

# Algebraic Geometry

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### 1. HOMEWORK #1: 7/19/2018

**Homework 1.1.** Let  $U_1 = \{[z_0, z_1] \in \mathbb{P}^1 : z_1 \neq 0\}$  where  $\mathbb{P}^1$  is the projective space of  $\mathbb{C}^2$  and  $[z_0, z_1]$  is the homogeneous coordinate. Prove that

$$\begin{aligned}\varphi_1 : U_1 &\rightarrow \mathbb{C}, \\ [z_0, z_1] &\mapsto w = z_0/z_1\end{aligned}$$

is a homeomorphism.

*Proof.* Since for all  $z_0, z_1, z'_0, z'_1 \in \mathbb{C}$  with  $z_1, z'_1 \neq 0$ ,  $[z_0, z_1] = [z'_0, z'_1] \Leftrightarrow [z_0/z_1, 1] = [z'_0/z'_1, 1] \Leftrightarrow z_0/z_1 = z'_0/z'_1$ . For each open set  $V \in \mathbb{C}$ ,  $\varphi_1^{-1}(V)$  is the collection of equivalent classes of all elements in  $V$ . By the definition of quotient topology,  $\varphi_1^{-1}(V)$  automatically is open. So  $\varphi_1$  is well-defined, continuous, one-to-one, and onto. Note that  $\varphi_1^{-1}$  is given by  $\varphi_1^{-1}(z) = \pi(z, 1)$  for each  $z \in \mathbb{C}$  where  $\pi$  is the projection from  $\mathbb{C}^2$  to  $\mathbb{P}^1$ . So  $\varphi_1^{-1}$  is continuous. Hence,  $\varphi_1$  is a homeomorphism.  $\square$

**Homework 1.2.** Let  $f$  be a holomorphic function on  $\mathbb{C}^n$  and  $\text{grad} f$  has no zero on  $\mathbb{C}^n$ , that is,  $\text{rank}(\partial f/\partial z_1, \dots, \partial f/\partial z_n) = 1$  on  $X$ . Prove that  $X = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$  is a complex manifold.

*Proof.* Assume that  $X \neq \emptyset$ . For  $\alpha = (\alpha_1, \dots, \alpha_n) \in X$ , assume that  $f_{z_1}(\alpha) \neq 0$ . Then by implicit function theorem, there exists an open set  $U'_\alpha \subset \mathbb{C}^n$  containing  $\alpha$  and an open set  $W_\alpha \subset \mathbb{C}^{n-1}$  together with a holomorphic function  $g_\alpha : W_\alpha \rightarrow \mathbb{C}$  such that  $f(z_1, \dots, z_n) = 0$  on  $U'_\alpha \Leftrightarrow z_1 = g_\alpha(z_2, \dots, z_n)$  on  $W_\alpha$ . Let  $U_\alpha = \{(z_1, \dots, z_n) \in U'_\alpha : f(z_1, \dots, z_n) = 0\}$  and  $\varphi_\alpha : U_\alpha \rightarrow \varphi_\alpha(U_\alpha) \subset \mathbb{C}^{n-1}$  by  $(z_1, z_2, \dots, z_n) = (g_\alpha(z_2, \dots, z_n), z_2, \dots, z_n) \mapsto (z_2, \dots, z_n)$ . It's obvious that  $\varphi_\alpha$  is a homeomorphism. Clearly, all  $U_\alpha$  covers  $X$ .

For arbitrary  $\alpha, \beta \in X$  with  $U_\alpha \cap U_\beta \neq \emptyset$ , assume  $f_{z_1}(\alpha) \neq 0$  and  $f_{z_i}(\beta) \neq 0$ . If  $i = 1$ , then it's trivial that  $\varphi_\beta \circ \varphi_\alpha^{-1}$  is a biholomorphism. So we may assume that  $i = 2$ , then the homeomorphism  $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$  is given by

$$(z_2, \dots, z_n) \mapsto (z_1, z_3, \dots, z_n) = (g_\alpha(z_2, \dots, z_n), z_3, \dots, z_n).$$

So  $\varphi_\alpha \circ \varphi_\beta^{-1}$  is holomorphic since  $g_\alpha$  is holomorphic. Similarly,  $\varphi_\alpha \circ \varphi_\beta^{-1}$  is holomorphic and then  $\varphi_\beta \circ \varphi_\alpha^{-1}$  is a biholomorphism.

Since  $X \subset \mathbb{C}^n$ , so the subspace topology is obviously Hausdorff and second countable. Hence,  $X$  is a complex manifold.  $\square$

**Homework 1.3.** Let  $\Gamma$  be a full rank lattice. Prove that  $\mathbb{C}^n/\Gamma$  is a complex manifold.

*Proof.* Suppose  $\Gamma$  is generated by linearly independent elements  $\alpha_1, \dots, \alpha_{2n} \in \mathbb{C}^n$ , then  $\Gamma = \{k_1\alpha_1 + \dots + k_{2n}\alpha_{2n} : k_1, \dots, k_{2n} \in \mathbb{Z}\}$ .

Let  $m \in \{1, \dots, 2n\}$  and  $I = \{i_k\}_{k=1}^{2n}$  be a rearrangement of  $\{k\}_{k=1}^{2n}$ . Let  $U_I^m = \{[k_1\alpha_1 + \dots + k_{2n}\alpha_{2n}] : 0 < k_{i_1}, \dots, k_{i_m} < 1, 1/4 < k_{i_{m+1}}, \dots, k_{i_n} < 5/4\}$ . Then the union of  $U_I^m$  for all  $m, I$  obviously covers  $\mathbb{C}^n/\Gamma$ . Let  $\varphi_I^m : U_I^m \rightarrow \varphi(U_I^m) \subset \mathbb{C}^n$  be defined by

$$[k_1\alpha_1 + \dots + k_{2n}\alpha_{2n}] \mapsto k_1\alpha_1 + \dots + k_{2n}\alpha_{2n}.$$

Since  $\alpha_i$ 's are linearly independent, it's clear that  $\varphi_I^m(U_I^m)$  is an open subset of  $\mathbb{C}^n$  and  $\varphi_I^m$  is a homeomorphism.

For all  $m, m', I, I'$  with  $U_I^m \cap U_{I'}^{m'} \neq \emptyset$ , it's easy to see that

$$\varphi_{U_{I'}^{m'}} \circ \varphi_{U_I^m}^{-1} : \varphi_{U_I^m}(U_I^m \cap U_{I'}^{m'}) \rightarrow \varphi_{U_{I'}^{m'}}(U_I^m \cap U_{I'}^{m'})$$

is an identity map and, also,  $\varphi_{U_I^m}(U_I^m \cap U_{I'}^{m'}) = \varphi_{U_{I'}^{m'}}(U_I^m \cap U_{I'}^{m'})$  by the construction of  $U_I^m$  and  $U_{I'}^{m'}$ . Hence,  $\varphi_{U_{I'}^{m'}} \circ \varphi_{U_I^m}^{-1}$  is a biholomorphism.

Since the quotient topology on  $\mathbb{C}^n/\Gamma$  is inherited from  $\mathbb{C}^n$  under  $\Gamma$ , then  $\mathbb{C}^n/\Gamma$  is Hausdorff and second countable. Thus,  $\mathbb{C}^n/\Gamma$  is a complex manifold.  $\square$

## 2. HOMEWORK #2: 7/20/2018

**Homework 2.1.** Prove that  $\mathbb{P}^n$  is compact.

*Proof.* Since  $S^{2n+1}$  is the unit sphere in  $\mathbb{C}^n$ , let  $f : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$  be defined by  $z \mapsto [z]$ , and then  $g := f|_{S^{2n+1}} : S^{2n+1} \rightarrow \mathbb{P}^n$  is the restriction of  $f$  to  $S^{2n+1}$ . By the derivation of  $\mathbb{P}^n$ , the topology on  $\mathbb{P}^n$  is induced by  $f$ . Then  $f$  is a quotient map and so it's continuous. Thus,  $g$  is continuous.

For each  $[z_0, z_1, \dots, z_n] \in \mathbb{P}^n$ , assume that  $z_0 \neq 0$ , then  $(z_0/c, z_1/c, \dots, z_n/c) \in S^{2n+1}$  and

$$g\left(\frac{z_0}{c}, \frac{z_1}{c}, \dots, \frac{z_n}{c}\right) = f\left(\frac{z_0}{c}, \frac{z_1}{c}, \dots, \frac{z_n}{c}\right) = \left[\frac{z_0}{c}, \frac{z_1}{c}, \dots, \frac{z_n}{c}\right] = [z_0, z_1, \dots, z_n].$$

So  $g$  is onto. Note that  $S^{2n+1}$  is compact. Thus,  $\mathbb{P}^n = g(S^{2n+1})$  is compact.  $\square$

### 3. HOMEWORK #3: 7/24/2018

**Homework 3.1.** Given an open cover  $\{U_\alpha\}$  of the complex manifold  $X$  and a collection of  $g_{\alpha\beta}$ 's which are holomorphic functions from  $U_\alpha \cap U_\beta$  to  $GL(r, \mathbb{C})$  satisfying  $g_{\alpha\alpha} = 1$ ,  $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$  on  $U_\alpha \cap U_\beta$ , and  $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$  on  $U_\alpha \cap U_\beta \cap U_\gamma$ . Construct a holomorphic vector bundle  $E$  over  $X$ :

$$E = \Pi_\alpha U_\alpha \times \mathbb{C}^r / \sim$$

where  $(x, v_\alpha) \sim (y, v_\beta)$  iff  $x = y$  and  $v_\alpha = g_{\alpha\beta}v_\beta$ .

- (a) Show that  $\sim$  is an equivalent relationship.
- (b) Prove that  $E$  is a well-defined holomorphic vector bundle over  $X$ .
- (c) Consider the Cartier divisor  $\{(U_\alpha, f_\alpha)\}$  and let  $g_{\alpha\beta} = f_\alpha/f_\beta$ . Show that how this Cartier divisor induces a line bundle uniquely.

*Solution.* (a) For each  $x \in U_\alpha$ , since  $x = x$  and  $v_\alpha = v_\alpha = g_{\alpha\alpha}v_\alpha$ , we have  $(x, v_\alpha) \sim (x, v_\alpha)$ .

If  $(x, v_\alpha) \sim (y, v_\beta)$  where  $x \in U_\alpha$  and  $y \in U_\beta$ , then  $x = y$  and  $v_\alpha = g_{\alpha\beta}v_\beta$ . Hence  $y = x$  and  $v_\beta = g_{\alpha\beta}^{-1}v_\alpha = g_{\beta\alpha}v_\alpha$ , which implies that  $(y, v_\beta) \sim (x, v_\alpha)$ .

If  $(x, v_\alpha) \sim (y, v_\beta)$  and  $(y, v_\beta) \sim (z, v_\gamma)$ , then  $x = y = z$ ,  $v_\alpha = g_{\alpha\beta}v_\beta$ , and  $v_\beta = g_{\beta\gamma}v_\gamma$ . Thus we derive  $x = z$ ,  $v_\alpha = g_{\alpha\beta}g_{\beta\gamma}v_\gamma = g_{\alpha\gamma}v_\gamma$ . So  $(x, v_\alpha) \sim (z, v_\gamma)$ .

In all, we have shown that  $\sim$  is indeed an equivalent relationship.

(b) Given any  $x \in X$ , there exists an open set  $U_\alpha$  containing  $x$  since  $\{U_\alpha\}$  is an open cover of  $X$ . Denote the element of  $\Pi_\alpha U_\alpha \times \mathbb{C}^r / \sim$  by  $[(x, v)]$  where  $(x, v) \in \Pi_\alpha U_\alpha \times \mathbb{C}^r$ .

Define  $\pi : \Pi_\alpha U_\alpha \times \mathbb{C}^r / \sim \rightarrow X = \Pi_\alpha U_\alpha$  by  $[(x, v_\alpha)] \mapsto x$  and define  $h_\alpha : U_\alpha \times \mathbb{C}^r \rightarrow \pi^{-1}(U_\alpha)$  by  $(x, v_\alpha) \mapsto [(x, v_\alpha)]$ . It's clear that  $h_\alpha$  is continuous. Also, it's revealed that  $h_\alpha$  is a surjection. This is because for each  $[(x, v_\beta)] \in \pi^{-1}(U_\alpha)$  there exists a transition function  $g_{\alpha\beta}$  such that  $(x, g_{\alpha\beta}(x)v_\beta) = (x, v_\alpha)$  and then  $[(x, v_\beta)] = [(x, v_\alpha)]$ . Suppose  $h_\alpha(x_1, v_{\alpha_1}) = h_\alpha(x_2, v_{\alpha_2})$ , then  $[(x_1, v_{\alpha_1})] = [(x_2, v_{\alpha_2})]$ . So  $(x_1, v_{\alpha_1}) \sim (x_2, v_{\alpha_2}) \Rightarrow x_1 = x_2$  and  $v_{\alpha_1} = g_{\alpha_1\alpha_2}v_{\alpha_2}$ , which implies that  $h_\alpha$  is an injection. Now we have shown that  $h_\alpha$  is bijective. And by the derivation of the quotient topology on  $\pi^{-1}(U_\alpha)$ ,  $h_\alpha$  appears to be a one-to-one correspondence of open sets. Hence it's an homeomorphism. So  $E$  is a well-defined holomorphic vector bundle over  $X$  since  $\alpha$  is arbitrary.

(c) Since  $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ ,  $g_{\gamma\alpha}^{-1} = g_{\alpha\gamma} = g_{\alpha\beta}g_{\beta\gamma}$ . Define the relationship for  $(x, f_\alpha(x)) \in U_\alpha \times \mathbb{C}$  and  $(y, f_\beta(y)) \in U_\beta \times \mathbb{C}$  that  $(x, f_\alpha(x)) \sim (y, f_\beta(y))$  iff  $x = y$  and  $f_\alpha(x) = g_{\alpha\beta}f_\beta(y)$ . By (a),  $\sim$  is an equivalent relationship. So  $\Pi_\alpha(U_\alpha, f_\alpha)/\sim$  is a holomorphic vector bundle as well as a line bundle which is uniquely determined by the Cartier divisor under  $\sim$ .  $\square$

### 4. HOMEWORK #4: 7/26/2018

**Homework 4.1.** Suppose  $L_1, L_2 \in \text{Pic}(X)$ . Show that  $L_1 \otimes L_2$  is a line bundle.

*Proof.* Since  $L_1, L_2$  are line bundles, for any  $b \in X$  and its neighborhood  $U_b$ , there exists  $s_1$  as the only basis of sections on  $U_b$  of  $L_1$  and  $s_2$  as the only basis of sections on  $U_b$  of  $L_2$ . Therefore,  $s_1 \otimes s_2$  is exactly the basis of sections on  $U_b$ . So  $L_1 \otimes L_2$  is a line bundle.  $\square$

**Homework 4.2.** Let  $L_1, L_2$  be line bundles generated by  $\{U_\alpha, g_{\alpha\beta}^1\}$  and  $\{U_\alpha, g_{\alpha\beta}^2\}$  respectively. Show that the transition functions of  $L_1 \otimes L_2$  are  $\{g_{\alpha\beta}^1 g_{\alpha\beta}^2\}$ .

*Proof.* Suppose  $s_1^\alpha \otimes s_2^\alpha$  is a local basis of sections on  $U_\alpha$  and  $s_1^\beta \otimes s_2^\beta$  is a local basis of sections on  $U_\beta$ . It's known that  $s_1^\alpha = g_{\alpha\beta}^1 s_1^\beta$  and  $s_2^\alpha = g_{\alpha\beta}^2 s_2^\beta$ . So  $s_1^\alpha \otimes s_2^\alpha = g_{\alpha\beta}^1 s_1^\beta \otimes g_{\alpha\beta}^2 s_2^\beta = g_{\alpha\beta}^1 g_{\alpha\beta}^2 s_1^\beta \otimes s_2^\beta$ . Hence, the transition functions of  $L_1 \otimes L_2$  are  $g_{\alpha\beta}^1 g_{\alpha\beta}^2$ 's.  $\square$

## 5. HOMEWORK #5: 7/27/2018

**Homework 5.1.** Let  $S \subset X$  be a subvariety and  $\mathcal{L}_S(U) = \{f \in \mathcal{O}_X(U) : f|_{S \cap U} = 0\}$ . Prove that  $\mathcal{L}_S$  is a sheaf.

*Proof.* For each  $U \subset V$ , the restriction  $\gamma_{VU} : \mathcal{L}_S(V) \rightarrow \mathcal{L}_S(U)$  is a homomorphism. In fact, for all holomorphic functions  $f, g$  vanishing on  $V \cap S$ ,  $\gamma_{VU}(f + g) = f|_U + g|_U$  vanishes on  $U \cap S$ . Also,  $\gamma_{VU}(f + g) = f|_U + g|_U = \gamma_{VU}(f) + \gamma_{VU}(g)$ .

For all  $U \subset V \subset W$ , a holomorphic function  $f$  vanishing on  $W \cap S$  also vanishes on  $U \cap S$  and  $V \cap S$ . Thus  $\gamma_{WU}(f) = f|_U$ ,  $\gamma_{VU} \circ \gamma_{WV} = \gamma_{VU}(f|_V)$ , and  $f|_V$  vanishing on  $V \cap S$  also vanishes on  $U \cap S$ . So  $\gamma_{VU}(f|_V) = f|_U = \gamma_{WV}(f) = \gamma_{VU} \circ \gamma_{WV}(f) \Rightarrow \gamma_{WV} = \gamma_{VU} \circ \gamma_{WV}$ .

For a collection of open sets  $U_\alpha \subset X$  ( $\alpha \in I$ ) with  $U = \cup_{\alpha \in I} U_\alpha$ . If  $h \in \mathcal{O}(U)$  and  $\gamma_{U_\alpha}(h) = 0$ , then  $h$  vanishes on  $U_\alpha \cap S$  for all  $\alpha \in I$ . So  $h$  vanishes on  $\cup_{\alpha \in I} (U_\alpha \cap S) = U \cap S \Rightarrow h = 0$ .

For  $U = \cup_{\alpha \in I} U_\alpha$  define  $h : U_\alpha \rightarrow \mathbb{C}$  such that  $h(x) = f_\alpha(x)$  for all  $x \in U_\alpha$ . Since  $\gamma_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha) = \gamma_{U_\beta, U_\alpha \cap U_\beta}(f_\beta)$ , then  $h$  is holomorphic on  $U$ . Given a fixed  $\alpha \in I$ ,  $\gamma_{U_\alpha}(h) = f_\alpha$ . And it's obvious that  $\mathcal{F}(\emptyset) = 0$  and  $\gamma_{VU} = 1$ .

In all, we have shown that  $\mathcal{L}_S$  is indeed a sheaf.  $\square$

**Homework 5.2.** Consider the sheaf morphism  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ . This induces a morphism  $\varphi_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$ ,  $(U, f) \mapsto (U, \varphi_U(f))$  for each  $p \in X$ . Show that  $\varphi_p$  is well-defined.

*Proof.* If  $(U, f) \sim (V, g)$  where  $(U, f) \in \mathcal{F}_p$  and  $(V, g) \in \mathcal{F}_p$ , then  $\varphi_p^1(f) = f' \in \mathcal{O}_\mathcal{G}(U)$ ,  $\varphi_p(U, f) = (U, f')$ ,  $\varphi_p^2(g) = g' \in \mathcal{O}_\mathcal{G}(V)$ , and  $\varphi_p(V, g) = (V, g')$ . For any  $U \subset X$ ,  $f \in \mathcal{O}(U)$ , and  $W \subset U$ ,  $\gamma_{UW}^\mathcal{F}(f) = f|_W$ . Since there exists a homomorphism from  $\mathcal{F}(W)$  to  $\mathcal{G}(W)$ , denoted  $\varphi_{pW}^1$ , then  $\varphi_p^1(f|_W) = f'|_W$ . Also,  $\gamma_{UW}^\mathcal{G} \circ \varphi_p^1(f) = f'|_W = \varphi_p^2 \circ \gamma_{UW}^\mathcal{F}(f)$ ,  $\gamma_{UW}^\mathcal{G} \circ \varphi_p^1(g) = g'|_W = \varphi_p^2 \circ \gamma_{UW}^\mathcal{F}(g)$ , and  $f|_W = g|_W$ . So  $f'|_W = \varphi_p^2(f|_W) = \varphi_p^2(g|_W) = g'|_W$ . Therefore,  $(U, f') \sim (V, g')$  and  $\varphi_p$  is well-defined.  $\square$

**Homework 5.3.** Prove that  $\varphi$  defined in Homework 5.2 is injective iff for each  $U \subset X$ ,  $\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is injective.

*Proof.*  $\Leftarrow$  Suppose  $\varphi_U$  is injective. If  $\varphi_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$  is injective, then  $\varphi$  is injective. The aim is to prove that  $\varphi_p$  is injective.

$[(U, f)]$  is the equivalent class of  $(U, f)$  in  $\mathcal{F}_p$ , and let  $\varphi_p([(U, f)]) = [(U, f')]$  where  $[(U, f')]$  is the equivalent class of  $(U, f')$  in  $\mathcal{G}_p$ . Since there exists  $(V, g)$  such that  $\varphi_p([(V, g)]) = [(V, g')] = [(U, f')]$  and  $[(V, g')] = [(U, f')]$ , then  $(V, g') \sim (U, f')$ . So there is  $W \subset U \cap V$  containing  $p$  such that  $f'|_W = g'|_W$ . Since  $\varphi_U$  is injective for any open set  $U \subset X$ ,  $f|_W = \varphi_{pW}^{-1}(f'|_W) = \varphi_{pW}^{-1}(g'|_W) = g|_W$ . Hence  $(U, f) \sim (V, g)$  and  $[(U, f)] = [(V, g)]$ . So  $\varphi_p$  is injective.

$\Rightarrow$ ) Suppose  $\varphi$  is injective, then  $\varphi_p$  is injective. The aim is to prove that  $\varphi_U$  is injective. Suppose  $f, g$  are arbitrary holomorphic functions on  $U$ . If  $\varphi_U(f) = f' = g' = \varphi_U(g)$ , then  $(U, f') \sim (U, g')$ . For each  $p \in U$ , since  $\varphi_p$  is injective,  $\varphi_p^{-1}([(U, f')]) = [(U, f)] = [(U, g)] = \varphi_p^{-1}([(U, g')])$ . So  $(U, f) \sim (U, g)$  and there exists  $W_p \subset U$  containing  $p$  such that  $f|_{W_p} = g|_{W_p}$ . Hence  $f = g$  on  $U = \cup_{p \in U} W_p$  since  $p$  is optional and then  $\varphi_U$  is injective.  $\square$