ASSIGNMENTS FOR FUNCTIONAL ANALYSIS

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These assignments were arranged in Soochow's functional analysis class in Spring 2018, taught by Prof. Yisheng Huang. I live-TEXed them using sublime, and as such there may be typos; please send questions, comments, complaints, and corrections to xiaohao1096@163.com.

Contents

1. Assignment #1: 3/21/2018

Exercise 1.1. *Let $b \neq 0$. Prove that $\|a+b\| = \|a\| + \|b\|$ iff there exists a real constant $k \geq 0$ such that $a = kb$.*

Proof. From $\|a+b\|^2 = (a+b)\overline{(a+b)} = a\bar{a} + b\bar{b} + \bar{a}b + a\bar{b} = \|a\|^2 + \|b\|^2 + 2\mathrm{Re}(\bar{a}b)$, it follows that $\|a+b\| = \|a\| + \|b\| \Leftrightarrow \mathrm{Re}(\bar{a}b) = \|ab\|$. Since $b \neq 0$, put $a/b = k$ and then $\mathrm{Re}(\bar{a}b) = \|b\|^2\mathrm{Re}(\bar{k})$ and $\|ab\| = \|b\|^2\|k\|$. Now we see that $\mathrm{Re}(\bar{k}) = \|k\| \geq 0$, which is equivalent to k is real and $k \geq 0$. Thus, the statement holds. □

Exercise 1.2. *Let $p \geq 1$. Prove that $(\sum_{k=1}^n \|a_k\|)^p \leq n^{p-1} \sum_{k=1}^n \|a_k\|^p$.*

Proof. The inequality is trivial for $p = 1$. When $p > 1$, use Hölder's inequality and we derive

$$\sum_{k=1}^n \|a_k\| = \sum_{k=1}^n 1 \cdot \|a_k\| \leq \left(\sum_{k=1}^n 1^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\sum_{k=1}^n \|a_k\|^p\right)^{\frac{1}{p}} = n^{\frac{p-1}{p}} \left(\sum_{k=1}^n \|a_k\|^p\right)^{\frac{1}{p}},$$

which is equivalent to $(\sum_{k=1}^n \|a_k\|)^p \leq n^{p-1} \sum_{k=1}^n \|a_k\|^p$. □

Exercise 1.3. *In a metric space (X, d) prove that $|d(x,y) - d(x',y')| \leq d(x,x') + d(y,y')$ holds for all $x, y, x', y' \in X$.*

Proof. Since any metric satisfies triangle inequality, we have

$$\begin{aligned}|d(x,y)-d(x',y')| &= |(d(x,y)+d(x',y))-(d(x',y)+d(x',y'))| \\ &\leq |d(x,y)+d(x',y)|+|d(x',y)+d(x',y')| \\ &\leq |d(x,x')|+|d(y,y')| \\ &= d(x,x')+d(y,y').\end{aligned}$$

□

Exercise 1.4. *Check whether the following functions ρ are metrics on the $\mathbb{R}$.*

(a) $\rho(x,y)=(x-y)^2$. (b) $\rho(x,y)=\sqrt{|x-y|}$.

Solution. (a) Since $\rho(1,2)+\rho(2,3)=2<4=\rho(1,3)$, we know that $\rho(x,y)=(x-y)^2$ is not a metric on $\mathbb{R}$.

(b) It is obvious that for all x and y in $\mathbb{R}$, $\rho(x,y)\geq 0$ and $\rho(x,y)=0$ iff $x=y$. Also, $\rho(x,y)+\rho(y,z)=\sqrt{|x-y|}+\sqrt{|y-z|}\geq\sqrt{|x-y|+|y-z|+2\sqrt{|x-y|}\sqrt{|y-z|}}\geq\sqrt{|x-y|+|y-z|}\geq\sqrt{|x-z|}=\rho(x,z)$ for all x,y,z in $\mathbb{R}$. Now we see that $\rho(x,y)=\sqrt{|x-y|}$ is a metric on $\mathbb{R}$. □

Exercise 1.5. *Suppose that (X,d) is a metric space. Prove that each of the following functions ρ is a metric on the set X.*

(a) *For every $x,y\in X$, $\rho(x,y)=\frac{d(x,y)}{1+d(x,y)}$.*
(b) *For every $x,y\in X$, $\rho(x,y)=\min\{1,d(x,y)\}$.*

Proof. (a) It is obvious that for all x,y in X, $\rho(x,y)\geq 0$ and $\rho(x,y)=0$ iff $x=y$. Since $\varphi(t)=\frac{t}{1+t}$ increases on $[0,+\infty)$, apply triangle inequality here and we derive $\rho(x,y)+\rho(y,z)=\frac{d(x,y)}{1+d(x,y)}+\frac{d(y,z)}{1+d(y,z)}\geq\frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)}\geq\frac{d(x,z)}{1+d(x,z)}=\rho(x,z)$ for all x,y,z in X. Thus, ρ is a metric on X.

(b) It is obvious that for all x,y in X, $\rho(x,y)\geq 0$ and $\rho(x,y)=0$ iff $x=y$. For all $x,y,z\in X$, if $d(x,y)\geq 1$ or $d(y,z)\geq 1$, then

$$\rho(x,y)+\rho(y,z)=\min\{1,d(x,y)\}+\min\{1,d(y,z)\}\geq 1\geq\min\{1,d(x,z)\}=\rho(x,z);$$

if $d(x,y)<1$ and $d(y,z)<1$, then

$$\rho(x,y)+\rho(y,z)=d(x,y)+d(y,z)\geq d(x,z)\geq\min\{1,d(x,z)\}=\rho(x,z).$$

Hence, ρ is a metric on X. □

Exercise 1.6. *In a metric space (X,d), prove that $x_n\to x$ and $y_n\to y$ as $n\to\infty$ imply $d(x_n,y_n)\to d(x,y)$ as $n\to\infty$.*

Proof. From Exercise 1.3, we know that $0\leq|d(x_n,y_n)-d(x,y)|\leq d(x_n,x)+d(y_n,y)$ always holds. Since $x_n\to x$ and $y_n\to y$ as $n\to\infty$, $|d(x_n,y_n)-d(x,y)|\to 0$ as $n\to\infty$. Thus, $d(x_n,y_n)\to d(x,y)$ as $n\to\infty$. □

Exercise 1.7. *In a metric space (X,d), prove that if a Cauchy sequence has a convergent subsequence then the whole sequence is convergent.*

Proof. Suppose $\{x_n\}_{n=1}^\infty \subset X$ is a Cauchy sequence and $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ is a subsequence that converges to $x \in X$. Then for any $\varepsilon > 0$, there exists $K_1 > 0$ such that $d(x_n, x_{n_k}) < \varepsilon/2$ when $n, n_k > K_1$; there exists $K_2 > 0$ such that $d(x_{n_k}, x) < \varepsilon/2$ when $n_k > K_2$. Take $K = \max\{K_1, K_2\} > 0$ and then $0 \le d(x_n, x) \le d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ when $n, n_k > K$. In brief, we see that for any $\varepsilon > 0$, there exists $K > 0$ such that $0 \le d(x_n, x) \le \varepsilon$ when $n > K$, that is, $x_n \to x$ $(n \to \infty)$, which implies that the whole sequence $\{x_n\}_{n=1}^\infty$ is convergent. □

Exercise 1.8. *In (X, d), let triangle inequality be replaced by the axiom $d(x, z) \le \max\{d(x, y), d(y, z)\}$, but keep the same definition of Cauchy sequence. Prove that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence iff $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.*

Proof. $\Rightarrow$) If $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence, then it is obvious that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$ by definition.

$\Leftarrow$) Since $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, for any $\varepsilon > 0$, there exists $K > 0$ such that $d(x_n, x_{n+1}) < \varepsilon$ when $n > K$. Then for $m > n > K$, we derive $0 \le d(x_n, x_m) \le \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \cdots, d(x_{m-1}, x_m)\} = \max_{0 \le i \le m-n-1} d(x_{n+i}, x_{n+i+1})$ by the new axiom. Note that $d(x_{n+i}, x_{n+i+1}) < \varepsilon$ for all $0 \le i \le m-n-1$ and the index set $\{0, 1, \cdots, m-n-1\}$ is finite. Hence, $0 \le d(x_n, x_m) < \varepsilon$ when $n, m > K$ and $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. □

Exercise 1.9. *Let A and B be subsets of a metric space (X, d). Show that the following statements hold.*

(a) *If $A \subset B$ then $A' \subset B'$, $A^\circ \subset B^\circ$ and $\bar{A} \subset \bar{B}$.*
(b) *$(A \cup B)' \subset A' \cup B'$.*
(c) *A is open iff $A = A^\circ$.*
(d) *A is closed iff $A = \bar{A}$.*

Proof. (a) If $x \in A'$, then $(B(x, \varepsilon) \setminus \{x\}) \cap A \ne \varnothing$ for all $\varepsilon > 0$. Since $A \subset B$, $(B(x, \varepsilon) \setminus \{x\}) \cap B \ne \varnothing$ for all $\varepsilon > 0$, which indicates that $x \in B'$. Thus, $A' \subset B'$ since x is arbitrary.

If $x \in A^\circ$, then there is some $r > 0$ such that $B(x, r) \subset A$. Since $A \subset B$, we have $B(x, r) \subset B$. Thus, $A^\circ \subset B^\circ$ since x is arbitrary.

If $A \subset B$, we now know that $\bar{A} = A \cup A' \subset B \cup B' \subset \bar{B}$, that is, $\bar{A} \subset \bar{B}$.

(b) If $x \in A'$, then $(B(x, \varepsilon) \setminus \{x\}) \cap (A \cup B) \ne \varnothing$ for all $\varepsilon > 0$. Then at least one of the following claims is true:

- $(B(x, \varepsilon) \setminus \{x\}) \cap A \ne \varnothing$ for all $\varepsilon > 0$;
- $(B(x, \varepsilon) \setminus \{x\}) \cap B \ne \varnothing$ for all $\varepsilon > 0$,

which indicates that $x \in A'$ or $x \in B'$. Hence, $x \in A' \cup B'$ and then $(A \cup B)' \subset A' \cup B'$ since x is arbitrary.

(c) $\Leftarrow$) If $A = A^\circ$, then every point of A is also a point of A°, which implies that every point of A is an interior point of A by the definition of A°. Thus, A is open.

$\Rightarrow$) If A is open then every point of A is an interior point of A and so is a point of A°, which implies that $A \subset A^\circ$. Since $A^\circ \subset A$ by definition, then $A = A^\circ$.

(d) $\Leftarrow$) If $A = \bar{A}$, then $A' \subset A$ since $\bar{A} = A \cup A'$. Thus, A is closed by definition.

$\Rightarrow$) If A is closed, then $A' \subset A$ and so $A \cup A' \subset A$. Since $A \subset A \cup A'$, we deduce that $A = A \cup A' = \bar{A}$. □

Exercise 1.10. *Consider the metric space (X, d), where $X = [0,3) \cup [4,5] \cup (6,7) \cup \{8\}$ and d is the euclidean metric in $\mathbb{R}$ restricted to X. For each of the following subsets, check whether it is open or closed, and justify you assertions.*

(a) $[0,3)$. (b) $[4,5)$. (c) $(6,7)$. (d) $\{8\}$. (e) $[0,3) \cup [4,5)$. (f) $[0,3) \cup (6,7)$. (g) $(6,7) \cup \{8\}$. (h) $[1,2)$. (i) $(1,2)$. (j) $[1,2]$.

Solution. (a) $[0,3)$ is open and closed since $[0,3) = (-1,3) \cap X$ and $[0,3) = [0,3] \cap X$.

(b) $[4,5)$ is open since $[4,5) = (3.5,5) \cap X$. $[4,5)$ is not closed since $[4,5)$ does not contain its accumulation point 5.

(c) $(6,7)$ is open and closed since $(6,7) = (6,7) \cap X$ and $(6,7) = [6,7] \cap X$.

(d) $\{8\}$ is open and closed since $\{8\} = (7.5, 8.5) \cap X$ and $\{8\} = [7.5, 8.5] \cap X$.

(e) $[0,3) \cup [4,5)$ is open since $[0,3)$ and $[4,5)$ are open. $[0,3) \cup [4,5)$ is not closed since $[0,3) \cup [4,5)$ does not contain its accumulation point 5.

(f) $[0,3) \cup (6,7)$ is open and closed since both $[0,3)$ and $(6,7)$ are open and closed.

(g) $(6,7) \cup \{8\}$ is open and closed since both $(6,7)$ and $\{8\}$ are open and closed.

(h) $[1,2)$ is neither open nor closed since $1 \in [1,2)$ is not an interior point of $[1,2)$ and $[1,2)$ does not contain its accumulation point 2.

(i) $(1,2)$ is open since $(1,2) = (1,2) \cap X$. $(1,2)$ is not closed since $(1,2)$ does not contain its accumulation point 2.

(j) $[1,2]$ is closed since $[1,2] = [1,2] \cap X$. $[1,2]$ is not open since $1 \in [1,2]$ is not an interior point of $[1,2]$. □

Exercise 1.11. *Let A be a nonempty set of (X, d), show that A is an open set in X iff A is a union of some open balls.*

Proof. $\Rightarrow$) If A is an open set, then for all $x \in A$, there exists some $r_x > 0$ such that $B(x, r_x) \subset A$. Hence, we have $\cup_{x \in A} B(x, r_x) \subset A$. Note that the union takes over all elements in A. It is clear that $A \subset \cup_{x \in A} B(x, r_x)$. Then $A = \cup_{x \in A} B(x, r_x)$, that is, a union of open balls.

$\Leftarrow$) Suppose that $A = \cup_{i \in I} B(x_i, r_i)$, where I is an nonempty index set and $B(x_i, r_i)$ is an open ball centered at x_i with radius $r_i > 0$ for all $i \in I$. If $x \in A$, then there exists some $i \in I$ such that $x \in B(x_i, r_i)$. Put $r = r_i - d(x, x_i) > 0$ and then $B(x, r) \subset B(x_i, r_i) \subset A$, which shows that x is an interior point of A. Since x is arbitrary, we conclude that A is open. □

2. Assignment #2: 4/4/2018

Exercise 2.1. *If a metric space (X, d) is separable, prove that every subspace of X is separable.*

Proof. Since X is separable, there exists a countable subset $A \subset X$ such that $X \subset \bar{A}$.

The claim goes that $\mathscr{B} = \{B(a, q) : a \in A, q \in \mathbb{Q}^+\}$ is a base for the open sets in X. Basically, for any open set U of X and any $x \in U$, we see $x \in \bar{A}$ implies that there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset A$ which converges to x. Since U is open, there exists an open ball $B(x, r)$ centered at x of radius $r > 0$ such that $B(x, r) \subset U$. Choose $n \in \mathbb{N}_+$ satisfying $d(x_n, x) < r/2$ since $x_n \to x$ $(n \to \infty)$. Now we obtain a open ball $B(x_n, r/2) \subset B(x, r) \subset U$ which contains x. Let $q \in (d(x_n, x), r/2)$ be a rational number and then $x \in B(x_n, q) \subset B(x_n, r/2) \subset U$. Note that $B(x_n, q) \in \mathscr{B}$. This shows the claim.

Recall that A is countable. Thus, $\mathscr{B}$ is a countable base. Suppose Y is a subspace of X. Without loss of generality, we also assume that Y is nonempty. Let $\mathscr{B}' := \{B \cap Y : B \cap Y \neq \varnothing, B \in \mathscr{B}\}$ and then choose $x_B \in B$ for every $B \in \mathscr{B}'$. Now we derive a set $D = \{x_B : B \in \mathscr{B}'\}$ which is countable, obviously nonempty, and a subset of Y. We perform the following procedure:

- Choose $B_1 \in \mathscr{B}'$ such that $y \in B_1 \subset B(y, 1) \cap Y$;
- Choose $B_2 \in \mathscr{B}'$ such that $y \in B_2 \subset B(y, 1/2) \cap Y$;
- Choose $B_3 \in \mathscr{B}'$ such that $y \in B_3 \subset B(y, 1/3) \cap Y$;
- $\cdots\cdots$

The procedure works because $B(y, 1/n) \cap Y$ is open in Y for all $n \in \mathbb{N}_+$ and $\mathscr{B}'$ is obviously a base of Y. Now we obtain a sequence $\{x_{B_n}\}_{n=1}^{\infty} \subset D$ such that $d(x_{B_n}, y) < 1/n$ $(n \in \mathbb{N}_+)$, which implies that $x_{B_n} \to y$ as $n \to \infty$. Since y is arbitrary, we get $Y \subset \bar{D}$. This proves that Y is separable and so is every subspace of X. □

Exercise 2.2. *Prove that each Cauchy sequence of a metric space is bounded.*

Proof. Suppose $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence of a metric space (X, d). There exists a positive integer N such that $d(x_n, d_m) < 1$ when $n, m > N$. Fix $m = n_0 > N$ and then $d(x_n, x_{n_0}) < 1$ when $n > N$. Put $r = \max\{1, d(x_1, x_{n_0}), d(x_2, x_{n_0}), \cdots, d(x_N, x_{n_0})\} > 0$ and then $\{x_n\}_{n=1}^{\infty} \subset B(x_{n_0}, r)$, which implies $\{x_n\}_{n=1}^{n}$ is bounded. Hence, the claim holds since $\{x_n\}_{n=1}^{\infty}$ is arbitrary. □

Exercise 2.3. *Let X be the set of all continuous functions on $[0, 1]$. Show that X with the metric $\rho(x, y) = \int_0^1 |x(t) - y(t)| \mathrm{d}t$ (not the usual metric on $C[0, 1]$) is incomplete.*

Proof. Consider the sequence $\{x_n\}_{n=1}^{\infty}$ of (X, ρ) defined by

$$x_n(t) = \begin{cases} 0 & 0 \le t < 1/2 \\ nt - n/2 & 1/2 \le t \le 1/2 + 1/n \\ 1 & 1/2 + 1/n < t \le 1 \end{cases}$$

which is a Cauchy sequence which is not convergent in X. In fact, for any $\varepsilon > 0$, let $N = \frac{1}{2\varepsilon} > 0$ and then $\rho(x_n, x_m) = \int_0^1 |x_n(t) - x_m(t)| \mathrm{d}t = \frac{n-m}{2nm} < \frac{n}{2nm} = \frac{1}{2m} < \varepsilon$ when $n > m > N$.

However, the limit of $\{x_n\}_{n=1}^{\infty}$ is a discontinuous function

$$x(t) = \begin{cases} 0 & 0 \leq t < 1/2 \\ 1 & 1/2 \leq t \leq 1 \end{cases},$$

which is not in $C[0,1]$. Thus, X is not complete when equipped with the metric ρ. $\square$

Exercise 2.4. *Let (X, d) be a metric space. Show that (X, d) is complete iff each closed set sequence $\{A_n\}_{n=1}^{\infty}$ in X with $A_n \neq \varnothing$, $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}_+$ and*

$$\lim_{n\to\infty} \sup\{d(x,y) : x, y \in A_n\} = 0$$

implies that $\cap_{n=1}^{\infty} A_n$ is a set of a single point.

Proof. $\Rightarrow$) Suppose $\{A_n\}_{n=1}^{\infty}$ is an arbitrary closed set sequence in X with $A_n \neq \varnothing$, $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}_+$ and $\lim_{n\to\infty} \sup\{d(x,y) : x, y \in A_n\} = 0$. We choose $x_n \in A_n$ for every $n \in \mathbb{N}_+$ and obtain a sequence $\{x_n\}_{n=1}^{\infty}$ in X.

The claim goes that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Basically, since

$$\lim_{n\to\infty} \sup\{d(x,y) : x, y \in A_n\} = 0,$$

for every $\varepsilon > 0$ we can always choose $N > 0$ such than $d(x,y) \leq \varepsilon/2$ when $n > N$ and $x, y \in A_n$. Note the $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets. It follows that $d(x_n, x_m) < \varepsilon$ when $n, m > N$. This shows the claim. Now we see that $x_n \to x \in X$ as $n \to \infty$ since X is complete.

Since $\{x_n\}_{n=m}^{\infty} \subset A_m$ and A_m is closed, then $x_n \to x \in A_m$ $(n \to \infty)$ holds for all $m \in \mathbb{N}_+$. This shows that $x \in \cap_{n=1}^{\infty} A_n$, which means $\cap_{n=1}^{\infty} A_n$ is nonempty.

If there exist two distinct points in $\cap_{n=1}^{\infty} A_n$, say x, y, then $\sup\{d(x,y) : x, y \in A_n\} \geq d(x,y) > 0$ holds for all $n \in \mathbb{N}_+$. So $\lim_{n\to\infty} \sup\{d(x,y) : x, y \in A_n\} \geq d(x,y) > 0$, which is a contradiction. Thus, $\cap_{n=1}^{\infty} A_n$ contains at most one element. And hence $\cap_{n=1}^{\infty} A_n$ is a single point set.

$\Leftarrow$) Given an arbitrary Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X and we perform the following procedure:

- Choose integer $n_1 > 0$ such that $d(x_n, x_m) < 1$ when $n, m \geq n_1$;
- Choose integer $n_2 > n_1$ such that $d(x_n, x_m) < 1/2$ when $n, m \geq n_2$;
- Choose integer $n_3 > n_2$ such that $d(x_n, x_m) < 1/3$ when $n, m \geq n_3$;
- $\cdots\cdots$

Let $A_1 = \bar{B}(x_{n_1}, 1)$ and $A_k = A_{k-1} \cap \bar{B}(x_{n_k}, 1/n)$ when $n \geq 2$. Now we derive a decreasing closed set sequence $\{A_n\}_{n=1}^{\infty}$ with $A_n \neq \varnothing$, $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}_+$ and $\mathrm{diam}(A_n) \to 0$ as $n \to \infty$. Since $\cap_{n=1}^{\infty} A_n$ is a single point set, we denote the only element of $\cap_{n=1}^{\infty} A_n$ by x. It follows that $d(x_n, x) < 1/n \to 0$ as $n \to \infty$ and then

$\{x_n\}_{n=1}^{\infty}$ converges in X. This is also true for all the other Cauchy sequence in X. Now we conclude that X is complete. $\square$

Exercise 2.5. *Let S be a subset of a metric space (X, d). Prove that the following two statements are equivalent.*

(a) *S is nowhere dense in X.*
(b) *The complement $(\bar{S})^{\mathrm{C}}$ is dense in X.*

Proof. (a)⇒(b) For every $x \in X$ we have either $x \in \bar{S}$ or $x \in (\bar{S})^{\mathrm{C}}$. If $x \in \bar{S}$, since S is nowhere dense in X, that is, $(\bar{S})^{\circ} = \varnothing$, we can choose $x_n \in B(x, 1/n) \cap (\bar{S})^{\mathrm{C}}$ for all $n \in \mathbb{N}_+$. Then $\{x_n\}_{n=1}^{\infty} \subset (\bar{S})^{\mathrm{C}}$ and $x_n \to x$ $(n \to \infty)$. Hence x is in the closure of $(\bar{S})^{\mathrm{C}}$. Then $(\bar{S})^{\mathrm{C}}$ is dense in X since x is arbitrary.

(b)⇒(a) For every $x \in \bar{S}$, since $(\bar{S})^{\mathrm{C}}$ is dense in X, we have $B(x, r) \cap (\bar{S})^{\mathrm{C}} \neq \varnothing$ for all $\varepsilon > 0$. Then x is not an interior point of $\bar{S}$. Then $(\bar{S})^{\circ} = \varnothing$ since x is arbitrary. Hence, S is nowhere dense in X. $\square$

Exercise 2.6. *Use Barie category theorem to deduce that $[0, 1]$ in $\mathbb{R}$ is uncountable.*

Proof. Assume $[0, 1] = \{x_1, x_2, x_3, \cdots\}$ is countable. Since $\{x_n\}$ is nowhere dense for all $n \in \mathbb{N}_+$, $[0, 1] = \cup_{n=1}^{\infty}\{x_n\}$ is of the first category. However, $[0, 1]$ is nonempty and complete. Then $[0, 1]$ is of the second category by Barie category theorem, which is a contradiction. Thus, $[0, 1]$ is uncountable. $\square$

Exercise 2.7. *Let (X, d) be a metric space and $d(x, A) = \inf_{y \in A} d(y, x)$ for every subset $A \subset X$ and every point $x \in X$.*

(a) *Prove that if $x \in A$, then $d(x, A) = 0$.*
(b) *Is the converse of* (a) *true? Justify your assertion.*
(c) *Prove that $d(x, A) = d(x, \bar{A})$. In particular, $d(x, A) = 0$ iff $x \in \bar{A}$.*

Solution. (a) Since $x \in A$, we derive $0 \le d(x, A) = \inf_{y \in A} d(y, x) \le d(x, x) = 0$, that is, $d(x, A) = 0$.

(b) The converse of (a) is false in general. For instance, let $X = \mathbb{R}$ with the usual metric on $\mathbb{R}$ and $A = (0, 1)$. Then it is obvious that $d(0, (0, 1)) = 0$ but $0 \notin (0, 1)$.

(c) It is easy to see that $d(x, A) = \inf_{y \in A} d(y, x) \ge \inf_{y \in \bar{A}} d(y, x) = d(x, \bar{A})$ since $A \subset \bar{A}$. Obviously, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset \bar{A}$ such that $d(x_n, x) \to d(x, \bar{A})$ as $n \to \infty$. Now we discuss the problem under two cases.

- If $\{x_n\}_{n=1}^{\infty} \cap A'$ is a finite set, without loss of generality, we assume $\{x_n\}_{n=1}^{\infty} \subset A$. Hence $d(x, A) \le \lim_{n\to\infty} d(x_n, x) = d(x, \bar{A})$.
- If $\{x_n\}_{n=1}^{\infty} \cap A'$ is an infinite set, then $\{x_n\}_{n=1}^{\infty}$ contains a subsequence in A'. Without loss of generality, we can assume $\{x_n\}_{n=1}^{\infty} \subset A'$. Since $(B(x_n, 1/n) \setminus \{x_n\}) \cap A \neq \varnothing$, choose $y_n \in (B(x_n, 1/n) \setminus \{x_n\}) \cap A$ for all $n \in \mathbb{N}_+$. Since $\{y_n\}_{n=1}^{\infty} \subset A$, we derive $d(x, A) \le \lim_{n\to\infty} d(y_n, x) \le \lim_{n\to\infty} d(y_n, x_n) + \lim_{n\to\infty} d(x_n, x) \le \lim_{n\to\infty} 1/n + d(x, \bar{A}) = d(x, \bar{A})$.

So we always have $d(x,A) \le d(x,\bar{A})$. Thus, $d(x,A) = d(x,\bar{A})$.

In particular, when $x \in \bar{A}$, $d(x,A) = d(x,\bar{A}) = 0$ by (a). If $d(x,A) = 0$, assume $x \notin \bar{A}$ and then there is some $r > 0$ such that $B(x,r) \subset X \setminus A$. But $d(x,A) \ge r > 0$ which is a contradiction. Hence, $x \in \bar{A}$. We conclude that $d(x,A) = 0$ iff $x \in \bar{A}$. $\square$

Exercise 2.8. *Suppose that A is a nonempty subset of a metric space (X,d). Prove that the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x,A)$ is continuous on X. Furthermore, f is uniformly continuous on X, in the sense that for all $\varepsilon > 0$ there exists some $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ whenever $x_1, x_2 \in X$ and $d(x_1,x_2) < \delta$.*

Proof. For $\varepsilon > 0$, put $\delta = \varepsilon/2$. When $x_1, x_2 \in X$ and $d(x_1,x_2) < \delta$, there always exist sequences $\{y_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty} \subset A$ such that $\lim_{n\to\infty} d(y_n,x_1) = d(x_1,A)$ and $\lim_{n\to\infty} d(z_n,x) = d(x_2,A)$. Take $N > 0$ satisfying $|d(y_n,x_1) - d(x_1,A)| < \varepsilon/2$ and $|d(z_n,x_2) - d(x_2,A)| < \varepsilon/2$ if $n > N$. Hence, when $n > N$, we derive

$$
\begin{aligned}
d(x_1,A) &\le d(z_n,x_1) \le d(x_1,x_2) + d(z_n,x_2) \\
&< \delta + d(x_2,A) + \varepsilon/2 = d(x_2,A) + \varepsilon, \\
d(x_2,A) &\le d(y_n,x_2) \le d(x_1,x_2) + d(y_n,x_1) \\
&< \delta + d(x_1,A) + \varepsilon/2 = d(x_1,A) + \varepsilon.
\end{aligned}
$$

It is now clear that $|d(x_1,A) - d(x_2,A)| < \varepsilon$ when $d(x_1,x_2) < \delta$, which implies d is uniformly continuous on X since ε is arbitrary. Then automatically, d is continuous on X. $\square$

Exercise 2.9. *Suppose that (X,d) is a metric space, F_1 and F_2 are closed subsets of X with $F_1 \cap F_2 = \varnothing$. Prove that there exists a continuous function on X such that $f(x) = 0$ if $x \in F_1$, and $f(x) = 1$ if $x \in F_2$.*

Proof. Consider $f(x) = d(x,F_1)/(d(x,F_1) + d(x,F_2))$ which is a well-defined continuous function on X. In fact, if $d(x,F_1) = d(x,F_2) = 0$, then $x \in F_1, F_2$ by part (c) of Exercise 2.7 since F_1, F_2 are closed. But $F_1 \cap F_2 = \varnothing$, which is a contradiction. So $d(x,F_1)$ and $d(x,F_2)$ cannot be 0 at the same time. This shows that f is well-defined and the continuity comes from Exercise 2.8. It is easy to check that $f(x) = 0$ if $x \in F_1$, and $f(x) = 1$ if $x \in F_2$. Hence the claim holds. $\square$

Exercise 2.10. *Prove that a set in $\mathbb{R}^n$ is compact iff it is bounded and closed.*

Proof. $\Rightarrow$) Let $E \subset \mathbb{R}^n$ be compact.

If E is unbounded, then choose a fixed point $a \in E$ and there exists $x_n \in E$ such that $\|x_n - a\| > n$ for all $n \in \mathbb{N}_+$. By the compactness of E, some subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ would converge to $x_0 \in E$. But $\|x_{n_k} - x_0\| \ge \|x_{n_k} - a\| - \|x_0 - a\| \ge n_k - \|x_0 - a\| \to \infty$ as $k \to \infty$, which is a contradiction. Hence E is bounded.

Given $y_0 \in E'$ and then there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset E$ that converges to y_0. Since E is compact, $\{y_n\}_{n=1}^{\infty}$ has a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ which converges to $y_0' \in E$.

Obviously, $y_0 = y_0' \in E$. Then $y_0 \in E$ and hence $E' \subset E$ since y_0 is arbitrary, which shows E is closed.

$\Leftarrow$) Suppose E is bounded and closed in $\mathbb{R}^n$. By accumulation principle of bounded sequence in $\mathbb{R}^n$, for all $\{x_n\}_{n=1}^\infty \subset E$ there is a subsequence $\{x_{n_k}\}_{k=1}^\infty$ that converges to $x_0 \in \bar{E} = E$. Then E is compact since $\{x_n\}_{n=1}^\infty$ is arbitrary. $\square$

Exercise 2.11. *Let $X = [0,1] \cup \{2,3,\cdots\}$ with the metric $d(x,y) = |x-y|$ defined for all $x,y \in X$. Justify the following assertions.*

(a) *Is X complete?* (b) *Is X separable?* (c) *Is X compact?*

Solution. (a) It is clear that d is a well-defined metric on X since it is induced from the usual metric on $\mathbb{R}$. Then X can be regarded as a subspace of $\mathbb{R}$. Since $\mathbb{R}$ is complete and X is closed, it is obvious that X is complete.

(b) Note that $\overline{X \cap \mathbb{Q}} = X$ and $X \cap \mathbb{Q} \subset X$ is a countable subset. So X is separable.

(c) X is not compact since $\{x_n = n\}_{n=1}^\infty \subset X$ does not even have a convergent subsequence. $\square$

Exercise 2.12. *Suppose E is a nonempty compact set in a metric space (X,d). Prove that there exist $x,y \in E$ such that $d(x,y) = \sup_{u,v\in E} d(u,v)$.*

Proof. E is bounded by the compactness of E. Then $l := \sup_{u,v\in E} d(u,v) < \infty$ and there exists $u_n, v_n \in E$ such that $l - 1/n < d(u_n, v_n) \le l$ for all $n \in \mathbb{N}_+$. From the definition of compactness, there exist $\{u_{n_k}\}_{k=1}^\infty \subset \{u_n\}_{n=1}^\infty$ and $\{v_{n_k}\}_{k=1}^\infty \subset \{v_n\}_{n=1}^\infty$ such that $\{u_{n_k}\}_{k=1}^\infty$ converges to $u_0 \in E$, $\{v_{n_k}\}_{k=1}^\infty$ converges to $v_0 \in E$, and $l - 1/n_k < d(u_{n_k}, v_{n_k}) \le l$ for all $k \in \mathbb{N}_+$. Hence $d(u_0, v_0) = l$ and the claim holds. $\square$

3. Assignment #3: 4/18/2018

Exercise 3.1. *Given $M \subset C[a,b]$ for which there exist $m, L > 0$ and $x_0 \in [a,b]$ such that $|f(x_0)| \le m$ and $|f(x) - f(y)| \le L|x-y|$ for all $f \in M$ and $x,y \in [a,b]$. Prove that M is relatively compact in $C[a,b]$.*

Proof. From the assumption, it follows that for all $f \in M$ and $x \in [a,b]$, $|f(x)| \le |f(x) - f(x_0)| + |f(x_0)| \le L|x - x_0| + m \le L(b-a) + m$. This shows that M is uniformly bounded.

Also, M is equicontinuous. In fact, for all $\varepsilon > 0$, $|f(x_1) - f(x_2)| \le L|x_1 - x_2| < \varepsilon$ for all $f \in M$ whenever $x_1, x_2 \in [a,b]$ with $|x_1 - x_2| < \varepsilon/L$.

Hence, M is relatively compact by Arzelà-Ascoli Theorem. $\square$

Exercise 3.2. *Given $M \subset C^1[a,b]$ such that $\int_a^b (|f(x)|^2 + |f'(x)|^2)\mathrm{d}x \le k$ for all $f \in M$ where $k > 0$ is a constant. Prove that M is relatively compact in $C[a,b]$.*

Proof. We compute that

$$
\begin{aligned}
|f(a)| \le |f(x)-f(a)| + |f(x)| &= \left| \int_a^x f'(t)\mathrm{d}t \right| + |f(x)| \\
&\le \left(\int_a^x \mathrm{d}t \right)^{1/2} \left(\int_a^x |f'(t)|^2 \mathrm{d}t \right)^{1/2} + |f(x)| \\
&\le \sqrt{(b-a)k} + |f(x)|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(b-a)|f(a)| = \int_a^b |f(a)|\mathrm{d}x &\le (b-a)^{3/2}\sqrt{k} + \int_a^b |f(x)|\mathrm{d}x \\
&\le (b-a)^{3/2}\sqrt{k} + \sqrt{b-a} \left(\int_a^b |f(x)|^2 \mathrm{d}x \right)^{1/2} \\
&\le (b-a)^{3/2}\sqrt{k} + \sqrt{(b-a)k}.
\end{aligned}
$$

Set $k_1 = \frac{(b-a)^{3/2}\sqrt{k}+\sqrt{(b-a)k}}{b-a}$ and then $|f(a)| \le k_1$ for all $f \in M$. So for all $f \in M$ and $x \in [a,b]$, we have

$$
|f(x)| \le |f(x)-f(a)| + |f(a)| \le \sqrt{(b-a)k} + k_1.
$$

Then M is uniformly bounded.

And for all $\varepsilon > 0$, choose $\delta = \frac{\varepsilon^2}{k} > 0$, then for all $f \in M$ and $x_1, x_2 \in [a,b]$ with $x_1 \le x_2$ and $x_2 - x_1 < \delta$ we have

$$
\begin{aligned}
|f(x_2) - f(x_1)| = \left| \int_{x_1}^{x_2} f'(t)\mathrm{d}t \right| &\le \int_{x_1}^{x_2} |f'(t)|\mathrm{d}t \\
&\le \left(\int_{x_1}^{x_2} \mathrm{d}t \right)^{1/2} \left(\int_{x_1}^{x_2} |f'(t)|^2 \mathrm{d}t \right)^{1/2} \\
&\le \sqrt{(x_2-x_1)k} \\
&< \sqrt{\varepsilon^2/k \cdot k} = \varepsilon.
\end{aligned}
$$

Thus, M is equicontinuous.

Hence, M is relatively compact in $C[a,b]$ by Arzelà-Ascoli Theorem. □

Exercise 3.3. *Given $M \subset C^1[a,b]$ which satisfies the following properties:*

- *There exists $L > 0$ such that $|f'(x)| \le L$ for all $f \in M$ and $x \in [a,b]$;*
- *There is at least one solution to $f(x) = 0$ on $[a,b]$ for each $f \in M$.*

Prove that M is relatively compact in $C[a,b]$.

Proof. Since $|f'(x)| \le L$ for all $f \in M$ on $[a,b]$, we have $|f(x)-f(y)| = |f'(\xi)||x-y| \le L|x-y|$ for all $x, y \in [a,b]$ where ξ is some point on the line segment with x, y as endpoints. For each $f \in M$, let $x = x_0$ be the solution to $f(x) = 0$, then

$$|f(x)| = |f(x) - f(x_0)| \le L|x - x_0| \le L(b-a),$$

which indicates that M is uniformly bounded.

Also, M is equicontinuous. In fact, for all $\varepsilon > 0$, $|f(x_1) - f(x_2)| \le L|x_1 - x_2| < \varepsilon$ for all $f \in M$ whenever $x_1, x_2 \in [a,b]$ and $|x_1 - x_2| < \varepsilon/L$.

Hence, M is relatively compact by Arzelà-Ascoli Theorem. □

Exercise 3.4. *Determine whether or not the following sets of functions are relatively compact in $C[a,b]$:*

(i) $\{f_\alpha(x) = \sin \alpha x : \alpha \in \mathbb{R}\}$.

(ii) $\{f_\alpha(x) = \sin(x+\alpha) : \alpha \in \mathbb{R}\}$.

(iii) $\{f_\alpha(x) = \arctan(\alpha x) : \alpha \in \mathbb{R}\}$.

(iv) $\{f_\alpha(x) = \mathrm{e}^{x-\alpha} : \alpha \in [0,\infty)\}$.

Solution. (i) It is not relatively compact.

The set of functions is not equicontinuous. In fact, for all $\delta > 0$ there exists $\alpha > 0$ such that $2\pi/\alpha < \min\{(b-a)/2, \delta/2\}$. Since $2\pi/\alpha$ is the least positive period of f_α, there always exist $x_1, x_2 \in [a,b]$ with $|x_1 - x_2| < \delta$ such that $f_\alpha(x_1) = 1, f_\alpha(x_2) = -1 \Rightarrow |f_\alpha(x_1) - f_\alpha(x_2)| = 2$.

(ii) This set of functions is relatively compact.

In fact, it is obvious that $|f_\alpha| \le 1$ for all $\alpha \in \mathbb{R}$, which indicates that the set is uniformly bounded.

For all $\varepsilon > 0$ and $\alpha \in \mathbb{R}$, we have $|f_\alpha(x_1) - f_\alpha(x_2)| \le |\sin(x_1+\alpha) - \sin(x_2+\alpha)| = 2|\cos(\frac{x_1+x_2}{2} + \alpha)\sin(\frac{x_1-x_2}{2})| \le 2|\frac{x_1-x_2}{2}| = |x_1 - x_2| < \varepsilon$ whenever $x_1, x_2 \in [a,b]$ with $|x_1 - x_2| < \varepsilon$. So the set is equicontinuous.

Hence, the set is relatively compact by Arzelà-Ascoli Theorem.

(iii) The answer depends on whether or not $0 \in [a,b]$.

For all $\alpha \in \mathbb{R}$ and $x \in [a,b]$, $|f_\alpha(x)| \le \pi/2$. So the set is uniformly bounded.

If $0 \notin [a,b]$, set $m = \min\{|a|,|b|\} > 0$ and then $|f'_\alpha(x)| = \frac{\alpha}{1+(\alpha x)^2} \le \frac{\alpha}{1+(\alpha m)^2} = \frac{1}{m^2\alpha + 1/\alpha} \le \frac{1}{2m}$ for all $x \in [a,b]$. Thus, the set is equicontinuous by Exercise 3.3.

But when $0 \in [a,b]$, without loss of generality, we assume that $b > 0$. For all $\delta > 0$, choose $\alpha > \max\{1/b, 1/\delta\}$ then $0 < 1/\alpha < b$, $1/\alpha - 0 < \delta$, and $\arctan(\alpha \cdot 1/\alpha) - \arctan(0) = \arctan(1) = \pi/4$. So the set is not uniformly bounded.

Thus, the set is relatively compact iff $0 \notin [a,b]$ by Arzelà-Ascoli Theorem and not iff $0 \in [a,b]$.

(iv) It is relatively compact.

In fact, for all $\alpha \in [0,\infty)$ and $x \in [a,b]$, $|f_\alpha(x)| = |f'_\alpha(x)| \le \mathrm{e}^b$, which means that the set is uniformly bounded and equicontinuous by Exercise 3.3. So it is relatively compact by Arzelà-Ascoli Theorem. □

Exercise 3.5. *Let the mapping $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = x + \pi/2 - \arctan x$ for all $x \in \mathbb{R}$. Show that $d(Tx_1, Tx_2) < d(x_1, x_2)$ whenever $x_1, x_2 \in \mathbb{R}$ and $x_1 \neq x_2$, but T has no fixed-point.*

Proof. For all $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$, there exists ξ between x and y such that

$$|\arctan x_1 - \arctan x_2| = |f'(\xi)||x_1 - x_2| = \frac{|x_1 - x_2|}{1 + \xi^2}.$$

Consider the function $f(x) = x - \arctan x + c$ with an arbitrary constant c defined on $\mathbb{R}$. Then $f'(x) = \frac{x^2}{1+x^2}$ and $f'(x) = 0$ iff $x = 0$. So f strictly increases on $\mathbb{R}$ with $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$, which shows that f has a unique root on $\mathbb{R}$. Thus, the graphs of $y = x + c$ and $y = \arctan x$ intersect at a unique point.

If $\xi = 0$, then there is some constant c such that $(x_1, \arctan x_1)$ and $(x_2, \arctan x_2)$ lie on the graph of $y = x + c$, which is a contradiction since $x_1 \neq x_2$. Now we see that $\xi \neq 0$ if $x_1 \neq x_2$. Hence, $|\arctan x_1 - \arctan x_2| < |x_1 - x_2|$. Note that the signs of $x_1 - x_2$ and $\arctan x_1 - \arctan x_2$ are the same. So $d(Tx_1, Tx_2) = |Tx_1 - Tx_2| = |(x_1 - x_2) - (\arctan x_1 - \arctan x_2)| < |x_1 - x_2| = d(x_1, x_2)$.

From $Tx = x$, it follows that $\arctan x = \pi/2$. But $\arctan x < \pi/2$ for all $x \in \mathbb{R}$, which yields the contradiction. Thus, T has no fixed-point on $\mathbb{R}$. $\square$

Exercise 3.6. *Suppose that (X, d) is complete and $T : X \to X$. Prove that if*

$$\alpha_0 = \inf_{n \in \mathbb{N}_+} \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d(T^n x, T^n y)}{d(x, y)} < 1,$$

then T has a unique fixed-point on X.

Proof. Choose $c \in (\alpha_0, 1)$ and then there exists $n_0 \in \mathbb{N}_+$ such that

$$\alpha_0 \leq \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d(T^n x, T^n y)}{d(x, y)} < c,$$

which means $d(T^{n_0} x, T^{n_0} y) \leq c d(x, y)$ for all $x, y \in X$. Since T^{n_0} is a contraction with c, then $Tx = x$ has a unique solution, i.e. T has a unique fixed-point on X. $\square$

Exercise 3.7. *Let (X, d) be compact. Suppose the mapping $T : X \to X$ satisfies that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Prove that T has a unique fixed-point on X.*

Proof. Let $f(x) = d(Tx, x)$ for all $x \in X$ and f is continuous on X. In fact, for all $x \in X$ and all neighborhood V of $f(x)$, there exists $r > 0$ such that $B(f(x), r) \subset V$. For all $y \in B(x, r/2)$, we have $d(f(x), f(y)) = |d(Tx, x) - d(Ty, y)| \leq d(Tx, Ty) + d(x, y) < 2d(x, y) < r$ by Exercise 1.3, which shows that $f(B(x, r/2)) \subset B(f(x), r) \subset V$.

Since X is compact, there exists $x_0 \in X$ such that $f(x_0) = \min_{x \in X} f(x)$. We claim that $f(x_0) = 0$. If not, $f(Tx_0) = d(T^2x_0, Tx_0) < d(Tx_0, x_0) = \min_{x \in X} f(x)$, which is a contradiction. So $f(x_0) = d(Tx_0, x_0) = 0$. Thus, $Tx_0 = x_0$.

If there exists $x_1 \in X$ such that $Tx_1 = x_1$, then $d(x_1, x_0) = d(Tx_1, Tx_0) < d(x_1, x_0)$ if $x_1 \neq x_0$, which is impossible. Thus, $x_1 = x_0$ and T has a unique fixed-point. □

Remark 3.8. *The claim falls when X is not compact. Take $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x^2 + 1}$ for all $x \in \mathbb{R}$ as a counterexample.*

Exercise 3.9. *Suppose that $(X, \|\bullet\|)$ is a normed linear space. If $x \in X \setminus \{0\}$ and $r > 0$, find a real number $c \in \mathbb{R}$ such that $\|cx\| = r$.*

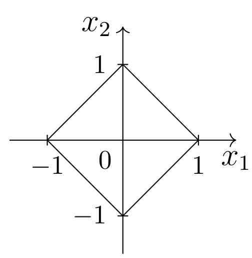
Solution. Let $c = r/\|x\|$ and then $\|cx\| = |c| \cdot \|x\| = \frac{r}{\|x\|} \cdot \|x\| = r$. So $c = r/\|x\|$ is a real number which we are looking for. □

Exercise 3.10. *Consider the real linear space $\mathbb{R}^2$. For every $x = (x_1, x_2) \in \mathbb{R}^2$, let $\|x\|_1 = |x_1| + |x_2|$.*

(a) *Show that $\|\bullet\|_1$ defines a norm on $\mathbb{R}^2$.*
(b) *Sketch the unit circle $\{x \in \mathbb{R}^2 : \|x\|_1 = 1\}$.*

Solution. (a) Assume that $x, y \in \mathbb{R}^2$ and $\alpha \in \mathbb{F}$. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Firstly, $\|x\|_1 = |x_1| + |x_2| \geq 0$ and $\|x\|_1 = 0$ iff $|x_1| + |x_2| = 0 \Leftrightarrow x_1 = x_2 = 0 \Leftrightarrow x = 0$. Secondly, $\|\alpha x\|_1 = |\alpha x_1| + |\alpha x_2| = |\alpha|(|x_1| + |x_2|) = |\alpha| \cdot \|x\|_1$. Thirdly, $\|x + y\|_1 = |x_1 + y_1| + |x_2 + y_2| \leq (|x_1| + |x_2|) + (|y_1| + |y_2|) \leq \|x\|_1 + \|y\|_1$. Then $\|\bullet\|_1$ is indeed a well-defined norm on $\mathbb{R}^2$ since x, y, α are arbitrary.

(b) Denote $x = (x_1, x_2)$ and note that $\|x\|_1 = 1 \Leftrightarrow |x_1| + |x_2| = 1$.



The graph of the unit circle $\{x \in \mathbb{R}^2 : \|x\|_1 = 1\} = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + |x_2| = 1\}$ is as above. □

Exercise 3.11. *Show that the discrete metric on a linear space $X \neq \{0\}$ cannot be obtained from a norm.*

Proof. If there exists a norm $\|\bullet\|$ on X such that the discrete metric d on X is induced from $\|\bullet\|$, then select $x_0 \in X \setminus \{0\}$. It is obvious that $\|x_0\| = d(x_0, 0) = 1$. Note that $d(x_0/2, 0) = \|x_0/2\| = \|x_0\|/2 = 1/2$ but $d(x_0/2, 0) = 1$ since $x_0/2 \neq 0$, which is a contradiction. Hence, the discrete metric cannot be induced from a norm. □

Exercise 3.12. *Suppose that $(X, \|\bullet\|_1)$ and $(Y, \|\bullet\|_2)$ are Banach spaces. Let $Z = X \times Y$ with the norm $\|(x, y)\| = \|x\|_1 + \|y\|_2$ for all $(x, y) \in X \times Y$.*

(a) *Prove that Z is a Banach space.*
(b) *Is $(Z, \|\bullet\|_0)$ a Banach space with the norm $\|(x, y)\|_0 = \max\{\|x\|_1, \|y\|_2\}$ defined for all $(x, y) \in Z$?*

Solution. (a) Suppose $\sum_{n=1}^{\infty} \|(x_n, y_n)\|$ is a convergent series. Since $\sum_{n=1}^{\infty} \|(x_n, y_n)\| = \sum_{n=1}^{\infty} \|x_n\|_1 + \sum_{n=1}^{\infty} \|y_n\|_2$, then $\sum_{n=1}^{\infty} \|x_n\|_1$ and $\sum_{n=1}^{\infty} \|y_n\|_2$ are both convergent.

From the fact that X, Y are both Banach spaces, it follows that $\sum_{n=1}^{\infty} x_n$ converges in X and $\sum_{n=1}^{\infty} y_n$ converges in Y. So $\sum_{n=1}^{\infty} (x_n, y_n)$ converges in Z since $\sum_{n=1}^{\infty} (x_n, y_n) = (\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} y_n)$.

Hence, Z is a Banach space since $\sum_{n=1}^{\infty} \|(x_n, y_n)\|$ is arbitrary.

(b) The answer is yes.

In fact, suppose $\sum_{n=1}^{\infty} \|(x_n, y_n)\|$ is a convergent series. Since $\sum_{n=1}^{\infty} \|(x_n, y_n)\| = \sum_{n=1}^{\infty} \max\{\|x_n\|_1, \|y_n\|_2\} \Rightarrow \sum_{n=1}^{\infty} \|x_n\|_1 \le \sum_{n=1}^{\infty} \|(x_n, y_n)\|$ and $\sum_{n=1}^{\infty} \|y_n\|_2 \le \sum_{n=1}^{\infty} \|(x_n, y_n)\|$, then $\sum_{n=1}^{\infty} \|x_n\|_1$ and $\sum_{n=1}^{\infty} \|y_n\|_2$ are both convergent.

From the fact that X, Y are both Banach spaces, it follows that $\sum_{n=1}^{\infty} x_n$ converges in X and $\sum_{n=1}^{\infty} y_n$ converges in Y. So $\sum_{n=1}^{\infty} (x_n, y_n)$ converges in Z since $\sum_{n=1}^{\infty} (x_n, y_n) = (\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} y_n)$.

Hence, Z is a Banach space since $\sum_{n=1}^{\infty} \|(x_n, y_n)\|$ is arbitrary. $\square$

Exercise 3.13. *Let $0 < \alpha < \beta$. For what value of p does the function $f(x) = 1/(x^\alpha + x^\beta)$ $(x \in (0, \infty))$ belong to $L^p(0, \infty)$?*

Solution. When $p < 1/\alpha$, $\int_0^1 1/(x^\alpha + x^\beta)^p \mathrm{d}x \le \int_0^1 1/x^{\alpha p} \mathrm{d}x$ is finite since $\alpha p < 1$. When $p > 1/\beta$, $\int_1^\infty 1/(x^\alpha + x^\beta)^p \mathrm{d}x \le \int_1^\infty 1/x^{\beta p} \mathrm{d}x$ is finite since $\beta p > 1$. So $f \in L^p(0, \infty)$ provided that $1/\beta < p < 1/\alpha$.

But if $p \ge 1/\alpha$, $\int_0^1 1/(x^\alpha + x^\beta)^p \mathrm{d}x \ge \int_0^1 1/(2x^\alpha)^p \mathrm{d}x \ge 2^{-p} \int_0^1 1/x^{\alpha p} \mathrm{d}x$ is infinite since $\alpha p \ge 1$. If $p \le 1/\beta$, $\int_1^\infty 1/(x^\alpha + x^\beta)^p \mathrm{d}x \ge \int_1^\infty 1/(2x^\beta)^p \mathrm{d}x \ge 2^{-p} \int_1^\infty 1/x^{\beta p} \mathrm{d}x$ is infinite since $\beta p \le 1$.

Hence, f belongs to $L^p(0, \infty)$ iff $p \in (1/\beta, 1/\alpha)$. $\square$

4. Assignment #4: 5/9/2018

Exercise 4.1. *Prove that ℓ^p $(p \ge 1)$ is a separable Banach space.*

Proof. Note that $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$ is separable. So take a countable dense subset, say, S of $\mathbb{F}$. Define for all $n \in \mathbb{N}_+$ that $A_n = \{(x_1, \cdots, x_n, 0, 0, \cdots) : x_i \in S, i \in [1, n] \cap \mathbb{Z}\}$. It's clear that each A_n is a countable subset of ℓ^p and so is $A = \cup_{n=1}^{\infty} A_n$.

The claim goes that A is dense in ℓ^p. Basically, for each $x = (x_1, x_2, \cdots) \in \ell^p$ and $\varepsilon > 0$, there exists an integer $N > 0$ such that $\sum_{n=N+1}^{\infty} |x_n|^p < \varepsilon^p/2$. Then choose $y = (y_1, \cdots, y_N, 0, 0, \cdots) \in A_N \subset A$ such that $|y_n - x_n|^p < \varepsilon^p/2N$ for all

$n \in [1, N] \cap \mathbb{Z}$. So

$$\|y - x\|^p \leq \sum_{n=1}^{N} |y_n - x_n|^p + \sum_{n=N+1}^{\infty} |x_n|^p < \varepsilon^p/2 + \varepsilon^p/2 = \varepsilon^p,$$

and then $\|y - x\| < \varepsilon$. Hence, $B(x, \varepsilon) \cap A \neq \varnothing$ for each $\varepsilon > 0$. It follows that $x \in \overline{A}$ and $\ell^p \subset \overline{A}$ since x is arbitrary. Thus, ℓ^p is separable.

For each Cauchy sequence $\{x_n = (x_{n,1}, x_{n,2}, \cdots)\}_{n=1}^{\infty} \subset \ell^p$ and $\varepsilon > 0$, there exists $N > 0$ such that $|x_{n,i} - x_{m,i}| \leq (\sum_{i=1}^{\infty} |x_{n,i} - x_{m,i}|^p)^{1/p} < \varepsilon$ for all $i \in \mathbb{N}_+$ when $n, m > N$. So each $\{x_{n,i}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}$. Let $y_i := \lim_{n\to\infty} x_{n,i}$ for each $i \in \mathbb{N}_+$ since $\mathbb{F}$ is complete. We claim that $\lim_{n\to\infty} x_n = y = (y_1, y_2, \cdots) \in \ell^p$.

Note that for all $k \in \mathbb{N}_+$, $\sum_{i=1}^{k} |x_{n,i} - x_{m,i}|^p (\leq \sum_{i=1}^{\infty} |x_{n,i} - x_{m,i}|^p) < \varepsilon^p$ when $n, m > N$. Send $m \to \infty$ and we will get that $\sum_{i=1}^{k} |x_{n,i} - y_i|^p \leq \varepsilon^p$ holds for all $k \in \mathbb{N}_+$ when $n > N$. Send $k \to \infty$ and we will get $\sum_{i=1}^{\infty} |x_{n,i} - y_i|^p \leq \varepsilon^p$ when $n > N$, which implies that $\|x_n - y\| < \varepsilon$ when $n > N$. So $\|x_n - y\| \to 0$ as $n \to \infty$. Also, since $\|y\| \leq \|x_n\| + \|x_n - y\| < \|x_n\| + \varepsilon$ when $n > N$, it follows that $y \in \ell^p$. Hence, ℓ^p is complete since $\{x_n\}_{n=1}^{\infty}$ is arbitrary. $\square$

Exercise 4.2. *Let E be a Lebesgue measurable set in $\mathbb{R}$ with $m(E) < \infty$. Denote the norms on $L^p(E)$ $(p \geq 1)$ and $L^\infty(E)$ by $\| \bullet \|_p$ and $\| \bullet \|_\infty$ respectively. Prove that $L^\infty(E) \subset L^p(E)$ for all $p \geq 1$ and $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.*

Proof. For each $x \in L^\infty(E)$, there exists $A \subset E$ with $m(A) = 0$ such that

$$\sup_{t\in E\setminus A} |x(t)| = \operatorname*{ess\,sup}_{t\in E} |x(t)| = \|x\|_\infty := M < \infty.$$

It follows that

$$\int_E |x(t)|^p \mathrm{d}t = \int_A |x(t)|^p \mathrm{d}t + \int_{E\setminus A} |x(t)|^p \mathrm{d}t = \int_{E\setminus A} |x(t)|^p \mathrm{d}t \leq M^p m(E) < \infty,$$

which shows that $x \in L^p(E)$. Thus, $L^\infty(E) \subset L^p(E)$ since x is arbitrary.

If $M = 0$, then $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$ would be really obvious. So we may assume that $M > 0$. By the definition of essential supremum, for each $\varepsilon \in (0, M)$, there exists $B \subset E$ with $m(B) > 0$ such that $|x(t)| > M - \varepsilon$ when $t \in B$. Then

$$\|x\|_p = \left(\int_E |x(t)|^p \mathrm{d}t\right)^{1/p} \geq \left(\int_B (M-\varepsilon)^p \mathrm{d}t\right)^{1/p} = (M - \varepsilon)(m(B))^{1/p} \to M - \varepsilon$$

as $p \to \infty$, which implies that $\lim_{p\to\infty} \|x\|_p \geq M$ since $\varepsilon \in (0, M)$ is arbitrary. Also, $\|x\|_p \leq M(m(E))^{1/p} \to M$ as $p \to \infty$. Hence, $\|x\|_p \to \|x\|_\infty$ as $p \to \infty$. $\square$

Exercise 4.3. *Let $P[0,1]$ denote the complex vector space of all complex-valued polynomials defined on $[0,1]$. This can be viewed as a linear subspace of $C([0,1], \mathbb{C})$. Show that the two norms $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$ and $\|f\|_1 = \int_0^1 |f(t)|\mathrm{d}t$ defined for all $f \in P[0,1]$ are not equivalent on $P[0,1]$.*

Proof. For each $n \in \mathbb{N}_+$, define $f_n(t) = t^n$ $(t \in [0,1])$. It follows that $f_n \in P[0,1]$, $\|f_n\|_\infty = 1$, and $\|f_n\|_1 = 1/(n+1)$ for all $n \in \mathbb{N}_+$. Suppose that $\|\bullet\|_\infty$ and $\|\bullet\|_1$ are equivalent on $P[0,1]$, then there exists a constant $K > 0$ such that $\|f\|_\infty \le K\|f\|_1$ whenever $f \in P[0,1]$. So $1 = \|f_n\|_\infty \le K\|f_n\|_1 = K/(n+1) \to 0$ as $n \to \infty$, which is a contradiction. Hence, $\|\bullet\|_\infty$ and $\|\bullet\|_1$ are not equivalent on $P[0,1]$. $\square$

Exercise 4.4.

(i) *If $(X, \|\bullet\|)$ is a normed linear space and $Y \subset X$ is a finite-dimensional linear subspace, prove that any element of X has a projection on Y, that is, for each $x \in X$, there exists $y_0 \in Y$ such that $\|x - y_0\| = d(x, Y) = \inf_{y \in Y} \|x - y\|$.*

(ii) *Is this projection unique? Give a proof or a counterexample.*

Solution. (i) Let $\delta := d(x, Y)$. By definition of infimum, there exists $\{y_n\}_{n=1}^\infty \subset Y$ such that $\|y_n - x\| \to \delta$ $(n \to \infty)$. Since $(y_n + y_m)/2 \in Y$ for each $n, m \in \mathbb{N}_+$, we derive

$$\begin{aligned} 0 \le \|y_n - y_m\|^2 &= \|(y_n - x) + (x - y_m)\|^2 \\ &= 2(\|y_n - x\|^2 + \|y_m - x\|^2) - \|2x - (y_n + y_m)\|^2 \\ &= 2(\|y_n - x\|^2 + \|y_m - x\|^2) - 4\left\|x - \frac{y_n + y_m}{2}\right\|^2 \\ &\le 2(\|y_n - x\|^2 + \|y_m - x\|^2) - 4\delta^2 \to 0 \ (n, m \to \infty), \end{aligned}$$

which implies that $\{y_n\}_{n=1}^\infty$ is a Cauchy sequence. Note that Y is complete since Y is finite dimensional. Hence there exists $y_0 \in Y$ such that $d(y_n, y_0) \to 0$ $(n \to \infty)$. Then $\|x - y_0\| = \|x - \lim_{n\to\infty} y_n\| = \lim_{n\to\infty} \|x - y_n\| = \delta$ since $\|\bullet\|$ is continuous.

(ii) If $y_1 \in Y$ and $\|x - y_1\| = \delta$, then $0 \le \|y_1 - y_0\| \le 2(\|y_1 - x\|^2 + \|y_0 - x\|^2) - 4\delta^2 = 0$. So $\|y_1 - y_0\| = 0 \Rightarrow y_1 = y_0$. Hence, the projection is unique. $\square$

Exercise 4.5. *Let $S = \{\{x_n\}_{n=1}^\infty \in \ell^2 :$ there exists $N \in \mathbb{N}_+$ such that $x_n = 0$ for all $n \ge N\}$ so that S is a linear subspace of ℓ^2 consisting of all sequences with only finitely many nonzero terms. Show that S is not closed.*

Proof. Let $x = (1, 1/2, 1/3, \cdots)$ and $x_n = (1, 1/2, \cdots, 1/n, 0, 0, \cdots) \in \ell^2$ for each $n \in \mathbb{N}_+$. Since $\sum_{n=1}^\infty 1/n^2 < \infty$, it follows that $x \in \ell^2$. And $\|x_n - x\| = \sum_{i=n+1}^\infty 1/i^2 \to 0$ as $n \to \infty$, which shows that $\{x_n\}_{n=1}^\infty \subset S$ is a convergent sequence in ℓ^p and then $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in S. But $x \notin S$. Thus, S is not closed. $\square$

Exercise 4.6. *Let X be a normed linear space, $x \in X \setminus \{0\}$, and Y be a linear subspace of X.*

(a) *If there exists $\eta > 0$ such that $\{y \in X : \|y\| < \eta\} \subset Y$, show that $\frac{\eta x}{2\|x\|} \in Y$ whenever $x \in X$.*

(b) *Suppose Y is open. Show that $Y = X$.*

Proof. (a) Since

$$\left\| \frac{\eta x}{2\|x\|} \right\| = \frac{\eta}{2\|x\|}\|x\| = \frac{\eta}{2} < \eta,$$

it follows that $\frac{\eta x}{2\|x\|} \in \{y \in X : \|y\| < \eta\}$. So $\frac{\eta x}{2\|x\|} \in Y$.

(b) Since $0 \in Y$ and Y is open, it follows that there exists $r > 0$ such that $\{y \in X : \|y\| < r\} = B(0, r) \subset Y$. By (a), $\frac{rx}{2\|x\|} \in Y$ for each $x \in X$. Note that Y is a linear subspace. Thus, $x = \frac{2\|x\|}{\eta} \cdot \frac{\eta x}{2\|x\|} \in Y$. So $X \subset Y$ and then $Y = X$. $\square$

Exercise 4.7. *Let X be a normed linear space with $X \neq \{0\}$. Show that X is a Banach space iff the set $S = \{x \in X : \|x\| = 1\}$ is complete in X.*

Proof. $\Rightarrow$) Since X is complete and S is closed, it's obvious that S is complete.

$\Leftarrow$) Suppose $\{x_n\}_{n=1}^{\infty} \subset X$ is an arbitrary Cauchy sequence. Then $|\|x_n\| - \|x_m\|| \leq \|x_n - x_m\| \to 0$ $(n \to \infty)$. So $\{\|x_n\|\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $c \in \mathbb{R}$ such that $\lim_{n\to\infty} \|x_n\| = c$. If $c = 0$, it's obvious that $x_n \to 0 \in X$ as $n \to \infty$. If $c > 0$, then there exists $N > 0$ such that $c/2 \leq \|x_n\| \leq 3c/2$ when $n > N$. So when $n, m > N$,

$$\begin{aligned}
\left\| \frac{x_n}{\|x_n\|} - \frac{x_m}{\|x_m\|} \right\| &= \frac{\| \|x_m\| x_n - \|x_n\| x_m \|}{\|x_n\| \|x_m\|} \\
&\leq \frac{4}{c^2} \| \|x_m\|(x_n - x_m) + (\|x_m\| - \|x_n\|)x_m \| \\
&\leq \frac{4}{c^2} (\|x_m\| \|x_n - x_m\| + |\|x_m\| - \|x_n\|| \|x_m\|) \to 0 \ (n, m \to \infty).
\end{aligned}$$

Thus, $\{x_n/\|x_n\|\}_{n=N+1}^{\infty} \subset S$ is a Cauchy sequence. Since S is complete, there is some $x \in S$ such that $x_n/\|x_n\| \to x$ as $n \to \infty$, which implies that $x_n \to cx$ as $n \to \infty$. It follows that X is complete since $\{x_n\}_{n=1}^{\infty}$ is arbitrary. $\square$

Exercise 4.8. *In an inner product space, suppose that $y \neq 0$. Prove that $\|x + y\| = \|x\| + \|y\|$ iff $x = py$ for some real $p \geq 0$.*

Proof. Since $\|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \overline{\langle x, y \rangle}$ and $(\|x\| + \|y\|)^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$, it follows that $\|x + y\| = \|x\| + \|y\| \Leftrightarrow \langle x, y \rangle + \overline{\langle x, y \rangle} = 0 \Leftrightarrow \mathrm{Re}(\langle x, y \rangle) = \|x\|\|y\|$. By Cauchy-Schwarz inequality, $|\langle x, y \rangle| \leq \|x\|\|y\|$. Since $|\langle x, y \rangle| = \sqrt{(\mathrm{Re}(\langle x, y \rangle))^2 + (\mathrm{Im}(\langle x, y \rangle))^2}$, it follows that

$$\|x\|\|y\| = \mathrm{Re}(\langle x, y \rangle) \leq \sqrt{(\mathrm{Re}(\langle x, y \rangle))^2 + (\mathrm{Im}(\langle x, y \rangle))^2} = |\langle x, y \rangle| \leq \|x\|\|y\|.$$

So $\mathrm{Re}(\langle x,y\rangle) = \langle x,y\rangle$ and $\mathrm{Im}(\langle x,y\rangle) = 0$. Thus, $\mathrm{Re}(\langle x,y\rangle) = \|x\|\|y\| \Leftrightarrow \langle x,y\rangle = \|x\|\|y\|$. Note that $y \neq 0$. Let $p := \langle x,y\rangle/\langle y,y\rangle$, then if $p \in \mathbb{R}$,

$$\begin{aligned}\langle x-py, x-py\rangle &= \langle x,x\rangle + p^2\langle y,y\rangle - p\overline{\langle x,y\rangle} - \bar{p}\langle x,y\rangle \\ &= \langle x,x\rangle + \frac{|\langle x,y\rangle|^2}{(\langle y,y\rangle)^2}\langle y,y\rangle - \frac{\langle x,y\rangle\overline{\langle x,y\rangle}}{\langle y,y\rangle} - \frac{\overline{\langle x,y\rangle}\langle x,y\rangle}{\langle y,y\rangle} \\ &= \langle x,x\rangle - \frac{|\langle x,y\rangle|^2}{\langle y,y\rangle} = \frac{\langle x,x\rangle\langle y,y\rangle - |\langle x,y\rangle|^2}{\langle y,y\rangle}.\end{aligned}$$

So $\langle x,y\rangle = \|x\|\|y\| \Leftrightarrow \langle x-py, x-py\rangle = 0$ and $p \in \mathbb{R} \Leftrightarrow x = py$ and $p \in \mathbb{R}$. □

Exercise 4.9. *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in an inner product space satisfying $\|x_n\|, \|y_n\| \leq 1$ for all $n \in \mathbb{N}_+$ and $\lim_{n\to\infty}\langle x_n, y_n\rangle = 1$. Show that $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Proof. The claim goes that $\lim_{n\to\infty}\|x_n\|^2 + \|y_n\|^2 = 2$. In fact, since $\|x_n\| \leq 1$ and $\|y_n\| \leq 1$ for all $n \in \mathbb{N}_+$, then $\lim_{n\to\infty}\|x_n\|^2 + \|y_n\|^2 \leq 2$. Also, $\|x_n\|^2 + \|y_n\|^2 \geq 2\|x_n\|\|y_n\| \geq 2|\langle x_n, y_n\rangle|^2 \to 2$ as $n \to \infty$. Hence, $\lim_{n\to\infty}\|x_n\|^2 + \|y_n\|^2 = 2$. Note that complex conjugate is uniformly continuous when regarded as a function. It follows that $\|x_n - y_n\|^2 = \langle x_n - y_n, x_n - y_n\rangle = (\|x_n\|^2 + \|y_n\|^2) - \langle x_n, y_n\rangle - \overline{\langle x_n, y_n\rangle} \to 2 - 1 - 1 = 0$ as $n \to \infty$. □

Exercise 4.10. *Let X be a real inner product space. If $\|x+y\|^2 = \|x\|^2 + \|y\|^2$, show that $x \perp y$. Is this result true if X is complex?*

Solution. Since $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(\langle x,y\rangle)$, it follows from $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ that $\mathrm{Re}(\langle x,y\rangle) = 0$. If X is real, then $\langle x,y\rangle = \mathrm{Re}(\langle x,y\rangle) = 0 \Rightarrow x \perp y$.

If X is complex, $x \perp y$ does not hold in general. As a counterexample, consider $X = \mathbb{C}$ equipped with the inner product $\langle x,y\rangle = x\bar{y}$ defined for all $x, y \in \mathbb{C}$. Let $x = 1 + \mathrm{i}$ and $y = 1 - \mathrm{i}$, then $\|x+y\| = 2$ and $\|x\| = \|y\| = \sqrt{2}$. So $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. But $\langle x,y\rangle = 2\mathrm{i} \neq 0$. Thus, $x \perp y$ fails to hold here. □

Exercise 4.11. *If an inner product space $\mathcal{H}$ is real, show that the condition $\|x\| = \|y\|$ implies $\langle x+y, x-y\rangle = 0$. What does this mean geometrically if $\mathcal{H} = \mathbb{R}^2$? What does the condition imply if $\mathcal{H}$ is complex?*

Solution. Since $\mathcal{H}$ is real and $\|x\| = \|y\|$, it follows that $\langle x+y, x-y\rangle = \|x\|^2 - \|y\|^2 - \langle x,y\rangle + \overline{\langle x,y\rangle} = -\langle x,y\rangle + \langle x,y\rangle = 0$.

If $\mathcal{H} = \mathbb{R}^2$, $\langle x+y, x-y\rangle = 0$ means that the two diagonals of a rhombus are perpendicular to each other.

If $\mathcal{H}$ is complex, then $\langle x+y, x-y\rangle = -\langle x,y\rangle + \overline{\langle x,y\rangle} = -2\mathrm{i}\cdot\mathrm{Im}(\langle x,y\rangle)$, which shows that $\mathrm{Re}(\langle x+y, x-y\rangle) = 0$. □

Exercise 5.1. *Let M be a convex subset of a Hilbert space $\mathcal{H}$ and let $\{x_n\}_{n=1}^{\infty} \subset M$ with $\|x_n\| \to d := \inf_{x\in M}\|x\|$ as $n \to \infty$. Prove that $\{x_n\}_{n=1}^{\infty}$ is convergent in $\mathcal{H}$.*

Proof. By parallelogram law, $\|x_n - x_m\|^2 = 2(\|x_n\|^2 + \|x_m\|^2) - \|x_n + x_m\|^2$. Note that the convexity of M assures that $(x_n + x_m)/2$ always belongs to M. So divide both sides of the identity by 4, then we derive

$$0 \le \frac{1}{4}\|x_n - x_m\|^2 = \frac{1}{2}(\|x_n\|^2 + \|x_m\|^2) - \left\|\frac{x_n + x_m}{2}\right\|^2$$
$$\le \frac{1}{2}(\|x_n\|^2 + \|x_m\|^2) - d^2 \to \frac{1}{2}(d^2 + d^2) - d^2 = 0 \ (n \to \infty),$$

which shows that $\|x_n - x_m\| \to 0$ as $n \to \infty$. Thus, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{H}$ and so is convergent in $\mathcal{H}$ since $\mathcal{H}$ is complete. □

Exercise 5.2. *Let X be an inner product space over $\mathbb{F}$ and $x, y \in X$. Prove that $x \perp y$ iff $\|x + \alpha y\| \ge \|x\|$ for all $\alpha \in \mathbb{F}$.*

Proof. Firstly we notice that $x \perp y \Leftrightarrow x \perp \operatorname{span}(\{y\}) \Leftrightarrow x \in (\operatorname{span}(\{y\}))^{\perp}$. Hence, $x \perp y$ is equivalent to $\|x - y'\| \ge \|x\|$ whenever $y' \in \operatorname{span}(\{y\})$. Since each $y' \in \operatorname{span}(\{y\})$ would be of the form $y' = -\alpha y$ where $\alpha \in \mathbb{F}$. So it follows that $x \perp y$ iff $\|x + \alpha y\| \ge \|x\|$ for all $\alpha \in \mathbb{F}$. □

Exercise 5.3. *Suppose $A = \{\{x_n\}_{n=1}^{\infty} \in \ell^2 : x_{2n} = 0$ for all $n \in \mathbb{N}_+\}$. Find $A^{\perp}$.*

Solution. Let $B = \{\{x_n\}_{n=1}^{\infty} \in \ell^2 : x_{2n-1} = 0$ for all $n \in \mathbb{N}_+\}$, then it's clear that $B \subset A^{\perp}$. For each $y = \{y_n\}_{n=1}^{\infty} \in A^{\perp}$, select $x = \{x_n\}_{n=1}^{\infty} \in A$ such that $x_{2n-1} = y_{2n-1}$ for all $n \in \mathbb{N}_+$. Then $\langle y, x\rangle = \sum_{n=1}^{\infty} |x_{2n-1}|^2 = 0 \Rightarrow x_{2n-1} = 0$ for all $n \in \mathbb{N}_+$, which indicates that $y \in B$. Hence, $A^{\perp} \subset B$ since y is arbitrary. So $A^{\perp} = B$. □

Exercise 5.4. *Let X be an inner product space and let $A \subset X$. Show that $A^{\perp} = \bar{A}^{\perp}$.*

Proof. Since $A \subset \bar{A}$, it's obvious that $\bar{A}^{\perp} \subset A^{\perp}$. For each $y \in A^{\perp}$, then for all $x \in \bar{A}$, there exists $\{x_n\}_{n=1}^{\infty} \subset A$ such that $x_n \to x$ as $n \to \infty$. Then $\langle y, x\rangle = \lim_{n\to\infty}\langle y, x_n\rangle = 0$ since $\langle \bullet, \bullet\rangle$ is continuous. So $y \in \bar{A}^{\perp}$ since x is arbitrary. Hence, $A^{\perp} \subset \bar{A}^{\perp}$. So we derive $A^{\perp} = \bar{A}^{\perp}$. □

Exercise 5.5. *Let X be a Hilbert space and let $A \subset X$ be nonempty. Show that:*

(a) $A^{\perp\perp} = \overline{\operatorname{span}(A)}$;
(b) $A^{\perp\perp\perp} = A^{\perp}$.

Proof. (a) Firstly it's known that $A \subset A^{\perp\perp}$ and $A^{\perp\perp}$ is a closed linear subspace of X. So $A \subset A^{\perp\perp} \Rightarrow \operatorname{span}(A) \subset \operatorname{span}(A^{\perp\perp}) = A^{\perp\perp}$ and then $\overline{\operatorname{span}(A)} \subset A^{\perp\perp}$.

Note that $A \subset \overline{\operatorname{span}(A)}$. So $A^{\perp} \supset \overline{\operatorname{span}(A)}^{\perp}$ and then $A^{\perp\perp} \subset \overline{\operatorname{span}(A)}^{\perp\perp}$. Since X is a Hilbert space and $\overline{\operatorname{span}(A)}$ is a closed linear subspace of X, it follows that $X =$

$\overline{\operatorname{span}(A)} \oplus \overline{\operatorname{span}(A)}^{\perp}$. For each $x \in \overline{\operatorname{span}(A)}^{\perp\perp} \subset X$, there exists unique $y \in \overline{\operatorname{span}(A)}$ and $z \in \overline{\operatorname{span}(A)}^{\perp}$ such that $x = y + z$. Noticing $x \perp z$, $\langle x, z\rangle = \langle y, z\rangle + \langle z, z\rangle = \langle z, z\rangle = \|z\|^2 = 0 \Rightarrow z = 0$. So $x = y \in \overline{\operatorname{span}(A)}$ and then $\overline{\operatorname{span}(A)}^{\perp\perp} \subset \overline{\operatorname{span}(A)}$. Since $\overline{\operatorname{span}(A)} \subset \overline{\operatorname{span}(A)}^{\perp\perp}$, it follows that $\overline{\operatorname{span}(A)}^{\perp\perp} = \overline{\operatorname{span}(A)}$. Now we see that $A^{\perp\perp} \subset \overline{\operatorname{span}(A)}$.

In all, it has been showed that $A^{\perp\perp} = \overline{\operatorname{span}(A)}$.

(b) By (a), $A^{\perp\perp\perp} = (A^{\perp})^{\perp\perp} = \overline{\operatorname{span}(A^{\perp})} = \overline{A^{\perp}} = A^{\perp}$. □

Exercise 5.6. *Use the Gram-Schmidt algorithm to orthogonalize and normalize the following vectors $x_0(t) \equiv 1$, $x_1(t) = t$, $x_2(t) = t^2$ in $L^2[-1, 1]$.*

Solution. We compute that $\|x_0\| = \sqrt{2}$, $\|x_1\| = \sqrt{2/3}$, $\|x_2\| = \sqrt{2/5}$, $\langle x_0, x_1\rangle = 0$, $\langle x_0, x_2\rangle = 2/3$, and $\langle x_1, x_2\rangle = 0$. So

$$\begin{aligned}
u_0(t) &= x_0(t) \equiv 1, \\
u_1(t) &= x_1(t) - \frac{\langle u_0, x_1\rangle}{\langle u_0, u_0\rangle} u_0(t) = t, \\
u_2(t) &= x_2(t) - \frac{\langle u_0, x_2\rangle}{\langle u_0, u_0\rangle} u_0(t) - \frac{\langle u_1, x_2\rangle}{\langle u_1, u_1\rangle} u_1(t) = t^2 - \frac{1}{3}.
\end{aligned}$$

Then normalize u_0, u_1, u_2, and we derive

$$\begin{aligned}
e_0(t) &= \frac{u_0(t)}{\|u_0\|} \equiv \frac{\sqrt{2}}{2}, \\
e_1(t) &= \frac{u_1(t)}{\|u_1\|} = \frac{\sqrt{6}}{2} t, \\
e_2(t) &= \frac{u_2(t)}{\|u_2\|} = \frac{\sqrt{10}}{4}(3t^2 - 1).
\end{aligned}$$

□

Exercise 5.7. *Show that an orthonormal sequence $\{e_n\}_{n\in\mathbb{N}_+}$ in a Hilbert space $\mathcal{H}$ cannot have a convergent subsequence.*

Proof. If $\{e_{n_k}\}_{k=1}^{\infty} \subset \{e_n\}_{n\in\mathbb{N}_+}$ is a convergent subsequence, then it is a Cauchy sequence in $\mathcal{H}$. So $\sqrt{2} = \sqrt{\|e_{n_k}\|^2 + \|e_{n_l}\|^2} = \|e_{n_k} - e_{n_l}\| \to 0$ as $n \to \infty$, which is a contradiction. Hence, $\{e_n\}_{n\in\mathbb{N}_+}$ has no convergent subsequence. □

Exercise 5.8. *Let $\mathcal{H}$ be a Hilbert space and let $\{e_n\}_{n\in\mathbb{N}_+}$ be an orthonormal sequence in $\mathcal{H}$. Determine whether the following series converge in $\mathcal{H}$.*

(a) $\sum_{n=1}^{\infty} \frac{e_n}{n}$. (b) $\sum_{n=1}^{\infty} \frac{e_n}{\sqrt{n}}$.

Solution. By Riesz-Fischer theorem, we derive:

(a) $\sum_{n=1}^{\infty} \frac{e_n}{n}$ converges since $\sum_{n=1}^{\infty} n^{-2} < \infty$;

(b) $\sum_{n=1}^{\infty} \frac{e_n}{\sqrt{n}}$ does not converge since $\sum_{n=1}^{\infty} n^{-1}$ diverges. □

Exercise 5.9. *Let $\{e_n\}_{n\in\mathbb{N}_+}$ be an orthonormal basis in a Hilbert space $\mathcal{H}$ and let $\{f_n\}_{n\in\mathbb{N}_+}$ be an orthonormal sequence in $\mathcal{H}$, satisfying $\sum_{n=1}^{\infty}\|e_n - f_n\|^2 < 1$. Show that $\{f_n\}_{n=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}$.*

Proof. Since $\{e_n\}_{n\in\mathbb{N}_+}$ is an orthonormal basis, it follows that $y = \sum_{n=1}^{\infty}\langle y, e_n\rangle e_n$ for each $y \in \{f_n\}_{n\in\mathbb{N}_+}^{\perp}$. Note that $\langle y, f_n\rangle = 0$, then $y = \sum_{n=1}^{\infty}\langle y, e_n\rangle e_n - \sum_{n=1}^{\infty}\langle y, f_n\rangle e_n = \sum_{n=1}^{\infty}\langle y, e_n - f_n\rangle e_n$. The claim goes that $y = 0$. If not, then

$$\begin{aligned}
\|y\|^2 &= \left\langle \sum_{n=1}^{\infty}\langle y, e_n - f_n\rangle e_n, \sum_{n=1}^{\infty}\langle y, e_n - f_n\rangle e_n \right\rangle \\
&= \sum_{n=1}^{\infty} |\langle y, e_n - f_n\rangle|^2 \langle e_n, e_n\rangle \\
&\le \sum_{n=1}^{\infty} \|y\|^2 \|e_n - f_n\|^2 \|e_n\|^2 \\
&= \|y\|^2 \sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \|y\|^2,
\end{aligned}$$

which is a contradiction. Hence $y = 0$ and so we derive $\{f_n\}_{n\in\mathbb{N}_+}^{\perp} = \{0\}$ since y is arbitrary. Thus, $\{f_n\}_{n\in\mathbb{N}_+}$ is an orthonormal basis in $\mathcal{H}$. □

Exercise 5.10. *Consider a linear functional $T : C[0,1] \to \mathbb{C}$, defined for every $x \in C[0,1]$ by $Tx = x(1)$.*

(a) *Show that T is continuous on $C[0,1]$ with respect to the standard norm.*
(b) *Determine whether T is continuous on $C[0,1]$ with respect to the norm $\|x\| = (\int_0^1 |x(t)|^2 \mathrm{d}t)^{1/2}$, and justify your assertion.*

Solution. (a) For each $x \in C[0,1]$, it's clear that $\|Tx\| = |x(1)| \le \max_{t\in[0,1]} |x(t)| = \|x\|$, which shows that T. Also, for each $\alpha, \beta \in \mathbb{C}$ and each $x, y \in C[0,1]$, we have $T(\alpha x + \beta y) = (\alpha x + \beta y)(1) = \alpha x(1) + \beta y(1) = \alpha Tx + \beta Ty$. So T is indeed linear and then it is bounded. Hence, T is continuous on $C[0,1]$.

(b) T is not continuous on $C[0,1]$. If not, assume that T is continuous on $C[0,1]$, then there exists $M > 0$ such that $\|Tx\| \le M\|x\|$ for each $x \in C[0,1]$. Consider the sequence $\{x_n : x_n(t) = t^{n/2}, t \in [0,1]\}_{n=1}^{\infty}$. It follows that $1 = x_n(1) = \|Tx_n\| \le M\|x\| = M/\sqrt{n+1} \to 0$ $(n \to \infty)$, which is a contradiction. Hence, the assumption fails and T is not continuous on $C[0,1]$. □

Exercise 5.11. *Let $h \in L^{\infty}[0,1]$.*

(a) *If f is in $L^2[0,1]$, show that $fh \in L^2[0,1]$.*
(b) *Let $T : L^2[0,1] \to L^2[0,1]$ be defined as $Tf = hf$. Show that T is a bounded linear operator.*

Proof. (a) Since $h \in L^\infty[0,1]$, there exists $A_0 \subset [0,1]$ such that $m(A_0) = 0$ and $\|h\| = \sup_{t\in[0,1]\setminus A_0} |x(t)| < \infty$. Let $M = \|h\|$, then

$$\int_{[0,1]} |f(t)h(t)|^2 \mathrm{d}t \leq M^2 \int_{[0,1]\setminus A_0} |f(t)|^2 \mathrm{d}t \leq M^2 \int_{[0,1]} |f(t)|^2 \mathrm{d}t < \infty$$

since $f \in L^2[0,1]$. Thus, $fh \in L^2[0,1]$.

(b) It follows from the proof of (a) that $\|Tf\| \leq M\|f\|$, where $M = \|h\|$ is a nonnegative constant. Hence, T is bounded since T is obviously linear. □

Exercise 5.12. *Consider the normed linear space ℓ^2 of all square summate infinite sequences of complex numbers, with norm $\|x\| = (\sum_{i=1}^\infty |x_i|^2)^{1/2}$ defined for all $x = \{x_i\}_{i=1}^\infty \in \ell^2$. For every $x = (x_1, x_2, x_3, \cdots) \in \ell^2$, let $Tx = (0, 4x_1, x_2, 4x_3, x_4, \cdots)$.*

(a) *Show that $Tx \in \ell^2$ for every $x \in \ell^2$.*
(b) *Show that $T : \ell^2 \to \ell^2$ is a bounded linear operator.*
(c) *Find the norm $\|T\|$.*

Solution. (a) For each $x = \{x_i\}_{i=1}^\infty \in \ell^2$, $\|Tx\| = (16\sum_{i=1}^\infty |x_{2i-1}|^2 + \sum_{i=1}^\infty |x_{2i}|^2)^{1/2} \leq (16\sum_{i=1}^\infty |x_{2i-1}|^2 + 16\sum_{i=1}^\infty |x_{2i}|^2)^{1/2} = 4(\sum_{i=1}^\infty |x_i|^2)^{1/2} = 4\|x\| < \infty$. Hence, $Tx \in \ell^2$ for each $x \in \ell^2$.

(b) It's clear by (a) that $T \in \mathscr{B}(\ell^2)$.

(c) It's clear by (a) that $\|T\| \leq 4$. For $e_1 = (1, 0, 0, \cdots) \in \ell^2$, $\|e_1\| = 1$ and $\|T\| = \sup_{\|x\|=1} \|Tx\| \geq \|Te_1\| = \|(0, 4, 0, \cdots)\| = 4$. So $\|T\| = 4$. □

Exercise 5.13. *Suppose that $\mathcal{H}$ is a Hilbert space over $\mathbb{F}$, and that $x_0 \in \mathcal{H}$ is fixed. For every $x \in \mathcal{H}$, let $Tx = \langle x, y\rangle z$. Show that $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, and find the norm $\|T\|$.*

Proof. For all $x_1, x_2 \in \mathcal{H}$ and $\alpha_1, \alpha_2 \in \mathbb{F}$, $T(\alpha_1 x_1 + \alpha_2 x_2) = \langle \alpha_1 x_1 + \alpha_2 x_2, y\rangle z = \alpha_1\langle x_1, y\rangle z + \alpha_2\langle x_2, y\rangle z = \alpha_1 Tx_1 + \alpha_2 Tx_2$. So T is linear.

For each $x \in \mathcal{H}$, $\|Tx\| = \|\langle x, y\rangle z\| = |\langle x, y\rangle|\|z\| \leq \|x\|\|y\|\|z\|$. So T is bounded and $\|T\| \leq \|y\|\|z\|$.

Note that $\|Ty\| = \|y\|^2\|z\|$ and then $\|T\| \geq \|y\|\|z\|$. Hence, $\|T\| = \|y\|\|z\|$. □

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