Chapter 6

Stability of Linear Feedback Systems

6.1 Concept of Stability

- Closed-loop feedback system that is unstable is of minimal value (exceptions: aircraft)
- Closed-loop feedback is used to

→Stabilize an unstable systems or adjust performance of a stable open-loop system

- Absolute stability

→ Stable/not stable

- Relative stability (given a stable closed-loop system)

→ Characterize the degree of stability

Relative Stability for Aircraft Design

- The more stable an aircraft was, the more difficult it was to maneuver (that is, to turn)
- A acrobatic aircraft is less stable than a commercial transport; hence it can maneuver

Stability Criterion

- A system is stable (in the absolute sense) if a bounded input yields a bounded response

→all transfer function poles lie in the left-half s-plane

→or all the eigenvalues of the system matrix A in state variable representation lie in the left-half s-plane.

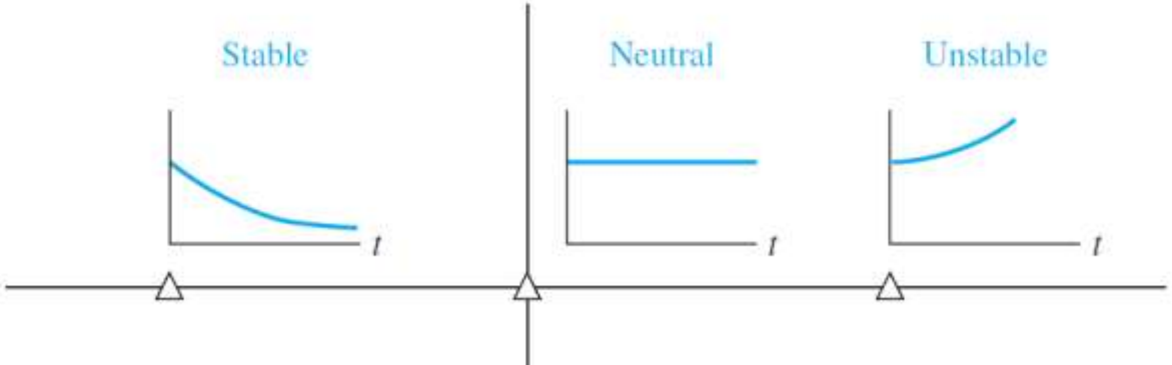
- Given that all the poles (or eigenvalues) are in the left-half s-plane

→ Examine the relative locations of the poles (or eigenvalues) for relative stability

Illustration of Stability

- The concept of stability can be illustrated by considering a right circular cone placed on a plane horizontal surface.

(a) Stable (b) Neutral (c) Unstable



Stability in terms of Location of Poles

- Closed-loop transfer function

$$T(s) = \frac{p(s)}{q(s)} = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^N \prod_{k=1}^{Q}(s+\sigma_k)\prod_{m=1}^{R}[s^2+2\alpha_m s+(\alpha_m^2+\omega_m^2)]},$$

- Output response for an impulse function input (when N = 0) is

$$y(t) = \sum_{k=1}^{Q} A_k\, e^{-\sigma_k t} + \sum_{m=1}^{R} B_m\left(\frac{1}{\omega_m}\right)e^{-\alpha_m t}\sin(\omega_m t+\theta_m),$$

Stability in terms of Location of Poles

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- To obtain a bounded response

→ poles of the closed-loop system must be in the left-hand portion of the s-plane.

$$T(s) = \frac{p(s)}{q(s)} = \frac{K \prod_{i=1}^{M} (s + z_i)}{s^N \prod_{k=1}^{Q} (s + \sigma_k) \prod_{m=1}^{R} [s^2 + 2\alpha_m s + (\alpha_m^2 + \omega_m^2)]},$$

Stability in terms of Location of Poles (Summary)

- Stable system

→all the poles of the transfer function are in the left-half s-plane

→bounded inputs yield bounded outputs

- Marginally stable system

→simple poles on the imaginary axis and all other poles in the left-half s-plane

→only certain bounded inputs (sinusoids of the frequency of the poles) will cause the output to become unbounded; other bounded inputs lead to oscillatory outputs

- Unstable system

→at least one pole in the right-half s-plane or repeated poles on the imaginary axis

→the output is unbounded for any input.

6.2 Routh–Hurwitz Stability Criterion

$$\Delta(s) = q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0.$$

- Factorization $\quad a_n (s - r_1)(s - r_2) \cdots (s - r_n) = 0,$

$$\begin{aligned} q(s) = a_n s^n &- a_n(r_1 + r_2 + \cdots + r_n)s^{n-1} \\ &+ a_n (r_1 r_2 + r_2 r_3 + r_1 r_3 + \cdots)s^{n-2} \\ &- a_n (r_1 r_2 r_3 + r_1 r_2 r_4 \cdots)s^{n-3} + \cdots \\ &+ a_n (-1)^n r_1 r_2 r_3 \cdots r_n = 0. \end{aligned}$$

Necessary conditions: all the coefficients are nonzero and have the same sign

- If the necessary condition is satisfied, we still need to proceed further to ascertain the stability of the system
- Example: $q(s) = (s + 2)(s^2 - s + 4) = (s^3 + s^2 + 2s + 8)$
- Routh–Hurwitz criterion is a necessary and sufficient criterion for the stability of linear systems.

Routh–Hurwitz Criterion

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 = 0$$

- Routh array

$$\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
s^0 & h_{n-1} & & &
\end{array}$$

- Butterfly notations:

$$b_{n-1} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix},$$

$$b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}, \cdots$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}, \quad \cdots$$

Four Distinct Cases for Routh Array Calculation Procedure

- Routh–Hurwitz criterion states that the number of roots with positive real parts is equal to the number of changes in sign of the first column of the Routh array.
- Four distinct cases or configurations of the first column array must be considered

→each case must be treated separately and requires suitable modifications of the array calculation procedure

- Four cases:

1. No element in the first column is zero;
2. There is a zero in the first column, but some other elements of the row containing the zero in the first column are nonzero;
3. One row with all-zero elements
4. As in the third case, but with repeated roots on the imaginary axis

Case 1: 2nd system

- No element in the first column is zero

$$q(s) = a_2 s^2 + a_1 s + a_0.$$

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & b_1 & 0 \end{array},$$

$$b_1 = \frac{a_1 a_0 - (0) a_2}{a_1} = \frac{-1}{a_1} \begin{vmatrix} a_2 & a_0 \\ a_1 & 0 \end{vmatrix} = a_0.$$

A second-order system is stable if and only if all the coefficients are positive (or all negative).

Case 1: 3rd system

$$q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0.$$

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & b_1 & 0 \\ s^0 & c_1 & 0 \end{array},$$

$$b_1 = \frac{a_2 a_1 - a_0 a_3}{a_2} \quad \text{and} \quad c_1 = \frac{b_1 a_0}{b_1} = a_0.$$

A third-order system is stable if and only if all the coefficients are positive (or all negative) and $a_2 a_1 > a_0 a_3$.

Example of unstable system: $q(s) = (s - 1 + j\sqrt{7})(s - 1 - j\sqrt{7})(s + 3) = s^3 + s^2 + 2s + 24.$

Case 2

- There is a zero in the first column, but some other elements of the row containing the zero in the first column are nonzero

→ Replace the zero with a small positive ε and allow it to approach zero

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10.$$

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & \epsilon & 6 & 0 \\ s^2 & c_1 & 10 & 0 \\ s^1 & d_1 & 0 & 0 \\ s^0 & 10 & 0 & 0 \end{array}, \quad c_1 = \frac{4\epsilon - 12}{\epsilon} \quad \text{and} \quad d_1 = \frac{6c_1 - 10\epsilon}{c_1}.$$

When $0 < \epsilon \ll 1$, we find that $c_1 < 0$ and $d_1 > 0$.

Case 3

- One row with all-zero elements

→Singularities about the origin, i.e., $(s+\sigma)(s-\sigma)$ or $(s+j\omega)(s-j\omega)$

→ System is at most marginally stable

→Use auxiliary polynomial that precedes the row with all-zero elements

$$q(s) = s^3 + 2s^2 + 4s + K,$$

Stable if $0 < K < 8.$

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & K \\ s^1 & \dfrac{8-K}{2} & 0 \\ s^0 & K & 0 \end{array}.$$

If K=8, auxiliary polynomial U(s)

$$U(s) = 2s^2 + Ks^0 = 2s^2 + 8 = 2(s^2+4) = 2(s+j2)(s-j2).$$

marginally stable

Case 4

- As in the third case, but with repeated roots on the imaginary axis

→ Check the details in your textbook

Example 6.4 (a combination of 2 cases)

- Consider the characteristic polynomial

$$q(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63.$$

- Routh array

$$\begin{array}{c|ccc} s^5 & 1 & 4 & 3 \\ s^4 & 1 & 24 & 63 \\ s^3 & -20 & -60 & 0. \\ s^2 & 21 & 63 & 0 \\ s^1 & 0 & 0 & 0 \end{array}$$

Case 3

$$\Rightarrow \quad U(s) = 21s^2 + 63 = 21(s^2 + 3) = 21(s + j\sqrt{3})(s - j\sqrt{3}),$$

Example 6.4

- To examine the remaining roots of $q(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63.$

$$U(s) = 21s^2 + 63 = 21(s^2 + 3) = 21(s + j\sqrt{3})(s - j\sqrt{3}),$$

$$\frac{q(s)}{s^2 + 3} = s^3 + s^2 + s + 21.$$

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1 & 21 \\ s^1 & -20 & 0 \\ s^0 & 21 & 0 \end{array}$$

Case 1

$\Rightarrow$ unstable

Videos for Quick Tips

- https://www.youtube.com/watch?v=WBCZBOB3LCA&t=9s
- https://www.youtube.com/watch?v=oMmUPvn6lP8
- Also please read the textbook for the detailed discussions.

6.3 Relative Stability of Feedback Control Systems

- Routh–Hurwitz criterion ascertains the absolute stability of a system by determining whether any of the roots of the characteristic equation lie in the right-half s-plane
- It is desirable to determine the relative stability

1) measured by the relative real part of each root or pair of roots
2) Damping ratio

- Axis shift

→ Determine the relative stability by shifting the axis

Example 6.6

$$q(s) = s^3 + 4s^2 + 6s + 4.$$

$$(s_n - 1)^3 + 4(s_n - 1)^2 + 6(s_n - 1) + 4 = s_n{}^3 + s_n{}^2 + s_n + 1$$

$$\begin{array}{c|cc} s_n{}^3 & 1 & 1 \\ s_n{}^2 & 1 & 1 \\ s_n{}^1 & 0 & 0 \\ s_n{}^0 & 1 & 0 \end{array}$$

After shifting the roots to the right by one unit, we have an unstable system.

https://www.youtube.com/watch?v=wGC5C_7Yy-s&t=1s

6.4 Stability of State Variable Systems

- System represented by a signal-flow graph

→Obtain the characteristic equation by evaluating the graph determinant (Mason's signal gain formula)

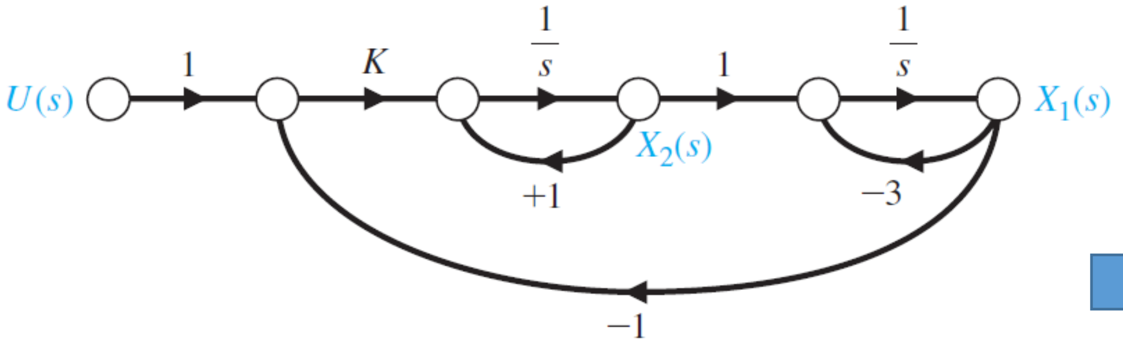
- System represented by a block diagram model

→Obtain the characteristic equation using the block diagram reduction methods

- System represented by a state-space model

→ Obtain the characteristic equation by evaluating the eigenvalues of the transition matrix A (why?)

Example 6.7 (Mason's signal flow gain formula)



$$\dot{x}_1 = -3x_1 + x_2 \quad \text{and} \quad \dot{x}_2 = +1x_2 - Kx_1 + Ku,$$

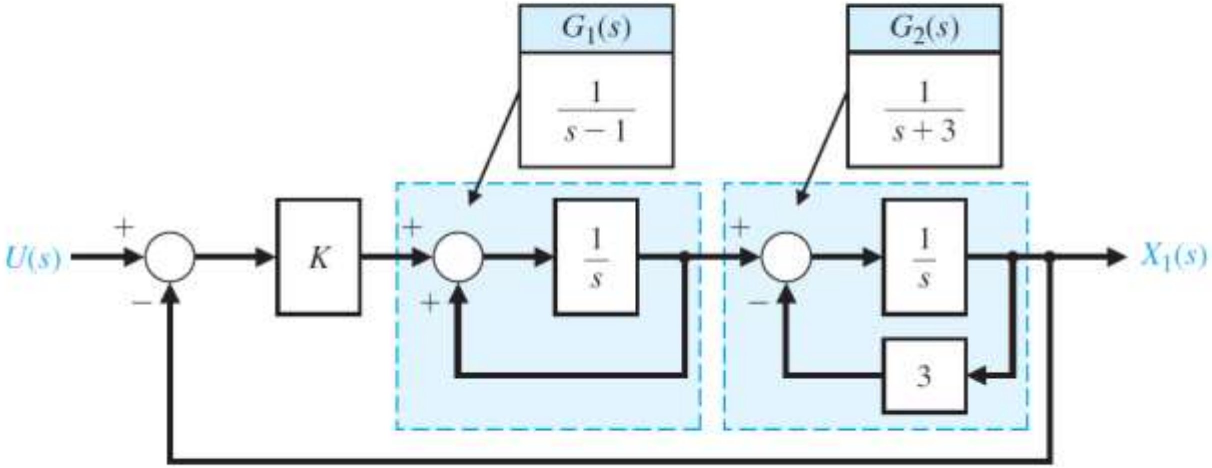
$$L_1 = s^{-1}, \qquad L_2 = -3s^{-1}, \quad \text{and} \quad L_3 = -Ks^{-2},$$

$$\Delta = 1 - (L_1 + L_2 + L_3) + L_1L_2 = 1 - (s^{-1} - 3s^{-1} - Ks^{-2}) + (-3s^{-2})$$

$$s^2 + 2s + (K - 3) = 0.$$

we require $K > 3$ for stability.

Example 6.7 (Block diagram reduction)



$$T(s) = \frac{KG_1(s)G_2(s)}{1 + KG_1(s)G_2(s)}.$$

$$\Delta(s) = 1 + KG_1(s)G_2(s) = 0,$$

$$\Delta(s) = (s - 1)(s + 3) + K = s^2 + 2s + (K - 3) = 0.$$

Example 6.8 (Eigenvalues of transition matrix)

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -\alpha & -\beta & 0 \\ \beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

$$\begin{aligned}\det(\lambda\mathbf{I} - \mathbf{A}) &= \det\left\{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -\alpha & -\beta & 0 \\ \beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{bmatrix}\right\} \\ &= \det\begin{bmatrix} \lambda + \alpha & \beta & 0 \\ -\beta & \lambda + \gamma & 0 \\ -\alpha & -\gamma & \lambda \end{bmatrix} \\ &= \lambda[\lambda^2 + (\alpha + \gamma)\lambda + (\alpha\gamma + \beta^2)] = 0.\end{aligned}$$

System is marginally stable when $\alpha + \gamma > 0$ and $\alpha\gamma + \beta^2 > 0$.