REVIEW: DENSITY OF $\{n^\lambda \sin n\}_{n=1}^\infty$ ON THE REAL LINE

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Abstract. In this short review, we collect a series of propositions on the density of the sequence $\{n^\lambda \sin n\}_{n=1}^\infty$ in $\mathbb{R}$ where λ is a positive number. It will be revealed that $\{n^\lambda \sin n\}_{n=1}^\infty$ is dense in $\mathbb{R}$ for $\lambda \in (0,1)$ but not for large λ. We then proceed to define the degree of rationality of $\alpha \in \mathbb{R}$, denoted $\deg_{\mathbb{Q}} \alpha$, and conjecture that $\deg_{\mathbb{Q}} \phi < \deg_{\mathbb{Q}} \pi$ where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. This review presents a route to peep at the extent to which the number π is irrational.

1. Degenerate Case of $\lambda = 0$

We first consider a degenerate case, that is, $\{\sin n\}_{n=1}^\infty$ is dense in $[-1,1]$, whose proof basically depends on Dirichlet's approximation theorem:

Lemma 1.1. ***Dirichlet's Approximation Theorem***
For any real number α and any positive integer n, there exists integers p, q such that $1 \leq q \leq n$ and $|q\alpha - p| < 1/n$.

Then we derive the following lemma, which is a generalized version of our main result in this review. In fact, this was, during an online discussion, mentioned by an associate professor in my faculty and later added to this review.

In this short review, $\mathbb{N}$ denotes all nonnegative integers and $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$.

Theorem 1.2. *Suppose a continuous periodic function f is defined on $\mathbb{R}$ and the least positive period of f exists, denoted T, then*

(a) *if T is rational, then $f(\mathbb{N}_+)$ is finite and $f|_{\mathbb{N}_+}$ is periodic;*
(b) *if T is irrational, then $f(\mathbb{N}_+)$ is dense in $f(\mathbb{R})$.*

Proof. (a) Let $T = p/q$ where $p, q \in \mathbb{N}_+$ and $\gcd(p,q) = 1$. It is obvious that $qT = p$ is a also period of f. If $p = 1$, then $f(\mathbb{N}_+) = \{f(1)\}$ is a single point set. If $p > 1$, for each $n \in \mathbb{N}_+$ there are nonnegative integers a, i with $1 \leq i \leq p$ such that $n = ap + i$. So $f(n) = f(ap+i) = f(i)$. So f is periodic, and since each element of $f(\mathbb{N}_+)$ is one of $f(1), f(2), \cdots, f(p)$, $f(\mathbb{N}_+)$ is always finite. Note that f is of period p, but it is possible that f has smaller periods, which, though, must divide p.

(b) By Lemma 1.1, there exists $a_n := p_n - q_n T \to 0$ as $n \to \infty$ where p_n, q_n are positive integers for all $n \in \mathbb{N}_+$. Each $p_n - q_n T$ is nonzero since T is irrational. Obviously, it suffices to consider $f([0,T))$. For each $\alpha \in f([0,T))$, choose $x_\alpha \in [0,T)$ satisfying $f(x_\alpha) = \alpha$. When $a_n > 0$, there is integer $k_n \geq 0$ and real number $\beta_n \in [0,1)$

such that $x_\alpha/a_n = k_n + \beta$ which implies $0 \le x_\alpha - k_n a_n = \beta a_n < a_n$. If $a_n < 0$, choose $l_n \in \mathbb{N}_+$ satisfying $x_\alpha - l_n T < 0$, then there is integer $k_n \ge 0$ and real number $\beta_n \in [0,1)$ such that $(x_\alpha - l_n T)/a_n = k_n + \beta$. So $0 \ge x_\alpha - k_n a_n - l_n T = \beta a_n > a_n$.

Set $P_n = k_n p_n$. Without loss of generality, we omit all zero terms in $\{P_n\}_{n=1}^\infty$ since k_n will be driven far away from 0 by a_n when n is large. Also, we delete all repeating terms in $\{P_n\}_{n=1}^\infty$ and rearrange it to be strictly increasing. This is possible for that there are infinite distinct terms in $\{P_n\}_{n=1}^\infty$, otherwise $a_n \nrightarrow 0$ since a_n never equals 0. Thus $\{f(P_n)\}_{n=1}^\infty \subset \{f(n)\}_{n=1}^\infty$. Let $Q_n = 0$ when $a_n > 0$ and $Q_n = l_n T$ when $a_n < 0$, then $P_n + Q_n \to x_\alpha$ as $n \to \infty$. Hence $f(P_n) = f(P_n + Q_n) \to f(x_\alpha) = \alpha$ as $n \to \infty$ by continuity of f. For $\alpha \in f([0,T))$ is arbitrary, $f(\mathbb{N}_+)$ is dense in $f(\mathbb{R})$. $\square$

Remark 1.3. *Part (b) of Theorem 1.2 can be interpreted through general topology. Let* $\langle T \rangle = T\mathbb{Z}$*, then* $\mathbb{N}_+/\langle T \rangle$*, regarded as a subspace of* $\mathbb{R}/\langle T \rangle$*, is dense in* $\mathbb{R}/\langle T \rangle$ *or simply* $\mathrm{cl}(\mathbb{N}_+/\langle T \rangle) = \mathbb{R}/\langle T \rangle$*. Since* f *induces a function* $\tilde{f} : \mathbb{R}/\langle T \rangle \to f(\mathbb{R})$ *and continuous function preserves limit points, we derive* $\mathrm{cl}(\tilde{f}(\mathbb{N}_+/\langle T \rangle)) = \tilde{f}(\mathrm{cl}(\mathbb{N}_+/\langle T \rangle)) = \tilde{f}(\mathbb{R}/\langle T \rangle) = f([0,T))$.

Remark 1.4. *Theorem 1.2 reveals a basic property in the motion of* ***two-dimensional harmonic oscillator****. Let* $x(t) = X \sin(\alpha(t - t_x))$ *and* $y(t) = Y \sin(\beta(t - t_y))$ *for* $t \ge 0$ *where* $X, Y, \alpha, \beta, t_x, t_y$ *are given and* $X, Y, \alpha, \beta \ne 0$*. Consider the orbit generated by* $(x(t), y(t))$ *when* t *varies.*

Since $y(t) = Y \sin(\beta(t - t_y + 2\pi k/\beta))$ *where* $k \in \mathbb{Z}$ *could be arbitrary, for fixed* y *we fix a corresponding* t *and consider* x *as a function of* k*, that is,* $x := \tilde{x}(k) = X \sin(\alpha(t - t_x) + 2\pi \frac{\alpha}{\beta} k)$*. Note that* $\tilde{x}(s)$ *for* $s \in \mathbb{R}$ *is of least positive period* $\frac{1}{|\alpha/\beta|}$ *which is independent of the choice of* y*. So by Theorem 1.2, if* α/β *is rational, we then get a finite number of values of* x *between* $-|X|$ *and* $|X|$ *for any value of* y *and the orbit is periodic; if* α/β *is irrational, we then get a infinite number of values of* x *whose collection is dense in* $[-|X|, |X|]$ *and therefore the orbit is dense in* $[-|X|, |X|] \times [-|Y|, |Y|]$ *when* y *varies.*

Now we are able to easily derive the main result of this section:

Corollary 1.5. $\{\sin n\}_{n=1}^\infty$ *is dense in* $[-1, 1]$.

Remark 1.6. *Although* $\{\sin n\}_{n=1}^\infty$ *is dense in* $[-1,1]$*, note that* $\{\sin n\}_{n=1}^\infty \subsetneqq [-1, 1]$.

2. Case of $\lambda \in (0, 1)$

Theorem 2.1. *Zero is a limit point of* $\{n^\lambda \sin n\}_{n=1}^\infty$ *for any fixed* $\lambda \in (0,1)$.

Proof. It suffices to show that $\{n^\lambda \sin n\}_{n=1}^\infty$ has a subsequence converging to 0. By Lemma 1.1, for each $n \in \mathbb{N}_+$ there is a positive integer p_n such that $|2\pi n - p_n| < 1/n$. Delete repeating terms in $\{p_n\}_{n=1}^\infty$ and rearrange it to be strictly increasing as in the proof of Theorem 1.2, then $\{\sin p_n\}_{n=1}^\infty \subset \{\sin n\}_{n=1}^\infty$ and $p_n \to \infty$ as $n \to \infty$ since $\{p_n\}_{n=1}^\infty$ is an integer sequence. So $|p_n^\lambda \sin p_n| = p_n^\lambda |\sin(p_n - 2\pi n)| \le p_n^\lambda |p_n - 2\pi n| \le p_n^\lambda / n \le p_n^{\lambda-1}(2\pi n + 1/n)/n \le p_n^{\lambda-1}(2\pi + 1/n^2) \to 0$ as $n \to \infty$ for that $\lambda < 1$. $\square$

Theorem 2.2. *$\{n^\lambda \sin n\}_{n=1}^\infty$ is dense in $\mathbb{R}$ for any fixed $\lambda \in (0,1)$.*

Proof. $\square$

We basically expect to generalize a "shifting" principle:

Conjecture 2.3. *For any fixed $\lambda > 0$, if zero is a limit point of $\{n^\lambda \sin n\}_{n=1}^\infty$, then $\{n^\lambda \sin n\}_{n=1}^\infty$ is dense in $\mathbb{R}$.*

Remark 2.4. *Whether $\{n \sin n\}_{n=1}^\infty$ is dense in $\mathbb{R}$ depends on better Diophantine approximation of π not the general one for all the irrationals. This is intuitively supported by Theorem 4.3: for $\{n^\lambda \sin n\}_{n=1}^\infty$ with larger λ, if we expect to prove the density of it in $\mathbb{R}$ through Diophantine approximation, there must exist a precedent result assuring some positive $\varepsilon > 0$ such that $|2\pi - p/q| < 1/p^{2+\varepsilon}$ for infinitely many fractions p/q, which probably ensures that at least zero is a limit point of the sequence, whereas $\varepsilon = 0$ is the best we can do for general irrationals.*

3. Case of Large λ

Initially in 1953, K. Mahler proved in [M] that Lemma 3.1 holds for $\nu = 30$.

Lemma 3.1. *There exists $q_0 \in \mathbb{N}_+$ such that if p and $q \geq q_0$ are positive integers, then $|\pi - p/q| > q^{-\nu}$.*

Quite contrary to Lemma 1.1, Lemma 3.1 actually figures as the lower bound of the rational approximation towards π.

In 2010, V. Kh. Salikhov gave the best known estimate of ν, which is $7.606308\cdots$, in [S]. Hereafter, we always use these digits for ν in this short review.

Lemma 3.2. ***Jordan's Inequality***

For $x \in [0, \pi/2]$, it follows that $\frac{2}{\pi}x \leq \sin x \leq x$.

Theorem 3.3. *$\{n^\lambda \sin n\}_{n=1}^\infty$ is not dense in $\mathbb{R}$ when $\lambda \geq \nu - 1 = 6.606308\cdots$.*

Proof. Obviously there exists integer $m_n > 0$ such that $|n - m_n\pi| \leq \pi/2$. By Lemma 3.2, we derive

$$|n^\lambda \sin n| = |n^\lambda \sin(n - m_n\pi)| \geq \frac{2}{\pi} n^\lambda |n - m_n\pi| = \frac{2}{\pi} n^\lambda m_n \left| \frac{n}{m_n} - \pi \right|.$$

From the definition of $\{m_n\}_{n=1}^\infty$, we have $m_n = n/\pi + O(n) = n(1/\pi + O(1))$ as $n \to \infty$ and $|O(n)| \leq 1/2$ here. Hence there is some n_0 such that $m_n \geq q_0$ when $n \geq n_0$. By Lemma 3.1, when $n \geq n_0$, it follows that

$$|n^\lambda \sin n| \geq \frac{2}{\pi} n^\lambda m_n^{1-\nu} = \frac{2}{\pi} n^{\lambda+1-\nu} (1/\pi + O(1))^{1-\nu} = 2n^{\lambda+1-\nu}(\pi + O(1))^\nu.$$

So when $\lambda + 1 - \nu \geq 0$, $\liminf_{n\to\infty} |n^\lambda \sin n| \geq 2(\pi - 1/2)^\nu$. Even zero is not a limit point of the sequence and hence the sequence is surly not dense in $\mathbb{R}$ when $\lambda \geq \nu - 1 = 6.606308\cdots$. $\square$

4. Personal Attempts

I started to consider the density of $\{n^\lambda \sin n\}_{n=1}^\infty$ on the real line and write this short review when a freshman of my faculty asked me how to show that $\{n^m \sin n\}_{n=1}^\infty$ diverges as $n \to \infty$ where $m = 1, 2$. It is a typical problem in freshman analysis.

Later on, I posted the answer in the faculty's online discussion group. Several minutes past, a professor in that group encouraged us to prove detailedly that $\{\sin n\}_{n=1}^\infty$ is dense in $[0, 1]$ and to consider the density of $\{n \sin n\}_{n=1}^\infty$ on the real line.

The former one is a well-known fact. I solved questions like, for instance, *prove that there are infinitely many choices for* $n \in \mathbb{N}_+$ *such that* $\cos n > 2013/2014$ when participating in high school mathematics competition years ago. This is obviously equivalent to the fact that 1 is a limit point of $\{\cos n\}_{n=1}^\infty$. The solution is easy if you have ever learned Dirichlet's approximation theorem, i.e., Lemma 1.1 in this review. There is actually a stronger version of Lemma 1.1, which shows the upper bound for the Diophantine approximations of any irrational number:

Theorem 4.1. ***Émile Borel (1903)***

For every irrational number α, there are infinitely many fractions p/q such that $|\alpha - p/q| < 1/\sqrt{5}q^2$.

However, this cannot be improved without excluding some irrationals.

Definition 4.2. ***Equivalence of Real Numbers***

*Two real numbers x, y are said to be **equivalent** if there are integers a, b, c, d with $ad - bc = \pm 1$ such that $y = (ax + b)/(cx + d)$, that is, x, y are the same with respect to a integer linear fractional transformation.*

Theorem 4.3. *Let ϕ denote the golden ratio $\frac{1+\sqrt{5}}{2}$, then*

- *for any real constant $c > \sqrt{5}$, there are only a finite number of rational numbers p/q such that $|\alpha - p/q| < 1/\sqrt{5}q^2$;*
- *for every irrational number α, which is not equivalent to ϕ, there are infinite many fractions p/q such that $|\alpha - p/q| < 1/\sqrt{8}q^2$.*

Motivated by the work of K. Mahler and V. Kh. Salikhov, we propose the following definition to measure the irrationality of real numbers:

Definition 4.4. ***Degree of Rationality***

*Given a real number α, its **degree of rationality** is defined to be*

$$\deg_{\mathbb{Q}} \alpha := \inf\{\lambda \geq 0 : \exists n_0 \in \mathbb{N}_+, \forall m, n \in \mathbb{Z} \text{ with } n \geq n_\alpha, |\alpha - m/n| \geq 1/n^\lambda\}.$$

Corollary 4.5. *Immediately derived from Definition 4.4, $\deg_{\mathbb{Q}} \alpha = \infty$ for all $\alpha \in \mathbb{Q}$ and $\deg_{\mathbb{Q}} \alpha \geq 2$ for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The latter, i.e., $\deg_{\mathbb{Q}} \alpha > 0$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ indicates that an irrational number cannot be "entirely irrational" for that all irrationals can be well-approximated by rationals (see Lemma 1.1 and Theorem 4.1).*

Especially, we have $2 \leq \deg_{\mathbb{Q}} \pi \leq \nu (= 7.606308 \cdots)$.

Remark 4.6. *Maybe it is just a coincidence that* $\deg_{\mathbb{Q}} \pi$ *is probably related to natural logarithm: we observe* $|\log \nu - 2| < 0.03$ *and* $|\nu - \frac{1}{2}\mathrm{e}^{\mathrm{e}}| < 0.03$*, that is,* ν *is close to* e^2 *and* $\frac{1}{2}\mathrm{e}^{\mathrm{e}}$*. So they are good approximations to* $\deg_{\mathbb{Q}} \pi$ *provided that V. Kh. Salikhov's* ν *is a nice and close estimate of* $\deg_{\mathbb{Q}} \pi$*. If so, this could be really mysterious:* π *and* e *are likely within some unknown tight relationship.*

`https://en.wikipedia.org/wiki/Liouville_number#cite_note-4`

$$\mathrm{e}^{i\pi} + 1 = 0$$

$$\mu(\pi) = 2?$$

It is indicated by Theorem 4.3 that ϕ and numbers equivalent to it are the "most irrational" numbers in some degree. Since π is not equivalent to ϕ, we address the following conjecture:

Conjecture 4.7. $\deg_{\mathbb{Q}} \phi < \deg_{\mathbb{Q}} \pi$*.*

Remark 4.8. *I initially conjectured that the degrees of rationality of algebraic numbers are generally greater that those of transcendental numbers which, to some extent, is intuitively contrary to Theorem 4.3. Note that the degree of rationality is defined from Diophantine approximation and barely possible to address directly its relation to the roots of polynomial functions with real coefficients. So it seems promising to make the conjecture while caring more about what we have already known in Diophantine approximation.*

References

[M] K. Mahler, *On the approximation of* π, Nederl. Akad. Wetensch. Proc. Ser. A **56** (1953), no. 1, 30–42.

[S] V. Kh. Salikhov, *On the Measure of Irrationality of the Number* π, Matematicheskie Zametki **88** (2010), no. 4, 583–593.

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