Algebraic Geometry – Homework

Name: Hao Xiao

School: Soochow University

1. Homework #1

Homework 1.1. *Let* $U_1 = \{[z_0, z_1] \in \mathbb{P}^1 : z_1 \neq 0\}$ *where* $\mathbb{P}^1$ *is the projective space of* $\mathbb{C}^2$ *and* $[z_0, z_1]$ *is the homogeneous coordinate. Prove that*

$$\begin{aligned} \varphi_1 &: U_1 \to \mathbb{C}, \\ [z_0, z_1] &\mapsto w = z_0/z_1 \end{aligned}$$

is a homeomorphism.

Proof. Since for all $z_0, z_1, z_0', z_1' \in \mathbb{C}$ with $z_1, z_1' \neq 0$, $[z_0, z_1] = [z_0', z_1'] \Leftrightarrow [z_0/z_1, 1] = [z_0'/z_1', 1] \Leftrightarrow z_0/z_1 = z_0'/z_1'$. For each open set $V \in \mathbb{C}$, $\varphi_1^{-1}(V)$ is the collection of equivalent classes of all elements in V. By the definition of quotient topology, $\varphi_1^{-1}(V)$ automatically is open. So φ_1 is well-defined, continuous, one-to-one, and onto. Note that φ_1^{-1} is given by $\varphi_1^{-1}(z) = \pi(z, 1)$ for each $z \in \mathbb{C}$ where π is the projection from $\mathbb{C}^2$ to $\mathbb{P}^1$. So φ_1^{-1} is continuous. Hence, φ_1 is a homeomorphism. □

Homework 1.2. *Let* f *be a holomorphic function on* $\mathbb{C}^n$ *and* $\text{grad} f$ *has no zero on* $\mathbb{C}^n$*, that is,* $\text{rank}(\partial f/\partial z_1, \cdots, \partial f/\partial z_n) = 1$ *on* X*. Prove that* $X = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : f(z_1, \cdots, z_n) = 0\}$ *is a complex manifold.*

Proof. Assume that $X \neq \varnothing$. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in X$, assume that $f_{z_1}(\alpha) \neq 0$. Then by implicit function theorem, there exists an open set $U_\alpha' \subset \mathbb{C}^n$ containing α and an open set $W_\alpha \subset \mathbb{C}^{n-1}$ together with a holomorphic function $g_\alpha : W_\alpha \to \mathbb{C}$ such that $f(z_1, \cdots, z_n) = 0$ on $U_\alpha' \Leftrightarrow z_1 = g_\alpha(z_2, \cdots, z_n)$ on W_α. Let $U_\alpha = \{(z_1, \cdots, z_n) \in U_\alpha' : f(z_1, \cdots, z_n) = 0\}$ and $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{C}^{n-1}$ by $(z_1, z_2, \cdots, z_n) = (g_\alpha(z_2, \cdots, z_n), z_2, \cdots, z_n) \mapsto (z_2, \cdots, z_n)$. It's obvious that φ_α is a homeomorphism. Clearly, all U_α covers X.

For arbitrary $\alpha, \beta \in X$ with $U_\alpha \cap U_\beta \neq \varnothing$, assume $f_{z_1}(\alpha) \neq 0$ and $f_{z_i}(\beta) \neq 0$. If $i = 1$, then it's trivial that $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a biholomorphism. So we may assume that $i = 2$, then the homeomorphism $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is given by

$$(z_2, \cdots, z_n) \mapsto (z_1, z_3, \cdots, z_n) = (g_\alpha(z_2, \cdots, z_n), z_3, \cdots, z_n).$$

So $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic since g_α is holomorphic. Similarly, $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic and then $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a biholomorphism.

Since $X \subset C^n$, so the subspace topology is obviously Hausdorff and second countable. Hence, X is a complex manifold. □

Homework 1.3. *Let* Γ *be a full rank lattice. Prove that* $\mathbb{C}^n/\Gamma$ *is a complex manifold.*

Proof. Suppose Γ is generated by linearly independent elements $\alpha_1, \cdots, \alpha_{2n} \in \mathbb{C}^n$, then $\Gamma = \{k_1\alpha_1 + \cdots + k_{2n}\alpha_{2n} : k_1, \cdots, k_{2n} \in \mathbb{Z}\}$.

Let $m \in \{1, \cdots, 2n\}$ and $I = \{i_k\}_{k=1}^{2n}$ be a rearrangement of $\{k\}_{k=1}^{2n}$. Let $U_I^m = \{[k_1\alpha_1 + \cdots + k_{2n}\alpha_{2n}] : 0 < k_{i_1}, \cdots, k_{i_m} < 1, 1/4 < k_{i_{m+1}}, \cdots, k_{i_n} < 5/4\}$. Then the union of U_I^m for all m, I obviously covers $\mathbb{C}^n/\Gamma$. Let $\varphi_I^m : U_I^m \to \varphi(U_I^m) \subset \mathbb{C}^n$ be defined by

$$[k_1\alpha_1 + \cdots + k_{2n}\alpha_{2n}] \mapsto k_1\alpha_1 + \cdots + k_{2n}\alpha_{2n}.$$

Since α_i's are linearly independent, it's clear that $\varphi_I^m(U_I^m)$ is an open subset of $\mathbb{C}^n$ and φ_I^m is a homeomorphism.

For all m, m', I, I' with $U_I^m \cap U_{I'}^{m'} \neq \varnothing$, it's easy to see that

$$\varphi_{U_{I'}^{m'}} \circ \varphi_{U_I^m}^{-1} : \varphi_{U_I^m}(U_I^m \cap U_{I'}^{m'}) \to \varphi_{U_{I'}^{m'}}(U_I^m \cap U_{I'}^{m'})$$

is an identity map and, also, $\varphi_{U_I^m}(U_I^m \cap U_{I'}^{m'}) = \varphi_{U_{I'}^{m'}}(U_I^m \cap U_{I'}^{m'})$ by the construction of U_I^m and $U_{I'}^{m'}$. Hence, $\varphi_{U_{I'}^{m'}} \circ \varphi_{U_I^m}^{-1}$ is a biholomorphism.

Since the quotient topology on $\mathbb{C}^n/\Gamma$ is inherited from $\mathbb{C}^n$ under Γ, then $\mathbb{C}^n/\Gamma$ is Hausdorff and second countable. Thus, $\mathbb{C}^n/\Gamma$ is a complex manifold. □

2. Homework #2

Homework 2.1. *Prove that $\mathbb{P}^n$ is compact.*

Proof. Since S^{2n+1} is the unit sphere in $\mathbb{C}^n$, let $f : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be defined by $z \mapsto [z]$, and then $g := f|_{S^{2n+1}} : S^{2n+1} \to \mathbb{P}^n$ is the restriction of f to S^{2n+1}. By the derivation of $\mathbb{P}^n$, the topology on $\mathbb{P}^n$ is induced by f. Then f is a quotient map and so it's continuous. Thus, g is continuous.

For each $[z_0, z_1, \cdots, z_n] \in \mathbb{P}^n$, assume that $z_0 \neq 0$, then $(z_0/c, z_1/c, \cdots, z_n/c) \in S^{2n+1}$ and

$$g\left(\frac{z_0}{c}, \frac{z_1}{c}, \cdots, \frac{z_n}{c}\right) = f\left(\frac{z_0}{c}, \frac{z_1}{c}, \cdots, \frac{z_n}{c}\right) = \left[\frac{z_0}{c}, \frac{z_1}{c}, \cdots, \frac{z_n}{c}\right] = [z_0, z_1, \cdots, z_n].$$

So g is onto. Note that S^{2n+1} is compact. Thus, $\mathbb{P}^n = g(S^{2n+1})$ is compact. □

3. Homework #3

Homework 3.1. *Given an open cover $\{U_\alpha\}$ of the complex manifold X and a collection of $g_{\alpha\beta}$'s which are holomorphic functions from $U_\alpha \cap U_\beta$ to $GL(r, \mathbb{C})$ satisfying $g_{\alpha\alpha} = 1$, $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ on $U_\alpha \cap U_\beta$, and $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Construct a holomorphic vector bundle E over X:*

$$E = \amalg_\alpha U_\alpha \times \mathbb{C}^r / \sim$$

where $(x, v_\alpha) \sim (y, v_\beta)$ iff $x = y$ and $v_\alpha = g_{\alpha\beta}v_\beta$.

(a) *Show that $\sim$ is an equivalent relationship.*
(b) *Prove that E is a well-defined holomorphic vector bundle over X.*
(c) *Consider the Cartier divisor $\{(U_\alpha, f_\alpha)\}$ and let $g_{\alpha\beta} = f_\alpha/f_\beta$. Show that how this Cartier divisor induces a line bundle uniquely.*

Solution. (a) For each $x \in U_\alpha$, since $x = x$ and $v_\alpha = v_\alpha = g_{\alpha\alpha} v_\alpha$, we have $(x, v_\alpha) \sim (x, v_\alpha)$.

If $(x, v_\alpha) \sim (y, v_\beta)$ where $x \in U_\alpha$ and $y \in U_\beta$, then $x = y$ and $v_\alpha = g_{\alpha\beta} v_\beta$. Hence $y = x$ and $v_\beta = g_{\alpha\beta}^{-1} = g_{\beta\alpha} v_\alpha$, which implies that $(y, v_\beta) \sim (x, v_\alpha)$.

If $(x, v_\alpha) \sim (y, v_\beta)$ and $(y, v_\beta) \sim (z, v_\gamma)$, then $x = y = z$, $v_\alpha = g_{\alpha\beta} v_\beta$, and $v_\beta = g_{\beta\gamma} v_\gamma$. Thus we derive $x = z$, $v_\alpha = g_{\alpha\beta} g_{\beta\gamma} v_\gamma = g_{\alpha\gamma} v_\gamma$. So $(x, v_\alpha) \sim (z, v_\gamma)$.

In all, we have shown that $\sim$ is indeed an equivalent relationship.

(b) Given any $x \in X$, there exists an open set U_α containing x since $\{U_\alpha\}$ is an open cover of X. Denote the element of $\amalg_\alpha U_\alpha \times \mathbb{C}^r / \sim$ by $[(x, v)]$ where $(x, v) \in \amalg_\alpha U_\alpha \times \mathbb{C}^r$.

Define $\pi : \amalg_\alpha U_\alpha \times \mathbb{C}^r / \sim \to X = \amalg_\alpha U_\alpha$ by $[(x, v_\alpha)] \mapsto x$ and define $h_\alpha : U_\alpha \times \mathbb{C}^r \to \pi^{-1}(U_\alpha)$ by $(x, v_\alpha) \mapsto [(x, v_\alpha)]$. It's clear that h_α is continuous. Also, it's revealed that h_α is a surjection. This is because for each $[(x, v_\beta)] \in \pi^{-1}(U_\alpha)$ there exists a transition function $g_{\alpha\beta}$ such that $(x, g_{\alpha\beta}(x) v_b) = (x, v_\alpha)$ and then $[(x, v_\beta)] = [(x, v_\alpha)]$. Suppose $h_\alpha(x_1, v_{\alpha_1}) = h_\alpha(x_2, v_{\alpha_2})$, then $[(x_1, v_{\alpha_1})] = [(x_2, v_{\alpha_2})]$. So $(x_1, v_{\alpha_1}) \sim (x_2, v_{\alpha_2}) \Rightarrow x_1 = x_2$ and $v_{\alpha_1} = g_{\alpha_1 \alpha_2} v_{\alpha_2}$, which implies that h_α is an injection. Now we have shown that h_α is bijective. And by the derivation of the quotient topology on $\pi^{-1}(U_\alpha)$, h_α appears to be a one-to-one correspondence of open sets. Hence it's an homeomorphism. So E is a well-defined holomorphic vector bundle over X since α is arbitrary.

(c) Since $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$, $g_{\gamma\alpha}^{-1} = g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$. Define the relationship for $(x, f_\alpha(x)) \in U_\alpha \times \mathbb{C}$ and $(y, f_\beta(y)) \in U_\beta \times \mathbb{C}$ that $(x, f_\alpha(x)) \sim (y, f_\beta(y))$ iff $x = y$ and $f_\alpha(x) = g_{\alpha\beta} f_\beta(y)$. By (a), $\sim$ is an equivalent relationship. So $\amalg_\alpha (U_\alpha, f_\alpha) / \sim$ is a holomorphic vector bundle as well as a line bundle which is uniquely determined by the Cartier divisor under $\sim$. □

4. Homework #4

Homework 4.1. *Suppose $L_1, L_2 \in \mathrm{Pic}(X)$. Show that $L_1 \otimes L_2$ is a line bundle.*

Proof. Since L_1, L_2 are line bundles, for any $b \in X$ and its neighborhood U_b, there exists s_1 as the only basis of sections on U_b of L_1 and s_2 as the only basis of sections on U_b of L_2. Therefore, $s_1 \otimes s_2$ is exactly the basis of sections on U_b. So $L_1 \otimes L_2$ is a line bundle. □

Homework 4.2. *Let L_1, L_2 be line bundles generated by $\{U_\alpha, g^1_{\alpha\beta}\}$ and $\{U_\alpha, g^2_{\alpha\beta}\}$ respectively. Show that the transition functions of $L_1 \otimes L_2$ are $\{g^1_{\alpha\beta} g^2_{\alpha\beta}\}$.*

Proof. Suppose $s_1^\alpha \otimes s_2^\alpha$ is a local basis of sections on U_α and $s_1^\beta \otimes s_2^\beta$ is a local basis of sections on U_β. It's known that $s_1^\alpha = g^1_{\alpha\beta} s_1^\beta$ and $s_2^\alpha = g^2_{\alpha\beta} s_2^\beta$. So $s_1^\alpha \otimes s_2^\alpha = g^1_{\alpha\beta} s_1^\beta \otimes g^2_{\alpha\beta} s_2^\beta = g^1_{\alpha\beta} g^2_{\alpha\beta} s_1^\beta \otimes s_2^\beta$. Hence, the transition functions of $L_1 \otimes L_2$ are $g^1_{\alpha\beta} g^2_{\alpha\beta}$'s. □

5. Homework #5

Homework 5.1. *Let $S \subset X$ be a subvariety and $\mathcal{L}_S(U) = \{f \in \mathcal{O}_X(U) : f|_{S \cap U} = 0\}$. Prove that $\mathcal{L}_S$ is a sheaf.*

Proof. For each $U \subset V$, the restriction $\gamma_{VU} : \mathcal{L}_S(V) \to \mathcal{L}_S(U)$ is a homomorphism. In fact, for all holomorphic functions f, g vanishing on $V \cap S$, $\gamma_{VU}(f + g) = f|_U + g|_U$ vanishes on $U \cap S$. Also, $\gamma_{VU}(f + g) = f|_U + g|_U = \gamma_{VU}(f) + \gamma_{VU}(g)$.

For all $U \subset V \subset W$, a holomorphic function f vanishing on $W \cap S$ also vanishes on $U \cap S$ and $V \cap S$. Thus $\gamma_{WU}(f) = f|_U$, $\gamma_{VU} \circ \gamma_{WV} = \gamma_{VU}(f|_V)$, and $f|_V$ vanishing on $V \cap S$ also vanishes on $U \cap S$. So $\gamma_{VU}(f|_V) = f|_U = \gamma_{WV}(f) = \gamma_{VU} \circ \gamma_{WV}(f) \Rightarrow \gamma_{WV} = \gamma_{VU} \circ \gamma_{WV}$.

For a collection of open sets $U_\alpha \subset X$ ($\alpha \in I$) with $U = \cup_{\alpha\in I} U_\alpha$. If $h \in \mathcal{O}(U)$ and $\gamma_{UU_\alpha}(h) = 0$, then h vanishes on $U_\alpha \cap S$ for all $\alpha \in I$. So h vanishes on $\cup_{\alpha\in I}(U_\alpha \cap S) = U \cap S \Rightarrow h = 0$.

For $U = \cup_{\alpha\in I} U_\alpha$ define $h : U_\alpha \to \mathbb{C}$ such that $h(x) = f_\alpha(x)$ for all $x \in U_\alpha$. Since $\gamma_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha) = \gamma_{U_\beta, U_\alpha \cap U_\beta}(f_\beta)$, then h is holomorphic on U. Given a fixed $\alpha \in I$, $\gamma_{UU_\alpha}(h) = f_\alpha$. And it's obvious that $\mathcal{F}(\varnothing) = 0$ and $\gamma_{VU} = \mathbb{1}$.

In all, we have shown that $\mathcal{L}_S$ is indeed a sheaf. □

Homework 5.2. *Consider the sheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$. This induces a morphism $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p, (U, f) \mapsto (U, \varphi_U(f))$ for each $p \in X$. Show that φ_p is well-defined.*

Proof. If $(U, f) \sim (V, g)$ where $(U, f) \in \mathcal{F}_p$ and $(V, g) \in \mathcal{F}_p$, then $\varphi_p^1(f) = f' \in \mathcal{O}_\mathcal{G}(U)$, $\varphi_p(U, f) = (U, f')$, $\varphi_p^2(g) = g' \in \mathcal{O}_\mathcal{G}(V)$, and $\varphi_p(V, g) = (V, g')$. For any $U \subset X$, $f \in \mathcal{O}(U)$, and $W \subset U$, $\gamma_{UW}^\mathcal{F}(f) = f|_W$. Since there exists a homomorphism from $\mathcal{F}(W)$ to $\mathcal{G}(W)$, denoted φ_{pW}^1, then $\varphi_p^1(f|_W) = f'|_W$. Also, $\gamma_{UW}^\mathcal{G} \circ \varphi_p^1(f) = f'|_W = \varphi_p^2 \circ \gamma_{UW}^\mathcal{F}(f)$, $\gamma_{UW}^\mathcal{G} \circ \varphi_p^1(g) = g'|_W = \varphi_p^2 \circ \gamma_{UW}^\mathcal{F}(g)$, and $f|_W = g|_W$. So $f'|_W = \varphi_p^2(f|_W) = \varphi_p^2(g|_W) = g'|_W$. Therefore, $(U, f') \sim (V, g')$ and φ_p is well-defined. □

Homework 5.3. *Prove that φ defined in Homework 5.2 is injective iff for each $U \subset X$, $\varphi_U : \mathcal{F}(\mathcal{U}) \to \mathcal{G}(U)$ is injective.*

Proof. $\Leftarrow$) Suppose φ_U is injective. If $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective, then φ is injective. The aim is to prove that φ_p is injective.

$[(U, f)]$ is the equivalent class of (U, f) in $\mathcal{F}_p$, and let $\varphi_p([(U, f)]) = [(U, f')]$ where $[(U, f')]$ is the equivalent class of (U, f') in $\mathcal{G}_p$. Since there exists (V, g) such that $\varphi_p([(V, g)]) = [(V, g')] = [(U, f')]$ and $[(V, g')] = [(U, f')]$, then $(V, g') \sim (U, f')$. So there is $W \subset U \cap V$ containing p such that $f'|_W = g'|_W$. Since φ_U is injective for any open set $U \subset X$, $f|_W = \varphi_{pW}^{-1}(f'|_W) = \varphi_{pW}^{-1}(g'|_W) = g|_W$. Hence $(U, f) \sim (V, g)$ and $[(U, f)] = [(V, g)]$. So φ_p is injective.

$\Rightarrow$) Suppose φ is injective, then φ_p is injective. The aim is to prove that φ_U is injective. Suppose f, g are arbitrary holomorphic functions on U. If $\varphi_U(f) = f' = g' = \varphi_U(g)$, then $(U, f') \sim (U, g')$. For each $p \in U$, since φ_p is injective, $\varphi_p^{-1}([(U, f')]) = [(U, f)] = [(U, g)] = \varphi_p^{-1}([(U, g')])$. So $(U, f) \sim (U, g)$ and there exists $W_p \subset U$ containing p such that $f|_{W_p} = g|_{W_p}$. Hence $f = g$ on $U = \cup_{p\in U} W_p$ since p is optional and then φ_U is injective. □