Chapter 5

Performance of Feedback Control Systems

5.1 Introduction

- Control systems are inherently dynamic

→ performance is specified in terms of both the transient response and the steady-state response

- Feedback control systems

→ability to adjust the transient and steady-state performance

- Transient response

→response that disappears with time

- Steady-state response

→ response that exists for a long time following an input signal initiation

Design Specs

- Design specifications (specs)

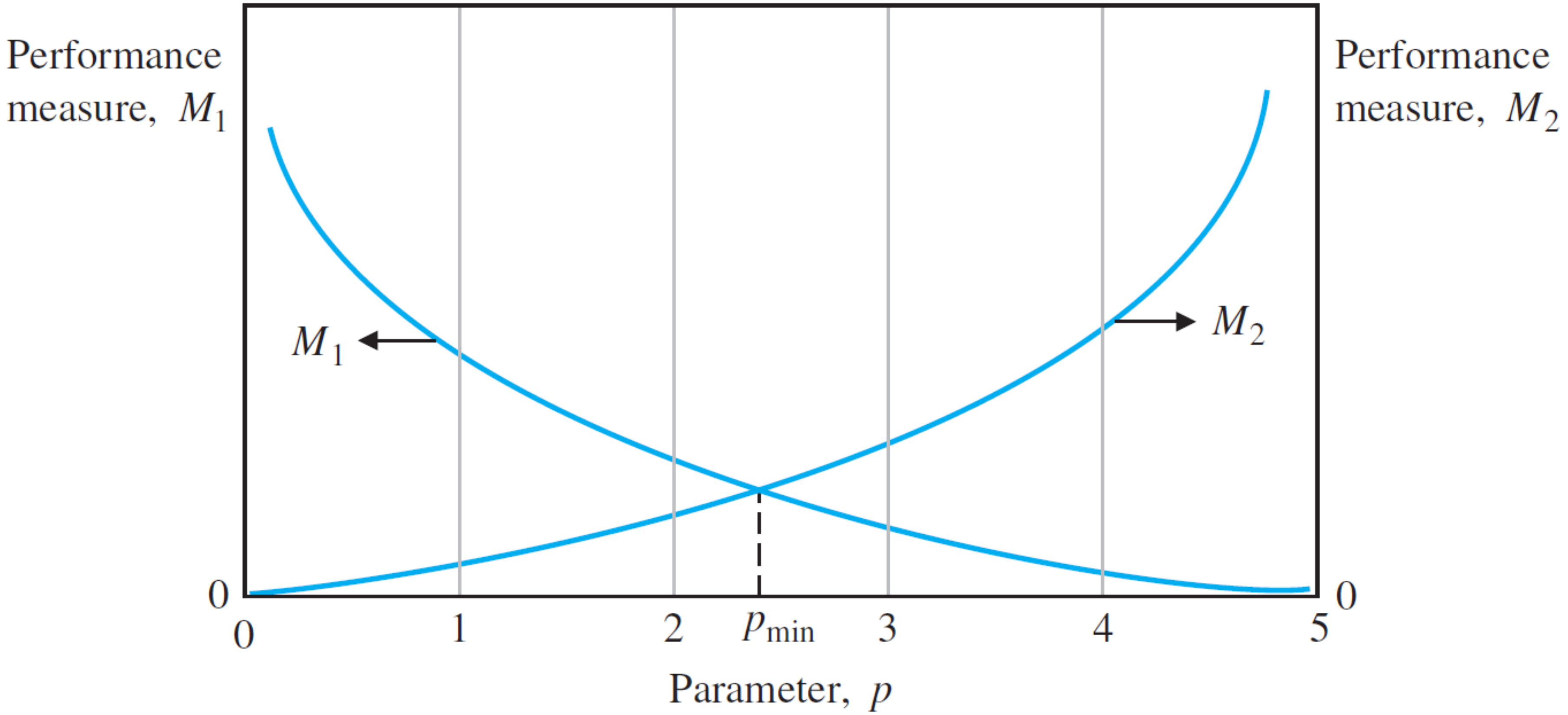
→including several time response indices for a specified input command, as well as a desired steady-state accuracy

- Specifications are

→seldom a rigid set of requirements (why?)

→often revised to effect a compromise

→an attempt to quantify the desired performance



Performance measure, M_1
Performance measure, M_2
M_1
M_2
0
0
0
1
2
p_{min}
3
4
5
Parameter, p

5.2 Test Input Signals

- Control systems are inherently time-domain systems

→Time-domain performance specifications are important indices

- If the system is stable, the response to a specific input signal will provide several measures of the performance (stability will be discussed later)
- Actual input signal of the system is usually unknown

→ standard test input signals are normally chosen.

Reasons for Using Test Signals

1. A reasonable correlation between the response of a system to a standard test input and the system’s ability to perform under normal operating conditions.
2. Allows the designer to compare several competing designs.
3. Many control systems experience input signals that are very similar to the standard test signals.

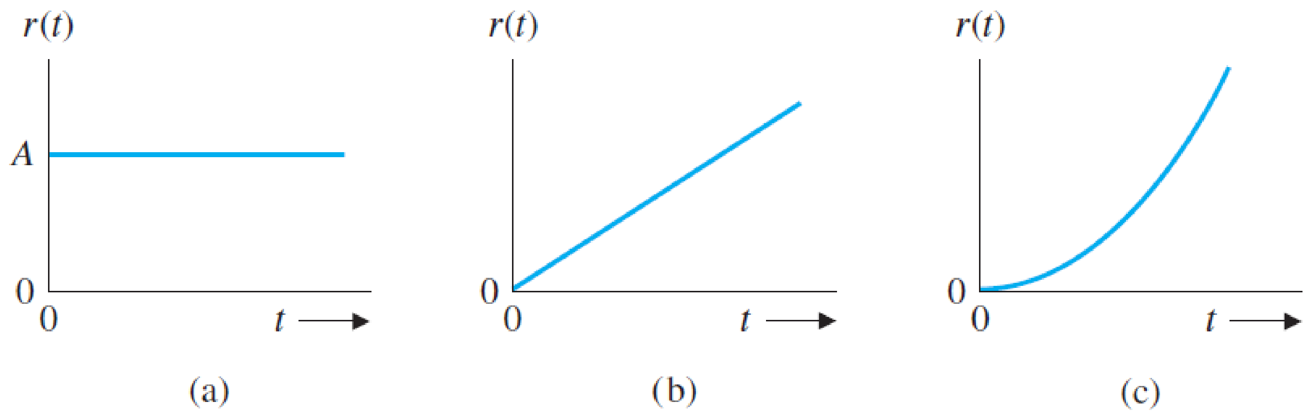
Standard Test Input Signals

For r(t) instead of u(t)

Table 5.1 Test Signal Inputs

Test Signal	$r(t)$	$R(s)$
Step	$r(t) = A, t > 0$ $= 0, t < 0$	$R(s) = A/s$
Ramp	$r(t) = At, t > 0$ $= 0, t < 0$	$R(s) = A/s^2$
Parabolic	$r(t) = At^2, t > 0$ $= 0, t < 0$	$R(s) = 2A/s^3$

Response to one test signal is related to the response of another test signal (why?)



Unit Impulse Function

$$f_\epsilon(t) = \begin{cases} 1/\epsilon, & -\dfrac{\epsilon}{2} \le t \le \dfrac{\epsilon}{2}; \\ 0, & \text{otherwise}, \end{cases}$$

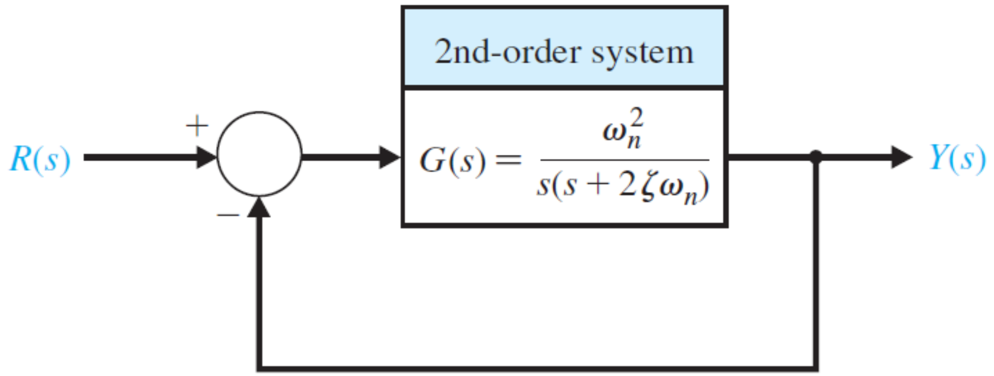
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t-a) g(t)\, dt = g(a).$$

$$y(t) = \int_{-\infty}^{t} g(t-\tau) r(\tau)\, d\tau = \mathscr{L}^{-1}\{G(s)R(s)\}.$$

TF is the LT of impulse response

5.3 Performance of 2nd Systems

$$Y(s) = \frac{G(s)}{1 + G(s)} R(s)$$



$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} R(s)$$

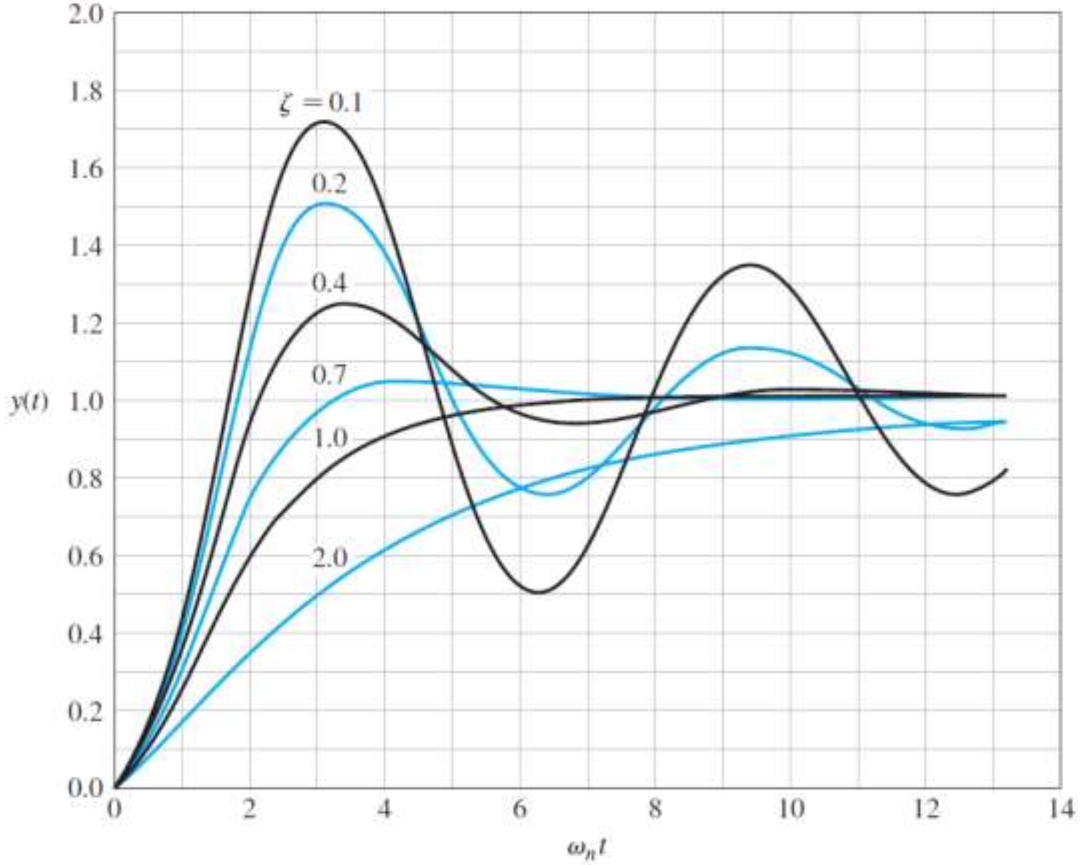
$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta),$$

Step response

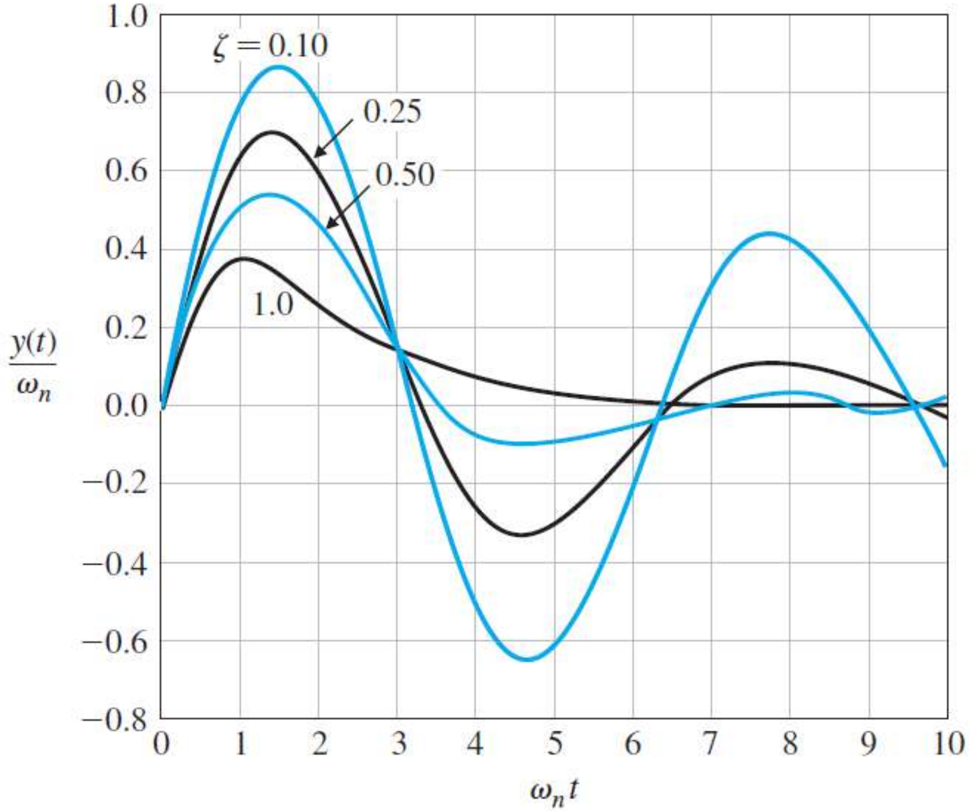
$$y(t) = \frac{\omega_n}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t)$$

Impulse response

Step response



Impulse response

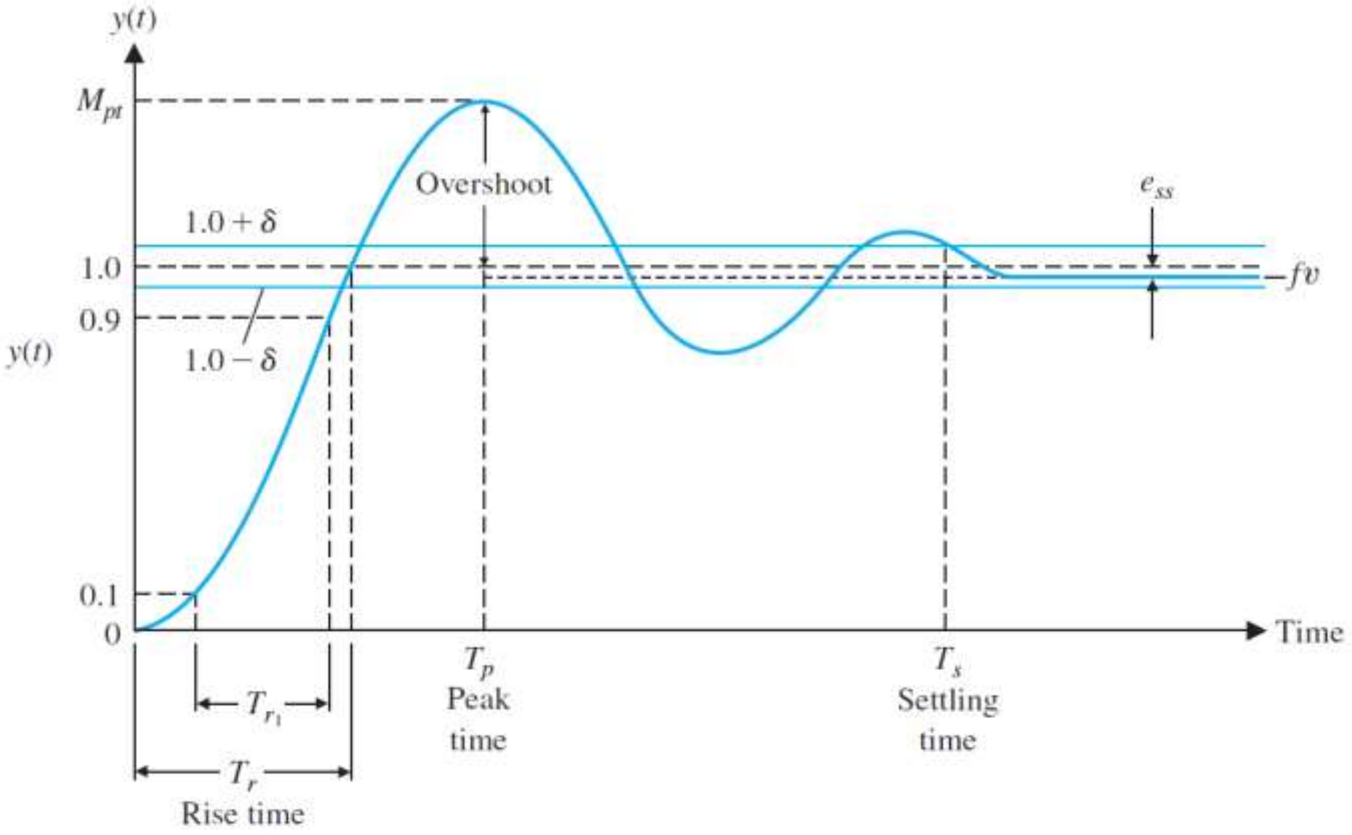


1. What is the relationship between the step and impulse responses?
2. In chapter 2, we examined the natural response of a 2nd system; here we examined the forced response of the system
3. If you are given a TF, then we can only focus on the forced response; the natural response is missing because the TF assumes the zero initial conditions

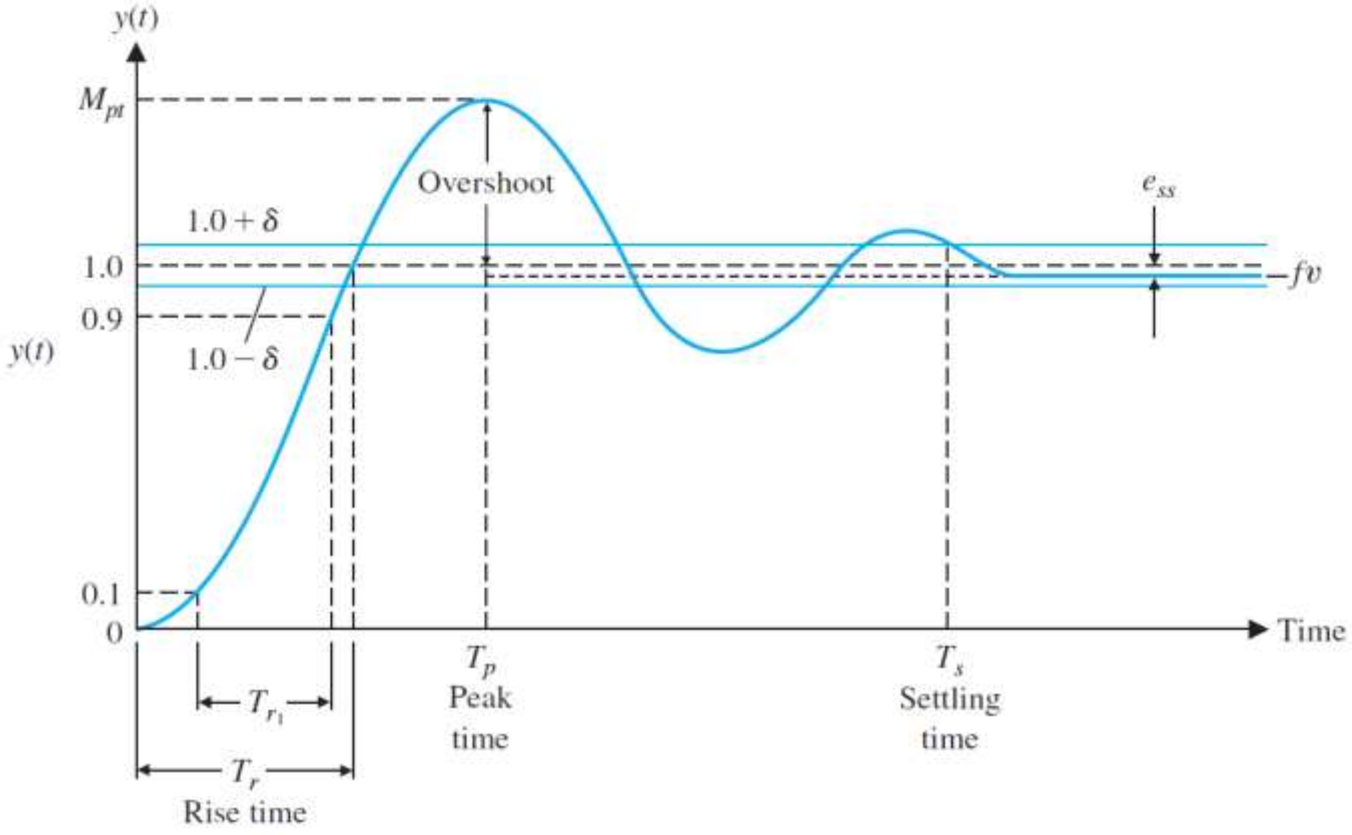
Standard Performance Measures

- Defined in terms of the step response of the closed-loop system

→ step input signal is the easiest to generate and evaluate



Transient Response



- Swiftness of the response

→rise time and peak time

(peak time always has

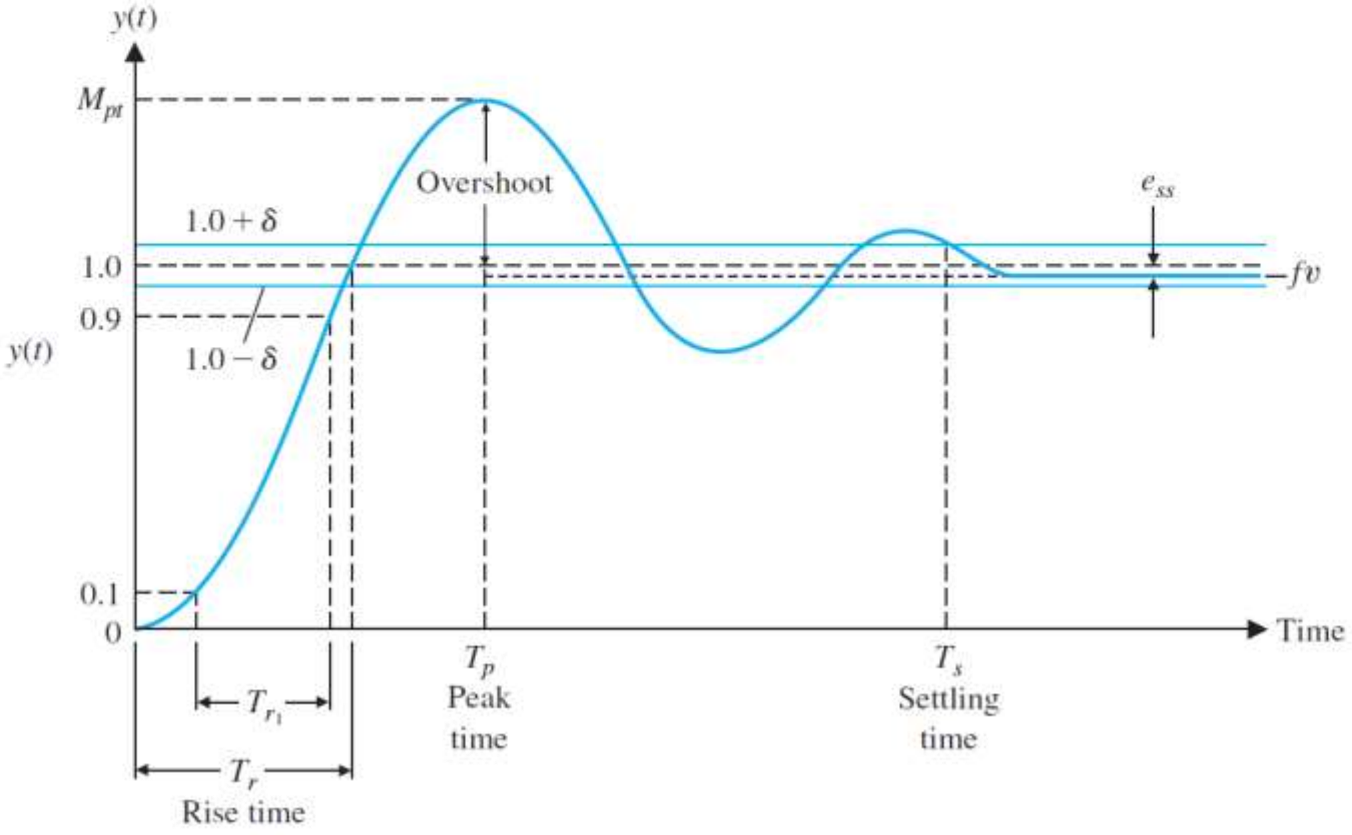
its own definition?)

- Closeness of the response

→Percent overshoot (P.O) and settling time

$$P.O. = \frac{M_{Pt} - fv}{fv} \times 100\%$$

why consider a percentage?



- Settling time

→the time required for the system to settle within a certain percentage of the input amplitude (band of $\pm\delta$)

Settling Time as Four Time Constants

- For the second-order system with closed-loop damping constant $\zeta\omega_n$

→Ts= the response remains within 2% of the final value

$$e^{-\zeta\omega_n T_s} < 0.02,$$

$$\zeta\omega_n T_s \cong 4.$$

$$T_s = 4\tau = \frac{4}{\zeta\omega_n}.$$ (time constant $\tau = 1/\zeta\omega_n$)

- General definition of the settling time

→ four time constants of the dominant roots of the characteristic equation

Summary of Performance Measures

- Transient response

→swiftness: rise time and peak time

→closeness: P.O. and settling time

- Steady-state response

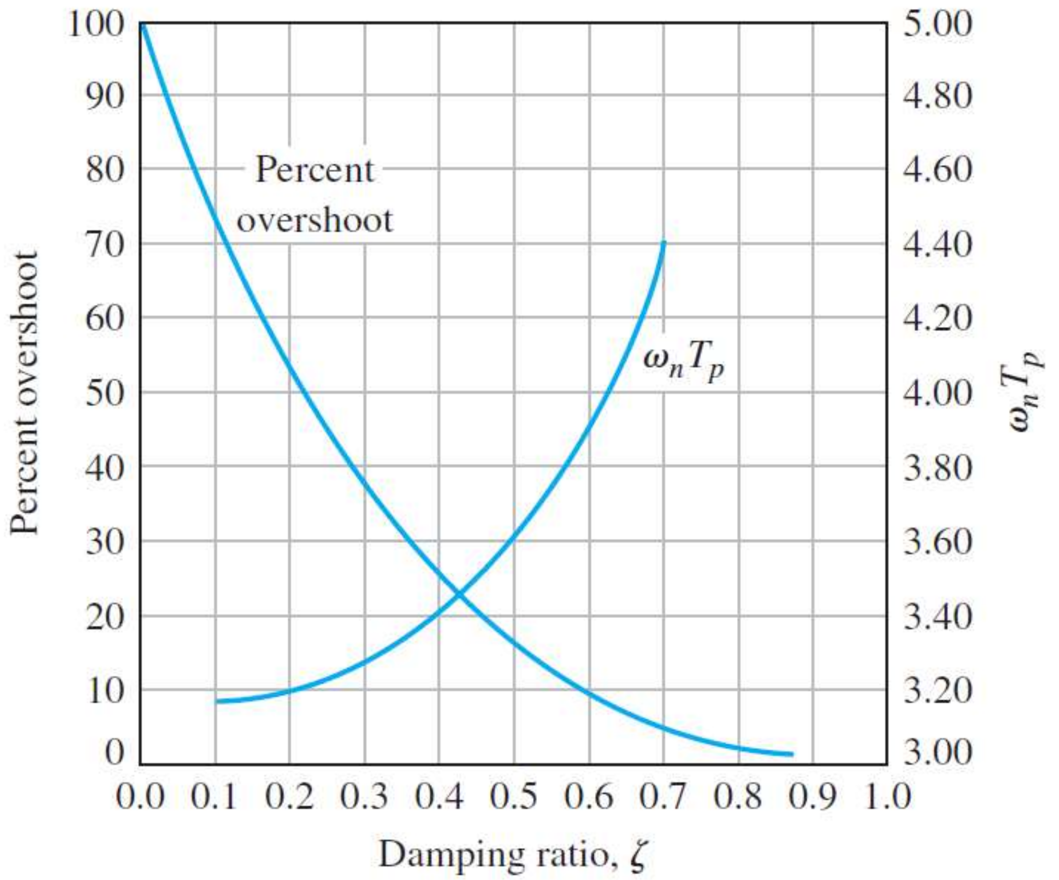
→Steady-state error

- For the closed-loop 2nd system

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}, \quad M_{pt} = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}.$$

$$P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}.$$

Compromise Between Swiftness and Closeness



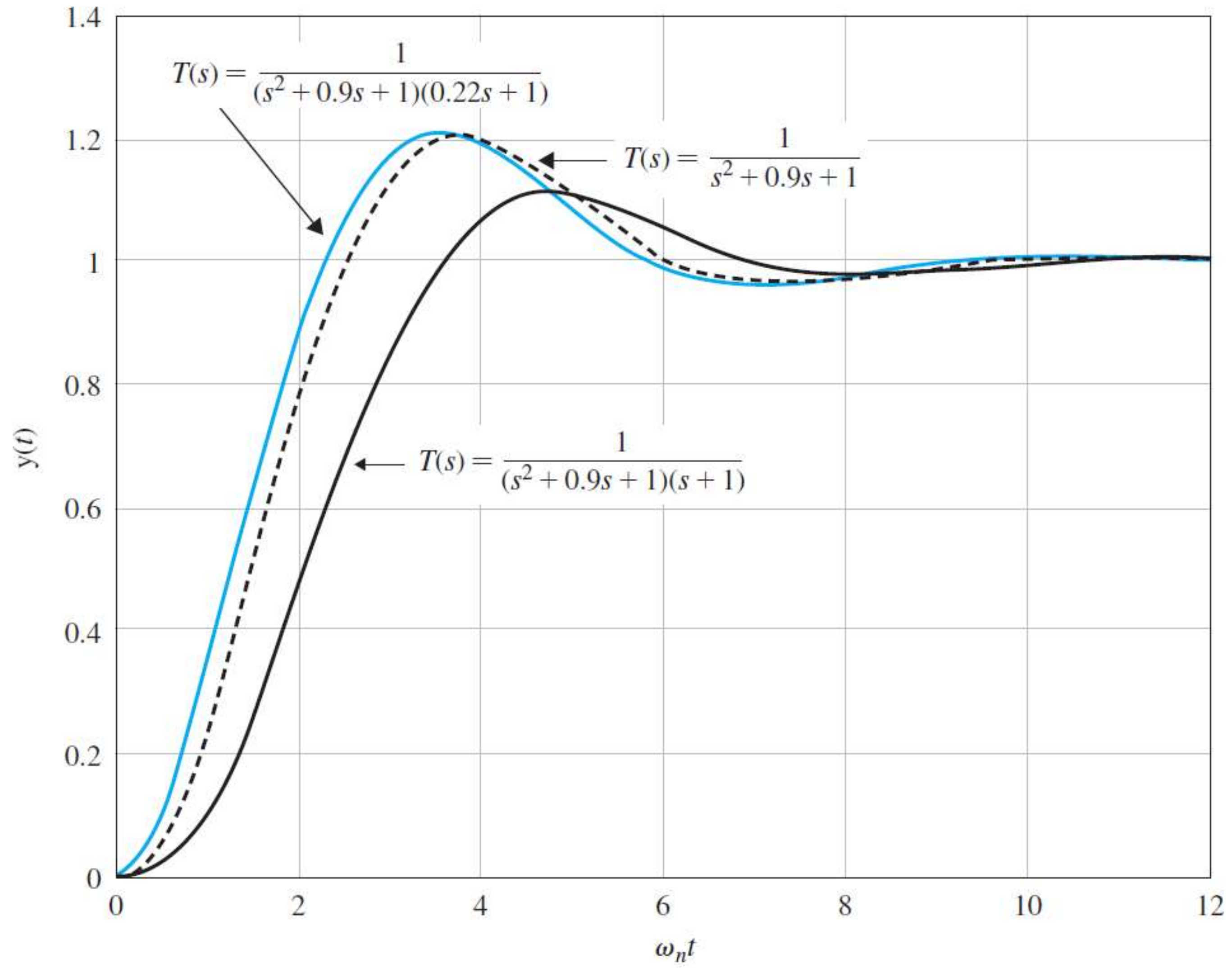
5.4 Effects of A Third Pole and a Zero

- Effect of a third pole

→a 3rd-order system can be approximated by the 2nd-order system when the real part of the dominant roots is less than one tenth of the real part of the third root

$$T(s) = \frac{1}{(s^2 + 2\zeta s + 1)(\gamma s + 1)},$$

$$|1/\gamma| \geq 10|\zeta \omega_n|.$$



Comparison of two third-order systems with a second-order system (dashed line) illustrating the concept of dominant poles

- Effect of a zero

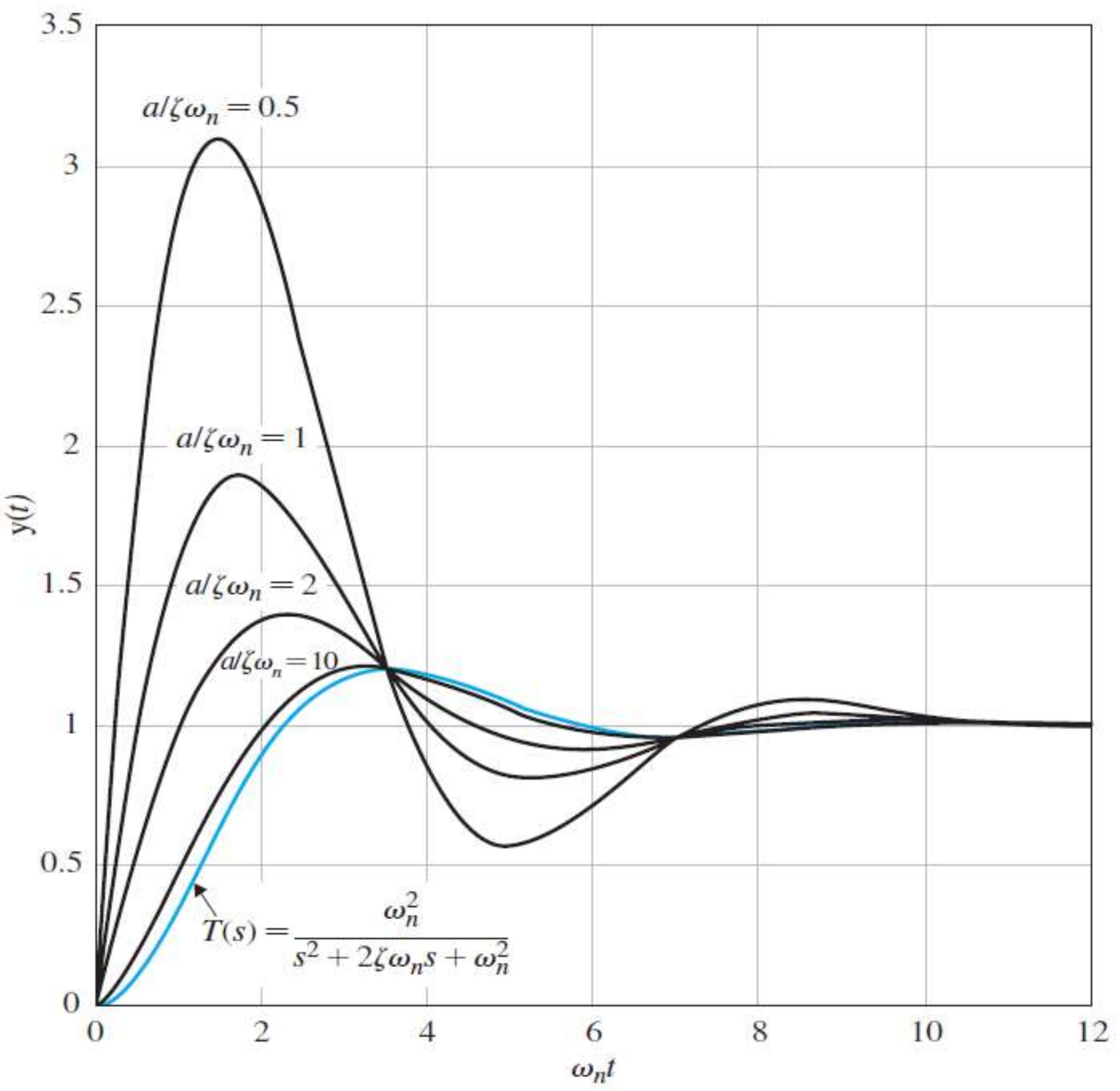
→affects the transient response of the system

- The zero is near the dominant complex poles

→pole-zero cancellation; much impact

- The zero is far from the dominant complex poles

→ little impact



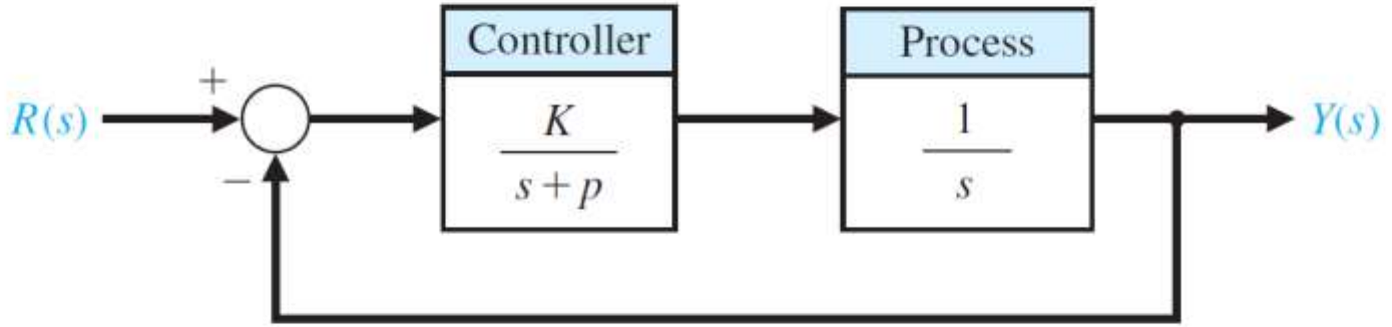
$$T(s) = \frac{(\omega_n^2/a)(s + a)}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Example 5.1 (First Design Example)

- Find K and p such that

  P.O.<= 5%

  Ts<= 4 seconds

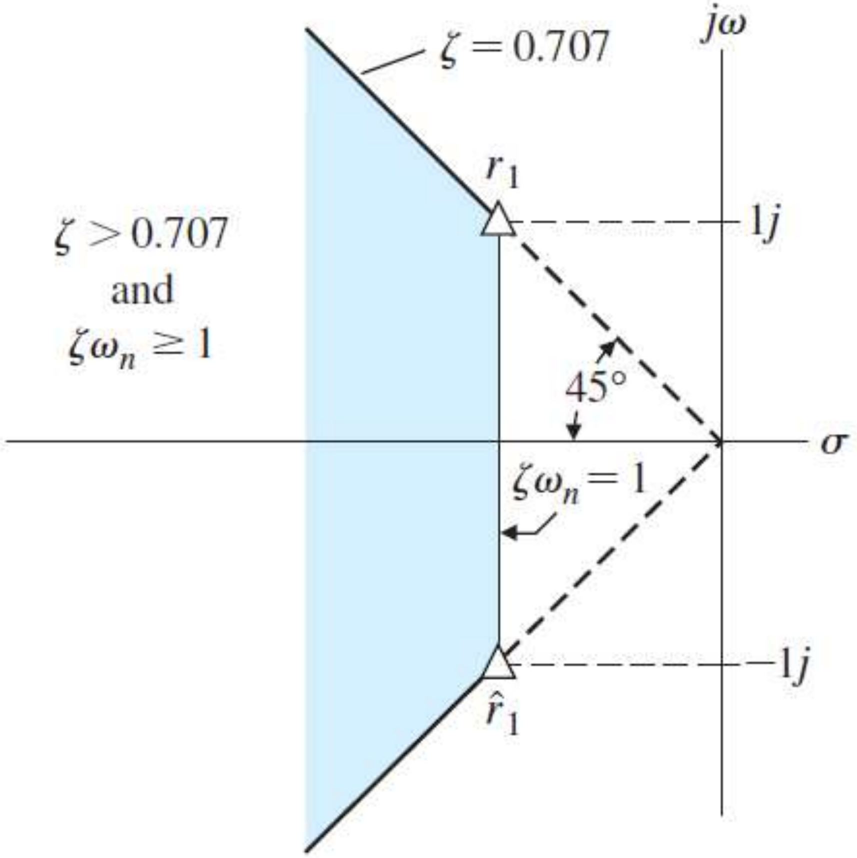


- P.O.<= 5% ➡ $P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$. <= 5%

➡ $\zeta = 0.707\ (P.O. = 4.3\%)$

- Ts<= 4 ➡ $\zeta\omega_n \geq 1$

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{K}{s^2 + ps + K} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$



$\zeta = 0.707$
$j\omega$
r_1
$1j$
$\zeta > 0.707$
and
$\zeta\omega_n \geq 1$
$45°$
σ
$\zeta\omega_n = 1$
$-1j$
$\hat{r}_1$

5.5 S-Plane Root Location and Transient Response

- Transient response of a closed-loop feedback control system

→Determined by the location of the poles and zeros of the transfer function

→Poles determine the response modes

→Zeros determine the weightings

- Step response of a system without repeated roots

$$Y(s) = \frac{1}{s} + \sum_{i=1}^{M} \frac{A_i}{s + \sigma_i} + \sum_{k=1}^{N} \frac{B_k s + C_k}{s^2 + 2\alpha_k s + (\alpha_k^2 + \omega_k^2)},$$

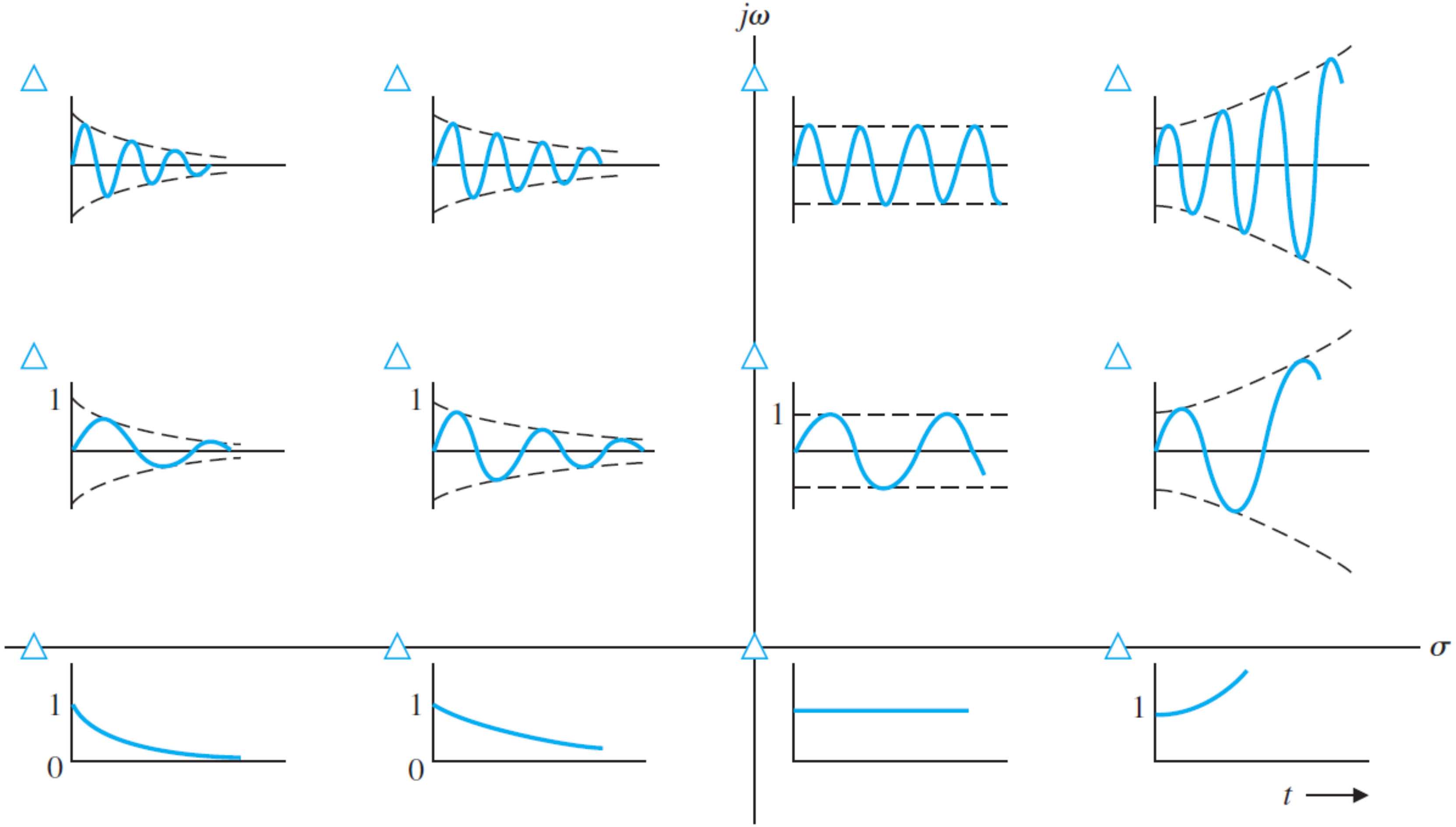
$s = -\sigma_i$

$s = -\alpha_k \pm j\omega_k$

$$y(t) = 1 + \sum_{i=1}^{M} A_i e^{-\sigma_i t} + \sum_{k=1}^{N} D_k e^{-\alpha_k t} \sin(\omega_k t + \theta_k)$$

steady-state output exponential terms damped sinusoidal terms

- Impulse response of a stable system



Important Relationship between Root Location and Transient Response

- Control system designer will envision the effects on the step and impulse responses of adding, deleting, or moving poles and zeros of the TF
- Poles determine the particular response modes
- Zeros establish the relative weightings of the individual mode functions

weightings

$$y(t) = 1 + \sum_{i=1}^{M} A_i e^{-\sigma_i t} + \sum_{k=1}^{N} D_k e^{-\alpha_k t} \sin(\omega_k t + \theta_k)$$

Response modes

5.6 Steady-State Error of Feedback Control Systems

- Feedback

→attendant improvement in the reduction of the steady-state error

- Steady-state error of a stable closed-loop system is usually several orders of magnitude smaller than the error of an open-loop system

→useful to determine the steady-state error of the closed-loop system using unity feedback for the three standard test inputs

- Consider a unity negative feedback system:

$$E(s) = \frac{1}{1 + G_c(s)G(s)} R(s)$$

Steady-State Error of Step Input

$$E(s) = \frac{1}{1 + G_c(s)G(s)} R(s)$$

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} s \frac{1}{1 + G_c(s)G(s)} R(s)$$

$$e_{ss} = \lim_{s\to 0} \frac{s(A/s)}{1 + G_c(s)G(s)} = \frac{A}{1 + \lim_{s\to 0} G_c(s)G(s)}$$

$$G_c(s)G(s) = \frac{K\prod_{i=1}^{M}(s + z_i)}{s^N \prod_{k=1}^{Q}(s + p_k)}$$

Loop transfer function $G_c(s)G(s)$ determines the steady-state error

Steady-State Error of Step Input (N=0)

- System type is defined as the number of integrations in the loop transfer function, i.e., N

$$G_c(s)G(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^N\prod_{k=1}^{Q}(s+p_k)}$$

- Type-zero system (N=0)

$$e_{ss} = \frac{A}{1+G_c(0)G(0)} = \frac{A}{1+K\prod_{i=1}^{M} z_i \Big/ \prod_{k=1}^{Q} p_k}$$

$$e_{ss} = \frac{A}{1+K_p}$$

position error constant

$$K_p = \lim_{s\to 0} G_c(s)G(s)$$

Steady-State Error of Step Input (N>0)

$$e_{ss} = \lim_{s \to 0} \frac{A}{1 + K \prod z_i / (s^N \prod p_k)} = \lim_{s \to 0} \frac{A s^N}{s^N + K \prod z_i / \prod p_k} = 0$$

- For the prototype 2nd-order system, the steady-state error of a step input is zero (why?)

Steady-State Error of Ramp Input

$$E(s) = \frac{1}{1 + G_c(s)G(s)} R(s)$$

$$\lim_{t \to \infty} e(t) = e_{ss} = \lim_{s \to 0} s \frac{1}{1 + G_c(s)G(s)} R(s)$$

$$e_{ss} = \lim_{s \to 0} \frac{s(A/s^2)}{1 + G_c(s)G(s)} = \lim_{s \to 0} \frac{A}{s + sG_c(s)G(s)} = \lim_{s \to 0} \frac{A}{sG_c(s)G(s)}$$

Define velocity constant $K_v = \lim_{s \to 0} sG_c(s)G(s)$

- For a type-1 system (N=1)

$$e_{ss} = \lim_{s \to 0} \frac{A}{sK\prod(s + z_i)/[s\prod(s + p_k)]} \Rightarrow e_{ss} = \frac{A}{K\prod z_i/\prod p_k} = \frac{A}{K_v}$$

- For a type-0 system (N=0)→steady-state error= infinite
- For a type-N system with N>1→ steady-state error= 0

Steady-State Error of Acceleration Input

$$E(s) = \frac{1}{1 + G_c(s)G(s)} R(s)$$

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} s \frac{1}{1 + G_c(s)G(s)} R(s)$$

$$e_{ss} = \lim_{s\to 0} \frac{s(A/s^3)}{1 + G_c(s)G(s)} = \lim_{s\to 0} \frac{A}{s^2 G_c(s)G(s)}$$

Define acceleration constant $K_a = \lim_{s\to 0} s^2 G_c(s)G(s)$

- For a type-2 system (N=2)

$$e_{ss} = \frac{A}{K\prod z_i/\prod p_k} = \frac{A}{K_a}$$

- For a type-N system with N<2 →steady-state error= infinite
- For a type-N system with N>2→ steady-state error= 0

Steady-State Error and System Type

Valid only for stable closed-loop systems

Table 5.2 Summary of Steady-State Errors

Number of Integrations in $G_c(s)G(s)$, Type Number	Input		
	Step, $r(t) = A$, $R(s) = A/s$	Ramp, $r(t) = At$, $R(s) = A/s^2$	Parabola, $r(t) = At^2/2$, $R(s) = A/s^3$
0	$e_{ss} = \dfrac{A}{1 + K_p}$	∞	∞
1	$e_{ss} = 0$	$\dfrac{A}{K_v}$	∞
2	$e_{ss} = 0$	0	$\dfrac{A}{K_a}$

Concept of Error Constants

- Error constants, Kp, Kv, and Ka, describe the ability of a system to reduce or eliminate the steady-state error.
- Error constants are utilized as numerical measures of the steady-state performance.
- The designer determines the error constants for a given system and attempts to determine methods of increasing the error constants while maintaining an acceptable transient response.
- However, an increase in error constants may result in an attendant decrease in the system damping ratio, thereby leading to a more oscillatory response to a step input.

→ seek a compromise that provides the largest error constant based on the smallest allowable damping ratio

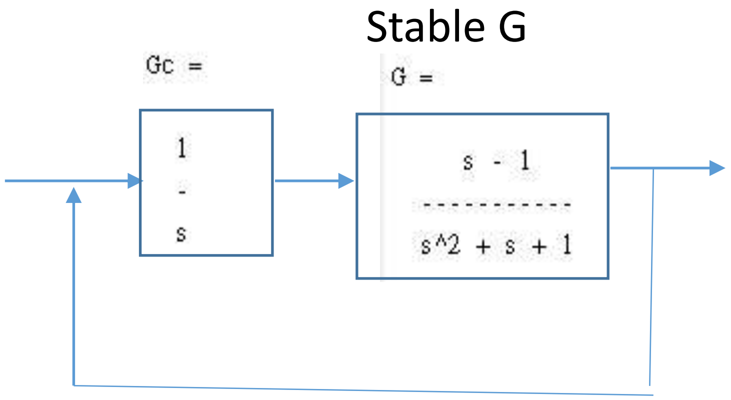
Caution!

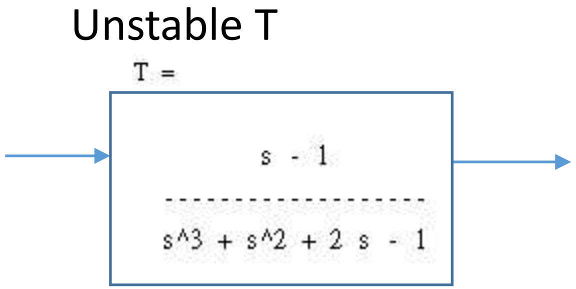
- It seems that we can always increase the system type by using Gc=1/s to reduce the steady-state error

→It is true if the resulting closed-loop system is still stable

→Sometimes increasing the system type makes the system unstable

Example:





Nonunity Negative Feedback

- We have addressed the steady-state error for unity feedback (error constant analysis). How about nonunity feedback?

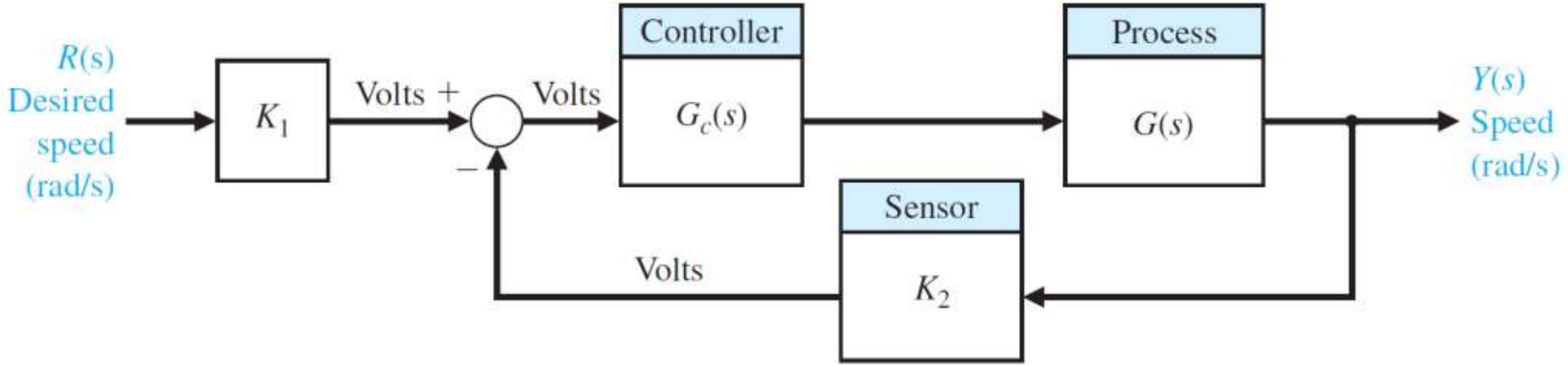
→Approach 1: transform a nonunity feedback into a unity feedback and use the error constant analysis

→Approach 2: derive the steady-state error directly without using the error constant analysis

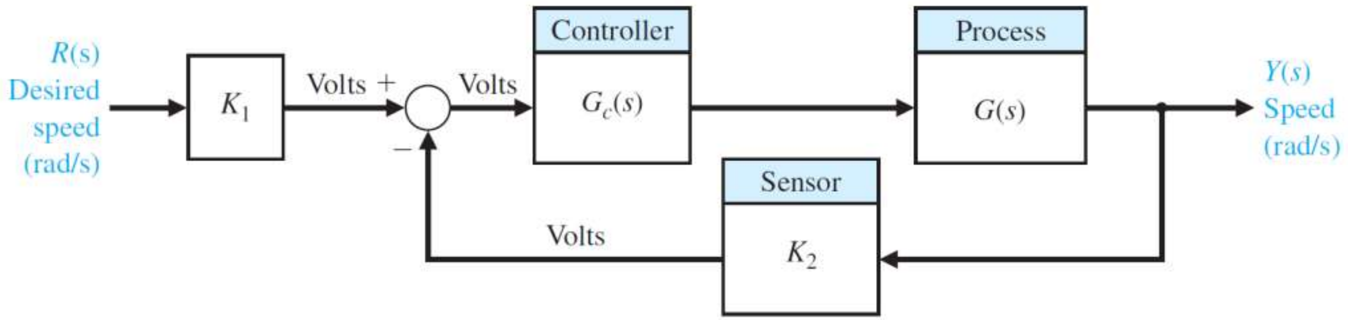
Example 1: Nonunity Feedback (using approach 1)

- For a system with a nonunity feedback, the units of the output are usually different from the output of the sensor.

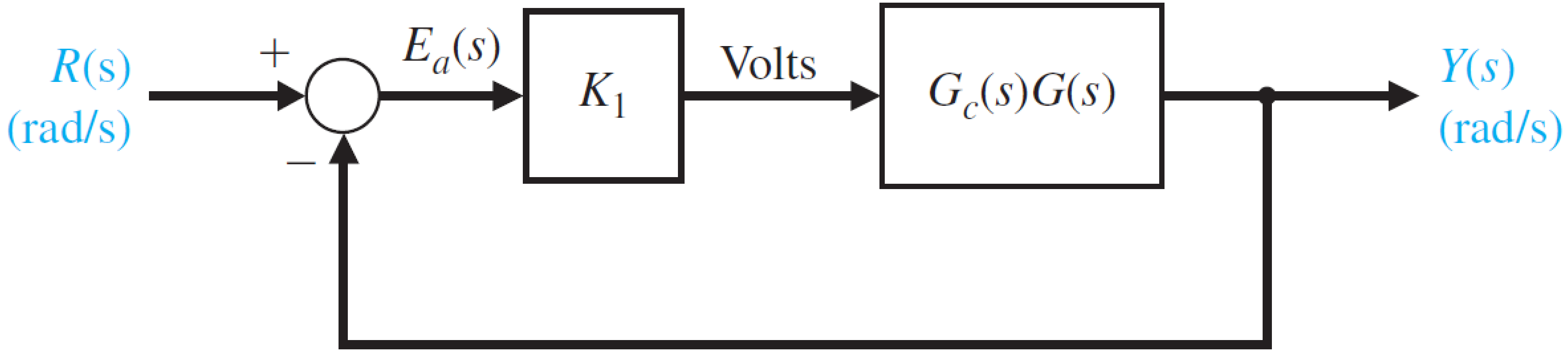
→ Constants K_1 and K_2 account for the conversion of one set of units to another set of units (from rad/s to volts)



Example 1: Nonunity Feedback (using approach 1)



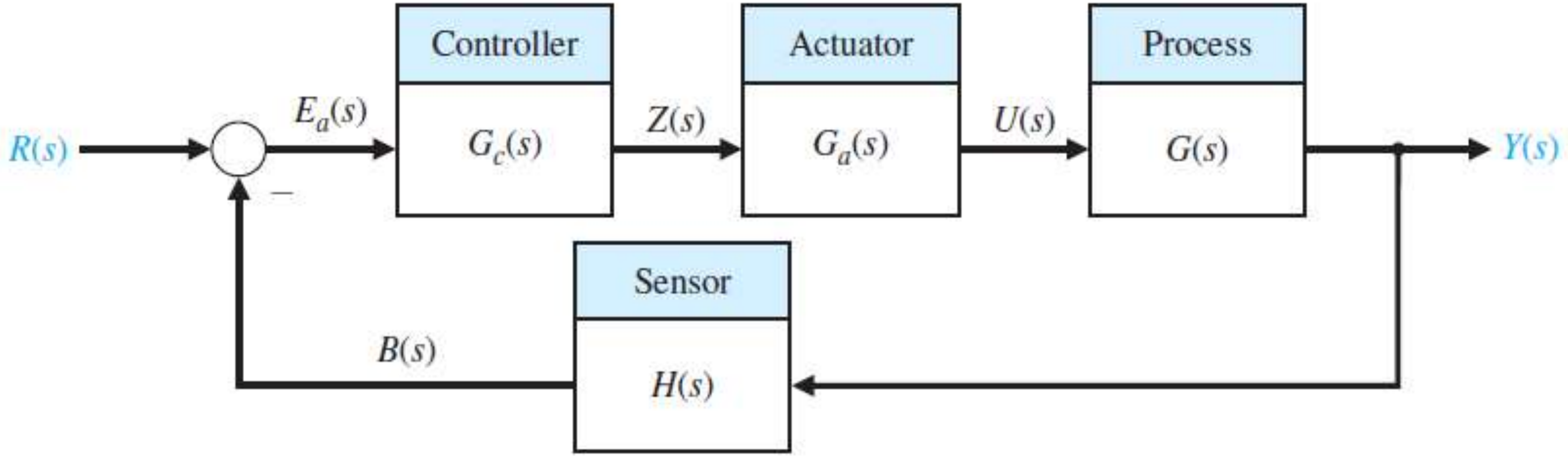
- Select K_1 such that $K_1 = K_2$
- Move the block for K_1 and K_2 past the summing node.
- Obtain the equivalent block diagram for a unity feedback system



Example 2: Nonunity Feedback (using approach 2)

$$H(s) = \frac{K_2}{\tau s + 1}$$

$$\lim_{s \to 0} H(s) = K_2$$



$$Y(s) = T(s)R(s)$$

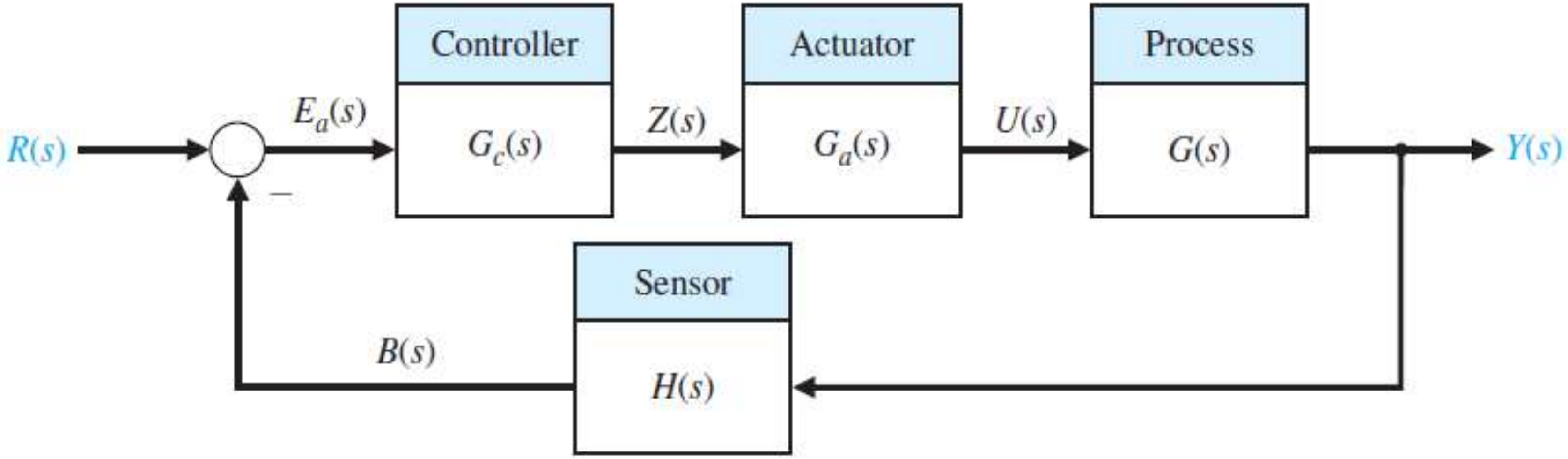
$$E(s) = R(s) - Y(s) = [1 - T(s)]R(s)$$

$$T(s) = \frac{K_1 G_c(s)G(s)}{1 + H(s)G_c(s)G(s)} = \frac{(\tau s + 1)K_1 G_c(s)G(s)}{\tau s + 1 + K_1 G_c(s)G(s)};$$

$$E(s) = \frac{1 + \tau s(1 - K_1 G_c(s)G(s))}{\tau s + 1 + K_1 G_c(s)G(s)} R(s)$$

$$e_{ss} = \lim_{s \to 0} s\,E(s) = \frac{1}{1 + K_1 \lim_{s \to 0} G_c(s)G(s)}$$

Example 3: Nonunity Feedback (using approach 1)



$$\frac{Y(s)}{R(s)} = T(s) = \frac{G_c(s)G(s)}{1 + H(s)G_c(s)G(s)}$$

$$\frac{Y(s)}{R(s)} = T(s) = \frac{Z(s)}{1 + Z(s)} \quad \text{where} \quad Z(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)(H(s) - 1)}$$

Z(s) is the equivalent loop transfer function of a unity feedback control system

Example 3: Nonunity Feedback (using approach 1)

- Error constants can be defined in terms of Z(s)

$$K_p = \lim_{s \to 0} Z(s),\ K_v = \lim_{s \to 0} sZ(s),\ \text{and}\ K_a = \lim_{s \to 0} s^2 Z(s)$$

5.7 Performance Indices

- A performance index

→to be calculated or measured and used to evaluate the system performance

- Optimum control system

→the system parameters are adjusted so that the performance index reaches an extremum, commonly a minimum value

- Integral of squared error $\mathrm{ISE} = \int_0^T e^2(t)\, dt.$
- Other performance indices:

$$\mathrm{IAE} = \int_0^T |e(t)|\, dt,$$

$$\mathrm{ITAE} = \int_0^T t|e(t)|\, dt$$

$$\mathrm{ITSE} = \int_0^T t e^2(t) dt.$$

The general form of the performance integral is

$$I = \int_0^T f(e(t), r(t), y(t), t)\, dt,$$

Summary of Chapter 5

- Control system performance is measured in terms of the transient and steady-state response
- Transient response

→Determined by poles and zeros (poles dictate the response modes and zeros dictate the weightings)

→Performance measures for swiftness: rise time and peak time

→Performance measures for closeness: P.O. and settling time

→Design method: location of poles and zeros in the s-plane (tentative)

- Steady-state response (of a stable system)

→Determined by the input and system type

→ Performance measure: steady-state error

→Design method: error constant analysis