NOTES ON FIBER BUNDLES & CHARACTERISTIC CLASSES

HAO XIAO

These notes were taken in the course of Fiber Bundles & Characteristic Classes, lectured by Prof. Xianzhe Dai[1] from UCSB, during the 1st International Fundamental Undergraduate Mathematics Summer School[2] held in Peking University. I live-TEXed them using sublime, and as such there may be typos; please send questions, comments, complaints, and corrections to `xiaohao1096@163.com`.

Contents

[1]Lecturer's email address: `dai@math.ucsb.edu`

[2]Summer school website: `http://www.math.pku.edu.cn/misc/puremath/summerschool/`

1. Introduction to the Course

1.1. **Warm-up of Vector Bundles.** Fiber bundles appeared in the name of this course are generalized vector bundles.

Simply, vector bundles are a continuous family of vector spaces on a parametrized topological space, which indicates that the vector spaces vary continuously with respect to the parameters that generate the topological space. The underlying topological space is, often in many cases, taken to be a smooth manifold.

Basically, vector bundles present a beautiful combination of geometry/topology and linear algebra.

Example 1.1. *Suppose M is a smooth manifold and TM is a tangent space on M, i.e., a tangent bundle on M. Then the parameter space of TM is M and the tangent space TM here actually linearize the manifold M.*

Example 1.2. ***Möbius Strip***

There are two important features of Möbius strip:

- *nearby vector spaces of a point on the strip can be identified, called "local triviality" in general; and*
- *global property: any vector bundle on the strip are "twisted".*

Question 1.3. *How do we describe the "twisting" of vector bundles?* ***Characteristic classes*** *is theory focused on this topic.*

1.2. **Some Reincarnations of Euler Characteristic.** In this subsection, M^2 will always denote a closed oriented surface.

Example 1.4. ***Triangulation***

Choose a arbitrary triangulation of M^2, then the Euler characteristic of M^2 is $\chi(M^2) = \#V - \#E + \#F$, where $\#V, \#E, \#F$ are the numbers of vertices, edges, faces respectively. A core combinatorial property of $\chi(M^2)$ goes that it's independent of the triangulation or, equivalently, all triangulations of M^2 yield the same Euler characteristic of M^2.

Example 1.5. ***Topological Invariant***

Suppose $b_i(M^2) := \dim(H^i(M^2))$ for $i = 0, 1, 2$ are Betti #'s of M^2, then the identity holds that $\chi(M^2) = b_0(M^2) - b_1(M^2) + b_2(M^2)$.

The algebraic fact of the identity relies on the finite-dimensional differential chain complex of M^2, where d's are the corresponding differential operators:

$$0 \longrightarrow C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots \xrightarrow{d} C^n \longrightarrow 0.$$

Hence, the identity can be expressed as $\sum_{i=0}^{n}(-1)^n \dim(C^i) = \sum_{i=0}^{n}(-1)^i \dim(H^i(M^2))$*, where* $H^i(M^2) = \mathrm{Ker}(d|_{C^i})/\mathrm{Im}(d|_{C^{i-1}})$ *for each* i*.*

This belongs to the content of cohomological theory. If the arrows are taken inversely and d*'s are replaced by boundary map* ∂*'s, the complex chain would be the (finite-dimensional) chain complex in homological theory.*

Example 1.4 and 1.5 are special expressions, failing to be generalized, of Euler characteristic.

Example 1.6. *Fix a Riemannian metric* g *on* M^2*, which is equal to the first fundamental form of* M^2*. By* ***Gauss-Bonnet formula*** *of* M^2*, we derive*

$$\chi(M^2) = \int_{M^2} \frac{K}{2\pi} \mathrm{d}v$$

where K *is the Gaussian curvature of the surface (i.e., one kind of local curvature of the surface) and* $\int \mathrm{d}v$ *denotes the surface integral.*

This geometric description of Euler characteristic in terms of curvature (that is, infinitesimal twisting) of the surface does generalize. ***Chern-Weil theory*** *researches the generalized version of this description.*

Example 1.7. *Suppose* X *is a vector field on a* n*-dimensional smooth manifold* M *with only isolated zeros (i.e., zero vectors). By* ***Poincaré-Hopf index formula*** *of* M^2*, we derive*

$$\chi(M^2) = \sum_{x \in \mathrm{Zero}(X)} \mathrm{Ind}_x(X)$$

where $\mathrm{ind}_x(X)$ *is defined to be the degree of the map* $u : \partial D \to S^{n-1}$ *from the boundary of* D*, which is a picked closed ball centered at* x*, to* S^{n-1} *given by* $u(z) = X(z)/|X(z)|$*.*

This investigates Euler characteristic from the differential viewpoint.

Theorem 1.8. *If* $\chi(M^2) \neq 0$ *(e.g.,* $\chi(S^2) = 2$*), then there exists no nowhere vanishing vector field on* M^2*.*

Question 1.9. *Theorem 1.8 motivates the investigation of the link between Euler characteristic and the existence of nowhere vanishing vector field. This dates back to the original definition of characteristic classes that nowadays is* ***obstruction theory****.*

Remark 1.10. *Euler characteristic only a partially describes the twisting of vector bundles on* M^2*.*

Theorem 1.11. ***Hairy Ball Theorem***

There exists no nowhere vanishing tangent vector field on S^2*.*

Remark 1.12. *Theorem 1.11 lively claims that one can never comb his hair without any cowlick (provided that the head is hairy everywhere). Also note that punk hairstyle is not in contradiction to Theorem 1.11 since the "punk vectors" are not tangent to the head surface.*

Example 1.13. *Comparison of Modern Approaches*

Axiomatic approach to Euler characteristic shows the following features:

- *elegant and general (good for topological spaces);*
- *no loss of torsion information; and*
- *computationally effective.*

But the shortcoming goes that existence and uniqueness are quite involved. Also, note that cohomology sometimes produces torsion characteristic classes.

Chern-Weil theory shows the following features when applied to Euler characteristic:

- *elegant and geometric;*
- *very direct; and*
- *so closely connected with geometry to give geometric applications.*

But Chern-Weil theory loses torsion information and not suitable for computation.

2. Vector Bundles

2.1. General Definitions and Properties.

Definition 2.1. *Vector bundle is a continuous family of vector spaces.*

More precisely, suppose B is a topological space. A ***real vector bundle*** *ξ on B is a triple (E, B, π) where $E = E(\xi)$ is a topological space, called the* ***total space*** *of ξ, and $\pi : E \to B$ is a continuous map, called* ***projection****, such that for each $b \in B$, $\pi^{-1}(b)$ is a real vector space or a* ***fiber*** *on b satisfying* ***local triviality****, that is, for each $b \in B$, there exists a nbh. $U \subset B$ of b, an integer $n \in \mathbb{N}_+$, and a homeomorphism $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ in such a way that for each $b' \in U$, $x \mapsto h(b', x) \in \pi^{-1}(b')$ for all $x \in \mathbb{R}^n$ defines a linear isomorphism between $\mathbb{R}^n$ and $\pi^{-1}(b')$.*

The property of h can be revealed through the following commutative diagram:

$$\begin{tikzcd} U \times \mathbb{R}^n \arrow[r, "h"] \arrow[dr, "\pi_1"'] & \pi^{-1}(U) \arrow[d, "\pi"] \\ & U \end{tikzcd}$$

Remark 2.2. *The integer n appears in Definition 2.1 is locally constant. If B is connected, then $n = \mathrm{const} =: \dim(\xi) =: \mathrm{rank}(\xi)$, called the* ***dimension*** *or* ***rank*** *of ξ respectively.*

Definition 2.3. *ξ is called a* ***line bundle*** *if $n = 1$.*

Example 2.4. *Let $E = [0,1] \times \mathbb{R}/\sim$ be the quotient space of $[0,1] \times \mathbb{R}$ under the equivalent relationship that identifies (x, s) with (x', s') provided that $x = x' \neq 0, 1, s = s'$ or $x = 0, x' = 1, s = -s'$ or $x = 1, x' = 0, s = -s'$.*

Let $\pi : E \to B = [0,1]/(0 \sim 1) = S^1$, then $\pi^{-1}(b)$ is a 1-dimensional real vector space for each $b \in B$.

Now we are going to verify the local triviality in this case. Set $B = U_1 \cup U_2$ where $U_1 = [(1/4, 3/4)]$ and $U_2 = [[0, 1/2) \cup (1/2, 1]]$. Here $[\bullet]$ denotes the equivalent class *of the object between the square brackets. Define* $s_1 : U_1 \to E$ *as* $[x] \mapsto [(x, 1)]$*, and define* $s_2 : U_2 \to E$ *as* $[x] \mapsto [(x, 1)]$ *if* $x \in [0, 1/2)$ *and* $[x] \mapsto [(x, -1)]$ *if* $x \in (1/2, 1]$*. It's obvious that* s_1, s_2 *are continuous and* $\pi \circ s_1 = \mathbb{1}_{U_1}, \pi \circ s_2 = \mathbb{1}_{U_2}$*. Hence the sections as continuous maps are non-vanishing. Then* $h_i : U_i \times \mathbb{R} \to \pi^{-1}(U_i),\ (b, t) \mapsto ts_i(b)$ *for* $i = 1, 2$ *are the desired isomorphisms. And also, the bundle consists of lines.*

Note that local triviality equals the existence of local basis of sections and $(E, B = S^1, \pi)$ *is a line bundle on* S^1*, i.e., the Möbius strip.*

Example 2.5. $E = B \times \mathbb{R}^n \xrightarrow{\pi_1} B$ *is called the* ***trivial vector bundle*** *on* B*.*

Example 2.6. *Let* $\mathbb{R}\mathrm{P}^n$ *be the space of all lines in* $\mathbb{R}^{n+1}$ *passing through* 0*. Also,* $\mathbb{R}\mathrm{P}^n = S^n/\mathbb{Z}_2$ *where the equivalent relationship identifies all pairs of antipodal points on the sphere.*

Let $E = E(\gamma_n^1) = \{(l, v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} : v \in l\} = \{([x], v) \in S^n/\mathbb{Z}_2 \times \mathbb{R}^{n+1} : v = cx \text{ for some } c \in \mathbb{R}\}$*. Define* $\pi : E \to \mathbb{R}\mathrm{P}^n$ *to be the map that projects the first component of each* $(l, v) \in E$ *to* $l \in \mathbb{R}\mathrm{P}^n$*. This is called the* ***tautological line bundle*** γ_n^1 *on* $\mathbb{R}\mathrm{P}^n$*.*

$(E, B = \mathbb{R}\mathrm{P}^n, \pi)$ *is a* 1*-dimensional real vector bundle (or a real vector bundle of rank* 1*). In particular, if* $n = 1$*,* γ_1^1 *on* $\mathbb{R}\mathrm{P}^1 = S^1$ *is a Möbius strip.*

Remark 2.7. *Can we drop the local triviality condition?*

Absolutely no! Vector bundle without local triviality is a "vector party", i.e., disordered collection of vector spaces which fails to be a vector bundle in many ways.

In general, local triviality can be verified by constructing local sections that form a basis at each point of the base space.

Definition 2.8. *A* ***section*** *of* ξ *is a continuous map* $s : B \to E(\xi)$ *such that* $\pi \circ s = \mathbb{1}_B$*, that is,* $s(b) \in \pi^{-1}(b)$ *for each* $b \in B$*.*

Example 2.9.

- *For a vector bundle* ξ *on* B*, the* 0***-section*** $s : B \to E(\xi)$*, defined by setting* $s(b) = 0 \in \pi^{-1}(b)$ *for each* $b \in B$*, is section of* ξ*.*
- *Suppose* M *is a smooth manifold and* TM *is its tangent bundle. Let the projection be* $\pi : TM \to M$*. Then any section* s *of this vector bundles is a vector filed on* M*.*

$$\begin{tikzcd} TM \arrow[r, bend left, "\pi"] & M \arrow[l, bend left, "s"] \end{tikzcd}$$

Definition 2.10. *A collection of sections* $s_1, \cdots, s_k$ *is called* ***linearly independent*** *if for each* $b \in B$*,* $s_1(b), \cdots, s_k(b) \in \pi^{-1}(b)$ *are l.i.. Moreover,* $s_1, \cdots, s_k$ *form a* ***basis*** *of* ξ *if* $s_1(b), \cdots, s_k(b)$ *form a basis at each* $b \in B$*.*

Definition 2.11. *As vector bundles on the same base space* B*,* ξ, η *are said to be* ***isomorphic*** *provided that there is a homeomorphism* $f : E(\xi) \to E(\eta)$ *such* *that for each point $b \in B$ the restriction of f to $\pi_\xi^{-1}(b)$ is a linear isomorphism $f|_{\pi^{-1}(b)} : \pi_\xi^{-1}(b) \to \pi_\eta^{-1}(b)$.*

Lemma 2.12. *Let ξ, η be vector bundles over B, and let $f : E(\xi) \to E(\eta)$ be a continuous function which maps each vector space $F_b(\xi)$ isomorphically onto the corresponding vector space $F_b(\eta)$. Then f is necessarily a homeomorphism. Hence ξ is isomorphic to η.*

Proposition 2.13. *Suppose ξ is a vector bundle on B, and $U \subset B$ is a subspace, then $\xi|_U = (\pi^{-1}(U), U, \pi|_{\pi^{-1}(U)})$ is a vector bundle.*

Theorem 2.14. *The following statements for vector bundles are equivalent.*

- *Local triviality.*
- *For each $b \in B$, there exists a nbh. U of b such that $\xi|_U$ is isomorphic to the trivial bundle $U \times \mathbb{R}^n$.*
- ***Local basis of sections**: for each $b \in B$, there exists a nbh. U of b such that the vector bundle $\xi|_U$ has a basis of sections. In particular, for a line bundle, there exists nowhere vanishing local sections.*

Definition 2.15. *A vector bundle on B is **trivial** if it is isomorphic to $(B \times \mathbb{R}^n, B, \pi_1)$.*

Theorem 2.16. *A vector bundle ξ is trivial iff there is a basis of sections of ξ.*

2.2. Twisting of Vector Bundles.

Example 2.17. *The trivial bundle $B \times \mathbb{R}^n$ does not twist. ξ isomorphic to $B \times \mathbb{R}^n$ is straightforward or does not twist.*

Example 2.18. *The tautological line bundle γ_n^1 on $\mathbb{R}\mathrm{P}^n$ is not isomorphic to the trivial bundle on $\mathbb{R}\mathrm{P}^n$. It suffices to check there is no nowhere vanishing section of γ_n^1. Suppose $s : \mathbb{R}\mathrm{P}^n \to E(\gamma_n^1)$ is a section, and let $\tilde{s}(x) := s([x]) = ([x], t(x)x)$ for each x on S^n that automatically defines a continuous real function t on S^n which satisfies $t(-x) = -t(x)$ for all x since $([x], t(x)x) = s([x]) = s([-x]) = ([-x], t(-x)(-x))$ yields that $t(-x) = -t(x)$. By Mean-Value Theorem, there exists $x_0 \in S^n$ such that $t(x_0) = 0$, which yields that $s([x_0])$ vanishes. So γ_n^1 is twisted.*

$$\begin{tikzcd} \mathbb{R}\mathrm{P}^n \arrow[r, "s"] & E(\gamma_n^1) \\ S^n \arrow[u, "\sim"] \arrow[ur, "\tilde{s}"'] & \end{tikzcd}$$

More generally, the question raises that whether one can find a collection l.i. sections? This leads to the theory of characteristic classes via obstruction theory, that is, expanding l.i. sections on 0-skeletons to 1-skeletons, then to 2-skeletons and continuing this process conductively.

2.3. **New Vectors Bundles out of Old Ones.** Vector space operations yield new parametrized setting.

Definition 2.19. *The* ***Whitney sum****, i.e., direct sum bundle (construction) of* ξ_1, ξ_2 *is the vector bundle* $\xi_1 \oplus \xi_2$ *on* B*, where* $E(\xi_1 \oplus \xi_2) = \{(e_1, e_2) \in E(\xi_1) \times E(\xi_2) : \pi_{\xi_1}(e_1) = \pi_{\xi_2}(e_2)\}$ *and then* $\pi_{\xi_1 \oplus \xi_2}(e_1, e_2) := \pi_{\xi_1}(e_1) = \pi_{\xi_2}(e_2)$ *for each* $(e_1, e_2) \in E(\xi_1 \oplus \xi_2)$*. Hence for all* $b \in B$*,* $\pi^{-1}_{\xi_1 \oplus \xi_2}(b) = \pi^{-1}_{\xi_1}(b) \times \pi^{-1}_{\xi_2}(b)$*.*

Definition 2.20. *The* ***tensor product bundle*** *of* ξ_1, ξ_2 *is the vector bundle* $\xi_1 \otimes \xi_2$ *on* B*, where* $E(\xi_1 \otimes \xi_2) = \bigcup_{b \in B} \pi^{-1}_{\xi_1}(b) \otimes \pi^{-1}_{\xi_2}(b)$ *and* $\pi_{\xi_1 \otimes \xi_2}(v) := b \in B$ *for all* $v \in E(\xi_1 \otimes \xi_2)$ *such that* $v \in \pi^{-1}_{\xi_1}(b) \otimes \pi^{-1}_{\xi_2}(b)$*. Hence for all* $b \in B$*,* $\pi^{-1}_{\xi_1 \otimes \xi_2}(b) = \pi_{\xi_1}(b)^{-1} \otimes \pi^{-1}_{\xi_2}(b)$*.*

Definition 2.21. *The* ***Hom-bundle*** *of* ξ_1, ξ_2 *is the vector bundle* $\mathrm{Hom}(\xi_1, \xi_2)$*, where* $E(\mathrm{Hom}(\xi_1, \xi_2)) = \bigcup_{b \in B} \mathrm{Hom}(\pi^{-1}_{\xi_1}(b), \pi^{-1}_{\xi_2}(b))$ *and* $\pi_{\mathrm{Hom}(\xi_1,\xi_2)}(f) := b \in B$ *for all* $f \in E(\mathrm{Hom}(\xi_1, \xi_2))$ *such that* $f \in \mathrm{Hom}(\pi^{-1}_{\xi_1}(b), \pi^{-1}_{\xi_2}(b))$*. Hence for all* $b \in B$*,* $\pi^{-1}_{\mathrm{Hom}(\xi_1,\xi_2)}(b) = \mathrm{Hom}(\pi^{-1}_{\xi_1}(b), \pi^{-1}_{\xi_2}(b))$*.*

Definition 2.22. *The* ***dual vector bundle*** ξ^* *of* ξ *on* B *is the Hom-bundle* $\mathrm{Hom}(\xi, \xi')$ *where* $\xi' = (B \times \mathbb{R}, B, \pi_1)$ *is the trivial line bundle on* B*. Suppose* η *is a vector bundle on* B*, then there is a canonical vector bundle isomorphism* $\mathrm{Hom}(\xi, \eta) \approx \xi^* \otimes \eta$*.*

Definition 2.23. *Suppose* ξ *is a vector bundle on* B *and* B' *is another topological space, let* $f : B' \to B$ *be continuous. The* ***pullback bundle*** *or* ***induced bundle*** $f^*\xi$ *is a vector bundle on* B' *such that* $E(f^*\xi) = \{(b', e) \in B' \times E(\xi) : f(b') = \pi_\xi(e)\}$ *where* $\pi_{f^*\xi} : E(f^*\xi) \to B'$ *maps each* (b', e) *to* b'*. In fact, for each* $b' \in B'$*,* $\pi^{-1}_{f^*\xi}(b') = \{b'\} \times \pi^{-1}_\xi(f(b'))$*. In other words,* $f^*\xi$ *is a reparametrization of* ξ *via* f*.*

Definition 2.24. *Suppose* ξ, η *are vector bundles on* B*,* $\xi \approx \eta$ *if there exists* $h : E(\xi) \to E(\eta)$ *and* h *restricts to a linear isomorphism* $\pi^{-1}_\xi(b) \to \pi^{-1}_\eta(b)$ *at each* $b \in B$*. This can be stated as the following commutative diagram.*

$$\begin{tikzcd} E(\xi) \arrow[r, "h"] \arrow[dr, "\pi_\xi"'] & E(\eta) \arrow[d, "\pi_\eta"] \\ & B \end{tikzcd}$$

Definition 2.25. *Suppose* ξ *is a vector bundle on* B *and* η *is a vector bundle on* B'*. A* ***bundle map*** *from* η *to* ξ *is a continuous map* $F : E(\eta) \to E(\xi)$ *such that for each* $b' \in B'$*, there exists* $b \in B$ *such that* F *restricts to* $\pi^{-1}_\eta(b')$ *a linear isomorphism* $F|_{\pi^{-1}_\eta(b')} : \pi^{-1}_\eta(b') \to \pi^{-1}_\xi(b)$*.*

Thus in fact, F induces $f : B' \to B$ by $b' \mapsto b$. In other words, there holds the following commutative diagram and we say that F ***covers*** *f.*

$$\begin{array}{ccc} E(\eta) & \xrightarrow{F} & E(\xi) \\ \big\downarrow{\scriptstyle \pi_\eta} & & \big\downarrow{\scriptstyle \pi_\xi} \\ B' & \xrightarrow{\quad f \quad} & B \end{array}$$

3. Stiefel-Whitney Classes

3.1. **Axioms.** Vector bundles yield topological invariant – cohomology.

Suppose B is a topological space, G is an abelian group (e.g., $\mathbb{Z}, \mathbb{Z}_2, \mathbb{R}$), and $H^i(B, G)$ is the i-th cohomological group of B with coefficients in G.

Here we are going to introduce the four axioms that characterize the Stiefel-Whitney cohomology class of a vector bundle.

- **A-I** To each real vector bundle ξ on B, there corresponds a sequence of cohomological classes $\{w_i(\xi) \in H^i(B; \mathbb{Z}_2)\}_{i=0}^{\infty}$, called **Stiefel-Whitney classes** of ξ, such that $w_0(\xi) = 1 \in H^0(B; \mathbb{Z}_2)$ and $w_i(\xi) = 0$ for each i greater than $n = \mathrm{rank}(\xi)$.
- **A-II Naturality**: For each $f : B(\xi) \to B(\eta)$ covered by a bundle map F from ξ to η, $w_i(\xi) = f^* w_i(\eta)$ for all i.
- **A-III Whitney Sum Formula**: Suppose ξ and η are vector bundles on B, let $w(\xi) = w_0(\xi) + w_1(\xi) + w_2(\xi) + \cdots \in H^*(B; \mathbb{Z}_2) = \oplus_{i=0}^{\infty} H^i(B; \mathbb{Z}_2)$, called the **total Stiefel-Whitney class**. Then for vector bundles ξ, η on B, $w(\xi \oplus \eta) = w(\xi) \cup w(\eta)$, i.e., $w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta)$ for each k. For example, $w_1(\xi \oplus \eta) = w_1(\xi) + w_1(\eta)$ and $w_2(\xi \oplus \eta) = w_2(\xi) + w_1(\xi) w_1(\eta) + w_2(\eta)$.
- **A-IV Normalization**: For the line bundle γ_1^1 on S^1, $w_1(\gamma_1^1) \neq 0$.

Theorem 3.1. *The Stiefel-Whitney classes of a vector bundle ξ satisfying Axiom I-IV exist and are unique.*

Remark 3.2. *In 1935, Stiefel constructed Stiefel-Whitney classes for tangent bundles of smooth manifolds. Whitney constructed Stiefel-Whitney classes for sphere bundles on simplicial complices.*

In 1940-1941, Whitney proved Whitney sum formula.

In 1948, Wu studied Stiefel-Whitney classes. In 1955, he established Wu classes.

In 1966, Hirzebruch proposed the axiomatic approach to Stiefel-Whitney classes.

3.2. **Consequences of the Axioms.** We first discuss two immediate consequences of A-II:

Proposition 3.3.

(a) *Suppose $\xi \approx \eta$, then $w_i(\xi) = w_i(\eta)$.*
(b) *Suppose ξ is trivial, then $w_i(\xi) = 0$ for all $i > 0$.*

Proof. (a) Since $\xi \approx \eta$, there is a homeomorphism $F : E(\xi) \to E(\eta)$ inducing the base map $f : B \to B$ as an identity map, that is, the homeomorphism F is a bundle map covering the identity map f. So $w_i(\xi) = f^* w_i(\eta) = w_i(\eta)$ for all i.

(b) Let B be the base space of ξ. Since ξ is trivial, ξ is isomorphic to $\xi' = (B \times \mathbb{R}^n, B, \pi_1)$, then $w_i(\xi) = w_i(\xi')$ for all i. Let $p \in \mathbb{R}^n$ and $\pi : \mathbb{R}^n \to \{p\}$. Consider the bundle map $\pi_2 : B \times \mathbb{R}^n \to \mathbb{R}^n$ where $\mathbb{R}^n$ is regarded as the total space of vector bundle $\eta = (\mathbb{R}^n, \{p\}, \pi)$, then π_2 induces a base map $f : B \to \{p\}$.

$$\begin{array}{ccc} B \times \mathbb{R}^n & \xrightarrow{\pi_2} & \mathbb{R}^n \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi} \\ B & \xrightarrow{f} & \{p\} \end{array}$$

Since all higher cohomological groups of a single-point set are trivial, $w_i(\eta) = 0$ and then $w_i(\xi') = f^* w_i(\eta) = 0$ for all $i > 0$. Hence, $w_i(\xi) = 0$ for all $i > 0$. $\square$

Remark 3.4. *Suppose ξ is a vector bundle on B. For $f : B' \to B$, $w_i(f^*\xi) = f^* w_i(\xi)$*

$$\begin{array}{ccc} E(f^*\xi) & \xrightarrow{F} & E(\xi) \\ \big\downarrow{\scriptstyle \pi_{f^*\xi}} & & \big\downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

If $B' = B$ and $f = \mathbb{1}_B$, then it's easy to see $f^*\xi \approx \xi$. Moreover, if ξ is trivial, then $f^*\xi$ is trivial and $w_i(f^*\xi) = 0$ for all $i > 0$.

The following property following immediately from (b) of Proposition 3.3 and A-III:

Proposition 3.5. *Suppose ε is trivial, then $w_i(\varepsilon \oplus \eta) = w_i(\eta)$ for all i.*

Proof. By A-III, that is, Whitney sum formula, $w_i(\varepsilon \oplus \xi) = \sum_{k=0}^{i} w_k(\varepsilon) w_{i-k}(\eta) = w_0(\xi) w_i(\eta) = w_i(\eta)$ for all i. $\square$

Previous to the discussion of further consequences, we introduce:

Definition 3.6. *A (euclidean) metric on a (real) v.b. ξ (on B) is a cont. function $g : E(\xi) \to \mathbb{R}$ such that for each $b \in B$, g restricts to a positive quadratic function on $\pi^{-1}(b)$, i.e., there exists an inner product $\langle \bullet, \bullet \rangle_b$ on $\pi^{-1}(b)$ such that $g(e) = \langle e, e \rangle_b$ for all $e \in \pi^{-1}(b)$. In other words, g consists of a cont. family of fiber-wise inner products.*

Example 3.7. *Suppose $M \subset \mathbb{R}^n$ is a submanifold, TM inherits the metric from the canonical inner product on $\mathbb{R}^n$.*

Definition 3.8. *Suppose M is smooth manifold, a (smooth) metric on TM is called a **Riemannian metric**.*

Example 3.9. *Suppose v.b. ε is trivial, then there automatically is a metric on ε.*

Definition 3.10. *A* ***refinement*** *of a cover of a space X is a new cover of the same space such that every set in the new cover is a subset of some set in the old cover.*

An open cover of a space X is ***locally finite*** *if every point of the space has a nbh. that intersects only finitely many sets in the cover.*

A topological space X is ***paracompact*** *if every open cover has a locally finite open refinement.*

Definition 3.11. *A* ***partition of unity*** *of a space X is a family R of continuous functions from X to the unit interval $[0, 1]$ such that for every $x \in X$:*

- *there is a nbh. of x where all but a finite number of the functions of R are 0;*
- *the sum of all the function values at x is 1, i.e., $\sum_{\rho \in R} \rho(x) = 1$.*

A partition of unity $\{f_i\}_{i \in I}$ on X is ***subordinate*** *to an open cover of X provided that for each f_i, there is an element U of the open cover such that $\operatorname{supp}(f_i) \subset U$.*

Lemma 3.12. *A Hausdorff space X is paracompact iff every open cover admits a subordinate partition of unity.*

Theorem 3.13. *If B is Hausdorff and paracompact, then B has partition of unity by continuous functions and any vector bundle on B has a metric.*

Theorem 3.14. *Suppose there is a metric on ξ, then the dual vector bundle ξ^* is isomorphic to ξ, i.e., $\xi^* \approx \xi$.*

Definition 3.15. *Suppose ξ, η are vector bundles on B with $E(\eta) \subset E(\xi)$, then η is a* ***subbundle****, i.e., each fiber of η is a subspace of the corresponding one of ξ.*

Lemma 3.16. *Let η_1 and η_2 be subbundles of ξ such that each vector space $\pi^{-1}(b)$ is equal to the direct sum of the subspaces $\pi_{\eta_1}^{-1}(b) \oplus \pi_{\eta_2}^{-1}(b)$. Then ξ is isomorphic to the Whitney sum $\eta_1 \oplus \eta_2$.*

Theorem 3.17. *Suppose there is a metric on ξ and η is a subbundle of ξ. Let $E(\eta^\perp)$ denote the union taken over each $b \in B$ of subspaces consisting of all vectors in $\pi_\xi^{-1}(b)$ perpendicular to $\pi_\eta^{-1}(b)$. Then $E(\eta^\perp)$ is the total space of a subbundle $\eta^\perp \subset \xi$. Furthermore, ξ is isomorphic to the Whitney sum $\eta \oplus \eta^\perp$.*

Definition 3.18. *As $\xi \approx \eta \oplus \eta^\perp$ illustrated in Definition 3.17, $\eta^\perp$ is called the* ***fiber-wise orthogonal complement*** *of η.*

Definition 3.19. *Suppose that $M \subset N \subset \mathbb{R}^n$ are smooth manifold, and suppose that N is provided with a Riemannian metric. The* ***tangent bundle*** *TM is a subbundle of the restriction $TN|_M$. In this case, the orthogonal complement $TM^\perp \subset TN|_M$ is called the* ***normal bundle*** *of M in N, denoted $\nu(TM)$.*

Corollary 3.20. *For any smooth submanifold M of a smooth Riemannian manifold N the normal bundle $\nu(M)$ is defined, and $TM \oplus \nu(M) \approx TN|_M$.*

Example 3.21. *Suppose $M \subset \mathbb{R}^n$ is a submanifold and let $\varepsilon = (M \times \mathbb{R}^n, M, \pi_1)$ be equipped with the natural metric, then $\varepsilon = TM \oplus (TM)^{\perp} = TM \oplus \nu(M)$.*

In particular, note that $S^n \subset \mathbb{R}^{n+1}$ is a submanifold. Let $\varepsilon^{n+1} = (S^n \times \mathbb{R}^{n+1}, S^n, \pi_1)$, then $\varepsilon^{n+1} = TS^n \oplus \nu(S^n)$. Actually, $\nu(S^n)$ is trivial.

Remark 3.22. *TS^n is not trivial in general by Theorem 1.11, but ε^{n+1} and $\nu(S^n)$ are trivial. By Whitney sum formula, $w_i(TS^n) = 0$ for $i > 0$, that is, TS^n cannot be distinguished from the trivial bundle over S^n by means of Stiefel-Whitney classes.*

It's revealed by this case that Stiefel-Whitney classes do not exist as a complete invariant of vector bundles.

Now we are able to show a further consequence of the axioms:

Proposition 3.23. *Suppose ξ is a vector bundle with a (euclidean) metric which possesses a nowhere vanishing section, then $w_n(\xi) = 0$ where $n = \mathrm{rank}(\xi)$.*

If ξ possesses k linearly independent sections, say, $s_1, \cdots, s_k$, then $w_{n-k+1}(\xi) = w_{n-k+2}(\xi) = \cdots = w_n(\xi) = 0$.

Proof. Let $\varepsilon = \langle s_1, \cdots, s_k \rangle$ be the subbundle of ξ generated (or spanned) by the k sections. By Theorem 2.16, ε is trivial. Since $\xi = \varepsilon \oplus \varepsilon^{\perp}$ and $\mathrm{rank}(\varepsilon^{\perp}) = n - k$, it follows that $w_i(\xi) = w_i(\varepsilon^{\perp}) = 0$ if $i > n - k$. $\square$

3.3. Vector Bundles on $\mathbb{R}\mathrm{P}^n$.

Definition 3.24. *$H^{\Pi}(B; \mathbb{Z}_2)$ will denote the ring consisting of all formal infinite series $a = a_0 + a_1 + a_2 + \cdots$ with $a_i \in H^i(B; \mathbb{Z}_2)$. The product operation in this ring is to be given by the formula $(a_0 + a_1 + a_2 + \cdots)(b_0 + b_1 + b_2 + \cdots) = (a_0 b_0) + (a_1 b_0 + a_0 b_1) + (a_2 b_0 + a_1 b_1 + a_0 b_2) + \cdots$. This product is commutative (since we are working modulo 2) and associative. Additively, $H^{\Pi}(B; \mathbb{Z}_2)$ is to be simply the cartesian product of the groups $H^i(B; \mathbb{Z}_2)$.*

First we need some info. about $H^i(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$:

Lemma 3.25. *The group $H^i(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ is cyclic of order 2 for $0 \leq i \leq n$ and is zero for higher values of i. Furthermore, if a denotes the non-zero element of $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$, then each $H^i(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ is generated by the $i-fold$ cup product $a^i = \underbrace{a \cup \cdots \cup a}_{i \text{ terms}}$.*

As a consequence, $H^{\Pi}(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) = \mathbb{Z}_2[a]/\langle a^{n+1} \rangle$. More precisely, $H^{\Pi}(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ can be described as the algebra with unit over $\mathbb{Z}_2$ having one generator a and one relation $a^{n+1} = 0$.

Remark 3.26. *This lemma can be used to compute the homomorphism*

$$f^* : H^n(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \to H^n(S^n; \mathbb{Z}_2)$$

providing that $n > 1$. In fact, $f^*(a^n) = (f^* a)^n$ is zero since $f^* a \in H^1(S^n; \mathbb{Z}_2) = 0$.

Example 3.27. *The total Stiefel-Whitney class of the canonical line bundle γ_n^1 over $\mathbb{R}\mathrm{P}^n$ is given by $w(\gamma_n^1) = 1 + a$.*

Consider the standard inclusion $j : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^n$ is clearly covered by a bundle map from γ_1^1 to γ_n^1. Therefore, $j^* w_1(\gamma_n^1) = w_1(\gamma_1^1) \neq 0$. This shows that $w_1(\gamma_n^1)$ cannot be zero, hence must equal to a. Then $w(\gamma_n^1) = w_0(\gamma_n^1) + w_1(\gamma_n^1) = 1 + a$.

Lemma 3.28. *The collection of all infinite series $w = 1 + w_1 + w_2 + \cdots \in H^{\Pi}(B; \mathbb{Z}_2)$ with leading term 1 forms a commutative group under multiplication. This is precisely the group of units of the ring $H^{\Pi}(B; \mathbb{Z}_2)$.*

Proof. The inverse $\bar{w} = 1 + \bar{w}_1 + \bar{w}_2 + \cdots$ of a given element w can be constructed inductively by the algorithm $\bar{w}_n = w_1 \bar{w}_{n-1} + w_2 \bar{w}_{n-2} + \cdots + w_{n-1}\bar{w}_1 + w_n$. Note that the coefficient of a Stiefel-Whitney class takes value in $\mathbb{Z}_2$. Thus one obtains:

$$\begin{aligned} \bar{w}_1 &= w_1, & \bar{w}_2 &= w_1^2 + w_2, \\ \bar{w}_3 &= w_1^3 + w_3, & \bar{w}_4 &= w_1^4 + w_1^2 w_2 + w_2^2 + w_4, \end{aligned}$$

and so on. This completes the proof.

Alternatively $\bar{w}$ can be computed by the power series expansion:

$$\begin{aligned} \bar{w} &= [1 + (w_1 + w_2 + w_3 + \cdots)]^{-1} \\ &= 1 - (w_1 + w_2 + w_3 + \cdots) + (w_1 + w_2 + \cdots)^2 - (w_1 + w_2 + \cdots)^3 + \cdots \\ &= 1 - w_1 + (w_1^2 - w_2) + (-w_1^3 + 2w_1 w_2 - w_3) + \cdots \end{aligned}$$

where the signs are of course irrelevant. This leads to the precise expression $(i_1 + \cdots + i_k)!/i_1! \cdots i_k!$ for the coefficient of $w_1^{i_1} \cdots w_k^{i_k}$ in $\bar{w}$. ☐

Remark 3.29. *Consider two vector bundles ξ and η over the same base space. It follows from Lemma 3.28 that $w(\xi \oplus \eta) = w(\xi) w(\eta)$ can be uniquely solved by $w(\eta) = \bar{w}(\xi) w(\xi \oplus \eta)$. In particular, if $\xi \oplus \eta$ is trivial, then $w(\eta) = \bar{w}(\xi)$.*

Theorem 3.30. ***Whitney Duality Theorem***

If TM is the tangent bundle of a manifold in euclidean space and ν is the normal bundle then $w_i(\nu) = \bar{w}_i(TM)$.

Example 3.31. *Let $\varepsilon^{n+1} := (\mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}, \mathbb{R}\mathrm{P}^n, \pi_1)$, then $\gamma_n^1 \subset \varepsilon^{n+1}$ is a subbundle and $\varepsilon^{n+1} = \gamma_n^1 \oplus (\gamma_n^1)^\perp$. Denote $(\gamma_n^1)^\perp$ by $\gamma^\perp$, and then $\mathrm{rank}(\gamma^\perp) = n$ and the total space $E(\gamma^\perp)$ consists of all pairs $([x], v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}$ with v perpendicular to x.*

In fact, since $\gamma_n^1 \oplus \gamma^\perp$ is trivial, we, by Theorem 3.30, have $w(\gamma^\perp) = \bar{w}(\gamma_n^1) = (1 + a)^{-1} = 1 + a + a^2 + \cdots + a^n$. Thus, this an example that all of the first n Stiefel-Whitney class of a n-dimensional bundle may be nonzero (e.g., $\gamma^\perp$).

Moreover, we can use this to verify that $w(\varepsilon^{n+1}) = w(\gamma_n^1) w(\gamma^\perp) = 1$.

Our main aim now is to prove $w(T(\mathbb{R}\mathrm{P}^n)) = (1 + a)^{n+1}$, but it precedes with some preparations. First we need to better understand $T(\mathbb{R}\mathrm{P}^n)$:

Lemma 3.32. $T(\mathbb{R}\mathrm{P}^n) \approx \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$.

Proof. $T(\mathbb{R}\mathrm{P}^n) = T(S^n/\mathbb{Z}_2) = TS^n/\mathbb{Z}_2 = \{(x,v) \in S^n \times \mathbb{R}^{n+1} : x \perp v\}/\mathbb{Z}_2$. Note that each $x \in S^n$ defines a vector as the basis of a fiber of γ_n^1, and each v such that $v \perp x$ is a vector in the corresponding fiber of $\gamma^\perp$.

Hence, all $(x,v) \in S^n \times \mathbb{R}^{n+1}$ with $v \perp x$ defines a linear homomorphism by $x \mapsto v$ from the corresponding fiber of γ_n^1 to that of $\gamma^\perp$ which is invariant under the $\mathbb{Z}_2$-action. Hence, $T(\mathbb{R}\mathrm{P}^n) \to \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$ given by $[(x,v)] \mapsto (x \mapsto v)$ is the desired homeomorphism. ☐

Lemma 3.33. *For any line bundle ξ on B, $\mathrm{Hom}(\xi, \xi)$ is trivial.*

Proof. Define $s : B \to \mathrm{Hom}(\xi, \xi)$ by $b \mapsto \mathbb{1}_{\pi_\xi^{-1}(b)}$, then this obviously is a nowhere vanishing section on B. Hence $\mathrm{Hom}(\xi, \xi)$ is trivial. ☐

Theorem 3.34. *The Whitney sum $T(\mathbb{R}\mathrm{P}^n) \oplus \varepsilon^1$ is isomorphic to the $(n+1)$-fold Whitney sum $\gamma_n^1 \oplus \cdots \oplus \gamma_n^1$. Hence the total Stiefel-Whitney class of $\mathbb{R}\mathrm{P}^n$ is given by $w(T(\mathbb{R}\mathrm{P}^n)) = (1+a)^{n+1} = 1 + \binom{n+1}{1}a + \binom{n+1}{2}a^2 + \cdots + \binom{n+1}{n}a^n$, where ε^1 is the trivial line bundle on $\mathbb{R}\mathrm{P}^n$ and a is the generator of $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ (see Lemma 3.25).*

Proof. Upon citing the previous preparations, we derive

$$T(\mathbb{R}\mathrm{P}^n) \oplus \varepsilon^1 \approx \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \approx \mathrm{Hom}(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) \approx \mathrm{Hom}(\gamma_n^1, \varepsilon^{n+1}).$$

By Theorem 3.14, $\mathrm{Hom}(\gamma_n^1, \varepsilon^1)$ is isomorphic to γ_n^1 since there is a metric on γ_n^1. Since $\mathrm{Hom}(\gamma_n^1, \varepsilon^{n+1}) \approx \mathrm{Hom}(\gamma_n^1, \underbrace{\varepsilon^1 \oplus \cdots \oplus \varepsilon^1}_{n+1 \text{ terms}}) \approx \underbrace{\mathrm{Hom}(\gamma_n^1, \varepsilon^1) \oplus \cdots \oplus \mathrm{Hom}(\gamma_n^1, \varepsilon^1)}_{n+1 \text{ terms}}$, we have $T(\mathbb{R}\mathrm{P}^n) \oplus \varepsilon^1 \approx \underbrace{\gamma_n^1 \oplus \cdots \oplus \gamma_n^1}_{n+1 \text{ terms}}$. By $w(T(\mathbb{R}\mathrm{P}^n)) = w(T(\mathbb{R}\mathrm{P}^n) \oplus \varepsilon^1) = w(\underbrace{\gamma_n^1 \oplus \cdots \oplus \gamma_n^1}_{n+1 \text{ terms}}) = (w(\gamma_n^1))^{n+1} = (1+a)^{n+1} = 1 + \binom{n+1}{1}a + \binom{n+1}{2}a^2 + \cdots + \binom{n+1}{n}a^n$. ☐

Remark 3.35. *Now we are able to calculate all $w(T(\mathbb{R}\mathrm{P}^n))$'s. By computing $\binom{n+1}{i}$ modulo 2 and noticing $H^{n+1}(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) = 0$, we derive the first few total Stiefel-Whitney classes of $w(T(\mathbb{R}\mathrm{P}^n))$'s:*

$$\begin{aligned}
&w(T(\mathbb{R}\mathrm{P}^1)) = 1, && w(T(\mathbb{R}\mathrm{P}^2)) = 1 + a + a^2,\\
&w(T(\mathbb{R}\mathrm{P}^3)) = 1, && w(T(\mathbb{R}\mathrm{P}^4)) = 1 + a + a^4,\\
&w(T(\mathbb{R}\mathrm{P}^5)) = 1 + a^2 + a^4, && w(T(\mathbb{R}\mathrm{P}^6)) = 1 + a + a^2 + a^3 + a^4 + a^5 + a^6.
\end{aligned}$$

3.4. **Geometric Applications.** One clear point is that nonzero Stiefel-Whitney classes are obstructions to the existence of the basis consisting of global linearly independent sections. This is because $w_i(\xi) = 0$ for all $i > 0$ if ξ is trivial.

In this subsection, we introduce other applications of Stiefel-Whitney classes in vector bundles. The first one is **orientation**.

Definition 3.36. *For a finite-dimensional vector space, two bases are said to be **equivalent** if their linear transformation matrix has positive determinant. This basically is an equivalent relationship and it divides the collection of bases into two parts. Each part is called an **orientation** on the vector space.*

Definition 3.37. *An **orientation** on a vector bundle ξ on B is a continuous choice of orientations on the fibers of ξ. Here the word "continuous" is used to indicate that the orientation can be represented by a basis of local sections. Say that ξ is **orientable** if ξ has an orientation.*

Example 3.38.

- *The trivial bundle $\xi = (B \times \mathbb{R}^n, B, \pi_1)$ is orientable.*
- *Möbius stripi, i.e., the nontrivial line bundle on S^1 is not orientable.*

Theorem 3.39. *ξ is orientable iff $w_1(\xi) = 0$.*

Corollary 3.40. *$\mathbb{RP}^n$ is orientable when n is odd and not when n is even.*

Theorem 3.41. *ξ has a n-spin structure iff $w_1(\xi) = 0$ and $w_2(\xi) = 0$.*

The second application of Stiefle-Whitney classes in vector bundles is **cobordism**, which studies when a closed manifold would become the boundary of another one. A typical example is $S^n \approx \partial D^{n+1}$.

In this subsection, we restrict our concern to a closed smooth manifold $M^n =: B$ as a base space of a vector bundle where a **closed manifold** is defined to be a compact one without boundary. Note that sometimes we omit the superscript of M^n which indicates the dimension of a manifold and simply write M. In this case, there exists a unique fundamental homology class $\mu_M \in H_n(M; \mathbb{Z}_2)$. So for any cohomology class $v \in H^n(M^n; \mathbb{Z}_2)$, it pairs with μ_M to define the **Kronecker index** $\langle v, \mu_M \rangle \in \mathbb{Z}_2$.

Definition 3.42. *Let ξ be a vector bundle on a closed, possibly disconnected, smooth manifold M of rank n, and let $r_1, \cdots, r_n$ be nonnegative integers with $r_1 + 2r_2 + \cdots + nr_n = n$. We then form the monomial $w_1(\xi)^{r_1} \cdots w_n(\xi)^{r_n} \in H^n(B; \mathbb{Z}_2)$.*

In particular, we carry out this construction if ξ is the tangent bundle on the manifold M. The corresponding integer modulo 2

$$\langle w_1(\xi)^{r_1} \cdots w_n(\xi)^{r_n}, \mu_M \rangle \in \mathbb{Z}_2$$

*or briefly $w_1^{r_1} \cdots w_n^{r_n}[M]$ is called the **Stiefel-Whitney number** of M associated with the monomial $w_1^{r_1} \cdots w_n^{r_n}$.*

Theorem 3.43. ***Pontrjagin***

If N is a smooth compact $(n+1)$-dimensional manifold such that $\partial N = M$, then all Stiefel-Whitney numbers of M are 0.

Proof. Choosing a euclidean metric on TN by Theorem 3.13, then there is a unique outward normal vector field along M, spanning a trivial line bundle $\nu(M) \approx \varepsilon^1$.

Consider the inclusion $i : M \hookrightarrow N$, then $i^*(TN|_M) \approx TM \oplus \nu(M) \approx TM \oplus \varepsilon^1$. It's clear that $w_i(TM) = w_i(TM \oplus \varepsilon^1) = w_i(i^*TN|_M) = i^*w_i(TN|_M) = w_i(TN|_M)$, which indicates that all Stiefel-Whitney classes of TM restricted to M are precisely equal to those of TN. Using the exact sequence

$$\cdots \longrightarrow H^n(N) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(N, M) \longrightarrow \cdots$$

and the natural homomorphism $\partial : H_{n+1}(N, M) \to H_n(M)$ that maps μ_N to μ_M. (There is no sign since we are working modulo 2.) For any class $v \in H^n(M)$, note the identity $\langle v, \partial\mu_B\rangle = \langle \delta v, \mu_B\rangle$. Immediately, we derive

$$\begin{aligned}
\langle w_1(TM)^{r_1} \cdots w_n(TM)^{r_n}, \partial\mu_N\rangle &= \langle w_1(TN|_M)^{r_1} \cdots w_n(TN|_M)^{r_n}, \partial\mu_N\rangle \\
&= \langle \delta(w_1(TN|_M)^{r_1} \cdots w_n(TN|_M)^{r_n}), \mu_N\rangle \\
&= \langle 0, \mu_N\rangle = 0.
\end{aligned}$$

Hence, all the Stiefel-Whitney numbers of M are zero.

Alternatively, this can be derived from the exact sequence

$$\cdots \longrightarrow H_{n+1}(N, M) \xrightarrow{\partial} H_n(M) \xrightarrow{i_*} H_n(N) \longrightarrow \cdots$$

Since $\mu_M = \partial\mu_N$ and $i_*\partial = 0$, $i_*\mu_M = i_*\partial\mu_N = 0$ and then

$$\begin{aligned}
\langle w_1(TM)^{r_1} \cdots w_n(TM)^{r_n}, \mu_M\rangle &= \langle i^*(w_1(TN|_M)^{r_1} \cdots w_n(TN|_M)^{r_n}), \partial\mu_N\rangle \\
&= \langle w_1(TN|_M)^{r_1} \cdots w_n(TN|_M)^{r_n}, i_*\partial\mu_N\rangle \\
&= \langle w_1(TN|_M)^{r_1} \cdots w_n(TN|_M)^{r_n}, 0\rangle = 0
\end{aligned}$$

□

Example 3.44. *Since $w(T(\mathbb{R}\mathrm{P}^n)) = (1+a)^{n+1} = a + \binom{n+1}{1}a + \binom{n+1}{2}a^2 + \cdots + \binom{n+1}{n}a^n$, $w_n(T(\mathbb{R}\mathrm{P}^n)) = a^n$ when n is even. Then $\langle w_n(T(\mathbb{R}\mathrm{P}^n)), \mu_{\mathbb{R}\mathrm{P}^n}\rangle = 1 \neq 0$. $\mathbb{R}\mathrm{P}^n$ can never be a boundary of some manifold when n is even.*

Essentially, it turns out that the converse of Theorem 3.43 is also true and, though, much harder to prove.

Theorem 3.45. ***Thom***

If all Stiefel-Whitney numbers of M are zero, then M can be realized as the boundary of some compact smooth manifold.

Example 3.46. *First we note that for odd n, the monomial $w_1^{r_1} \cdots w_n^{r_n}$ must contain a factor w_j where j is odd.*

For $\mathbb{R}\mathrm{P}^3$, all Stiefel-Whitney numbers of $\mathbb{R}\mathrm{P}^3$ are zero since $w(T(\mathbb{R}\mathrm{P}^3)) = 1$. Thus $\mathbb{R}\mathrm{P}^3$ is a boundary of some compact smooth manifold.

For $\mathbb{R}\mathrm{P}^5$, all Stiefel-Whitney numbers of $\mathbb{R}\mathrm{P}^5$ are zero since $w(T(\mathbb{R}\mathrm{P}^5)) = 1+a^2+a^4$. Thus $\mathbb{R}\mathrm{P}^5$ is a boundary of some compact smooth manifold.

Remark 3.47. *Stiefel-Whitney numbers of the tangent bundle of a smooth manifold are known to be cobordism invariants.*

One Stiefel-Whitney number of importance in surgery theory is the de Rham invariant of a $(4k+1)$-dimensional manifold, i.e., $w_2 w_{4k-1}$.

3.5. **Some General Remarks.** The proof of existence and uniqueness of Stiefel-Whitney classes uses heavy algebraic topology.

We now present a quick glance at universal bundle and space classification as the end of this subsection.

Example 3.48. *TS^n and Grassmannian*

Consider $S^n \subset \mathbb{R}^{n+1}$ and note that each $T_x S^n \subset \mathbb{R}^{n+1}$ is an n-dimensional subspace of $\mathbb{R}^{n+1}$ determined by $x \in S^n$. This motivates us to denote all n-dimensional subspaces (through the origin) of $\mathbb{R}^{n+1}$ by $G_n(\mathbb{R}^{n+1})$. If fact, fix $0 \le k \le n+1$, $G_k(\mathbb{R}^{n+1})$ is the collection of all k-dimensional subspaces (through the origin) of $\mathbb{R}^{n+1}$. All $G_k(\mathbb{R}^{n+1})$'s are compact smooth manifolds. Then ***Grassmann manifold*** *or* ***Grassmannian*** *is defined to be $G_k(\mathbb{R}^{n+1})$ with $0 \le k \le n+1$.*

Naturally, there is a tautological vector bundle γ^k on $G_k(\mathbb{R}^{n+1})$ of rank k. So let $f : S^n \to G_n(\mathbb{R}^{n+1})$ be the smooth function that maps each $x \in S^n$ to $T_x S^n \in G_n(\mathbb{R}^{n+1})$, and then it's obvious that $TS^n = f^*\gamma^n$. This shows that TS^n is a pullback of a tautological vector bundle.

Similarly, for $M \subset \mathbb{R}^n$ together with $f : M \to G_n(\mathbb{R}^{n+1})$ that maps each $x \in M$ to $T_x \in G_n(\mathbb{R}^{n+1})$, we derive $TM = f^*\gamma^n$.

Remark 3.49. *As we see, the pullback of tautological vector bundle on Grassmannian results in various specific vector bundles. Thus in some sense, $\gamma^k \to G_k(\mathbb{R}^{n+1})$ is "universal".*

The claim goes that any v.b. $\xi \to B$ can be realized as a pullback of the tautological v.b. (i.e., universal v.b.) on an infinite-dimensional Grassmannian. The following three stuffs then arise:

- *cohomology of infinite-dimensional Grassmannian;*
- *Leray-Hirsch theorem; and*
- *splitting principle.*

For vector bundles, Stiefel-Whitney classes are one out of the four kinds of characteristic classes:

- Stiefel-Whitney classes $w_i(\xi) \in H^i(B; \mathbb{Z}_2)$ for real vector bundles;
- Chern classes $c_i(\xi) \in H^{2i}(B; \mathbb{Z})$ for complex vector bundles;
- Pontryagin classes $p_i(\xi) \in H^{4i}(B; \mathbb{Z})$ for real vector bundles; and
- Euler classes $e(\xi) \in H^n(B; \mathbb{Z})$ for orientable real vector bundles of rank n.

The first two kinds are similar. But p_i's are a refinement of Stiefel-Whitney classes and e is a further refinement of Stiefel-Whitney classes.

Theorem 3.50. *There exists a unique sequence consisting of $c_i(\xi) \in H^{2i}(B; \mathbb{Z})$ for a complex v.b. ξ on B, called the* ***Chern classes*** *of ξ, such that*

A-I ***Dimension***: $c_0(\xi) = 1$ *and* $c_i(\xi) = 0$ *for* $i > \mathrm{rank}(\xi)$. *Note that the rank here is the complex dimension of the fibers;*

A-II ***Naturality***: *for any bundle map from* η *to* ξ *covering* $f : B' \to B$, $c_i(\eta) = f^* c_i(\xi)$;

A-III ***Whitney Duality***: *let* $c(\xi) := \sum_{i=0}^{\infty} c_i(\xi)$ *(i.e., the* ***total chern class****), and then* $c(\xi \oplus \eta) = c(\xi)c(\eta)$ *for any* ξ, η *on* B*; and*

A-IV ***Normalization***: *for the tautological* $\mathbb{C}$*-liner bundle* $\gamma^1_{\mathbb{C},1}$ *on* $\mathbb{CP}^1$, $c_1(\gamma^1_{\mathbb{C},1}) = -\mu_{\mathbb{CP}^1} \in H^2(\mathbb{CP}^1; \mathbb{Z})$.

Remark 3.51. *We list some basic properties of Chern classes here:*

- *an alternative expression of A-II is* $\xi \approx \eta \Rightarrow c_i(\xi) = c_i(\eta)$ *and* $c_i(f^*\xi) = f^* c_i(\xi)$ *for all* i*;*
- ξ *is trivial* $\Rightarrow c_i(\xi) = 0$ *for* $i > 0$*, i.e.,* $c(\xi) = 1$*;*
- *nonzero* $c_i(\xi)$*'s are obstructions to the existence of* $n - i + 1$ *l.i. sections;*
- $c(\gamma^1_{\mathbb{C},n}) = 1 - a$ *where* $a \in H^2(\mathbb{CP}^n; \mathbb{Z})$ *is the canonical generator;*
- *furthermore,* $c((\gamma^1_{\mathbb{C},n})^*) = 1 + a$ *and* $c(T_{\mathbb{C}}(\mathbb{CP}^n)) = (1 + a)^{n+1}$*; and*
- $H^*(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}[a]/\langle a^{n+1} \rangle$ *where* $a \in H^2(\mathbb{CP}^n; \mathbb{Z})$ *is the canonical generator.*

Remark 3.52. *Pontryagin classes focus on* ξ *real v.b.'s on* B *but constructed on complexification of* ξ*, that is,* $\xi_{\mathbb{C}} = \xi \otimes \mathbb{C}$ *as the complexification of* ξ*, i.e., each fiber* $\otimes\, \mathbb{C}$*. Then* $p_i(\xi) = (-1)^i c_{2i}(\xi_{\mathbb{C}}) \in H^{4i}(B; \mathbb{Z})$.

Note that $2c_{2i+1}(\xi_{\mathbb{C}}) = 0$ *produces* 2*-torsion! Similarly,* $p(\xi) := p_0(\xi) + p_1(\xi) + \cdots$ *is the* ***total Pontryagin class*** *of* ξ *and* $p(\xi \oplus \eta) = p(\xi)p(\eta)$ *up to* 2*-torsions.*

ξ *is a complex v.b. of rank* k *and a real v.b. of rank* $2k$*. The* ξ *is canonically orientated and* $e(\xi_{\mathbb{R}}) = c_k(\xi) \in H^{2k}(B; \mathbb{Z})$.

4. Chern-Weil Theory

Chern-Weil theory, quite different from Stiefel-Whitney classes, studies characteristic classes from the view of differential geometry.

4.1. Review of de Rham Cohomology Theory.

In this subsection, n will always denote the dimension of the manifold M.

Definition 4.1. *Suppose* M *is a closed smooth manifold, then*

- *let* TM *denote the* ***tangent vector bundle*** *of* M*;*
- *let* $T^*M := (TM)^*$ *denote the* ***cotangent vector bundle*** *of* M*;*
- *let* $\wedge^*(T^*M) := \bigoplus_{i=0}^{n} \wedge^i(T^*M)$ *be the* ***(complex) exterior algebra bundle*** *of* T^*M*; and*
- *let* $\Omega^*(M) := \Gamma(\wedge^*(T^*M))$ *be the* ***space of smooth sections*** *of* $\wedge^*(T^*M)$*. In particular, for any integer* p *such that* $0 \le p \le n$*, we denote by* $\Omega^p(M) := \Gamma(\wedge^p(T^*M))$ *the* ***space of smooth*** p***-forms*** *over* M*.*

Remark 4.2. *TM, T^*M, and $\wedge^*(T^*M)$ are all smooth bundles on M.*

Basically, $\Omega^*(M) = C^\infty(M, \wedge^*(T^*M))$ is the space of all differential forms over M and $\Omega^p(M) = C^\infty(M, \wedge^p(T^*M))$ is the space of all differential p-forms over M. For example, $\Omega^0(M) = C^\infty(M, \wedge^0(T^*M)) = C^\infty(M)$ and $\Omega^1(M) = C^\infty(M, \wedge^1(T^*M))$ is the space of all 1-forms.

Definition 4.3. *Let $\mathrm{d} : \Omega^*(M) \to \Omega^*(M)$ denote the **exterior differential operator**. Then d maps a p-form to a $(p+1)$-form. Furthermore, there holds the important formula $\mathrm{d}^2 = 0$.*

Remark 4.4. *The exterior derivative $\mathrm{d} : C^\infty(M) \to \Omega^1(M)$ maps f to $\mathrm{d}f$. It's clear that $(\mathrm{d}f)X = Xf$ here. $\Omega^1(M)$ is generated by $g\mathrm{d}f$ for all $f, g \in \Omega^0(M)$.*

For $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ where $\Omega^k(M)$ is generated by all $f_0\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$'s, we have $\mathrm{d}(f_0\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) = \mathrm{d}f_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$.

Definition 4.5. *The **de Rham complex** is defined by*

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \longrightarrow 0$$

where $\mathrm{d}^2 = 0$, denoted by $(\Omega^*(M), \mathrm{d})$.

Definition 4.6. *For any integer p such that $0 \leq p \leq n$, the **p-th de Rham cohomology** of M (with complex coefficients) is defined by*

$$H^p_{\mathrm{dR}}(M; \mathbb{C}) = \frac{\mathrm{Ker}(\mathrm{d}|_{\Omega^p(M)})}{\mathrm{d}\Omega^{p-1}(M)}.$$

*The **(total) de Rham cohomology** of M is then defined as*

$$H^*_{\mathrm{dR}}(M; \mathbb{C}) = \bigoplus_{p=0}^{n} H^p_{\mathrm{dR}}(M; \mathbb{C}).$$

Theorem 4.7. ***de Rham Theorem***

If M is a orientable closed smooth manifold, then for any integer p such that $0 \leq p \leq n$, we have

- *$\dim(H^p_{\mathrm{dR}}(M; \mathbb{C})) < \infty$; and*
- *$H^p_{\mathrm{dR}}(M; \mathbb{C}) \approx H^p(M; \mathbb{C})$ canonically, i.e., the p-th singular cohomology of M.*

4.2. **Connections on Vector Bundles.**

Definition 4.8. *Suppose (E, M, π) is a smooth complex vector bundle on a compact smooth manifold M. We denote by $\Omega^*(M; E)$ the space of smooth sections of the tensor product vector bundle $\wedge^*(T^*M) \otimes E$ obtained from $\wedge^*(T^*M)$ and E, that is, $\Omega^*(M; E) := \Gamma(\wedge^*(T^*M) \otimes E)$.*

Definition 4.9. *A **connection** ∇^E on E is a $\mathbb{C}$-linear operator $\nabla^E : \Gamma(E) \to \Omega^1(M; E)$ such that for any $f \in C^\infty(M)$ and $X \in \Gamma(E)$, the following **Leibniz's rule** holds, i.e.,*

$$\nabla^E(fX) = (\mathrm{d}f)X + f\nabla^E X.$$

Remark 4.10. *A connection on E may be though of as an extension of the exterior differential operator* d *to include the coefficient of smooth sections on E, or, roughly, an approach to "differentiating" the smooth sections.*

For the trivial line bundle on M where $E = M \times \mathbb{R}$, $\mathrm{d} : \Gamma(E)(= C^\infty(M)) \to \Omega^1(M; E)(= \Omega^1(M))$ implies that $(\mathrm{d}f)X = Xf$ for all $f \in C^\infty(M)$. It's clear that d *satisfies that* $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$. *For the general case, it fails to multiply sections. But we can multiply a C^∞ section by a C^∞ function through $C^\infty(M) \times \Gamma(E) \to C^\infty(M, E)$ that maps each (f, s) to fs, i.e., a number times a vector fiber-wisely.*

It's better to set a similar product rule for the way of differentiating sections. The satisfactory solution will be 1*-forms: $\Omega^1(M; E) = \Gamma(\wedge^1(T^*M) \otimes E)$ generated by $w \otimes s$ where $w \in \Omega^1(M)$ and $s \in \Gamma(E)$.*

Example 4.11. *For the trivial line bundle $E = M \times \mathbb{R} \xrightarrow{\pi_1} M$, $\mathrm{d} : \Gamma(E)(= C^\infty(M)) \to \Omega^1(M; E)(= \Omega^1(M))$ is called the **trivial connection** on E. But are there other connections on E?*

In fact, fix $\omega \in \Omega^1(M)$, define $\nabla = \mathrm{d} + \omega$ as $\nabla s = \mathrm{d}s + s\omega$ for each $s \in \Gamma(E)$. Then $\nabla = \mathrm{d} + \omega$ gives all connections on E when ω varies. Basically, for an arbitrary connection on E, say, ∇', we have $(\nabla' - \mathrm{d})(fs) = \nabla'(fs) - \mathrm{d}(fs) = (\mathrm{d}f)s + f\nabla' s - (\mathrm{d}f)s - f\mathrm{d}s = f(\nabla' - \mathrm{d})s$ where $f \in C^\infty(M)$ and $s \in \Gamma(E)$. Now set $s \equiv 1$, then $(\nabla' - \mathrm{d})f = f(\nabla' - \mathrm{d})1$. Since $(\nabla' - \mathrm{d})1 = \omega \in \Omega^1(M)$, it's easy to see that all connections on E are given by $\nabla = \mathrm{d} + \omega$.

Example 4.12. *Consider $E = M \times \mathbb{R}^k \xrightarrow{\pi_1} M$. Let $d : (f_1, \cdots, f_k)^T \mapsto (\mathrm{d}f_1, \cdots, \mathrm{d}f_k)^T$ be the **trivial connection** on E. Set $\omega = (\omega_{ij})_{k\times k}$ and $\omega_{ij} \in \Omega^1(M)$, then*

$$\nabla = \mathrm{d} + \omega : \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{d}f_1 \\ \vdots \\ \mathrm{d}f_k \end{bmatrix} + \omega \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix}$$

gives all connections on E when ω varies.

Proposition 4.13. *Suppose ∇^1, ∇^2 are two connections on $E \to M$, then*

- *$f\nabla^1 + (1 - f)\nabla^2$ is a connection on E for each $f \in C^\infty(M)$; and*
- *$\nabla^2 = \nabla^1 + \omega$ for some $\omega \in \Omega^1(M, \mathrm{End}(E)) = C^\infty(M, T^*M \otimes \mathrm{End}(E))$ where $\mathrm{End}(E) = \mathrm{Hom}(E, E)$. Locally, we have $\nabla|_{U_\alpha} = \mathrm{d} + \omega_\alpha$, $\nabla|_{U_\beta} = \mathrm{d} + \omega_\beta$ on U_β, and $\omega_\alpha = \mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\beta g_{\alpha\beta}^{-1}$ on $U_\alpha \cap U_\beta$ where $\omega_\alpha, \omega_\beta$ are $k \times k$ matrices and $g_{\alpha\beta}$ is the transition matrix.*

Proposition 4.14. ***Local Description of a Connection***

Let $E \to M$ be a real vector bundle and ∇ be a connection on E. The local triviality condition assures that there exists an open cover $\{U_\alpha\}$ such that $E|_{U_\alpha}$ is trivial, i.e., there exist $s_1^\alpha, \cdots, s_k^\alpha$ as the basis of local sections on U_α for each α. Then $\nabla|_{U_\alpha} = \mathrm{d} + \omega_\alpha$ where $\omega_\alpha = (\omega_{ij}^\alpha)_{k\times k}$ and $\omega_{ij}^\alpha \in \Omega^1(U_\alpha)$.

Moreover, on $U_\alpha \cap U_\beta$, $s_i^\alpha = g_{ij}^{\alpha\beta} s_j^\beta$ where $g_{ij}^{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta)$. Then $g_{\alpha\beta} = (g_{ij}^{\alpha\beta})_{k\times k}$ is called the ***transition matrix****. Hence, $\mathrm{d}s_i^\alpha + \omega_\alpha s_i^\alpha = \nabla s_i^\alpha = \nabla(g_{ij}^{\alpha\beta} s_j^\beta) = (\mathrm{d}g_{ij}^{\alpha\beta})s_j^\beta + g_{ij}^{\alpha\beta}\nabla s_j^\beta \Rightarrow \omega_{im}^\alpha = (\mathrm{d}g_{ij}^{\alpha\beta})g_{\alpha\beta}^{jm} + g_{il}^{\alpha\beta}\omega_{lj}^\beta g_{\alpha\beta}^{jm}$ where $(g_{\alpha\beta}^{ij})_{k\times k} = (g_{ij}^{\alpha\beta})_{k\times k}^{-1}$. More succinctly, $\omega_\alpha = (\mathrm{d}g_{\alpha\beta})g_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\beta g_{\alpha\beta}^{-1}$. In other words, locally $\nabla|_{U_\alpha} = \mathrm{d} + \omega_\alpha$, and conversely any $\omega_\alpha = (\omega_{ij}^\alpha)_{k\times k} \in \Omega^1(U_\alpha)$ satisfying the above identity gives a connection on E.*

Definition 4.15. *For the vector bundle $E \to M$, a smooth function $f : M' \to M$ induces $f^*M \to M'$. Let ∇^E be a connection on E, then there exists an open cover $\{U_\alpha\}$ of M such that $\nabla^E|_{U_\alpha} = \mathrm{d} + \omega_\alpha$. So $\{f^{-1}(U_\alpha)\}$ is an open cover of M'.*

If $s_1^\alpha, \cdots, s_k^\alpha$ is a basis of local sections on U_α with transition matrix $g_{\alpha\beta}$, then $s_1^\alpha \circ f, \cdots, s_k^\alpha \circ f$ is a basis of local sections on $f^{-1}(U_\alpha)$ with transition matrix $g_{\alpha\beta}\circ f$. If ω_α satisfies $\omega_\alpha = (\mathrm{d}g_{\alpha\beta})g_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\beta g_{\alpha\beta}^{-1}$, then $f^*\omega_\alpha$ satisfies $f^*\omega_\alpha = [\mathrm{d}(f^*g_{\alpha\beta})](f^*g_{\alpha\beta})^{-1} + (f^*g_{\alpha\beta})\omega_\beta(f^*g_{\alpha\beta})^{-1}$ where $f^*g_{\alpha\beta} = g_{\alpha\beta}\circ f$. So there is indeed a connection ∇^{f^*E} on f^*E such that $\nabla|_{U_\alpha} = \mathrm{d} + f^*\omega_\alpha$ for each α, called the ***pullback connection*** *of ∇^E along f.*

Remark 4.16. *Suppose $E, F \to M$ are smooth vector bundles and ∇^E, ∇^F are connections on E, F respectively. Then $\nabla^{E\oplus F}$ is a connection on $E \oplus F$.*

Definition 4.17. *If X is a vector field on M, then*

$$\begin{aligned}\nabla_X^E : C^\infty(M,E) &\to C^\infty(M,E)\\ s &\mapsto \nabla_X^E s = (\nabla^E s)X\end{aligned}$$

is called the ***covariant derivative****.*

4.3. **The Curvature of a Connection.** The notion of curvature originates from the study of surfaces by Gauss and Riemann. Basically, Gaussian curvature $\approx$ non-commutativity of the second derivatives.

Definition 4.18. *The* ***curvature*** *R^E of a connection ∇^E is defined by*

$$R^E = \nabla^E \circ \nabla^E : \Gamma(E) \to \Omega^2(M;E),$$

which, for brevity, is written as $R^E = (\nabla^E)^2$.

Definition 4.19. *A* ***differential form of degree*** *k, a k**-form**, on a differentiable manifold M is a k-times covariant tensor field on M.*

Proposition 4.20. ***Tensorial Property***

The curvature R^E is $C^\infty(M)$-linear, that is, for any $f \in C^\infty(M)$ and $X \in \Gamma(E)$, one has $R^E(fX) = fR^EX$.

Proof. One simply computes that $R^E(fX) = \nabla^E((\mathrm{d}f)X + f\nabla^E X) = (-1)^{\deg \mathrm{d}f}\mathrm{d}f \wedge \nabla^E X + \mathrm{d}f \wedge \nabla^E X + f(\nabla^E)^2 X = fR^EX$. □

Remark 4.21. *Let* $\mathrm{End}(E)$ *denote the vector bundle over* M *formed by the fiber-wise endomorphisms of* E*. Then by proposition 4.20,* R^E *may be thought of as an element of* $\Gamma(\mathrm{End}(E))$ *with coefficients in* $\Omega^2(M)$*, that is,* $R^E \in \Omega^2(M; \mathrm{End}(M))$*. For give a more precise formal alternative definition of curvature (cf. Definition 4.18), suppose* $X, Y \in \Gamma(TM)$ *are two smooth sections of* TM*, then* $R^E(X,Y)$ *is an element in* $\Gamma(\mathrm{End}(E))$ *given by* $R^E(X,Y) = \nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X - \nabla^E_{[X,Y]}$*, where* $[X,Y] \in \Gamma(TM)$ *is the* ***Lie bracket*** *of* X *and* Y *defined through the formula* $[X,Y]f = X(Yf) - Y(Xf) \in C^\infty(M)$ *holding for all* $f \in C^\infty(M)$*. Note that* $\Omega^k(M;E) \xrightarrow{\nabla^E} \Omega^{k+1}(M;E)$ *and by Leibniz's rule,* $\nabla^E(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^{\deg \omega} \omega \wedge \nabla^E s$*. So the alternative definition of curvature yields* $R^E = (\nabla^E)^2$*. Hence the two definitions are equivalent.*

Actually, the latter formal definition naturally yields the fact that $R^E \in \Omega^2(M; \mathrm{End}(M))$*. Also, Proposition 4.20 now can be rephrased as* $R^E(X,Y) fZ = fR^E(X,Y)Z$ *for all* $Z \in \Gamma(E)$*. Moreover, we may proceed to define* $(R^E)^k = \underbrace{R^E \circ R^E \circ \cdots \circ R^E}_{k \text{ terms}} : \Gamma(E) \to \Omega^{2k}(M;E)$*, which is a well-defined element in* $\Omega^{2k}(M; \mathrm{End}(E))$*.*

Example 4.22. *Let* $E : M \times \mathbb{R}^k \xrightarrow{\pi} M$ *and* $\nabla = \mathrm{d} + \omega$ *where* $\omega = (\omega_{ij})_{k \times k}$ *and* $\omega_{ij} \in \Omega^1(M)$*. Then for any* $s \in \Gamma(E)$*,* $\nabla_X s = Xs + \omega X s$*. Compute that*

$$\begin{aligned} R(X,Y)s &= \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s \\ &= \nabla_X(Ys + \omega Y s) - \nabla_Y(Xs + \omega X s) - [(XY - YX)s + \omega(XY - YX)s] \\ &= (X(\omega Y) - Y(\omega X) - \omega[X,Y]) + ((\omega X)(\omega Y) - (\omega Y)(\omega X))s \\ &= \mathrm{d}\omega(X,Y)s + (\omega \wedge \omega)(X,Y)s. \end{aligned}$$

Therefore, $R(X,Y) = \mathrm{d}\omega(X,Y) + (\omega \wedge \omega)(X,Y)$ *or* $R = \mathrm{d}\omega + \omega \wedge \omega$*, a* $k \times k$ *matrix consisting of* 2*-forms, or* $R = (\mathrm{d} + \omega)^2 = (\mathrm{d} + \omega) \wedge (\mathrm{d} + \omega) = \nabla \wedge \nabla = \nabla^2 = \Delta$*.*

Theorem 4.23. ***Bianchi Identity***

$[\nabla^E, (R^E)^k] = [\nabla^E, (\nabla^E)^{2k}] = 0$ *(or simply* $\nabla \circ R^k = R^k \circ \nabla$*) for all integer* $k \geq 0$*.*

Remark 4.24. *One can prove Theorem 4.23 for the trivial case as proposed in Example 4.22. The proof of the general case relies on* ***Jacobi identity****.*

4.4. **Complex Line Bundles and the First Chern Class.** Suppose $L \xrightarrow{\pi} M$ is a complex line bundle, then there exists an open cover $\{U_\alpha\}$ of M and corresponding s_α's constituting a local basis of sections. On each $U_\alpha \cap U_\beta$, we have $s_\alpha = g_{\alpha\beta} s_\beta$ where $g_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, \mathbb{C})$.

If ∇ is connection on L, then each U_α corresponds an $\omega_\alpha \in \Omega^1_{\mathbb{C}}(U_\alpha) := \Omega^1(U_\alpha; \mathbb{C})$ such that $\omega_\alpha = (\mathrm{d}g_{\alpha\beta}) g_{\alpha\beta}^{-1} + \omega_\beta$ on $U_\alpha \cap U_\beta$. On U_α, $R = \mathrm{d}\omega_\alpha + \omega_\alpha \wedge \omega_\alpha = \mathrm{d}[(\mathrm{d}g_{\alpha\beta}) g_{\alpha\beta}^{-1} + \omega_\beta] = \mathrm{d}\omega_\beta$. In other words, $\{\mathrm{d}\omega_\alpha\}$ can be regarded as a globally-defined 2-form. Clearly, it is closed, i.e., $\mathrm{d}R = 0$ by Theorem 4.23.

Definition 4.25. ***The First Chern Class***

Suppose $L \xrightarrow{\pi} M$ is a complex line bundle and ∇ is a connection on L, then $R = \nabla \circ \nabla \in \Omega^2_{\mathbb{C}}(M)$ is closed and

$$c_1(L) := \left[\frac{\sqrt{-1}}{2\pi} R\right] \in H^2_{\mathrm{dR}}(M; \mathbb{C})$$

*is called **the first Chern class** of L.*

Proposition 4.26. *$c_1(L)$ is independent of the connection ∇.*

Proof. Let ∇' be another connection on L, then $\nabla' = \nabla + \omega$ where $\omega \in \Omega^1(M; \mathrm{End}(L))$ and here $\mathbb{C} = \mathrm{End}(L)$. Let $\{U_\alpha\}$ be an open cover of M such that $s_\alpha : U_\alpha \to L|_{U_\alpha}$ is local section whose collection with respect to α is a basis. So $\nabla|_{U_\alpha} = \mathrm{d} + \omega_\alpha$ where $\omega_\alpha \in \Omega^*_{\mathbb{C}}(U_\alpha)$ and $R = \mathrm{d}\omega_\alpha$ on U_α. Then $\nabla'|_{U_\alpha} = \nabla|_{U_\alpha} + \omega = \mathrm{d} + (\omega_\alpha + \omega) \Rightarrow R' = \mathrm{d}(\omega_\alpha + \omega) = \mathrm{d}\omega_\alpha + \mathrm{d}\omega = R + \mathrm{d}\omega$ on U_α. So $\frac{\sqrt{-1}}{2\pi} R' = \frac{\sqrt{-1}}{2\pi} R + \mathrm{d}(\frac{\sqrt{-1}}{2\pi}\omega)$ differs with $\frac{\sqrt{-1}}{2\pi} R$ by an exact form which has nothing to do with α. Thus $[\frac{\sqrt{-1}}{2\pi} R'] = [\frac{\sqrt{-1}}{2\pi} R]$ and $c_1(L)$ is independent of ∇. ☐

Remark 4.27. *By universal coefficient theorem,*

$$H^*_{\mathrm{dR}}(M; \mathbb{C}) \approx H^*(M; \mathbb{C}) \approx H^*(M; \mathbb{Z}) \otimes \mathbb{C}$$

which kills the torsion. Note that $\mathbb{C}$ here can be replaced by $\mathbb{R}$ or $\mathbb{Q}$.

Proposition 4.28. *Suppose $c_1(f^*L) = f^*c_1(L)$, then $L \approx L'$ and $c_1(L) = c_1(L')$, i.e., the first Chern class satisfies the naturality axiom (see Theorem 3.50).*

Proof. Let $\{U_\alpha\}$ be an open cover of M such that $s_\alpha : U_\alpha \to L|_{U_\alpha}$ constitutes a basis of local sections. Let ∇^L be a connection on L, then $\nabla|_{U_\alpha} = \mathrm{d} + \omega_\alpha$ where $\omega_\alpha \in \Omega^1_{\mathbb{C}}(U_\alpha)$. For the pull back connection ∇^{f^*L}, we have $\nabla^{f^*L}|_{f^{-1}(U_\alpha)} = \mathrm{d} + f^*\omega_\alpha$ for each α. Then $R^{f^*L} = \mathrm{d}(f^*\omega_\alpha) = f^*(\mathrm{d}\omega_\alpha) = f^*R^L \Rightarrow c_1(f^*L) = f^*c_1(L)$.

As for the second part, let $F : L \to L'$ be a bundle isomorphism, then $F(s_\alpha) = s'_\alpha$ constitutes a basis of local sections. So $\{\omega_\alpha\}$ defines a connection $\nabla^{L'}$ on L'. Hence $R^{L'} = R^L \Rightarrow c_1(L) = c_1(L')$. ☐

In order to check whether c_1 satisfies the normalization axiom, we need to introduce more structures:

Definition 4.29. *Suppose $E \xrightarrow{\pi} M$ is a $\mathbb{C}$-vector bundle. An **hermitian metric** on E is a smooth assignment of hermitian metrics on the fibers of E. More precisedly, for each $x \in M$, $h(x) = \langle \bullet, \bullet \rangle_x$ an hermitian metric on $E_x := \pi^{-1}(x)$, i.e.,*

- *$\langle v, w \rangle_x = \overline{\langle w, v \rangle_x}$ for each $v, w \in E_x$,*
- *$\mathbb{C}$-linear in the first variable $\Rightarrow$ conj. linear in the second variable, and*
- *$\langle v, v \rangle_x \geq 0$ and $= 0$ iff $v = 0$.*

Moreover, the family $h(x) = \langle \bullet, \bullet \rangle_x$ is smooth in the sense, for any smooth sections s, s' of E, $\langle s(x), s'(x) \rangle_x$ is C^∞ in x.

Example 4.30. *Suppose $L \xrightarrow{\pi} M$ is a $\mathbb{C}$-line bundle equipped with $\{(U_\alpha, s_\alpha)\}$ as exemplified before. Then $h_\alpha = \langle s_\alpha, s_\alpha\rangle \in \mathbb{C}^\infty(U_\alpha)$ and $h_\alpha \geq 0$ for each α. On $U_\alpha \cap U_\beta$, $s_\alpha = g_{\alpha\beta}s_\beta$ with $g_{\alpha\beta}$ as the transition function. Therefore, $h_\alpha = |g_{\alpha\beta}|^2 h_\beta$. Conversely, any $\{h_\alpha\} \subset C^\infty(U_\alpha)$ defines an hermitian metric on L provided that $h_\alpha \geq 0$, $h_\alpha = |g_{\alpha\beta}|^2 h_\beta$, and for each $x \in M$ there is h_α such that $g_\alpha(x) > 0$.*

Example 4.31. *Consider the tautological complex line bundle $\gamma^1_{\mathbb{C},1} \xrightarrow{\pi} \mathbb{C}\mathrm{P}^1$ (see Theorem 3.50 for normalization), and elements in $\mathbb{C}\mathrm{P}^1$ are represented by $[z_0, z_1]$ where z_0, z_1 never both equal zero.*

Let $U_0 = \{[z_0, z_1] | z_1 \neq 0\}$, $U_1 = \{[z_0, z_1] | z_0 \neq 0\}$, $s_0 : U_0 \to \gamma^1_{\mathbb{C},1}|_{U_0}$, $s_0([z, 1]) = ([z, 1], (z, 1))$, $s_1 : U_1 \to \gamma^1_{\mathbb{C},1}|_{U_1}$, and $s_1([1, w]) = ([1, w], (1, w))$. Then $s_0 = g_{01}s_1$ where $g_{01}([z, 1]) = \frac{1}{z}$ for all $z \neq 0$. Then connection on $\gamma^1_{\mathbb{C},1}$ is given by $\omega_0 \in \Omega^1_{\mathbb{C}}(U_0)$, $\omega_1 \in \Omega^1_{\mathbb{C}}(U_1)$ such that $\omega_0 = \mathrm{d}(g_{01})g_{01}^{-1} + \omega_1$.

An hermitian metric helps us in picking out such $\{\omega_\alpha\}$'s. An hermitian metric on $\gamma^1_{\mathbb{C},1}$ means $h_0 \in C^\infty(U_0)$, $h_0 \geq 0$, $h_1 \in C^\infty(U_1)$, $h_1 \geq 0$, and $h_0 = |g_{01}|^2 h_1 = \frac{h_1}{|z|^2}$. For example, let $h_0(z) = h_0([z, 1]) = (1 + |z|^2)$ and $h_1(w) = (1 + |w|^2)$.

Definition 4.32. *Let M be a complex manifold, then $E \to M$ is said to be a **holomorphic vector bundle** if there exists an open holomorphic coordinate cover $\{U_\alpha\}$ and a basis of local sections consisting of s_α's on corresponding U_α's where all transitions on $U_\alpha \cap U_\beta$ is holomorphic.*

Example 4.33. *$\gamma^1_{\mathbb{C},1} \to \mathbb{CP}^1$ is a holomorphic line bundle.*

Theorem 4.34. ***Shiing-Shen Chern***

*Let $E \to M$ be a holomorphic vector bundle equipped with an hermitian metric h, then there exists a unique connection on E, i.e., **Chern connection** such that $\omega_\alpha = \partial H_\alpha \cdot H_\alpha^{-1}$ where $H_\alpha = (\langle s_i^{(\alpha)}, s_j^{(\alpha)}\rangle)_{k\times k}$ and $k = \mathbb{C}\text{-rank}(E)$ for each α. In particular, for $\mathbb{C}$-line bundles, we have $\omega_\alpha = \partial h_\alpha \cdot h_\alpha^{-1}$.*

Remark 4.35. *The existence and uniqueness of Chern connection comes from*

- *the compatiblity with hermitian metrics, and*
- *the compatiblity with the complex structures.*

Example 4.36. *For $\gamma^1_{\mathbb{C},1} \to \mathbb{CP}^1$, $h_0(z) = 1 + |z|^2$, $U_0 \cap U_1 = \{z : z \neq 0\}$, and $\omega_0 = \partial h_0 \cdot h_0^{-1} = \partial(\log h_0)$, we have $R = \mathrm{d}\omega_0 = (\partial + \bar\partial)\partial(\log h_0) = \bar\partial\partial \log h_0 = -\frac{\mathrm{d}z\wedge\mathrm{d}\bar z}{(1+|z|^2)^2}$, $\partial(|z|^2) = \partial(z\bar z) = \bar z \mathrm{d}z = 0 + 0$, and $\int_{\mathbb{CP}^1} \frac{\sqrt{-1}}{2\pi} R = \int_{\mathbb{C}} -\frac{\sqrt{-1}}{2\pi}\frac{\mathrm{d}z\wedge\mathrm{d}\bar z}{(1+|z|^2)^2} = -1$, which is the correct normalization.*

Remark 4.37. *The remaining axioms require higer Chern classes.*

4.5. **Chern-Weil Theorem.** We consider the $\mathbb{C}$-vector bundle $E \to M$, and ∇ is a connection on E. Then $R^E \in \Omega^2(M, \mathrm{End}(E))$. So how do we get an honest form out of this?

Recall that for a $\mathbb{C}$-matrix $A = (a_{ij})_{k\times k}$, we have $\mathrm{tr}(A) = \sum_{i=1}^k a_{ii} \in \mathbb{C}$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, i.e., $\mathrm{tr}([A,B]) = 0$. Also, $\mathrm{tr}(B^{-1}AB) = \mathrm{tr}(A)$ for each invertible B and $\mathrm{tr} : \mathrm{End}(V) \to \mathbb{C}$ well-defined for any finite-dimensional real or complex vector space V. Trace over fiber-wisely, we derive $\mathrm{tr} : C^\infty(M, \mathrm{End}(E)) \to C^\infty(M)$. In fact, this extends to $\mathrm{tr} : \Omega^*(M; \mathrm{End}(E)) \to \Omega^*(M)$, $\alpha \otimes A \mapsto \alpha\mathrm{tr}(A)$.

Example 4.38. *For $E = M \times \mathbb{C}^k$ and $\omega = (\omega_{ij})_{k\times k} \in \Omega^*(M; \mathrm{End}(E))$ where each $\omega_{ij} \in \Omega^*(M)$, we have $\mathrm{tr}(\omega) = \sum_{i=1}^k \omega_{ii}$.*

Recall that $R^E \in \Omega^2(M; \mathrm{End}(E))$. If $f(x) = a_0 + a_1x + a_2x^2 + \cdots \in \mathbb{C}[[x]]$ is a formal power series, then let $f(R^E) =: a_0I + a_1R^E + a_2(R^E)^2 + \cdots \in \Omega^{\mathrm{even}}(M; \mathrm{End}(E))$ which terminates with finite terms.

For each $\omega \in \Omega^*(M; \mathrm{End}(E))$, let

$$[\nabla^E, \omega] := \nabla^E \circ \omega - (-1)^{\deg \omega}\omega \circ \nabla^E \in \Omega^*(M; \mathrm{End}(E)).$$

Example 4.39. *Consider $E = M \times \mathbb{C}^r \to M$, $\nabla = \mathrm{d} + \omega$ where $\omega = (\omega_{ij})_{k\times k}$ and $\omega_{ij} \in \Omega^*_{\mathbb{C}}(M)$. Choose $\tilde{\omega} = (\tilde{\omega}_{ij})_{k\times k} \in \Omega^*(M; \mathrm{End}(E))$ where $\tilde{\omega}_{ij} \in \Omega^*_{\mathbb{C}}(M)$. Then*

$$\begin{aligned}[\nabla, \tilde{\omega}]s &= \nabla(\tilde{\omega}s) - (-1)^{\deg \tilde{\omega}}\tilde{\omega}(\nabla s) \\ &= (\mathrm{d} + \omega)(\tilde{\omega}s) - (-1)^{\deg \tilde{\omega}}\tilde{\omega}(\mathrm{d}s + \omega s) \\ &= (\mathrm{d}\tilde{\omega})s + (-1)^{\deg \tilde{\omega}}\tilde{\omega}\mathrm{d}s + \omega\tilde{\omega}s - \cdots \\ &= [\mathrm{d}\tilde{\omega} + \omega\tilde{\omega} - (-1)^{\deg \tilde{\omega}}\tilde{\omega}\omega]s.\end{aligned}$$

For general k, $[\omega, \tilde{\omega}] \neq 0$, but $\mathrm{tr}([\omega, \tilde{\omega}]) = 0$. So $\mathrm{tr}([\nabla, \tilde{\omega}]) = \mathrm{d}\tilde{\omega} = \mathrm{d}(\mathrm{tr}(\tilde{\omega}))$.

Example 4.40. *For $R \in \Omega^2(M; \mathrm{End}(E))$, we have $\nabla \circ R - (-1)^2 R \circ \nabla = [\nabla, R] \equiv 0$ by Theorem 4.23.*

We then derive the following lemma which states a local equation. Since any vector bundle is locally trivial so the proof reduces to the aforementioned computations.

Lemma 4.41. *For any connection ∇^E on E and $\tilde{\omega} \in \Omega^*(M; \mathrm{End}(E))$, we have $\mathrm{d}(\mathrm{tr}(\tilde{\omega})) = \mathrm{tr}([\nabla^E, \tilde{\omega}])$.*

By Lemma 4.41, we derive:

Theorem 4.42. ***Chern-Weil Theorem***

- *The differential form $\mathrm{tr}(f(R^E)) \in \Omega^{even}_{\mathbb{C}}(M)$ is closed.*
- *The cohomology class $[\mathrm{tr}(f(R^E))] \in H^{even}_{\mathrm{dR}}(M, \mathbb{C})$, the characteristic class defined by $f \in \mathbb{C}[[x]]$, is independent of connection.*

Definition 4.43. ***Total Chern Class***

*The **total Chern class** is defined to be*

$$c(E) := c\left(\frac{\sqrt{-1}}{2\pi}R^E\right) = \det\left(I + \frac{\sqrt{-1}}{2\pi}R^E\right) = \exp\left[\mathrm{tr}\left(\log\left(I + \frac{\sqrt{-1}}{2\pi}R^E\right)\right)\right].$$

Remark 4.44. *Check that for any* $\omega \in \Omega^{even}(M; \mathrm{End}(E))$*, it follows that*

$$\det(I+\omega) = \exp[\mathrm{tr}(\log(I+\omega))].$$

Since $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$, we have $\mathrm{tr}(\log(I+\omega)) = \mathrm{tr}(\omega) - \frac{1}{2}\mathrm{tr}(\omega^2) + \frac{1}{3}\mathrm{tr}(\omega^3) - \cdots$. Then we derive

$$\begin{aligned}\exp[\mathrm{tr}(\log(I+\omega))] &= 1 + \left[\mathrm{tr}(\omega) - \frac{1}{2}\mathrm{tr}(\omega^2) + \frac{1}{3}\mathrm{tr}(\omega^3) - \cdots\right] \\ &+ \frac{1}{2}\left[\mathrm{tr}(\omega) - \frac{1}{2}\mathrm{tr}(\omega^2) + \frac{1}{3}\mathrm{tr}(\omega^3) - \cdots\right]^2 + \frac{1}{6}\left[\mathrm{tr}(\omega) - \frac{1}{2}\mathrm{tr}(\omega^2) + \frac{1}{3}\mathrm{tr}(\omega^3) - \cdots\right]^3.\end{aligned}$$

It follows that

$$c_1(E) = \frac{\sqrt{-1}}{2\pi}\mathrm{tr}(R^E),\ c_2(E) = \frac{\sqrt{-1}}{8\pi^2}\left[\mathrm{tr}((R^E)^2) - (\mathrm{tr}(R^E))^2\right].$$

One the other hand, if $\omega = (\omega_{ij})_{k\times k}$ where $\omega_{ij} \in \Omega^{\mathrm{even}}(M)$, then by the usual formula for determinants

$$\det(\omega) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot \omega_{1\sigma(1)} \cdots \omega_{k\sigma(k)}.$$

Similarly, for $(\omega_{ij})_{k\times k}$ and $\omega' = (\omega'_{ij})_{l\times l}$, we have $\det(\mathrm{diag}(\omega, \omega')) = \det(\omega)\det(\omega')$.

Now we have verified the rest of the axioms of Chern class (see Theorem 3.50):

- dimension: if $c_0(E) = 1$ and $c_i(E) = 0$ when $i > \mathrm{rank}(E)$;
- Whitney sum formula: $c(E \oplus F) = c(E)c(F)$.

What has been discussed above presents a construction of the rational Chern classes for complex vector bundles which suffices in most cases of application. Moreover, given a connection, the construction produces smooth differential forms representing the Chern classes. This is thus another refinement of the Chern classes in the smooth category. Also, the Pontryagin classes for real vector bundles can then be defined via complexification, as mentioned before.

4.6. **Principle Bundles, Chern-Weil Homomorphism, and to Inifintity and Beyond.** Due to the limited sessions, the speaker failed to lecture on this part.

Department of Mathematics, Soochow University, Suzhou, Jiangsu 215000, China