NOTES ON ALGEBRAIC TOPOLOGY I: HOMOLOGY THEORY

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These notes were taken while reading Allen Hatcher's *Algebraic Topology*[1] book in Spring 2018. I live-TEXed them using sublime, and as such there may be typos; please send questions, comments, complaints, and corrections to `xiaohao1096@163.com`.

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1. Prerequisites and Preliminaries

1.1. Homotopy Notions.

Definition 1.1. *A* ***deformation retraction*** *of a space X onto a subspace A is a family of maps $f_t : X \to X,\ t \in I$, such that $f_0 = \mathbb{1}$, $f_1(X) = A$, and $f_t|_A = \mathbb{1}$ for all t. The family f_t should be continuous in the sense that the associated map $X \times I \to X,\ (x,t) \mapsto f_t(x)$, is continuous.*

Definition 1.2. *A deformation retraction $f_t : X \to X$ is a special case of the general notion of a* ***homotopy****, which is simply any family of maps $f_t : X \to X,\ t \in I$, such that the associated map $F : X \times I \to Y$ given by $F(x,t) = f_t(x)$ is continuous.*

One says that two maps $f_0, f_1 : X \to Y$ are ***homotopic*** *if there exists a homotopy f_t connecting them, and one writes $f_0 \simeq f_1$.*

[1] Book source: `http://www.math.cornell.edu/~hatcher/AT/AT.pdf`

Definition 1.3. *In these terms, a deformation retraction of X onto a subspace A is a homotopy from the identity map of X to a* ***retraction*** *of X onto A, a (continuous) map $r : X \to X$ such that $r(X) = A$ and $r|_A = \mathbb{1}$.*

From a more formal viewpoint a retraction is a map $r : X \to X$ with $r^2 = r$, since this equation says exactly that r is the identity on its image.

Remark 1.4. *Once could equally well regard a retraction as a map $X \to A$ restricting to the identity on the subspace $A \subset X$. Restrictions are the topological analogs of projection operators in other parts of mathematics.*

Definition 1.5. *A homotopy $f_t : X \to X$ that gives a deformation of X onto a subspace A has the property that $f_t|_A = \mathbb{1}$ for all t. In general, a homotopy $f_t : X \to Y$ whose restriction to a subspace $A \subset X$ is independent of t is called a* ***homotopy relative to*** *A, or more concisely, a homotopy* rel *A.*

Remark 1.6. *Immediately, we know that a deformation retraction of X onto A is a homotopy* rel *A from the identity map of X to a retraction of X onto A.*

Definition 1.7. *A map is called a* ***homotopy equivalence*** *if there is a map $g : Y \to X$ such that $f \circ g \simeq \mathbb{1}$ and $g \circ f \simeq \mathbb{1}$. The space X and Y are said to be* ***homotopy equivalent*** *or have the same* ***homotopy type****. The notion is $X \simeq Y$.*

Exercise 1.8. *Homotopy equivalence is an equivalence relation, in contrast with the nonsymmetric notion of deformation retraction.*

Definition 1.9. *A space having the homotopy type of a point is called* ***contractible****. This amounts to requiring that the identity map of the space be* ***nullhomotopic****, that is, homotopic to a constant map.*

Exercise 1.10. *In general, contractibility is slightly weaker than saying the space deformation retracts to a point.*

1.2. **Cell Complexes.**

Proposition 1.11. *An orientable surface M_g of genus g can be constructed from a polygon with $4g$ sides by identifying pairs of edges.*

Definition 1.12. *Construct a space by the following procedure:*

- *Start with a discrete set X^0, whose points are regarded as 0-cells.*
- *Inductively, form the* ***n-skeleton*** *X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D^n_\alpha$ of X^{n-1} with a collection of n disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \amalg_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.*
- *One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n <$ inf, or one can continue indefinitely, setting $X = \cup_n X^n$. In the latter case X is given the weak topology: a set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.*

*A space constructed in this way is called a **cell complex** or **CW complex**.*

*If $X = X^n$ for some n, then X is said to be finite-dimensional, and the smallest such n is the **dimension** of X, the maximum dimension of cells of X.*

Definition 1.13. *Each cell e^n_α in a cell complex X has a **characteristic map** $\Phi_\alpha : D^n_\alpha \to X$ which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α. Namely, we can take Φ_α to be the composition $D^n_\alpha \hookrightarrow X^{n-1} \amalg_\alpha D^n_\alpha \to X^n \hookrightarrow X$ where the middle map is the quotient map defining X^n.*

Definition 1.14. *A 1-dimensional cell complex $X = X^1$ is what is called a **graph** in algebraic topology.*

1.3. **Other Notions.**

Definition 1.15. ***Euler characteristic** for a cell complex with finitely many cells is defined to be the number of even-dimensional cells minus the number of odd-dimensional cells.*

Exercise 1.16. *The sphere S^n has the structure of a cell complex with just two cells, e^0 and e^n, the n cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n/\partial D^n$.*

Definition 1.17. ***Complex projective n-space** $\mathbb{CP}^n$ is the space of complex lines through the origin in $\mathbb{C}^{n+1}$, that is, 1-dimensional vector subspaces of $\mathbb{C}^{n+1}$.*

2. Δ-Complex

Definition 2.1. *A **hyperplane** is the set of solutions of a system of linear equations.*

Definition 2.2. *The **n-simplex** is the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $v_0, \cdots, v_n$ that do not lie in a hyperplane of dimension less than n. The points v_i are the **vertices** of the simplex, and the simplex itself will be denoted $[v_0, \cdots, v_n]$.*

Definition 2.3. *The **standard n-simplex** is*

$$\Delta^n = \{(t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \Sigma_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}$$

whose vertices are the unit vectors along the coordinate axes.

Definition 2.4. *The **ordering** of the vertices of a simplex $[v_0, \cdots, v_n]$ is defined to be orientations of the edges $[v_i, v_j]$ according to increasing subscripts.*

Definition 2.5. *Ordering of the vertices determines a canonical linear homeomorphism from the standard n-simplex Δ^n onto any other n-simplex $[v_0, \cdots, v_n]$, preserving the ordering of vertices, namely, $(t_0, \cdots, t_n) \mapsto \Sigma_i t_i$. The coefficients t_i are the **barycentric coordinates** of the point $\Sigma_i t_i$ in $[v_0, \cdots, v_n]$.*

Definition 2.6. *If we delete one of the $n+1$ vertices of an n-simplex $[v_0, \cdots, v_n]$, then the remaining n vertices span an $(n-1)$-simplex, called a **face** of $[v_0, \cdots, v_n]$.*

We adopt the convention that the vertices of a face will always be ordered according to their order in the larger simplex.

Definition 2.7. *The union of all the faces of Δ^n is the **boundary** of Δ^n, written $\partial\Delta^n$. The **open simplex** $\mathring{\Delta}^n$ is $\Delta^n - \partial\Delta^n$, the interior of Δ^n.*

Remark 2.8. *Δ^0 as a single point is an open simplex.*

Definition 2.9. *A **Δ-complex** structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that:*

(i) *The restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$.*
(ii) *Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.*
(iii) *A set $A \subset X$ is open iff $\sigma^{-1}(A)$ is open in Δ^n for each σ_α.*

Remark 2.10. *The last condition of Definition 2.9 rules out trivialities like regarding all the points of X as individual vertices.*

Remark 2.11. *The consequence of the condition (iii) is that X can be built as a quotient space of a collection of disjoint simplices Δ^n_α, one for each $\sigma_\alpha : \Delta^n \to X$, the quotient space obtained in the way described in condition (ii), starting with a discrete set of vertices.*

Example 2.12. *Torus, projective plane, and Klein bottle can be built in the way as described in Remark 2.11.*

*If one starts with a single 2 simplex and identifies all three edges to a single edge, preserving the orientations given by the ordering of the vertices, this produces a Δ-complex known as the **dunce cap**.*

Proposition 2.13. *Thinking of a Δ-complex X as a quotient space of a collection of disjoint simplices, it is not hard to see that X must be a Hausdorff space. Condition (iii) then implies that each restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is a homeomorphism onto its image, which is thus an open simplex in X.*

These open simplices $\sigma_\alpha(\mathring{\Delta}^n)$ are the cells e^n_α of a CW complex structure on X with the σ_α's as characteristic maps.

And also, the restrictions of each characteristic map $\sigma_\alpha : \Delta^n \to X$ to $(n-1)$-dimensional faces of Δ^n are characteristic maps σ_β for open simplices e^{n-1}_β of X.

3. Simplicial Homology

Definition 3.1. *Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices e^n_α of X. Elements of $\Delta_n(X)$, called **n-chains**, can be written as finite formal* *sums $\Sigma_\alpha n_\alpha e^n_\alpha$ with coefficients $n_\alpha \in \mathbb{Z}$. Equivalently, we could write $\Sigma_\alpha n_\alpha \sigma_\alpha$ where $\sigma_\alpha : \Delta^n \to X$ is the characteristic map of e^n_α, with image the closure of e^n_α.*

Definition 3.2. *We apply the notation $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ for $(n-1)$-dimensional simplices as faces of n-simplex $[v_0, \cdots, v_n]$, where the hat symbol over v_i indicates that this vertex is deleted from the sequence $v_0, \cdots, v_n$.*

Having taken orientations into account, we let the ***boundary*** *of $[v_0, \cdots, v_n]$ be*

$$\Sigma_i (-1)^i [v_0, \cdots, \hat{v}_i, \cdots, v_n],$$

instead of Definition 2.7, so that all the faces of a simplex are coherently oriented.

Definition 3.3. *We define for a general Δ-complex X a* ***boundary homomorphism*** *$\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by specifying its values on basis elements:*

$$\partial_n(\sigma_\alpha) = \Sigma_i (-1)^i \sigma_\alpha|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]}.$$

Lemma 3.4. *The composition $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-1}(X)$ is zero.*

Proof. We have

$$\begin{aligned}
\partial_{n-1}\partial_n(\sigma) &= \Sigma_{j<i}(-1)^j(-1)^i \sigma|_{[v_0, \cdots, \hat{v_j}, \cdots, \hat{v_i}, \cdots, v_n]} + \Sigma_{j>i}(-1)^{j-1}(-1)^i \sigma|_{[v_0, \cdots, \hat{v_i}, \cdots, \hat{v_j}, \cdots, v_n]} \\
&= \Sigma_{j<i}(-1)^{i+j} \sigma|_{[v_0, \cdots, \hat{v_j}, \cdots, \hat{v_i}, \cdots, v_n]} - \Sigma_{j<i}(-1)^{i+j} \sigma|_{[v_0, \cdots, \hat{v_j}, \cdots, \hat{v_i}, \cdots, v_n]} \\
&= 0,
\end{aligned}$$

which proves the lemma. □

Definition 3.5. *The algebraic situation we have now is a sequence of homomorphisms of abelian groups*

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial_n \partial_{n+1} = 0$ for each n. Such a sequence is called a ***chain complex****.*

From $\partial_n \partial_{n+1} = 0$, it follows $\partial_{n+1} \subset \mathrm{Ker}\partial_n$. So we define the n^{th} ***homology group*** *of the chain complex to be the quotient group $H_n = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$.*

Elements of $\mathrm{Ker}\partial_n$ are called ***cycles*** *and* ***elements*** *of $\mathrm{Im}\partial_{n+1}$ are* ***boundaries****. Elements of $H_n = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$ are cosets of $\mathrm{Im}\partial_{n+1}$, called* ***homology classes****.*

Two cycles representing the same homology class are said to be ***homologous****. This means their difference is a boundary.*

Definition 3.6. *For $C_n = \Delta_n(X)$, the homology group $\mathrm{Ker}\partial_\mathrm{n} / \mathrm{Im}\partial_{n+1}$ will be denoted $H^\Delta_n(X)$ and called the n^{th}* ***simplicial homology group*** *of X.*

Example 3.7. *Let X be a Δ-complex. Some examples of the simplicial homology groups of X are as follows:*

- *$X = S^1$, with on vertex v and one edge e. Then*

$$H^\Delta_n(S^1) \approx \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & n \geq 2 \end{cases}$$

- *$X = T$, with one vertex, three edges a, b, c, and two 2-simplices U and L. Then*

$$H_n^\Delta(T) \approx \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & n \geq 3 \end{cases}$$

- *$X = \mathbb{R}\mathrm{P}^2$, with two vertices v and w, three edges a, b, c, and two 2-simplices U and L. Then*

$$H_n^\Delta(\mathbb{R}\mathrm{P}^2) \approx \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

Example 3.8. *We can obtain a Δ-complex structure on S^n by taking two copies of Δ^n and identifying their boundaries via the identity map. Labeling these two n-simplices U and L, then it is obvious that $\mathrm{Ker}\partial_n$ is infinite cyclic generated by $U - L$. Thus $H_n^\Delta(Sn) \approx Z$ for this Δ-complex structure on S^n.*

Remark 3.9. *Many similar examples like other closed orientable and nonorientable surfaces. But the calculations do tend to increase in complexity, especially for higher-dimensional complexes.*

Remark 3.10. *There are very some important questions we need to consider:*

- *Are the groups $H^n_\Delta(X)$ independent of the choice of Δ-complex structure on X?*
- *If two Δ-complexes are homeomorphic, do they have isomorphic homology groups?*
- *If two Δ-complexes are merely homotopy equivalent, do they have isomorphic homology groups?*

Actually, after some theory has been developed, we will show that simplicial and singular homology groups coincide for Δ-complex.

Example 3.11. *Traditionally, simplicial homology is defined for* ***simplicial complexes****, which are the Δ-complexes whose simplices are uniquely determined by their vertices. The only requirement is that each $(k+1)$-element subset of the vertices of an n-simplex in X_n is a k-simplex, in X_k.*

Exercise 3.12. *Every Δ-complex can be subdivided to be a simplicial complex. In particular, every Δ-complex is then homeomorphic to a simplicial complex.*

Remark 3.13. *Compared with simplicial complexes, Δ-complexes have the advantage of simpler computations since every simplices are required.*

4. Singular Homology

Definition 4.1. *A* ***singular n-simplex*** *in a space X is a map $\sigma : \Delta \to X$.*

Remark 4.2. *The word 'singular' is used here to express the idea that σ need not be a nice embedding but can have 'singularities' where its image does not look at all like a simplex. All that is required is that σ be continuous.*

Definition 4.3. *Let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X. Elements of $C_n(X)$, called n-chains, or more precisely singular* ***n-chains****, are finite formal sums $\Sigma_i n_i \sigma_i$ for $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$. A boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by*

$$\partial_n(\sigma) = \Sigma_i (-1)^i \sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]}.$$

The canonical identification of $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ with Δ^{n-1} preserves the ordering of vertices and $\sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]}$ is regarded as a map Δ^{n-1}, that is, a singular $(n-1)$-simplex.

Definition 4.4. *Write ∂_n from $C_n(X)$ to $C_{n-1}(X)$ simply as ∂ and then $\partial_n \partial_{n+1} = 0$ is concisely $\partial^2 = 0$. The Lemma 3.4 holds for singular simplices as well, and so we can define the* ***singular homology group*** *$H_n(X) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$.*

Remark 4.5. *It is evident from the definition that homeomorphic spaces have isomorphic singular homology groups H_n, in contrast with the situation for H_n^Δ. On the other hand, since the groups $C_n(X)$ are so large, the number of singular n-simplices in X usually being uncountable, it is not at all clear that for a Δ-complex X with finitely many simplices, $H_n(X)$ should be finitely generated for all n, or that $Hn(X)$ should be zero for n larger than the dimension of X—two properties that are trivial for $H_n^\Delta(X)$.*

Remark 4.6. *For an arbitrary space X, define the* ***singular complex*** *$S(X)$ to be the Δ-complex with one n-simplex Δ^n_σ for each singular n-simplex $\sigma : \Delta^n \to X$, with Δ^n_σ attached in the obvious way to the $(n-1)$-simplices of $S(X)$ that are the restrictions of σ to the various $(n-1)$-simplices in $\partial\Delta^n$. It is clear from the definitions that $H_n^\Delta(S(X))$ is identical with $H_n(X)$ for all n, and in this sense the singular homology group $H_n(X)$ is a special case of a simplicial homology group. One can regard $S(X)$ as a Δ-complex model for X, although it is usually an extremely large object compared to X.*

Remark 4.7. *Cycles in singular homology are defined algebraically, but they can be given a somewhat more geometrically interpretation in terms of of maps from finite Δ-complexes, which involves the knowledge of manifold and a sort of homology theory built from manifolds, called* ***bordism*** *(see page 106-107 of Allen Hatcher's book).*

Proposition 4.8. *Corresponding to the decomposition of a space X into its path-components X_α there is an isomorphism of $H_n(X)$ with the direct sum $\oplus_\alpha H_n(X_\alpha)$.*

Proof. Since a singular simplex always has path-connected image, $C_n(X)$ splits as the direct sum of its subgroups $C_n(X_\alpha)$. The boundary maps ∂_n preserve this direct sum decomposition, taking $C_n(X_\alpha)$ to $C_{n-1}(X)$, so $\mathrm{Ker}\partial_n$ and $\mathrm{Im}\partial_{n+1}$ split similarly as direct sums, hence the homology groups also split, $H_n(X) \approx \oplus_\alpha H_n(X_\alpha)$. $\square$

Proposition 4.9. *If X is nonempty and path connected, then $H_0(X) \approx \mathbb{Z}$. Hence for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path-connected component of X.*

Proof. By definition, $H_0(X) = C_0(X)/\mathrm{Im}(\partial_1)$ since $\partial_0 = 0$. Define a homomorphism $\varepsilon : C_0(X) \to \mathbb{Z}$ by $\varepsilon(\Sigma_i n_i \sigma_i) = \Sigma_i n_i$. This is obviously surjective if X is nonempty.

The claim is that $\mathrm{Ker}(\varepsilon) = \mathrm{Im}(\partial_1)$ if X path-connected, and hence ε induces an isomorphism $H_0(X) \approx \mathbb{Z}$.

Observe that for a singular 1-simplex $\sigma : \Delta^1 \to X$ we have $\varepsilon \circ \partial_1(\sigma) = \varepsilon(\sigma|_{[v_1]} - \sigma|_{[v_0]}) = 1 - 1 = 0$, so $\mathrm{Im}(\partial_1) \subset \mathrm{Ker}(\varepsilon)$. Suppose that $\varepsilon(\Sigma_i n_i \sigma_i) = 0$, so $\Sigma_i n_i = 0$. Then σ_i's are singular 0-simplices, which are simply points of X. Choose a path $\tau_i : I \to X$ from a base point x_0 to $\sigma_i(v_0)$ and let σ_0 be the singular 0-simplex with image x_0. We can view τ_i as a singular 1-simplex, a map $\tau_i : [v_0, v_1] \to X$, and then we have $\partial \circ \tau_i = \sigma_i - \sigma_0$. Hence $\square$

Proposition 4.10. *If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \approx \mathbb{Z}$.*

Proof. Since X is a point, there is a unique singular n-simplex σ_n for each n, and $\partial_n(\sigma_n) = \Sigma_i(-1)^i \sigma_{n-1}$, a sum of $n+1$ terms, which is therefore 0 for n odd and σ_{n-1} for n even when $n \neq 0$. Then we have the chain complex

$$\cdots \to \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

with boundary maps alternatively isomorphisms and trivial maps, except at the last $\mathbb{Z}$. The homology groups of this complex are trivial except for $H_0 = \mathbb{Z}$. $\square$

It is often very convenient to have a slightly modified version of homology for which a point has trivial homology groups in all dimensions, including zero. This motivates the following definition.

Definition 4.11. *The* ***reduced homology group*** $\tilde{H}_n(X)$ *is defined to be the homology groups of the augmented chain complex*

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\varepsilon(\Sigma_i n_i \sigma_i) = \Sigma_i n_i$ as in the proof of Proposition 4.10.

Remark 4.12. *In Definition 4.11, we had better require X to be nonempty, to avoid having a nontrivial homology group in dimension -1.*

Remark 4.13. *Since $\varepsilon\partial_1 = 0$, ε vanishes on $\mathrm{Im}(\partial_1)$ and hence induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\tilde{H}_0(X)$, so $H_0(X) \approx \tilde{H}_0(X) \oplus \mathbb{Z}$. Obviously $H_n(X) \approx \tilde{H}_n(X)$ for $n > 0$.*

Remark 4.14. *It can be shown that $H_1(X)$ is the abelianization of $\pi_1(X)$ whenever X is path-connected (see page 166-167 of Allen Hatcher's book).*

Remark 4.13 is derived from the lemmas in algebra below. We list them as an ending of this section.

Lemma 4.15. *If $f : G \to H$ is a homomorphism of groups and N is a normal subgroup of G contained in the kernel of f, then there is a unique homomorphism $\bar{f} : G/N \to H$ such that $\bar{f}(aN) = f(a)$ for all $a \in G$.* $\mathrm{Im}(f) = \mathrm{Im}(\bar{f})$ *and* $\mathrm{Ker}(\bar{f}) = \mathrm{Ker}(f)/N$. *$\bar{f}$ is an isomorphism iff f is an epimorphism and $N = \mathrm{Ker}(f)$.*

Proof. If $b \in aN$, then $b = an$, $n \in N$, and $f(b) = f(an) = f(a)f(n) = f(a)e = f(a)$, since $N < \mathrm{Ker}(f)$. Therefore, f has the same effect on every element on the coset aN and the map $\bar{f} : (aNbN) = \bar{f}(abN) = f(ab) = f(a)f(b) = \bar{f}(aN)\bar{f}(bN)$, $\bar{f}$ is a homomorphism. Clearly $\mathrm{Im}(\bar{f}) = \mathrm{Im}(f)$ and

$$aN \in \mathrm{Ker}(\bar{f}) \Leftrightarrow f(a) = e \Leftrightarrow a \in \mathrm{Ker}(f),$$

whence $\mathrm{Ker}(\bar{f}) = \{aN : a \in \mathrm{Ker}(f)\} = \mathrm{Ker}(f)/N$. $\bar{f}$ is unique since it is completely determined by f. Finally it is clear that $\bar{f}$ is an epimorphism iff f is. $\bar{f}$ is a monomorphism iff $\mathrm{Ker}(\bar{f}) = \mathrm{Ker}(f)/N$ is the trivial subgroup of G/N, which occurs iff $\mathrm{Ker}(f) = N$. □

Remark 4.16. *The essential part of the conclusion may be rephrased: there exists a unique homomorphism $\bar{f} : G/N \to H$ such that the diagram*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"] & H \\ G/N \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

is commutative, where π is the canonical projection from G to G/N.

Lemma 4.17. ***First Isomorphism Theorem***
If $f : G \to H$ is a homomorphism of groups, then f induces an isomorphism $G/\mathrm{Ker}(f) \approx \mathrm{Im}(f)$.

Proof. $f : G \to \mathrm{Im}(f)$ is an epimorphism. Apply Lemma 4.15 with $N = \mathrm{Ker}(f)$. □

Lemma 4.18. *Let $A \xrightarrow{f'} A'$ be a surjective homomorphism of abelian groups, and assume that A' is free. Let B be the kernel of f. Then there exists a subgroup C of A such that the restriction of f to C induces an isomorphism of C with A', and such that $A = B \oplus C$.*

Proof. Let $\{x'_i\}_{i \in I}$ be a basis of A', and for each $i \in I$, let x_i be an element of A such that $f(x_i) = x'_i$. Let C be the subgroup of A generated by all elements x_i $(i \in I)$. If we have a relation $\Sigma_{i \in I} n_i x_i = 0$ with integers n_i, almost all of which are equal to 0, then applying f yields

$$0 = \Sigma_{i \in I} n_i f(x_i) = \Sigma_{i \in I} n_i x'_i,$$

whence all $n_i = 0$. Similarly, one sees that if $z \in C$ and $f(z) = 0$ then $z = 0$. Hence $B \cap C = 0$. Let $x \in A$. Since $f(x) \in A'$ there exists integers n_i $(i \in I)$, such that

$$f(x) = \Sigma_{i \in I} n_i x'_i.$$

Applying f to $x - \Sigma_{i \in I} n_i x_i = b \in B$. From this we see that $x \in B + C$, and hence finally that $A = B \oplus C$ is a direct sum, as contended. □

5. Homotopy Invariance

Definition 5.1. *For a map $f : X \to Y$, an induced homomorphism $f_\sharp : C_n(X) \to C_n(Y)$ is defined by composing each singular n-simplex $\sigma : \Delta^n \to X$ with f to get a singular n-simplex $f_\sharp(\sigma) = f\sigma : \Delta^n \to Y$, then extending $f_\sharp$ via $f_\sharp(\Sigma_i n_i \sigma_i) = \Sigma_i n_i f_\sharp(\sigma_i) = \sigma_i n_i f \sigma_i$.*

Remark 5.2. *The maps $f_\sharp : C_n(X) \to C_n(Y)$ satisfy $f_\sharp \partial = \partial f_\sharp$ since*

$$f_\sharp \partial(\sigma) = f_\sharp(\Sigma_i (-1)^i \sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]}) = \Sigma_i (-1)^i f\sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]} = \partial f_\sharp(\sigma).$$

Thus we have a diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \cdots \\
 & & \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \\
\cdots & \longrightarrow & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \xrightarrow{\partial} & \cdots
\end{array}$$

such that in each square the composition $f_\sharp \partial$ equals the composition $\partial f_\sharp$.

Definition 5.3. *A diagram of maps with the property that any two compositions of maps starting at one point in the diagram and ending at another are equal is called a* ***commutative digram****.*

Remark 5.4. *In the present case, commutativity of the diagram is equivalent to the commutativity relation $f_\sharp \partial = \partial f_\sharp$, but commutative diagrams can contain commutative triangles, pentagons, etc., as well as commutative squares.*

Remark 5.5. *The fact that the maps $f_\sharp : C_n(X) \to C_n(Y)$ satisfy $f_\sharp \partial = \partial f_\sharp$ is also expressed by saying that the $f_\sharp$'s define a* ***chain map*** *from the singular chain complex of X to that of Y.*

Lemma 5.6. *If $f : G \to H$ is a homomorphism of groups, $N \triangleleft G$, $M \triangleleft H$, and $f(N) < M$, then f induces a homomorphism $\bar{f} : G/N \to H/M$, given by $aN \mapsto f(a)M$.*

$\bar{f}$ is an isomorphism iff $\mathrm{Im}(f) \vee M = H$ and $f^{-1}(M) \subset N$. In particular, if f is an epimorphism such that $f(N) = M$ and $\mathrm{Ker}(f) \subset N$, then $\bar{f}$ is an isomorphism.

Proof. Consider the composition $G \xrightarrow{f} H \xrightarrow{\pi} H/M$ and verify that $N \subset f^{-1}(M) = \mathrm{Ker}(\pi f)$. By Lemma 4.15 (applied to πf), the map $G/N \to H/M$ given by $aN \to \pi f(a) = f(a)M$ is a homomorphism that is an isomorphism iff πf is an epimorphism and $N = \mathrm{Ker}(\pi f)$. But the latter conditions hold iff $\mathrm{Im}(f) \vee M = H$ and $f^{-1}(M) \subset N$. If f is an epimorphism, then $H = \mathrm{Im}(f) = \mathrm{Im}(f) \vee M$. If $f(N) = M$ and $\mathrm{Ker}(f) \subset N$, then $f^{-1}(M) \subset N$, whence $\bar{f}$ is an isomorphism. $\square$

Remark 5.7. *The relation $f_\sharp \partial = \partial f_\sharp$ implies that $f_\sharp$ takes cycles to cycles since $\partial\alpha = 0$ implies $\partial(f_\sharp \alpha) = f_\sharp(\partial\alpha) = 0$. Also, $f_\sharp$ takes boundaries to boundaries since $f_\sharp(\partial\beta) = \partial(f_\sharp\beta)$. By Lemma 5.6, $f_\sharp$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$.*

An algebraic statement of what has just been proved is:

Proposition 5.8. *A chain map between chain complexes induces homomorphisms between the boundary groups of the two complexes.*

Two basic properties of induced homomorphisms which are important in spite of being rather trivial are:

(i) $(fg)_* = f_* g_*$ *for a composed mapping* $X \xrightarrow{g} Y \xrightarrow{f} Z$. *This follows from associativity of compositions* $\Delta^n \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z$.

(ii) $\mathbb{1}_* = \mathbb{1}$ *where* $\mathbb{1}$ *denotes the identity map of a space or a group.*

Theorem 5.9. *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$.*

Proof. Let $\Delta^n \times \{0\} = [v_0, \cdots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \cdots, w_n]$ where v_i and w_i have the same image under the projection $\Delta^n \times I \to \Delta^n$.

The n-simplex $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$ is the graph of the linear function $\varphi_i : \Delta^n \to I$ defined in barycentric coordinates by $\varphi_i(t_0, \cdots, t_n) = t_{i+1} + \cdots + t_n$ since the vertices of this simplex $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$ are on the graph of φ_i and the simplex projects homeomorphically onto Δ^n under the projection $\Delta^n \times I \to \Delta^n$. The graph of φ_i lies below the graph of φ_{i-1} since $\varphi_i \leq \varphi_{i-1}$, and the region between these two graphs is the simplex $[v_0, \cdots, v_i, w_i, \cdots, w_n]$, a true $(n+1)$-simplex since w_i is not on the graph of φ_i and hence is not in the n-simplex $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$.

From the string of inequalities $0 = \varphi_n \leq \varphi_{n-1} \leq \cdots \leq \varphi_0 \leq \varphi_{-1} = 1$, we deduce that $\Delta^n \times I$ is the union of the $(n+1)$-simplices $[v_0, \cdots, v_i, w_i, \cdots, w_n]$, each intersecting the next in a n-simplex face.

Given a homotopy $F : X \times I \to Y$ from f to g, we can define **prism operators** $P : C_n(X) \to C_{n+1}(Y)$ by

$$P(\sigma) = \Sigma_i (-1)^i F \circ (\sigma \times \mathbb{1})|_{[v_0, \cdots, v_i, w_i, \cdots, w_n]}$$

for $\sigma : \Delta^n \to X$, where $F \circ (\sigma \times \mathbb{1})$ is the composition $\Delta^n \times I \to X \times I \to Y$. There prism operators satisfy the basic relation

$$\partial P = g_\sharp - f_\sharp - P\partial.$$

Geometrically, the left side of the equation represents the boundary of the prism. To prove the relation we calculate

$$\begin{aligned}\partial P(\sigma) &= \Sigma_{j\leq i}(-1)^i(-1)^j F\circ(\sigma\times\mathbb{1})|_{[v_0,\cdots,\hat{v}_j,\cdots,v_i,w_i,\cdots,w_n]}\\ &\quad+\Sigma_{j\geq i}(-1)^i(-1)^{j+1}F\circ(\sigma\times\mathbb{1})|_{[v_0,\cdots,v_i,w_i,\cdots,\hat{w}_j,\cdots,w_n]}.\end{aligned}$$

The terms with $i=j$ in the two sums cancel except for $F\circ(\sigma\times\mathbb{1})|_{[\hat{v}_0,w_0,\cdots,w_n]}$, which is $g\circ\sigma=g_\sharp(\sigma)$, and $-F\circ(\sigma\times\mathbb{1})|_{[v_0,\cdots,v_n,\hat{w}_n]}$, which is $-f\circ\sigma=f_\sharp(\sigma)$. The terms with $i\neq j$ are exactly $-P\partial(\sigma)$ since

$$\begin{aligned}P(\partial\sigma) &= \Sigma_{i<j}(-1)^i(-1)^j F\circ(\sigma\times\mathbb{1})|_{[v_0,\cdots,v_i,w_i,\cdots,\hat{w}_j,\cdots,w_n]}\\ &\quad+\Sigma_{j\geq i}(-1)^{i-1}(-1)^j F\circ(\sigma\times\mathbb{1})|_{[v_0,\cdots,\hat{v}_j,\cdots,v_i,w_i,\cdots,w_n]}.\end{aligned}$$

If $\alpha\in C_n(X)$ is a cycle, then we have $g_\sharp(\alpha)-f_\sharp(\alpha)=\partial P(\alpha)+P\partial(\alpha)=\partial P(\alpha)$ since $\partial\alpha=0$. Thus $g_\sharp(\alpha)-f_\sharp(\alpha)$ is a boundary, so $g_\sharp(\alpha)$ and $f_\sharp(\alpha)$ determine the same homology class, which means that g_* equals f_* on the homology class of α. $\square$

Remark 5.10. *The relationship $\partial P+P\partial=g_\sharp-f_\sharp$ is expressed by saying P is a **chain homotopy** between the chain maps $f_\sharp$ and $g_\sharp$.*

We have just shown:

Proposition 5.11. *Chain-homotopic chain maps induce the same homomorphism on homology.*

Remark 5.12. *There are also induced homomorphisms $f_*:\tilde{H}_n(X)\to\tilde{H}_n(Y)$ for reduced homology groups since $f_\sharp\varepsilon=\varepsilon f_\sharp$. The properties of induced homomorphisms we proved above hold equally well in the setting of reduced homology, with the same proofs.*

By Proposition 5.8 and Theorem 5.9, it immediately follows that:

Corollary 5.13. *The maps $f_*:H_n(X)\to H_n(Y)$ induced by a homotopy equivalence $f:X\to Y$ are isomorphisms for all n.*

Example 5.14. *If X is contractible (see Definition 1.9), then $\tilde{H}_n(X)=0$ for all n.*

6. Exact Sequence and Relative Homology Group

Remark 6.1. *Every space X can be embedded as a subspace of a space with trivial homology groups, namely the cone $CX=(X\times I)/(X\times\{0\})$, which is contractible. Hence it turns out that $H_n(X)/H_n(A)$ has little hope to be isomorphic to $H_n(X/A)$ in general, where A is regarded as a subspace of X. Otherwise, homology theory would then collapse totally.*

Remark 6.2. *The actual relation is that it involves the groups $H_n(X)$, $H_n(A)$, and $H_n(X/A)$ for all values of n simultaneously. It has the side effect of sometimes allowing higher-dimensional homology groups to be computed in terms of lower-dimensional groups, which may already be known by induction of examples.*

Definition 6.3. *A sequence of homomorphisms*

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

*is said to be **exact** if* $\mathrm{Ker}(\alpha_n) = \mathrm{Im}(\alpha_{n+1})$ *for each* n.

The inclusions $\mathrm{Im}(\alpha_{n+1}) \subset \mathrm{Ker}(\alpha_n)$ *are equivalent to* $\alpha_n \alpha_{n+1} = 0$, *so the sequence is a chain complex (see Definition 3.5), and the opposite inclusions* $\mathrm{Ker}(\alpha_n) \subset \mathrm{Im}(\alpha_{n+1})$ *say that the homology groups of this chain complex are trivial.*

Proposition 6.4. *A number of algebraic concepts can be expressed in terms of exact sequences:*

(i) $0 \to A \xrightarrow{\alpha} B$ *is exact iff* $\mathrm{Ker}(\alpha) = 0$, *i.e.,* α *is injective;*
(ii) $A \xrightarrow{\alpha} B \to 0$ *is exact iff* $\mathrm{Im}(\alpha) = B$, *i.e.,* α *is surjective;*
(iii) $0 \to A \xrightarrow{\alpha} B \to 0$ *is exact iff* α *is isomorphism, by (i) and (ii);*
(iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *iff* α *is injective and* β *is surjective, and* $\mathrm{Ker}(\beta) = \mathrm{Im}(\alpha)$, *so* B *induces an isomorphism* $C = B/\mathrm{Im}(\alpha)$. *This can be written* $C = B/A$ *if we think of* α *as an inclusion of* A *as a subgroup of* B.

Definition 6.5. *An exact sequence* $0 \to A \to B \to C \to 0$ *as shown in (iv) of Proposition 6.4 is called a **short exact sequence**.*

Exact sequences provide the right tool to relate the homology groups of a space, a subspace, and the associated quotient space:

Theorem 6.6. *If* X *is a space and* A *is nonempty closed subspace that is a deformation retract of some neighborhood in* X, *then there is an exact sequence*

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \to \cdots$$
$$\cdots \to \tilde{H}_0(X/A) \to 0$$

where i *is the inclusion* $A \hookrightarrow X$ *and* j *is the quotient map* $X \to X/A$.

Remark 6.7. *The map* ∂ *will be constructed in the course of the proof. The idea is that an element* $x \in \tilde{H}_n(X/A)$ *can be represented by a chain* α *in* X *with* $\partial\alpha$ *a cycle in* A *whose homology class is* $\partial x \in \tilde{H}_{n-1}(A)$.

Remark 6.8. *Pairs of spaces* (X, A) *satisfying the hypothesis of the theorem will be called **good pairs**. For example, if* X *is a CW complex and* A *is a nonempty subcomplex, then* (X, A) *is a good pair (see page 523 of Allen Hatcher's book).*

The following two lemmas are necessary for the two corollaries of Theorem 6.6.

Lemma 6.9. *Let* X *be a topological space with equivalent relation* R, *and let* $f : X \to Y$ *be a continuous map with properties*

- $f(a) = f(b)$ *iff* $(a, b) \in R$;
- f *is surjective;*

- *V is open in Y iff $f^{-1}(V)$ is open in X.*

Then $X/R \approx Y$.

Lemma 6.10. *$D^n/S^{n-1} \approx S^n$.*

Proof. Define a mapping $f : D^n \to S^n$ as $y = (y_0, y_1, \cdots, y_n) = f(x)$ for each $x = (x_1, \cdots, x_n) \in D^n$, where $y_0 = 2\|x\| - 1$ and $y_k = tx_k$ for all $k \in [1, n] \cap \mathbb{Z}$.

Now we are going to discuss t appeared above. If $\|x\| \neq 0$, then t is defined to be $\sqrt{4/\|x\| - 4}$. Note that $\|x\| \leq 1$ if $x \in D^n$, so t is well-defined when $\|x\| \neq 0$. Send $\|x\| \to 0^+$, then $x_k \to 0$ and so

$$0 \leq |tx_k|^2 \leq \frac{4x_k^2}{\|x\|} - 4x_k^2 \leq \frac{4\|x\|}{\|x\|}|x_k| - 4x_k^2 = 4|x_k| - 4x_k^2 \to 0,$$

which implies $y_k \to 0$. Thus we can define $y_k = \lim_{\|x\| \to 0^+} tx_k = 0$ for each $k \in [1, n] \cap \mathbb{Z}$ when $\|x\| = 0$. This would make f continuous.

It's obvious that $\|y\| = 1$ when $\|x\| = 0$. If $\|x\| \neq 0$, then

$$\|y\|^2 = \sum_{k=0}^{n} y_k^2 = 4\|x\|^2 - 4\|x\| + 1 + \left(\frac{4}{\|x\|} - 4\right)\|x\|^2 = 1.$$

Hence, $f(D^n) \subset S^n$ and f is a well-defined mapping. Also, if $x \in S^{n-1}$, $f(x) = (1, 0, \cdots, 0)$. Note that f is bijective from $D^n \setminus S^{n-1}$ to $S^n \setminus \{(1, 0, \cdots, 0)\}$, then f is a quotient map. It follows that $D^n/S^{n-1} \approx S^n$ by Lemma 6.9. ☐

Remark 6.11. *Basically, the homeomorphism don't depend on the space that the unit disk and sphere are embedded in. We omit the details here.*

Corollary 6.12. *$\tilde{H}_n(S^n) \approx \mathbb{Z}$ and $\tilde{H}_i(S^n) \approx 0$ for $i \neq n$.*

Proof. For $n > 0$, take $(X, A) = (D^n, S^{n-1})$ so $X/A = S^n$. The terms $\tilde{H}_i(D^n)$ in the long exact sequence for this pair are zero since D^n is contractible. Exactness of the sequence then implies that the maps $\tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1})$ are isomorphisms for $i > 0$ and that $\tilde{H}_0(S^n) = 0$. The result now follows by induction on n, starting with the case of S^0 where the result holds by Proposition 4.8 and 4.10. ☐

Remark 6.13. *Note that $S^0 \approx \{[v_0], [v_1]\}$, so $\tilde{H}_0(S^0) = \mathbb{Z}$ and $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$.*

Corollary 6.14. ***Brouwer Fixed-Point Theorem***

∂D^n is not a retract of D^n. Hence every map $f : D^n \to D^n$ has a fixed-point.

Proof. Suppose on the contrary that $f(x) \neq x$ for all $x \in D^n$. Then we can define a map $r : D^n \to S^{n-1}$ by letting $r(x)$ be the point of S^{n-1} where the ray in $\mathbb{R}^n$ starting at $f(x)$ and passing through x leaves D^n. Continuity of r is clear since small perturbation of x produce small perturbation of $f(x)$, hence also small perturbations of the ray through theses two points. The crucial property of r, besides continuity, is that $r(x) = x$ if $x \in S^{n-1}$. Thus r is a retraction of D^n onto S^{n-1}.

If $r : D^n \to \partial D^n$ is a retraction, then $ri = \mathbb{1}$ for $i : \partial D^n \to D^n$ the inclusion map. The composition $\tilde{H}_{n-1}(\partial D^n) \xrightarrow{i_*} \tilde{H}_{n-1}(D^n) \xrightarrow{r_*} \tilde{H}_{n-1}(\partial D^n)$ is then the identity map on $\tilde{H}_{n-1}(\partial D^n) \approx \mathbb{Z}$. But i_* and r_* are both zero since $\tilde{H}_{n-1}(D^n) = 0$, and we have a contradiction. $\square$

The derivation of the exact sequence of homology groups for a good pair (X, A) will be rather a long story. We are going to derive a more general exact sequence which holds for arbitrary pairs (X, A).

Remark 6.15. *It sometimes happens that by ignoring a certain amount of data or structure one obtains a simpler, more flexible theory which, almost paradoxically, can give results not readily obtainable in the original setting. A familiar instance of this is arithmetic* mod n*, where one ignores multiples of* n*.*

Definition 6.16. *Given a space* X *and a subspace* $A \subset X$*, let* $C_n(X, A)$ *be the quotient group* $C_n(X)/C_n(A)$*. Thus chains in* A *are trivial in* $C_n(X, A)$*. Since the boundary map* $\partial : C_n(X) \to C_{n-1}(X)$ *takes* $C_n(A)$ *to* $C_{n-1}(A)$*, it induces a quotient map* $\partial : C_n(X, A) \to C_{n-1}(X, A)$*. Letting* n *vary, we have a sequence of boundary maps*

$$\cdots \to C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \to \cdots$$

The relation $\partial^2 = 0$ *holds for theses boundary map since it holds before passing to quotient groups. So we have a chain complex, and the homology groups* $\mathrm{Ker}(\partial)/\mathrm{Im}(\partial)$ *of the chain complex are by definition the* ***relative homology groups*** $H_n(X/A)$*.*

Proposition 6.17. *By considering the definition of relative boundary map we see:*

- *Elements of* $H_n(X, A)$ *are represented by* ***relative cycles****:* n*-chains* $\alpha \in C_n(X)$ *such that* $\partial\alpha \in C_{n-1}(A)$*.*
- *A relative cycle* α *is trivial in* $H_n(X, A)$ *iff it is a* ***relative boundary****:* $\alpha = \partial\beta + \gamma$ *for some* $\beta \in C_{n+1}(X)$ *and* $\gamma \in C_n(A)$*.*

Remark 6.18. *These properties make precise the intuitive idea that* $H_n(X, A)$ *is "homology of* X *modulo* A*".*

Remark 6.19. *The quotient group* $C_n(X)/C_n(A)$ *could be viewed as a subgroup of* $C_n(X)$*, the subgroup with the basis the singular* n*-simplices* $\sigma : \Delta^n \to X$ *whose images are not contained in* A*. However, the boundary map does not take this subgroup of* $C_n(X)$ *to the corresponding subgroup of* $C_{n-1}(X)$*, so it usually better to regard* $C_n(X, A)$ *as a quotient rather than a subgroup of* $C_n(X)$*.*

The goal now is to show that the relative homology groups $H_n(X, A)$ for any pair (X, A) fit into a long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$
$$\cdots \to H_0(X, A) \to 0.$$

This will be entirely a matter of algebra. To start the process, consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_n(A) & \xrightarrow{i} & C_n(X) & \xrightarrow{j} & C_n(X,A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \longrightarrow & C_{n-1}(A) & \xrightarrow{i} & C_{n-1}(X) & \xrightarrow{j} & C_{n-1}(X,A) & \longrightarrow & 0
\end{array}$$

where i is inclusion and j is the quotient map. The diagram is commutative by the definition of the boundary maps. Letting n vary, and drawing these short exact sequences vertically rather than horizontally, we have a large commutative diagram of the form shown below, where the columns are exact (by Proposition 6.4) and the rows are chain complexes which we denote A, B, and C.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \\
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Such a diagram is called a **short exact sequence of chain complexes**.

Remark 6.20. *We will show that when we pass to homology groups, this short exact sequence of chain complexes stretches out into a long exact sequence of homology groups, that is*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

where $H_n(A)$ denotes the homology group $\mathrm{Ker}(\partial)/\mathrm{Im}(\partial)$ *at A_n in the chain complex A, and $H_n(B)$ and $H_n(C)$ are defined similarly.*

The commutativity of the squares in the short exact sequence of chain complexes means that i and j are chain maps. They therefore induce maps i_* and j_* on homology.

To define the boundary map $\partial : H_n(C) \to H_{n-1}(A)$, let $c \in C_n$ be a cycle. Since j is onto, $c = j(b)$ for some $b \in B_n$. The element $\partial b \in B_{n-1}$ is in $\mathrm{Ker}(j)$ since $j(\partial b) = \partial j(b) = \partial c = 0$. So $\partial b = i(a)$ for some $a \in A_{n-1}$ since $\mathrm{Ker}(j) = \mathrm{Im}(i)$. Note that $\partial a = 0$ since $i(\partial a) = \partial i(a) = \partial\partial b = 0$ and i is injective.

$$\begin{tikzcd}
 & & a \arrow[dd, mapsto] & \\
 & & & A_{n-1} \arrow[dd, "i"] \\
b \arrow[rr, mapsto] \arrow[dd, mapsto] & & \partial b & \\
 & B_n \arrow[rr, "\partial"'] \arrow[dd, "j"] & & B_{n-1} \\
c & & & \\
 & C_n & &
\end{tikzcd}$$

We define $\partial : H_n(C) \to H_{n-1}(A)$ be sending the homology class of c to the homology class of a, $\partial[c] = [a]$. This well-defined since:

- The element a is uniquely determined by ∂b since i is injective;
- A different choice b' for b would have $j(b') = j(b)$, so $b' - b$ is in $\mathrm{Ker}(j) = \mathrm{Im}(i)$. Thus $b' - b = i(a')$ for some a', hence $b' = b + i(a')$. The effect of replaceing b by $b + i(a')$ is to change a to the homologous element $a + \partial a'$ since $i(a + \partial a') = i(a) + i(\partial a') = \partial b + \partial i(a') = \partial(b + i(a'))$.
- A different choice of c within its homology class would have the form $c + \partial c'$. Since $c' = j(b')$ for some b', we then have $c + \partial c' = c + \partial j(b') = c + j(\partial b') = j(b + \partial b')$, so b is replaced by $b + \partial b'$, which leaves ∂b and therefore also a unchanged.

The map $\partial : H_n(C) \to H_{n-1}(A)$ is a homomorphism since if $\partial[c_1] = [a_1]$ and $\partial[c_2] = [a_2]$ via elements b_1 and b_2 as above, then $j(b_1 + b_2) = j(b_1) + j(b_2) = c_1 + c_2$ and $i(a_1 + a_2) = i(a_1) + i(a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1 + b_2)$, so $\partial([c_1] + [c_2]) = [a_1] + [a_2]$.

Theorem 6.21. *The sequence of homology groups*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

is exact.

Proof. There are six things to verify:

- $\mathrm{Im}(i_*) \subset \mathrm{Ker}(j_*)$. This immediate since $ji = 0$ implies $j_* i_* = 0$.
- $\mathrm{Im}(j_*) \subset \mathrm{Ker}(\partial)$. We have $\partial j_* = 0$ since in this case $\partial b = 0$ in the definition of ∂.
- $\mathrm{Im}(\partial) \subset \mathrm{Ker}(i_*)$. Here $i_*\partial = 0$ since $i_*\partial$ takes $[c]$ to $[\partial b] = 0$.
- $\mathrm{Ker}(j_*) \subset \mathrm{Im}(i_*)$. A homology class in $\mathrm{Ker}(j_*)$ is represented by a cycle $b \in B_n$ with $j(b)$ a boundary, so $j(b) = \partial c'$ for some $c' \in C_{n+1}$. Since j is surjective, $c' = j(b')$ for some $b' \in B_{n+1}$. We have $j(b - \partial b') = j(b) - j(\partial b') = j(b) - \partial j(b') = 0$ since $\partial j(b') = \partial c' = j(b)$. So $b - \partial b' = i(a)$ for some $a \in A_n$ since $\mathrm{Ker}(j) = \mathrm{Im}(i)$. This a is a cycle since $i(\partial a) = \partial i(a) = \partial(b - \partial b') = \partial b = 0$ and i is injective. Thus $i_*[a] = [b - \partial b'] = [b]$, showing that i_* maps onto $\mathrm{Ker}(j_*)$.

- $\operatorname{Ker}(\partial) \subset \operatorname{Im}(j_*)$. In the notation used in the definition of ∂, if c represents a homology class in $\operatorname{Ker}\partial$, then $a = \partial a'$ for some $a' \in A_n$. The element $b - i(a')$ is a cycle since $\partial(b - i(a')) = \partial b - \partial i(a') = \partial b - i(\partial a') = \partial b - i(a) = 0$. And $j(b - i(a')) = j(b) - ji(a') = j(b) = c$, so j_* maps $[b - i(a')]$ to $[c]$.
- $\operatorname{Ker}(i_*) \subset \operatorname{Im}(\partial)$. Given a cycle $a \in A_{n-1}$ such that $i(a) = \partial b$ for some $b \in B_n$, then $j(b)$ is a cycle since $\partial j(b) = j(\partial b) = ji(a) = 0$, and ∂ takes $[j(b)]$ to $[a]$. $\square$

Remark 6.22. *This theorem represents the beginning of the subject of homological algebra. The method of proof is sometimes called **diagram chasing**.*

Remark 6.23. *The preceding algebraic theorem yields a long exact sequence groups:*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots$$
$$\cdots \to H_0(X,A) \to 0.$$

The boundary map $\partial : H_n(X,A) \to H_{n-1}(A)$ has a very simple description: if a class $[\alpha] \in H_n(X,A)$ is represented by a relative cycle α, then $\partial[\alpha]$ is in the class of the cycle $\partial\alpha$ in $H_{n-1}(A)$. This is immediate from the algebraic definition of the boundary homomorphism in the long exact sequence of homology groups associated to a short exact sequence of chain complexes.

Lemma 6.24. *For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective.*

Lemma 6.24 is rather trivial, and immediately we have the following remark.

Remark 6.25. *This long exact sequence makes precise the idea that the groups $H_n(X,A)$ measure the difference between the groups $H_n(X)$ and $H_n(A)$. In particular, exactness implies that if $H_n(X,A) = 0$ for all n, then the inclusion $A \hookrightarrow X$ induces isomorphisms $H_n(A) \approx H_n(X)$ for all n, by the (iii) of Remark 6.4 following Definition 6.3. The converse is also true by Lemma 6.24.*

Remark 6.26. *There is a completely analogous long exact sequence of reduced homology groups for a pair (X,A) with $A \neq \varnothing$. This comes from applying the preceding algebraic machinery to the short exact sequence of chain complexes formed by the short exact sequences $0 \to C_n(X) \to C_n(X) \to C_n(X,A) \to 0$ in nonnegative dimensions, augmented by the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\mathbb{1}} \mathbb{Z} \to 0 \to 0$ in dimension -1. In particular this means that $\tilde{H}_n(X,A) \approx H_n(X,A)$ for all n, when $A \neq \varnothing$.*

Example 6.27. *In the long exact sequence of reduced homology groups for the pair $(D^n, \partial D^n)$, the maps $H_i(D^n, \partial D^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1})$ are isomorphisms for all $i > 0$ since the remaining terms $\tilde{H}_i(D^n)$ are zero for all i. Thus we obtain the calculation*

$$H_i(D^n, \partial D^n) \approx \begin{cases} \mathbb{Z} & i = n \\ 0 & \textit{otherwise} \end{cases}$$

Example 6.28. *Applying the long exact sequence of reduced homology groups to a pair $(X, \{x_0\})$, simply denoted (X, x_0), with $x_0 \in X$ yields isomorphisms $H_n(X, x_0) \approx \tilde{H}_n(X)$ for all n since $\tilde{H}_n(x_0) = 0$ for all n.*

There are induced homomorphisms for relative homology just as there are in the nonrelative, or "absolute" case. A map $f : X \to Y$ with $f(A) \subset B$, or more concisely $f : (X, A) \to (Y, B)$, induces homomorphisms $f_\sharp : C_n(X, A) \to C_n(Y, B)$ since the chain map takes $f_\sharp : C_n(X) \to C_n(Y)$ takes $C_n(A)$ to $C_n(B)$, so we get a well-defined map on quotients, $f_\sharp : C_n(X, A) \to C_n(Y, B)$. The relation $f_\sharp \partial = \partial f_\sharp$ holds for relative chains since it holds for absolute chains. By Proposition 5.8 we then have induced homomorphisms $f_* : H_n(X, A) \to H_n(Y, B)$.

Proposition 6.29. *If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs $(X, A) \to (Y, B)$, then $f_* = g_* : H_n(X, A) \to H_n(Y, B)$.*

Proof. The prism operator P from the proof of Theorem 5.9 takes $C_n(A)$ to $C_{n+1}(B)$, hence induces a relative prism operator $P : C_n(X, A) \to C_{n+1}(Y, B)$. Since we are just passing to quotient groups, the formula $\partial P + P\partial = g_\sharp - f_\sharp$ remains valid. Thus the maps $f_\sharp$ and $g_\sharp$ on relative chains groups are chain homotopic, and hence they induce the same homomorphism on relative homology groups. $\square$

An easy generalization of the long exact sequence of a pair (X, A) is the long exact sequence of a triple (X, A, B), where $B \subset A \subset X$:

$$\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \cdots$$

This is the long exact sequence of homology groups associated to the short exact sequence of chain complexes formed by the short exact sequences

$$0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0$$

For example, taking B to be a single point set, the long exact sequence of the triple (X, A, B) becomes the long exact sequence of reduced homology for the pair (X, A).

7. Excision

Theorem 7.1. ***Excision Theorem***

Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - Z, A - Z) \to H_n(X, A)$ for all n. Equivalently, for subspaces $A, B \subset X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$ for all n.

Remark 7.2. *Theorem 7.1 is a fundamental property of relative homology groups, describing when the relative groups $H_n(X, A)$ are unaffected by deleting, or excising, a subset $Z \subset A$.*

Remark 7.3. *The translation between the two versions is obtained by setting* $B = X - Z$ *and* $Z = X - B$. *Then* $A \cap B = A - Z$ *and the condition* $\mathrm{cl}(Z) \subset int(A)$ *is equivalent to* $X = \mathrm{int}(A) \cup \mathrm{int}(B)$ *since* $X - \mathrm{int}(B) = \mathrm{cl}(Z)$.

8. Naturality

To be continued...

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