Control Systems HW5

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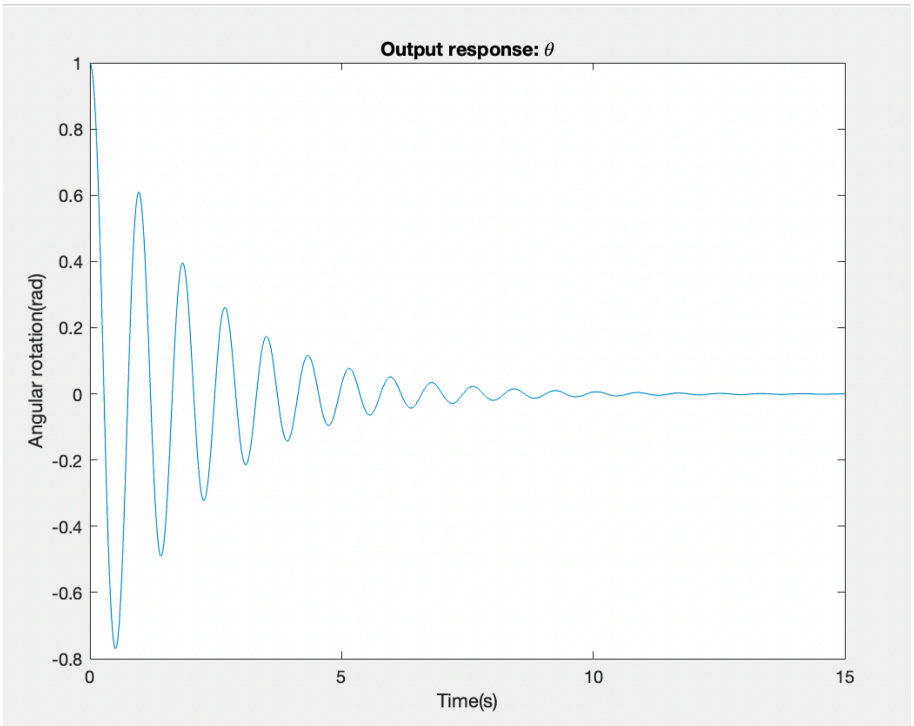
1. Simulations by script (use the function of ode45)

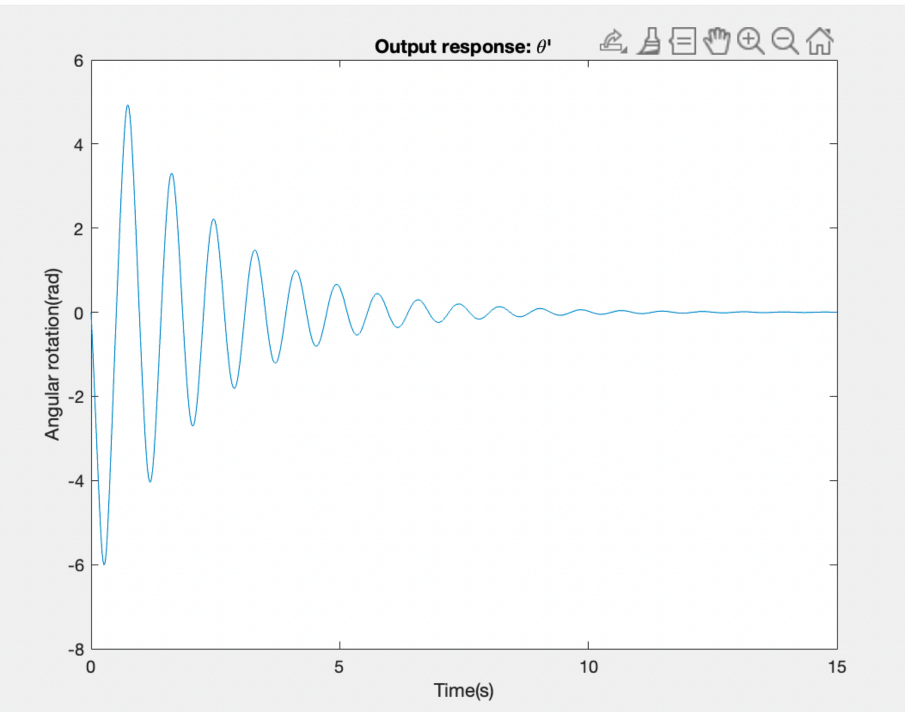
Assume that $M = 2.5\ kg$, $m = 10\ kg$, $l = 1\ m$ and $g = 9.81\ m/s^2$

$K_P = -7000,\ K_I = -100,\ K_D = -100$

$\theta(t)$ v.s t :



$\dot{\theta}(t)$ v.s t :



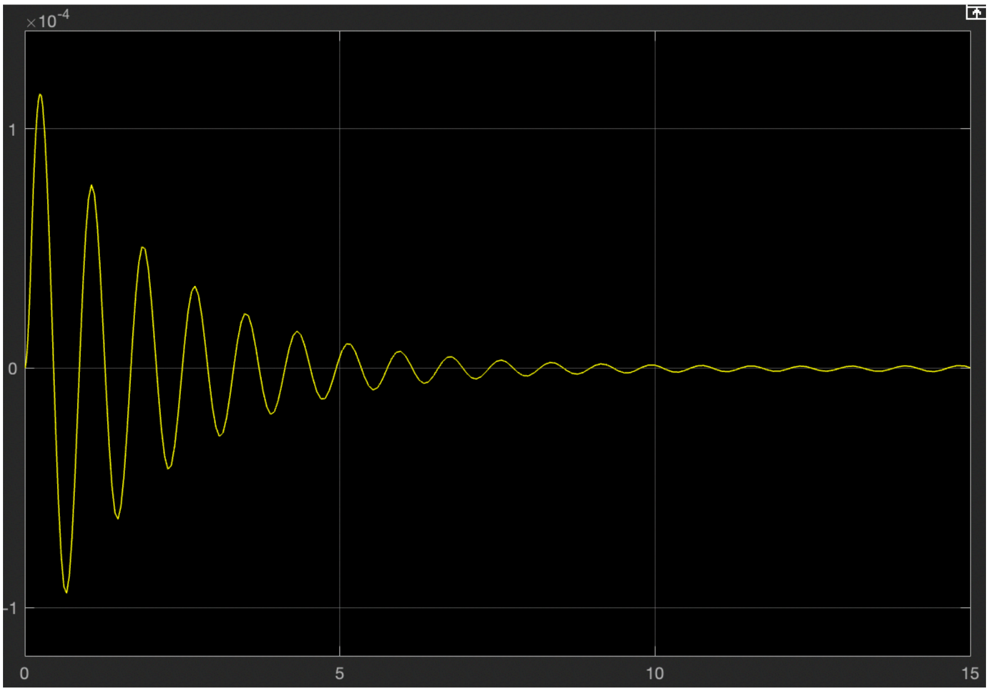
The above is the stable system since it satisfies that $\theta(t) = 0$, $\dot{\theta}(t) = 0$, and we can also see that it approximately takes 9s to 10s to reach its steady state.

2. Simulation by Simulink

Assume that $M = 2.5\ kg$, $m = 10\ kg$, $l = 1\ m$ and $g = 9.81\ m/s^2$ (The same as part 1)

$K_P = -7000,\ K_I = -100,\ K_D = -100$

$\theta(t)$ v.s t :



The above system approximately takes 9s to 10s to reach its steady state.

3. Analysis

Problem 1

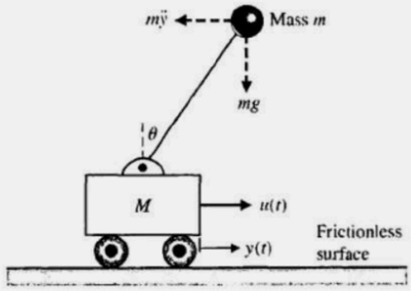


Figure 1 A cart and an inverted pendulum.

Force (horizontal):

$$(\mathrm{M}+\mathrm{m})\ddot{y}(t) = u(t) - \left(mL\ddot{\theta}(t)\cos\theta(t) - mL\dot{\theta}(t)^2\sin\theta(t)\right)$$

Torque:

$$\mathrm{mL}^2\ddot{\theta}(t) = mgL\sin\theta(t) - mL\ddot{y}(t)\cos\theta(t)$$

Assume that M=100Kg, m=10Kg, L=1m, g=9.81m/s^2 and $\mathrm{G}_c(s) = K_P + \frac{K_I}{s} + sK_D$,

From the above, we obtain that

$$\ddot{y}(t) = \frac{1}{(M+m)}\left(u(t) - \left(mL\ddot{\theta}(t)\cos\theta(t) - mL\dot{\theta}^2(t)\sin\theta(t)\right)\right)$$

$$\ddot{\theta}(t) = \frac{1}{ML^2}\left(mgL\sin\theta(t) - mL\ddot{y}(t)\cos\theta(t)\right)$$

And setting that $x_1 = y,\ x_2 = \dot{y},\ x_3 = \int \theta dt,\ x_4 = \theta,\ x_5 = \dot{\theta}$

So we can get the following relation:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{u(t)}{(M+m)} - \dfrac{mL\dot{x}_5\cos x_4 - mLx_5^2\sin x_4}{(M+m)} \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = x_5 \\ \dot{x}_5 = \dfrac{g}{L}\sin x_4 - \dfrac{\dot{x}_2}{L}\cos x_4 \end{cases}$$

and

$$U(s) = G_c(s)(R(s) - \theta(s)) \quad -(PID\ control)$$

$$\Rightarrow u(t) = -K_P\theta(t) - K_I\int\theta(t)dt - K_D\frac{d\theta(t)}{dt}$$

$$= -K_Px_4 - K_Ix_3 - K_Dx_5$$

Next, we have to rearrange the equations of $\dot{x}_2$ and $\dot{x}_5$ since the equations are mixed with the derivative part of x_2 and x_5. After arrangement, we obtain that

$$\dot{x}_2 = \frac{1}{M + m(1 - \cos^2 x_4)}\left(u(t) - mg\cos x_4 \sin x_4 + mLx_5^2 \sin x_4\right)$$

$$\dot{x}_5 = \frac{1}{M + m(1 - \cos^2 x_4)}\left(\frac{(M+m)g \sin x_4}{L} - \frac{u(t)\cos x_4}{L} - m x_5^2 \cos x_4 \sin x_4\right)$$

By plotting the state-space parameters ($\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$, $\dot{x}_4$, $\dot{x}_5$) obtained above using MATLAB function ode45, we can get the output response of $\theta(t)$ and $\dot{\theta}(t)$. To stabilize the system, we have to check if poles of the system transfer function are all in the left-half plain, and use PID controller to adjust parameters (K_P , K_I , K_D) in an efficient way. However, the equations are too complex to solve in this part, so the characteristic function has to be found, which is based on some assumptions. Next, Problem 2 will mention it in detail.

Problem 2

2) Use Simulink to simulate the output response in s-domain.

Assume that $\theta(t) \to 0^o \Longrightarrow \sin\theta(t) \approx \theta(t), \cos\theta(t) \approx 1, \dot{\theta}(t) \approx 0,$

$$(\mathrm{M} + \mathrm{m})\ddot{y}(t) + mL\ddot{\theta}(t) = u(t)$$

$$\ddot{y}(t) + \mathrm{L}\ddot{\theta}(t) - g\theta(t) = 0$$

$$\mathrm{G(s)} = \frac{-1/_{ML}}{s^2 - \frac{(M+m)g}{ML}}$$

Applying PID control to the inverted pendulum.

Assume that $T_d(s) = 1(T_d(t) = \delta(t))$

By the above assumptions, we get G(s). The following uses MATLAB to find the characteristic equation:

```
%% Characteristic equation
clc; clear all; close all;

syms s KP KI KD
m=10;
M=100;
L=1;
g=9.81;

G = (-1/(M*L))/(s^2-(M+m)*g/(M*L));
Gc = (KD*s^2+KP*s+KI)/s;
(1+G*Gc)
```

```
ans =

1 - (KD*s^2 + KP*s + KI)/(100*s*(s^2 - 10791/1000))
```

So the characteristic equation is

$$100s^3 - K_D s^2 - (K_P + 1079.1)s - K_I = 0$$

By using Routh-Hurwitz stability criterion,

* Routh array

s^3	100	$-1079.1-K_P$
s^2	$-K_D$	$-K_I$
s^1	a_1	0
s^0	$-K_I$	0

$$a_1 = \frac{100K_I + K_D(1079.1+K_P)}{-K_D}$$

For a stable system, it must be satisfied that

$$-K_D > 0 \ , -K_I > 0 \ , a_1 = \frac{100K_I + K_D(1079.1 + K_P)}{-K_D} > 0$$

$$\Rightarrow K_D < 0 \ , K_I < 0 \ , K_P < -1079.1 - \frac{100K_I}{K_D}$$

We choose $K_I = K_D = -100 \ , K_P = -7000$ to get a stable system.

It is obvious to see that the simulations by ode45 in part 1 and by Simulink in part 2 basically come to a stable system and almost have the same results. This homework gives the basic idea about how to use PID controller to get a stable system, and how to tune the parameters in a systematic way.