RESEARCH NOTES ON FOCK SPACE ANALYSIS

HAO XIAO

These notes were taken as a warm-up of the research on Fock space analysis supervised by Prof. Shengzhao Hou at Soochow University in Autumn 2018, which tightly follows Kehe Zhu's *Analysis on Fock Spaces*[1]. I live-TEXed them using sublime, and as such there may be typos; please send questions, comments, complaints, and corrections to `xiaohao1096@163.com`.

Contents

1. Entire Function

We are concerned about certain spaces of entire functions and certain operators defined on there spaces. So this section is mainly about the review of some elementary results about entire functions.

Definition 1.1. *Let $\mathbb{C}$ denote the complex plane. If a function f is analytic on the entire complex plane $\mathbb{C}$, we say that f is an **entire function**.*

Theorem 1.2. ***Identity Theorem***

If f is entire and the zero set of f, $Z(f) = \{z \in \mathbb{C} :\}$, has a limit point in $\mathbb{C}$, then $f \equiv 0$ on $\mathbb{C}$. Or equivalently, suppose f is an entire function. If there is a point $a \in \mathbb{C}$ such that $f^{(n)}(a) = 0$ for all $n \geq 0$, then $f \equiv 0$ on $\mathbb{C}$.

Note that actually the zeros of an entire function are countable at most. So when we say that $\{z_n\}$ is the zero sequence of an entire function, we always assume that any zero of multiplicity k is repeated k times in $\{z_n\}$. As a consequence of the identity theorem, we see that the zero set of an entire function that is not identically zero cannot have any finite limit point and no value occurs infinitely many times in the sequence. Consequently, the zero sequence $\{z_n\}$ of an entire function is either finite

[1]Book source: `https://www.springer.com/la/book/9781441988003`.

or satisfies the condition that $|z_n| \to \infty$ as $n \to \infty$. In particular, we can always arrange the zeros so that $|z_1| \le |x_2| \le \cdots \le |z_n| \le \cdots$.

From the subharmonicity of the function $|f(p)|$ in $|z-a| < R(>0)$, it follows the mean value theorem:

Theorem 1.3. ***Mean Value Theorem***

Suppose f is entire and $0 < p < \infty$. Then

$$|f(a)|^p \le \frac{1}{2\pi}\int_0^{2\pi} |f(a + r\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta$$

for all $a \in \mathbb{C}$ and all $r \in [0, \infty)$.

Since r is arbitrary, multiply both sides of the above inequality by r and integrate from 0 to $R(>0)$, the result is

$$|f(a)|^p \le \frac{1}{\pi R^2}\int_{|z-a|<R} |f(z)|^p \mathrm{d}A(z),$$

where $z = x + \mathrm{i}y$ and $\mathrm{d}A(z) = \mathrm{d}x\mathrm{d}y$ is the Lebesgue area measure.

Theorem 1.4. ***Liouville's Theorem***

A bounded entire function is necessarily constant.

More generally, if a complex-valued harmonic function defined on the entire complex plane is bounded, then it must be constant.

Theorem 1.5. *A bounded entire function is necessarily constant. More generally, if a complex-valued harmonic function defined on the entire complex plane is bounded, then it must be constant.*

Remark 1.6. *The lack of the bounded entire functions is one of the key differences between the theory of Fock spaces and the more classical theories of Hardy and Bergman spaces.*

An important tool in the study of zeros of analytic functions in specific spaces is the classical formula below:

Theorem 1.7. ***Jensen's Formula***

Suppose that

- *f is analytic on the closed disk $|z| \le z$,*
- *f does not vanish on $|z| = r$,*
- *$f(0) = 1$, and*
- *the zeros of f in $|z| < r$ are $\{z_1, \cdots, z_N\}$, with multiple zeros repeated according to multiplicity.*

Then

$$\sum_{k=1}^{N} \log \frac{r}{|z_k|} = \frac{1}{2\pi}\int_0^{2\pi} \log |f(\mathrm{e}^{\mathrm{i}\theta})| \mathrm{d}\theta.$$

If $f(0)$ is nonzero but not necessarily 1. Jensen's formula takes the form

$$\log|f(0)| = -\sum_{k=1}^{N} \log\frac{r}{|z_k|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta,$$

where $\{z_1, \cdots, z_N\}$ are zeros of f in $0 < |z| < r$. More generally, if f has a zero of order k at the origin, then Jensen's formula takes the following form

$$\log\frac{|f^{(k)}(0)|}{k!} + k\log r = -\sum_{k=1}^{N} \log\frac{r}{|z_k|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta,$$

where $\{z_1, \cdots, z_N\}$ are zeros of f in $0 < |z| < r$.

We can factor out the zeros of f in a canonical way, a process that is usually referred to as Weierstrass factorization. The basis for the theorem is a collection of simple entire functions called **elementary factors**. More specifically, we define $E_0(z) = 1 - z$, and for any positive integer n, $E_n(z) = (1-z)\exp(z + \frac{z^2}{2} + \cdots + \frac{z^n}{n})$. If a is any nonzero complex number, it is clear that $E_n(z/a)$ $(n \geq 0)$ has a unique, simple zero at $z = a$.

Theorem 1.8. *Let $\{z_n\}$ be a sequence of nonzero complex numbers such that the sequence $\{|z_n|\}$ is nondecreasing and tends to ∞. Then it is possible to choose a sequence $\{p_n\}$ of nonnegative integers such that*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{p_n+1} < \infty \tag{1.1}$$

for all > 0. Furthermore, the infinite product

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

converges uniformly on every compact subset of $\mathbb{C}$, the function P is entire, and the zeros of P are exactly $\{z_n\}$, counting multiplicity.

Note that $p_n = n - 1$ will always satisfies (1.1). In many cases, however, there are "better" choices.

Definition 1.9. *If $\{z_n\}$ is the zero sequence of an entire function f and if there exits an integer p such that*

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{p+1}}, \tag{1.2}$$

we say that f is of ***finite rank****. If p is the smallest integer such that (1.2) is satisfied, then f is said to be of* ***rank*** *p. A function with only a finite number of zero has rank 0. A function is of* ***infinite rank*** *if it is not of finite rank.*

Definition 1.10. *If f is not of finite rank and $\{z_n\}$ is the zero sequence of f, then (1.2) is satisfied with $p_n = p$. The product $P(z)$ associated with the canonical choice of $\{p_n\}$ will be called the* ***standard form****.*

Theorem 1.11. ***Weierstrass Factorization Theorem***
Let f be an entire function of finite rank p. If P is the standard product associated with the zeros of f, then there exist a nonzero integer m and an entire function g such that

$$f(z) = z^m P(z) \mathrm{e}^{g(z)}. \tag{1.3}$$

The integer m is unique, and the entire function g is unique up to an additive constant of the form $2k\pi\mathrm{i}$ ($k \in \mathbb{Z}$).

Definition 1.12. *For an entire function of finite rank, we say that (1.3) is the* ***standard factorization*** *of f, or the* ***Weierstrass factorization*** *of f.*

Definition 1.13. *Let f be an entire function of finite rank p. If the entire function g in the standard factorization (1.3) is a polynomial of degree q, then we say that f has* ***finite genus****. In this case, the number $\mu = \max\{p, q\}$ is called the* ***genus*** *of f.*

Definition 1.14. *Let f be an entire function. For any $r > 0$, we write*

$$M(r) = M_f(r) = \sup_{|z|=r} |f(z)|.$$

We say that f is of ***order*** *ρ if*

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

It is clear that $0 \le p \le \infty$. When $\rho < \infty$, f is said to be ***finite order****.*

There are two useful characterization for entire functions to be finite order:

Theorem 1.15. *An entire function f is of finite order iff there exist positive constants a and r such that*

$$|f(z)| < \exp(|z|^a) \ (|z| > r)$$

. In this case, the order of f is the infimum of the set of all such numbers a.

Theorem 1.16. ***Hadamard Factorization Theorem***
An entire function f is of finite order ρ iff it is of finite genus μ.

2. Lattice in the Complex Plane

Definition 2.1. *The simplest lattice in $\mathbb{C}$ is the* ***standard integer lattice***

$$\mathbb{Z}^2 = \{m + \mathrm{i}n : m \in \mathbb{Z}, n \in \mathbb{Z}\},$$

where $\mathbb{Z}$ is the integer group. All lattices used in these notes are isomorphic to $\mathbb{Z}^2$.

Definition 2.2. *Let ω be any complex number, and let ω_1 and ω_2 be any two nonzero complex numbers such that their ratio is not real. For any integers m and n, let $\omega_{mn} = \omega + m\omega_1 + n\omega_2$. The set*

$$\Lambda = \Lambda(\omega, \omega_1, \omega_2) = \{\omega_{mn} : m \in \mathbb{Z}, n \in \mathbb{Z}\}$$

is then called the lattice ***generated*** *by $\omega, \omega_1, \omega_2$.*

Definition 2.3. *The initial parallelogram at ω spanned by ω_1 and ω_2 has vertices*

$$\omega, \omega + \omega_1, \omega + \omega_2, \omega + \omega_1 + \omega_2,$$

and is centered at

$$c = \omega + \frac{1}{2}(\omega_1 + \omega_2).$$

We shift this parallelogram so that the center becomes ω and the vertices become

$$\omega - \frac{1}{2}(\omega_1 + \omega_2), \omega + \frac{1}{2}(\omega_1 - \omega_2), \omega + \frac{1}{2}(\omega_2 - \omega_1), \omega + \frac{1}{2}(\omega_1 + \omega_2).$$

We denote this new parallelogram by R_{00} and call it the ***fundamental region*** *of $\Lambda(\omega, \omega_1, \omega_2)$. For any integers m and n, let $R_{mn} = R_{00} + \omega_{mn}$, with ω being the center of R_{mn}.*

Lemma 2.4. *Let $\Lambda = \Lambda(\omega, \omega_1, \omega_2)$ be any lattice in $\mathbb{C}$. For any positive number δ, there exists a positive constant C such that*

$$\sum_{z \in \Lambda} \mathrm{e}^{-\delta|z-w|^2} \leq C$$

for all $w \in \mathbb{C}$.

Proof. By translation invariance, it suffices to prove the desired inequality for w in the fundamental region R_{00} of Λ. If w is in the relatively compact set R_{00}, then $|w/z| < 1/2$ for all but a finite number of points $z \in \Lambda$. For all such points z, we have

$$|z - w|^2 = |z|^2 |1 - (w/z)|^2 \geq \frac{1}{4}|z|^2.$$

Since $\sum_{z \in \Lambda} \mathrm{e}^{-\frac{\delta}{4}|z|^2}$ is obvious convergent, we obtain the desired result. $\square$

Lemma 2.5. *With notation from above, we have*

$$\mathbb{C} = \bigcup_{m,n \in Z} R_{mn},$$

and

$$\int_{\mathbb{C}} f(z) \mathrm{d}A(z) = \sum_{m,n \in \mathbb{Z}} \int_{R_{mn}} f(z) \mathrm{d}A(z)$$

for every $f \in L^1(\mathbb{C}, \mathrm{d}A)$.

Proof. The decomposition of $\mathbb{C}$ into the union of congruent parallelograms is obvious. Since any two different R_{mn} only overlap on a set of zero area, the desired integral decomposition follows immediately. $\square$

Lemma 2.6. *Let $\Lambda = \Lambda(\omega, \omega_1, \omega_2)$ be a lattice in $\mathbb{C}$. For any positive number R, there exists a positive integer N such that we can decompose Λ into the disjoint union of N sublattices,*

$$\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N,$$

such that the distance between any two points in each of the sublattices is at least R.

Proof. Fix a positive integer k such that $k|\omega_1| > R$ and $k|\omega_2| > R$. For each $j = (j_1, j_2)$ with $0 \le j_1 \le k$ and $0 \le j_2 \le k$, let

$$\begin{aligned}\Lambda_j &= \Lambda(\omega + j_1\omega_1 + j_2\omega_2, k\omega_1, k\omega_2)\\ &= \{(\omega + j_1\omega_1 + j_2\omega_2) + (mk\omega_1 + nk\omega_2) : m, n \in \mathbb{Z}\}.\end{aligned}$$

Then each Λ_j is a sublattice of Λ; the distance between any two points in Λ_j is at least R, and $\Lambda = \cup\Lambda_j$. There are a few duplicates among Λ_j caused by points from the boundary of the parallelogram at ω spanned by $k\omega_1$ and $k\omega_2$. After these duplicates are deleted, we arrive at the desired decomposition for Λ. $\square$

Definition 2.7. *Most lattices used in these notes are* ***square*** *ones. More specifically, for any given positive parameter r, we consider the case when $\omega = 0$, ω_2, and $\omega_2 = \mathrm{i}r$. The resulting lattice is*

$$r\mathbb{Z}^2 = \{rm + \mathrm{i}rn : m, n \in \mathbb{Z}\}.$$

Example 2.8. *We mention two particular cases.*

- *For $r = \sqrt{\pi/\alpha}$, where α is a positive parameter, the resulting lattices are used in Weierstrass σ-functions.*
- *For $r = 1/N$, where N is a positive integer, the resulting lattices will be employed in Hankel and Toeplitz operators in Schatten classes.*

Definition 2.9. *For any two points $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ in $r\mathbb{Z}^2$, we let $\gamma(z, w)$ denote the following* ***path*** *in $r\mathbb{Z}^2$: we first move horizontally from z to $u + \mathrm{i}y$ and then vertically from $u + \mathrm{i}y$ to $u + \mathrm{i}v$. When $z = 0$, we write $\gamma(w)$ in place of $\gamma(0, w)$. The path $\gamma(z, w)$ if of course discrete. We use $|\gamma(z, w)|$ to denote the number of points in $\gamma(z, w)$ and call it the* ***length*** *of $\gamma(z, w)$.*

Lemma 2.10. *For any positive r and σ, there exists a positive constant $C = C_{r,\sigma}$ such that*

$$\sum_{z\in r\mathbb{Z}^2} \sum_{w\in r\mathbb{Z}^2} \mathrm{e}^{-\sigma|z-w|^2} \chi_{\gamma(z,w)}(u) \le C$$

for all $u \in r\mathbb{Z}^2$, where $\chi_{\gamma(z,w)}$ is the characteristic function of $\gamma(z, w)$.

Proof. Without loss of generality, we may assume that $r = 1$. Adjusting the constant σ will then produce the general case.

Also, it is obvious that $u + \gamma(z, w) = \gamma(u + z, u + w)$, which implies that the sum

$$S = \sum_{z\in\mathbb{Z}^2} \sum_{w\in\mathbb{Z}^2} \mathrm{e}^{-|z-w|^2} \chi_{\gamma(z,w)}(u)$$

is actually independent of u. For convenience, we will assume that $u = 0$.

For any z and w, the path $\gamma(z, w)$ consists of a horizontal segment and a vertical segment (one or both are allowed to degenerate). From the definition of $\gamma(z, w)$, we see that the origin 0 lies on the horizontal segment of $\gamma(z, w)$ iff one the following is true:

- z is on the negative x-axis and w is in the first or fourth quadrant: $z = -n$, $w = m + \mathrm{i}k$, where n and m are nonnegative integers and k is an integer.
- z is on the positive x-axis and w is in the second or third quadrant: $z = n$, $w = -m + \mathrm{i}k$, where n and m are nonnegative integers and k is an integer.

Similarly, 0 lies on the vertical segment of $\gamma(z, w)$ iff one of the following is true:

- w is on the positive y-axis and z is in the third or fourth quadrant: $w = \mathrm{i}n$, $z = k - \mathrm{i}m$, where n and m are nonnegative integers and k is an integer.
- w is on the negative y-axis and z is in the first or second quadrant: $w = -\mathrm{i}n$, $z = k + \mathrm{i}m$, where n and m are nonnegative integers and k is an integer.

In each of the cases above, we have

$$|z - w|^2 = (n + m)^2 + k^2 \geq n^2 + m^2 + k^2.$$

Therefore,

$$\begin{aligned} S &\leq 4 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\sigma(n^2+m^2+k^2)} \\ &= 4 \sum_{n=0}^{\infty} \mathrm{e}^{-\sigma n^2} \sum_{m=0}^{\infty} \mathrm{e}^{-\sigma m^2} \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\sigma k^2} < \infty. \end{aligned}$$

This proves the lemma. □

Remark 2.11. *In the proof above, when it comes to positive/negative x-axis and the four quadrants, we always include their boundaries implicitly.*

3. Weierstrass σ-Function

To be continued...

Department of Mathematics, Soochow University, Suzhou, Jiangsu 215000, China