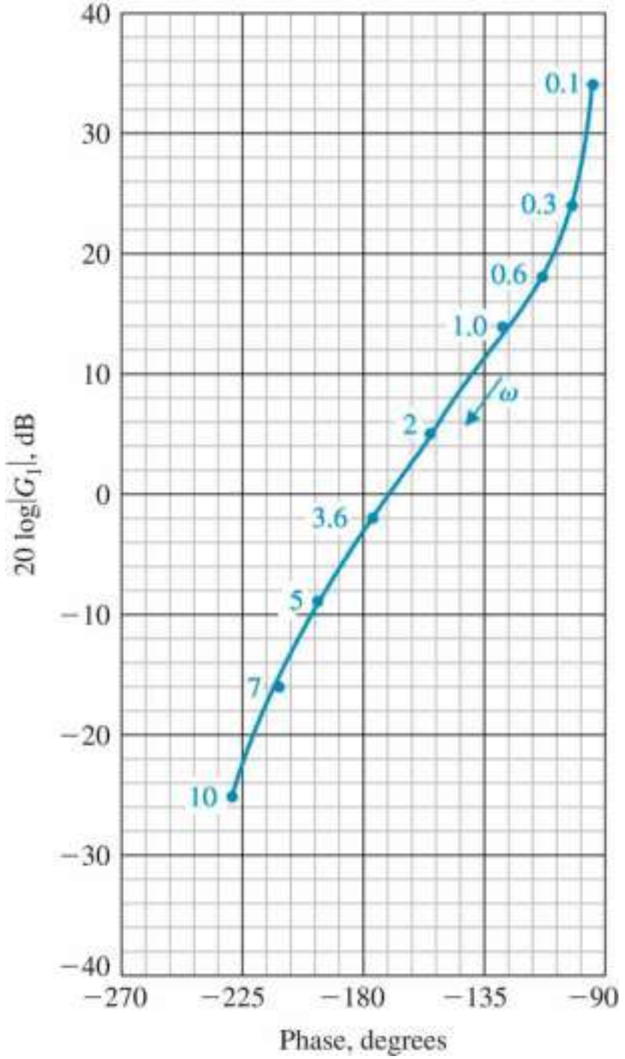
→ The magnitude and phase plots versus w

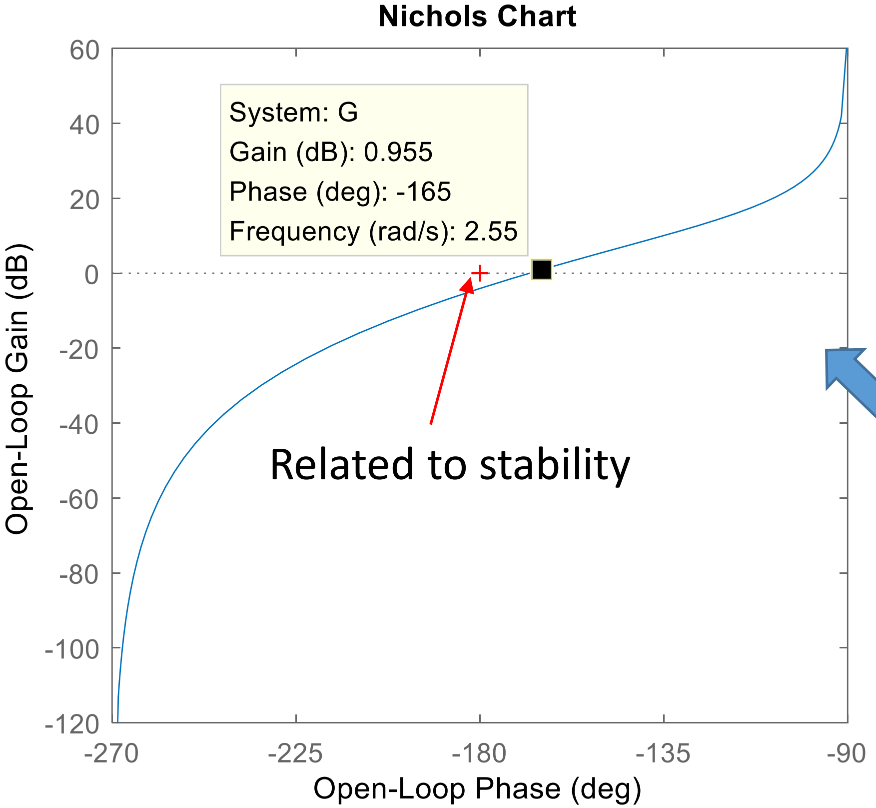
3. Log-magnitude–phase diagram (+M and N circles= Nichols chart)

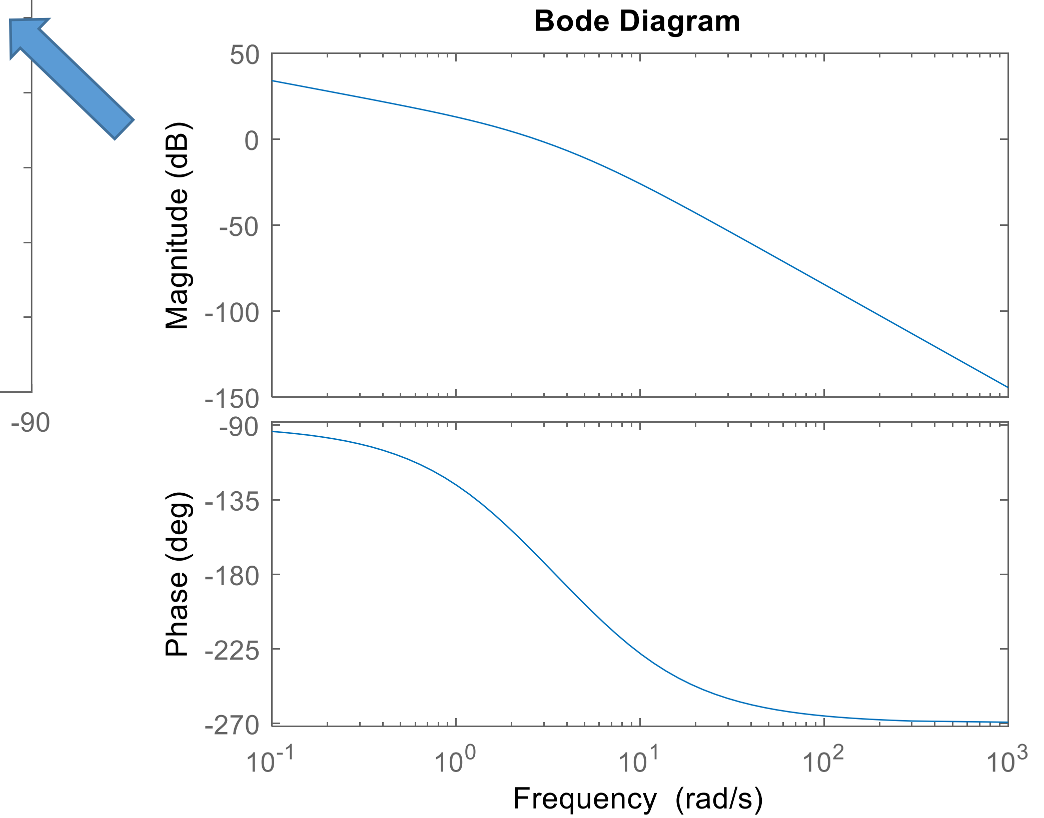
→ a plot of logarithmic magnitude in dB versus phase angle for a range of frequencies w

$$G_1(j\omega) = \frac{5}{j\omega(0.5j\omega + 1)(j\,\omega/6 + 1)}$$

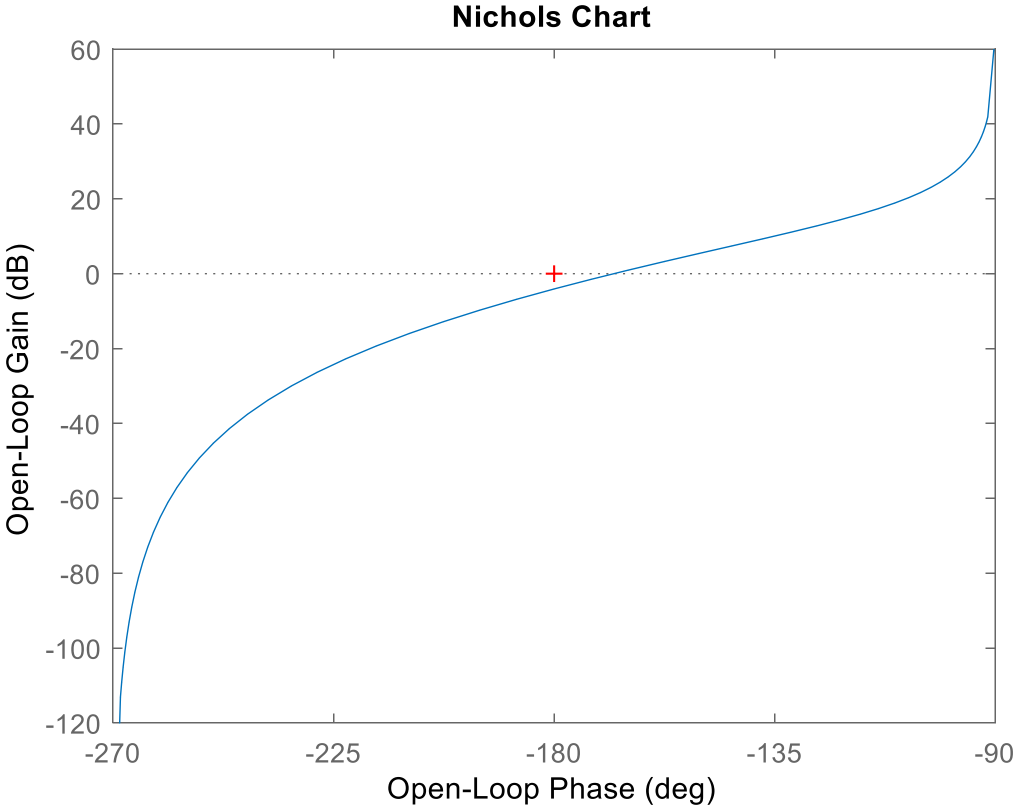
```
num=5;
dec=conv([1 0],[0.5 1]);
dec=conv(dec,[1/6 1]);
G=tf(num,dec);
nichols(G)
```

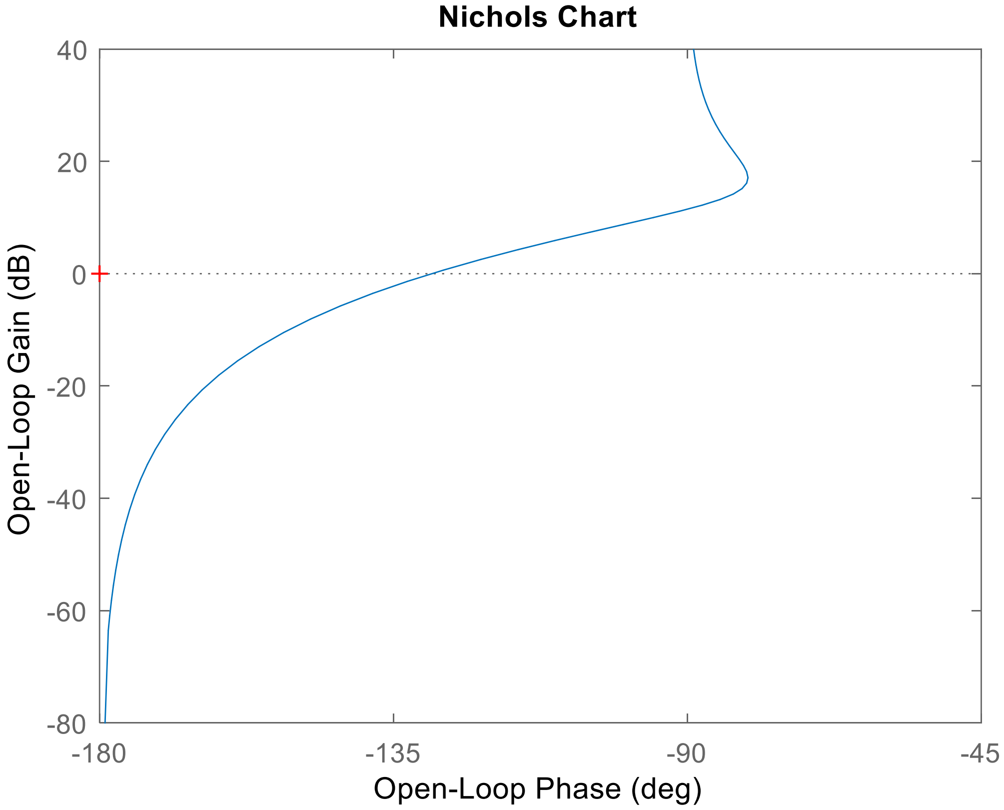






Addition of a zero at -1





- Difficult to use for design

→ Bode plot is often used rather than polar plot and Log-magnitude–phase diagram