# Chapter 2

Mathematical Models of Systems

# 2.1 Introduction

- Systems under consideration are dynamic in nature

→Differential equations

- Linear systems are so important because we can solve them

→Linearization and Laplace transform

# 2.2 Differential Equations of Physical Systems

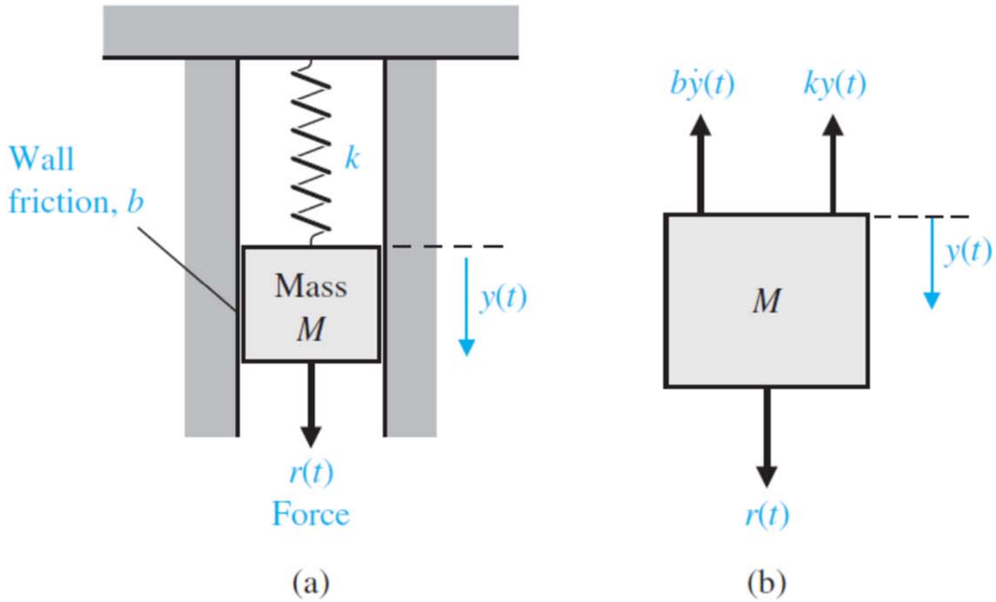



**FIGURE 2.2**
(a) Spring-mass-damper system.
(b) Free-body diagram.

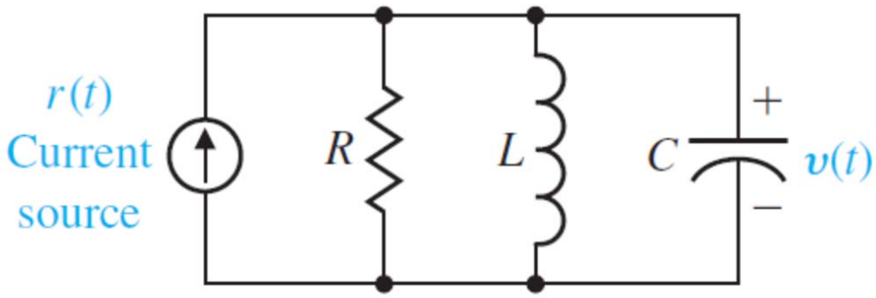
$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t)$$

# 2nd-order linear differential equation with constant coefficients

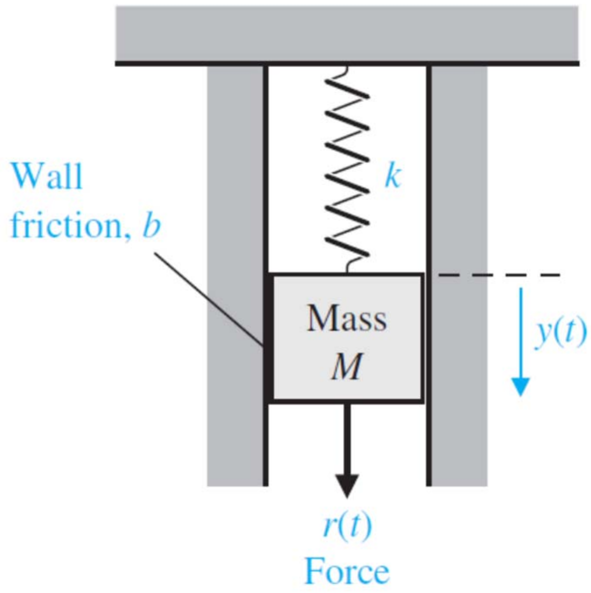
$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t)$$

- M: mass
- k: spring constant of the ideal spring
- b: friction constant

# Analogous Systems




$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)\,dt = r(t)$$




$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t)$$

# Analogous Variables

- Voltage–velocity analogy

(also called force–current analogy)

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t)$$

- Force– voltage analogy

→analogy that relates

velocity and current variables

$$v(t) = \frac{dy(t)}{dt}.$$

$$M\frac{dv(t)}{dt} + bv(t) + k\int_0^t v(t)\,dt = r(t).$$

# 2.3 Linear Approximations of Physical Systems

- A great majority of physical systems are linear

→within some range of the variables

- A system is defined as linear in terms of the system excitation (input) and response (output)
- Linear system

→superposition + homogeneity

- $y(t) = mx(t) + b$ is a linear function? A linear system? A linear transformation?

# Different Perspective

- May be considered linear about an operating point $x_0, y_0$ for small changes $\Delta x$ and $\Delta y$.

$$y(t) = mx(t) + b$$

$$x(t) = x_0 + \Delta x(t) \qquad y(t) = y_0 + \Delta y(t)$$

$$y_0 + \Delta y(t) = mx_0 + m\Delta x(t) + b$$

$$\Delta y(t) = m\Delta x(t)$$

"We are all in the gutter, but some of us are looking at the stars."
– Oscar Wilde, Writer

# Taylor Series Expansion (Linear Approximation)

$$y(t) = g(x(t)) = g(x_0) + \left.\frac{dg}{dx}\right|_{x(t)=x_0} \frac{(x(t) - x_0)}{1!} + \left.\frac{d^2g}{dx^2}\right|_{x(t)=x_0} \frac{(x(t) - x_0)^2}{2!} + \cdots . \tag{2.7}$$

$$m = \left.\frac{dg}{dx}\right|_{x(t)=x_0},$$

$$y(t) = g(x_0) + \left.\frac{dg}{dx}\right|_{x(t)=x_0} (x(t) - x_0) = y_0 + m(x(t) - x_0). \tag{2.8}$$

# 2.4 Laplace Transform

- Ability to obtain LTI approximations of physical systems

→Laplace transformation

- Laplace transformation

→Substitute relatively easily solved algebraic equations for the more difficult differential equations

- Inverse Laplace transformation

→ Heaviside partial fraction expansion

Oliver Heaviside (/ˈhɛvisaɪd/; 18 May 1850 – 3 February 1925) was an English self-taught electrical engineer, mathematician, and physicist.

# Illustration of Laplace Transform

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t). \qquad (2.18)$$

$$M\left(s^2Y(s) - sy(0^-) - \frac{dy}{dt}(0^-)\right) + b(sY(s) - y(0^-)) + kY(s) = R(s).$$

Initial conditions and zero input: $r(t) = 0, \quad \text{and} \quad y(0^-) = y_0, \quad \text{and} \quad \left.\frac{dy}{dt}\right|_{t=0^-} = 0,$

$$Ms^2Y(s) - Msy_0 + bsY(s) - by_0 + kY(s) = 0.$$

$$Y(s) = \frac{(Ms + b)y_0}{Ms^2 + bs + k} = \frac{p(s)}{q(s)}.$$

$$Y(s) = \frac{(Ms + b)y_0}{Ms^2 + bs + k} = \frac{p(s)}{q(s)}.$$

- *q(s)=0*

→Characteristic equation (roots of this equation determine the character of the time response)

- Critical frequencies

→ poles: roots of *q(s)=0*

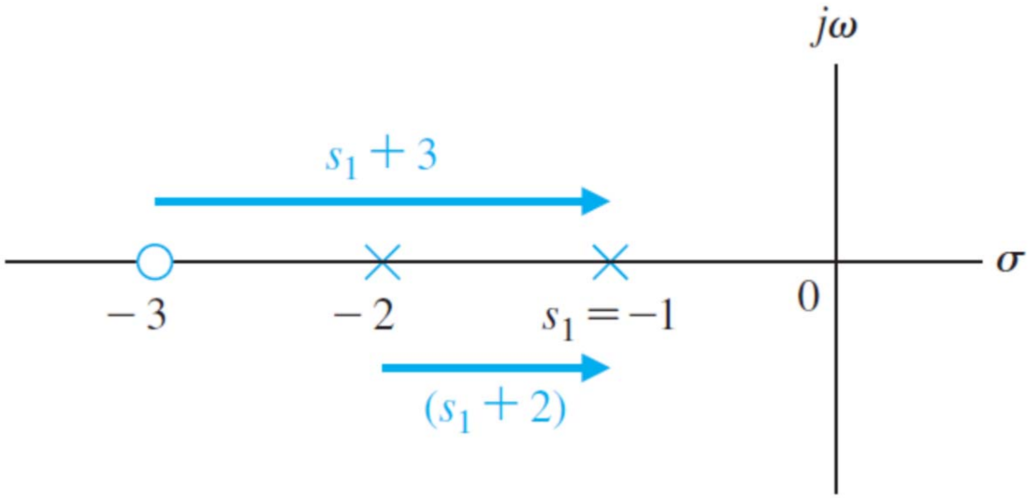
→ zeros: roots of *p(s)=0*

- Y(s) becomes infinite at poles and zero at the zeros.
- Complex frequency *s*-plane plot of the poles and zeros

→ graphically portray the character of the natural transient response of the system

# Residues

$$k/M = 2 \text{ and } b/M = 3.$$




$$Y(s) = \frac{(s+3)y_0}{(s+1)(s+2)}.$$

$$Y(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2},$$ Partial fraction expansion

Evaluated algebraically

Residues: $$k_1 = \left.\frac{(s-s_1)p(s)}{q(s)}\right|_{s=s_1}$$

$$= \left.\frac{(s+1)(s+3)}{(s+1)(s+2)}\right|_{s_1=-1} = 2$$

Evaluated graphically

$$k_1 = \left.\frac{s+3}{s+2}\right|_{s=s_1=-1}$$

$$= \left.\frac{s_1+3}{s_1+2}\right|_{s_1=-1} = 2.$$

# Steady-state or Final value of the Response

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-1}{s+2}\right\}.$$

$$y(t) = 2e^{-t} - 1e^{-2t}.$$

# Final Value Theorem

$$\boxed{\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s),}$$

Always zero?

- All poles of *Y(s)* strictly in the left half-plane except for at most one simple pole at the origin

→poles on the imaginary axis and in the right half-plane (not allowed)

→repeated poles at the origin (not allowed)

# Damping Ratio and Natural Frequency

- Second-order spring-mass-damper system

$$Y(s) = \frac{(s + b/M)y_0}{s^2 + (b/M)s + k/M} = \frac{(s + 2\zeta\omega_n)y_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{2.30}$$

$\zeta$ is the dimensionless **damping ratio**

$\omega_n$ is the **natural frequency**

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}, \quad \omega_n = \sqrt{k/M} \text{ and } \zeta = b/(2\sqrt{kM})$$

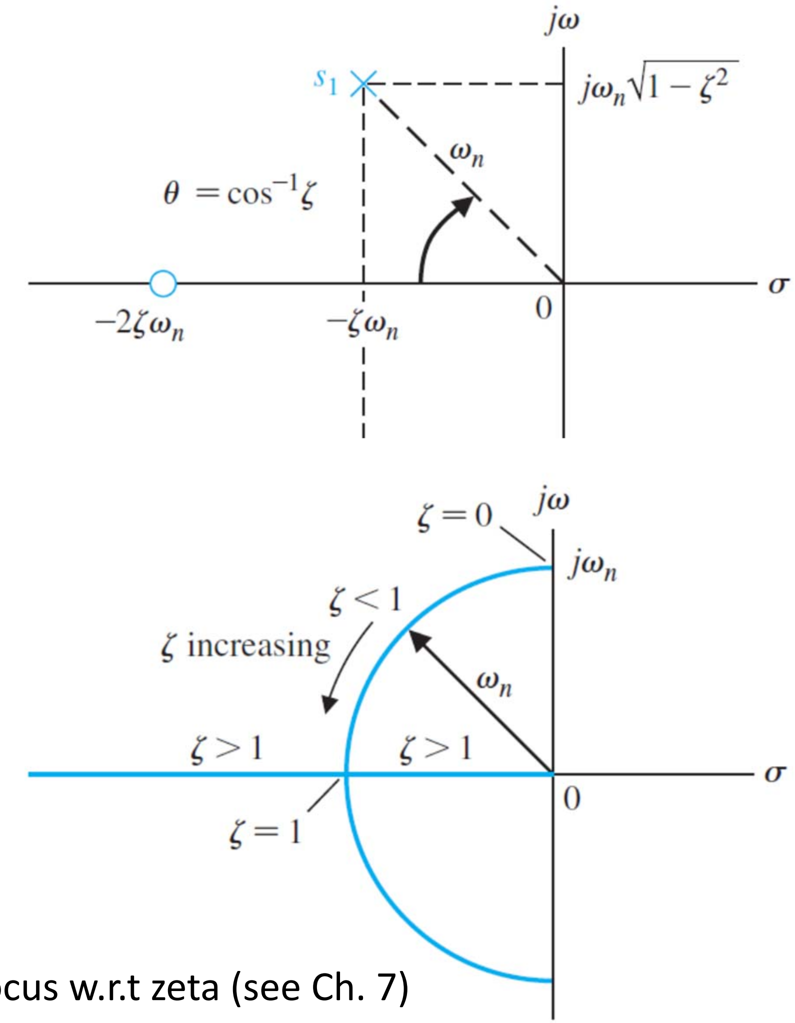
# Natural Frequency = Frequency ?

$$
\begin{aligned}
y(t) &= k_1 e^{s_1 t} + k_2 e^{s_2 t} \\
&= \frac{y_0}{2\sqrt{1-\zeta^2}} \left( e^{j(\theta-\pi/2)} e^{-\zeta\omega_n t} e^{j\omega_n \beta t} + e^{j(\pi/2-\theta)} e^{-\zeta\omega_n t} e^{-j\omega_n \beta t} \right) \\
&= \frac{y_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \theta).
\end{aligned}
$$

# Damping in Frequency Domain

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}.$$

- Overdamped $\zeta > 1$
- Underdamped $\zeta < 1$
- Critically damping $\zeta = 1$

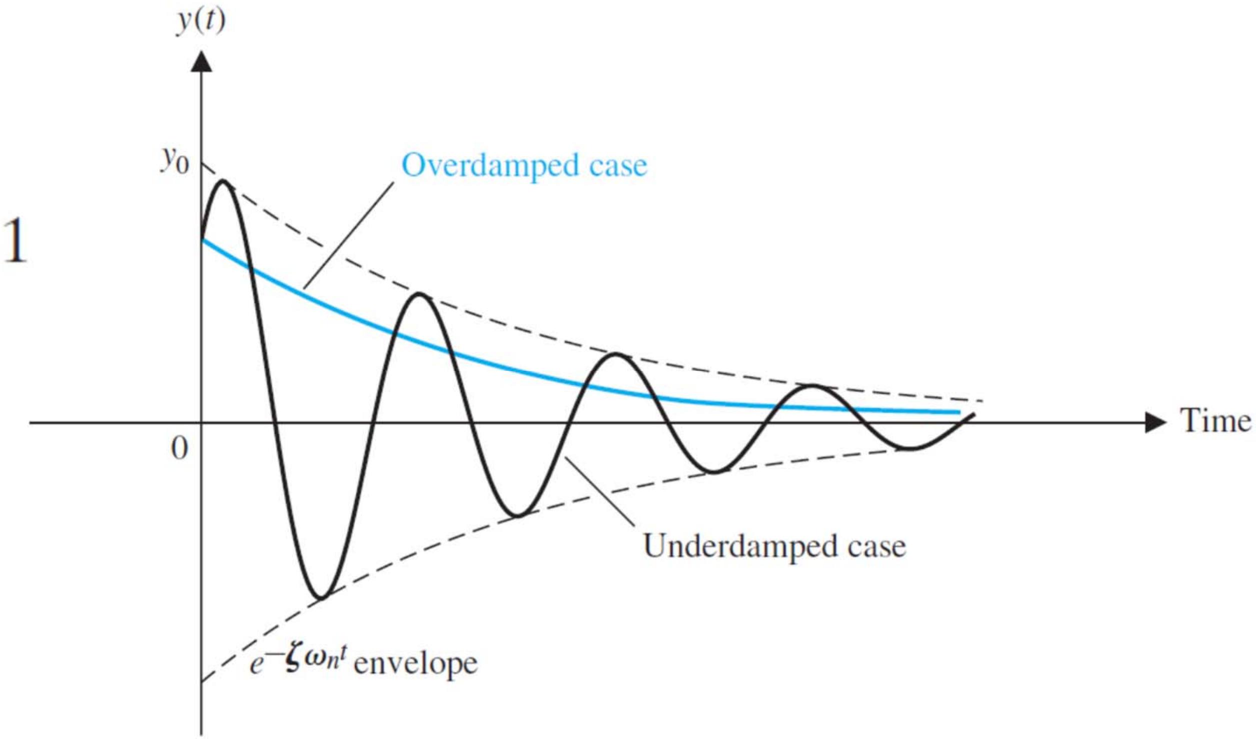



Root locus w.r.t zeta (see Ch. 7)

# Damping in Time Domain

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}.$$

- Overdamped $\zeta > 1$
- Underdamped $\zeta < 1$
- Critically damped $\zeta = 1$



# 2.5 Transfer Function of Linear Systems

- Transfer function of a linear system

→Definition: the ratio of the Laplace transform (LT) of the output variable to the Laplace transform of the input variable, with zero initial conditions

→an input–output description of the behavior of a system. It does not include any information concerning the internal structure of the system and its behavior

→TF is the Laplace transform of the impulse response

- LTI systems (stationary, constant parameter)→ OK for LT
- Time-varying systems (nonstationary, time-varying parameters) → X

# Transfer Function of Spring-Mass-Damper System

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t)$$

$$Ms^2Y(s) + bsY(s) + kY(s) = R(s).$$

$$G(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + bs + k}.$$

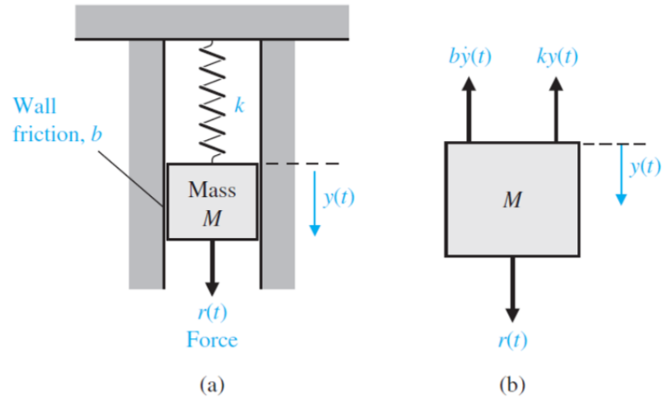
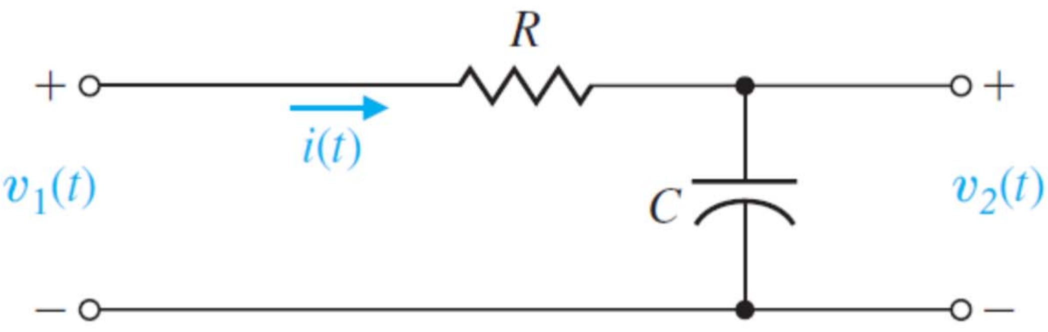



**FIGURE 2.2**
(a) Spring-mass-damper system.
(b) Free-body diagram.

# Transfer Function of RC Network




$$V_1(s) = \left(R + \frac{1}{Cs}\right) I(s),$$

$$V_2(s) = I(s)\left(\frac{1}{Cs}\right). \qquad V_2(s) = \frac{(1/Cs)V_1(s)}{R + 1/Cs}.$$

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs + 1} = \frac{1}{\tau s + 1} = \frac{1/\tau}{s + 1/\tau},$$

$\tau = RC$, the **time constant** of the network.

# Long-term System Behavior

$$\frac{d^n y(t)}{dt^n} + q_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + q_0 y(t)$$
$$= p_{n-1}\frac{d^{n-1}r(t)}{dt^{n-1}} + p_{n-2}\frac{d^{n-2}r(t)}{dt^{n-2}} + \cdots + p_0 r(t),$$

Transform equation: $q(s)Y(s) - m(s) = p(s)R(s)$ $m(s)$ is indecued by initial conditions

Transfer function (zero initial conditions, $m(s) = 0$):

$$Y(s) = G(s)R(s) = \frac{p(s)}{q(s)}R(s) = \frac{p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0}{s^n + q_{n-1}s^{n-1} + \cdots + q_0}R(s).$$

# Long-term System Behavior

$$\frac{d^n y(t)}{dt^n} + q_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + q_0 y(t)$$
$$= p_{n-1}\frac{d^{n-1}r(t)}{dt^{n-1}} + p_{n-2}\frac{d^{n-2}r(t)}{dt^{n-2}} + \cdots + p_0 r(t),$$

Transform equation: $q(s)Y(s) - m(s) = p(s)R(s)$ $m(s)$ is indecued by initial conditions

System output: $Y(s) = \frac{m(s)}{q(s)} + \frac{p(s)}{q(s)}R(s)$ with rational function $R(s) = \frac{n(s)}{d(s)}$

$$Y(s) = \frac{m(s)}{q(s)} + \frac{p(s)}{q(s)}\frac{n(s)}{d(s)} = Y_1(s) + Y_2(s) + Y_3(s)$$

# Long-term System Behavior

$$Y(s) = \frac{m(s)}{q(s)} + \frac{p(s)}{q(s)}\frac{n(s)}{d(s)} = Y_1(s) + Y_2(s) + Y_3(s)$$

$Y_1(s)$ partial fraction expansion of the natural response.

$Y_2(s)$ partial fraction expansion of the terms involving factors of $q(s)$

$Y_3(s)$ partial fraction expansion of the terms involving factors of $d(s)$

$$y(t) = y_1(t) + y_2(t) + y_3(t).$$

Natural response (determined by the initial conditions): $y_1(t)$
Forced response (determined by the input): $y_2(t) + y_3(t)$
Transient response: $y_1(t)+y_2(t)$
Steady-state response: $y_3(t)$

Example 2.2 $$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2r(t).$$

initial conditions are $y(0) = 1, \frac{dy}{dt}(0) = 0$, and $r(t) = 1, t \geq 0.$

$$[s^2Y(s) - sy(0)] + 4[sY(s) - y(0)] + 3Y(s) = 2R(s).$$

Since $R(s) = 1/s$ and $y(0) = 1$, we obtain

$$Y(s) = \frac{s + 4}{s^2 + 4s + 3} + \frac{2}{s(s^2 + 4s + 3)},$$

$$Y(s) = \left[\frac{3/2}{s + 1} + \frac{-1/2}{s + 3}\right] + \left[\frac{-1}{s + 1} + \frac{1/3}{s + 3}\right] + \frac{2/3}{s} = Y_1(s) + Y_2(s) + Y_3(s).$$

$$Y(s) = \left[\frac{3/2}{s+1} + \frac{-1/2}{s+3}\right] + \left[\frac{-1}{s+1} + \frac{1/3}{s+3}\right] + \frac{2/3}{s} = Y_1(s) + Y_2(s) + Y_3(s).$$

$$y(t) = \left[\frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}\right] + \left[-1e^{-t} + \frac{1}{3}e^{-3t}\right] + \frac{2}{3}, \qquad \lim_{t\to\infty} y(t) = \frac{2}{3}.$$

3 ways of calculating the steady-state response

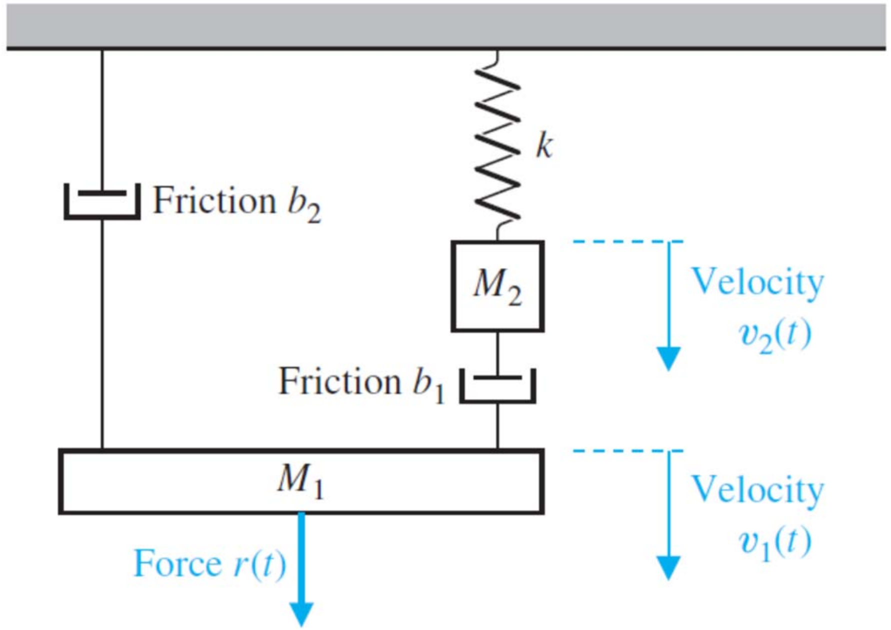
1. Use Laplace transform and inverse Laplace transform
2. Apply FVT
3. Time domain perspective: steady-state response satisfies the DE and thus dy(t)/dt=0; we have 3y(t)=2r(t) → y(t)=2r(t)/3=2/3 for step input

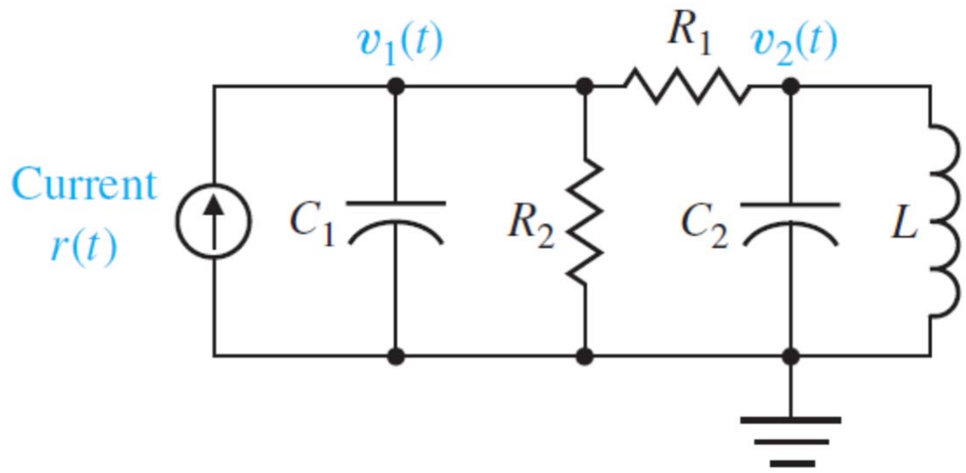
(There is a caveat!)

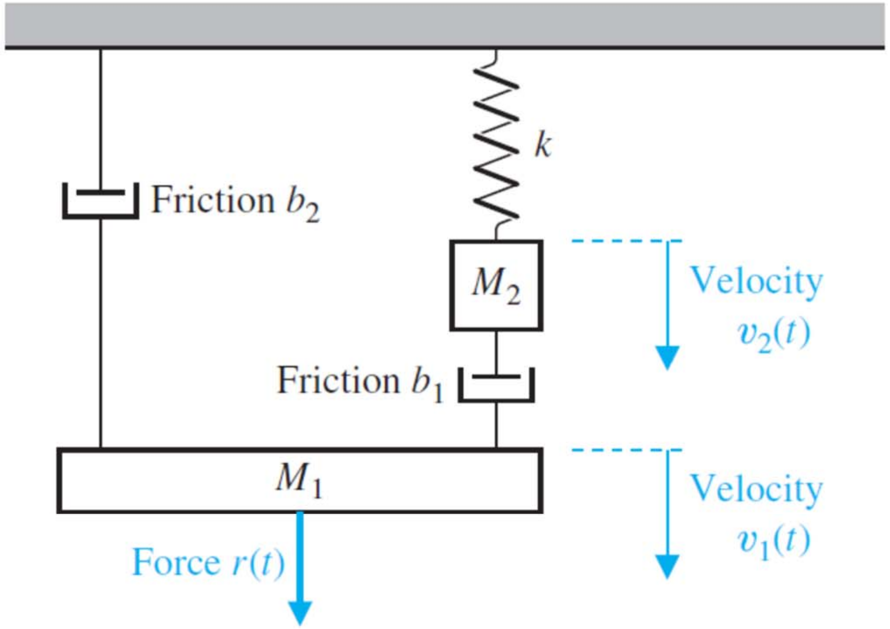
$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2r(t).$$

# Example 2.4

- Velocity-voltage analogy (force-current analogy)






$$M_1 s V_1(s) + (b_1 + b_2)V_1(s) - b_1 V_2(s) = R(s),$$

$$M_2 s V_2(s) + b_1(V_2(s) - V_1(s)) + k\frac{V_2(s)}{s} = 0.$$

$$\begin{bmatrix} M_1 s + b_1 + b_2 & -b_1 \\ -b_1 & M_2 s + b_1 + \dfrac{k}{s} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix}$$

Assuming that the velocity of $M_1$ is the output variable

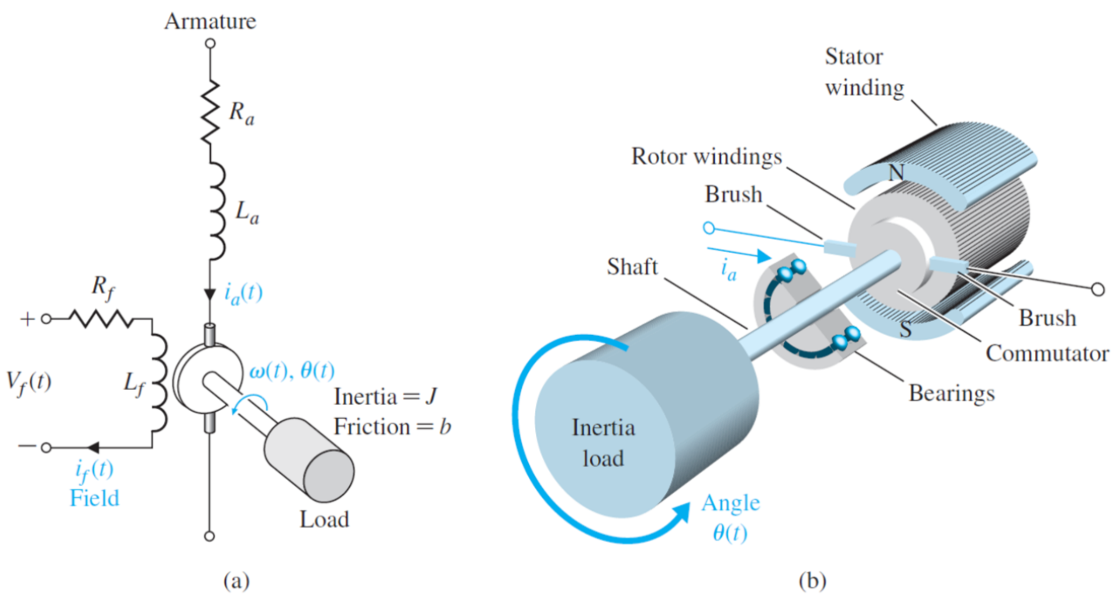
$$G(s) = \frac{V_1(s)}{R(s)} = \frac{(M_2 s + b_1 + k/s)}{(M_1 s + b_1 + b_2)(M_2 s + b_1 + k/s) - b_1^2}$$

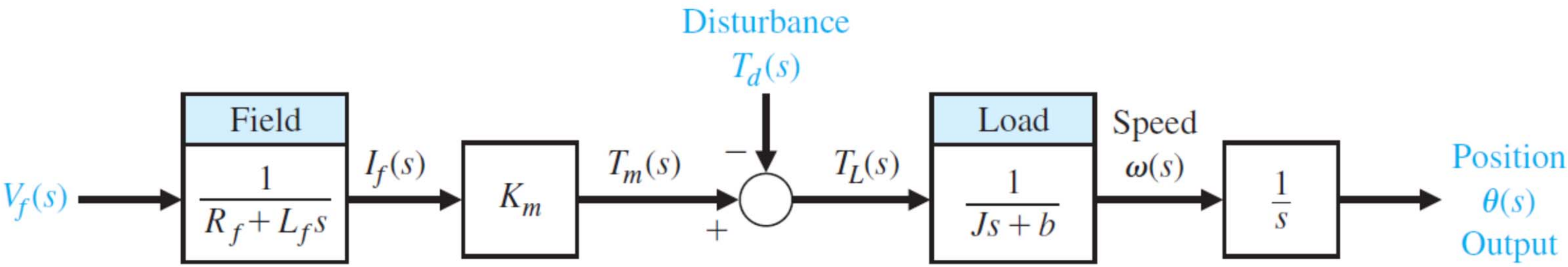
$$= \frac{(M_2 s^2 + b_1 s + k)}{(M_1 s + b_1 + b_2)(M_2 s^2 + b_1 s + k) - b_1^2 s}.$$

What is the transfer function of $X_1(s)/R(s)$?

# Example 2.5 DC Motor

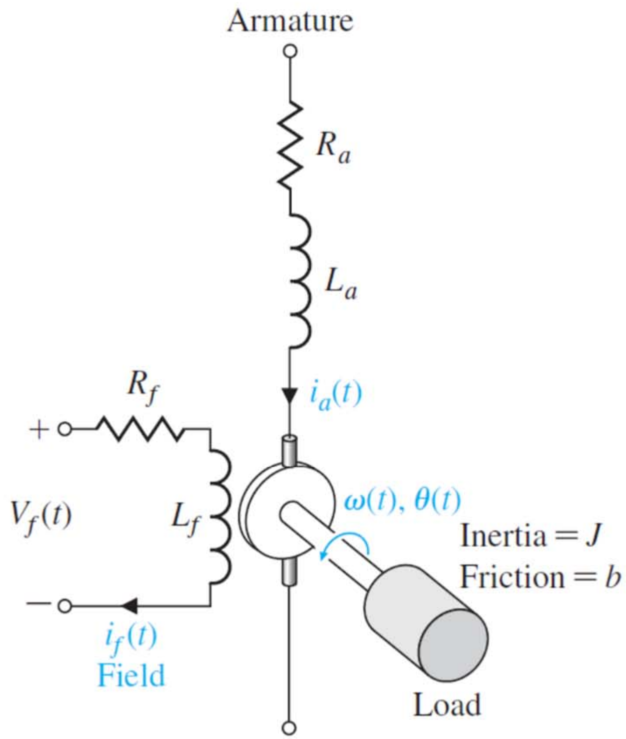
- DC motor moves loads
- An actuator is a device that provides the motive power to the process
- DC motor is an example of an actuator






$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_fs + R_f)} = \frac{K_m/(JL_f)}{s(s + b/J)(s + R_f/L_f)}.$$

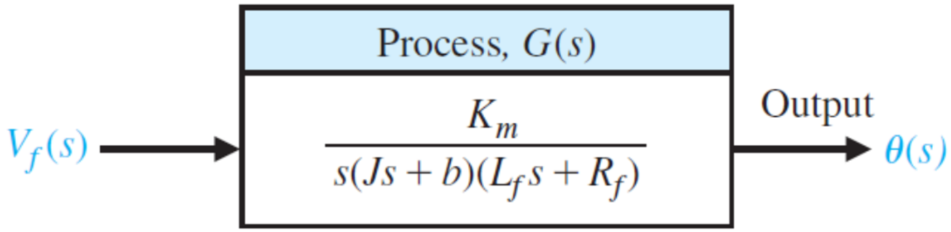
1. No need for considering the inner structure
2. How to achieve a desired position?

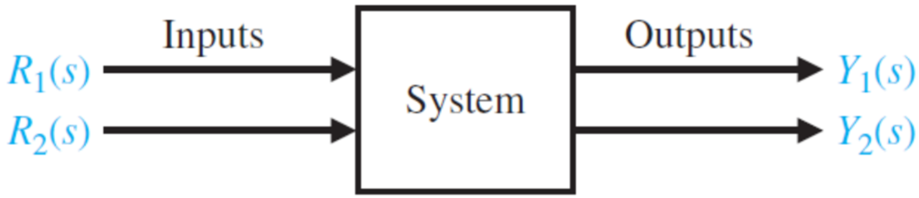


# 2.6 Block Diagram Models

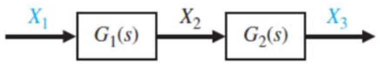
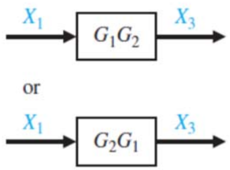
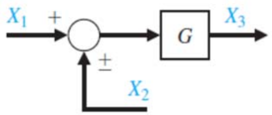
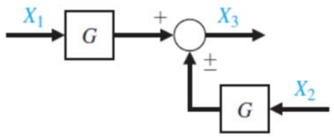
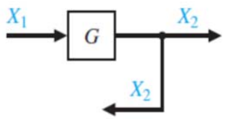
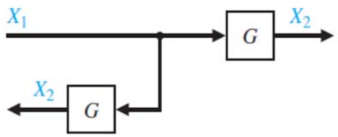
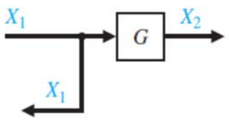
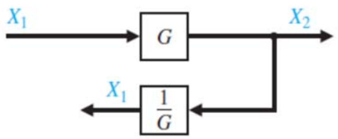
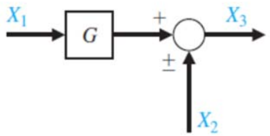
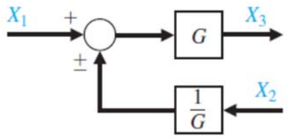
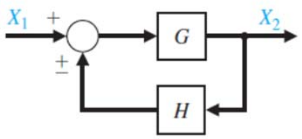
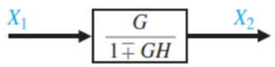
- Block diagram

→graphical representation of the relationship between the outputs (controlled variables, dependent variables) and inputs (controlling variables, independent variables)



# Block Diagram Transformations

| Transformation | Original Diagram | Equivalent Diagram |
|---|---|---|
| 1. Combining blocks in cascade |   |   |
| 2. Moving a summing point behind a block |   |   |
| 3. Moving a pickoff point ahead of a block |   |   |
| 4. Moving a pickoff point behind a block |   |   |
| 5. Moving a summing point ahead of a block |   |   |
| 6. Eliminating a feedback loop |   |   |

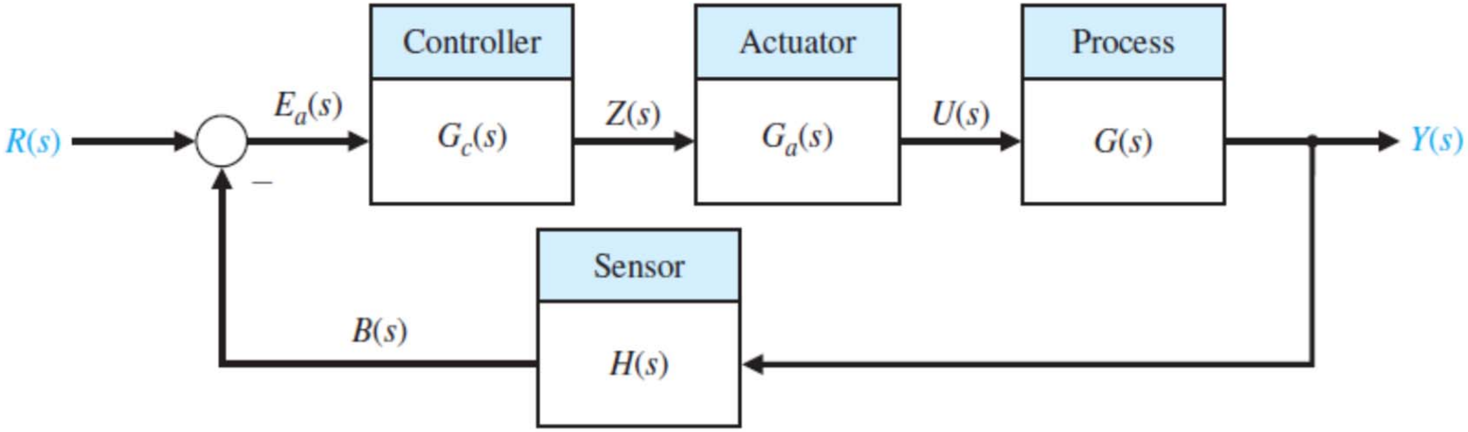
# Assumption

- No loading effect is assumed

→Loading and interaction between interconnected components or systems may occur

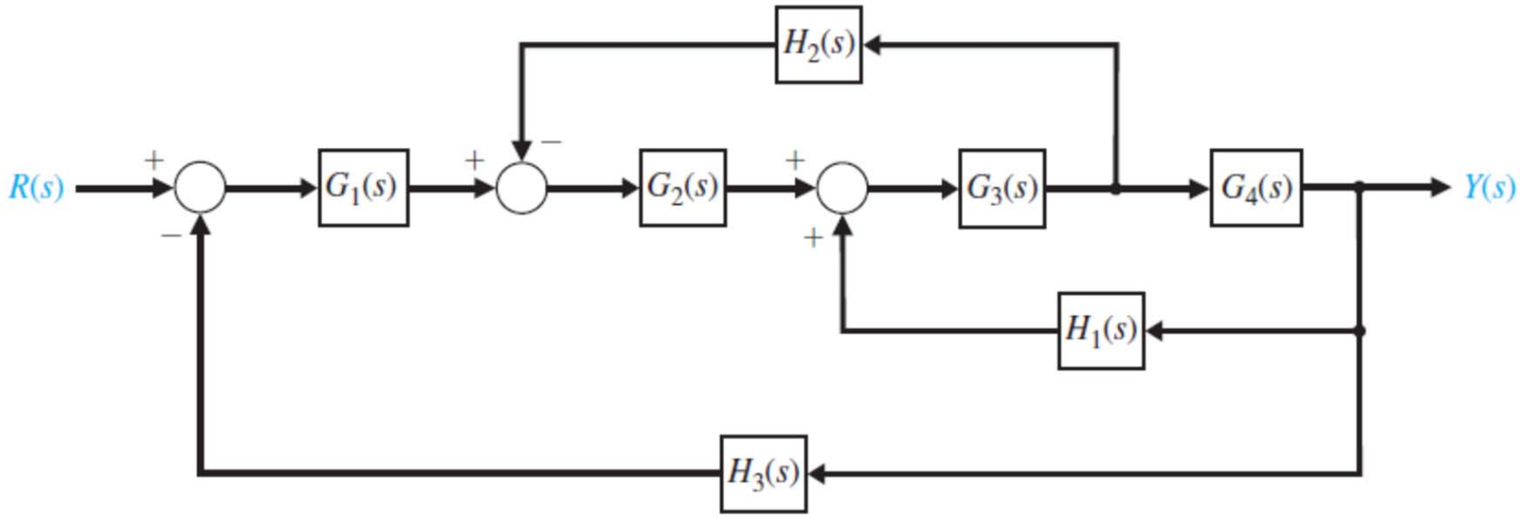
→If the loading of interconnected devices does occur, the engineer must account for this change in the transfer function and use the corrected transfer function in subsequent calculations

# Example




$$\frac{Y(s)}{R(s)} = \frac{G(s)G_a(s)G_c(s)}{1 + G(s)G_a(s)G_c(s)H(s)}.$$

# Example 2.6




loop $G_3(s)G_4(s)H_1(s)$ is a **positive feedback loop**.

# 2.7 Signal-flow Graph Models

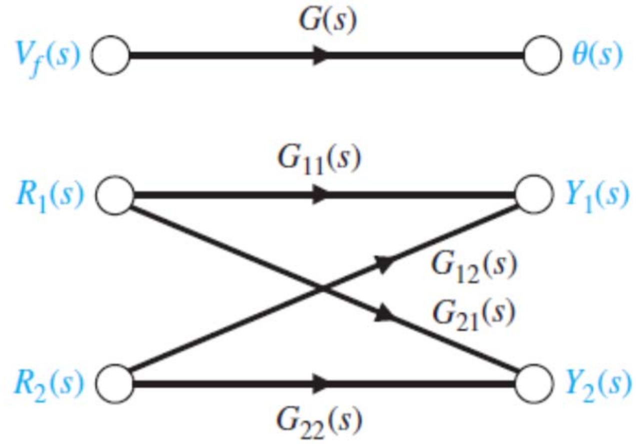
- Signal-flow graph

→ an alternative method for graphically determining the relationship between system variables

→developed by Mason

→ advantage: signal-flow gain formula

# Models




- Signal-flow graph

→a diagram consisting of **nodes** that are connected by several directed **branches** and a graphical representation of a set of linear relations

- Branch (equivalent to a block in block diagram)

→a unidirectional path segment

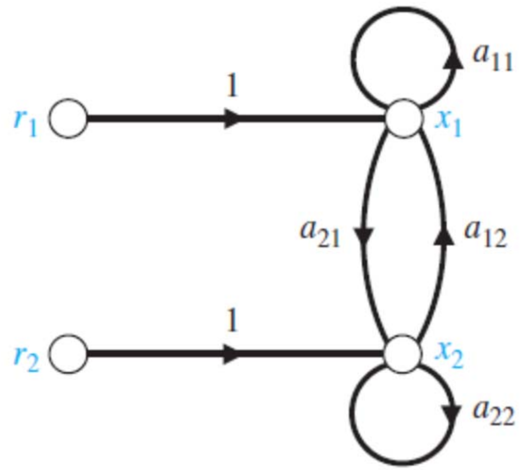
- Nodes

→input and output points or junctions

- Path

→ a branch or a continuous sequence of branches that can be traversed from one signal (node) to another signal (node).

# Models




- Loop

→a closed path that originates and terminates on the same node, with no node being met twice along the path

- Nontouching loops

→ Loops do not have a common node

# From Cramer's Rule to Mason's Gain Formula

$$a_{11}x_1 + a_{12}x_2 + r_1 = x_1$$

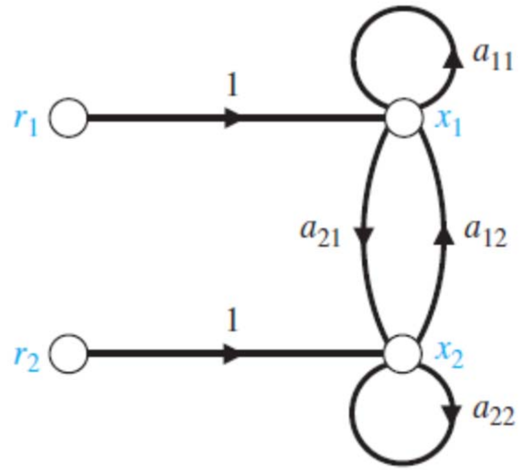
$$a_{21}x_1 + a_{22}x_2 + r_2 = x_2.$$

$$x_1(1 - a_{11}) + x_2(-a_{12}) = r_1,$$

$$x_1(-a_{21}) + x_2(1 - a_{22}) = r_2.$$

$$x_1 = \frac{(1 - a_{22})r_1 + a_{12}r_2}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} = \frac{1 - a_{22}}{\Delta}r_1 + \frac{a_{12}}{\Delta}r_2,$$

$$x_2 = \frac{(1 - a_{11})r_2 + a_{21}r_1}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} = \frac{1 - a_{11}}{\Delta}r_2 + \frac{a_{21}}{\Delta}r_1.$$



# From Cramer’s Rule to Mason’s Gain Formula

$$x_1 = \frac{(1 - a_{22})r_1 + a_{12}r_2}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} = \frac{1 - a_{22}}{\Delta}r_1 + \frac{a_{12}}{\Delta}r_2,$$

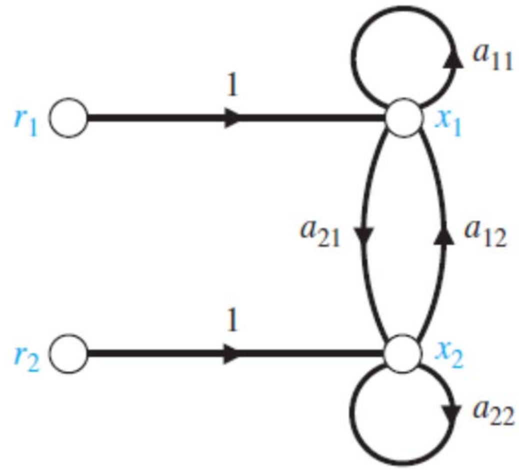
$$x_2 = \frac{(1 - a_{11})r_2 + a_{21}r_1}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} = \frac{1 - a_{11}}{\Delta}r_2 + \frac{a_{21}}{\Delta}r_1.$$

Nontouching loop

$$\Delta = (1 - a_{11})(1 - a_{22}) - a_{12}a_{21} = 1 - a_{11} - a_{22} + a_{11}a_{22} - a_{12}a_{21}.$$

Self-loop

$\Delta =$ 1- self-loop gains +nontouching loop gains



# Mason's Gain Formula

Simplified version

$$T_{ij}(s) = \frac{\sum_k P_{ijk}(s)\,\Delta_{ijk}(s)}{\Delta(s)},$$

$$\boxed{T(s) = \frac{\Sigma_k P_k(s)\Delta_k(s)}{\Delta(s)},}$$

$P_{ijk}(s)$ = gain of $k$th path from variable $x_i$ to variable $x_j$,

$\Delta(s)$ = determinant of the graph,

$\Delta_{ijk}(s)$ = cofactor of the path $P_{ijk}(s)$,

Explanations:

$$\Delta(s) = 1 - \sum_{n=1}^{N} L_n(s) + \sum_{\substack{n,\,m \\ \text{nontouching}}} L_n(s)L_m(s) - \sum_{\substack{n,\,m,\,p \\ \text{nontouching}}} L_n(s)L_m(s)L_p(s) + \cdots,$$

$\Delta$ = 1 − (sum of all different loop gains)
 + (sum of the gain products of all combinations of two nontouching loops)
 −(sum of the gain products of all combinations of three nontouching loops)
 + $\cdots$.

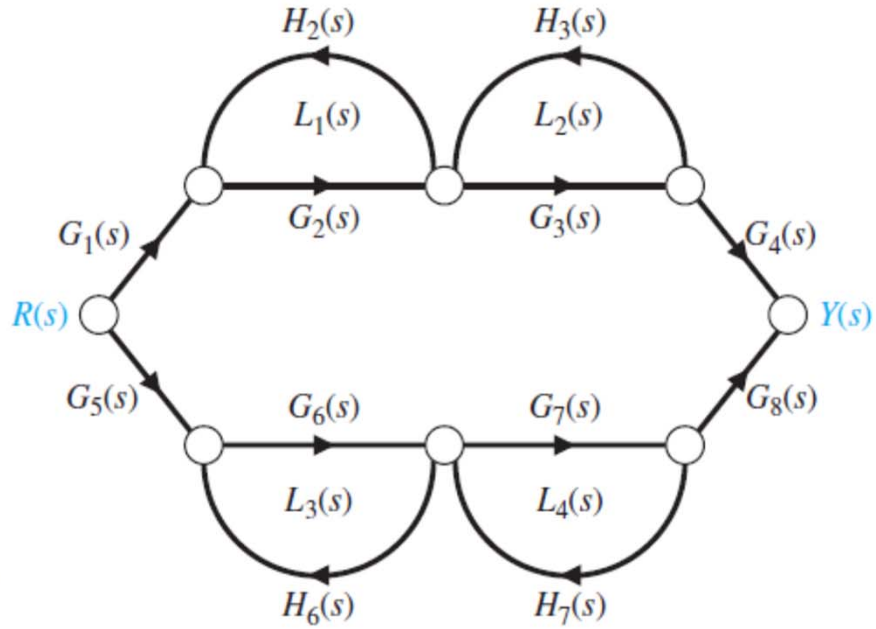
The cofactor $\Delta_{ijk}(s)$ is the determinant with the loops touching the $k$th path removed.

# Example 2.7

Paths:

$$P_1(s) = G_1(s)G_2(s)G_3(s)G_4(s) \text{ (path 1)}$$

$$P_2(s) = G_5(s)G_6(s)G_7(s)G_8(s) \text{ (path 2).}$$




Self-loops:

$$L_1(s) = G_2(s)H_2(s), \qquad L_2(s) = H_3(s)G_3(s),$$

$$L_3(s) = G_6(s)H_6(s), \qquad \text{and} \qquad L_4(s) = G_7(s)H_7(s).$$

Determinant:

$$\Delta(s) = 1 - (L_1(s) + L_2(s) + L_3(s) + L_4(s)) + (L_1(s)L_3(s) + L_1(s)L_4(s) + L_2(s)L_3(s) + L_2(s)L_4(s)).$$

Cofactors:

$$\Delta_1(s) = 1 - (L_3(s) + L_4(s)).$$

$$\Delta_2(s) = 1 - (L_1(s) + L_2(s)).$$

Transfer function:

$$\frac{Y(s)}{R(s)} = T(s) = \frac{P_1(s)\Delta_1(s) + P_2(s)\Delta_2(s)}{\Delta(s)}$$

# Caution!

- Calculate $T(s)=X_1(s)/R_1(s)$

Path: P=1

Self-loops: $a_{11}$ $a_{22}$ $a_{12}a_{21}$.

Determinant: $\Delta = 1 - a_{11} - a_{22} - a_{12}a_{21} + a_{11}a_{22}$

Cofactor: $1 - a_{22}$

Transfer function: $\dfrac{1 - a_{22}}{\Delta}$.

Relationship: $x_1 = \dfrac{1 - a_{22}}{\Delta} r_1$ Correct?