Chapter 3

State Variable Models

3.1 Introduction

- System models can be described in

→Frequency domain (classical control)

→Time domain (modern control)

- State variable methods (time domain)

→Can investigate the inner structure of control systems

→Readily extend to nonlinear systems

→Suitable for MIMO systems

3.2 State Variables of a Dynamic System

- System state

→a set of variables whose values, together with the input signals and the equations describing the dynamics, will provide the future state and output of the system

→ state variables $\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$

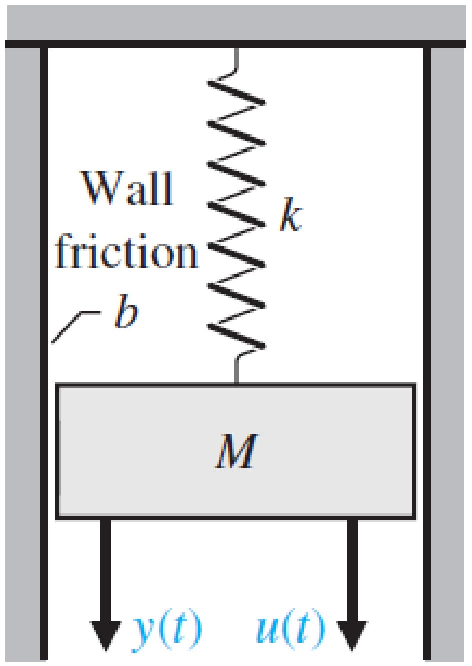
$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = u(t).$$

$$x_1(t) = y(t) \quad \text{and} \quad x_2(t) = \frac{dy(t)}{dt}.$$

$$M\frac{dx_2(t)}{dt} + bx_2(t) + kx_1(t) = u(t).$$

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = \frac{-b}{M}x_2(t) - \frac{k}{M}x_1(t) + \frac{1}{M}u(t).$$



Unique set of state variables?

3.3 State Differential Equation

$$\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_{11}u_1(t) + \cdots + b_{1m}u_m(t),$$
$$\dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_{21}u_1(t) + \cdots + b_{2m}u_m(t),$$
$$\vdots$$
$$\dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_{n1}u_1(t) + \cdots + b_{nm}u_m(t),$$

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{bmatrix} b_{11} \cdots & b_{1m} \\ \vdots & \vdots \\ b_{n1} \cdots & b_{nm} \end{bmatrix} \begin{pmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{pmatrix}.$$

State-space or

State-variable Representation

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

State vector

State differential equation $\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t).}$

Output equation $\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),}$

Solution of State Differential Equation

- Similar to the method for solving a first-order differential

$$\dot{x}(t) = ax(t) + bu(t),$$

$$sX(s) - x(0) = aX(s) + bU(s);$$

$$X(s) = \frac{x(0)}{s - a} + \frac{b}{s - a}U(s).$$

$$x(t) = e^{at}x(0) + \int_0^t e^{+a(t-\tau)}bu(\tau)d\tau.$$

Viewpoint 1: Generalization from a scalar case

- Solution of the general state differential equation

→ matrix exponential function

$$\boxed{e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^kt^k}{k!} + \cdots,}$$

$$x(t) = e^{at}x(0) + \int_0^t e^{+a(t-\tau)}bu(\tau)d\tau.$$

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0) + \int_0^t \exp[\mathbf{A}(t-\tau)]\mathbf{B}\mathbf{u}(\tau)d\tau.$$

Viewpoint 2: Direct Laplace Transform

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t).}$$

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s),$$

Let $[s\mathbf{I} - \mathbf{A}]^{-1} = \mathbf{\Phi}(s)$

$$\boxed{\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau.}$$

fundamental or **state transition matrix** $\mathbf{\Phi}(t)$

3.4 Signal-flow Graph and Block Diagram Models

- Many electric circuits, electromechanical systems, and other control systems are complex

→difficult to determine a set of first-order differential equations

- How to get a state-space model? (thus signal-flow graph)

→derive transfer function of the system

→derive the state-space model

- Two state-variable representations: phase variable canonical form; input feedforward canonical form

Phase Variable Canonical Form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$G(s) = \frac{b_m s^{-(n-m)} + b_{m-1} s^{-(n-m+1)} + \cdots + b_1 s^{-(n-1)} + b_0 s^{-n}}{1 + a_{n-1} s^{-1} + \cdots + a_1 s^{-(n-1)} + a_0 s^{-n}}.$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_k P_k(s)\Delta_k(s)}{\Delta(s)}.$$

$$G(s) = \frac{\sum_k P_k(s)}{1 - \sum_{q=1}^{N} L_q(s)} = \frac{\text{Sum of the forward-path factors}}{1 - \text{sum of the feedback loop factors}}.$$

Example 1: from TF to SS

Method 1: intuitive approach

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$\frac{d^4(y(t)/b_0)}{dt^4} + a_3 \frac{d^3(y(t)/b_0)}{dt^3} + a_2 \frac{d^2(y(t)/b_0)}{dt^2} + a_1 \frac{d(y(t)/b_0)}{dt} + a_0(y(t)/b_0) = u(t).$$

$$\begin{aligned} x_1(t) &= y(t)/b_0 \\ x_2(t) &= \dot{x}_1(t) = \dot{y}(t)/b_0 \\ x_3(t) &= \dot{x}_2(t) = \ddot{y}(t)/b_0 \\ x_4(t) &= \dot{x}_3(t) = \dddot{y}(t)/b_0. \end{aligned}$$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t), \\ \dot{x}_3(t) &= x_4(t), \end{aligned}$$

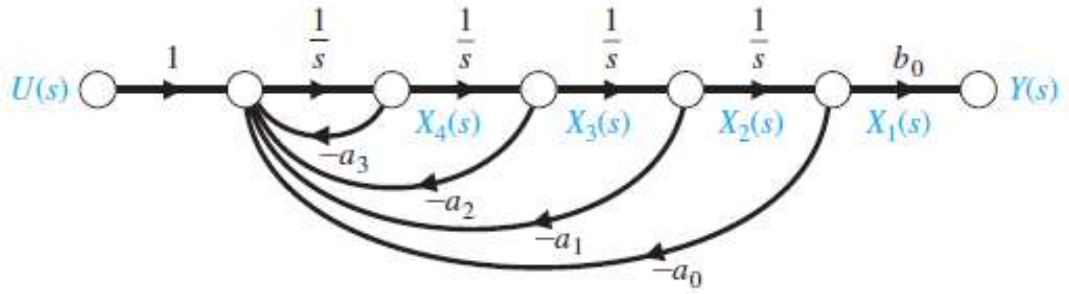
$$\dot{x}_4(t) = -a_0 x_1(t) - a_1 x_2(t) - a_2 x_3(t) - a_3 x_4(t) + u(t);$$

$$y(t) = b_0 x_1(t)$$

Example 1: from TF to SS

Method 2: Mason's signal-flow gain formula

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$
$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$



$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = x_3(t),$$
$$\dot{x}_3(t) = x_4(t),$$

$$\dot{x}_4(t) = -a_0 x_1(t) - a_1 x_2(t) - a_2 x_3(t) - a_3 x_4(t) + u(t);$$

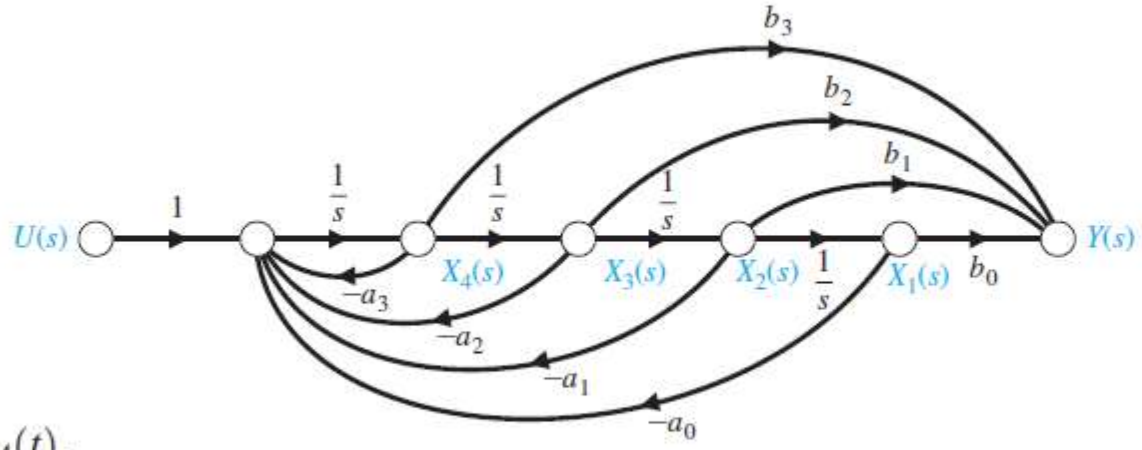
$$y(t) = b_0 x_1(t)$$

Example 2: from TF to SS

Method 2: Mason's signal-flow gain formula
with feedforward of phase variables
(phase variable canonical form)

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_3 s^{-1} + b_2 s^{-2} + b_1 s^{-3} + b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$



$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = x_3(t), \qquad \dot{x}_3(t) = x_4(t),$$

$$\dot{x}_4(t) = -a_0 x_1(t) - a_1 x_2(t) - a_2 x_3(t) - a_3 x_4(t) + u(t).$$

In this equation, $x_1(t), x_2(t), \ldots\ x_n(t)$ are the n **phase variables.**

Example 2: from TF to SS

Method 3: Intermediate variable Z(s)
(phase variable canonical form)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \frac{Z(s)}{Z(s)}.$$

$$Y(s) = [b_3 s^3 + b_2 s^2 + b_1 s + b_0] Z(s)$$

$$U(s) = [s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0] Z(s).$$

$$y(t) = b_3 \frac{d^3 z(t)}{dt^3} + b_2 \frac{d^2 z(t)}{dt^2} + b_1 \frac{dz(t)}{dt} + b_0 z(t)$$

$$u(t) = \frac{d^4 z(t)}{dt^4} + a_3 \frac{d^3 z(t)}{dt^3} + a_2 \frac{d^2 z(t)}{dt^2} + a_1 \frac{dz(t)}{dt} + a_0 z(t).$$

$$\begin{aligned} x_1(t) &= z(t) \\ x_2(t) &= \dot{x}_1(t) = \dot{z}(t) \\ x_3(t) &= \dot{x}_2(t) = \ddot{z}(t) \\ x_4(t) &= \dot{x}_3(t) = \dddot{z}(t). \end{aligned}$$

$$\dot{x}_4(t) = -a_0 x_1(t) - a_1 x_2(t) - a_2 x_3(t) - a_3 x_4(t) + u(t),$$

$$y(t) = b_0 x_1(t) + b_1 x_2(t) + b_2 x_3(t) + b_3 x_4(t).$$

Example 2: from TF to SS

Method 2 and Method 3 yield the phase variable canonical form

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$\Rightarrow \quad \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t).$$

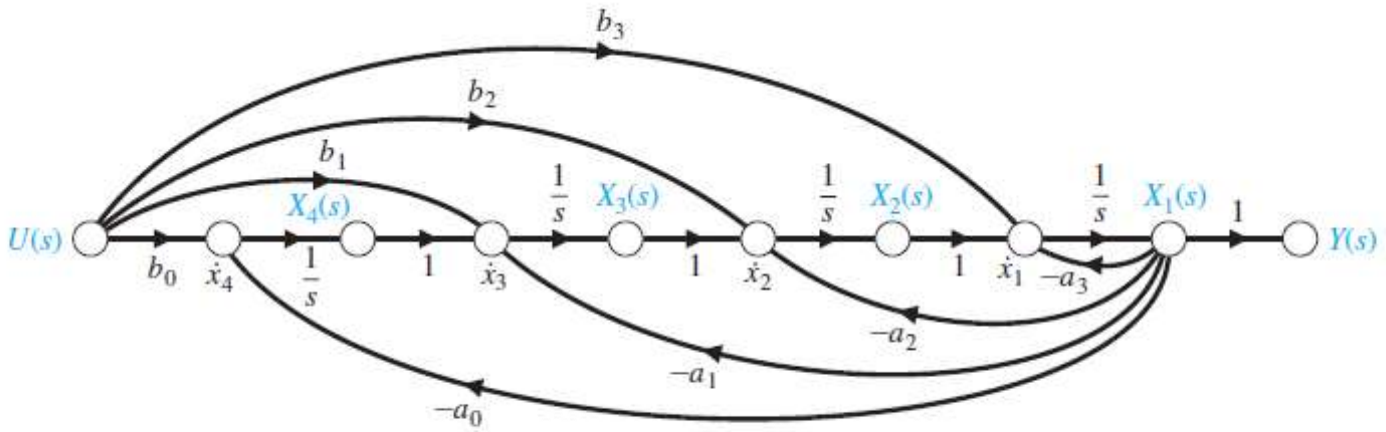
$$y(t) = \mathbf{C}\mathbf{x}(t) = [b_0 \quad b_1 \quad b_2 \quad b_3]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Example 2: from TF to SS

Method 4: Mason's signal-flow gain formula
with feedforward of the input
(input feedforward canonical form)

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_3 s^{-1} + b_2 s^{-2} + b_1 s^{-3} + b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$



$$\dot{x}_1(t) = -a_3 x_1(t) + x_2(t) + b_3 u(t), \qquad \dot{x}_2(t) = -a_2 x_1(t) + x_3(t) + b_2 u(t),$$

$$\dot{x}_3(t) = -a_1 x_1(t) + x_4(t) + b_1 u(t), \quad \text{and} \quad \dot{x}_4(t) = -a_0 x_1(t) + b_0 u(t). \qquad (3.52)$$

$$\frac{d\mathbf{x}(t)}{dt} = \begin{bmatrix} -a_3 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ -a_1 & 0 & 0 & 1 \\ -a_0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} b_3 \\ b_2 \\ b_1 \\ b_0 \end{bmatrix} u(t)$$

$$y(t) = [1 \quad 0 \quad 0 \quad 0]\mathbf{x}(t) + [0]u(t).$$

Summary of Methods: TF to SS

(incomplete!)

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t).$$

$$y(t) = \mathbf{C}\mathbf{x}(t) = [b_0 \quad b_1 \quad b_2 \quad b_3]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

- Phase variable canonical form

→Intuitive approach (no derivatives on the input signal)

→Mason's signal-flow gain formula with feedforward of phase variables

→Use of Intermediate variable

- Input feedforward canonical form

→ Mason's signal-flow gain formula with feedforward of the input

$$\frac{d\mathbf{x}(t)}{dt} = \begin{bmatrix} -a_3 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ -a_1 & 0 & 0 & 1 \\ -a_0 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} b_3 \\ b_2 \\ b_1 \\ b_0 \end{bmatrix} u(t)$$

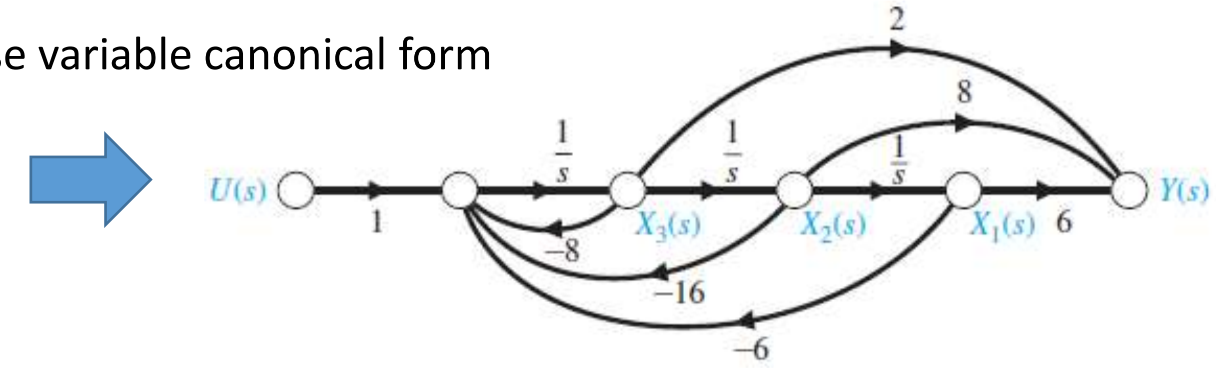
$$y(t) = [1 \quad 0 \quad 0 \quad 0]\mathbf{x}(t) + [0]u(t).$$

Example 3.2

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}.$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}.$$

Phase variable canonical form



$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t),$$

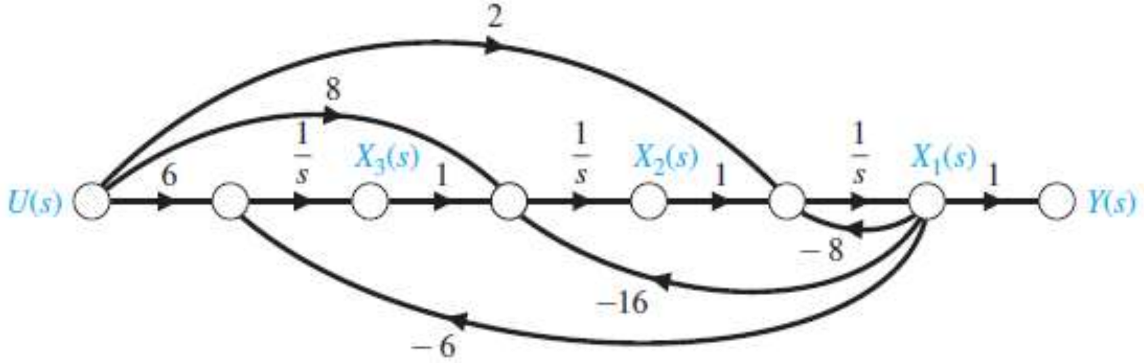
$$y(t) = [6 \quad 8 \quad 2]\mathbf{x}(t).$$

Example 3.2

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}.$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}.$$

Input feedforward canonical form



$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -8 & 1 & 0 \\ -16 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2 \\ 8 \\ 6 \end{bmatrix} u(t),$$

$$y(t) = [1 \quad 0 \quad 0]\mathbf{x}(t).$$

Remarks

When using the phase variable or input feedforward canonical form

- No need to factor the denominator or numerator to obtain the SS model
- The number of state variables = the order of the system
- One set of state variables is not identical but related to the other set of state variables by an appropriate linear transformation of variables

3.5 Alternative Signal-flow Graph and Block Diagram Models

- Select the physical variables as the state variables

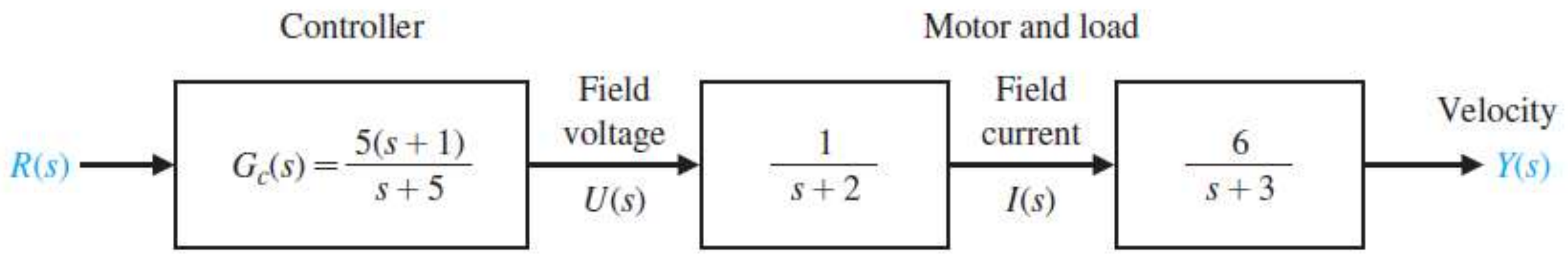
→Jordan canonical form (diagonal canonical form for distinct poles)

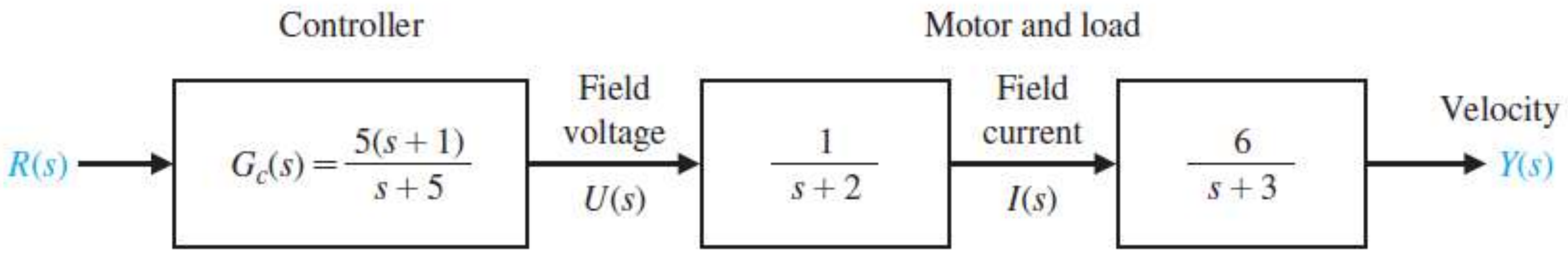
$x_1(t) = y(t)$ Velocity

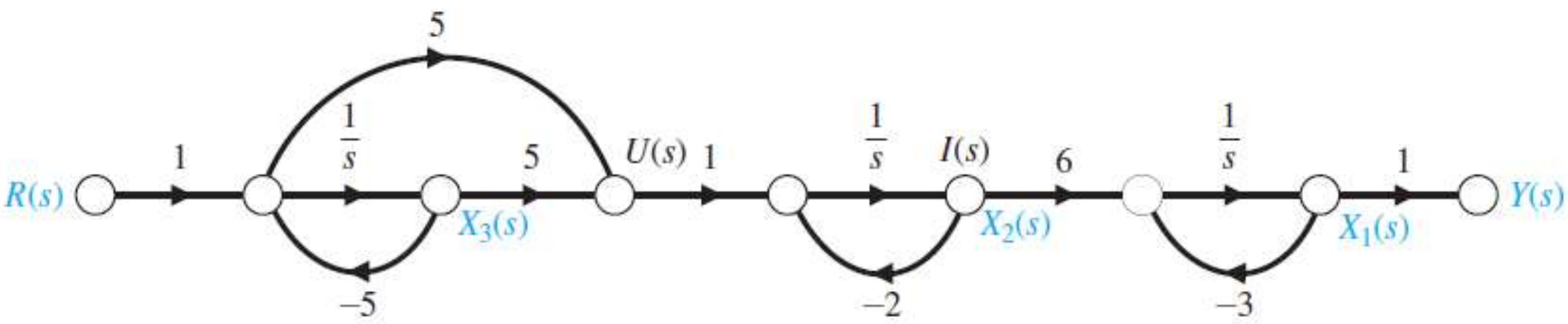
$x_2(t) = i(t),$ Field current

$x_3(t) = \frac{1}{4}r(t) - \frac{1}{20}u(t)$

$$\frac{U(s)}{R(s)} = G_c(s) = \frac{5(s+1)}{s+5} = \frac{5+5s^{-1}}{1+5s^{-1}},$$



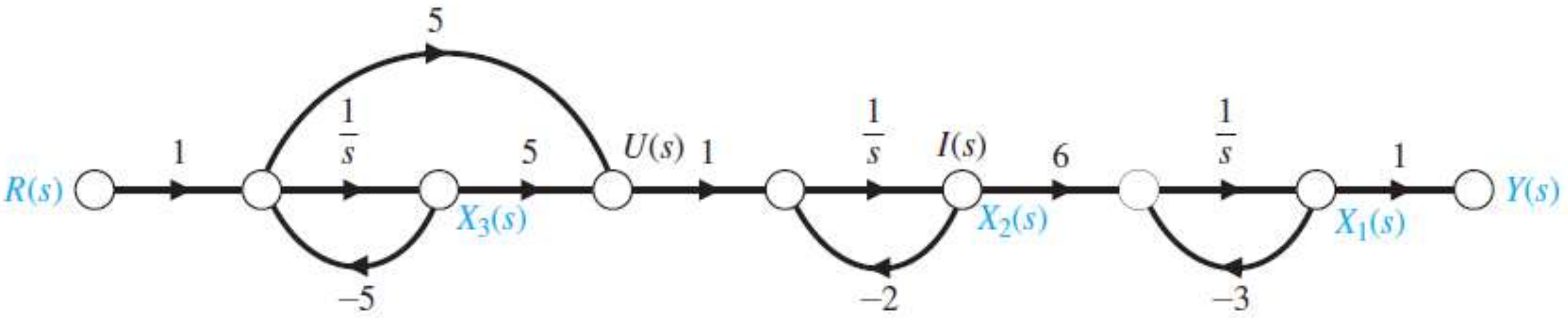




Jordan canonical form

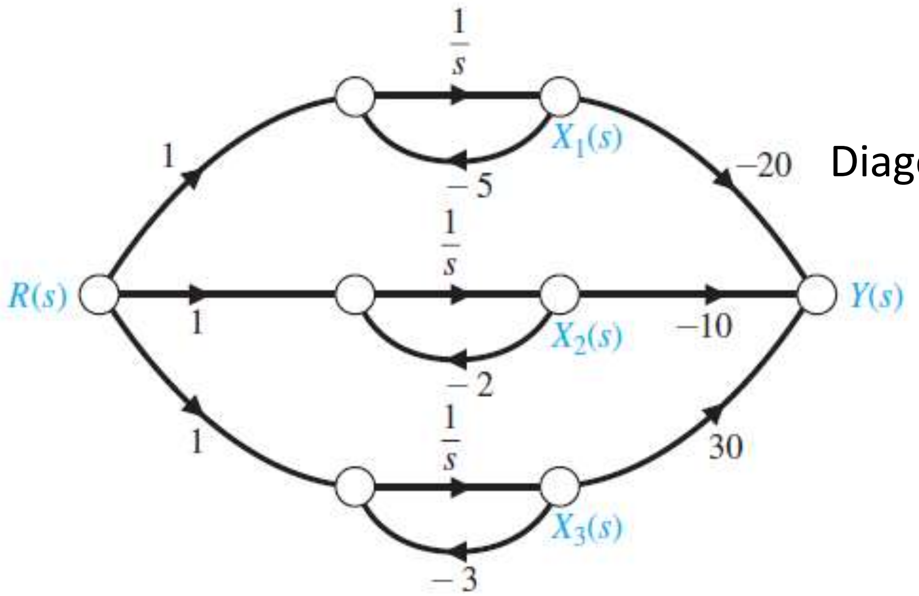
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -3 & 6 & 0 \\ 0 & -2 & -20 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} r(t)$$

$$y = [1 \quad 0 \quad 0]\mathbf{x}(t).$$



$$\frac{Y(s)}{R(s)} = T(s) = \frac{30(s+1)}{(s+5)(s+2)(s+3)} = \frac{q(s)}{(s-s_1)(s-s_2)(s-s_3)},$$

$$\frac{Y(s)}{R(s)} = T(s) = \frac{k_1}{s+5} + \frac{k_2}{s+2} + \frac{k_3}{s+3}, \qquad k_1 = -20,\ k_2 = -10, \text{ and } k_3 = 30.$$



Diagonal canonical form

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = [-20 \quad -10 \quad 30]\mathbf{x}(t).$$

Summary of Methods: TF to SS

- Phase variable canonical form

→Intuitive approach (no derivatives on the input signal)

→Mason’s signal-flow gain formula with feedforward of phase variables

→Use of Intermediate variable

- Input feedforward canonical form

→Mason’s signal-flow gain formula with feedforward of the input

- Jordan canonical form (diagonal canonical form for distinct poles)

→ Partial fraction expansion + signal-flow graph

3.6 Transfer Function from State Equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}U(s)$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s).$$

$$\mathbf{X}(s) = \mathbf{\Phi}(s)\mathbf{B}U(s).$$

$$Y(s) = [\mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + \mathbf{D}]U(s).$$

$$\boxed{G(s) = \mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + \mathbf{D}}$$

3.7 Time Response and State Transition Matrix

- To obtain the time response for performance evaluation

→ Need to know the state transition matrix

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\, d\tau.$$

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t) = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}$$

3 Ways of Finding State Transition Matrix

1. Calculate inverse Laplace transform (matrix inversion can be difficult)

$$\mathbf{\Phi}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}. \Rightarrow \mathbf{\Phi}(t) = \mathscr{L}^{-1}\{\mathbf{\Phi}(s)\}.$$

2. Use a truncated version of $\mathbf{\Phi}(t) = \exp(\mathbf{A}t) = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}$ (approximation)
3. Use signal-flow graph and Mason's gain formula

$$\mathbf{u}(\tau) = 0, \Rightarrow \mathbf{X}(s) = \mathbf{\Phi}(s)\mathbf{x}(0).$$

Example 3.5

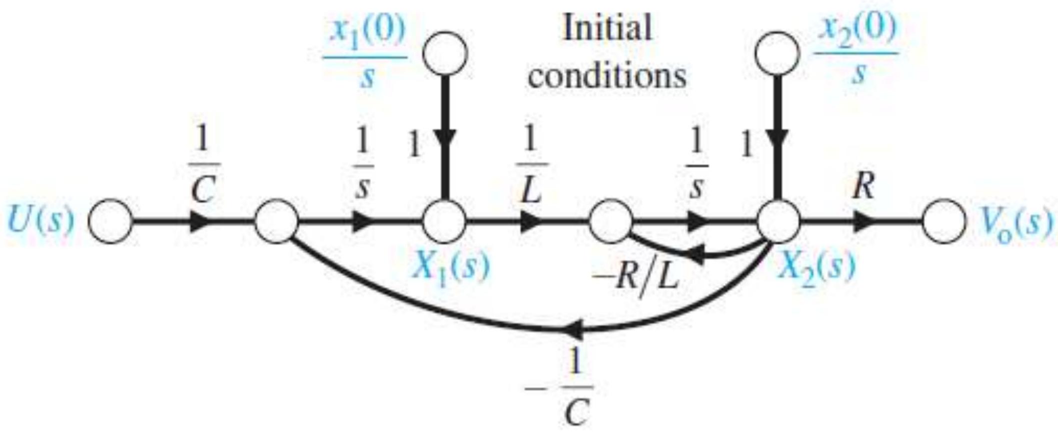


FIGURE 3.20
Flow graph of the *RLC* network.

Pay attention to how initial conditions are incorporated into the signal-flow graph

$$\mathbf{X}(s) = \mathbf{\Phi}(s)\mathbf{x}(0).$$

$$X_1(s) = \frac{1 \cdot \Delta_1(s) \cdot [x_1(0)/s]}{\Delta(s)},$$

When $R = 3$, $L = 1$, and $C = 1/2$,

$\Delta(s) = 1 + 3s^{-1} + 2s^{-2}$. $\Delta_1 = 1 + 3s^{-1}$

$$\phi_{11}(s) = \frac{(1 + 3s^{-1})(1/s)}{1 + 3s^{-1} + 2s^{-2}} = \frac{s + 3}{s^2 + 3s + 2}.$$

$$X_1(s) = \frac{(-2s^{-1})(x_2(0)/s)}{1 + 3s^{-1} + 2s^{-2}}.$$

$$\phi_{12}(s) = \frac{-2}{s^2 + 3s + 2}.$$

$$\mathbf{X}(s) = \mathbf{\Phi}(s)\mathbf{x}(0).$$

$$\phi_{11}(s) = \frac{(1 + 3s^{-1})(1/s)}{1 + 3s^{-1} + 2s^{-2}} = \frac{s + 3}{s^2 + 3s + 2}. \qquad \phi_{12}(s) = \frac{-2}{s^2 + 3s + 2}.$$

$$\phi_{21}(s) = \frac{(s^{-1})(1/s)}{1 + 3s^{-1} + 2s^{-2}} = \frac{1}{s^2 + 3s + 2}. \qquad \phi_{22}(s) = \frac{1(1/s)}{1 + 3s^{-1} + 2s^{-2}} = \frac{s}{s^2 + 3s + 2}.$$

$$\mathbf{\Phi}(s) = \begin{bmatrix} (s + 3)/(s^2 + 3s + 2) & -2/(s^2 + 3s + 2) \\ 1/(s^2 + 3s + 2) & s/(s^2 + 3s + 2) \end{bmatrix}.$$

$$\mathbf{\Phi}(t) = \mathcal{L}^{-1}\{\mathbf{\Phi}(s)\} = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}.$$

Caution!

- Different state-variable representations result in different matrices A, and hence different state transition matrices $\Phi(s)$

Ex:

A =

```
0   -2
1   -3
```

$\xrightarrow{\text{Diagonal canonical form}}$

D =

```
-1    0
 0   -2
```

$\xrightarrow{\text{State transition matrix}}$

$\Phi_D(t)$=

```
[ exp(-t),          0]
[       0, exp(-2*t)]
```

A $\xrightarrow{\text{State transition matrix}}$

$\Phi_A(t)$=

```
[ 2*exp(-t) - exp(-2*t), 2*exp(-2*t) - 2*exp(-t)]
[   exp(-t) - exp(-2*t),   2*exp(-2*t) - exp(-t)]
```

```matlab
function [phi] = statetrans(A)
    t = sym('t');
    phi = expm(A * t);
end
```