Notes on non-linear spinwave theory using two expansions for the Holstein-Primakoff square root

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(Dated: February 12, 2020)

I. GENERAL SPIN-WAVE THEORY: MAGNON HAMILTONIANS TO SIXTH ORDER

Any bilinear spin Hamiltonian can be written

$$H = \sum_{i,j} S_i^a \Lambda_{ij}^{ab} S_j^b, \tag{1}$$

where Λ_{ij} is the spin-spin interaction matrix, and $a, b = x, y, z$ are Cartesian indices. In the following we will assume that $\Lambda_{ii}^{ab} = 0$ for all a, b. For noncollinear magnetic configurations we will want to apply an SO(3) spin rotation from the global coordinate system (x, y, z) to a local coordinate system $(\tilde{x}_i, \tilde{y}_i, \tilde{z}_i)$ for each site i, such that $\hat{z}_i$ points along the direction of the magnetic moment $\langle \mathbf{S}_i \rangle$. By introducing the spin rotation $S_i^a = R_i^{ab} \tilde{S}_i^b$ we rewrite H as

$$H = \sum_{i,j} \tilde{S}_i^a \left[R_i^T \Lambda_{ij} R \right]^{ab} \tilde{S}_j^b \equiv \sum_{i,j} \tilde{S}_i^a \tilde{\Lambda}_{ij}^{ab} \tilde{S}_j^b, \tag{2}$$

where all information about interactions and spin states are now contained in $\tilde{\Lambda}_{ij}$. Next we introduce the Holstein-Primakoff magnon representation[1,2],

$$\tilde{S}_i^+ = \sqrt{2S} \sqrt{1 - \frac{a_i^\dagger a_i}{2S}} a_i, \tag{3}$$

$$\tilde{S}_i^- = \sqrt{2S} a_i^\dagger \sqrt{1 - \frac{a_i^\dagger a_i}{2S}}, \tag{4}$$

$$\tilde{S}_i^z = S - a_i^\dagger a_i, \tag{5}$$

where S is the size of the magnetic moment, and deviations from the classic ground state are described by bosonic creation and annihilation operators satisfying $[a_i, a_j^\dagger] = \delta_{i,j}$. The physical Hilbert space is defined by $a_i^\dagger a_i < 2S$. Thus spin-wave theory is a low-temperature theory, particularly in the case $S = 1/2$.

A. Taylor expansion approach

In the standard Taylor expansion approach, one uses

$$\sqrt{1 - \frac{a^\dagger a}{2S}} \approx 1 - \frac{a^\dagger a}{4S} \tag{6}$$

to write (to first order)

$$\tilde{S}^+ \approx \sqrt{2S} \left(a - \frac{a^\dagger a^2}{4S} \right), \quad \tilde{S}^- \approx \sqrt{2S} \left(a^\dagger - \frac{\left(a^\dagger\right)^2 a}{4S} \right). \tag{7}$$

This is usually considered an expansion in $1/S$, or sometimes in $1/(zS)$, where z is the coordination number[3]. However, as Roderich Moessner pointed out to me, the validity of spin-wave theory is controlled by the size of the effective magnetic field at a given spin site, not by the size of zS. To see this, it's enough to note that both the square lattice and the pyrochlore lattice have $z = 4$, but one is considerably more frustrated than the other.

By calculating all bilinears $\tilde{S}_i^a \tilde{S}_j^b$ and grouping terms based on the number of bosonic operators, one finds that the resulting Hamiltonian can be written

$$H = H^{(0)} + H^{(1)} + H^{(2)} + H^{(3)} + H^{(4)} + H^{(5)} + H^{(6)} + \ldots \tag{8}$$

where $\ldots$ signify the higher-order terms we have neglected.

B. Dyson-Maleev representation

The Dyson-Maleev representation[4–6] can be obtained by applying the similarity transform[78]

$$a_i \to \sqrt{1 - \frac{a_i^\dagger a_i}{2S}} a_i, \tag{9}$$

$$a_i^\dagger \to a_i^\dagger \frac{1}{\sqrt{1 - \frac{a_i^\dagger a_i}{2S}}}, \tag{10}$$

which yields

$$\tilde{S}_i^+ = \sqrt{2S} \left(1 - \frac{a_i^\dagger a_i}{2S} \right) a_i, \quad \tilde{S}_i^- = \sqrt{2S} a_i^\dagger. \tag{11}$$

This procedure allows us to sidestep the issue of dealing with the square root. The transformation is not unitary, but preserves the number operator $a_i^\dagger a_i$ and the commutation relations within the physically relevant Hilbert space $n \leq 2S$. Note also that the Dyson-Maleev representation results in a Hamiltonian in the form of Eq. (8) that is exact to sixth order in the boson operators. The price we pay is to introduce terms that generically make the Hamiltonian non-Hermitian.

C. New expansion

In our proposed alternate expansion we have

$$\tilde{S}^+ \approx \sqrt{2S} \left[a + \left(\sqrt{1 - \frac{1}{2S}} - 1 \right) a^\dagger a^2 \right], \tag{12}$$

$$\tilde{S}^- \approx \sqrt{2S} \left[a^\dagger + \left(\sqrt{1 - \frac{1}{2S}} - 1 \right) \left(a^\dagger\right)^2 a \right], \tag{13}$$

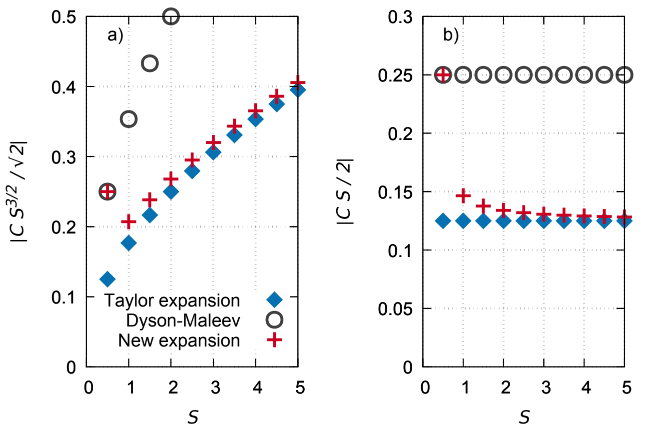

FIG. 1. The prefactors of the modified terms a) $H_2^{(3)}$ and b) $H_2^{(4)}$. The difference between the new and Taylor expansions is significant for $S = 1/2$, but rapidly disappears for large S. The generically larger prefactors for the Dyson-Maleev representation stem from a lower number of Hamiltonian terms.

which results in modified coefficients for some interaction terms compared to the Taylor expansion approach, as shown in Fig. 1. For large S we recover the prefactors found through the Taylor expansion. For non-large S this implies that Eq. (8) is no longer a series in powers of S. Thus this expansion should be considered only as an asymptotic expansion in powers of the bosonic operators. We also note that, for the particularly interesting case of $S = 1/2$, the Hamiltonian (8) is exact to sixth order in the boson operators, like in the Dyson-Maleev case. However, the new expansion preserves hermiticity.

D. General expressions for magnon Hamiltonians

We first recognize that, in all three approaches the spin operators may be written

$$\tilde{S}^+ \simeq \sqrt{2S}\left(a + C_1 a^\dagger a^2\right), \tag{14}$$

$$\tilde{S}^- \simeq \sqrt{2S}\left(a + C_2 \left(a^\dagger\right)^2 a\right), \tag{15}$$

$$\tilde{S}^z = S - a_i^\dagger a_i, \tag{16}$$

where $C_1 = C_2 \equiv C$ for the new and Taylor expansions, where C depends on the expansion used and the value of S. Similarly, $C_1 \equiv C$, $C_2 = 0$ for the Dyson-Maleev representation. We have

$$C = \begin{cases} -\frac{1}{4S} & \text{for Taylor exp. to order } 1/S \\ -\frac{1}{2S} & \text{for Dyson-Maleev rep.} \\ \sqrt{1-\frac{1}{2S}} - 1 & \text{for the new expansion} \end{cases} \tag{17}$$

One then readily finds that all three approaches to the square root yield the same answers on the level of linear (i.e. non-interacting) spin-wave theory. Explicitly, we have

$$H^{(0)} = S^2 \sum_{i,j} \tilde{\Lambda}_{ij}^{zz}, \tag{18}$$

$$H^{(1)} = \frac{S^{3/2}}{\sqrt{2}} \sum_{i,j} \left[a_i\left(\tilde{\Lambda}_{ij}^{xz} - i\tilde{\Lambda}_{ij}^{yz}\right) + a_j\left(\tilde{\Lambda}_{ij}^{zx} - i\tilde{\Lambda}_{ij}^{zy}\right) + \text{H.c.}\right], \tag{19}$$

$$\begin{aligned} H^{(2)} = \frac{S}{2} \sum_{i,j} \Big[& a_i a_j \left(\tilde{\Lambda}_{ij}^{xx} - \tilde{\Lambda}_{ij}^{yy} - i\tilde{\Lambda}_{ij}^{xy} - i\tilde{\Lambda}_{ij}^{yx}\right) \\ & + a_i^\dagger a_j \left(\tilde{\Lambda}_{ij}^{xx} + \tilde{\Lambda}_{ij}^{yy} - i\tilde{\Lambda}_{ij}^{xy} + i\tilde{\Lambda}_{ij}^{yx}\right) \\ & - \tilde{\Lambda}_{ij}^{zz}\left(a_i^\dagger a_i + a_j^\dagger a_j\right) + \text{H.c.}\Big], \end{aligned} \tag{20}$$

where H.c. denotes Hermitian conjugate, and where we have assumed that $\tilde{\Lambda}_{ij}^{\mu\nu} \in \mathbb{R}$, which is guaranteed to hold if the original spin Hamiltonian has $\Lambda_{ij}^{\mu\nu} \in \mathbb{R}$.

For the cubic terms we find

$$H^{(3)} = H_1^{(3)} + H_2^{(3)} + \eta\left(H_2^{(3)}\right)^\dagger, \tag{21}$$

where $\eta = 0$ for the (now manifestly non-hermitian) Dyson-Maleev representation, and $\eta = 1$ for both the Taylor and new expansions. Above,

$$\begin{aligned} H_1^{(3)} = \sqrt{\frac{S}{2}} \sum_{i,j} \Big[& a_i^\dagger a_j^\dagger a_j \left(-\tilde{\Lambda}_{ij}^{xz} - i\tilde{\Lambda}_{ij}^{yz}\right) \\ & + a_i^\dagger a_i a_j \left(-\tilde{\Lambda}_{ij}^{zx} + i\tilde{\Lambda}_{ij}^{zy}\right) + \text{H.c.}\Big], \end{aligned} \tag{22}$$

$$\begin{aligned} H_2^{(3)} = \frac{S^{3/2} C}{\sqrt{2}} \sum_{i,j} \Big[& a_i^\dagger a_i^2 \left(\tilde{\Lambda}_{ij}^{xz} - i\tilde{\Lambda}_{ij}^{yz}\right) \\ & + a_j^\dagger a_j^2 \left(\tilde{\Lambda}_{ij}^{zx} - i\tilde{\Lambda}_{ij}^{zy}\right)\Big]. \end{aligned} \tag{23}$$

Similarly, the quartic terms become

$$H^{(4)} = H_1^{(4)} + H_2^{(4)} + \eta\left(H_2^{(4)}\right)^\dagger, \tag{24}$$

where

$$H_1^{(4)} = \sum_{i,j} a_i^\dagger a_i a_j^\dagger a_j \tilde{\Lambda}_{ij}^{zz}, \tag{25}$$

$$\begin{aligned} H_2^{(4)} = \frac{SC}{2} \sum_{i,j} \Big[& a_i a_j^\dagger a_j^2 \left(\tilde{\Lambda}_{ij}^{xx} - \tilde{\Lambda}_{ij}^{yy} - i\tilde{\Lambda}_{ij}^{xy} - i\tilde{\Lambda}_{ij}^{yx}\right) \\ & + a_i^\dagger a_j^\dagger a_j^2 \left(\tilde{\Lambda}_{ij}^{xx} + \tilde{\Lambda}_{ij}^{yy} - i\tilde{\Lambda}_{ij}^{xy} + i\tilde{\Lambda}_{ij}^{yx}\right) \\ & + a_i^\dagger a_i^2 a_j \left(\tilde{\Lambda}_{ij}^{xx} - \tilde{\Lambda}_{ij}^{yy} - i\tilde{\Lambda}_{ij}^{xy} - i\tilde{\Lambda}_{ij}^{yx}\right) \\ & + a_i^\dagger a_i^2 a_j^\dagger \left(\tilde{\Lambda}_{ij}^{xx} + \tilde{\Lambda}_{ij}^{yy} + i\tilde{\Lambda}_{ij}^{xy} - i\tilde{\Lambda}_{ij}^{yx}\right)\Big]. \end{aligned} \tag{26}$$

The quintic terms are simply

$$H^{(5)} = H_1^{(5)} + \eta\left(H_1^{(5)}\right)^\dagger, \tag{27}$$

where

$$\begin{aligned} H_1^{(5)} = C\sqrt{\frac{S}{2}} \sum_{i,j} \Big[& a_i^\dagger a_i^2 a_j^\dagger a_j \left(-\tilde{\Lambda}_{ij}^{xz} + i\tilde{\Lambda}_{ij}^{yz}\right) \\ & + a_i^\dagger a_i a_j^\dagger a_j^2 \left(-\tilde{\Lambda}_{ij}^{zx} + i\tilde{\Lambda}_{ij}^{zy}\right)\Big]. \end{aligned} \tag{28}$$

Finally, for the sextic terms we find

$$H^{(6)} = H_1^{(6)} + \eta \left(H_1^{(6)}\right)^\dagger + \eta H_2^{(6)}, \tag{29}$$

where

$$H_1^{(6)} = \frac{SC^2}{2} \sum_{i,j} a_i^\dagger a_j^\dagger a_i^2 a_j^2 \left(\tilde{\Lambda}_{ij}^{xx} - \tilde{\Lambda}_{ij}^{yy} - i\tilde{\Lambda}_{ij}^{xy} - i\tilde{\Lambda}_{ij}^{yx}\right), \tag{30}$$

$$H_2^{(6)} = \frac{SC^2}{2} \sum_{i,j} \Big[\left(a_i^\dagger\right)^2 a_j^\dagger a_i a_j^2 \left(\tilde{\Lambda}_{ij}^{xx} + \tilde{\Lambda}_{ij}^{yy} - i\tilde{\Lambda}_{ij}^{xy} + i\tilde{\Lambda}_{ij}^{yx}\right) + \text{H.c.}\Big]. \tag{31}$$

E. Other representations

For completeness we note that a number of other bosonic representations of the spin operators have been proposed, with various properties and uses. The most prominent of these is the Schwinger boson representation[1], which is commonly used to describe disordered and quantum spin liquid ground states, since it does not rely on a specific quantization axis. The Holstein-Primakoff and Dyson-Maleev representations can be obtained from the Schwinger representation by appropriate gauge choices[9]. Further representations were proposed by Goldhirsch et al.[10,11], Villain[12], and Garbaczewski[13,14]. However, the three representations considered above can be seen to come from a common starting point, and lead to the same Hamiltonian expressions. For the mean-field theory discussion to come, we will focus on the new and Taylor expansions for the square root.

II. APPLICATION TO THE EASY-PLANE XXZ ANTIFERROMAGNET

We will now specialize to the nearest-neighbor XXZ antiferromagnet on the triangular lattice,

$$H = J \sum_{\langle i,j \rangle} \left(S_i^x S_j^x + S_i^y S_j^y + \Delta S_i^z S_j^z\right), \tag{32}$$

with $J > 0$, and $0 < \Delta < 1$ for an easy-plane anisotropy. This systems orders magnetically in the non-collinear 120° spin structure shown in Fig. 2. The classical spin configurations are

$$\mathbf{S}_A = S\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\right), \tag{33}$$

$$\mathbf{S}_B = S\left(+\frac{\sqrt{3}}{2}, +\frac{1}{2}, 0\right), \tag{34}$$

$$\mathbf{S}_C = S\,(0, +1, 0), \tag{35}$$

where S is the magnitude of the moment, and A, B, C are the three magnetic sublattices, colored like the spin state in Fig. 2. The $S = 1/2$ isotropic ($\Delta = 1$) triangular lattice Heisenberg antiferromagnet is a paradigmatic model for the study of

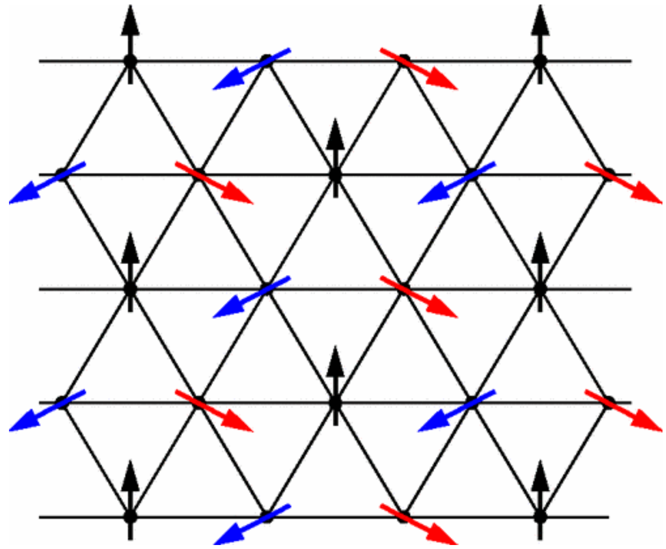

FIG. 2. The 120° order on the triangular lattice. The magnetic unit cell contains three sites. Figure from Ref.[15].

frustration, as well as magnon decay effects. The easy-plane anisotropy relieves a lot of the frustration, allowing for a more controlled spin wave study. Several works have studied these models up to quartic order in the $1/S$ expansion,[15–23] and so it represents a good proving ground for our new expansion.

To follow the notation of Chernyshev and Zhitomirsky[15,21] we first cyclically permute coordinates, and assume that the spins lie in the xz plane. This can be achieved by relabeling the XXZ model as one with "XZX" anisotropy, for which the lab-frame Hamiltonian is written

$$H = J \sum_{\langle i,j \rangle} \left(S_i^{x_0} S_j^{x_0} + S_i^{z_0} S_j^{z_0} + \Delta S_i^{y_0} S_j^{y_0}\right). \tag{36}$$

The spins are transformed into a rotating frame (x, y, z) by

$$S_i^{z_0} = S_i^z \cos\theta_i - S_i^x \sin\theta_i, \tag{37}$$

$$S_i^{x_0} = S_i^z \sin\theta_i + S_i^x \cos\theta_i, \tag{38}$$

and $S_i^y = S_i^{y_0}$, where $\theta_i = \mathbf{Q}\cdot\mathbf{r}_i$, and $\mathbf{Q} = (4\pi/3, 0)$ is the magnetic ordering vector for 120° configuration (corresponding to the K point in reciprocal space). After this transformation, the spin Hamiltonian becomes

$$H = J \sum_{\langle i,j \rangle} \Big[\Delta S_i^y S_j^y + \cos\left(\theta_i - \theta_j\right)\left(S_i^x S_j^x + S_i^z S_j^z\right) + \sin\left(\theta_i - \theta_j\right)\left(S_i^z S_j^x - S_i^x S_j^z\right)\Big]. \tag{39}$$

In the 120° configuration, $\theta_i - \theta_j = \pm 2\pi/3$.. By comparison to Eq. (2) we identify

$$0 = \tilde{\Lambda}^{xy} = \tilde{\Lambda}^{yx} = \tilde{\Lambda}^{yz} = \tilde{\Lambda}^{zy}, \quad \tilde{\Lambda}^{yy} = \Delta \tag{40}$$

$$\tilde{\Lambda}^{xx} = \tilde{\Lambda}^{zz} = J\cos\left(\theta_i - \theta_j\right), \tag{41}$$

$$-\tilde{\Lambda}^{xz} = +\tilde{\Lambda}^{zx} = J\sin\left(\theta_i - \theta_j\right). \tag{42}$$

In matrix form, we have

$$\Lambda_{ij} = \begin{bmatrix} J & 0 & 0 \\ 0 & J\Delta & 0 \\ 0 & 0 & J \end{bmatrix}, \tag{43}$$

$$\tilde{\Lambda}_{ij} = J \begin{bmatrix} \cos\left(\theta_i - \theta_j\right) & 0 & -\sin\left(\theta_i - \theta_j\right) \\ 0 & \Delta & 0 \\ \sin\left(\theta_i - \theta_j\right) & 0 & \cos\left(\theta_i - \theta_j\right) \end{bmatrix} \tag{44}$$

From this matrix and the general magnon Hamiltonians derived in the previous section, we can immediately see that the anisotropy only enters in terms even in the number of boson operators. Another way of looking at it involves decomposing the Hamiltonian (39) as

$$H = H_{\text{coll}} + H_{\text{non-coll}}, \tag{45}$$

where the collinear and non-collinear parts are

$$H_{\text{coll}} = J \sum_{\langle i,j \rangle} \left[\Delta S_i^y S_j^y + \cos\left(\theta_i - \theta_j\right) \left(S_i^x S_j^x + S_i^z S_j^z \right) \right], \tag{46}$$

$$H_{\text{non-coll}} = J \sum_{\langle i,j \rangle} \left[\sin\left(\theta_i - \theta_j\right) \left(S_i^z S_j^x - S_i^x S_j^z \right) \right]. \tag{47}$$

H_{coll} produces terms with $2, 4, 6, \ldots$ magnon operators, while $H_{\text{non-coll}}$ produces terms with $1, 3, 5, \ldots$ magnon operators. The latter vanish completely in the collinear case.

A. Sublinear terms

Combining Eqs. (18) and (108) we find

$$H^{(0)} = JS^2 \sum_{\langle i,j \rangle} \cos\left(\theta_i - \theta_j\right) = -\frac{JS^2}{2} \sum_{\langle i,j \rangle} 1. \tag{48}$$

Introducing the NN vector $\vec{\delta}$, we can write

$$H^{(0)} = -\frac{JS^2}{2} \sum_i \sum_{\vec{\delta}} 1 = -\frac{zJS^2N}{4} = -\frac{3JS^2N}{2}, \tag{49}$$

where N is the total number of sites, $z = 6$ is the coordination number, and a factor $1/2$ was introduced to avoid double counting. The classical energy is independent of the anisotropy, reflecting that the spins lie in the plane.

Using Eq. (19) for the linear terms we have

$$H^{(1)} = \frac{JS^{3/2}}{\sqrt{2}} \sum_{\langle i,j \rangle} \sin\left(\theta_i - \theta_j\right) \left(a_j - a_i + \text{H.c.} \right) = 0, \tag{50}$$

where H.c. denotes the Hermitian conjugate. The linear terms sum to zero as expected, since the $120°$ spin configuration is the classical ground state of H. This is easy to see by fixing i and summing over NNs, noting that $\sin\left(\theta_i - \theta_j\right) = \pm\sqrt{3}/2$.

B. Linear spin wave theory

Using Eq. (20) the quadratic terms are written

$$H^{(2)} = \frac{S}{2} \sum_{\langle i,j \rangle} \Big\{ \left(a_i a_j \tilde{\Lambda}_{ij}^- + a_i a_j^\dagger \tilde{\Lambda}_{ij}^+ + \text{H.c.} \right) - 2\tilde{\Lambda}_{ij}^{zz} \left(a_i^\dagger a_i + a_j^\dagger a_j \right) \Big\}, \tag{51}$$

where $\tilde{\Lambda}_{ij}^\pm \equiv \tilde{\Lambda}_{ij}^{xx} \pm \tilde{\Lambda}_{ij}^{yy}$. It is convenient to introduce a unit cell index μ, and a sublattice index α. After Fourier transforming[24] and summing over μ,

$$H^{(2)} = \frac{S}{2\cdot 3} \sum_{\mathbf{k}} \sum_{\alpha} \sum_{\vec{\delta}} \Big\{ \tilde{\Lambda}^-_{\alpha,\alpha+\vec{\delta}} e^{i\mathbf{k}\cdot\vec{\delta}} \left(a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + a_{-\mathbf{k}} a_{\mathbf{k}} \right) + \left(\tilde{\Lambda}^+_{\alpha,\alpha+\vec{\delta}} e^{i\mathbf{k}\cdot\vec{\delta}} - 2\tilde{\Lambda}^{zz}_{\alpha,\alpha+\vec{\delta}} \right) \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}} \right) \Big\} \tag{52}$$

Counting all bonds, and introducing a factor $1/2$ to avoid double counting, we obtain

$$\sum_{\alpha,\vec{\delta}} \tilde{\Lambda}^+_{\alpha,\alpha+\vec{\delta}} e^{i\mathbf{k}\cdot\vec{\delta}} = 9J\left(\Delta - \frac{1}{2}\right)\gamma_{\mathbf{k}}, \tag{53}$$

$$\sum_{\alpha,\vec{\delta}} \tilde{\Lambda}^-_{\alpha,\alpha+\vec{\delta}} e^{i\mathbf{k}\cdot\vec{\delta}} = -9J\left(\Delta + \frac{1}{2}\right)\gamma_{\mathbf{k}}, \tag{54}$$

$$\sum_{\alpha,\vec{\delta}} \left(-2\tilde{\Lambda}^{zz}_{\alpha,\alpha+\vec{\delta}} \right) = 9J, \tag{55}$$

where (see Appendix A for alternative definitions of $\gamma_{\mathbf{k}}$)

$$\gamma_{\mathbf{k}} = \frac{1}{3}\left[\cos k_x + 2\cos\frac{k_x}{2}\cos\frac{\sqrt{3}}{2}k_y \right]. \tag{56}$$

Thus the Fourier-transformed quadratic Hamiltonian can be written

$$H^{(2)} = \sum_{\mathbf{k}} \Big\{ A_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - \frac{B_{\mathbf{k}}}{2} \left(a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + a_{-\mathbf{k}} a_{\mathbf{k}} \right) \Big\}, \tag{57}$$

$$A_{\mathbf{k}} = 3JS\left[1 + \left(\Delta - \frac{1}{2}\right)\gamma_{\mathbf{k}} \right], \tag{58}$$

$$B_{\mathbf{k}} = 3JS\left(\Delta + \frac{1}{2}\right)\gamma_{\mathbf{k}}, \tag{59}$$

These expressions match those in the Supplemental Material of Zhu et al.[25] up to the (for them!) inconsequential sign of $B_{\mathbf{k}}$. The same sign discrepancy is found in the work by Maksimov et al., Ref.[22]. Is this a result of the spin rotations used? In either case, the expressions for $A_{\mathbf{k}}$, $B_{\mathbf{k}}$ precisely match those of Merdan and Xian[23]. and reduce to those of Chubukov et al.[26] and Chernyshev and Zhitomirsky[15] for $\Delta = 1$. These three references certainly use the same rotation that is employed here. We may note that the coefficients $A_{\mathbf{k}}, B_{\mathbf{k}}$ are real-valued and even in $\mathbf{k}$.

We now apply a standard Bogoliubov transformation

$$a_{\mathbf{k}} = u_{\mathbf{k}} b_{\mathbf{k}} + v_{\mathbf{k}} b_{-\mathbf{k}}^\dagger, \tag{60}$$

where $u_{\mathbf{k}} = u_{-\mathbf{k}} \in \mathbb{R}$, $v_{\mathbf{k}} = v_{-\mathbf{k}} \in \mathbb{R}$. Requiring $[b_{\mathbf{k}}, b^\dagger_{\mathbf{k}}] = 1$ gives the condition

$$u^2_{\mathbf{k}} - v^2_{\mathbf{k}} = 1. \tag{61}$$

The off-diagonal components vanish if we also have

$$u^2_{\mathbf{k}} + v^2_{\mathbf{k}} = \frac{A_{\mathbf{k}}}{\sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}}}, \quad 2u_{\mathbf{k}} v_{\mathbf{k}} = \frac{B_{\mathbf{k}}}{\sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}}}. \tag{62}$$

Under these conditions the diagonal Hamiltonian may be written

$$H^{(2)} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} b^\dagger_{\mathbf{k}} b_{\mathbf{k}}, \tag{63}$$

where

$$\epsilon_{\mathbf{k}} = \sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}} = 3JS\,\omega_{\mathbf{k}}, \tag{64}$$

with the dimensionless frequency,

$$\omega_{\mathbf{k}} = \sqrt{(1-\gamma_{\mathbf{k}})(1+2\Delta\gamma_{\mathbf{k}})}, \tag{65}$$

in agreement with Refs.[15,18,19,23].

We may also read off the explicit expressions for the Bogoliubov prefactors from Ref.[18],

$$u_{\mathbf{k}} = \sqrt{\frac{A_{\mathbf{k}} + \epsilon_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}}}, \quad v_{\mathbf{k}} = \frac{B_{\mathbf{k}}}{|B_{\mathbf{k}}|} \sqrt{\frac{A_{\mathbf{k}} - \epsilon_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}}}. \tag{66}$$

C. Cubic terms

The cubic terms are given by Eq. (21). We first note that the on-site term H^3_2 in Eq. (23) vanishes. The reason is that the $\sin\left(\theta_i - \theta_j\right) = \pm\frac{\sqrt{3}}{2}$ factors cancel in the sum when multiplying single-site terms. Thus we have that $H^{(3)} = H^{(3)}_1$, and using Eq. (22),

$$H^{(3)} = J\sqrt{\frac{S}{2}} \sum_{\langle i,j\rangle} \sin\left(\theta_i - \theta_j\right) \left[\left(a_i + a^\dagger_i\right) a^\dagger_j a_j - a^\dagger_i a_i \left(a_j + a^\dagger_j\right)\right], \tag{67}$$

matching Eq. (19) in Ref.[15], or Eq. (11) in Ref.[23]. The fact that $H^{(3)}_2$ vanishes means that $H^{(3)}$ is independent of C, and thus of the specific expansion used for the square root. We also note that the ground state expectation value $\langle 0|H^{(3)}|0\rangle = 0$, where $|0\rangle$ is the magnon vacuum. Thus cubic terms only contribute to second and higher orders in perturbation theory.

We carry out the Fourier transform, the Bogoliuobov transform and collect terms in the same manner as Chernyshev and Zhitomirsky[15],

$$\begin{aligned} H^{(3)} = \sum_{\mathbf{k},\mathbf{q}} \Bigg[& \frac{1}{2!}\Gamma_1\left(\mathbf{q}, \mathbf{k}-\mathbf{q}; \mathbf{k}\right) b^\dagger_{\mathbf{q}} b^\dagger_{\mathbf{k}-\mathbf{q}} b_{\mathbf{k}} \\ & + \frac{1}{3!}\Gamma_2\left(\mathbf{q}, -\mathbf{k}-\mathbf{q}, \mathbf{k}\right) b^\dagger_{\mathbf{q}} b^\dagger_{-\mathbf{k}-\mathbf{q}} b^\dagger_{\mathbf{k}} + \text{H.c.}\Bigg]. \end{aligned} \tag{68}$$

The first interaction vertex Γ_1 represents (possibly virtual) magnon decay, and is symmetric under exchange of $\mathbf{q}$ and $\mathbf{k}-\mathbf{q}$. In contrast, the source term Γ_2 represents spontaneous magnon creation, and is symmetric under exchange of all momenta. The interaction vertices are made dimensionless,

$$\Gamma_{1,2}\left(1,2;3\right) = 3iJ\sqrt{\frac{3S}{2N}}\tilde{\Gamma}_{1,2}\left(1,2;3\right), \tag{69}$$

where

$$\begin{aligned} \tilde{\Gamma}_1\left(1,2;3\right) = {} & \bar{\gamma}_1\left(u_1+v_1\right)\left(u_2u_3+v_2v_3\right) + \bar{\gamma}_2\left(u_2+v_2\right)\left(u_1u_3\right. \\ & \left. + v_1v_3\right) - \bar{\gamma}_3\left(u_3+v_3\right)\left(u_1v_2+v_1u_2\right), \end{aligned} \tag{70}$$

$$\begin{aligned} \tilde{\Gamma}_2\left(1,2,3\right) = {} & \bar{\gamma}_1\left(u_1+v_1\right)\left(u_2v_3+v_2u_3\right) + \bar{\gamma}_2\left(u_2+v_2\right)\left(u_1v_3\right. \\ & \left. + v_1u_3\right) + \bar{\gamma}_3\left(u_3+v_3\right)\left(u_1v_2+v_1u_2\right), \end{aligned} \tag{71}$$

and

$$\bar{\gamma}_{\mathbf{k}} = \frac{1}{3}\left[\sin k_x - 2\sin\left(\frac{k_x}{2}\right)\cos\left(\frac{\sqrt{3}}{2}k_y\right)\right]. \tag{72}$$

These expressions for the vertices match those of C& Z[15] up to the $1/\sqrt{N}$ factors. These are canceled anyway in the perturbation theory step.

D. Quartic terms

The quartic terms are given in Eqs. (24)-(26). When specializing to the 120° order on the triangular lattice, we may write (26) more compactly as

$$\begin{aligned} H^{(4)}_2 = \frac{SC}{2}\sum_{ij} \Big[& \tilde{\Lambda}^+_{ij}\left(a^\dagger_i a^\dagger_j a^2_j + a^\dagger_i a^2_i a^\dagger_j\right) \\ & + \tilde{\Lambda}^-_{ij}\left(a_i a^\dagger_j a^2_j + a^\dagger_i a^2_i a_j\right)\Big] \end{aligned} \tag{73}$$

Using

$$\tilde{\Lambda}^{zz}_{ij} = J\cos\left(\pm\frac{2\pi}{3}\right) = -\frac{J}{2}, \tag{74}$$

$$\tilde{\Lambda}^{\pm}_{ij} = J\left[\cos\left(\pm\frac{2\pi}{3}\right) + \pm\Delta\right] = -\frac{J}{2} \pm J\Delta \tag{75}$$

we arrive at

$$H^{(4)}_1 = -\frac{J}{2}\sum_{i,j} a^\dagger_i a_i a^\dagger_j a_j, \tag{76}$$

$$\begin{aligned} H^{(4)}_2 = \frac{-SCJ}{2}\sum_{ij} \Bigg[& \left(\Delta + \frac{1}{2}\right)\left(a_i a^\dagger_j a^2_j + a^\dagger_i a^2_i a_j\right) \\ & - \left(\Delta - \frac{1}{2}\right)\left(a^\dagger_i a^\dagger_j a^2_j + a^\dagger_i a^2_i a^\dagger_j\right)\Bigg], \end{aligned} \tag{77}$$

such that $H^{(4)}$ reduces to Chernyshev and Zhitomirsky's[15] Eq. (24) when $\Delta = 1$, $C = -1/2$.

After Fourier and Bogoliubov transformations, application of Wick's theorem, and normal ordering, we expect to have

$$H^{(4)} = \delta E^{(4)} + \delta\tilde{H}^{(2)} + \tilde{H}^{(4)}, \tag{78}$$

where $\delta E^{(4)}$ is the Hartree-Fock correction to the ground state energy, $\delta\tilde{H}^{(2)}$ is the correction to the magnon self-energy, and $\tilde{H}^{(4)}$ describes two-particle magnon-magnon scattering processes. Chernyshev and Zhitomirsky cite Harris et al.[27] to argue that the last term only contains higher-order $1/S$ corrections, and may therefore be neglected. This does not seem justified in the new expansion, where $1/S$ is no longer (even formally) considered a small parameter. Naively, one would still expect a mean-field treatment to work at low temperatures, but we might have to include all terms of a given order in C to achieve consistent results. We will return to this point in the next section.

Introducing the Hartree-Fock decomposition before the Bogoliubov transformation is reportedly easier technically, so we will do the same here. We introduce the following Hartree-Fock parameters

$$\bar{n} = \langle a_i^\dagger a_i\rangle, \quad \bar{m} = \langle a_i^\dagger a_j\rangle, \quad \bar{\Delta} = \langle a_i a_j\rangle, \quad \bar{\delta} = \langle a_i^2\rangle. \tag{79}$$

Following Appendix A in Chernyshev and Zhitomirsky[15], but restoring the normalization $1/N$,

$$\bar{n} = \frac{1}{N}\sum_{\mathbf{k}} v_{\mathbf{k}}^2 = \frac{1}{2N}\sum_{\mathbf{k}}\left[\frac{1+\left(\Delta-\frac{1}{2}\right)\gamma_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right] - \frac{1}{2}, \tag{80}$$

$$\bar{m} = \frac{1}{N}\sum_{\mathbf{k}} \gamma_{\mathbf{k}} v_{\mathbf{k}}^2 = \frac{1}{2N}\sum_{\mathbf{k}}\left[\frac{\gamma_{\mathbf{k}}+\left(\Delta-\frac{1}{2}\right)\gamma_{\mathbf{k}}^2}{\omega_{\mathbf{k}}}\right], \tag{81}$$

where in derivation for $\bar{m}$ it is used that $\sum_{\mathbf{k}}\gamma_{\mathbf{k}} = 0$. We also have

$$\bar{\Delta} = \frac{1}{N}\sum_{\mathbf{k}} \gamma_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} = \frac{\Delta+1/2}{2N}\sum_{\mathbf{k}}\frac{\gamma_{\mathbf{k}}^2}{\omega_{\mathbf{k}}}, \tag{82}$$

$$\bar{\delta} = \frac{1}{N}\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} = \frac{\Delta+1/2}{2N}\sum_{\mathbf{k}}\frac{\gamma_{\mathbf{k}}}{\omega_{\mathbf{k}}}. \tag{83}$$

See Appendix B for more details on the derivation of the above expressions. The three two-dimensional integrals

$$c_l = \frac{1}{N}\sum_{\mathbf{k}}\frac{(\gamma_{\mathbf{k}})^l}{\omega_{\mathbf{k}}}, \quad l = 0,1,2 \tag{84}$$

can be used to rewrite the averages

$$\bar{n} = \frac{1}{2}\left[c_0 + \left(\Delta-\frac{1}{2}\right)c_1\right] - \frac{1}{2}, \tag{85}$$

$$\bar{m} = \frac{1}{2}\left[c_1 + \left(\Delta-\frac{1}{2}\right)c_2\right], \tag{86}$$

$$\bar{\Delta} = \frac{\Delta+1/2}{2}c_2, \quad \bar{\delta} = \frac{\Delta+1/2}{2}c_1. \tag{87}$$

Note that the anisotropy Δ enters c_l through $\omega_{\mathbf{k}}$. Chernyshev and Zhitomirsky evaluated the integrals numerically for the case of the isotropic Heisenberg model, finding

$$c_0 = 1.574\,733\,4, \quad c_1 = -0.104\,253\,9, \quad c_2 = 0.344\,445\,8.$$

In the case of $\Delta = 0.89$ I obtain (using Mathematica's adaptive Monte Carlo algorithm)

$$c_0 = 1.336\,104\,2 \pm 2.326\,997283 \times 10^{-8},$$
$$c_1 = 0.018\,104\,1 \pm 3.037\,602\,078 \times 10^{-8},$$
$$c_2 = 0.299\,121\,1 \pm 2.003\,767\,129 \times 10^{-8},$$

where the $\pm$ value is the internal error estimate, which should be based on the standard deviation. However, when using the Cuba integration package (which algorithm?) Hao got

$$c_0 = 1.336\,025\,4, \quad c_1 = 0.018\,561\,1, \quad c_2 = 0.298\,991\,6.$$

TODO: Resolve discrepancy, get these values properly converged, and put here.

For the correction to the ground state energy, we find (see Appendix C)

$$\delta E^{(4)} = -\frac{3J}{2}\left[\bar{n}^2 + \bar{m}^2 + \bar{\Delta}^2 - 4SC\left(\Delta+\frac{1}{2}\right)\left(2\bar{n}\bar{\Delta}+\bar{m}\bar{\Delta}\right) + 4SC\left(\Delta-\frac{1}{2}\right)\left(2\bar{n}\bar{m}+\bar{\Delta}\bar{\delta}\right)\right]. \tag{88}$$

The factor 3 in front of the brackets comes from the fact that the expressions in the Appendix are averages that here get multiplied by $z/2$, where z is the coordination number and $1/2$ is a factor to avoid double counting. Plugging in $\Delta = 1$, $S = 1/2$, $C = 1/2$, this reproduces Eq. (A4) of Chernyshev and Zhitomirsky. Plugging in Eqs. (85)-(87) results in a long and not very illuminating expression, which we omit here. However, I have checked that it reduces to the appropriate expression in the same limit.

The corrections to the harmonic spin-wave Hamiltonian take the form

$$\delta\tilde{H}^{(2)} = \frac{J}{2}\sum_{\langle i,j\rangle}\left[a_i a_j \Xi^- + a_i a_j^\dagger \Xi^+ + \frac{\Xi^{zz}}{2}\left(a_i^\dagger a_i + a_j^\dagger a_j\right) + \frac{\Xi'}{2}\left(a_i a_i + a_j a_j\right) + \mathrm{H.c.}\right], \tag{89}$$

which almost has the same form as Eq. (51). Here the coefficients are

$$\Xi^- = -\bar{\Delta} - 4SC\left(\Delta+\frac{1}{2}\right)\bar{n} + 2SC\left(\Delta-\frac{1}{2}\right)\bar{\delta}, \tag{90}$$

$$\Xi^+ = -\bar{m} - 2SC\left(\Delta+\frac{1}{2}\right)\bar{\delta} + 4SC\left(\Delta-\frac{1}{2}\right)\bar{n}, \tag{91}$$

$$\Xi^{zz} = -\bar{n} - 4SC\left(\Delta+\frac{1}{2}\right)\bar{\Delta} + 4SC\left(\Delta-\frac{1}{2}\right)\bar{m}, \tag{92}$$

$$\Xi' = 2SC\left[-\left(\Delta+\frac{1}{2}\right)\bar{m} + \left(\Delta-\frac{1}{2}\right)\bar{\Delta}\right]. \tag{93}$$

After a Fourier transform, we get

$$\delta\tilde{H}^{(2)} = \frac{J}{2}\sum_{k}\sum_{\vec{\delta}}\Big[\Xi^- e^{i\mathbf{k}\cdot\vec{\delta}}\left(a_k^\dagger a_{-k}^\dagger + a_{-k}a_k\right) + \Xi^+ e^{i\mathbf{k}\cdot\vec{\delta}}\left(a_k^\dagger a_k + a_{-k}a_{-k}^\dagger\right) + \Xi^{zz}\left(a_k^\dagger a_k + a_{-k}a_{-k}^\dagger\right) + \Xi'\left(a_k^\dagger a_{-k}^\dagger + a_{-k}a_k\right)\Big] \tag{94}$$

Since the Ξ parameters only contain mean-field parameters and constants, the relevant sublattice sums are simply (avoiding double counting, see Appendix A)

$$\sum_{\vec{\delta}} e^{i\mathbf{k}\cdot\vec{\delta}} = 3\gamma_{\mathbf{k}}, \quad \sum_{\vec{\delta}} 1 = 3, \tag{95}$$

which means that we can write

$$\delta\tilde{H}^{(2)} = \sum_{\mathbf{k}} \left[\delta A_{\mathbf{k}} a_k^\dagger a_k - \frac{1}{2}\delta B_k \left(a_k^\dagger a_{-k}^\dagger + a_{-k} a_k \right) \right], \tag{96}$$

where

$$\delta A_k = +3J\left(\Xi^+\gamma_k + \Xi^{zz}\right), \tag{97}$$

$$\delta B_k = -3J\left(\Xi^-\gamma_k + \Xi'\right). \tag{98}$$

These expressions reduce to Chernyshev and Zhitomirsky's Eq. (A5) in the limit $\Delta = 1$, $S = 1/2$, $C = -1/(4S)$.

Then, using the same Bogoliubov transformation as before, we can write

$$\delta\tilde{H}^{(2)} = \sum_{k} \left[\epsilon_k^{(4)} b_k^\dagger b_k - \frac{1}{2} B_k^{(4)} \left(b_k^\dagger b_{-k}^\dagger + b_{-k} b_k \right) \right], \tag{99}$$

where

$$\epsilon_k^{(4)} = \left(u_k^2 + v_k^2\right)\delta A_k - 2u_k v_k \delta B_k \tag{100}$$

$$= \frac{1}{\omega_k}\left(\left[1 + \left(\Delta - \frac{1}{2}\right)\gamma_k \right] \delta A_k - \left(\Delta + \frac{1}{2}\right)\gamma_k \delta B_k \right), \tag{101}$$

$$B_k^{(4)} = \left(u_k^2 + v_k^2\right)\delta B_k - 2u_k v_k \delta A_k \tag{102}$$

$$= \frac{1}{\omega_k}\left(\left[1 + \left(\Delta - \frac{1}{2}\right)\gamma_k \right] \delta B_k - \left(\Delta + \frac{1}{2}\right)\gamma_k \delta A_k \right). \tag{103}$$

These expressions reduce to Eqs. (28)-(29) of Ref.[15] in the limit $\Delta = 1$, $S = 1/2$, $C = -1/(4S)$.

III. TAKING A PERTURBATION THEORY SHORTCUT

The most straight-forward non-linear spin-wave theory approaches treat the nonlinear terms as perturbation effects around the LSWT result. In the Taylor expansion approach for noncollinear systems, one typically keeps only effects from the cubic and quartic terms. The former gives a contribution to second order in perturbation theory, but not the first order. On the other hand, the quartic terms do have a contribution at first order in perturbation theory. Hence, for a consistent perturbation theory, we need to account for both types of terms at first order in $1/S$. A good litmus test is to consider the effects at the point of a Goldstone mode, which is symmetry-protected and should be preserved by the NLSWT calculation. In the Taylor expansion approach, one remarkably finds that the cubic and quartic contributions cancel at these points.

In our new expansion, $1/S$ is no longer (even formally) a small expansion parameter. Indeed, the most consistent expansion would appear to be in orders of C. One would thus need to include the effects of $H^{(5)}$ in an expansion to first order in C. We have not attempted to do so here. Instead, we note the formal similarity between the expressions for the spin raising and lowering operators in Eqs. (14) — (16). One may note that the $C_{\rm new}(S)$ in the new expansion can be replicated by using an effective spin moment, $S_{\rm eff}$, in the Taylor expansion, such that $C_{\rm Taylor}(S_{\rm eff}) = C_{\rm new}(S)$. For $S = 1/2$ we get $S_{\rm eff} = 1/4$. Substituting $S \to S_{\rm eff}$ means that we also have to rescale the $\sqrt{2S}$ factors, which mostly amounts to a rescaling of the spin-spin interaction parameters. Explicitly, we get

$$\tilde{S}^{\pm}(S) \to \sqrt{\frac{S}{S_{\rm eff}}}\tilde{S}^{\pm}(S_{\rm eff}), \tag{104}$$

$$\tilde{S}^{z}(S) \to \tilde{S}^{z}(S_{\rm eff}) + (S - S_{\rm eff}). \tag{105}$$

Now consider our rotated spin Hamiltonian, Eq. (39). With the above rescaling we get

$$H \to H' + \delta H', \tag{106}$$

where

$$H' = J\sum_{\langle i,j\rangle}\left[\frac{S\Delta}{S_{\rm eff}}\tilde{S}_i^y\tilde{S}_j^y + \cos\left(\theta_i - \theta_j\right)\left(\frac{S}{S_{\rm eff}}\tilde{S}_i^x\tilde{S}_j^x + \tilde{S}_i^z\tilde{S}_j^z \right) + \sin\left(\theta_i - \theta_j\right)\sqrt{\frac{S}{S_{\rm eff}}}\left(\tilde{S}_i^z\tilde{S}_j^x - \tilde{S}_i^x\tilde{S}_j^z \right) \right]. \tag{107}$$

Note that all spin operators in H' are now $S_{\rm eff}$ operators. We may construct the rescaled interaction parameter matrix,

$$\tilde{\Lambda}_{ij} = J\begin{bmatrix} \cos\left(\theta_i - \theta_j\right)\frac{S}{S_{\rm eff}} & 0 & -\sin\left(\theta_i - \theta_j\right)\sqrt{\frac{S}{S_{\rm eff}}} \\ 0 & \frac{S\Delta}{S_{\rm eff}} & 0 \\ \sin\left(\theta_i - \theta_j\right)\sqrt{\frac{S}{S_{\rm eff}}} & 0 & \cos\left(\theta_i - \theta_j\right) \end{bmatrix} \tag{108}$$

Above, the correction $\delta H'$ contains the terms that are not mere rescalings of spin-spin interaction terms. We have

$$\delta H' = J\sum_{\langle i,j\rangle}\left[\cos\left(\theta_i - \theta_j\right)(\delta S)^2 + \cos\left(\theta_i - \theta_j\right)\delta S\left(\tilde{S}_i^z + \tilde{S}_j^z \right) + \sin\left(\theta_i - \theta_j\right)\delta S\sqrt{\frac{S}{S_{\rm eff}}}\left(\tilde{S}_j^x - \tilde{S}_i^x \right) \right], \tag{109}$$

where $\delta S = S - S_{\rm eff}$, and all spin operators are $S_{\rm eff}$ operators. The first term in Eq. (109) is a constant, and may be neglected. The third term vanishes, for the same reason that $H^{(1)}$ and $H_2^{(3)}$ vanished earlier. To see this, fix i and sum over neighboring sites j, recalling that $\sin\left(\theta_i - \theta_j\right) = \pm\sqrt{3}/2$. The middle term, however, will remain,

$$\delta H' = J\sum_{\langle i,j\rangle}\left[\cos\left(\theta_i - \theta_j\right)\delta S\left(\tilde{S}_i^z + \tilde{S}_j^z \right) \right]. \tag{110}$$

It enters the problem at the LSWT level. Note that it should not be considered a correction to linear spin-wave theory (if we truncate the theory at this order in the boson number), but that consistency demands it be included in the quadratic Hamiltonian used as a starting point for NLSWT calculations, if $S_{\rm eff}$ is introduced to rescale terms higher in boson operator number.

The Fourier-transformed quadratic Hamiltonian retains the form in Eq. (57), but now Eq. (58)-(59) become

$$A_{\mathbf{k}} = 3JS_{\rm eff}\left[1 + \frac{\delta S}{S_{\rm eff}} + \left(\Delta - \frac{1}{2}\right)\gamma_{\mathbf{k}}\right], \tag{111}$$

$$B_{\mathbf{k}} = 3JS_{\rm eff}\left(\Delta + \frac{1}{2}\right)\gamma_{\mathbf{k}}. \tag{112}$$

Thus $A_{\mathbf{k}}$ contains a new term, which vanishes when $S_{\rm eff} \to S$. This modifies the energy, Eq. (64) and the dimensionless frequency, Eq. (65) into

$$\epsilon_{\mathbf{k}} = \sqrt{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2} = 3JS_{\rm eff}\omega_{\mathbf{k}}, \tag{113}$$

$$\omega_{\mathbf{k}} = \sqrt{\left(1 - \gamma_{\mathbf{k}} + \frac{\delta S}{S_{\rm eff}}\right)\left(1 + 2\Delta\gamma_{\mathbf{k}} + \frac{\delta S}{S_{\rm eff}}\right)}. \tag{114}$$

If $S_{\rm eff} = S$ we recover Eqs. (64), (65). The cubic vertices retain their previous form. Just note that the Bogoliubov prefactors, Eq. (66) must be evaluated using the $A_{\mathbf{k}}$, $B_{\mathbf{k}}$ and $\epsilon_{\mathbf{k}}$ in Eqs. (111)-(113).

The modified quadratic Hamiltonian also modified the harmonic averages of the Hartree-Fock parameters, Eqs. (85), (86), (87). We now get

$$\bar{n} = \frac{1}{2}\left[\left(1 + \frac{\delta S}{S_{\rm eff}}\right)c_0 + \left(\Delta - \frac{1}{2}\right)c_1\right] - \frac{1}{2}, \tag{115}$$

$$\bar{m} = \frac{1}{2}\left[\left(1 + \frac{\delta S}{S_{\rm eff}}\right)c_1 + \left(\Delta - \frac{1}{2}\right)c_2\right], \tag{116}$$

$$\bar{\Delta} = \frac{\Delta + 1/2}{2}c_2, \quad \bar{\delta} = \frac{\Delta + 1/2}{2}c_1. \tag{117}$$

That is, the form of $\bar{\Delta}$ and $\bar{\delta}$ does not change. However, since Eq. (114) depends on $\delta S/S_{\rm eff}$ we need to reevaluate the c_l integrals.

Appendix A: A note on lattice sums

Following the literature)[15,18] we define

$$\gamma_{\mathbf{k}} = \frac{1}{6}\sum_{\vec{\delta}} e^{i\mathbf{k}\cdot\vec{\delta}}, \tag{A1}$$

where the sum runs over all six NN vectors connected to a given site. It is straightforward to verify that,

$$\gamma_{\mathbf{k}} = \frac{1}{3}\left[\cos k_x + 2\cos\frac{k_x}{2}\cos\frac{\sqrt{3}}{2}k_y\right]. \tag{A2}$$

Note that, for the lion's share of these notes, we avoid double counting by only considering bonds pointing to the right. For the various lattice sums, we symmetrize the bonds and introduce a factor $1/2$ to account for the double counting. This allows a manifestly real-valued expression for $\gamma_{\mathbf{k}}$ and similar sums.

Then the lattice sum $\sum_\alpha \sum_{\vec{\delta}} e^{i\mathbf{k}\cdot\vec{\delta}}$ is calculated as follows

$$\sum_\alpha\sum_{\vec{\delta}} e^{i\mathbf{k}\cdot\vec{\delta}} \equiv \sum_\alpha \sum_{\vec{\delta}\in\{\vec{\delta}_1,\vec{\delta}_2,\vec{\delta}_3\}} e^{i\mathbf{k}\cdot\vec{\delta}} = \frac{1}{2}\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} e^{i\mathbf{k}\cdot(\mathbf{r}_\alpha - \mathbf{r}_\beta)} = 9\gamma_{\mathbf{k}}. \tag{A3}$$

Note that the answer is not $18\gamma_{\mathbf{k}}$ as one would obtain by summing over six NNs in the sum over $\vec{\delta}$.

Similarly,

$$\sum_\alpha\sum_{\vec{\delta}} 1 \equiv \sum_\alpha \sum_{\vec{\delta}\in\{\vec{\delta}_1,\vec{\delta}_2,\vec{\delta}_3\}} 1 = \frac{1}{2}\times 3\times 6 = 3\times 3 = 9, \tag{A4}$$

where $\frac{1}{2}$ is from double counting, 3 is the number of magnetic sublattices α, and 6 is the number of NNs for each α.

Appendix B: Expectation values in the harmonic approximation

For future reference, here we provide derivations for the Hartree-Fock expectation values in the harmonic approximation (i.e. expectation values for a system described by the linear spin-wave Hamiltonian). First we have

$$\begin{aligned}\bar{n} &= \left\langle a_i^\dagger a_i\right\rangle = \frac{1}{N}\sum_i \left\langle a_i^\dagger a_i\right\rangle = \frac{1}{N}\sum_{\mathbf{k}}\left\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\right\rangle, \\ &= \frac{1}{N}\sum_{\mathbf{k}}\left\langle\left(u_{\mathbf{k}}b_{\mathbf{k}}^\dagger + v_{\mathbf{k}}b_{-\mathbf{k}}\right)\left(u_{\mathbf{k}}b_{\mathbf{k}} + v_{\mathbf{k}}b_{-\mathbf{k}}^\dagger\right)\right\rangle \\ &= \frac{1}{N}\sum_{\mathbf{k}} v_{\mathbf{k}}^2.\end{aligned} \tag{B1}$$

Plugging in $v_{\mathbf{k}}$ from Eq. (66), and then $A_{\mathbf{k}}$ from Eq. (58) and $\epsilon_{\mathbf{k}}$ from Eq. (64), we get

$$\bar{n} = \frac{1}{N}\sum_{\mathbf{k}}\frac{A_{\mathbf{k}} - \epsilon_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}} = \frac{1}{N}\sum_{\mathbf{k}}\left[\frac{1 + \left(\Delta - \frac{1}{2}\right)\gamma_{\mathbf{k}}}{2\omega_{\mathbf{k}}} - \frac{1}{2}\right]. \tag{B2}$$

Let z be the coordination number ($z = 6$ for the triangular lattice). Then

$$\begin{aligned}\bar{m} &= \langle a_i^\dagger a_j\rangle = \frac{1}{N}\sum_i \frac{1}{z}\sum_{\vec{\delta}}\left\langle a_i^\dagger a_{i+\vec{\delta}}\right\rangle = \frac{1}{zN}\sum_{\mathbf{k}}\left\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\right\rangle\sum_{\vec{\delta}} e^{i\mathbf{k}\cdot\vec{\delta}} \\ &= \frac{1}{N}\sum_{\mathbf{k}} v_{\mathbf{k}}^2\frac{1}{z}\sum_{\vec{\delta}} e^{i\mathbf{k}\cdot\vec{\delta}} = \frac{1}{N}\sum_{\mathbf{k}} v_{\mathbf{k}}^2\gamma_{\mathbf{k}} \\ &= \frac{1}{2N}\sum_{\mathbf{k}}\left[\frac{\gamma_{\mathbf{k}} + \left(\Delta - \frac{1}{2}\right)\gamma_{\mathbf{k}}^2}{\omega_{\mathbf{k}}} - \gamma_{\mathbf{k}}\right],\end{aligned} \tag{B3}$$

where $\sum_{\mathbf{k}}\gamma_{\mathbf{k}} = 0$.

Using Eq. (62) we also obtain

$$\begin{aligned}\bar{\delta} &= \langle a_i a_i\rangle = \frac{1}{N}\langle a_i a_i\rangle = \frac{1}{N}\sum_{\mathbf{k}}\langle a_{-\mathbf{k}}a_{\mathbf{k}}\rangle = \frac{1}{N}\sum_{\mathbf{k}} u_{\mathbf{k}}v_{\mathbf{k}} \\ &= \frac{1}{2N}\sum_{\mathbf{k}}\frac{B_{\mathbf{k}}}{\sqrt{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2}} = \frac{\Delta + \frac{1}{2}}{2N}\sum_{\mathbf{k}}\frac{\gamma_{\mathbf{k}}}{\omega_{\mathbf{k}}},\end{aligned} \tag{B4}$$

and

$$\bar{\Delta} = \left\langle a_i a_j \right\rangle = \frac{1}{zN} \sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \sum_{\vec{\delta}} e^{i\mathbf{k}\cdot\vec{\delta}} = \frac{1}{N} \sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \gamma_{\mathbf{k}}$$
$$= \frac{\Delta + \frac{1}{2}}{2N} \sum_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}^2}{\omega_{\mathbf{k}}}. \quad \text{(B5)}$$

Appendix C: Contractions from Wick's theorem

The fully contracted terms relevant to the Hartree-Fock correction to the classical energy are

$$\langle a_i^\dagger a_i a_j^\dagger a_j \rangle \Rightarrow \langle a_i^\dagger a_i a_j^\dagger a_j \rangle + \langle a_i^\dagger a_i a_j^\dagger a_j \rangle + \langle a_i^\dagger a_i a_j^\dagger a_j \rangle$$
$$= \langle a_i^\dagger a_i \rangle \langle a_j^\dagger a_j \rangle + \langle a_i^\dagger a_j \rangle \langle a_j^\dagger a_i \rangle + \langle a_i^\dagger a_j^\dagger \rangle \langle a_i a_j \rangle$$
$$= \bar{n}^2 + \bar{m}^2 + \bar{\Delta}^2, \quad \text{(C1)}$$

$$\langle a_i^\dagger a_i a_i a_j \rangle \Rightarrow \langle a_i^\dagger a_i a_i a_j \rangle + \langle a_i^\dagger a_i a_i a_j \rangle + \langle a_i^\dagger a_i a_i a_j \rangle$$
$$= 2\bar{n}\bar{\Delta} + \bar{m}\bar{\delta}, \quad \text{(C2)}$$

$$\langle a_j^\dagger a_j^\dagger a_j a_i \rangle \Rightarrow \langle a_j^\dagger a_j^\dagger a_j a_i \rangle + \langle a_j^\dagger a_j^\dagger a_j a_i \rangle + \langle a_j^\dagger a_j^\dagger a_j a_i \rangle$$
$$= 2\bar{n}\bar{m} + \bar{\Delta}\bar{\delta} \quad \text{(C3)}$$

The corrections to the harmonic spin-wave Hamiltonian, and thus to the magnon self-energy, come from the one-contraction, one-normal ordering terms

$$\langle a_i^\dagger a_i a_j^\dagger a_j \rangle \Rightarrow \langle a_i^\dagger a_i a_j^\dagger a_j \rangle + \langle a_i^\dagger a_i a_j^\dagger a_j \rangle + \langle a_i^\dagger a_i a_j^\dagger a_j \rangle$$
$$+ \langle a_i^\dagger a_i a_j^\dagger a_j \rangle + \langle a_i^\dagger a_i a_j^\dagger a_j \rangle + \langle a_i^\dagger a_i a_j^\dagger a_j \rangle \quad \text{(C4)}$$
$$= \bar{n}{:}a_i^\dagger a_i + a_j^\dagger a_j{:} + \bar{m}{:}a_j^\dagger a_i + a_i^\dagger a_j{:} + \bar{\Delta}{:}a_i a_j + a_i^\dagger a_j^\dagger{:},$$

$$\langle a_i^\dagger a_i a_i a_j \rangle \Rightarrow 2\bar{n}{:}a_i a_j{:} + \bar{m}{:}a_i a_i{:} + 2\bar{\Delta}{:}a_i^\dagger a_i{:} + \bar{\delta}{:}a_i^\dagger a_j{:}, \quad \text{(C5)}$$
$$\langle a_j^\dagger a_j a_j a_i \rangle \Rightarrow 2\bar{n}{:}a_j a_i{:} + \bar{m}{:}a_j a_j{:} + 2\bar{\Delta}{:}a_j^\dagger a_j{:} + \bar{\delta}{:}a_j^\dagger a_i{:}, \quad \text{(C6)}$$
$$\langle a_j^\dagger a_j^\dagger a_j a_i \rangle \Rightarrow 2\bar{n}{:}a_j^\dagger a_i{:} + 2\bar{m}{:}a_j^\dagger a_j{:} + \bar{\Delta}{:}a_j^\dagger a_j^\dagger{:} + \bar{\delta}{:}a_i a_j{:}, \quad \text{(C7)}$$
$$\langle a_j^\dagger a_i^\dagger a_i a_i \rangle \Rightarrow 2\bar{n}{:}a_j^\dagger a_i{:} + 2\bar{m}{:}a_i^\dagger a_i{:} + \bar{\Delta}{:}a_i a_i{:} + \bar{\delta}{:}a_j^\dagger a_i^\dagger{:} \quad \text{(C8)}$$

where $:\ldots:$ denotes normal ordering.

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