NOTES ON FUNCTIONAL ANALYSIS: AN INTRODUCTION

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These notes were taken in Soochow's functional analysis class in Spring 2018, taught by Yisheng Huang. The textbook used was *Functional Analysis: An Introduction*[1] by the lecturer. I live-TEXed them using sublime, and as such there may be typos; please send questions, comments, complaints, and corrections to `xiaohao1096@163.com`.

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[1]`http://www.ecsponline.com/goods.php?id=10907`.

1. Preface and Elementary Inequalities: 3/7/2018

1.1. **Preface to the Course.** Huang used the first hour on the "art" of mathematical English. He listed some frames of mathematical speaking:

- Implication sentence: It follows/implies/deduces that ...
- Grammar mistake: Since ..., hence/therefore/thus ...
- Proof/Solution: ...
- Hölder's Inequality/The Hölder Inequality

English is really a good language for math work, better than my mother tongue. I also recommend an excellent paper[2] on mathematical writing by Kleiman.

1.2. **A Brief Introduction to Functional Analysis.** He then used the next one hour for a brief introduction to **functional analysis**.

A mapping $f : A \to B$ is called a function provided $B \subset \mathbb{R}$ or $\mathbb{C}$. Recall that we use $A \subset \mathbb{R}$ or $\mathbb{R}^n$ and $B \subset \mathbb{R}$ in **mathematical analysis**. If A is an arbitrary "set" and $B \subset \mathbb{R}$ then f is said to be a **functional**.

For instance, let $A = C[a, b]$ (the collection of all continuous real-valued functions on $[a, b]$), $B \subset \mathbb{R}$, and $f : A \to B$, then f is a functional.

In linear algebra, $(+, \cdot)$ (i.e., addition and scalar multiplication) defined on a vector space could be regarded as an **algebraic structure**. As we shall see in algebra courses, this kind of structure involves only finite many times of operations.

We somehow expect to construct a **geometric structure** on A, which permits infinite many times of operations.

By convention, we keep both of the two structures on domain A.

1.3. **Elementary Inequalities.** Following the usage of English in mathematical writing, we commented on some useful but rather elementary inequalities.

Theorem 1.1. ***Triangle Inequality***

[2] `http://www.math.harvard.edu/graduate/advise/kleiman.pdf`.

$|a+b| \le |a|+|b|$ *for all* $a, b \in \mathbb{F} = \mathbb{C}$*or* $\mathbb{R}$*. Also, there is another version:*

$$\frac{|a+b|}{1+|a+b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}.$$

Theorem 1.2. ***Young's Inequality***
Given $p > 1$ *and* q *s.t.* $1/p + 1/q = 1$*, then for all* $a, b \ge 0$*,*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Remark 1.3. *A pair of real numbers* p, q *s.t.* $p > 1$ *and* $1/p + 1/q = 1$ *are referred to be* ***mutually conjugated****.*

Theorem 1.4. ***Hölder's Inequality***
Given $\{\xi_k\}_{k=1}^n, \{\eta_k\}_{k=1}^n \subset \mathbb{F}$*,* $p > 1$*,* $1/p + 1/q = 1$*, then*

$$\sum_{k=1}^n |\xi_k \eta_k| \le \left(\sum_{k=1}^n |\xi_k|^p\right)^{1/p} \left(\sum_{k=1}^n |\eta_k|^p\right)^{1/p}.$$

Given $x(t), y(t) \in L(E)$*,* $E \subset \mathbb{R}$ *or* $\mathbb{R}^n$*,* $p > 1$*,* $1/p + 1/q = 1$*, then*

$$\int_E |x(t)y(t)| \mathrm{d}t \le \left(\int_E |x(t)|^p \mathrm{d}t\right)^{1/p} \left(\int_E |x(t)|^q \mathrm{d}t\right)^{1/q}.$$

When $p = q = 1/2$*, Hölder's inequality is also called* ***Cauchy's inequality****.*

Remark 1.5. *In application, we expect* $\left(\int_E |x(t)|^p \mathrm{d}t\right)^{1/p} \left(\int_E |x(t)|^q \mathrm{d}t\right)^{1/q}$ *to be finite.*

Theorem 1.6. ***Minkowski's Inequality***
Given $\{\xi_k\}_{k=1}^n, \{\eta_k\}_{k=1}^n \subset \mathbb{F}$ *and* $p \ge 1$*,*

$$\left(\sum_{k=1}^n |\xi_k + \eta_k|^p\right)^{1/p} \le \left(\sum_{k=1}^n |\xi_k|^p\right)^{1/p} + \left(\sum_{k=1}^n |\eta_k|^q\right)^{1/q}.$$

Given $x(t), y(t)$ *measurable on* $E \subset \mathbb{R}$ *or* $\mathbb{R}^n$*,* $p \ge 1$*, then*

$$\left(\int_E |x(t)+y(t)|^p \mathrm{d}t\right)^{1/p} \le \left(\int_E |x(t)|^p \mathrm{d}t\right)^{1/p} + \left(\int_E |y(t)|^q \mathrm{d}t\right)^{1/q}.$$

Theorem 1.7. *Hölder's inequality implies Minkowski's inequality.*

Proof. From

$$|x(t)+y(t)|^p \le (|x(t)|+|y(t)|)^p \le 2^p(\max\{|x(t)|, |y(t)|\})^p \le 2^p(|x(t)|^p + |y(t)|^p),$$

we see that if $\int_E |x(t)+y(t)|^p \mathrm{d}t$ is infinite, then at least one of $\int_E |x(t)|^p \mathrm{d}t$, $\int_E |y(t)|^p \mathrm{d}t$ is infinite. We accept the inequality under this condition by convention.

Now we assume that both $\int_E |x(t)|^p \mathrm{d}t$, $\int_E |y(t)|^p \mathrm{d}t$ are finite and $p > 1$. We compute

$$\begin{aligned}\int_E |x(t)+y(t)|^p \mathrm{d}t &\le \int_E |x(t)+y(t)||x(t)+y(t)|^{p-1}\mathrm{d}t \\ &\le \int_E |x(t)||x(t)+y(t)|^{p-1}\mathrm{d}t + \int_E |y(t)||x(t)+y(t)|^{p-1}\mathrm{d}t \\ &\le \left[\left(\int_E |x(t)|^p \mathrm{d}t\right)^{1/p} + \left(\int_E |y(t)|^p \mathrm{d}t\right)^{1/p}\right]\left(\int_E |x(t)+y(t)|^{(p-1)q}\mathrm{d}t\right)^{1/q}.\end{aligned}$$

Since $(p-1)q = p$, we derive

$$\left(\int_E |x(t)+y(t)|^p \mathrm{d}t\right)^{1/p} \le \left(\int_E |x(t)|^p \mathrm{d}t\right)^{1/p} + \left(\int_E |y(t)|^p \mathrm{d}t\right)^{1/p},$$

when $\int_E |x(t)+y(t)|^p \mathrm{d}t \neq 0$. If $\int_E |x(t)+y(t)|^p \mathrm{d}t = 0$, then the claim is trivial. $\square$

Remark 1.8. *Minkowski's inequality implies Höder's inequality as well.*

2. Metric Space and Examples: 3/9/2018

I didn't attend the class because I overslept that day... These notes were provided by Wenyi Cai. She believed that Huang made some logical mistakes in the lecture. Of course, Cai rewrote those parts to fit in the correct logic.

Definition 2.1. *Let X be a nonempty set, a **metric** on X is a function $d : X \times X \to \mathbb{R}$ such that for all x, y, z in X,*

(M1) *$d(x, y) \ge 0$ and $d(x, y) = 0$ iff $x = y$;*
(M2) *$d(x, y) = d(y, x)$;*
(M3) *$d(x, y) + d(y, z) \ge d(x, z)$.*

*X equipped with a metric d is called a **metric space**.*

Remark 2.2. *(M2), (M3), and $d(x, x) = 0$ for all x in X implies (M1).*

Proof. $d(x, y) + d(y, x) \ge d(x, x) = 0 \Rightarrow 2d(x, y) = 0 \Rightarrow d(x, y) \ge 0$. $\square$

Example 2.3. *We list some important examples of metric and metric space:*

- *Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be $d(x, y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$ for all $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in $\mathbb{R}^n$, then d is a metric on $\mathbb{R}^n$.*
- *Similarly, let $d : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{R}$ be $d(x, y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$ for all $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in $\mathbb{F}^n$, then d is a metric on $\mathbb{F}^n$. Note that $\mathbb{F}$ might be $\mathbb{C}$, so we use the norm notation here.*
- *Let s be the set of all sequences of real or complex numbers. For all $x = \{\xi_i\}_{i=1}^\infty$ and $y = \{\eta_i\}_{i=1}^\infty$ in s. Define $d : s \times s \to \mathbb{R}$ as*

$$d(x, y) = \sum_{i=1}^\infty \frac{1}{2^i}\frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}.$$

Note that the sum of nonnegative terms in the definition always converges because $\frac{\xi_i-\eta_i|}{1+|\xi_i-\eta_i|} \leq 1$ *and* $\sum_{i=1}^{\infty} \frac{1}{2^i}$ *converges. Hence,* d *is indeed a well-defined function, satisfying (M1).*

It's obvious that (M2) holds for d. *With the other version of triangle inequality (see Theorem 1.1), we obtain*

$$\begin{aligned}\frac{|\xi_i-\zeta_i|}{1+|\xi_i-\zeta_i|} &= \frac{|(\xi_i-\eta_i)+(\eta_i-\zeta_i)|}{1+|(\xi_i-\eta_i)+(\eta_i-\zeta_i)|} \\ &\leq \frac{|\xi_i-\eta_i|}{1+|\xi_i-\eta_i|} + \frac{|\eta_i-\zeta_i|}{1+|\eta_i-\zeta_i|},\end{aligned}$$

where $x = \{\xi_i\}_{i=1}^{n}$, $y = \{\eta_i\}_{i=1}^{n}$, $z = \{\zeta_i\}_{i=1}^{n}$. *Thus (M3) holds and* d *is a metric on* s.

- *Let* $\ell^\infty \subset s$ *be the set of all bounded sequences in* $\mathbb{F}$. *Note that bounded sequences always have upper bound, so we define* $d : l^\infty \times l^\infty \to \mathbb{R}$ *as* $d(x, y) = \sup_{i\in\mathbb{N}_+} |\xi_i - \eta_i|$ *for all* $x = \{\xi_i\}_{i=1}^{\infty}$, $y = \{\eta_i\}_{i=1}^{\infty}$ *in* s.
- *Let* $C[a, b]$ *be the set of all real-valued continuous functions on* $[a, b]$ *and* $d : C[a, b] \times C[a, b] \to \mathbb{R}$ *as* $d(x, y) = \max_{t\in[a,b]} |x(t) - y(t)|$ *for all* $x(t)$ *and* $y(t)$ *in* $C[a, b]$. *Recall that any continuous function can touch its maximum on a closed bounded interval, that is, a compact set in* $\mathbb{R}$. *Since* $|x(t)-y(t)| \in C[a, b]$, d *is a well-defined metric on* $C[a, b]$.
- *Let* X *be a nonempty set, define* $d : X \times X \to \mathbb{R}$ *for all* (x, y) *in* $X \times X$ *as* $d(x, y) = 0$ *iff* $x = y$ *and* $d(x, y) = 1$ *when* $x \neq y$. *It's easy to check that (M1) and (M2) hold.*

  For all $x, y, z \in X$, *when* $x = z$, $d(x, y) + d(y, z) \geq d(x, z) = 0$ *always holds. If* $x \neq z$, *then* y *does not equal to at least one of* x, z, *say* $y \neq x$. *Thus we derive* $d(x, y) + d(y, z) \geq d(x, y) = 1 \geq d(x, z)$. *We now conclude that* d *satisfies (M3). Hence,* d *is a metric on* X.

  The metric observed here is called ***discrete metric****, which is of little value in theory useful as a counterexample in many cases, and the corresponding space is called* ***discrete metric space****.*
- *Let* S *be the set of all measurable functions on a measurable* E *with* $0 < m(E) < \infty$, *define* $d : S \times S \to \mathbb{R}$ *as* $d(x, y) = \int_E \frac{|x(t)-y(t)|}{1+|x(t)-y(t)|}\mathrm{d}t$ *for all* $(x, y) \in S \times S$.

  Obviously, (M2) and (M3) hold here. But $d(x, y) = 0$ *only implies* $x(t) = y(t)$ *a.e. on* E. *Thus* d *is not a metric on* S. d *is sometimes referred to as a semi-metric on* S *(see Definition 34.1).*

  However, we can still derive a metric here with some modification. We define an equivalent relationship on S, *that is,* $x \sim y$ *iff* $x(t) = y(t)$ *a.e. on* E. *The relation partitions* S *into equivalent classes. Let* E_x *stand for the equivalence class of* x, *which means that* $z(t) = x(t)$ *a.e. on* E *for all* $z \in E_x$.

The subscript x of E_x is a representative element of E_x and we can surely choose other element in E_x as well.

Let $\tilde{d} : \tilde{S} \times \tilde{S} \to \mathbb{R}$ be $\tilde{d}(E_x, E_y) = d(x, y)$ for all $E_x, E_y \in \tilde{S}$, where $\tilde{S}$ is the set of all equivalent classes in S. Now $\tilde{d}$ is a metric on $\tilde{S}$.

We can simply identify $\tilde{S}$ with S with some abuse of notation, but then S would be seen as a set consisting of equivalent classes instead of functions on E. When investigating an element in S, we can take an arbitrary function in that element as a representation.

Remark 2.4. *Functions equal a.e. to each other on their domain in common are regarded as the same when dealing with spaces of functions hereafter.*

Definition 2.5. *Let (X, d) be a metric space and $N \subset X$. N is called a subspace of X if the metric on N $d_N : N \times N \to \mathbb{R}$ is defined in such a way that $d_N(x, y) = d(x, y)$ for all $x, y \in N$. Then d_N is called the metric on N **induced** by the metric d on X.*

Remark 2.6. *Since d_N is induced from d, it is easy to check that d_N do is a metric on N from the definition.*

Example 2.7. *Let $c \subset \ell^\infty$ be the space of all convergent sequences in $\mathbb{F}$ and $c_0 \subset \ell^\infty$ the space of all null-sequences in $\mathbb{F}$, which means that the sequences converge to 0. Both c and c_0 are subspaces of ℓ^∞ with that metric induced from that on ℓ^∞.*

We can use $\rho(x, y) = \max_{i \in \mathbb{N}_+} |\xi_i - \eta_i|$ for c_0 as an alternative to $d(x, y) = \sup_{i \in \mathbb{N}_+} |\xi_i - \eta_i|$ since they are the same for c_0.

3. Convergence and Sets in Metric Space: 3/14/2018

I nearly overslept again... Hurried to the classroom and forgot to bring my laptop. I had to take notes with pen and paper. These notes were typed later that day.

3.1. **Convergence in Metric Space.**

Definition 3.1. *Assume that (X, d) is a metric space. Let $\{x_n\}_{n=1}^\infty \subset X$ and $x \in X$. Then $\{x_n\}_{n=1}^\infty$ **converges** to x as $n \to \infty$ iff $d(x_n, x) \to 0$ as $n \to \infty$, denoted $x_n \to x$ $(n \to \infty)$ or $\lim_{n\to\infty} x_n = x$. Or simply, $\{x_n\}_{n=1}^\infty$ is **convergent** in X.*

Remark 3.2. *In $\mathbb{R}$, $x_n \to x$ $(n \to \infty) \Leftrightarrow x_n - x \to 0$ $(n \to \infty)$. But in (X, d), the distance is measured by d. So the analogous version is invalid here. Hence, the classification of the notions and notations is worth paying attention to.*

Definition 3.3. *$\{x_n\}_{n=1}^\infty \subset X$ is called a **Cauchy sequence** in (X, d) if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.*

Proposition 3.4. *Three important properties follow immediately from Definition 3.1 and Definition 3.3:*

(1) *If $x_n \to x$ $(n \to \infty)$, then the limit x is unique;*

(2) *If $x_n \to x$ $(n \to \infty)$, then all subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ converges to x as $k \to \infty$;*
(3) *If $x_n \to x$ $(n \to \infty)$, then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.*

Proof. We only prove (1) and (3) since (2) is obvious.

(1) Assume $x_n \to x$ $(n \to \infty)$ and $x_n \to y$ $(n \to \infty)$, and then $0 \le d(x,y) \le d(x,x_n) + d(x_n,y) \to 0$ $(n \to \infty)$, which implies $d(x,y) = 0$ and hence $x = y$. So the limit of a convergent sequence is unique.
(3) Since $0 \le d(x_n,x_m) \le d(x_n,x) + d(x,x_m) \to 0$ $(n,m \to \infty)$, we derive $d(x_n,x_m) \to 0$ as $n,m \to \infty$, i.e., $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Hence the proof. □

Remark 3.5. *The converse of (3) in Proposition 3.4 may not be true. As a counterexample, let $X = (0,1] \subset \mathbb{R}$ with the metric induced from $\mathbb{R}$. $\{1/n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence for that $d(x_n,x_m) = |1/n - 1/m| \le 1/n + 1/m \to 0$ $(n,m \to 0)$. However, $\{1/n\}_{n=1}^{\infty}$ does not converge in X.*

It's clear that $\{1/n\}_{n=1}^{\infty}$ converges in $X \cup \{0\}$ and every Cauchy sequence in $X \cup \{0\}$ converges in $X \cup \{0\}$.

Generally, a noncomplete metric space can be made into a complete one with some points added, which is the so-called completion that we are going to talk about in the following sessions.

Remark 3.6. *Cauchy sequence is used to describe those sequences that actually converge in some way but might does not converge in the given space.*

Example 3.7. *Important examples of convergent sequence are given as follows. Please pay great attention to how convergence actually act in concrete cases!*

- *In $\mathbb{F}^N$, i.e., $\mathbb{R}^N$ or $\mathbb{C}^N$, let $\{x_n = (\xi_1^{(n)}, \cdots, \xi_N^{(n)})\}_{n=1}^{\infty} \subset \mathbb{F}^N$ and $x = (\xi_1, \cdots, \xi_N) \in \mathbb{F}^N$. Then we have the following equivalent statements:*
$$\begin{aligned} x_n \to x \ (n \to \infty) &\Leftrightarrow d(x_n,x) \to 0 \ (n \to \infty) \\ &\Leftrightarrow \sqrt{\sum_{i=1}^{N} |\xi_i^{(n)} - \xi_i|^2} \to 0 \ (n \to \infty) \\ &\Leftrightarrow \xi_i^{(n)} \to \xi_i \ (n \to \infty) \text{ for all } i \in \{1, \cdots, N\}. \end{aligned}$$
*Now we say that $\{x_n\}_{n=1}^{\infty}$ converges to x **in coordinate**.*
- *Let $\{x_n\}_{n=1}^{\infty} \subset C[a,b]$ and $x \in C[a,b]$, then*
$$\begin{aligned} x_n \to x \ (n \to \infty) &\Leftrightarrow d(x_n,x) \to 0 \ (n \to \infty) \\ &\Leftrightarrow \max_{t \in [a,b]} |x_n(t) - x(t)| \to 0 \ (n \to \infty) \\ &\Leftrightarrow \{x_n\}_{n=1}^{\infty} \text{ converges uniformly to } x \text{ on } [a,b] \text{ as } n \to \infty. \end{aligned}$$

- *In* s *as defined in Example 2.3, let* $\{x_n = (\xi_1^{(n)}, \xi_2^{(n)}, \cdots)\}_{n=1}^{\infty} \subset s$ *and* $x = (\xi_1, \xi_2, \cdots) \in s$*, then*

$$\begin{aligned}
&x_n \to x\ (n \to \infty) \Leftrightarrow d(x_n, x) \to 0\ (n \to \infty)\\
&\Leftrightarrow \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} \to 0\ (n \to \infty)\\
&\Leftrightarrow \xi_i^{(n)} \to \xi_i\ (n \to \infty) \text{ for all } i \in \mathbb{N}_+.
\end{aligned}$$

Indeed, the last "$\Leftrightarrow$*" requires some detailed explanation. Since* $\sum_{i=1}^{\infty} 1/2^i < \infty$*, then for all* $\varepsilon > 0$ *there is some integer* $M > 0$ *s.t.* $\sum_{i=M+1}^{n} 1/2^i < \varepsilon/2$ *and hence* $\sum_{i=M+1}^{n} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1+|\xi_i^{(n)} - \xi_i|} < \varepsilon/2$*. If* $\xi_i^{(n)} \to \xi_i$ $(n \to \infty)$ *for all* $i \in \mathbb{N}_+$*, then there is* $K_i > 0$ *for each* $i \in \{1, 2, \cdots, M\}$ *s.t.* $|\xi_i^{(n)} - \xi_i| < \varepsilon/2$ *when* $n > K_i$*. Take* $K = \max\{M, K_1, K_2, \cdots, K_M\} > 0$*, then when* $n \geq K$*. We obtain*

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} &= \sum_{i=1}^{M} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} + \sum_{i=M+1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|}\\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ for all } i \in \mathbb{N}_+.
\end{aligned}$$

Whence the "$\Leftarrow$*" holds and "*$\Rightarrow$*" is quite obvious.*

Now we are able to conclude that $\{x_n\}_{n=1}^{\infty}$ *converges in coordinate as well.*

- *In* S *as defined in Example 2.3, assume* $\{x_n\}_{n=1}^{\infty} \subset S$ *and* $x \in S$*, then*

$$\begin{aligned}
&x_n \to x\ (n \to \infty) \Leftrightarrow d(x_n, x) \to 0\ (n \to \infty)\\
&\Leftrightarrow \int_E \frac{|x_n(t) - x(t)|}{1 + |x_n(t) - x(t)|} \mathrm{d}t \to 0\ (n \to \infty)\\
&\Leftrightarrow \{x_n\}_{n=1}^{\infty} \text{ converges to } x \text{ in measure as } n \to \infty \text{ on } E.
\end{aligned}$$

The last statement is equivalent to that $m(E[|x_n(t) - x(t)| > \varepsilon]) \to 0$ *as* $n \to \infty$ *for all* $\varepsilon > 0$*.*

In fact, if $x_n \to x$ $(n \to \infty)$ *in* S*, then for all* $\varepsilon > 0$*,*

$$\begin{aligned}
0 &\leq \frac{\varepsilon}{1+\varepsilon} m(E[|x_n(t) - x(t)| > \varepsilon])\\
&\leq \int_{E[|x_n(t)-x(t)|>\varepsilon]} \frac{|x_n(t) - x(t)|}{1 + |x_n(t) - x(t)|} \mathrm{d}t\\
&\leq d(x_n, x) \to 0\ (n \to \infty).
\end{aligned}$$

Now we see that $\{x_n\}_{n=1}^{\infty}$ *converges to* x *in measure as* $n \to \infty$ *on* E*.*

Then we show the other side of the claim. If $\varepsilon > 0$*, we choose* $\sigma > 0$ *such that* $\frac{\sigma}{1+\sigma} m(E) < \varepsilon/2$ *and choose* $N > 0$ *such that* $m(E[|x_n(t) - x(t)| > \sigma]) <$

$\varepsilon/2$ *when* $n > N$. *We derive*

$$\begin{aligned} d(x_n, x) &= \int_E \frac{|x_n(t) - x(t)|}{1 + |x_n(t) - x(t)|} \mathrm{d}t \\ &= \left(\int_{E[|x_n(t)-x(t)|\leq\sigma]} + \int_{E[|x_n(t)-x(t)|>\sigma]} \right) \frac{|x_n(t) - x(t)|}{1 + |x_n(t) - x(t)|} \mathrm{d}t \\ &\leq \frac{\sigma}{1+\sigma} m(E) + m(E[|x_n(t) - x(t)| > \sigma]) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence, $d(x_n, x) \to 0$ $(n \to \infty)$.

3.2. **Sets in Metric Space.**

Definition 3.8. *Let* (X, d) *be a metric space,* $a \in X$*, and* $r > 0$.

- ***Open Ball***: $B(a, r) = \{x \in X : d(x, a) < r\}$.
- ***Closed Ball***: $S(a, r) = \{x \in X : d(x, a) \leq r\}$.
- ***Sphere***: $\tilde{S} = \{x \in X : d(x, a) = r\}$.

Remark 3.9. *Note that all the three kinds of sets defined above are subsets of the underlying metric space* (X, d).

Example 3.10. *In a discrete metric space* D, $B(a, 1) = \{a\}$ *while* $S(a, 1) = D$.

4. Topology in Metric Space: 3/16/2018

Huang always gave some lively explanations of the mathematical concepts, making it easier for us to get a clear picture of the course.

For instance, he refers to an accumulation point as such a super film star that every small circle centered at him would contain a fan of him.

Definition 4.1. *Let* $E \subset (X, d)$*, then* E *is said to be* ***bounded*** *if* $E \subset B(a, r)$ *for some* $a \in X$ *and* $r > 0$ *or* $\mathrm{diam}(E) := \sup_{x,y\in E} d(x, y) < \infty$.

Definition 4.2. *Let* $G \subset (X, d)$*, then* $x \in G$ *is said to be an* ***interior point*** *of* G *provided that there exists* $r > 0$ *such that* $B(x, r) \subset G$. *If every point of* G *is an interior point of* G*, then we say that* G *is an* ***open set*** *in* X. *If we denote the set of all interior points of* G *by* G°*, then it is clear that* G *is open iff* $G = G^\circ$.

Theorem 4.3. *In a metric space* (X, d), $B(x, r)$ *is open for all* $x \in X$ *and* $r > 0$.

Proof. It's obvious that x is an interior point of $B(x, r)$. For all $y \in B(x, r)$ with $y \neq x$, since $0 < d(y, x) < r$, consider $B(y, r - d(y, x))$. For each $z \in B(y, r - d(y, x))$, $d(z, x) \leq d(z, y) + d(y, x) < r - d(y, x) + d(y, x) = r$. Hence, $B(y, r - d(y, x)) \subset B(x, r)$. So y is an interior point of $B(x, r)$ and then $B(x, r)$ is open since y is arbitrary. $\square$

Theorem 4.4. *Let* (X, d) *be a metric space and then we have the following results:*

- $\varnothing$ *and* X *are open;*
- *All unions of open sets are open;*
- *Each finite intersection of open sets is open.*

Definition 4.5. *Let* F *be a subset of a metric space* (X,d). $x_0 \in X$ *is said to be an* ***accumulation point*** *of* F *provided that* $(B(x_0,\varepsilon) \setminus \{x_0\}) \cap F \neq \varnothing$ *for all* $\varepsilon > 0$.

If we denote the set of all accumulation points of F *by* F', *then* F *is said to be* ***closed*** *iff* $F' \subset F$. *The* ***closure*** *of* F *is defined to be* $\bar{F} := F \cup F'$.

Remark 4.6. *In the definition above, if* F *is closed, then it's clear that* $F \cup F' \subset F \subset F \cup F'$, *i.e* $F = F \cup F'$. *Hence,* F *is closed iff* $F = \bar{F}$. *Also,*

$$\bar{F} = \{x \in X : \exists \{x_n\}_{n=1}^{\infty} \subset F \text{ such that } x_n \to x \text{ as } n \to \infty\}.$$

Theorem 4.7. *The following statements are equivalent.*

1. x_0 *is an accumulation point of* F.
2. $B(x_0,\varepsilon)$ *contains infinitely many points of* F *for all* $\varepsilon > 0$.
3. *There exists* $\{x_n\}_{n=1}^{\infty} \subset F$ *such that* $x_n \neq x_0$ *for infinitely many* $n \in \mathbb{N}_+$ *and* $x_n \to x_0$ $(n \to \infty)$.

Theorem 4.8. *A set* $F \subset (X,d)$ *is closed iff* $F^{\mathrm{C}} := X \setminus F$ *is open.*

Proof. $\Rightarrow$) Assume that F is closed. For all $x \in F^{\mathrm{C}}$, x is not an accumulation point of F, i.e., there is some $r > 0$ such that $B(x,r) \cap F = \varnothing$, that is, $B(x,r) \subset F^{\mathrm{C}}$. It follows that x is an interior point of F^{C}, thus F^{C} is open.

$\Leftarrow$) If F^{C} is open, then for all $x \in F^{\mathrm{C}}$ (i.e., $x \notin F$), x is an interior point of F^{C}. So there exists $r > 0$ such that $B(x,r) \subset F^{\mathrm{C}}$, i.e., $B(x,r) \cap F = \varnothing$ or equally $(B(x,r) \setminus \{x\}) \cap F = \varnothing$, which implies that x is not an accumulation point of F. Whence F^{C} contains no accumulation point of F and $F' \subset F$. So F is closed. $\square$

Remark 4.9. *The idea in the second part of the proof of Theorem 4.8 is to show that every* $x \notin F$ *is not an accumulation point of* F. *Thus all accumulation points of* F, *if exist, must be contained in* F. *This valid since* $F' \subset F$ *implies* F *is closed by Definition 4.5.*

Theorem 4.10. *Let* (X,d) *be a metric space, then*

1. $\varnothing$ *and* X *are closed;*
2. *All intersections of closed sets are closed;*
3. *Each finite union of closed sets is closed.*

Example 4.11. *In the discrete metric space* D, *all sets in* D *are open and closed at the same time. In fact, if* $E \subset D$, *then* $B(x,1) = \{x\} \subset E$ *for all* $x \in E$, *which implies* E *is open and so is* E^{C}. *Thus* $E = (E^{\mathrm{C}})^{\mathrm{C}}$ *is closed by Theorem 4.8.*

Theorem 4.12. *Let* A *be a subspace of* X. *Then a subset* B *of* X *is closed in* A *iff there exists a closed set* F *in* X *such that* $B = A \cap F$; *a subset* C *of* X *is open in* A *iff there exists an open set* G *in* X *such that* $C = A \cap G$.

Proof. Denote the set of all accumulation points of B in X by B'.

If B is closed in A, we have $B \subset A$ and $B' \cap A \subset B$. Let $F = B \cup B'$, then F is closed in X and $A \cap F = (A \cap B) \cup (A \cap B') = B \cup (A \cap B') \subset B$. Also, it's clear that $B \subset A$ and $B \subset F$. Thus $B \subset A \cap F$ and $B = A \cap F$.

If there exists a closed set F in X such that $B = A \cap F$, then $B \subset F \Rightarrow B' \subset F$. From $B' \cap A \subset F \cap A = B$, it follows that B is closed in A. Hence, the first part of the claim holds and we can use this fact to prove the second one.

A subset C of X is open in A iff $C \subset A$ and $A \setminus C$ is closed in A. This is equivalent to the statement that $C \subset A$ and there exists a closed set F in X such that $A \setminus C = A \cap F \Leftrightarrow A \cap C = A \cap F^{\mathrm{C}}$. It's equal to say that $C = A \cap F^{\mathrm{C}}$. Then the second part of the claim holds since F^{C} is open in X. $\square$

Example 4.13. *Let (X, d), where $X = [0, 3) \cup [4, 5] \cup (6, 7) \cup \{8\}$, be the space equipped with the metric induced from the euclidean one on $\mathbb{R}$, then $[0, 3)$ is open and closed in X.*

Remark 4.14. *The above definitions, theorems, remarks and examples indicate that **metric induces topology**, and hence the results in topology are valid here.*

5. Separation and Completeness: 3/21/2018

Definition 5.1. *Let (X, d) be a metric space and $A \subset X$. $c \in X$ is said to be a **boundary point** of A provided that $B(c, \varepsilon) \cap A \neq \varnothing$ and $B(c, \varepsilon) \cap A^{\mathrm{C}} \neq \varnothing$ for all $\varepsilon > 0$. The set of all boundary points of A is called the **boundary** of A, denoted ∂A.*

Definition 5.2. *Let (X, d) be a metric space and $A, B \subset X$ with $B \subset A$. B is said to be **dense** in A provided that $A \subset \bar{B}$. Particularly, B is dense in X iff $X = \bar{B}$.*

Remark 5.3. *Of course, the metrics on A and B in Definition 5.2 are the ones induced naturally from the metric on X.*

Also, Definition 5.2 is equivalent to

- *For all $x \in A$, there exists $\{x_n\}_{n=1}^{\infty} \subset B$ such that $x_n \to x$ $(n \to \infty)$; or*
- *For all $x \in A$ and all $\varepsilon > 0$, there exists $y \in B$ such that $d(y, x) < \varepsilon$, i.e., there exists $y \in B$ such that $y \in B(x, \varepsilon)$ (or $x \in B(y, \varepsilon)$).*

Definition 5.4. *(X, d) is called to be **separable** if X has a countable dense subset.*

Lemma 5.5. *Weierstrass Approximation Theorem*

Suppose f is a continuous real-valued function defined on the real interval $[a, b]$. For every $\varepsilon > 0$, there exists a polynomial P (with rational coefficients) such that for all x in $[a, b]$, we have $|f(x) - P(x)| < \varepsilon$, or equivalently, the supremum norm $\|f - P\| < \varepsilon$.

Example 5.6. *Now we give some typical examples of dense set in metric space.*

- *$\mathbb{R}^n$ is separable since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.*

- *$C[a,b]$ is separable.*

  In fact, note that for $\{x_n\}_{n=1}^{\infty} \subset C[a,b]$, $x_n \to x \in C[a,b] \Leftrightarrow x_n \rightrightarrows x$ on $[a,b]$. For each $x \in C[a,b]$, there exists $\{P_n\}_{n=1}^{\infty} \subset P_{\mathbb{Q}}[a,b]$, the set of all polynomials on $[a,b]$ with rational coefficients, such that $P_n \rightrightarrows x$ on $[a,b]$. It is obvious that $P_n \to x$ as $n \to \infty$. The claim is derived by Lemma 5.5, which shows that $P_{\mathbb{Q}}[a,b]$ is a countable dense subset of $C[a,b]$.
- *ℓ^{∞} is inseparable.*

  In fact, let $E = \{x = \{\xi_i\}_{i=1}^{\infty} : \xi_i \in \{0,1\}$ for all $i \in \mathbb{N}_+\}$, then E is uncountable. $E \subset \ell^{\infty}$ and $d(x,y) = \sup_{i\in\mathbb{N}_+} |\xi_i - \eta_i| = 1$ iff $x \neq y = \{\eta_i\}_{i=1}^{\infty} \in E$. Suppose that ℓ^{∞} is separable, then ℓ^{∞} has a dense subset A which is countable and so is the set $\{B(x,1/2) : x \in A\}$. Since A is dense in ℓ^{∞}, then $\ell^{\infty} = \bar{A}$. We see that for each $y \in E$, $y \in B(x,1/2)$ holds for some $x \in A$. Since E is uncountable, there exists $y' \in E$ with $y' \neq y$ such that $y' \in B(x,1/2)$. So $1 = d(y,y') \leq d(y,x) + d(x,y') < 1/2 + 1/2 = 1$, which is a contradiction.

Definition 5.7. *Let (X,d) be a metric space, then X is said to be **complete** if each Cauchy sequence in X converges. Equivalently, (X,d) is complete iff for all $\{x_n\}_{n=1}^{\infty} \subset X$ with $d(x_n,x_m) \to 0$ $(n,m \to \infty)$ always implies $d(x_n,x) \to 0$ $(n \to \infty)$ for some $x \in X$.*

Example 5.8. *$\mathbb{F}^n$, $C[a,b]$, and ℓ^{∞} are complete.*

Definition 5.9. *Suppose A is a subset of X, then A is said to be **complete** if every Cauchy sequence in A converges to some $x \in A$.*

Theorem 5.10. *Suppose (X,d) is a metric space and $A \subset X$.*

(a) *If A is complete, then A is closed.*
(b) *If X is complete and A is closed, then A is complete.*

Proof. (a) For all $x \in \bar{A}$, there is $\{x_n\}_{n=1}^{\infty} \subset A$ such that $x_n \to x$ $(n \to \infty)$, which means that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in A. Then $x_n \to y$ $(n \to \infty)$ for some $y \in A$ since A is complete. By the uniqueness of limit, we have $x = y \in A$. Hence, $\bar{A} \subset A$ and then A is closed.

(b) Suppose $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in A. Since X is complete, it follows that $x_n \to x$ for some $x \in X$ as $n \to \infty$. Note that x is a limit point of A. This yields that $x \in \bar{A}$. So $x \in A$ because A is closed. Thus, $\{x_n\}_{n=1}^{\infty}$ converges in A and then A is complete since $\{x_n\}_{n=1}^{\infty}$ is arbitrary. $\square$

Remark 5.11. *An incomplete metric space may contain complete subspaces.*

A closed subset of a complete metric space is a complete metric subspace.

All complete subspace of a metric space are closed.

6. Nest and Sets of the First/Second Category: 3/23/2018

Example 6.1. *$\mathbb{F}^n$, $C[a,b]$, ℓ^{∞}, c, and c_0 are complete.*

$\mathbb{Q}$ *and* $(0,1)$ *as subspaces of* $\mathbb{R}$ *are incomplete.*

However, if we replace the metric on $C[a,b]$ *by another one, that is,*

$$d'(x,y) = \left(\int_a^b |x(t) - y(t)|^p \mathrm{d}t \right)^{1/p}$$

for all x, y *in* $C[a,b]$*, where* $p \geq 1$ *is some fixed real number, then* $C[a,b]$ *is incomplete. As a counterexample, consider* $\{f_n\}_{n=1}^\infty \subset C[0,1]$ *where* $f_n(x) = x^n$ *for all* $x \in [0,1]$ *and* $n \in \mathbb{N}_+$*.*

Definition 6.2. *A **nest** of closed balls is a sequence closed balls, say* $\{S_n\}_{n=1}^\infty$*, such that* $S_1 \supset S_2 \supset S_3 \supset \cdots$ *and* $\mathrm{diam}(S_n) \to 0$ $(n \to \infty)$*.*

Theorem 6.3. *Let* (X,d) *be a metric space, then* (X,d) *is complete iff all nests of closed balls in* X *have nonempty intersections. Moreover, the intersections are all single point sets.*

Proof. $\Rightarrow$) For each nest of closed balls $\{S_n\}_{n=1}^\infty \subset X$, let x_n be the center of S_n for all $n \in \mathbb{N}_+$. For all $\varepsilon > 0$, there exists $N > 0$ such that $d(x_n, x_m) \leq \mathrm{diam}(S_n) < \varepsilon$ when $m > n > N$, which implies that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in X. Hence, there exists some $x \in X$ such that $x_n \to x$ $(n \to \infty)$ since X is complete. Moreover, $x \in \bar{S}_n = S_n$ since each S_n is closed. Now we have $x \in \cap_{n=1}^\infty S_n \Rightarrow \cap_{n=1}^\infty S_n \neq \varnothing$. If $y \in \cap_{n=1}^\infty S_n$, then $0 \leq d(x,y) \leq \mathrm{diam}(S_n) \to 0$ $(n \to \infty) \Rightarrow x = y$, that is, $\cap_{n=1}^\infty S_n = \{x\}$. So the intersection is nonempty and of cardinality 1.

$\Leftarrow$) For any Cauchy sequence $\{x_n\}_{n=1}^\infty$ in X: $d(x_n, x_m) \to 0$ $(n, m \to \infty)$, we perform the following procedure:

- Choose $n_1 > 0$ such that $d(x_n, x_{n_1}) < 1/2$ when $n > n_1$;
- Choose $n_2 > n_1$ such that $d(x_n, x_{n_2}) < 1/4$ when $n > n_2$;
- Choose $n_3 > n_2$ such that $d(x_n, x_{n_3}) < 1/8$ when $n > n_3$;
- $\cdots\cdots$
- Choose $n_k > n_{k-1}$ such that $d(x_n, x_{n_k}) < 1/2^k$ when $n > n_k$;
- $\cdots\cdots$

Now we obtain a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ and a nest of closed balls $S_k = \{x \in X : d(x, x_{n_k}) \leq 1/2^k\}$. We see that there exists $x \in X$ such that $\cap_{k=1}^\infty S_k = \{x\}$. Hence, $x_{n_k} \to x$ $(k \to \infty)$. Then the whole sequence is convergent, i.e., $x_n \to x$ $(n \to \infty)$, which yields the completeness of X. $\square$

Definition 6.4. *Let* (X,d) *be a metric space and* $S \subset X$*.* S *is said to be **nowhere dense** if* $(\bar{S})^\circ = \varnothing$*.* S *is said to be of **the first category** if* S *can be expressed as a countable union of nowhere dense sets.* S *is said to be of **the second category** if* S *is not of the first category.*

Lemma 6.5. *Let* X *be a complete metric space and* $\{G_n\}_{n=1}^\infty$ *be a sequence of open dense subsets of* X*. Then* $\cap_{n=1}^\infty G_n \neq \varnothing$*.*

Proof. It's clear that each G_n is nonempty provided that X is nonempty. Take $x_1 \in G_1$ and then there exists an open ball $B(x_1) \subset G_1$ (the radius is omitted here since we are not interested in it). Choose a closed ball $S(x_1, r_1) \subset B(x_1)$ where $r_1 > 0$. Since G_2 is dense in X, there exists $x_2 \in G_2 \cap S(x_1, r_1/2)$. Pick an open ball $B(x_2) \subset G_2$ and then there is a closed ball $S(x_2, r_2) \subset B(x_2)$ with radius r_2 less than $r_1/2$. Hence, $r_2 < r_1/2$, $S(x_2, r_2) \subset S(x_1, r_1)$, and $S(x_2, r_2) \subset G_2$.

Continue the procedure and we will obtain a sequence of closed balls $\{S(x_n, r_n)\}_{n=1}^{\infty}$ such that $S(x_n, r_n) \subset G_n$, $S(x_{n+1}, r_{n+1}) \subset S(x_n, r_n)$ for all $n \in \mathbb{N}_+$. Since $r_n \to 0$ as $n \to \infty$, we know $\{S(x_n, r_n)\}_{n=1}^{\infty}$ is a nest. Note that X is complete. By Theorem 6.3, there exists $x \in \cap_{n=1}^{\infty} S(x_n, r_n) \subset \cap_{n=1}^{\infty} G_n$. Thus, $\cap_{n=1}^{\infty} G_n \neq \varnothing$. □

Theorem 6.6. ***Barie Category Theorem***

Let (X, d) be nonempty and complete, then X is of the second category.

Proof. If the claim in false, then there exists a sequence of nowhere dense sets $\{E_n\}_{n=1}^{\infty}$ such that $X = \cup_{n=1}^{\infty} E_n = \cup_{n=1}^{\infty} \bar{E}_n \Rightarrow \cap_{n=1}^{\infty} (\bar{E}_n)^{\mathrm{C}} = \varnothing$. From the fact that each E_n is nowhere dense, it follows that $B(x, \varepsilon) \cap (\bar{E}_n)^{\mathrm{C}} \neq \varnothing$ for all $x \in X$ whenever $\varepsilon > 0$. Thus, each open set $(\bar{E}_n)^{\mathrm{C}}$ is dense, which implies $\cap_{n=1}^{\infty} (\bar{E}_n)^{\mathrm{C}} \neq \varnothing$ by Lemma 6.5. This is a contradiction and hence the claim holds. □

7. Continuous Mapping on Metric Space: 3/28/2018

Example 7.1. *A completion of $C[a, b]$ under the metric d' as in Example 6.1 is $L^p[a, b]$, where $p \geq 1$ is some fixed real number. Note that d' actually has been extended to be well-defined on $L^p[a, b] \times L^p[a, b]$ here.*

Now we extend the notion of continuity of a function f on $\mathbb{R}$ to $\mathbb{R}$, i.e., $\lim_{x \to x_0} f(x) = f(x_0) (= f(\lim_{x \to x_0} x))$, to the version in metric spaces.

Definition 7.2. *Let (X, d) and (Y, ρ) be two metric spaces and $T : X \to Y$ be a mapping. T is said to be* ***continuous at*** *$x_0 \in X$ if for each $\varepsilon > 0$ there exists some $\delta > 0$ such that $\rho(Tx, Tx_0) < \varepsilon$ whenever $x \in X$ and $d(x, x_0) < \delta$, denoted $Tx \to Tx_0$ as $x \to x_0$ or $\lim_{x \to x_0} Tx = Tx_0$.*

Remark 7.3. *For mappings like T between metric spaces in Definition 7.2, $T(x)$ is often written as Tx for all elements in X since it's a better notation for operators and too many brackets would be annoying if there are several compositions.*

By Definition 7.2, we immediately derive the following proposition.

Proposition 7.4. *Let $(X, d), (Y, \rho)$ be two metric spaces and $T : X \to Y$, then the following statements are equivalent.*

- *T is continuous at $x_0 \in X$.*
- *For each $\varepsilon > 0$ there exists some $\delta > 0$ such that $T(B(x_0, \delta)) \subset B(Tx_0, \varepsilon)$.*

Remark 7.5. *In Proposition 7.4, note that $B(x_0, \delta) \subset X$ and $B(Tx_0, \varepsilon) \subset Y$. With a little abuse of notation, balls in various metric spaces will always be denoted as $B(x, r)$ when the context is clear. If not, balls in a metric space, say (X, d), will be denoted as $B_d(x, r)$.*

Definition 7.6. *T is said to be **continuous on** X if T is continuous at all $x \in X$.*

Theorem 7.7. *Let $T : (X, d) \to (Y, \rho)$, then the following statements are equivalent.*

(a) *T is continuous on X.*

(b) *The preimage of T for each open (resp. closed) set in Y is open (resp. closed) in X, i.e., $T^{-1}(G) = \{x \in X : Tx \in G\}$ is open (resp. closed) for each open (resp. closed) subset $G \subset Y$.*

Proof. It suffices to prove the part for open sets since the other one can be derived from the former easily.

(a)⇒(b) Suppose G is open in Y. Without loss of generality, we assume $G \neq \varnothing$. If $x \in T^{-1}(G)$, then $Tx \in G$. So there exists some $\varepsilon > 0$ such that $B(Tx, \varepsilon) \subset G$. By assumption, T is continuous at x. Thus, we can pick $\delta > 0$ such that $T(B(x, \delta)) \subset B(Tx, \varepsilon) \subset G$, which implies that $B(x, \delta) \subset T^{-1}(G)$. Hence, $T^{-1}(G)$ is open since x is arbitrary and then (b) holds since G is also arbitrary.

(b)⇒(a) If $x \in X$, then $T^{-1}(B(Tx, \varepsilon))$ is open in X for all $\varepsilon > 0$. So $x \in T^{-1}(B(Tx, \varepsilon))$ and there exists $\delta > 0$ such that $B(x, \delta) \subset T^{-1}(B(Tx, \varepsilon))$. This shows $T(B(x, \delta)) \subset B(Tx, \varepsilon)$. Thus, T is continuous at x since ε is arbitrary and then T is continuous on X since x is also arbitrary. □

Remark 7.8. *Theorem 7.7 actually proves the equivalence of continuity defined with respect to metric and topology provided that the topology is induced from the metric.*

Corollary 7.9. *Suppose that (X, d) is a metric space and $f : X \to \mathbb{R}$ is a real-valued function on X. Then f is continuous on X iff for each $r \in \mathbb{R}$, the sets $\{x \in X : f(x) \leq r\}$ and $\{x \in X : f(x) \geq r\}$ are all closed in X.*

Theorem 7.10. *Given a metric space (X, d), then $d : X \times X \to \mathbb{R}$ is continuous.*

Proof. We are going to use some topological techniques implicitly in this proof.

It suffices to prove $d^{-1}(a, b)$ is open for each interval $(a, b) \subset \mathbb{R}$. Without loss of generality, we assume $d^{-1}(a, b) \neq \varnothing$. For each $(x, y) \in d^{-1}(a, b)$, $d(x, y) \in (a, b)$ implies that there exists $\varepsilon > 0$ such that $B(d(x, y), \varepsilon) \subset (a, b)$. Then for all $(x', y') \in B(x, \varepsilon/2) \times B(y, \varepsilon/2)$, we have

$$d(x', y') \leq d(x', x) + d(x, y) + d(y, y') < \varepsilon/2 + d(x, y) + \varepsilon/2 = d(x, y) + \varepsilon,$$

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y) < \varepsilon/2 + d(x', y') + \varepsilon/2 = d(x', y') + \varepsilon.$$

Thus, $|d(x', y') - d(x, y)| < \varepsilon$ and then $d(B(x, \varepsilon/2) \times B(y, \varepsilon/2)) \subset B(d(x, y), \varepsilon) \subset (a, b)$, which shows that $B(x, \varepsilon/2) \times B(y, \varepsilon/2) \subset d^{-1}(a, b)$. Since $(x, y) \in d^{-1}(a, b)$ is arbitrary, we conclude that $d^{-1}(a, b)$ is open. □

Theorem 7.11. ***Heine's Theorem***

Suppose $T : (X, d) \to (Y, \rho)$ is a mapping and $x_0 \in X$, then the following statements are equivalent.

(a) ***Cauchy-continuity***: *T is continuous at x_0.*

(b) ***Heine-continuity***: *For each $\{x_n\}_{n=1}^{\infty} \subset X$ with $x_n \to x_0$ $(n \to \infty)$ in X, we have $Tx_n \to Tx_0$ $(n \to \infty)$ in Y.*

Proof. (a)⇒(b) Let $\{x_n\}_{n=1}^{\infty} \subset X$ with $x_n \to x_0$ $(n \to \infty)$ in X. Since T is continuous at $x_0 \in X$, then for each $\varepsilon > 0$ there exists some $\delta > 0$ such that $\rho(Tx, Tx_0) < \varepsilon$ when $d(x, x_0) < \delta$. Now we see that $x_n \to x_0$ $(n \to \infty)$ yields some $N > 0$ satisfying $d(x_n, x_0) < \delta$ when $n > N$ and hence $\rho(Tx_n, Tx_0) < \varepsilon$ when $n > N$, that is, $Tx_n \to Tx_0$ $(n \to \infty)$.

(b)⇒(a) If T is discontinuous at $x_0 \in X$, then there is $\varepsilon_0 > 0$ and $\{x_n\}_{n=1}^{\infty} \subset X$ with $d(x_n, x_0) < 1/n$ but $\rho(Tx_n, Tx_0) \geq \varepsilon_0$ for all $n \in \mathbb{N}_+$. Clearly, $x_n \to x_0$ in X and so $Tx_n \to Tx_0$ in Y as $n \to \infty$ by assumption. The contradiction goes that $0 = \rho(T_{x_0}, T_{x_0}) = \rho(\lim_{n\to\infty} Tx_n, Tx_0) = \lim_{n\to\infty} \rho(Tx_n, Tx_0) \geq \varepsilon_0$ since ρ is continuous by Theorem 7.10. Thus, T has to be continuous at x_0. □

Remark 7.12. *Theorem 7.11 shows that Cauchy-continuity is equivalent to Heine-continuity in metric spaces.*

Remark 7.13. *Mappings between metric spaces are always continuous at isolated points. As in Theorem 7.11, if no sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with infinitely number of $x_n \neq x_0$ converges to $x_0 \in X$, then x_0 is isolated in X. This automatically yields that T is continuous at x_0.*

Example 7.14. *Let $C^1[a, b]$ be the set of all continuously differentiable functions on $[a, b]$, then $C^1[a, b]$ as a subspace of $C[a, b]$ is a metric space. Define*

$$T : C^1[a, b] \to C[a, b],$$

$$x(t) \mapsto Tx(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t},\ t \in [a, b].$$

Then T is discontinuous on $C[a, b]$. Let $x_n(t) = \frac{1}{n}\mathrm{e}^{-n(t-a)}$ $(t \in [a, b])$ for all $n \in \mathbb{N}_+$. Then $\{x_n\}_{n=1}^{\infty} \subset C^1[a, b]$ and $\max_{t\in[a,b]} |x_n(t) - x_0(t)| = 1/n \to x_0$ $(n \to \infty)$ yields that $x_n \to x_0$ $(n \to \infty)$ in $C^1[a, b]$ where $x_0(t) \equiv 0$ on $[a, b]$. But $Tx_n = -\mathrm{e}^{-n(t-a)}$ and $\max_{t\in[a,b]} |Tx_n(t) - x_0(t)| = 1$ for all $n \in \mathbb{N}_+$. So $\{Tx_n\}_{n=1}^{\infty}$ does not converge to Tx_0 as $n \to \infty$. Hence, T is discontinuous at x_0 by Theorem 7.11.

However, if we replace the induced metric on $C^1[a, b]$ by $d'(x, y) = \max_{t\in[a,b]} |x(t) - y(t)| + \max_{t\in[a,b]} |x'(t) - y'(t)|$ (or $d''(x, y) = \max\{\max_{t\in[a,b]} |x(t) - y(t)|, \max_{t\in[a,b]} |x'(t) - y'(t)|\}$) defined for all $x, y \in C^1[a, b]$, then T is continuous on $C^1[a, b]$.

In fact, $d(Tx, Ty) = \max_{t\in[a,b]} |Tx(t) - Ty(t)| = \max_{t\in[a,b]} |x'(t) - y'(t)| \leq d'(x, y)$ (or $d''(x, y)$) for all $x, y \in [a, b]$. Then it's easy to show T is continuous on $C^1[a, b]$ under the metric d' or d''.

I went to Peking that day. These notes were again provided by Wenyi Cai.

Definition 8.1. *A topological space is* ***sequentially compact*** *if every infinite sequence has a convergent subsequence.*

Lemma 8.2. ***Bolzano-Weierstrass Theorem***

Each bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. Or equivalently, a subset of $\mathbb{R}^n$ is sequentially compact iff it is closed and bounded.

Remark 8.3. *Lemma 8.2 is also called the* ***sequential compactness theorem****.*

Theorem 8.4. ***Extreme Value Theorem***

Let f be a continuous real-valued function on a bounded and closed subset E of $\mathbb{R}^n$, then f is bounded on E and attains its infimum and supremum on E.

Proof. Without loss of generality, we assume $E \neq \varnothing$. Suppose f is unbounded on E, then there is a sequence $\{x_n\}_{n=1}^{\infty} \subset E$ such that $f(x_n) > n$ for all $n \in \mathbb{N}_+$. By Lemma 8.2 select a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ which converges to $x_0 \in \bar{E} = E$ since E is bounded and closed. Hence there is some positive integer N such that $f(x_{n_k}) \in B(f(x_0), 1)$ when $k > N$ by the continuity of f on E, that is, $f(\{x_{n_k}\}_{k=N+1}^{\infty})$ is bounded. But $f(x_{n_k}) > n_k$ for all $k > N$, which yields the contradiction. Thus, f is bounded on E. Then $\alpha = \inf_{x\in E} f(x)$ and $\beta = \sup_{x\in E} f(x)$ are both finite.

From the definition of infimum, we can choose $y_n \in E$ such that $\alpha \leq f(y_n) < \alpha + 1/n$ for all $n \in \mathbb{N}_+$ and surely there is a subsequence $\{y_{n_k}\}_{k=1}^{\infty} \subset \{y_n\}_{n=1}^{\infty}$ that converges to some $y_0 \in E$. Since $|f(y_{n_k}) - \alpha| < 1/n_k \to 0$ $(k \to \infty)$ and $f(y_{n_k}) \to f(y_0)$ $(k \to \infty)$, we derive $f(y_0) = \alpha$ by the uniqueness of limit. Thus, f attains its infimum on E. Similarly, f attains its supremum on E. □

Remark 8.5. *Note that if E is bounded and closed in an arbitrary metric space X, then f may not be bounded on E. For a counterexample, we take the metric d on $(0, 1)$ induced from the usual one on $\mathbb{R}$. Then $X = (0, 1)$ is bounded and closed. Let $f(x) = 1/x$ $(x \in X)$, then f continuous but not bounded on X.*

Remark 8.6. *In the proof of Theorem 8.4, the uniqueness of limit in metric spaces plays an important role. But here we recall that the limit of a sequence may not be unique in a general topological space. As a counterexample, consider the set $X = \{a, b, c\}$ together with the topology $\mathscr{T} = \{\varnothing, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then $\{x_n = b\}_{n=1}^{\infty}$ converges to a, b, c. But in Hausdorff spaces, the limit of each convergent sequence is always unique. And metric spaces are all Hausdorff ones.*

Definition 8.7. *In topology and other related topics, a* ***totally bounded*** *space is one that can be covered by finitely many subsets of every fixed "size" (where the kind of subsets and the meaning of "size" depend on the given context).*

As for metric spaces, subsets denote open balls and "size" denotes the radii of them. So a metric space is totally bounded iff for every $\varepsilon > 0$, there exists a finite collection *of open balls in X of radius ε whose union contains X. Equivalently, the metric space X is totally bounded iff for every $\varepsilon > 0$, there exists a finite cover such that the radius of each element of the cover is at most ε.*

Theorem 8.8. *Equivalent Forms of Compactness in Metric Space*
Let (X, d) be a metric space. The following statements are equivalent.

(a) *X is compact.*
(b) *X is sequentially compact.*
(c) *X is complete and totally bounded.*

Proof. (a)$\Rightarrow$(b) Assume that an arbitrary sequence $\{x_n\}_{n=1}^{\infty} \subset X$ does not have a convergent subsequence. So for every $x \in X$, there exists a neighborhood U_x of x such that $\{n \in \mathbb{N}_+ : x_n \in U_x\}$ is finite. Note that $\{U_x : x \in X\}$ is an open cover of X. By compactness of X, there is a finite subcover, say $U_{x_1}, U_{x_2}, \cdots, U_{x_k}$. Then $\mathbb{N}_+ = \{n \in \mathbb{N}_+ : x_n \in X\} = \cup_{i=1}^{k}\{n \in \mathbb{N}_+ : x_n \in U_i\}$. But $\cup_{i=1}^{k}\{n \in \mathbb{N}_+ : x_n \in U_i\}$ is finite, which is a contradiction. Hence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence and X is sequentially compact.

(b)$\Rightarrow$(c) By sequential compactness of X, every Cauchy sequence in X has a convergent subsequence, and so the whole sequence is convergent. Thus, X is complete.

Suppose X is not totally bounded, then there exists $\varepsilon > 0$ such that no finite collection of open balls of radius ε covers X. So we are able to perform the following procedure. Let $B_1 = B(x_1, \varepsilon)$ be one such ball where $x_1 \in X$. Choose $x_2 \in X \setminus B_1$ and let $B_2 = B(x_2, \varepsilon)$. Choose $x_3 \in X \setminus (B_1 \cup B_2)$ and let $B_3 = B(x_3, \varepsilon)$. Inductively, we will obtain a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_{n+1} \in X \setminus \cup_{i=1}^{n} B_i$ for all $n \in \mathbb{N}_+$. Thus, $d(x_n, x_m) \geq \varepsilon$ for all $n, m \in \mathbb{N}_+$ whenever $n \neq m$, and $\{x_n\}_{n=1}^{\infty}$ has no convergent subsequence, which is a contradiction. So X is totally bounded.

(c)$\Rightarrow$(a) If X is not compact, then there exists a collection of open set $\{U_i : i \in I\}$ covering X but it has no finite subcover. Here I is the index set.

Since X is totally bounded, it follows that X can be covered by finite number of subsets (the type of the subsets is not of our interest here and so is omitted) in X with diameters less that 1, say $C_1^1, C_2^1, \cdots, C_{p_1}^1$. Let $C^1 = C_{k_1}^1$ be the one that cannot be covered by finitely many U_i's. Then C^1 can be covered by finite number of subsets in C^1 with diameters less that $1/2$, say $C_1^2, C_2^2, \cdots, C_{p_2}^2$. Let $C^2 = C_{k_2}^2$ be the one that cannot be covered by finitely many U_i's.

Proceed this procedure and we will obtain a set sequence $C^1 \supset C^2 \supset \cdots$ such that $\operatorname{diam}(C^n) < 1/n$ for each $n \in \mathbb{N}_+$. Select $x_n \in C^n$ for each $n \in \mathbb{N}_+$ since it's obvious that they are nonempty. Then $\{x_n\}_{n=1}^{\infty}$ is surely a Cauchy sequence. From the compactness of X, it follows that $\{x_n\}_{n=1}^{\infty}$ converges to some $x \in X$. Obviously, there exists $i \in I$ such that $x \in U_i$. So there is some $\delta > 0$ satisfying $B(x, \delta) \subset U_i$. Choose a positive integer N sufficiently large such that $d(x_N, x) < \delta/2$ and $1/N < \delta/2$. Hence, $C^N \subset B(x_N, 1/N) \subset B(x_N, \delta/2) \subset B(x, \delta) \subset U_i$, which violates the construction of C^N. So the assumption fails and X is compact. $\square$

Remark 8.9. *In general there exist sequentially compact spaces that are not compact (such as the first uncountable ordinal with the order topology), and compact spaces that are not sequentially compact (such as the product of $2^{\aleph_0} = \mathfrak{c}$ copies of the closed unit interval).*

By Lemma 8.2 and Theorem 8.8, we derive the following corollary.

Corollary 8.10. ***Heine-Borel Theorem***
For a subset S of euclidean space $\mathbb{R}^n$, the following two statements are equivalent.

- *S is compact.*
- *S is closed and bounded.*

By Theorem 8.8, it would then be appropriate to propose the following definition of compactness in a metric space.

Definition 8.11. *Let X be a metric space, then a subset E of X is said to be* ***compact*** *if every sequence in E has subsequence that converges to an element of X, i.e., there is $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ for every $\{x_n\}_{n=1}^{\infty} \subset E$ such that $x_{n_k} \to x_0 \in E$ as $k \to \infty$. And E is defined to be* ***relatively compact*** *if $\bar{E}$ is compact.*

Remark 8.12. *Compared with what is proposed in Remark 8.5, let f be a continuous real-valued function on a compact subset E of a metric space X, then f is bounded and attains its infimum and supremum on E by Theorem 9.1.*

Theorem 8.13. *Let (X, d) be a metric space and $E \subset X$.*

(a) *If E is compact, then E is complete while the inverse may not be true.*
(b) *If E is compact, then E is bounded and closed in X while the inverse may not be true.*
(c) *If X is compact and E is closed, then E is compact.*

Proof. (a) Suppose $\{x_n\}_{n=1}^{\infty}$ is an arbitrary Cauchy sequence in E. Since E is compact, $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which converges to some $x_0 \in E$. Then $\{x_n\}_{n=1}^{\infty}$ converges to $x_0 \in E$ as well. Thus E is complete.

Conversely, take $X = E = \mathbb{R}$ with the usual metric as a counterexample.

(b) If E is compact, then E is complete by (a). Hence E is closed by Theorem 5.10. Suppose E is not bounded, then choose a fixed point $a \in E$ and there exists $x_n \in E$ such that $d(x_n, a) > n$ for all $n \in \mathbb{N}_+$. By compactness of E, $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence, which is a contradiction. So E has to be bounded.

Conversely, take $X = E = (0, 1]$ with the metric induced from the usual one on $\mathbb{R}$, then E is bounded and closed. But $\{x_n = 1/n\}_{n=1}^{\infty}$ has no subsequence that converges in E, which is a counterexample.

(c) Get an arbitrary sequence $\{x_n\}_{n=1}^{\infty} \subset E$ and then $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to, say, $x_0 \in X$ since X is compact. Note that E is closed. So $x_0 \in E$. Hence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to $x_0 \in E$, which shows that E is compact. $\square$

Definition 8.14. *Suppose (X, d) and (Y, ρ) are metric spaces. Mapping $f : X \to Y$ is said to be* ***uniformly continuous*** *on X if for all $\varepsilon > 0$, there exists some $\delta > 0$ such that $\rho(x_1, x_2) < \varepsilon$ when $d(x_1, x_2) < \delta$ where $x_1, x_2 \in X$.*

Theorem 8.15. ***Heine-Cantor Theorem***

Let E be a compact set in (X, d) and $T : E \to (Y, \rho)$ be continuous, then T is uniformly continuous on E.

Proof. Fix $\varepsilon > 0$ and for each $x \in X$, there exists $\delta_x > 0$ such that $\rho(Ty, Tx) < \varepsilon/2$ whenever $y \in X$ and $d(y, x) < \delta_x$ by the continuity of T on X. Then let $U_x = \{y \in X : d(y, x) < \delta_x/2\}$ for each $x \in X$, and $\{U_x : x \in X\}$ surely covers X. Since X is compact, there exist finitely many U_x's whose union contains X, say $U_{x_1}, U_{x_2}, \cdots, U_{x_n}$. Set $\delta = \min\{\delta_{x_1}/2, \delta_{x_2}/2, \cdots, \delta_{x_n}/2\}$, and then for all $x, y \in X$ with $d(x, y) < \delta$, we have $d(x, x_i) < \delta_{x_i}/2$ and $d(x_i, y) \leq d(x_i, x) + d(x, y) < \delta_{x_i}/2 + \delta \leq \delta_{x_i}$ for some $i \in [1, n] \cap \mathbb{Z}$. So $\rho(f(x), f(y)) \leq \rho(f(x), f(x_i)) + \rho(f(x_i), f(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This proves the theorem for that ε is actually arbitrary. □

9. Arzelà-Ascoli Theorem: 4/4/2018

The wind was picking up. Perhaps it's the weeping from the Tomb-Sweeping Day.

Theorem 9.1. *The continuous image of a compact set is compact.*

Proof. Suppose $f : X \to Y$ is a continuous mapping between topological spaces and X is compact. Say $\{V_i : i \in I\}$ is an arbitrary open cover of $f(X)$, then $\{f^{-1}(V_i) : i \in I\}$ is an open cover of X.

By compactness of X, there are finitely many $f^{-1}(V_i)$'s, say

$$f^{-1}(V_{n_1}), f^{-1}(V_{n_2}), \cdots, f^{-1}(V_{n_k}),$$

whose union contains X. The union of $V_{n_1}, V_{n_2}, \cdots, V_{n_k}$ contains $f(X)$ since $X \subset \cup_{i=1}^k f^{-1}(V_{n_i}) \Rightarrow f(X) \subset f(\cup_{i=1}^k f^{-1}(V_{n_i})) = \cup_{i=1}^k f(f^{-1}(V_{n_i})) \subset \cup_{i=1}^k V_{n_i}$. Hence, $f(X)$ is compact. □

Let K be a compact subset of a metric space, say, X. By Theorem 9.1, it follows that $d(x, y) = \max_{t \in K} |x(t) - y(t)|$ defined for all $x, y \in C(K)$ is indeed a metric on $C(K)$, which is called **the standard metric on** $C(K)$. Note that elements in $C(K)$ are all real-valued. $C(K)$ sometimes would be denoted as $C(X, \mathbb{R})$ to avoid confusion.

In this section, we will always use $C(K)$ to denote the same stuff as proposed here.

Theorem 9.2. *If $C(K)$ is equipped with the standard metric, then $C(K)$ is complete.*

Proof. Suppose $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in $C(K)$. Then $\{x(t)\}_{n=1}^\infty \subset \mathbb{R}$ converges to some unique, say, $x_t \in \mathbb{R}$ for each $t \in K$. Let $x(t) := x_t$ for each $t \in K$.

We first claim that $x \in C(K)$. In fact, for all $t_0 \in K$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|x_n(t) - x_n(t_0)| < \varepsilon/3$ when $t \in K$ and $d'(t, t_0) < \delta$ where d' is the metric on K. For each t with $d'(t, t_0) < \delta$, there is $K_t > 0$ such that $|x(t) - x_n(t)|, |x_n(t_0) -$

$x(t_0)| < \varepsilon/3$ when $n > K_t$. Then $|x(t) - x(t_0)| \le |x(t) - x_n(t)| + |x_n(t) - x_n(t_0)| + |x_n(t_0) - x(t_0)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ when $d'(t, t_0) < \delta$ and $n > K_t$. It follows that $|x(t) - x(t_0)| < \varepsilon$ whenever $t \in K$ and $d'(t, t_0) < \delta$. So x is continuous at t_0 and then continuous on K since $t_0 \in K$ is arbitrary. Hence, $x \in C(K)$.

The second claim goes that $x_n \to x$ $(n \to \infty)$. In fact, for all $\varepsilon > 0$, there exists $m \in \mathbb{N}_+$ such that $d(x_m, x_n) < \varepsilon/2$ when $n > m$. Then for each $t \in K$, $|x_m(t) - x_n(t)| < \varepsilon/2$ when $n > m$. Send $n \to \infty$ and we derive $|x_m(t) - x(t)| \le \varepsilon/2$ for all $t \in K$. Hence, $d(x_m, x) = \max_{t \in K} |x_m(t) - x(t)| \le \varepsilon/2$. So when $n > m$, $d(x_n, x) \le d(x_n, x_m) + d(x_m, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. It follows that $x_n \to x$ $(n \to \infty)$. $\square$

Theorem 9.3. ***Arzelà-Ascoli Theorem***

If $M \subset C(K)$ is infinite, then M is relatively compact iff

- *M is **uniformly bounded**, that is, there exists $m > 0$ such that for all $f \in M$, $|f(x)| \le m$ whenever $x \in K$; and*
- *M is **uniformly equicontinuous**, that is, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$, for all $f \in M$ whenever $x_1, x_2 \in K$ and $d(x_1, x_2) < \delta$.*

Proof. $\Leftarrow$) Based on diagonalization argument, let $\{x_n\}_{n=1}^{\infty}$ be an enumeration of rational numbers in K. Since M is uniformly bounded, the set of points $\{f(x_1)\}_{f \in M}$ is bounded, and hence by Lemma 8.2, there is a sequence $\{f_{n_1}\}$ such that $\{f_{n_1}(x_1)\}$ converges. Repeating the same argument for the sequence $\{f_{n_1}(x_2)\}$, there is a subsequence $\{f_{n_2}\} \subset \{f_{n_1}\}$ such that $\{f_{n_2}(x_2)\}$ converges.

By induction this process can be continued forever, and so there is a chain of subsequences $\{f_{n_1}\} \supset \{f_{n_2}\} \supset \cdots$ such that $\{f_{n_k}\}$ converges at $x_1, x_2, \cdots, x_k$ for each $k \in \mathbb{N}_+$. Now form the diagonal subsequence $\{f_m\}$ whose m-th term is the m-th term in the m-th subsequence $\{f_{n_m}\}$. By construction, f_m converges at every rational point of K.

Hence given any $\varepsilon > 0$ and rational number $x_k \in K$, there is an integer $N(\varepsilon, x_k) > 0$ such that $|f_n(x_k) - f_m(x_k)| < \varepsilon/3$ when $n, m > N$.

Since the family $C(K)$ is equicontinuous, for this fixed ε and for every x in K, there is an open interval U_x containing x such that $|f(s) - f(t)| < \varepsilon/3$ for all $f \in F$ and $s, t \in K \cap U_x$.

The collection of intervals $\{U_x\}_{x \in K}$ forms an open cover of K. Since K is compact, this covering admits a finite subcover, say, $U_1, U_2, \cdots, U_J$. There exists an integer M such that each open interval U_j $(1 \le j \le J)$ contains a rational x_k with $1 \le k \le M$.

Finally, for any $t \in K$, there are j and then k such that t and x_k belong to the same interval U_j. For the choice of k, we have

$$\begin{aligned}|f_n(t) - f_m(t)| &\le |f_n(t) - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(t)| \\ &\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon\end{aligned}$$

for all $n, m > N = \max_{1\le k\le M} N(\varepsilon, x_k)$. Consequently, the sequence $\{f_m\}$ is a Cauchy sequence in M, and therefore converges to a continuous function in $\bar{M}$.

$\Rightarrow$) Since $\bar{M}$ is compact, then $\bar{M}$ is bounded and so is M. We claim that M is uniformly bounded. In fact, by Definition 4.1, there exist $g \in C(K)$ and $r > 0$ such that $|f(x) - g(x)| \le r$ for all $f \in M$ and $x \in K$. Since K is compact and g is continuous, there is $L > 0$ such that $|g(x)| \le L$ for all $x \in K$. Hence, $|f(x)| \le |f(x) - g(x)| + |g(x)| \le r + L$ when $f \in M$ and $x \in K$. So M is uniformly bounded.

Suppose M is not uniformly equicontinuous, then there is some $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}_+$, there exist $x_n, y_n \in K$ and $f_n \in M$ with $|x_n - y_n| < 1/n$ satisfying $|f_n(x_n) - f_n(y_n)| \ge \varepsilon_0$. The claim is that $\{f_n\}_{n=1}^{\infty}$ has no convergent subsequence in $\bar{M}$. In fact, we may assume that there is $f \in \bar{M}$ such that $f_n \to f$ $(n \to \infty)$. For the above $\varepsilon_0 > 0$, there is $N > 0$ such that $\max_{x\in K} |f_n(x) - f(x)| < \varepsilon_0/3$ when $n > N$. So $|f_n(x_n) - f(x_n)| < \varepsilon_0/3$ and $|f_n(y_n) - f(y_n)| < \varepsilon_0/3$ whenever $n > N$. Note that f is a continuous function on a compact set K. Then f is uniformly continuous on K and there exists $\delta_0 > 0$ such that $|f(x) - f(y)| < \varepsilon_0/3$ whenever $x, y \in K$ with $|x - y| < \delta_0$. Thus, take a positive integer $n_0 > N$ such that $n_0 > 1/\delta_0$. Now we see that

$$\begin{aligned}|f_{n_0}(x_{n_0}) - f_{n_0}(y_{n_0})| &\le |f_{n_0}(x_{n_0}) - f(x_{n_0})| + |f(x_{n_0}) - f(y_{n_0})| \\ &\quad + |f(y_{n_0}) - f_{n_0}(y_{n_0})| < \varepsilon_0/3 + \varepsilon_0/3 + \varepsilon_0/3 = \varepsilon_0\end{aligned}$$

since $|x_{n_0} - y_{n_0}| < 1/n_0 < \delta_0$. But $|f_{n_0}(x_{n_0}) - f_{n_0}(y_{n_0})| \ge \varepsilon_0$, which is a contradiction. Thus, M is uniformly equicontinuous. $\square$

Remark 9.4. *Note that if a real-valued function f on K is bounded, then there exists $m_f > 0$ such that $|f(x)| \le m_f$ whenever $x \in K$, which indicates that the bound of f on K actually depends on f in general. This is where boundedness varies from uniform boundedness.*

Example 9.5. *Let $M = \{f(: [a,b] \to \mathbb{R}) : f \in C^1[a,b], |f(a)| \le K_1, \int_a^b |f'(t)|^2 \mathrm{d}t \le K_2\}$, where K_1, K_2 are positive constants. Then M is relatively compact in $C[a,b]$.*

In fact, for all $f \in M$ and $x \in [a,b]$, we have

$$\begin{aligned}|f(x)| &= |f(x) - f(a) + f(a)| \le |f(x) - f(a)| + |f(a)| \\ &\le \left|\int_a^x f'(t)\mathrm{d}t\right| + |f(a)| \le \int_a^b |f'(t)|\mathrm{d}t + |f(a)| \\ &\le \sqrt{b-a}\left(\int_a^b |f'(t)|^2\mathrm{d}t\right)^{1/2} + |f(a)| \\ &\le \sqrt{(b-a)K_2} + K_1,\end{aligned}$$

then M is uniformly bounded.

And for all $\varepsilon > 0$, choose $\delta = \frac{\varepsilon^2}{K_2} > 0$, then for all $x_1, x_2 \in [a,b]$, $x_1 \le x_2$, and $x_2 - x_1 < \delta$ we have

$$|f(x_2) - f(x_1)| = \left|\int_{x_1}^{x_2} f'(t)\mathrm{d}t\right| \le \int_{x_1}^{x_2} |f'(t)|\mathrm{d}t \le \sqrt{x_2 - x_1}\left(\int_a^b |f'(t)|^2\mathrm{d}t\right)^{1/2}$$
$$\le \sqrt{(x_2 - x_1)K_2} < \sqrt{\varepsilon^2/K_2 \cdot K_2} = \varepsilon.$$

Thus, M is uniformly equicontinuous.

Now we see that M is relatively compact in $C[a,b]$ by Theorem 9.3.

Example 9.6. *Let $M = \{f(: [a,b] \to \mathbb{R}) : f \in C^1[a,b], |f'(x)| \le L\ (x \in [a,b]), f(x) = 0$ has at least one solution on $[a,b]\}$, where L is a positive constant. Then M is relatively compact in $C[a,b]$.*

Indeed, for all $f \in M$ and $x \in [a,b]$, there exists $x_0 \in [a,b]$ such that $f(x_0) = 0$ and $|f(x)| = |f(x) - f(x_0)| = |f'(\xi_x)||x - x_0| \le L(b-a)$, where ξ_x is some point on the line segment with x_0, x as endpoints. Hence, M is uniformly bounded on $[a,b]$.

Also, for all $\varepsilon > 0$, choose $\delta = \varepsilon/L$, then for all $x_1, x_2 \in [a,b]$ with $|x_1 - x_2| < \delta$, $|f(x_1) - f(x_2)| = |f'(\eta)||x_1 - x_2| \le L|x_1 - x_2| < \varepsilon$, where η is some point on the line segment with x_1, x_2 as endpoints. Thus, M is uniformly equicontinuous.

Now we see that M is relatively compact in $C[a,b]$ by Theorem 9.3.

Example 9.7. *$M_1 = \{f_n(: [0,1] \to \mathbb{R}) : f_n(x) = x^n, n \in \mathbb{N}_+\}$ is not relatively compact in $C[0,1]$ nor is $M_2 = \{f_n(: [0,1] \to \mathbb{R}) : f_n(x) = \sin(nx)\ (x \in [0,1]), n \in \mathbb{N}_+\}$.*

In fact, M_1 is not uniformly equicontinuous. Otherwise, for all $\varepsilon > 0$ there exists $\delta > 0$ independent of x and n such that $|x_1^n - x_2^n| < \varepsilon$ when $x_1, x_2 \in [0,1]$ and $|x_1 - x_2| < \delta$. Choose $N > 0$ which is sufficiently large such that $1/N < \delta$. Let $x_1 = 1 - 1/n$ and $x_2 = 1$, then $|x_1 - x_2| = 1/n < \delta$ when $n > N$. So $|(1-1/n)^n - 1| < \varepsilon\ (n > N) \Rightarrow |1/\mathrm{e} - 1| \le \varepsilon$, which is a contradiction since ε is arbitrary.

M_2 is not uniformly equicontinuous as well. In fact, for every $\delta > 0$, choose $n > 0$ sufficiently large such that $\pi/(3n) < \min\{\delta, 1\}$, then select $x \in (\pi/(4n), \pi/(3n))(\subset [0,1])$. We have

$$|\sin(nx) - \sin(n \cdot 0)| = nx\cos(n\xi) \ge nx\cos(\pi/3) \ge nx/2 \ge \pi/8,$$

where ξ is some point on the line segment with $0, x$ as endpoints, while $|x - 0| < \delta$.

Remark 9.8. *Note that $\{f_n\}_{n=1}^\infty (= M_2)$ is not a Cauchy sequence in $C[0,1]$. In fact, choose an integer $m > \max\{n, \log_{3/4}(1/4)^n\}$ for each $n \in \mathbb{N}_+$. Let $x = (3/4)^{1/n}$, then*

$$x^n - x^m = 3/4 - (3/4)^{m/n} \ge 3/4 - (3/4)^{\frac{1}{n}\log_{3/4}(1/4)^n} = 3/4 - 1/4 = 1/2.$$

Or equivalently, if $\{f_n\}_{n=1}^\infty$ converges in $C[0,1]$, then $\lim_{n\to\infty} f_n$ is a continuous function on $[0,1]$. But this obviously fails.

In mathematics, we are frequently encountered with solving equations in the form of $Ax = B$ for x in some universal domain X. When addition and subtraction are defined, the equation is equivalent to $Ax - B + x = x \Leftrightarrow Tx = x$, where $T : X \to X$ is defined by $Tx = Ax - B + x$ for all $x \in X$. Then each solution to the equation would be a **fixed-point** of T. It's a warm-up for the next session.

10. The Contraction Mapping Principle: 4/8/2018

This session was shifted from 4/6/2018 because of the Tomb-Sweeping Day. I didn't attend the class because I went to the travel agency for my Thailand visa. These notes were provided by Wenyi Cai.

Definition 10.1. *Let (X, d) be a metric space and let $T : X \to X$ be a mapping. T is said to be a* ***contraction*** *if there is some $c \in [0, 1)$ such that for all $x, y \in X$, $d(Tx, Ty) \le cd(x, y)$.*

Remark 10.2. *Contractions on a metric space are surely uniformly continuous. This can be derived immediately from its definition.*

Theorem 10.3. ***Banach Fixed-Point Theorem***

Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction. Then T has a unique fixed-point, i.e., $Tx = x$ has exactly one solution on X.

Proof. Choose an arbitrary point $x_0 \in X$ and let $\{x_n\}_{n=0}^{\infty}$ defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}_+$. Compute that

- $d(x_2, x_1) = d(Tx_1, Tx_0) \le cd(x_1, x_0)$,
- $d(x_3, x_2) = d(Tx_2, Tx_1) \le cd(x_2, x_1) \le c^2 d(x_1, x_0)$,
- $d(x_4, x_3) = d(Tx_3, Tx_2) \le cd(x_3, x_2) \le c^2 d(x_2, x_1) \le c^3 d(x_1, x_0)$,
- $\cdots\cdots$
- $d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \le cd(x_n, x_{n-1}) \le \cdots \le c^n d(x_1, x_0)$,
- $\cdots\cdots$

For all $p > 1$, $d(x_{n+p}, x_n) \le d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \cdots + d(x_{n+1}, x_n) \le (c^{n+p-1} + c^{n+p-2} + \cdots + c^n) d(x_1, x_0) = \frac{c^n(1-c^p)}{1-c} d(x_1, x_0) \le \frac{c^n}{1-c} d(x_1, x_0)$. Since $0 \le c < 1$, we derive $d(x_{n+p}, x_n) \to 0$ as $n \to \infty$, which means $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Then there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$ since X is complete. Note that T is continuous. We obtain $Tx = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x$.

If $y \in X$ satisfies $Ty = y$, then $d(y, x) = d(Ty, Tx) \le cd(y, x) < d(y, x)$. It follows that $d(y, x) = 0$ and so $y = x$. □

Remark 10.4. *From the proof of Theorem 10.3, we see that for each point $x \in X$, $\lim_{n\to\infty} T^n x$ is always the unique fixed-point of T. So upon computing the unique fixed-point, it would make things easier if x_0 is chosen to be as simple as possible when the metric space X is some complicated one.*

Example 10.5. *Let $f : [0, 1] \to [0, 1]$ be continuous and then f has a fixed-point.*

Proof. If $f(0) = 0$ or $f(1) = 1$, then the claim is correct. Otherwise, if $f(0) > 0$ and $f(1) < 1$, set $F(x) = x - f(x)$. Since $F(0) < 0$ and $F(1) > 0$, there exists $\xi \in (0, 1)$ such that $F(\xi) = 0$, i.e., $f(\xi) = \xi$. $\square$

Example 10.6. *The equation $x^3 + 4x - 2 = 0$ has a unique real root on $[0, 1]$.*

Proof. Indeed, define $T : [0, 1] \to [0, 1]$ by $Tx = \frac{2-x^3}{4}$ for all $x \in [0, 1]$. Firstly, $[0, 1]$ is complete as a subspace of $\mathbb{R}$. For all $x, y \in [0, 1]$, we have $d(Tx, Ty) = |Tx - Ty| = \frac{1}{4}|x^3 - y^3| = \frac{1}{4}|x - y||x^2 + xy + y^2| \le \frac{3}{4}|x - y| = \frac{3}{4}d(x, y)$, which means that T is a contraction with $c = \frac{3}{4}$. By Theorem 10.3, $Tx = x$ has a unique solution $\xi \in [0, 1]$, that is, the unique real root of $x^3 + 4x - 2 = 0$ on $[0, 1]$. $\square$

Remark 10.7. *The idea of the proof is constructing a contraction on a complete metric space where we can apply Theorem 10.3. In addition, by Theorem 7.10 and Theorem 10.3, $|x_n - \xi| \le \frac{(3/4)^n}{1-3/4}|x_1 - x_0| = 2 \cdot (3/4)^n$ where $x_0 = 0$. This is the estimation of the absolute error between the n-th iterated point and the actual root.*

Theorem 10.8. ***Cauchy Initial Value Problem***

Let $f(x, t)$ be continuous on $\mathbb{R}^2$, satisfying $|f(x_1, t) - f(x_2, t)| \le L|x_1 - x_2|$ for all $x_1, x_2, t \in \mathbb{R}$ where L is a positive constant. The Cauchy initial value problem $\begin{cases} \dfrac{dx}{dt} = f(x, t) \\ x(t_0) = x_0 \end{cases}$ has a unique continuous solution near t_0.

Proof. In fact, the Cauchy initial value problem is equivalent to $x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s$. Define $Tx(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s$. Clearly, $T : C[t_0 - \delta, t_0 + \delta] \to C[t_0 - \delta, t_0 + \delta]$ where δ is a positive constant to be chosen. Now for all $x_1, x_2 \in C[t_0 - \delta, t_0 + \delta]$,

$$\begin{aligned} d(Tx_1, Tx_2) &= \max_{|t-t_0|\le\delta} |Tx_1(t) - Tx_2(t)| \\ &= \max_{|t-t_0|\le\delta} \left| \int_{t_0}^{t} [f(x_1(s), s) - f(x_2(s), s)]\mathrm{d}s \right| \\ &\le \max_{|t-t_0|\le\delta} \int_{t_0}^{t} |f(x_1(s), s) - f(x_2(s), s)|\mathrm{d}s \\ &\le \max_{|t-t_0|\le\delta} L \int_{t_0}^{t} |x_1(s) - x_2(s)|\mathrm{d}s \\ &\le L\delta \max_{|t-t_0|\le\delta} |x_1(t) - x_2(t)| = L\delta d(x_1, x_2), \end{aligned}$$

that is, $d(Tx_1, Tx_2) \le L\delta d(x_1, x_2)$.

Choose δ such that $L\delta < 1$, and then T is a contraction with $c = L\delta$. Then by Theorem 10.3, the Cauchy initial value problem has a unique solution x since $C[t_0 - \delta, t_0 + \delta]$ is complete. $\square$

Remark 10.9. *Just as proposed in Remark 10.4, we can always choose* x_0 *whose form is as simple as possible, then the solution to the Cauchy initial value problem would be obtained by* $\lim_{n\to\infty} T^n x_0$*. Solving an ODE like this is sometimes called the* ***Picard iterative process****.*

Theorem 10.10. ***Implicit Function Theorem***

Let f *be continuous on* $E = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b\}$*. Suppose that* $f_y(x, y)$ *exists for all* $(x, y) \in E$*, and there exist constants* m, M *such that* $0 < m < M$ *satisfying* $m \leq f_y(x, y) \leq M$ *for all* $(x, y) \in E$*. Then the equation* $f(x, y) = 0$ *has a unique solution* $y = g(x)$ *which is continuous on* $[a, b]$*.*

Proof. Consider $T : C[a, b] \to C[a, b]$ defined by $T\varphi(x) = \varphi(x) - \frac{1}{M} f(x, \varphi(x))$ for all $x \in [a, b]$ and $\varphi \in C[a, b]$, then T is a contraction on $C[a, b]$ since

$$\begin{aligned}
|T\varphi_1(x) - T\varphi_2(x)| &= \left|\varphi_1(x) - \varphi_2(x) - \frac{1}{M} f(x, \varphi_1(x)) + \frac{1}{M} f(x, \varphi_2(x))\right| \\
&= \left|\varphi_1(x) - \varphi_2(x) - \frac{1}{M} f_y(x, \varphi_2(x) + \theta(\varphi_1(x) - \varphi_2(x)))(\varphi_1(x) - \varphi_2(x))\right| \\
&\leq (1 - m/M)|\varphi_1(x) - \varphi_2(x)|
\end{aligned}$$

for all $x \in [a, b]$, which implies that $d(T\varphi_1, T\varphi_2) \leq cd(\varphi_1, \varphi_2)$ with $c = 1 - m/M \in [0, 1)$. Since $C[a, b]$ is complete, by Theorem 10.3, T has a unique fixed-point in $C[a, b]$, say g, such that $f(x, g(x)) \equiv 0$ on $[a, b]$. Hence the proof. ☐

Theorem 10.11. *Let* (X, d) *be complete and* $T : X \to X$*. If* $T^{n_0} = \underbrace{T \circ T \circ \cdots \circ T}_{n_0}$ *is a contraction for some* $n_0 \geq 1$*, then* $Tx = x$ *has a unique solution.*

Proof. By Theorem 10.3, T^{n_0} has a unique fixed-point, say x. Note that $T^{n_0}(Tx) = T(T^{n_0}x) = Tx$, which indicates Tx is also a fixed-point of T^{n_0}. By uniqueness of the fixed-point, we get $Tx = x$.

If y is a fixed-point of T, then $T^{n_0}y = T^{n_0-1}(Ty) = T^{n_0-1}y = \cdots = Ty = y$, which gives that y is a fixed-point of T^{n_0}. So y has to be equal to x. ☐

Theorem 10.12. ***Volterra Integral Equation***

For each $\lambda \in \mathbb{R}$*, the equation* $x(t) = \varphi(t) + \lambda \int_a^t K(t, s)x(s)\mathrm{d}s$ *has a unique continuous solution* $x = x(t)$ *on* $[a, b]$ *where* φ *is continuous on* $[a, b]$ *and* K *is continuous on* $[a, b] \times [a, b]$*.*

Proof. Let $T : C[a, b] \to C[a, b]$ defined by $Tx(t) = \varphi(t) + \lambda \int_a^t K(t, s)x(s)\mathrm{d}s$ for all $t \in [a, b]$. Then for all $x_1, x_2 \in C[a, b]$,

$$\begin{aligned}
|Tx_1(t) - Tx_2(t)| &\leq |\lambda| \int_a^t |K(t, s)||x_1(s) - x_2(s)|\mathrm{d}s \\
&\leq |\lambda| M \int_a^t |x_1(s) - x_2(s)|\mathrm{d}s \leq |\lambda| M(t - a)d(x_1, x_2)
\end{aligned}$$

where $M = \max_{(t,s)\in[a,b]\times[a,b]} |K(t,s)|$. Then

$$\begin{aligned}|T^2x_1(t) - T^2x_2(t)| &\le |\lambda| M \int_a^t |Tx_1(s) - Tx_2(s)|\mathrm{d}s \\ &\le |\lambda|^2 M^2 d(x_1,x_2) \int_a^t (s-a)\mathrm{d}s \le \lambda^2 M^2 \frac{(t-a)^2}{2} d(x_1,x_2).\end{aligned}$$

Perform the computation inductively and we can find that $|T^n x_1(t) - T^n x_2(t)| \le \lambda^n M^n \frac{(t-a)^n}{n!} d(x_1,x_2)$ for all $x \in [a,b]$ and $n \in \mathbb{N}_+$. Since $\lambda^n M^n \frac{(t-a)^n}{n!} \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}_+$ such that T^{n_0} is a contraction. Thus by Theorem 10.11, T has a unique fixed-point in $C[a,b]$, which is the unique continuous solution to the integral equation. □

11. Normed Linear Space: 4/11/2018

I went to Thailand that day. These notes were provided by Wenyi Cai.

Definition 11.1. *Let X be a linear space. A **norm** on X is a function $\|\bullet\| : X \to \mathbb{R}$ such that for all $x, y \in X$ and $\alpha \in \mathbb{F}$,*

- (N1) $\|x\| \ge 0$ *and* $\|x\| = 0$ *iff* $x = 0$;
- (N2) *Absolute homogeneity:* $\|\alpha x\| = |\alpha| \cdot \|x\|$;
- (N3) *Triangle inequality:* $\|x + y\| \le \|x\| + \|y\|$.

*X together with a norm is called a **normed linear space**, denoted $(X, \|\bullet\|)$.*

Theorem 11.2. *Let (X, d) be a normed linear space, then $\|\bullet\|$ automatically gives a metric d on X defined by $d(x,y) = \|x - y\|$ for all $x, y \in X$.*

Proof. We show that d satisfies (M1), (M2), and (M3) in Definition 2.1:

- (M1) $d(x,y) = \|x-y\| \ge 0$ and $d(x,y) = \|x-y\| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$;
- (M2) $d(x,y) = \|x-y\| = \|(-1)(y-x)\| = |-1| \cdot \|y-x\| = d(y,x)$;
- (M3) $d(x,z) = \|x-z\| = \|(x-y)+(y-z)\| \le \|x-y\| + \|y-z\| = d(x,y) + d(y,z)$.

Hence, d satisfies the axioms of metric and then (X,d) is a metric space. □

Remark 11.3. *If (X,d) is a metric space, can we define a norm $\|\bullet\|$ on X such that d is induced by the norm like showed above?*

However, it is not generally true even the metric space is a vector space. As counterexamples, consider the space s and S defined in Example 2.3.

Remark 11.4. *Let (X,d) be a vector space, and if $\begin{cases} d(x-y,0) = d(x,y) \\ d(\alpha x, 0) = |\alpha| d(x,0) \end{cases}$, then there is a norm $\|\bullet\|$ on X defined by $\|x\| = d(x,0)$, which induces the metric d on X. Some examples are as follows:*

- $\mathbb{F}^N$: $\|x\| = d(x,0) = \sqrt{\sum_{i=1}^N |x_i|^2}$;
- $C[a,b]$: $\|x\| = d(x,0) = \max_{t\in[a,b]} |x(t)|$;

- ℓ^∞: $\|x\| = d(x,0) = \sup_{i\geq 1} |\xi_i|$.

Remark 11.5. *In a normed linear space $(X, \|\bullet\|)$, for all $\{x_n\}_{n=1}^\infty \subset X$ and $x \in X$, as $n \to \infty$, we have $x_n \to x \Leftrightarrow \|x_n - x\| \to 0 \Rightarrow \|x_n\| \to \|x\|$ by (N3) in Definition 11.1, which indicates that $\|\bullet\|$ is continuous.*

Definition 11.6. *$(X, \|\bullet\|)$ is said to be a Banach space if $(X, \|\bullet\|)$ is complete, i.e., $\|x_n - x_m\| \to 0$ $(n, m \to \infty)$ implies $\{x_n\}_{n=1}^\infty$ is convergent for each $\{x_n\}_{n=1}^\infty \subset X$.*

Definition 11.7. *Let $(X, \|\bullet\|)$ be a normed linear space. A series $\sum_{n=1}^\infty x_n = x_1 + x_2 + \cdots$ ($x_n \in X$ for all $n \in \mathbb{N}_+$) **converges** to $s \in X$ if the partial sum sequence $\{s_n\}_{n=1}^\infty$ of the series defined by $s_n = x_1 + x_2 + \cdots + x_n$ for each $n \in \mathbb{N}_+$ converges to s as $n \to \infty$, i.e., $\|s_n - s\| \to 0$ as $n \to \infty$, denoted $\sum_{n=1}^\infty x_n = s$. A series $\sum_{n=1}^\infty x_n$ is **absolutely convergent** if the real series $\sum_{n=1}^\infty \|x_n\|$ is convergent.*

Remark 11.8. *Note that in a normed linear space, an absolutely convergent sequence does not converge in general. We give two counterexamples here.*

- *Let $x_1 = 1$ and $x_n = 2^{1-n}$ for each $n \geq 2$. Then $\sum_{n=1}^\infty x_n$ converges to 0 in $\mathbb{R}$. So $\sum_{n=1}^\infty x_n$ is not convergent in $\mathbb{R} \setminus \{0\}$. But $\sum_{n=1}^\infty \|x_n\|$ converges to 2.*
- *Let $c_{00} \subset \ell^\infty$ be the subspace consisting of all sequences with only finitely many nonzero terms. For each $n \in \mathbb{N}_+$, let e_n be the sequence consisting of zeros but a one as the n-th term. Consider the sequence $\{x_n = e_n/n^2\}_{n=1}^\infty$. Clearly, $\sum_{n=1}^\infty x_n$ is not convergent in c_{00}. But $\sum_{n=1}^\infty \|x_n\| = \sum_{n=1}^\infty 1/n^2$ converges as a real series.*

Note that in the above two counterexamples, $\mathbb{R} \setminus \{0\}$ and c_{00} are all incomplete spaces.

Theorem 11.9. *$(X, \|\bullet\|)$ is a Banach space iff all absolutely convergent series in X are convergent.*

Proof. $\Rightarrow$) If $\sum_{n=1}^\infty \|x_n\|$ is convergent, then for all $k, m \in \mathbb{N}_+$ with $k > m$, $\|s_k - s_m\| = \|x_{m+1} + x_{m+2} + \cdots + x_k\| \leq \|x_{m+1}\| + \|x_{m+2}\| + \cdots + \|x_k\| \to 0$ $(m \to \infty)$, where $\{s_n\}_{n=1}^\infty$ is the partial sum sequence of $\sum_{n=1}^\infty x_n$. Hence, $\{s_n\}_{n=1}^\infty$ is a Cauchy sequence. So it is convergent since X is complete, i.e., $\sum_{n=1}^\infty x_n$ is convergent.

$\Leftarrow$) Suppose that every absolutely convergent series is convergent. For all Cauchy sequence $\{x_n\}_{n=1}^\infty \subset (X, \|\bullet\|)$, we have $\|x_n - x_m\| \to 0$ $(n, m \to \infty)$. Choose integers $n_1 < n_2 < \cdots$ such that $\|x_{n_{k+1}} - x_{n_k}\| < 1/2^k$ for all $k \in \mathbb{N}_+$. Hence, $\sum_{k=1}^\infty \|x_{n_{k+1}} - x_{n_k}\| < \infty$. By assumption, $\sum_{k=1}^\infty (x_{n_{k+1}} - x_{n_k})$ converges, which means the partial sum $\sum_{k=1}^j (x_{n_{k+1}} - x_{n_k}) = x_{n_{j+1}} - x_{n_1}$ converges as $j \to \infty$. Then $\{x_{n_k}\}_{k=1}^\infty$ is convergent. In all, the Cauchy sequence $\{x_n\}_{n=1}^\infty$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$. So the whole sequence $\{x_n\}_{n=1}^\infty$ converges. Thus, X is complete. $\square$

12. L^p Space: 4/13/2018

I was in Bangkok that day. These notes were provided by Wenyi Cai and Jiaqi Li.

Example 12.1. $L^p(E)$ $(1 \le p < \infty)$

Let $L^p(E) = \{x(: E \to \mathbb{F}) : \int_E |x(t)|^p \mathrm{d}t < \infty\}$ *where* E *is a measurable set of* $\mathbb{R}^n$. *For all* $x, y \in L^p(E)$ *and* $\alpha \in \mathbb{F}$, *define* $(x+y)(t) = x(t) + y(t)$ *and* $(\alpha x)(t) = \alpha x(t)$. *Now* $L^p(E)$ *is a linear space. Then* $\| \bullet \|$ *defined by* $\|x\|_{L^p} = (\int_E |x(t)|^p \mathrm{d}t)^{1/p}$ *for all* $x \in L^p(E)$ *is a norm on* $L^p(E)$, *which is easy to be verified. So* $L^p(E)$ *is a normed linear space.*

Lemma 12.2. ***Levi's Monotone Convergence Theorem***

Suppose that $\{f_n\}_{n=1}^{\infty}$ *is a nondecreasing sequence of nonnegative measurable functions on* E. *If* $\lim_{n\to\infty} f_n = f$, *then*

$$\lim_{n\to\infty} \int_E f_n(x)\mathrm{d}x = \int_E f(x)\mathrm{d}x \left(= \int_E \lim_{n\to\infty} f_n(x)\mathrm{d}x \right).$$

Lemma 12.3. ***Lebesgue's Dominated Convergence Theorem***

Suppose that $\{f_n\}_{n=1}^{\infty} \subset L(E)$ *is a sequence of measurable functions on* E *with* $m(E) < \infty$. *If* $f_n \xrightarrow{a.e.} f$ $(n \to \infty)$ *on* E *and there exists* $g \in L(E)$ *such that* $|f_n| \le g$ *a.e. on* E *for all* $n \in \mathbb{N}_+$, *then*

$$\lim_{n\to\infty} \int_E f_n(x)\mathrm{d}x = \int_E f(x)\mathrm{d}x \left(= \int_E \lim_{n\to\infty} f_n(x)\mathrm{d}x \right).$$

Remark 12.4. *Pay attention to the difference between Lemma 12.2 and Lemma 12.3.*

Lemma 12.5. ***Fatou's Lemma***

Suppose that $\{f_n\}_{n=1}^{\infty}$ *is sequence of nonnegative measurable functions on* E, *then*

$$\int_E \liminf_{n\to\infty} f_n(x)\mathrm{d}x \le \liminf_{n\to\infty} \int_E f_n(x)\mathrm{d}x.$$

Theorem 12.6. $L^p(E)$ *is a Banach space.*

Proof. For all $\{x_n\}_{n=1}^{\infty} \subset L^p(E)$ with $\sum_{n=1}^{\infty} \|x_n\|_{L^p} < \infty$, there exists $N \in \mathbb{N}_+$ for all $\varepsilon > 0$ such that $\sum_{k=m+1}^{\infty} \|x_k\|_{L^p} < \varepsilon$ when $m > N$.

Let $E_1 = B(0,1) \cap E$, we see that $m(E_1) < \infty$. When $p = 1$, $\int_{E_1} |x_k(t)|\mathrm{d}t \le \|x_k\|_{L^p}$. When $p > 1$, $\int_{E_1} |x_k(t)|\mathrm{d}t \le (\int_{E_1} |x_k(t)|^p \mathrm{d}t)^{1/p} (\int_{E_1} \mathrm{d}t)^{1/p'} \le (m(E_1))^{1/p'} \|x_k\|_{L^p}$ where $1/p + 1/p' = 1$. Take $c = \max\{1, (m(E_1))^{1/p'}\} > 0$. Then by Lemma 12.2, we have $\int_{E_1} \sum_{k=1}^{\infty} |x_k(t)|\mathrm{d}t = \sum_{k=1}^{\infty} \int_{E_1} |x_k(t)|\mathrm{d}t \le c \sum_{k=1}^{\infty} \|x_k\|_{L^p} < \infty$.

Now we see that $\sum_{k=1}^{\infty} |x_k(t)| < \infty$ a.e. on E_1. Similarly, $\sum_{k=1}^{\infty} |x_k(t)| < \infty$ a.e. on $E_n = B(0,n) \cap E$ for all $n \in \mathbb{N}_+$. Then $\sum_{k=1}^{\infty} |x_k(t)| < \infty$ a.e. on $E = \cup_{n=1}^{\infty} E_n$. So $\sum_{k=1}^{\infty} x_k(t) < \infty$ a.e. on E and there exists a function S such that $S_m(t) := \sum_{k=1}^{m} x_k(t) \to S(t)$ a.e. on E as $m \to \infty$, that is, $S(t) = \sum_{k=1}^{\infty} x_k(t)$ a.e. on E.

When $m > N$,

$$\begin{aligned}
\|S - S_m\|_{L^p} &= \left(\int_E |S(t) - S_m(t)|^p \mathrm{d}t\right)^{1/p} = \left(\int_E \left|\sum_{k=1}^{\infty} x_k(t) - \sum_{k=1}^{m} x_k(t)\right|^p \mathrm{d}t\right)^{1/p} \\
&= \left(\int_E \lim_{n\to\infty} \left|\sum_{k=1}^{n} x_k(t) - \sum_{k=1}^{m} x_k(t)\right|^p \mathrm{d}t\right)^{1/p} = \left(\int_E \lim_{n\to\infty} \left|\sum_{k=m+1}^{n} x_k(t)\right|^p \mathrm{d}t\right)^{1/p} \\
&\overset{\text{Lemma 12.5}}{\leq} \left(\liminf_{n\to\infty} \int_E \left|\sum_{k=m+1}^{n} x_k(t)\right|^p \mathrm{d}t\right)^{1/p} = \left(\liminf_{n\to\infty} \left\|\sum_{k=m+1}^{n} x_k\right\|_{L^p}^p\right)^{1/p} \\
&\leq \left(\liminf_{n\to\infty} \left(\sum_{k=m+1}^{n} \|x_k\|_{L^p}\right)^p\right)^{1/p} \leq \left(\liminf_{n\to\infty} \varepsilon^p\right)^{1/p} = \varepsilon.
\end{aligned}$$

This shows that for all $\varepsilon > 0$, there exist $N \in \mathbb{N}_+$ such that $\|S - S_m\|_{L^p} \leq \varepsilon$ where $m > N$. Hence, $S_m \to S$ in $L^p(E)$ as $m \to \infty$. Now choose a fixed integer $m_0 > N$, then $\|S\|_{L^p} \leq \|S - S_{m_0}\|_{L^p} + \|S_{m_0}\|_{L^p} < \varepsilon + \|S_{m_0}\|_{L^p} = \varepsilon + \|\sum_{k=1}^{m_0} x_k\|_{L^p} \leq \varepsilon + \sum_{k=1}^{m_0} \|x_k\|_{L^p} < \infty$. So $S \in L^p(E)$ and $\sum_{k=1}^{\infty} x_k$ converges to $S \in L^p(E)$. Thus, $L^p(E)$ is a Banach space. $\square$

Remark 12.7. *In the proof of Theorem 12.6, we set no limit on the measure of E. Namely, the measure of E can be infinite, which is also permitted in Example 12.1.*

Lemma 12.8. ***Lusin's Theorem***

Suppose f is a complex measurable function on E, $A \subset E$, $m(A) < \infty$, $f(x) = 0$ if $x \neq A$, and $\varepsilon > 0$. Then there exists $g \in C_{\mathrm{C}}(E)$ such that $m(\{x \in E : g(x) \neq f(x)\}) < \varepsilon$ and $\sup_{x\in E} |g(x)| \leq \sup_{x\in E} |f(x)|$.

Remark 12.9. *The collection of all continuous complex functions on E whose support is compact is denoted $C_{\mathrm{C}}(E)$.*

Lemma 12.10. *Suppose $E = \cap_{n=1}^{\infty} E_n$ and $\{E_n\}_{n=1}^{\infty}$ is a nonincreasing set sequence with $m(E_1) < \infty$. Let f be nonnegative, measurable, and simple on E_1, then*

$$\lim_{n\to\infty} \int_{E_n} f(x)\mathrm{d}x = \int_E f(x)\mathrm{d}x \left(= \int_{\lim_{n\to\infty} E_n} f(x)\mathrm{d}x\right).$$

Proof. Suppose $f(x) = \sum_{i=1}^{m} \alpha_i \chi_{A_i}(x)$ on E_1 where $E_1 = \cup_{i=1}^{m} A_i$ and all A_i's are mutually disjoint. Since $\{E_n \cap A_i\}_{n=1}^{\infty}$ is nonincreasing with $m(E_1 \cap A_i) < \infty$ for each $i \in [1, m] \cap \mathbb{Z}$, we have $m(\lim_{n\to\infty}(E_n \cap A_i)) = \lim_{n\to\infty} m(E_n \cap A_i)$ for each

$i \in [1, m] \cap \mathbb{Z}$. Then

$$\begin{aligned}\int_E f(x)\mathrm{d}x &= \sum_{i=1}^m \alpha_i m(E \cap A_i) = \sum_{i=1}^m \alpha_i m\left(\lim_{n\to\infty}(E_n \cap A_i)\right) \\ &= \lim_{n\to\infty}\sum_{i=1}^m \alpha_i m(E_n \cap A_i) = \lim_{n\to\infty}\int_{E_n} f(x)\mathrm{d}x. \qquad \square\end{aligned}$$

Theorem 12.11. *$L^p(E)$ is separable.*

Proof. Here we only prove the case where E is a closed interval of $\mathbb{R}$, say $[a, b]$.

Define $x_n(t) = \begin{cases} x(t) & |x(t)| \le n \\ 0 & |x(t)| > n \end{cases}$ for all $x \in L^p[a, b]$ and $n \in \mathbb{N}_+$. We see that $x_n \in L^p[a, b]$ is bounded for each $n \in \mathbb{N}_+$. Note that

$$n^p m(\{t \in [a, b] : |x(t)| > n\}) \le \int_{\{t\in[a,b]:|x(t)|>n\}} |x(t)|^p \mathrm{d}t \le \int_{[a,b]} |x(t)|^p \mathrm{d}t = \|x\|^p_{L^p},$$

which means $m(\{t \in [a, b] : |x(t)| > n\}) \to 0$ $(n \to \infty)$. Then

$$\begin{aligned}\|x_n - x\|_{L^p} &= \left(\int_{[a,b]} |x_n(t) - x(t)|^p \mathrm{d}t\right)^{1/p} \\ &= \left(\int_{\{t\in[a,b]:|x(t)|>n\}} |x(t)|^p \mathrm{d}t\right)^{1/p} \xrightarrow{\text{Lemma 12.10}} 0 \ (n \to \infty).\end{aligned}$$

Let B be the set of all bounded functions on $[a, b]$, then for all $x \in L^p[a, b]$, there exists $\{x_n\}_{n=1}^\infty \subset B$ such that $d(x_n, x) = \|x_n - x\|_{L^p} \to 0$ $(n \to \infty)$, which indicates $L^p[a, b] \subset \bar{B}$. So $L^p[a, b] = \bar{B}$ since $B \subset L^p[a, b]$.

Let $C = \{x(: [a, b] \to \mathbb{F}) : x$ is continuous on $[a, b]\}$. By Lemma 12.8, for all $y \in B$, there exist $\{y_n\}_{n=1}^\infty \subset C$ and $\{A_n\}_{n=1}^\infty \subset [a, b]$ such that $0 < m(A_n) < 1/n$, $y(t) = y_n(t)$ when $t \in [a, b]\backslash A_n$, and $\max_{t\in[a,b]} |y_n(t)| \le \sup_{t\in[a,b]} |y(t)| = M > 0$. Then $0 \le d(y_n, y) = \|y_n - y\|_p = (\int_{[a,b]} |y_n(t) - y(t)|^p \mathrm{d}t)^{1/p} = (\int_{A_n} |y_n(t) - y(t)|^p \mathrm{d}t)^{1/p} \le 2M(m(A_n))^{1/p} \to 0$ $(n \to \infty)$. Thus, $B \subset \bar{C} \Rightarrow \bar{B} \subset \bar{C} \Rightarrow L^p[a, b] \subset \bar{C}$ and $L^p[a, b] = \bar{C}$ since $C \subset L^p[a, b]$.

Let $P_{\mathbb{Q}} = \{P(: [a, b] \to \mathbb{F}) : P$ is a ploynomial with rational coefficients$\}$. By Theorem 5.5, for all $z \in C$, there exists $\{P_n\}_{n=1}^\infty \subset P_{\mathbb{Q}}$ such that $\max_{t\in[a,b]} |P_n(t) - z(t)| \to 0$ $(n \to \infty)$. Then $d(P_n, z) = \|P_n - z\|_{L^p} = (\int_{[a,b]} |P_n(t) - z(t)|^p \mathrm{d}t)^{1/p} \le (b-a)^{1/p} \max_{t\in[a,b]} |P_n(t)-z(t)| \to 0$ $(n \to \infty)$. So $C \subset \bar{P}_{\mathbb{Q}} \Rightarrow \bar{C} \subset \bar{P}_{\mathbb{Q}} \Rightarrow L^p[a, b] = \bar{P}_{\mathbb{Q}}$ since $P_{\mathbb{Q}} \subset L^p[a, b]$.

Hence, $P_{\mathbb{Q}}$ is a countable dense subset of $L^p[a, b]$ and $L^p[a, b]$ is separable. $\square$

Remark 12.12. *The idea of the proof is finding a countable dense subset of $L^p(E)$. The procedure has been divided into several steps since it is not easy to get the expected set all of a sudden. Also, we have used convergent sequences heavily in the proof.*

Remark 12.13. *The measure of E can be infinite. The proof only involves the case where $m(E) < \infty$ and this fact, together with the closedness of $E = [a, b]$, is actually used in the proof.*

13. Examples of Normed Linear Space: 4/18/2018

I flew back to my university from Bangkok today and part of these notes was modified by Jiaqi Li.

Definition 13.1. *$x : E \to \mathbb{F}$ is said to be **essentially bounded** on E if there exists $M > 0$ and $A \subset E$ with $m(A) = 0$ such that $|x(t)| \leq M$ for all $t \in E \setminus A$.*

Example 13.2. *Let $L^\infty(E) = \{x(: E \to \mathbb{F}) : x \text{ is essentially bounded on } E\}$. $L^\infty(E)$ is a linear space over $\mathbb{F}$ with the addition and scalar multiplication of functions. For all $x \in L^\infty(E)$, define*

$$\|x\|_{L^\infty} = \inf_{\substack{A \subset E \\ m(A)=0}} \sup_{t \in E \setminus A} |x(t)| := \operatorname*{ess\,sup}_{t \in E} |x(t)|.$$

The claim goes that for all $x \in L^\infty(E)$, there exists $A_0 \subset E$ with $m(A_0) = 0$ such that $\|x\|_{L^\infty} = \sup_{t \in E \setminus A_0} |x(t)|$. In fact, there exists $A_n \subset E$ with $m(A_n) = 0$ such that for all $n \in \mathbb{N}_+$,

$$\sup_{t \in E \setminus A_n} |x(t)| \leq \|x\|_{L^\infty} + 1/n.$$

Set $A_0 = \cup_{n=1}^\infty A_n$, we see that $m(A_0) = 0$ and

$$\|x\|_{L^\infty} \leq \sup_{t \in E \setminus A_0} |x(t)| \leq \sup_{t \in E \setminus A_n} |x(t)| \leq \|x\|_{L^\infty} + 1/n$$

for all $n \in \mathbb{N}_+$. So $\|x\|_{L^\infty} = \sup_{t \in E \setminus A_0} |x(t)|$. This shows that the infimum in the definition of $\| \bullet \|_{L^\infty}$ actually can be attached.

Hence, L^∞ is a normed linear space. Also it is a Banach space but not separable.

Remark 13.3. *Note that $\overline{\lim} = \limsup$, $\underline{\lim} = \liminf$. Also,*

$$\limsup_{n \to \infty}(x_n + y_n) \leq \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n,$$
$$\liminf_{n \to \infty}(x_n + y_n) \geq \liminf_{n \to \infty} x_n + \liminf_{n \to \infty} y_n.$$

Remark 13.4. *Now the p appeared in $L^p(E)$ has been extended to $1 \leq p \leq \infty$.*

In fact, when $m(E) < \infty$, $\|x\|_{L^p} \to \|x\|_{L^\infty}$ as $p \to \infty$ for all $x \in L^p(E)$.

Remark 13.5. *If $1 < p < q < \infty$ and $m(E) < \infty$, then*

$$L^\infty(E) \subsetneqq L^q(E) \subsetneqq L^p(E) \subsetneqq L^1(E).$$

In fact, for all $x \in L^q(E)$, $\int_E |x(t)|^q \mathrm{d}t < \infty$ and

$$\int_E |x(t)|^p \mathrm{d}t \leq \left(\int_E |x(t)|^q \mathrm{d}t \right)^{p/q} \left(\int_E \mathrm{d}t \right)^{(q-p)/q} < \infty.$$

Hence, $L^q(E) \subset L^p(E)$ $(p < q)$.

Example 13.6. ℓ^p $(1 \le p \le \infty)$

When $p < \infty$, $\ell^p = \{x = \{\xi_n\}_{n=1}^\infty \subset \mathbb{F} : \sum_{n=1}^\infty |\xi_n|^p < \infty\}$. Then $\| \bullet \|_{\ell^p}$ is defined by $\|x\|_{\ell^p} = (\sum_{n=1}^\infty |\xi_n|^p)^{1/p}$ for all $x = \{\xi_n\}_{n=1}^\infty \in \ell^p$.

When $p = \infty$, $\ell^p = \{x = \{\xi_n\}_{n=1}^\infty \subset \mathbb{F} : \sup_{n\in\mathbb{N}_+} |\xi_n| < \infty\}$. Then $\| \bullet \|_{\ell^p}$ is defined by $\|x\|_{\ell^p} = \sup_{n\in\mathbb{N}_+} |\xi_n|$ for all $x = \{\xi_n\}_{n=1}^\infty \in \ell^p$.

ℓ^p is a separable Banach space for $p < \infty$ but an inseparable Banach space for $p = \infty$. Also, $\|x\|_{\ell^p} \to \|x\|_{\ell^\infty}$ as $p \to \infty$ for all $x \in \ell^p$.

Remark 13.7. *If $1 < p < q < \infty$, then $\ell^1 \subsetneqq \ell^p \subsetneqq \ell^q \subsetneqq \ell^\infty$.*

In fact, for all $x = \{\xi_n\}_{n=1}^\infty \in \ell^p$, $\sum_{n=1}^\infty |\xi_n|^p < \infty$. Then there exists $N \in \mathbb{N}_+$ such that $|\xi_n| < 1$ when $n > N$. Since

$$\sum_{n=1}^\infty |\xi_n|^q = \sum_{n=1}^N |\xi_n|^q + \sum_{n=N+1}^\infty |\xi_n|^q \le \sum_{n=1}^N |\xi_n|^q + \sum_{n=N+1}^\infty |\xi_n|^p < \infty.$$

Hence, $\ell^p \subset \ell^q$ $(p < q)$.

14. Finite Dimensional Normed Linear Space: 4/20/2018

I didn't attend the class because I was on an operation to remove the mucous cyst in my lips. These notes were provided by Shuning Gu and Wenyi Cai.

In $\mathbb{F}^n$, for all $x = (\xi_1, \xi_2, \cdots, \xi_N) \in \mathbb{F}^n$, we define the norms $\|x\| = (\sum_{n=1}^N |\xi_n|^2)^{1/2}$, $\|x\|_p = (\sum_{n=1}^N |\xi_n|^p)^{1/p}$ $(1 \le p < \infty)$, and $\|x\|_\infty = \max_{1\le n\le N} |\xi_n|$. And $\| \bullet \|$, sometimes denoted $\|x\|_0$ under the canonical basis of $\mathbb{F}^N$ (see Theorem 14.4), is called the **standard norm** of $\mathbb{F}^N$.

Definition 14.1. *Let $(X, \|\bullet\|_1)$ and $(X, \|\bullet\|_2)$ be two normed linear spaces. Then $\|\bullet\|_1$ and $\| \bullet \|_2$ are* ***equivalent*** *if there exist $k_1, k_2 > 0$ such that $k_1\|x\|_1 \le \|x\|_2 \le k_2\|x\|_1$ for all $x \in X$.*

Remark 14.2. *If $k_1\|x\|_1 \le \|x\|_2 \le k_2\|x\|_1$ for all $x \in X$, then $k_2^{-1}\|x\|_2 \le \|x\|_1 \le k_1^{-1}\|x\|_2$ for all $x \in X$.*

Theorem 14.3. *Let X be a linear space equipped with two norms $\| \bullet \|_1$ and $\| \bullet \|_2$ which are equivalent. Then for every $\{x_n\}_{n=1}^\infty \subset (X, \| \bullet \|_1)$,*

(a) *$\{x_n\}_{n=1}^\infty$ converges in $(X, \| \bullet \|_1)$ iff $\{x_n\}_{n=1}^\infty$ converges in $(X, \| \bullet \|_2)$;*
(b) *$\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in $(X, \| \bullet \|_1)$ iff $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in $(X, \| \bullet \|_2)$;*
(c) *$(X, \| \bullet \|_1)$ is a Banach space iff $(X, \| \bullet \|_2)$ is a Banach space.*

Theorem 14.4. *Let X be a finite-dimensional linear space with basis $\{e_1, e_2, \cdots, e_N\}$. Then each norm $\| \bullet \|$ on X is equivalent to $\| \bullet \|_0$ defined by $\|x\|_0 = (\sum_{n=1}^N |\xi_n|^2)^{1/2}$ for every $x = \sum_{n=1}^N \xi_n e_n \in X$.*

Proof. Let $K = (\sum_{n=1}^{N} \|e_i\|^2)^{1/2}$ and for all $= \sum_{n=1}^{N} \xi_n e_n \in X$, we have

$$\|x\| = \left\| \sum_{n=1}^{N} \xi_n e_n \right\| \leq \sum_{n=1}^{N} |\xi_n| \cdot \|e_n\| \leq \left(\sum_{n=1}^{N} |\xi_n|^2 \right)^{1/2} \left(\sum_{n=1}^{N} \|e_n\|^2 \right)^{1/2} = K\|x\|_0.$$

For all $x = (\xi_1, \xi_2, \cdots, \xi_N) \in \mathbb{F}^N$, define $f : \mathbb{F}^N \to \mathbb{F}$ by $f(x) = \| \sum_{n=1}^{N} \xi_n e_n \| (= \|x\|)$. Then f is continuous on $\mathbb{F}^N$ since $\| \bullet \|$ is continuous by Remark 11.5. Note that the unit circle $S = \{(\xi_1, \xi_2, \cdots, \xi_N) \in \mathbb{F}^n : \sum_{n=1}^{N} |\xi_n|^2 = 1\}$ is bounded and closed in $\mathbb{F}^n$. Hence, S is compact and there exists $(a_1, a_2, \cdots, a_N) \in S$ such that $k := f(a_1, a_2, \cdots, a_N) \leq f(\xi_1, \xi_2, \cdots, \xi_N)$ for all $(\xi_1, \xi_2, \cdots, \xi_N) \in S$. Note that $k > 0$. Otherwise, $a_1 = a_2 = \cdots = a_N = 0$, which is impossible since $(a_1, a_2, \cdots, a_N) \in S$.

Now for all $x = (\xi_1, \xi_2, \cdots, \xi_N) \in \mathbb{F}^N$ with $x \neq (0, 0, \cdots, 0)$, we have

$$\left(\frac{\xi_1}{\|x\|_0}, \frac{\xi_2}{\|x\|_0}, \cdots, \frac{\xi_N}{\|x\|_0} \right) \in S \Rightarrow k \leq f \left(\frac{\xi_1}{\|x\|_0}, \frac{\xi_2}{\|x\|_0}, \cdots, \frac{\xi_N}{\|x\|_0} \right) = \frac{\|x\|}{\|x\|_0}.$$

Hence, $k\|x\|_0 \leq \|x\|$ for all $x \in X$.

So $k\|x\|_0 \leq \|x\| \leq K\|x\|_0$ for all $x \in X$, which proves the theorem. $\square$

Corollary 14.5. $\| \bullet \|$*,* $\| \bullet \|_p$ $(1 \leq p < \infty)$*, and* $\| \bullet \|_\infty$ *are equivalent on* $\mathbb{F}^n$*.*

Corollary 14.6. *If there exist two norms* $\| \bullet \|_1$ *and* $\| \bullet \|_2$ *on a normed linear space* X *such that* $\| \bullet \|_1$ *and* $\| \bullet \|_2$ *are not equivalent, then* X *is infinite-dimensional.*

The following theorem is easy to prove by dealing with the vectors expressed in coordinate with respect to the basis of the space.

Theorem 14.7. *Suppose that* X *is a finite-dimensional linear space over* $\mathbb{F}$ *with basis* $\{e_1, \cdots, e_n\}$ *and the norm* $\| \bullet \|_0$*. Then* X *is complete, i.e., a Banach space, under the metric induced by* $\| \bullet \|_0$*.*

Corollary 14.8. *Any finite dimensional normed linear space over* $\mathbb{F}$ *is complete with respect to the metric induced by its norm, i.e., a Banach space, and hence is closed.*

Example 14.9. *Here we recall* c_{00} *raised in Remark 11.8 as an example of nonclosed linear subspace of a normed linear space.*

In ℓ^∞*,* c_{00} *is defined to be* $\{x = \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{F} : \xi_n \neq 0$ *for only a finite number of* $n \in \mathbb{N}_+\}$*.* c_{00} *is obvious a subspace of* ℓ^∞*. Then let* $\{x_n = (1, 1/2, \cdots, 1/n, 0, 0, \cdots)\}_{n=1}^{\infty} \subset c_{00}$ *and* $x = (1, 1/2, \cdots, 1/n, 1/(n+1), \cdots) \in \ell^\infty \setminus c_{00}$*. Note that* $\|x_n - x\|_{\ell^\infty} = \|(0, \cdots, 0, 1/(n+1), 1/(n+2), \cdots)\|_{\ell^\infty} = 1/(n+1) \to 0$ $(n \to \infty)$*. So* $c_{00} \neq \bar{c}_{00}$ *and* c_{00} *is not closed in* ℓ^∞*, where* $\bar{c}_{00}$ *equals to* $\{x \in \ell^\infty : \exists \{x_n\}_{n=1}^{\infty} \subset c_{00}$ *such that* $x_n \to x$ *as* $n \to \infty\}$*.*

Theorem 14.10. *Suppose* X *is a normed linear space over* $\mathbb{F}$*. If* A *is a linear subspace of* X*, then* $\bar{A}$ *is a closed linear subspace of* X*.*

Proof. For all $x, y \in \bar{A}$, there exist $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset A$ such that $x_n \to x, y_n \to y$ as $n \to \infty$. Then $\{x_n + y_n\}_{n=1}^{\infty} \subset A$ and $\lim_{n\to\infty}(x_n + y_n) = x + y$. Thus, $x + y \in \bar{A}$. Similarly, for all $\alpha \in \mathbb{F}$, $\alpha x \in \bar{A}$. $\square$

Definition 14.11. *Suppose A is a nonempty subset of $(X, \|\bullet\|)$, then linear space* ***spanned*** *by A is* $\operatorname{span}(A) = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n : n \in \mathbb{N}_+,\ \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{F},\ x_1, x_2, \cdots, x_n \in A\}$. *Obviously, $A \subset \operatorname{span}(A) \subset \overline{\operatorname{span}(A)}$ and $\overline{\operatorname{span}(A)}$ is a closed normed linear subspace of X.*

15. Subspace of Normed Linear Space: 4/25/2018

Theorem 15.1. *Let A be a nonempty subset of a normed linear space X over $\mathbb{F}$, then*

(a) $\operatorname{span}(A) = \{\sum_{i=1}^{n} \alpha_i x_i : \alpha_i \in \mathbb{F}, x_i \in A, 1 \le i \le n, n \in \mathbb{N}_+\}$ *is a linear subspace of X;*
(b) $\operatorname{span}(A) = \cap W$, *where W runs through all subspaces of X containing A;*
(c) $\overline{\operatorname{span}(A)} = \cap W$, *where W runs over all closed subspaces of X containing A.*

Proof. We only prove (c) here since (a) is trivial and the proof of (b) is similar to the following one of (c).

(c) Denote by F the set $\cap W$, where W runs over all closed subspaces of X containing A. For all $W \supset A$, where W is a closed linear subspace of X, $\operatorname{span}(A) \subset W \Rightarrow \overline{\operatorname{span}(A)} \subset \bar{W} = W \Rightarrow \overline{\operatorname{span}(A)} \subset F$. Note that $\overline{\operatorname{span}(A)} \supset A$ is a closed linear subspace of X, then $\overline{\operatorname{span}(A)} \supset F$. So it follows that $\overline{\operatorname{span}(A)} = F$. $\square$

Theorem 15.2. ***Riesz's Lemma***

Let $(X, \|\bullet\|)$ be a normed linear space and let Y be a closed linear subspace of X such that $Y \neq X$. Then for each $\alpha \in (0, 1)$, there exists $x_\alpha \in S = \{x \in X : \|x\| = 1\}$ such that $\|x_\alpha - y\| > \alpha$ for all $y \in Y$.

Proof. Let $x \in X \setminus Y$, then obviously $x \neq 0$ and $0 \le d := d(x, Y) = \inf_{y\in Y} d(x, y) = \inf_{y\in Y} \|x - y\|$. We claim that $d > 0$. Otherwise, $x \in \bar{Y} = Y$, which is a contradiction. By the definition of infimum, for every $\alpha \in (0, 1)$, there exists $y_\alpha \in Y$ such that $d \le \|x - y_\alpha\| < \frac{d}{\alpha}$. Note that $y_\alpha \neq x$. Set $x_\alpha = (x - y_\alpha)/\|x - y_\alpha\| \in S$ and we have

$$\begin{aligned}\|x_\alpha - y\| &= \left\| \frac{x - y_\alpha}{\|x - y_\alpha\|} - y \right\| \\ &= \frac{1}{\|x - y_\alpha\|} \|x - (y_\alpha + \|x - y_\alpha\| y)\| \\ &\ge \frac{1}{\|x - y_\alpha\|} d > \frac{\alpha}{d} d = \alpha\end{aligned}$$

since $y_\alpha + \|x - y_\alpha\| y \in Y$, which holds for all $y \in Y$. $\square$

Lemma 15.3. *Let $(X, \|\bullet\|)$ be a infinite-dimensional normed linear space, then $S = \{x \in X : \|x\| = 1\}$ is not compact. Hence it follows that every subset of X containing S like the unit closed ball is not compact.*

Proof. Fix $x_1 \in S$ and $\text{span}(\{x_1\}) = \{\alpha x_1 : \alpha \in \mathbb{F}\}$ is a 1-dimensional linear subspace of X. Note that $\text{span}(\{x_1\})$ is closed and $\text{span}(\{x_1\}) \neq X$. By Lemma 15.2, there exists $x_2 \in S$ such that $\|x_2 - x_1\| > 1/2$. Now $\text{span}(\{x_1, x_2\}) = \{\alpha_1 x_1 + \alpha_2 x_2 : \alpha_1, \alpha_2 \in \mathbb{F}\}$ is a closed linear subspace of X and $X \neq \text{span}(\{x_1, x_2\})$, hence there exists $x_3 \in S$ such that $\|x_3 - x_2\| > 1/2$ and $\|x_3 - x_1\| > 1/2$. By induction this process can be continued forever, and so we get a sequence $\{x_n\}_{n=1}^{\infty} \subset S$ such that $\|x_n - x_m\| > 1/2$ for all $n, m \in \mathbb{N}_+$ with $n \neq m$, and hence it does not have a convergent subsequence. So S is not compact. $\square$

Remark 15.4. *If S is not compact in a normed linear space X, then X is infinite-dimensional. Also, we recall Corollary 14.6.*

Lemma 15.5. *In a finite-dimensional normed linear space, a set is compact iff it is closed and bounded.*

Proof. $\Rightarrow$) It's trivial by Theorem 8.13.

$\Leftarrow$) Let $(X, \|\bullet\|)$ be a finite-dimensional normed linear space (over $\mathbb{F}$) and let A be an arbitrary subset of X which is bounded and closed. Without loss of generality, we assume $A \neq \varnothing$. It suffices to prove that A is compact in X.

Suppose that $\{x_m\}_{m=1}^{\infty}$ is an arbitrary sequence in A. If $\{e_1, \cdots, e_n\}$ is a basis of X where $n = \dim(X)$, then for each x_m, there exist unique $x_{m,1}, \cdots, x_{m,n} \in \mathbb{F}$ such that $x_m = x_{m,1}e_1 + \cdots + x_{m,n}e_n$. By Theorem 14.4, there exists $K > 0$ such that

$$\|x_m\| \geq K\|x_m\|_0 = K\sqrt{x_{m,1}^2 + \cdots + x_{m,n}^2} \geq K|x_{m,i}|$$

holds for all $m \in \mathbb{N}_+$ and $i \in [1, n] \cap \mathbb{Z}$.

Since A is bounded, there exists $M > 0$ such that $\|x\| \leq M$ whenever $x \in A$. Then $|x_{m,i}| \leq M/K$ for each $m \in \mathbb{N}_+$ and $i \in [1, n] \cap \mathbb{Z}$. Hence $\{x_{m,i}\}_{m=1}^{\infty}$ is bounded for each $i \in [1, n] \cap \mathbb{Z}$. Based on diagonalization argument, we perform the following procedure from the fact that each bounded sequence of $\mathbb{R}$ has a convergent subsequence:

- Choose a convergent subsequence $\{x_{m_k^{(1)},1}\}_{k=1}^{\infty}$ of $\{x_{m,1}\}_{m=1}^{\infty}$ which converges to $a_1 \in \mathbb{R}$;
- Choose a convergent subsequence $\{x_{m_k^{(2)},2}\}_{k=1}^{\infty}$ of $\{x_{m_k^{(1)},2}\}_{k=1}^{\infty}$ which converges to $a_2 \in \mathbb{R}$;
- Choose a convergent subsequence $\{x_{m_k^{(3)},3}\}_{k=1}^{\infty}$ of $\{x_{m_k^{(2)},3}\}_{k=1}^{\infty}$ which converges to $a_3 \in \mathbb{R}$;
- $\cdots\cdots$
- Choose a convergent subsequence $\{x_{m_k^{(n)},n}\}_{k=1}^{\infty}$ of $\{x_{m_k^{(n-1)},n}\}_{k=1}^{\infty}$ which converges to $a_n \in \mathbb{R}$.

By construction, $\{x_{m_k^{(n)},i}\}_{k=1}^{\infty}$ converges to a_i for each $i \in [1,n] \cap \mathbb{Z}$. It follows that $\{x_{m_k^{(n)}} = x_{m_k^{(n)},1}e_1 + x_{m_k^{(n)},2}e_2 + \cdots + x_{m_k^{(n)},n}e_n\}_{k=1}^{\infty}$ as a subsequence of $\{x_m\}_{m=1}^{\infty}$ converges to $a = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in X$. Hence A is sequentially compact since $\{x_m\}_{m=1}^{\infty}$ is arbitrary. Then A is compact by Theorem 8.8. ☐

Remark 15.6. *One can compare this proof with the that of Theorem 9.3. But here we just need the chain of sequences appeared in diagonalization argument since we work in a finite-dimensional space.*

Theorem 15.7. $(X, \|\bullet\|)$ *is finite-dimensional iff every bounded subset of X is relatively compact.*

Proof. $\Rightarrow$) It's trivial by Lemma 15.5.

$\Leftarrow$) If not, suppose that X is infinite-dimensional. By Lemma 15.3, S is a bounded and closed subset of X, but S is not compact, which is a contradiction. Hence, X is finite-dimensional. ☐

Consider the following progressive spaces as the motivation for Definition 15.8:

- $X = \{x(: E \to \mathbb{R}) : x$ is Lebesgue measurable and $\int_E |x(t)|^p \mathrm{d}t < \infty\}$;
- $N_1 = \{x(: E \to \mathbb{R}) : x$ is Lebesgue measurable and $x(t) = 0$ a.e. on $E\}$;
- $\pi(x) \in L^p(E) \Leftrightarrow \pi(x) = x + N_1 := \{x + y : y \in N_1\}$;
- $L^p(E) = \{\pi(x) : x \in X\} = X/N_1$.

Definition 15.8. *Let N be a subspace of a vector space X. For every $x \in X$, let $\pi(x)$ be the* ***coset*** *of N containing x, thus $\pi(x) = x + N$ which is defined to be the set $\{v : v = x + u, u \in N\}$. Then $X/N = \{\pi(x) : x \in X\}$ is called the* ***quotient space*** *of X modulo N, in which addition and multiplication are defined by*

$$\pi(x) + \pi(y) = \pi(x + y),\ \ \alpha\pi(x) = \pi(\alpha x)$$

for all $x, y \in X$ and $\alpha \in \mathbb{F}$.

Theorem 15.9. *Quotient spaces are well-defined.*

Theorem 15.10. *Let $(X, \|\bullet\|)$ be a normed linear space and let N be a closed linear subspace of X. $\|\bullet\|_0$ is defined by $\|\pi(x)\|_0 = \inf_{z\in N} \|x - z\| (= \inf_{z\in N} \|x + z\|)$ for every $x \in X$, then $(X/N, \|\bullet\|_0)$ is a normed linear space. Moreover, $(X/N, \|\bullet\|_0)$ is a Banach space if $(X, \|\bullet\|)$ is a Banach space.*

16. Inner Product Space: 4/27/2018

The murderer (midterm examination) is coming.

Definition 16.1. *Let X be a linear space over $\mathbb{F}$. An* ***inner product*** *on X is a mapping $\langle \bullet, \bullet \rangle : X \times X \to \mathbb{F}$ such that*

(I1) $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0$ *iff* $x = 0$,

(I2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$, *and*

(I3) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

*hold for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$. A linear space X equipped with an inner product is called an **inner product space**, denoted $(X, \langle \bullet, \bullet\rangle)$.*

Remark 16.2. *In an inner product space $(X, \langle \bullet, \bullet\rangle)$, it follows from (I2) and (I3) that $\langle x, \alpha y + \beta z\rangle = \bar{\alpha}\langle x, y\rangle + \bar{\beta}\langle x, z\rangle$ for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$.*

In a metric space (X, d), to show $x = y$, it suffices to prove $d(x, y) = 0$. In a normed linear space $(X, \|\bullet\|)$, to show $x = y$, it suffices to prove $\|x - y\| = 0$. However, in a inner product space, the analogy does not hold. In fact, it has to be modified.

Theorem 16.3. *Let $(X, \langle \bullet, \bullet\rangle)$ be an inner product space and $x, y \in X$. Then*

(1) *$\langle x, z\rangle = \langle y, z\rangle$ (or $\langle z, x\rangle = \langle z, y\rangle$) for all $z \in X$ iff $x = y$;*
(2) ***Cauchy-Schwarz inequality**: $|\langle x, y\rangle|^2 \le \langle x, x\rangle\langle y, y\rangle$ for all $x, y \in X$;*
(3) *$\|x\| = \sqrt{\langle x, x\rangle}$ (for all $x \in X$) defines a norm on X.*

Proof. (1) $\Leftarrow$) Trivial.

$\Rightarrow$) In fact, we have $\langle x - y, z\rangle = 0$ or $\langle z, x - y\rangle = 0$ for all $z \in Z$. In particular, $z = x - y \Rightarrow x - y = 0 \Rightarrow x = y$.

(2) Suppose $x, y \in X$ and $\alpha \in \mathbb{F}$, then

$$0 \le \langle x + \alpha y, x + \alpha y\rangle = \langle x, x\rangle + \bar{\alpha}\langle x, y\rangle + \alpha\overline{\langle x, y\rangle} + |\alpha|^2\langle y, y\rangle.$$

If $y = 0$, then the inequality obviously holds, hence we may assume $y \neq 0$. Take $\alpha = -\langle x, y\rangle/\langle y, y\rangle$ and we have

$$0 \le \langle x, x\rangle - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle},$$

which is equivalent to $|\langle x, y\rangle|^2 \le \langle x, x\rangle\langle y, y\rangle$. Thus the inequality holds since x, y are arbitrary.

(3) We just need to verify if the definition of norm holds for $\|\bullet\|$ defined above. Actually, for all $x, y \in X$ and $\alpha \in \mathbb{F}$, we have

(N1) $\langle x, x\rangle \ge 0$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$ trivially hold;
(N2) $\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{|\alpha|^2\langle x, x\rangle} \le |\alpha|\sqrt{\langle x, x\rangle} = \alpha\|x\|$;
(N3) $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + 2\mathrm{Re}(\langle x, y\rangle) + \|y\|^2 \le \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$.

Hence, $\|\bullet\|$ defined above is indeed a norm on X. $\square$

Remark 16.4. *(3) of Theorem 16.3 indicates that **inner product induces norm**.*

The following corollary follows immediately from (a) of Theorem 16.3.

Corollary 16.5. *Suppose $\mathcal{H}$ is a Hilbert space, $\{u_n\}_{n=1}^{\infty} \subset \mathcal{H}$, and $u \in \mathcal{H}$. Then $\lim_{n\to\infty} u_n = u$ iff $\lim_{n\to\infty}\langle u_n, z\rangle = \langle u, z\rangle$ for all $z \in \mathcal{H}$.*

Theorem 16.6. *Let $(X, \langle \bullet, \bullet \rangle)$ be an inner product space and $\| \bullet \|$ is the norm on X induced by $\langle \bullet, \bullet \rangle$, then for all $x, y \in X$, we have*

- ***Parallelogram law**: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\| + \|y\|)^2$;*
- ***Polarization identity I**: if X is real, i.e., $\mathbb{F} = \mathbb{R}$, then*

$$\langle x, y \rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2);$$

- ***Polarization identity II**: if X is complex, i.e., $\mathbb{F} = \mathbb{C}$, then*

$$\langle x, y \rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + \mathrm{i}\|x+\mathrm{i}y\|^2 - \mathrm{i}\|x-\mathrm{i}y\|^2).$$

Proof. For all $x, y \in X$, add up the following identities

$$\begin{aligned}
\|x+y\|^2 &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle, \\
\|x-y\|^2 &= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle,
\end{aligned}$$

and we derive $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$. If we subtract the identities, then we obtain

$$\begin{aligned}
2(\langle x, y \rangle + \langle y, x \rangle) &= \|x+y\|^2 - \|x-y\|^2, \\
2(\langle x, \mathrm{i}y \rangle + \langle \mathrm{i}y, x \rangle) &= \|x+\mathrm{i}y\|^2 - \|x-\mathrm{i}y\|^2.
\end{aligned}$$

If X is real, then $\langle x, y \rangle = \langle y, x \rangle$, and hence $\langle x, y \rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$.
If X is complex, then

$$\begin{aligned}
4\langle x, y \rangle &= 2(\langle x, y \rangle + \langle y, x \rangle) + 2(\langle x, y \rangle - \langle y, x \rangle) \\
&= 2(\langle x, y \rangle + \langle y, x \rangle) + 2\mathrm{i}(\langle x, \mathrm{i}y \rangle + \mathrm{i}\langle y, x \rangle) \\
&= \|x+y\|^2 - \|x-y\|^2 + \mathrm{i}\|x+\mathrm{i}y\|^2 - \mathrm{i}\|x-\mathrm{i}y\|^2.
\end{aligned}$$

Thus, the claim holds. $\square$

Remark 16.7. *The parallelogram law can be used to verify whether or not a norm $\| \bullet \| : X \to \mathbb{F}$ is induced from some inner product on X.*

For now, we conclude that **inner product induces norm**, and **norm induces metric**. With abuse of terminology, this can be revealed through set language:

$$\{\text{Metrics}\} \supset \{\text{Norms}\} \supset \{\text{Inner Products}\}.$$

Inversely, if a norm $\| \bullet \|$ on X satisfies the parallelogram law, then $\| \bullet \|$ produces some inner product. This will be proved in the next session.

I skipped the class for reviewing my paper on algebraic geometry. There notes were provided by Luyao Huang.

Theorem 17.1. *Let $(X, \|\bullet\|)$ be a normed linear space. Then there exists an inner product $\langle \bullet, \bullet \rangle$ on X such that $\|\bullet\|$ can be induced by $\langle \bullet, \bullet \rangle$, i.e., $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in X$, iff $\|\bullet\|$ satisfies parallelogram law (see Theorem 16.6).*

Proof. $\Rightarrow$) Trivial.

$\Leftarrow$) It's can be verified easily that

$$\langle x, y \rangle = \begin{cases} \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) & \text{if } X \text{ is real} \\ \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + \mathrm{i}\|x+\mathrm{i}y\|^2 - \mathrm{i}\|x-\mathrm{i}y\|^2) & \text{if } X \text{ is complex} \end{cases}$$

is a desired inner product. $\square$

Remark 17.2. *Note that there is a tiny trick here. By parallelogram law,*

$$\begin{aligned}
&\|x+y+z\|^2 - \|x+y-z\|^2 \\
&\quad = (\|(x+z)+y\|^2 + \|(x+z)-y\|^2) - (\|x-(y-z)\|^2 + \|x+(y-z)\|^2) \\
&\quad = 2(\|x+z\|^2 + \|y\|^2) - 2(\|x\|^2 + \|y-z\|^2) \\
&\quad = 2(\|x+z\|^2 - \|y-z\|^2 + \|y\|^2 - \|x\|^2), \\
&\|x+y+z\|^2 - \|x+y-z\|^2 \\
&\quad = (\|(y+z)+x\|^2 + \|(y+z)-x\|^2) - (\|(x-z)-y\|^2 + \|(x-z)+y\|^2) \\
&\quad = 2(\|y+z\|^2 + \|x\|^2) - 2(\|x-z\|^2 + \|y\|^2) \\
&\quad = 2(\|y+z\|^2 - \|x-z\|^2 + \|x\|^2 - \|y\|^2).
\end{aligned}$$

So $\|x+y+z\|^2 - \|x+y-z\|^2 = (\|x+z\|^2 - \|x-z\|^2) + (\|y+z\|^2 + \|y-z\|^2)$. This would result in $\langle \bullet, \bullet \rangle$ defined above satisfying (N3) in the definition of a norm.

Example 17.3. *Consider $(C[0,1], \|\bullet\|)$, where $\|x\| = \max_{t \in [0,1]} |x(t)|$ for all $x \in C[0,1]$. Take $x(t) \equiv 1$ and $y(t) = t$ $(t \in [0,1])$, then $\|x\| = \|y\| = 1$, $\|x+y\| = 2$, and $\|x-y\| = 1$. So $\|x+y\|^2 + \|x-y\|^2 = 5 \neq 4 = 2(\|x\|^2 + \|y\|^2)$. Hence, there is no inner product $\langle \bullet, \bullet \rangle$ on $C[0,1]$ such that $\|\bullet\|$ can be induced by $\langle \bullet, \bullet \rangle$, which can also be derived by considering the functions $x(t) = \sin(t)$ and $y(t) = \cos(t)$ $(t \in [0,1])$. So $C[0,1]$ will never be an inner product space if equipped with the usual norm on $C[0,1]$*

Example 17.4. *Consider $(\ell^p, \|\bullet\|)$, where $\|x\| = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$ for all $x = \{x_n\}_{n=1}^{\infty} \in \ell^p$. Take $x = (1, 1, 0, \cdots)$ and $y = (1, -1, 0, \cdots)$, then $\|x\| = \|y\| = 2^{1/p}$ and $\|x+y\| = \|x-y\| = 2$. So $\|x+y\|^2 + \|x-y\|^2 = 2^3 \neq 2^{2+2/p} = 2(\|x\|^2 + \|y\|^2)$ if $p \neq 2$. Hence, there is no inner product $\langle \bullet, \bullet \rangle$ on ℓ^p such that $\|\bullet\|$ can be induced by*

$\langle \bullet, \bullet \rangle$. *So ℓ^p will never be an inner product space for $p \neq 2$ if equipped with the usual norm on ℓ^p.*

Definition 17.5. *A complete inner product space $(X, \langle \bullet, \bullet \rangle)$ is called a **Hilbert space**.*

Example 17.6. *We give some examples of Hilbert space here.*

- $(\mathbb{F}^N, \langle \bullet, \bullet \rangle)$ *is a Hilbert space, where* $\langle x, y \rangle = \sum_{n=1}^N x_n \bar{y}_n$ *for all* $x = \{x_n\}_{n=1}^N$ *and* $\{y_n\}_{n=1}^N$ *of* $\mathbb{F}^N$.
- $(\ell^2, \langle \bullet, \bullet \rangle)$ *is a Hilbert space, where* $\langle x, y \rangle = \sum_{n=1}^\infty x_n \bar{y}_n$ *for all* $x = \{x_n\}_{n=1}^\infty$ *and* $\{y_n\}_{n=1}^\infty$ *of* ℓ^2.
- $(L^2(E), \langle \bullet, \bullet \rangle)$ *is a Hilbert space, where* $L^2(E) = \{x(: E \to \mathbb{F}) : \int_E |x(t)|^2 \mathrm{d}t < \infty\}$, $\langle x, y \rangle = \int_E x(t)\overline{y(t)} \mathrm{d}t$ *for all* $x, y \in L^2(E)$. *The induced norm* $\| \bullet \|$ *is defined by* $\|x\| = (\int_E |x(t)|^2)^{1/2}$ *for all* $x \in L^2(E)$.

Theorem 17.7. *Let $(X, \langle \bullet, \bullet \rangle)$ be an inner product space and suppose that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are sequences in X, and $x_n \to x, y_n \to y$ as $n \to \infty$. Then $\langle x_n, y_n \rangle \to \langle x, y \rangle$ as $n \to \infty$.*

Proof. It's easy to see that

$$\begin{aligned} |\langle x_n, y_n \rangle - \langle x, y \rangle| &\leq |\langle x_n, y_n \rangle - \langle x, y_n \rangle| + |\langle x, y_n \rangle - \langle x, y \rangle| \\ &\leq |\langle x_n - x, y_n \rangle| + |\langle x, y_n - y \rangle| \\ &\leq \|x_n - x\| \|y_n\| + \|x\| \|y_n - y\|, \end{aligned}$$

where $\| \bullet \|$ is the norm induced by $\langle \bullet, \bullet \rangle$ on X. Since $x_n \to x$ and $y_n \to y$ as $n \to \infty$, it follows that $\langle x_n, y_n \rangle \to \langle x, y \rangle$ as $n \to \infty$. □

Lemma 17.8. *If d_n is the metric on X_n for $n = 1, 2, \cdots, N$, then $\rho(x, y) = \max\{d_1(x_1, y_1), \cdots, d_N(x_N, y_N)\}$ (for all $x = (x_1, \cdots, x_N)$ and $y = (y_1, \cdots, y_N)$ of $X_1 \times \cdots \times X_N$) is a metric on $X_1 \times \cdots \times X_N$.*

By Theorem 7.11 and Lemma 17.8, it follows the continuity of inner product.

Corollary 17.9. *Let $(X, \langle \bullet, \bullet \rangle)$ be an inner product space, then $\langle \bullet, \bullet \rangle$ is a continuous mapping on $X \times X$ to $\mathbb{F}$.*

Example 17.10. $H_0^1(\Omega)$

Let Ω be an open subset of $\mathbb{R}^n$ and let $C_0^\infty(\Omega)$ be the vector space of all infinitely differentiable functions which, together with all of their partial derivatives, have compact support sets on Ω ($\operatorname{supp}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}$ *is called the **support set** of f on Ω. A typical example of an element in $C_0^\infty(\Omega)$ is the function ϕ on $\Omega = \mathbb{R}^n$ defined by $\phi(x) = 0$ when $|x| \geq 1$ and $\phi(x) = \exp(\frac{1}{|x|^2 - 1})$ when $|x| < 1$).*

Now we define

$$\langle f, g \rangle_1 = \int_\Omega \left(f(x)\overline{g(x)} + \sum_{i=1}^n f_{x_i}(x)\overline{g_{x_i}(x)} \right) \mathrm{d}x$$

where $f_{x_i}(x) = \frac{\partial f(x_1,\cdots,x_i,\cdots,x_n)}{\partial x_i}$. Then $\langle f, g\rangle_1$ is an inner product on $C_0^\infty(\Omega)$ and $C_0^\infty(\Omega)$ is an incomplete inner product space equipped with this inner product. Its completion space is a Hilbert space, denoted $H_0^1(\Omega)$ which is the so-called ***Sobolev space****. $H_0^1(\Omega)$ is a normed linear space with the norm induced by the above inner product. Similarly, we can define $H_0^k(\Omega)$ for each $k \in \mathbb{N}_+$.*

Definition 17.11. *Let $(X, \langle \bullet, \bullet\rangle)$ be an inner product space. For $x, y \in X$, x and y are said to be* ***orthogonal*** *if $\langle x, y\rangle = 0$, denoted $x \perp y$.*

Definition 17.12. *Let $(X, \langle \bullet, \bullet\rangle)$ be an inner product space. For $x \in X$ and $A \subset X$, x and A are said to be* ***orthogonal*** *if $\langle x, y\rangle = 0$ for all $y \in A$, denoted $x \perp A$.*

Definition 17.13. *Let $(X, \langle \bullet, \bullet\rangle)$ be an inner product space. For $M \in X$ and $N \subset X$, M and N are said to be* ***orthogonal*** *if $\langle x, y\rangle = 0$ for all $x \in M$ and $y \in N$, denoted $M \perp N$. In particular, if $A \subset X$, then $A^\perp := \{y \in X : \langle y, x\rangle = 0 \text{ for all } x \in A\}$.*

18. Midterm Examination: 5/4/2018

Question 18.1. *Find a metric space in which every subset of this space is open and closed, and justify your assertion.*

Question 18.2. *Solve the following questions.*

(a) *In a metric space (X, d), prove that if a Cauchy sequence has a convergent subsequence, then the whole sequence is convergent.*

(b) *Give the definitions of completeness and compactness for a metric space. Show that a compact metric space must be complete, but not conversely in general.*

Question 18.3. *Prove that the set $M \subset C[a, b]$ in which there exists $m, L > 0$ and $x_0 \in [a, b]$ such that $|f(x_0)| \le m$ for all $f \in M$ and $|f(x) - f(y)| \le L|x - y|$ for all $f \in M$ and $x, y \in [a, b]$, is relatively compact in $C[a, b]$.*

Question 18.4. *Solve the following questions.*

(a) *Let the mapping $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = x + \pi/2 - \arctan(x)$. Show that T satisfies that $d(Tx, Ty) < d(x, y)$ for all $x, y \in \mathbb{R}$ with $x \ne y$, but T has no fixed-point.*

(b) *Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \ne y$. Prove that T has a unique fixed-point on X.*

Question 18.5. *If $(X, \|\bullet\|)$ is a normed linear space and $Y \subset X$ is a finite-dimensional linear subspace, prove that each element of X has a projection on Y, that is, for each $x \in X$, there exists $y_0 \in Y$ such that $\|x - y_0\| = d(x, Y) = \inf_{y\in Y} \|x - y\|$.*

Question 18.6. *Solve the following questions.*

(a) *Suppose $1 < p < q < \infty$, show that $\ell^p \subsetneqq \ell^q \subsetneqq \ell^\infty$.*

(b) *Suppose* $\|\bullet\|_p$ *and* $\|\bullet\|_\infty$ *are norms on* ℓ^p $(p \geq 1)$ *and* ℓ^∞ *respectively. Prove that for each* $x \in \ell^p$*, it follows that* $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

19. Orthogonality: 5/9/2018

Theorem 19.1. *Suppose* $(X, \langle \bullet, \bullet \rangle)$ *is an inner product space with* $\|\bullet\|$ *as the induced norm. Let* A *be a nonempty subset of* X*, then*

(a) *if* $x \perp y$*, then* $\|x+y\|^2 = \|x\|^2 + \|y\|^2$*;*
(b) $0 \in A^\perp$ *(and hence* $A^\perp$ *is nonempty);*
(c) *if* $0 \in A$*, then* $A \cap A^\perp = \{0\}$*;*
(d) $\{0\}^\perp = X$ *and* $X^\perp = \{0\}$*;*
(e) *if* $A^\circ \neq \varnothing$*, then* $A^\perp = \{0\}$*;*
(f) *if* $\bar{A} = X$*, then* $A^\perp = \{0\}$*;*
(g) *if* $A \subset B$*, then* $A^\perp \supset B^\perp$*;*
(h) $A^\perp$ *is a closed linear subspace of* X*;*
(i) $A \subset A^{\perp\perp}$.

Proof. We only prove (e),(h),(i) here since others are trivial.

(e) Since $A^\circ \neq \varnothing$, there exists $a \in A$ and $r > 0$ such that $B(a,r) \subset A$. If $x \in A^\perp$, assume that $x \neq 0$. Then $x \perp A \Rightarrow x \perp B(a,r) \Rightarrow \langle x, a\rangle = 0$. So $\langle x, a + rx/(2\|x\|)\rangle = 0$ since $\|rx/(2\|x\|)\| = r/2 < r \Rightarrow a + rx/(2\|x\|) \in B(a,r)$. Thus,

$$0 = \left\langle x, a + \frac{rx}{2\|x\|} \right\rangle = \langle x, a\rangle + \frac{r}{2\|x\|}\langle x, x\rangle = \frac{r\|x\|}{2}.$$

So $r = 0$, which is a contradiction. Hence, $x = 0 \Rightarrow A^\perp \subset \{0\} \Rightarrow A = \{0\}$ by (b).

(h) For each $x, y \in A^\perp$ and $\alpha, \beta \in \mathbb{F}$, $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle = 0$ for all $z \in A$, which shows that $\alpha x + \beta y \in A^\perp$ and then $A^\perp$ is a linear subspace of X.

For each $x \in \overline{A^\perp}$, there exists $\{x_n\}_{n=1}^\infty \subset A^\perp$ such that $x_n \to x$ $(n \to \infty)$. Compute that $\langle x, a\rangle = \langle \lim_{n\to\infty} x_n, a\rangle = \lim_{n\to\infty}\langle x_n, a\rangle = 0$ holds for all $a \in A$, which indicates that $x \in A^\perp$. Then $\overline{A^\perp} \subset A^\perp \Rightarrow A^\perp$ is closed in X.

(i) For each $x \in A$, $\langle x, y\rangle$ for all $y \in A^\perp$. It's easy to see that $x \in A^{\perp\perp}$ and then $A \subset A^{\perp\perp}$. ☐

Remark 19.2. *Note that* $A \neq A^{\perp\perp}$ *in general. For instance,*

If A *is a closed linear subspace of* X*, then a natural question goes that what assures* $A = A^{\perp\perp}$*. (Note that if* A *is open, then* A *equals to the whole space.) We propose a sufficient condition here.*

Assume that $X = A \oplus A^\perp$*. For all* $x \in A^{\perp\perp} \subset X$*,* $x = y + z$ *for some* $y \in A$ *and* $z \in A^\perp$*, then* $0 = \langle x, z\rangle = \langle y+z, z\rangle = \langle y, z\rangle + \langle z, z\rangle = \langle z, z\rangle \Rightarrow z = 0 \Rightarrow x = y \in A$*. Hence,* $A = A^{\perp\perp}$*.*

Also, note that if $x \in A^\perp$*, then by (a) of Theorem 19.1,* $\|x - y\| \geq \|x\|$ *holds for all* $y \in A \Leftrightarrow d(x, A) \geq \|x\|$*, where* d *is the induced metric on* X*.*

Lemma 19.3. *Let Y be a linear subspace of $(X, \langle \bullet, \bullet \rangle)$, then $x \in Y^\perp \Leftrightarrow \|x-y\| \geq \|x\|$ for all $y \in Y$.*

Proof. $\Rightarrow$) If $x \in Y^\perp$, then $x \perp Y$. So $\|x-y\|^2 = \|x\|^2 + \|y\|^2 \geq \|x\|^2$ for each $y \in Y$ by (a) of Theorem 19.1.

$\Leftarrow$) For all $y \in Y$ and $\alpha \in \mathbb{F}$, it holds that

$$\begin{aligned}\|x\|^2 &\leq \|x - \alpha y\|^2 \\ &= \langle x - \alpha y, x - \alpha y \rangle \\ &= \|x\|^2 + |\alpha|^2 \|y\|^2 - \alpha \overline{\langle x, y \rangle} - \bar{\alpha} \langle x, y \rangle.\end{aligned}$$

Set $\alpha = \langle x, y \rangle / \|y\|^2$ since α is arbitrary. Then

$$\|x\|^2 + |\alpha|^2 \|y\|^2 - \alpha \overline{\langle x, y \rangle} - \bar{\alpha} \langle x, y \rangle = \|x\|^2 - |\langle x, y \rangle| / \|y\|^2.$$

So we derive $|\langle x, y \rangle| / \|y\|^2 \leq 0$, which implies $\langle x, y \rangle = 0$ and then $x \in Y^\perp$ since $y \in Y$ is arbitrary. □

Definition 19.4. *A subset A of a vector space X is said to be* ***convex*** *if for each $x, y \in A$, $[x, y] := \{\lambda x + (1-\lambda) y : 0 \leq \lambda \leq 1\} \subset A$.*

Theorem 19.5. *Let $(X, \langle \bullet, \bullet \rangle)$ be an inner product space and let A be a complete convex subset of X, then for all $x \in X$, there exists a unique element $q \in A$ such that $d(x, q) = \inf_{z \in A} d(x, z) = d(x, A)$ or $\|x - q\| = \inf_{z \in A} \|x - z\| (:= \delta)$.*

Proof. By the definition of infimum, there exists $\{q_n\}_{n=1}^\infty \subset A$ such that $\|q_n - x\| \to \delta$ $(n \to \infty)$. For each $n, m \in \mathbb{N}_+$, $(q_n + q_m)/2 \in A$ by convexity of A. So

$$\begin{aligned}0 \leq \|q_n - q_m\|^2 &= \|(q_n - x) + (x - q_m)\|^2 \\ &= 2(\|q_n - x\|^2 + \|x - q_m\|^2) - \|2x - (q_n + q_m)\|^2 \\ &= 2(\|x - q_n\|^2 + \|x - q_m\|^2) - 4 \left\| x - \frac{q_n + q_m}{2} \right\|^2 \\ &\leq 2(\|x - q_n\|^2 + \|x - q_m\|^2) - 4\delta^2 \to 0 \ (n, m \to \infty),\end{aligned}$$

which shows that $\{q_n\}_{n=1}^\infty$ is a Cauchy sequence in A. Hence, there exists $q \in A$ such that $q_n \to q$ $(n \to \infty)$ since A is complete. Thus $\|x - q\| = \|x - \lim_{n\to\infty} q_n\| = \|\lim_{n\to\infty}(x - q_n)\| = \lim_{n\to\infty} \|x - q_n\| = \delta$.

If $q_0 \in A$ and $\|x - q_0\| = \delta$, then $0 \leq \|q - q_0\|^2 = \|(q - x) + (x - q_0)\|^2 \leq 2(\|x - q\|^2 + \|x - q_0\|^2) - 4\delta^2 = 0$, which indicates that $q_0 = q$. □

20. Orthonormal System: 5/11/2018

Theorem 20.1. *Let $\mathcal{H}$ be a Hilbert space and let Y be a closed linear subspace of $\mathcal{H}$, then for each $x \in \mathcal{H}$, there exist unique $y \in Y$ and unique $z \in Y^\perp$ such that $x = y + z$, or equivalently, $\mathcal{H} = Y \oplus Y^\perp$.*

Proof. Since Y is a closed linear subspace of $\mathcal{H}$, it follows that Y is complete and convex. By Theorem 19.5, for each $x \in \mathcal{H}$, there exists unique $y \in Y$ such that $d(x, y) = d(x, Y) = \inf_{q \in Y} d(x, q)$. Let $z := x - y \in \mathcal{H}$, then for each $u \in Y$, $y + u \in Y$ and $\|z - u\| = \|x - y - u\| = \|x - (y + u)\| \geq \|x - y\| = \|z\|$, which implies $z \in Y^{\perp}$ by Lemma 19.3. Now we see that $x = y + z$ with $y \in Y$ and $z \in Y^{\perp}$.

If $x = y_1 + z_1$ with $y_1 \in Y$ and $z_1 \in Y^{\perp}$, then $y + z = y_1 + z_1 \Rightarrow y - y_1 = z_1 - z \in Y \cap Y^{\perp} = \{0\} \Rightarrow y - y_1 = z_1 - z = 0 \Rightarrow y_1 = y, z_1 = z$. $\square$

Corollary 20.2. *If Y is a closed linear subspace of a Hilbert space $\mathcal{H}$, then $Y = Y^{\perp\perp}$.*

Corollary 20.3. *If A is a linear subspace of a Hilbert space $\mathcal{H}$, then $\bar{A} = A^{\perp\perp}$. If A is only a subset of $\mathcal{H}$, then $\overline{\operatorname{span}(A)} = A^{\perp\perp}$ and $A^{\perp} = A^{\perp\perp\perp}$.*

Definition 20.4. *A collection of nonzero vectors $\{x_i\}_{i \in I} \subset (X, \langle \bullet, \bullet \rangle)$ is said to be an* ***orthogonal system*** *provided that $\langle x_i, x_j \rangle = \alpha \delta_{ij}$ whenever $i, j \in I$, where I is the index set, α is a real positive constant, and δ_{ij} is Kronecker delta. If $\alpha = 1$, $\{x_i\}_{i=1}^{\infty}$ is called an* ***orthonormal system****. Particularly, if I is countable (or simply equals to $\mathbb{N}_+$), then $\{x_i\}_{i \in I}$ is said to be an* ***orthonormal sequence****.*

Example 20.5. *Here comes some examples of orthonormal sequence.*

- *In ℓ^2, the inner product is defined by $\langle x, y \rangle = \sum_{i=1}^{\infty} \xi_i \bar{\eta}_i$ for each $x = \{\xi_i\}_{i=1}^{\infty}$ and $y = \{\eta_i\}_{i=1}^{\infty}$ in ℓ^2. Then $\{e_i = \{\delta_{ij}\}_{j=1}^{\infty}\}_{i=1}^{\infty}$ is an orthonormal sequence in ℓ^2.*
- *In $L^2[-\pi, \pi]$, $\langle x, y \rangle = \int_{-\pi}^{\pi} x(t) \overline{y(t)} \mathrm{d}t$ for all $x, y \in L^2[-\pi, \pi]$. Let*
$$f_0(t) \equiv \frac{1}{\sqrt{2\pi}}, \; f_n(t) = \frac{1}{\sqrt{\pi}} \cos(nt), \; \text{and } g_n(t) = \frac{1}{\sqrt{\pi}} \sin(nt),$$
where $t \in [0, 1]$ and $n \in \mathbb{N}_+$. It's easy to verify that $\{f_0, f_n, g_n\}_{n \in \mathbb{N}_+}$ is an orthonormal sequence in $L^2[-\pi, \pi]$.

Theorem 20.6. *Suppose $\{x_i\}_{i \in I}$ is an orthogonal system in an inner product space X where I is the index set, then*

(a) *for every finitely many elements $x_1, x_2, \cdots, x_n$ in $\{x_i\}_{i \in I}$,*
$$\|x_1 + x_2 + \cdots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2;$$
(b) *$\{x_i\}_{i \in I}$ is a linearly independent subset of X, i.e., every finitely many elements in X are linearly independent.*

Theorem 20.7. *Every infinite-dimensional inner product space contains an orthonormal sequence.*

Remark 20.8. *Theorem 20.7 can be derived by using* ***Gram-Schmidt process****.*

Given a linearly independent set of vectors $\{v_i\}_{i=1}^{n}$ of a inner product space X. For each $u, v \in X$, define $\operatorname{proj}_u(v) = \frac{\langle u, v \rangle}{\langle u, u \rangle} u$ where proj is called the ***projection operator****.*

Then Gram-Schmidt process works as follows:

$$\begin{aligned}
u_1 &= v_1, \\
u_2 &= v_2 - \mathrm{proj}_{u_1}(v_2), \\
u_3 &= v_3 - \mathrm{proj}_{u_1}(v_3) - \mathrm{proj}_{u_2}(v_3), \\
u_4 &= v_4 - \mathrm{proj}_{u_1}(v_4) - \mathrm{proj}_{u_2}(v_4) - \mathrm{proj}_{u_3}(v_4), \\
&\cdots\cdots\cdots\cdots \\
u_n &= v_n - \sum_{k=1}^{n-1} \mathrm{proj}_{u_k}(v_n).
\end{aligned}$$

Let $e_i = u_i/\|u_i\|$ for $i = 1, 2, \cdots, n$, then $\{u_i\}_{i=1}^n$ is an orthogonal system and $\{e_i\}_{i=1}^n$ is an orthonormal system.

Remark 20.9. *Here goes a natural question about orthonormal sequence. If $\{e_i\}_{i=1}^\infty$ is an orthonormal sequence in an inner product space $(X, \langle \bullet, \bullet \rangle)$, then for each $x \in X$, is it possible that there exists $\{c_i\}_{i=1}^\infty \subset \mathbb{F}$ such that $x = \sum_{i=1}^\infty c_i e_i$?*

If so, $\langle x, e_i \rangle = \langle \sum_{i=1}^\infty c_i e_i, e_i \rangle = \sum_{j=1}^\infty c_j \langle e_j, e_i \rangle = c_i$ for each $i \in \mathbb{N}_+$. Then we derive $x = \sum_{i=1}^\infty \langle x, e_i \rangle e_i$. And here comes another two natural questions:

(a) *For each $x \in X$, does $\sum_{i=1}^\infty \langle x, e_i \rangle e_i$ converge?*
(b) *If (a) holds, then does $\sum_{i=1}^\infty \langle x, e_i \rangle e_i$ converge to x?*

If X is a Hilbert space. Then the answer to (a) is yes with respect to norm, that is,

$$\left\| x - \sum_{i=1}^n \langle x, e_i \rangle e_i \right\| \to 0 \ (n \to \infty).$$

In $L^2[-\pi, \pi]$, $\{f_0, f_n, g_n\}_{n \in \mathbb{N}_+}$ is an orthonormal sequence. For each $f \in L^2[-\pi, \pi]$, we have

$$\begin{aligned}
\langle f, f_0 \rangle f_0 &= \left(\int_{-\pi}^{\pi} f(t) \frac{1}{\sqrt{2\pi}} \mathrm{d}t \right) \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \mathrm{d}t, \text{ and for all } n \in \mathbb{N}_+, \\
\langle f, f_n \rangle f_n &= \left(\int_{-\pi}^{\pi} f(t) \frac{1}{\sqrt{\pi}} \cos(nt) \mathrm{d}t \right) \frac{1}{\sqrt{\pi}} \cos(nt) = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \mathrm{d}t \right) \cos(nt), \\
\langle f, g_n \rangle g_n &= \left(\int_{-\pi}^{\pi} f(t) \frac{1}{\sqrt{\pi}} \sin(nt) \mathrm{d}t \right) \frac{1}{\sqrt{\pi}} \sin(nt) = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \mathrm{d}t \right) \sin(nt).
\end{aligned}$$

Set $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \mathrm{d}t$ and $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \mathrm{d}t$, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \mathrm{d}t$ for all $n \in \mathbb{N}_+$. Then

$$\langle f, f_0 \rangle f_0 + \sum_{n=1}^\infty (\langle f, f_n \rangle f_n + \langle f, g_n \rangle g_n) = \frac{a_0}{2} + \sum_{i=1}^\infty (a_n \cos(nt) + b_n \sin(nt)),$$

which is actually the Fourier series of f.

Lemma 21.1. ***Bessel's Inequality***

Let $(X, \langle \bullet, \bullet \rangle)$ be an inner product space and let $\{e_i\}_{i=1}^\infty$ be an orthonormal sequence in X, then for all $x \in X$, $\sum_{i=1}^m |\langle x, e_i\rangle|^2 \le \|x\|^2$ for $m \in \mathbb{N}_+$, and hence $\sum_{i=1}^\infty |\langle x, e_i\rangle|^2 \le \|x\|^2$.

Proof. Set $S_k = \sum_{i=1}^k \langle x, e_i\rangle e_i$ for each $k \in \mathbb{N}_+$. Since

$$\begin{aligned}\|x - S_k\|^2 &= \langle x - S_k, x - S_k\rangle = \|x\|^2 - \langle x, S_k\rangle - \langle S_k, x\rangle + \langle S_k, S_k\rangle \\ &= \|x\|^2 - \left\langle x, \sum_{i=1}^k \langle x, e_i\rangle e_i \right\rangle - \left\langle \sum_{i=1}^k \langle x, e_i\rangle e_i, x \right\rangle + \left\langle \sum_{i=1}^k \langle x, e_i\rangle e_i, \sum_{i=1}^k \langle x, e_i\rangle e_i \right\rangle \\ &= \|x\|^2 - 2\sum_{i=1}^k |\langle x, e_i\rangle|^2 + \sum_{i=1}^k \sum_{j=1}^k \langle x, e_i\rangle \overline{\langle x, e_j\rangle} \langle e_i, e_j\rangle = \|x\|^2 - \sum_{i=1}^k |\langle x, e_i\rangle|^2,\end{aligned}$$

it follows that $\sum_{i=1}^k |\langle x, e_i\rangle|^2 = \|x\|^2 - \|x - S_k\|^2 \le \|x\|^2$ for each $k \in \mathbb{N}_+$. □

Remark 21.2. *If $\sum_{i=1}^\infty |\langle x, e_i\rangle|^2 = \|x\|^2$, then it's called* ***Parseval's identity****.*

Lemma 21.3. ***Riesz-Fischer Theorem***

Let $\mathcal{H}$ be a Hilbert space and let $\{e_i\}_{i=1}^\infty$ be an orthonormal sequence of $\mathcal{H}$. Then for each $\{\alpha_i\}_{i=1}^\infty \subset \mathbb{F}$, the series $\sum_{i=1}^\infty \alpha_i e_i$ converges $\Leftrightarrow$ $\{\alpha_i\}_{i=1}^\infty \in \ell^2$, i.e., $\sum_{i=1}^\infty |\alpha_i|^2 < \infty$. Moreover, if the series converges, then $\|\sum_{i=1}^\infty \alpha_i e_i\|^2 = \sum_{i=1}^\infty |\alpha_i|^2$.

Proof. $\Rightarrow$) If $\sum_{i=1}^\infty \alpha_i e_i$ converges to, say, $x \in \mathcal{H}$, then $\alpha_i = \langle x, e_i\rangle$ $(i \in \mathbb{N}_+)$. So $\{\langle x, e_i\rangle\}_{i=1}^\infty \in \ell^2$ by Lemma 21.1.

$\Leftarrow$) Let $S_k = \sum_{i=1}^k \alpha_i e_i$ and $\sigma_k = \sum_{i=1}^k |\alpha_i|^2$ for each $k \in \mathbb{N}_+$. When $n > m$,

$$\|S_n - S_m\|^2 = \left\| \sum_{i=m+1}^n \alpha_i e_i \right\|^2 = \sum_{i=m+1}^n |\alpha_i|^2 = \sigma_n - \sigma_m = |\sigma_n - \sigma_m|$$

by Theorem 20.6. If follows that $\{S_k\}_{k=1}^\infty$ is a Cauchy sequence in $\mathcal{H}$ iff $\{\sigma_k\}_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{F}$. Since $\{\sigma_k\}_{k=1}^\infty$ is convergent in $\mathbb{F}$, $\{\sigma_k\}_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{F}$. So $\{S_k\}_{k=1}^\infty$ is a Cauchy sequence in $\mathcal{H}$. Hence, there exists $x \in \mathcal{H}$ such that $S_k \to x \in \mathcal{H}$ $(k \to \infty)$ because $\mathcal{H}$ is complete. So $\sum_{i=1}^\infty \alpha_i e_i$ converges to x. □

Corollary 21.4. *Let $\mathcal{H}$ be a Hilbert space and $\{e_i\}_{i=1}^\infty$ be an orthonormal sequence in $\mathcal{H}$, then for each $x \in \mathcal{H}$, $\sum_{i=1}^\infty \langle x, e_i\rangle e_i$ converges.*

Remark 21.5. *Let $\{e_i\}_{i=1}^\infty$ be an orthonormal sequence in $\mathcal{H}$, then $\{e_{2i}\}_{i=1}^\infty$ is also an orthonormal sequence in $\mathcal{H}$. However, $e_1 \ne \sum_{i=1}^\infty \langle e_1, e_{2i}\rangle e_{2i}$, though $\sum_{i=1}^\infty \langle x, e_{2i}\rangle e_{2i}$ converges in $\mathcal{H}$.*

Theorem 21.6. *Let $\mathcal{H}$ be a Hilbert space and let $\{e_i\}_{i=1}^\infty$ be an orthonormal sequence in $\mathcal{H}$, then the following statements are equivalent.*

(a) *For each* $x \in \mathcal{H}$, $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$.
(b) *For each* $x \in \mathcal{H}$, $\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 = \|x\|^2$.
(c) $\overline{\operatorname{span}(\{e_i : i \in \mathbb{N}_+\})} = \mathcal{H}$.
(d) $\{e_i\}_{i \in \mathbb{N}_+}^{\perp} = \{0\}$.

Remark 21.7. *Cauchy-Schwarz inequality and Bessel inequality are equivalent in inner product spaces.*

Definition 21.8. *Let* $\mathcal{H}$ *be a Hilbert space and* $S = \{e_i\}_{i \in I} \subset \mathcal{H}$ *be an orthonormal system.* S *is said to be an* ***orthonormal basis*** *of* $\mathcal{H}$ *if* S *cannot be extended to a larger orthonormal system.*

Proposition 21.9. *Suppose* $S = \{e_i\}_{i \in I}$ *is an orthonormal system in a Hilbert space* $\mathcal{H}$*, then* S *is an orthonormal basis of* $\mathcal{H}$ *iff* $S^{\perp} = \{0\}$.

Theorem 21.10. *Every nonzero Hilbert space has an orthonormal basis.*

Theorem 21.11. *Let* $\mathcal{H}$ *be an infinite-dimensional Hilbert space, then* $\mathcal{H}$ *has a countable orthonormal basis iff* $\mathcal{H}$ *is separable.*

Definition 21.12. *Suppose* $(X, \| \bullet \|_X)$ *and* $(Y, \| \bullet \|_Y)$ *are normed linear spaces, then* $X = Y$ *in the sense of* ***isometric isomorphism*** *provided that there exists a bijection* $\varphi : X \to Y$ *such that* $\varphi(\alpha x + \beta y) = \alpha \varphi(x) + \beta \varphi(y)$ *and* $\|\varphi(x)\|_Y = \|x\|_X$ *for all* $x, y \in X$ *and* $\alpha, \beta \in \mathbb{F}$.

Remark 21.13. *Actually, for an isometric isomorphism* φ, φ *and* φ^{-1} *are continuous by isometry. Hence,* φ *is a homeomorphism.*

Theorem 21.14. *Let* $\mathcal{H}$ *be an infinite-dimensional separable Hilbert space, then* $\mathcal{H} = \ell^2$ *in the sense of isometric isomorphism.*

Proof. If $\dim(\mathcal{H}) = \infty$, set $\varphi : \mathcal{H} \to \ell^2$ defined by $\varphi(x) = \{\langle x, e_i \rangle\}_{i=1}^{\infty} \in \ell^2$ which is well-defined according to Lemma 21.1, and the existence of the orthonormal basis $\{e_i\}_{i=1}^{\infty}$ is given by Theorem 21.11. Then for each $\{\alpha_i\}_{i=1}^{\infty} \in \ell^2$, there exists $y \in \mathcal{H}$ such that $\sum_{i=1}^{\infty} \alpha_i e_i = y$ since $\mathcal{H}$ is complete, and then $\alpha_i = \langle y, e_i \rangle$ ($i \in \mathbb{N}_+$). Note that φ is a surjective linear map. Hence it is injective. So φ is an isomorphism. Note that for each $x, y \in \mathcal{H}$,

$$\langle x, y \rangle = \left\langle \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i, \sum_{i=1}^{\infty} \langle y, e_i \rangle e_i \right\rangle = \sum_{i=1}^{\infty} \langle x, e_i \rangle \overline{\langle y, e_i \rangle} = \langle \varphi(x), \varphi(y) \rangle,$$

that is, φ preserves the inner product, and then φ is isometric. Thus, $\mathcal{H} = \ell^2$ in the sense of isometric isomorphism. □

Remark 21.15. *Suppose* $\mathcal{H}$ *is an* n*-dimensional inner product space over* $\mathbb{F}$*, then* $\mathcal{H} = \mathbb{F}^n$ *in the sense of isometric isomorphism.*

Remark 21.16. *Actually, there exists a continuous function whose Fourier series diverge at all rational multiples of* 2π *(and hence on a dense set).*

22. Linear Operator: 5/18/2018

Definition 22.1. *Let X and Y be linear spaces over the same scalar field $\mathbb{F}$, and let $T : X \to Y$ be a mapping. T is said to be a **linear operator** if $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$ for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$.*

*The **kernel** and **image** of T are defined to be* $\operatorname{Ker}(T) := \{x \in X : Tx = 0\}$ *and* $\operatorname{Im}(T) := TX = \{Tx : x \in X\}$ *respectively.*

*If $Y = \mathbb{F}$, then T is called a **linear functional**.*

Definition 22.2. *Let $T : X \to X$ be a mapping from a linear space X over the scalar field $\mathbb{F}$ to itself. If T is given by $Tx = \lambda x$ for each $x \in X$ where $\lambda \in \mathbb{F}$ is some fixed scalar, then T is called a **similar operator** or **scalar operator**. If $\lambda = 1$, T is called an **identity operator**, denoted I_X. If $\lambda = 0$, T is called a **zero operator**, denoted* $\mathbf{0}$.

Definition 22.3. *Let X, Y be two normed linear spaces, and let $T : X \to Y$ be a mapping. T is said to be **bounded** provided that there exists a constant $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ whenever $x \in X$.*

Remark 22.4. *Note that we have involved several norms of different spaces in Definition 22.3. We use this abuse of notation when there is no confusion in the context.*

Remark 22.5. *Note that the definition of boundedness for linear operators are different from that of the usual functions in calculus. It's kind of some "relative boundedness" or "local boundedness".*

Theorem 22.6. *Let X, Y be two normed linear spaces, and let $T : X \to Y$ be a linear operator. Then the following statements are equivalent:*

(a) *T is uniformly continuous on X;*
(b) *T is continuous on X;*
(c) *T is continuous at $x = 0$;*
(d) *$\{\|Tx\| : x \in X, \|x\| \leq 1\}$, as a subset of $\mathbb{R}$, is bounded; and*
(e) *T, as a linear operator, is bounded.*

Proof. It's obvious that (a)⇒(b)⇒(c).

(c)⇒(d) Since T is continuous at $x = 0$, there exists $\delta > 0$ such that for all $x \in X$, $\|x - 0\| = \|x\| \leq \delta \Rightarrow \|Tx\| = \|Tx - T0\| \leq 1$. Then for each $x \in X$ with $\|x\| \leq 1$, $\|\delta x\| \leq \delta$ and $\|Tx\| = \|T(\delta x)\|/\delta \leq 1/\delta$.

(d)⇒(e) Since there exists $M > 0$ such that $\|Tx\| \leq M$ holds for all $x \in X$ with $\|x\| \leq 1$. Then for each $x \in X$ with $x \neq 0$, $\|x/\|x\|\| = 1$ and $\|T(x/\|x\|)\| \leq M \Rightarrow \|Tx\|/\|x\| \leq M \Rightarrow \|Tx\| \leq M\|x\|$, which is also true for $x = 0$.

(e)⇒(a) Since T is bounded, there exists $M > 0$ such that $\|Tx\| \leq M\|x\|$ holds for each $x \in X$. For every $\varepsilon > 0$, let $\delta = \varepsilon/M$, then for all $x, y \in X$ with $\|x - y\| < \delta$, $\|Tx - Ty\| = \|T(x - y)\| \leq M\|x - y\| < M\delta = \varepsilon$. □

Definition 22.7. *Let X, Y be normed linear spaces over $\mathbb{F}$, and let $T : X \to Y$ be a bounded linear operator. Then*

$$\|T\| := \sup_{x \in X,\ x \neq 0} \frac{\|Tx\|}{\|x\|}$$

*is called the **norm of operator** T.*

Remark 22.8. *Note that $\|T\|$ is well-defined provided that T is linear and bounded.*

Theorem 22.9. *Let X, Y be normed linear spaces, and let $T : X \to Y$ be a bounded linear operator. Then $\|T\|$ is well-defined and $\|Tx\| \leq \|T\|\|x\|$ whenever $x \in X$. Moreover, $\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|$.*

Proof. Since T is a bounded linear operator, $\{\|Tx\|/\|x\| : x \in X, x \neq 0\}$ is bounded. If follows that $\|T\|$ is well-defined and Then $\|Tx\| \leq \|T\|\|x\|$ whenever $x \in X$. Hence, $\sup_{\|x\|=1} \|Tx\| \leq \sup_{\|x\| \leq 1} \|Tx\| \leq \|T\|$.

For all $x \in X$ with $x \neq 0$, $\|Tx\|/\|x\| = \|T(x/\|x\|)\| \leq \sup_{\|x\|=1} \|Tx\|$, which implies $\|T\| \leq \sup_{\|x\|=1} \|Tx\|$.

Now we see that $\sup_{\|x\|=1} \|Tx\| \leq \sup_{\|x\| \leq 1} \|Tx\| \leq \|T\| \leq \sup_{\|x\|=1} \|Tx\|$. Thus, $\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|$. $\square$

Remark 22.10. *It's usually hard to compute the norm of a given bounded linear operator. It would not be easier than computing the sum of a given convergent series.*

23. Examples of Bounded Linear Operator I: 5/23/2018

Example 23.1. *Suppose $T : X \to Y$ is a similar operator (see Definition 22.2), i.e., there exists $\lambda \in \mathbb{F}$ such that $Tx = \lambda x$ whenever $x \in X$.*

Clearly, T is linear. For each $x \in X$, $\|Tx\| = \|\lambda x\| = |\lambda|\|x\|$. So T is bounded and $\|T\| \leq |\lambda|$. When $x \neq 0$, $\|Tx\|/\|x\| = |\lambda|$. So $\|T\| = \sup_{x \neq 0}(\|Tx\|/\|x\|) \geq |\lambda|$. Hence, $\|T\| = |\lambda|$.

Example 23.2. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Tx = Ax$ for each $x \in \mathbb{R}^n$ where A is a matrix with $A^T = A^{-1}$.*

Clearly, T is linear. Note that A is an orthogonal matrix which is regarded as a rotoinversion geometrically when operated on an element of $\mathbb{R}^n$, preserving the norm of a vector in $\mathbb{R}^n$. Thus, it follows that $\|Tx\| = \|Ax\| = \|x\|$. So T is bounded and $\|T\| \leq 1$. When $x \neq 0$, $\|Tx\|/\|x\| = 1$. Hence, $\|T\| = \sup_{x \neq 0}(\|Tx\|/\|x\|) \geq 1$. So $\|T\| = 1$. Note that the norm $\| \bullet \|$ on $\mathbb{R}^n$ here is the usual norm on $\mathbb{R}^n$ which can be induced from some inner product on $\mathbb{R}^n$.

Example 23.3. *Suppose $f : C[a, b] \to \mathbb{R}$ is defined by $f(x) = \int_a^b x(t)\mathrm{d}t$ for each $x \in C[a, b]$. Clearly, f is linear. $\|f(x)\| = |f(x)| = |\int_a^b x(t)\mathrm{d}t| \leq (b-a)\max_{a \leq t \leq b} |x(t)| = (b-a)\|x\|$. It follows that f is bounded and $\|f\| \leq b-a$. Choose $x_0 \in C[a, b]$ such that*

$x_0(t) \equiv 1$ on $[0,1]$, then $\|x_0\| = 1$ and $|f(x_0)| = b-a$. So $\|f\| = \sup_{\|x\|=1} |f(x)| \geq |f(x_0)| = b-a \Rightarrow \|f\| = b-a$.

Example 23.4. *Define a functional $f : \ell^2 \to \mathbb{R}$ as $f(x) = \sum_{n=1}^{\infty} a_n x_n$ for each $x = \{x_n\}_{n=1}^{\infty} \in \ell^2$ where $a = \{a_n\}_{n=1}^{\infty} \in \ell^2$ is given.*

Since $\sum_{n=1}^{\infty} |a_n x_n| \leq (\sum_{n=1}^{\infty} |a_n|^2)^{1/2} (\sum_{n=1}^{\infty} |x_n|^2)^{1/2} < \infty$ for each $x = \{x_n\}_{n=1}^{\infty} \in \ell^2$, f is well-defined.

Clearly, f is linear. For each $x = \{x_n\}_{n=1}^{\infty}$, $|f(x)| = |\sum_{n=1}^{\infty} a_n x_n| \leq \sum_{n=1}^{\infty} |a_n x_n| \leq \|a\|\|x\| \Rightarrow f$ is bounded and $\|f\| \leq \|a\|$. Without loss of generality, we may assume that $\|a\| \neq 0$. Choose $x_0 = \bar{a}/\|a\| \in \ell^2$, then $\|x_0\| = 1$ and $|f(x_0)| = \|a\|$. $\|f\| = \sup_{\|x\|=1} |f(x)| \geq |f(x_0)| = \|a\| \Rightarrow \|f\| = \|a\|$.

Example 23.5. *Define a functional $f : \ell^1 \to \mathbb{R}$ as $f(x) = \sum_{n=1}^{\infty} a_n x_n$ for each $x = \{x_n\}_{n=1}^{\infty} \in \ell^1$ where $\{a_n\}_{n=1}^{\infty} := a \in \ell^{\infty}$ is given.*

Since $\sum_{n=1}^{\infty} |a_n x_n| \leq \|a\| \sum_{n=1}^{\infty} |x_n|$, $\|a\| < \infty$, and $\sum_{n=1}^{\infty} |x_n| < \infty$ for each $x \in \{x_n\}_{n=1}^{\infty}$, f is well-defined.

Clearly, f is linear. For all $x \in \ell^1$, $|f(x)| = |\sum_{n=1}^{\infty} a_n x_n| \leq \sum_{n=1}^{\infty} |a_n x_n| \leq \|a_n\|\|x\| \Rightarrow f$ is bounded and $\|f\| \leq \|a\|$. Choose $\{e_n\}_{n=1}^{\infty}$ as the canonical basis of ℓ^1, then $\|e_n\|_{\ell^1} = 1$ and $|f(e_n)| = |a_n|$ for each $n \in \mathbb{N}_+$. $\|f\| = \sup_{\|x\|=1} |f(x)| \geq |f(e_n)| = |a_n|$ for each $n \in \mathbb{N}_+ \Rightarrow \|f\| \geq \sup_{n \geq 1} |a_n| = \|a\|$. Hence, $\|f\| = \|a\|$.

Proposition 23.6. *For f defined in Example 23.5, it attains its norm on the closed unit ball iff $\|a\| \in \{|a_n| : n \in \mathbb{N}_+\}$.*

Proof. $\Rightarrow$) If not, then $\|a\| \neq |a_n|$ for all $n \in \mathbb{N}_+$. Note that there exists $x_0 = \{x_n\}_{n=1}^{\infty} \in \ell^1$ such that $\|x_0\| = \sum_{n=1}^{\infty} |x_n| \leq 1$, and then $\|a\| = \|f\| = |f(x_0)| = |\sum_{n=1}^{\infty} a_n x_n| \leq \sum_{n=1}^{\infty} |a_n x_n| \leq \|a\| \sum_{n=1}^{\infty} |x_n| \leq \|a\|$. This shows that $\sum_{n=1}^{\infty} |a_n||x_n| = \|a\| \sum_{n=1}^{\infty} |x_n| \Rightarrow \sum_{n=1}^{\infty} (\|a\| - |a_n|)|x_n| = 0$. So $(\|a\| - |a_n|)|x_n| = 0$ for each $n \in \mathbb{N}_+ \Rightarrow |x_n| = 0$ for each $n \in \mathbb{N}_+$. Hence, $\|a\| = 0 \Rightarrow a_n = 0$ for each $n \in \mathbb{N}_+ \Rightarrow \|a\| = |a_n|$ for each $n \in \mathbb{N}_+$, which is a contradiction. Thus, $\|a\| = |a_n|$ for some $n \in \mathbb{N}_+$.

$\Leftarrow$) If $\|a\| \in \{|a_n| : n \in \mathbb{N}_+\}$, then we may assume that $\|a\| = |a_1|$. Taking $x_0 = (\bar{a}_1/|a_1|, 0, \cdots, 0) \in \ell^1$, $f(x_0) = |a_1| = \|a\| = \|f\|$. ☐

24. Examples of Bounded Linear Operator II: 5/25/2018

Example 24.1. *Let $f : c_0 \to \mathbb{R}$ be $f(x) = \sum_{n=1}^{\infty} \xi_n / n!$ for each $x = \{\xi_n\}_{n=1}^{\infty} \in c_0$.*

Clearly, f is well-defined and linear.

For each $x = \{\xi_n\}_{n=1}^{\infty} \in c_0$,

$$|f(x)| = |\sum_{n=1}^{\infty} \xi_n/n!| \leq \sum_{n=1}^{\infty} |\xi_n|/n! \leq \max_{n \geq 1} |\xi_n| \sum_{n=1}^{\infty} 1/n! = \|x\| \sum_{n=1}^{\infty} 1/n!.$$

Hence, f is bounded and $\|f\| \leq \sum_{n=1}^{\infty} 1/n!$. For each $n \in \mathbb{N}_+$, let $x_n \in c_0$ such that the first n terms in x_n are 1 while the rest are 0, then $\|x_n\| = 1$ and $f(x_n) = \sum_{k=1}^{n} 1/k!$.

Thus, $\|f\| = \sup_{\|x\|=1} |f(x)| \geq |f(x_n)| = f(x_n) = \sum_{k=1}^{n} 1/k!$ *holds for each* $n \in \mathbb{N}_+$*. So we derive* $\|f\| \geq \sum_{n=1}^{\infty} 1/n! \Rightarrow \|f\| = \sum_{n=1}^{\infty} 1/n!$*.*

Example 24.2. *Let* $T : L^1[a,b] \to L^1[a,b]$ *be defined by* $Tx(t) = \int_a^t x(s)\mathrm{d}s$ $(t \in [a,b])$ *for each* $x \in L^1[a,b]$*.*

Clearly, T *is linear.*

For each $x \in L^1[a,b]$*,*

$$\begin{aligned} \|Tx\| &= \int_a^b \left| \int_a^t x(s)\mathrm{d}s \right| \mathrm{d}t \leq \int_a^b \left(\int_a^b |x(s)|\mathrm{d}s \right) \mathrm{d}t \\ &= (b-a) \int_a^b |x(t)|\mathrm{d}t = (b-a)\|x\|. \end{aligned}$$

Hence, T *is bounded and* $\|T\| \leq b - a$*. Choose* $n \in \mathbb{N}_+$ *such that* $a + 1/n < b$*, then let* $x_n(t) = n$ *when* $t \in [a, a+1/n]$ *and* $x_n(t) = 0$ *when* $t \in (a+1/n, b]$*. Note that* $x_n \in L^1[a,b]$ *and* $\|x_n\| = 1$*. Hence,* $\|T\| = \sup_{\|x\|=1} \|Tx\| \geq \|Tx_n\| = \int_a^b |\int_a^t x_n(s)\mathrm{d}s|\mathrm{d}t = \int_a^{a+1/n} |\int_a^t n\mathrm{d}s|\mathrm{d}t + \int_{a+1/n}^b |\int_a^{a+1/n} n\mathrm{d}s|\mathrm{d}t = n\int_a^{a+1/n}(t-a)\mathrm{d}t + \int_{a+1/n}^b \mathrm{d}t = 1/(2n) + b - a - 1/n = b - a - 1/(2n)$ *holds whenever* $n \in \mathbb{N}_+$ *and* $a + 1/n < b \Rightarrow \|T\| \geq b - a$*. Hence,* $\|T\| = b - a$*.*

Theorem 24.3. *Let* X *and* Y *be two normed linear spaces and let* $\dim(X) < \infty$*, then each linear operator* $T : X \to Y$ *is bounded.*

Proof. Define a new norm on X by $\|x\|_1 = \|x\|_X + \|Tx\|_Y$ for each $x \in X$. It's easy to verify that $\| \bullet \|_1$ is indeed a norm on X. Hence, $\| \bullet \|_1$ and $\| \bullet \|_X$ are equivalent since $\dim(X) < \infty$. Then it's easy to see that there exists $M > 0$ satisfying $\|Tx\|_Y \leq M\|X\|_X$ for each $x \in X$. $\square$

Theorem 24.4. *Let* f *be a linear functional defined on a normed linear space* X*. Then* f *is bounded (and hence continuous) iff* $\mathrm{Ker}(f) = \{x \in X : f(x) = 0\}$ *is closed.*

Proof. $\Rightarrow$) For each $x \in \overline{\mathrm{Ker}(f)}$, there exists $\{x_n\}_{n=1}^{\infty} \subset \mathrm{Ker}(f)$ such that $x_n \to x$ $(n \to \infty)$. Then $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = 0$, which implies that $x \in \mathrm{Ker}(f)$.

$\Leftarrow$) If f is unbounded, then there exists $\{x_n\}_{n=1}^{\infty} \subset X$ with $\|x_n\| = 1$ such that $|f(x_n)| > n$ for each n. Let $y_n = x_n/f(x_n) - x_1/f(x_1)$ and then $f(y_n) = 0 \Rightarrow y_n \in \mathrm{Ker}(f)$ for each n. Clearly, $0 \leq \|x_n/f(x_n)\| = \|x_n\|/|f(x_n)| < 1/n \to 0$ $(n \to \infty) \Rightarrow x_n/f(x_n) \to 0$ $(n \to \infty)$. So $y_n \to -x_1/f(x_1)$ $(n \to \infty)$. But $f(-x_1/f(x_1)) = -1 \neq 0$. Hence, $-x_1/f(x_1) \notin \mathrm{Ker}(f)$, which is a contradiction with respect to the closedness of $\mathrm{Ker}(f)$. Thus, f is bounded. $\square$

25. Space of Bounded Linear Operators: 5/30/2018

Definition 25.1. *Let* X *and* Y *be normed linear spaces over* $\mathbb{F}$*. We denote the set of all bounded (i.e., continuous) linear operators from* X *to* Y *by* $\mathscr{B}(X, Y)$*.*

Theorem 25.2. *$\mathscr{B}(X,Y)$ is a normed linear space equipped with the linear operation $(\alpha T_1 + \beta T_2)(x) = \alpha T_1 x + \beta T_2 y$ defined for all $\alpha, \beta \in \mathbb{F}$, $x, y \in X$, and $T_1, T_2 \in \mathscr{B}(X,Y)$. The norm on $\mathscr{B}(X,Y)$ is defined by $\|T\| = \sup_{x\neq 0} \|Tx\|/\|x\|$ for each $T \in \mathscr{B}(X,Y)$.*

Proof. It's clear that for all $\alpha, \beta \in \mathbb{F}$ and $T_1, T_2 \in \mathscr{B}(X,Y)$, $\alpha T_1 + \beta T_2 : X \to Y$ is linear. For each $x \in X$, $\|(\alpha T_1 + \beta T_2)(x)\| = \|\alpha T_1 x + \beta T_2 x\| \leq (|\alpha|\|T_1\| + |\beta|\|T_2\|)\|x\| = M\|x\|$, where $M = |\alpha|\|T_1\| + |\beta|\|T_2\|$ is a nonnegative constant. Hence, $\alpha T_1 + \beta T_2$ is bounded. So $\alpha T_1 + \beta T_2 \in \mathscr{B}(X,Y)$ and then $\mathscr{B}(X,Y)$ is a linear space.

It's obvious that $\|T\| \geq 0$ and $\|T\| = 0 \Leftrightarrow \|Tx\| = 0$ for all $x \in X \Leftrightarrow Tx = 0$ for all $x \in X \Leftrightarrow T = 0$. Also, $\sup_{x\neq 0} \|\alpha Tx\|/\|x\| = |\alpha| \sup_{x\neq 0} \|Tx\|/\|x\| = |\alpha|\|T\|$ for all $\alpha \in \mathbb{F}$. Since $\|(T_1+T_2)(x)\| = \|T_1 x + T_2 x\| \leq \|T_1 x\| + \|T_2 x\| \leq (\|T_1\| + \|T_2\|)\|x\|$ for all $T_1, T_2 \in \mathscr{B}(X,Y)$ and $x \in X$. So $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$ for all $T_1, T_2 \in \mathscr{B}(X,Y)$. Now we see that $\mathscr{B}(X,Y)$ is a normed linear space. □

Definition 25.3. *In particular, if $Y = \mathbb{F}$, then $\mathscr{B}(X,Y)$ is denoted by X^*, which is called the* ***dual space*** *of X.*

Theorem 25.4. *$\mathscr{B}(X,Y)$ is complete iff Y is complete.*

Proof. $\Leftarrow$) For each Cauchy sequence $\{T_n\}_{n=1}^\infty$ in $\mathscr{B}(X,Y)$, $\{T_n x\}_{n=1}^\infty$ is a Cauchy sequence in Y for each $x \in X$. Since Y is complete, there exists $y \in Y$ such that $T_n x \to y$ as $n \to \infty$.

Define $T : X \to Y$ by $Tx = y = \lim_{n\to\infty} T_n x$ for each $x \in X$, then $T \in \mathscr{B}(X,Y)$ and $\|T_n - T\| \to 0$ as $n \to \infty$.

For all $\alpha \in \mathbb{F}$ and $x_1, x_2 \in X$,

$$\begin{aligned} T(\alpha x_1 + \beta x_2) &= \lim_{n\to\infty} T_n(\alpha x_1 + \beta x_2) = \lim_{n\to\infty} (\alpha T_n x_1 + \beta T_n x_2) \\ &= \alpha \lim_{n\to\infty} T_n x_1 + \beta \lim_{n\to\infty} T_n x_2 = \alpha T x_1 + \beta T x_2. \end{aligned}$$

So T is linear. Since $\{T_n\}_{n=1}^\infty$ is a Cauchy sequence, there exists $M > 0$ such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}_+$. And for all $x \in X$, $\|Tx\| = \|\lim_{n\to\infty} T_n x\| = \lim_{n\to\infty} \|T_n x\| \leq \|x\| \lim_{n\to\infty} \|T_n\| \leq M\|x\|$. So T is bounded. Thus, $T \in \mathscr{B}(X,Y)$.

$\Rightarrow$) Assume $X \neq \{0\}$. Let $x_0 \in X$ and $\|x_0\| = 1$, then there exists $f \in X^*$ such that $f(x_0) = 1$ by Corollary 29.6. For each Cauchy sequence $\{y_n\}_{n=1}^\infty \subset Y$, define $\{T_n\}_{n=1}^\infty \subset \mathscr{B}(X,Y)$ by $T_n x = f(x) y_n$ for each $x \in X$ and $n \in \mathbb{N}_+$.

$\|T_n - T_m\| = \sup_{\|x\|=1} \|(T_n - T_m)(x)\| = \|y_n - y_m\| \sup_{\|x\|=1} |f(x)| = \|y_n - y_m\|\|f\| \Rightarrow \{T_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{B}(X,Y)$. Hence, there exists $T \in \mathscr{B}(X,Y)$ such that $T_n \to T$ $(n \to \infty) \Leftrightarrow \|T_n - T\| = \sup_{\|x\|=1} \|T_n x - Tx\| \to 0$ $(n \to \infty)$. In particular, $\|y_n - Tx_0\| = \|T_n x_0 - Tx_0\| \to 0$ $(n \to \infty)$, that is, $y_n \to Tx_0 \in Y$ $(n \to \infty)$. So Y is complete. □

Corollary 25.5. *Suppose X is a normed linear space, then the dual space X^* is a Banach space.*

Theorem 25.6. ***Riesz-Fréchet Theorem***

If $\mathcal{H}$ is a Hilbert space and $f \in \mathcal{H}^*$ is fixed, then there exists a unique $x_f \in \mathcal{H}$ such that $f(x) = \langle x, x_f \rangle$ for all $x \in \mathcal{H}$. Moreover, $\|f\| = \|x_f\|$.

Proof. If $f = 0$, then $x_f = 0$ is the desired one and it's obvious that $\|f\| = \|x_f\|$.

So we may assume that $f \neq 0$. Then $\operatorname{Ker}(f) = \{x \in \mathcal{H} : f(x) = 0\} \neq \mathcal{H} = \operatorname{Ker}(f) \oplus \operatorname{Ker}(f)^{\perp}$ since $\operatorname{Ker}(f)$ is a closed linear subspace of $\mathcal{H}$ by Theorem 24.4. Here, $\mathcal{H} = \operatorname{Ker}(f) \oplus \operatorname{Ker}(f)^{\perp}$ is derived by Theorem 20.1. Since it's clear that f is linear and $\operatorname{Ker}(f)$ is a proper subspace of $\mathcal{H}$, there exists $z \in \operatorname{Ker}(f)^{\perp}$ such that $f(z) = 1$. Then for all $x \in \mathcal{H}$, $x - f(x)z \in \operatorname{Ker}(f)$ since $f(x - f(x)z) = f(x) - f(f(x)z) = f(x) - f(x)f(z) = 0$. Thus, $\langle x - f(x)z, z \rangle = 0 \Rightarrow \langle x, z \rangle - f(x)\|z\|^2 = 0$, i.e., $f(x) = \langle x, z/\|z\|^2 \rangle$. So $x_f = z/\|z\|^2$ is a desired one.

Since $|f(x)| = |\langle x, x_f \rangle| \leq \|x\|\|x_f\|$ for each $x \in \mathcal{H}$, we derive $\|f\| \leq \|x_f\|$. Choose $y_0 = x_f/\|x_f\| \in \mathcal{H}$, then $\|y_0\| = 1$ and $f(y_0) = \|x_f\|$. So $\|f\| = \sup_{\|y\|=1} |f(y)| \geq |f(y_0)| = |\langle y_0, x_f \rangle| = \|x_f\|$. Hence, $\|f\| = \|x_f\|$.

If $\tilde{x} \in \mathcal{H}$ such that $f(x) = \langle x, \tilde{x} \rangle = \langle x, x_f \rangle$ for all $x \in \mathcal{H}$, then $\langle x, \tilde{x} - x_f \rangle = 0$ for all $x \in \mathcal{H}$. Thus, $\tilde{x} = x_f$ by (1) of Theorem 16.3. $\square$

Corollary 25.7. *Suppose $\mathcal{H}$ is a Hilbert space over $\mathbb{F}$.*

(a) *If $f \in \mathcal{H}$, then there exists a unique $y \in \mathcal{H}$ such that $f(x) = \langle x, y \rangle$ for all $x \in \mathcal{H}$. Moreover, $\|f\| = \|y\|$.*
(b) *For any given $y \in \mathcal{H}$, $\langle x, y \rangle$ defines an element of $\mathcal{H}^*$.*
(c) *If $\mathbb{F} = \mathbb{R}$, then $\mathcal{H}^* = \mathcal{H}$ in the sense of isometrical isomorphism.*

Proof. (a) Immediately derived from Theorem 25.6.

(b) Trivial.

(c) Define $T : \mathcal{H} \to \mathcal{H}^*$ by $Ty = f_y$ for each $y \in \mathcal{H}$, where $f_y(x) = \langle x, y \rangle$ for all $x \in \mathcal{H}$. T is well-defined by (b) and T is surjective by (a). Basically, for each $f \in \mathcal{H}^*$, there is a unique $x_f \in \mathcal{H}$ such that $f(x) = \langle x, x_f \rangle$ for all $x \in \mathcal{H}$. Moreover, $\|Ty\| = \|f_y\| = \|y\|$ for each $y \in \mathcal{H}$. So T is an isometry.

Note that $\mathbb{F} = \mathbb{R}$. So for each $x \in \mathcal{H}$, $T(\alpha y_1 + \beta y_2)(x) = f_{\alpha y_1 + \beta y_2}(x) = \langle x, \alpha y_1 + \beta y_2 \rangle = \bar{\alpha}\langle x, y_1 \rangle + \bar{\beta}\langle x, y_2 \rangle = \alpha f_{y_1}(x) + \beta f_{y_2}(x) = (\alpha f_{y_1} + \beta f_{y_2})(x) = (\alpha T y_1 + \beta T y_2)(x)$. Then $T(\alpha y_1 + \beta y_2) = f_{\alpha y_1 + \beta y_2} = \alpha f_{y_1} + \beta f_{y_2} = \alpha T y_1 + \beta T y_2$. Thus, T is one-to-one and then an isomorphism. $\square$

Remark 25.8. *A Hilbert space $\mathcal{H}$ with $\mathcal{H}^* = \mathcal{H}$ is called* ***self-dual****.*

26. Dual Space: 6/1/2018

Theorem 26.1.

(a) *For each $f \in (\ell^1)^*$, there exists $c = \{c_n\}_{n=1}^{\infty} \in \ell^{\infty}$ such that $f(x) = \sum_{n=1}^{\infty} c_n x_n$ and $\|f\| = \|c\|$ for each $x = \{x_n\}_{n=1}^{\infty} \in \ell^1$.*
(b) *For each given $c = \{c_n\}_{n=1}^{\infty} \in \ell^{\infty}$, $f(x) = \sum_{n=1}^{\infty} c_n x_n$ ($x = \{x_n\}_{n=1}^{\infty} \in \ell^1$) defines an element of $(\ell^1)^*$.*

(c) $(\ell^1)^* = \ell^\infty$ *in the sense of isometrical isomorphism.*

Proof. (a) Let $\{e_n\}_{n=1}^\infty \subset \ell^1$ be the canonical orthonormal basis of ℓ^1, then for each $x = \{x_n\}_{n=1}^\infty \in \ell^1$, $x = \sum_{k=1}^\infty x_k e_k$ by Theorem 21.6. Now for each $f \in (\ell^1)^*$,

$$f(x) = \lim_{n\to\infty} f\left(\sum_{k=1}^n x_k e_k\right) = \lim_{n\to\infty} \sum_{k=1}^n x_k f(e_k) = \sum_{k=1}^\infty f(e_k) x_k.$$

Let $c_k = f(e_k)$ for all $k \in \mathbb{N}_+$, then $f(x) = \sum_{k=1}^\infty c_k x_k$ for each $x = \{x_n\}_{n=1}^\infty \in \ell^1$. Note that $|c_k| = |f(e_k)| \le \|f\|\|e_k\| = \|f\|$ for each $k \in \mathbb{N}_+$. Then $c = \{c_k\}_{k=1}^\infty \in \ell^\infty$ and $\|c\| = \sup_{k\in\mathbb{N}_+} |c_k| \le \|f\|$. So $|f(x)| \le \sum_{k=1}^\infty |c_k||x_k| \le \sup_{k\in\mathbb{N}_+} |c_k| \sum_{k=1}^\infty |x_k| = \|c\|\|x\|$. So $\|f\| \le \|c\|$ and then $\|f\| = \|c\|$.

(b) Trivial.

(c) Define $T : \ell^\infty \to (\ell^1)^*$ by $Tc = f_c$ for all $c = \{c_n\}_{n=1}^\infty \in \ell^\infty$, where $f_c(x) = \sum_{n=1}^\infty c_n x_n$ for each $x = \{x_n\}_{n=1}^\infty \in \ell^1$. It's easy to see that T is an isometrical isomorphism by (a) and (b). □

Similarly, we can prove the following analogous theorem.

Theorem 26.2. *Assume that $p, q \in \mathbb{R}$ satisfying $p > 1$ and $1/p + 1/q = 1$.*

(a) *For each $f \in (\ell^p)^*$, there exists $c = \{c_n\}_{n=1}^\infty \in \ell^q$ such that $f(x) = \sum_{n=1}^\infty c_n x_n$ for each $x = \{x_n\}_{n=1}^\infty \in \ell^p$ and $\|f\| = \|c\|$.*

(b) *For each given $\{c_n\}_{n=1}^\infty \in \ell^q$, $f(x) = \sum_{n=1}^\infty c_n x_n$ $(x = \{x_n\}_{n=1}^\infty \in \ell^p)$ defines an element of $(\ell^p)^*$.*

(c) *$(\ell^p)^* = \ell^q$ in the sense of isometrical isomorphism.*

27. Banach-Steinhaus Theorem: 6/6/2018

Theorem 27.1. ***Uniform Boundedness Theorem***

Let $\{T_n\}_{n=1}^\infty \subset \mathscr{B}(X, Y)$, where X is a Banach space and Y is a normed linear space. If for each $x \in X$, there exists $c_x > 0$ such that $\|T_n x\| \le c_x$ whenever $n \in \mathbb{N}_+$, then there exists $c > 0$ such that $\|T_n\| \le c$ for each $n \in \mathbb{N}_+$, which indicates that $\{T_n\}_{n=1}^\infty$ is uniformly bounded.

Proof. Set $A_k = \{x \in X : \|T_n x\| \le k$ for each $n \in \mathbb{N}_+\}$, then each A_k is closed in X. So there exists $k_0 \in \mathbb{N}_+$ for each $x \in X$ such that $c_x \le k_0$, i.e., $x \in A_{k_0}$, which implies that $X = \cup_{k=1}^\infty A_k$. Since X is nonempty (by assumption) and complete, by Theorem 6.6, there exists $k^* \in \mathbb{N}_+$ such that $(\bar{A}_{k^*})^\circ = A_{k^*}^\circ \ne \varnothing$, i.e., A_{k^*} contains $B(x_0, \delta)$ for some $x_0 \in X$ and $\delta > 0$. Also, A_{k^*} contains $S(x_0, \delta)$, that is, $\|T_n x\| \le k^*$ holds for all $x \in S(x_0, \delta)$ (or $\|x - x_0\| \le \delta$) and $n \in \mathbb{N}_+$. So whenever $x \in S(0, 1)$ (or $\|x\| \le 1$), $\|T_n x\| = \|T_n(\delta x)\|/\delta = \|T_n(\delta x + x_0) - T_n x_0\|/\delta \le 2k^*/\delta$. Then $\|T_n\| = \sup_{\|x\|\le 1} \|T_n x\| \le 2k^*/\delta$ for all $n \in \mathbb{N}_+$. □

Corollary 27.2. *If X is a Banach space, $\{f_n\}_{n=1}^\infty \subset X^*$. Suppose that for each $x \in X$, $\{f_n(x)\}_{n=1}^\infty$ is bounded, then $\{f_n\}_{n=1}^\infty$ is uniformly bounded.*

Corollary 27.3. ***Banach-Steinhaus Theorem***

Let $\{T_n\}_{n=1}^{\infty} \subset \mathscr{B}(X,Y)$, where X is a Banach space. Suppose that $T : X \to Y$ and $\lim_{n\to\infty} T_n x = Tx$ for all $x \in X$, then $T \in \mathscr{B}(X,Y)$ and $\|T\| \le \liminf_{n\to\infty} \|T_n\|$.

Proof. By Theorem 27.1, there exists $c > 0$ such that $\|T_n\| \le c$ for all $n \in \mathbb{N}_+$, then $\liminf_{n\to\infty} \|T_n\| \le c$. Hence,

$$\|Tx\| = \left\| \lim_{n\to\infty} T_n x \right\| = \lim_{n\to\infty} \|T_n x\| = \liminf_{n\to\infty} \|T_n x\| \le \left(\liminf_{n\to\infty} \|T_n\| \right) \|x\|,$$

for all $x \in X$. $\square$

Corollary 27.4. *For $x = \{x_n\}_{n=1}^{\infty} \in \ell^1$, the series $\sum_{n=1}^{\infty} a_n x_n$ converges iff $\{a_n\}_{n=1}^{\infty} \in \ell^{\infty}$.*

Proof. $\Leftarrow$) Trivial.

$\Rightarrow$) Define $f_n(x) = \sum_{k=1}^{n} a_k x_k$ for all $x = \{x_n\}_{n=1}^{\infty} \in \ell^1$ and $n \in \mathbb{N}_+$, then each f_n is well-defined and linear, and $f_n \in (\ell^1)^*$. Since $\sum_{n=1}^{\infty} a_n x_n$ converges, and then $f(x) := \sum_{n=1}^{\infty} a_n x_n = \lim_{n\to\infty} f_n(x)$ for all $x = \{x_n\}_{n=1}^{\infty} \in \ell^1$.

By Corollary 27.3, $f \in (\ell^1)^*$ and $\|f\| \le M$ for some $M > 0$. Hence,

$$|f(x)| = \left| \sum_{n=1}^{\infty} a_n x_n \right| \le M \sum_{n=1}^{\infty} |x_n| = M\|x\|$$

for all $x \in \ell^1$. Let $x_n = 1$ and $x_k = 1$ for $k \ne n$, then we derive $|a_n| \le M$ whenever $n \in \mathbb{N}_+$. So $\{a_n\}_{n=1}^{\infty} \in \ell^{\infty}$. $\square$

Remark 27.5. *Suppose $T : X \to Y$ is a bijective, then there exists $S : Y \to X$ such that $S \circ T = I_X$ and $T \circ S = I_Y$, denoted $S = T^{-1}$. A natural question raises that if X, Y are linear spaces and T is linear, is T^{-1} also linear? In fact, $T \circ T^{-1}(\alpha y_1 + \beta y_2) = \alpha y_1 + \beta y_2 = \alpha T \circ T^{-1} y_1 + \beta T \circ T^{-1} y_2 = T(\alpha T^{-1} y_1 + \beta T^{-1} y_2)$, and then T^{-1} is linear.*

Moreover, if X and Y are normed linear spaces with $T \in \mathscr{B}(X,Y)$, then is $T^{-1} \in \mathscr{B}(X,Y)$? Recall that T^{-1} is continuous iff $G \subset X$ is open $\Rightarrow T(G)$ is open in Y.

28. Banach Theorem: 6/8/2018

Lemma 28.1. ***Open Mapping Theorem***

Let X and Y be Banach spaces and let $T \in \mathscr{B}(X,Y)$ be surjective, then T is an open mapping.

Remark 28.2. *For each open set $G \subset X$ and each $y \in T(G)$, there exists $x \in G$ such that $y = Tx$. Then there is $\delta > 0$ satisfying $B(x,\delta) \subset G \Rightarrow T(B(x,\delta)) \subset T(G)$. Provided that we can find $B_Y(y) \subset T(B(x,\delta))$, then $T(G)$ is open. What we want is $B_Y(0) + y \subset T(B(0,\delta)) + Tx \Leftrightarrow B_Y(0) \subset T(B(0,\delta)) = \delta T(B(0,1))$.*

Remark 28.3. *Let A be a subset of a linear space X over $\mathbb{F}$. Suppose $x \in X$ and $\alpha \in \mathbb{F}$. Define $A+x := \{a+x : a \in A\}$ and $\alpha A := \{\alpha a : a \in A\}$, then $B(y) = B(0)+y$, $B(x,\delta) = \delta B(0,1) + x$, and $T(B(x,\delta)) = \delta T(B(0,1)) + Tx$.*

Lemma 28.1 immediately yields the following important theorem.

Theorem 28.4. ***Banach's Theorem***

Let $T \in \mathscr{B}(X,Y)$ be a bijective where X and Y are Banach spaces, then T^{-1} exists and $T^{-1} \in \mathscr{B}(X,Y)$, i.e., T is ***invertible****.*

Theorem 28.5. *Let X be a Banach space and $T \in \mathscr{B}(X) := \mathscr{B}(X,X)$, then T is invertible $\Leftrightarrow$ $R(T)$ is dense in X and there exists $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$ for all $x \in X$.*

Proof. Without loss of generality, assume $X \neq \{0\}$.

$\Rightarrow$) $T(X) = X$ since T is invertible. Also, $\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\|\|Tx\|$. Take any $x \neq 0$ and it's easy to see $\|T^{-1}\| > 0$. Let $\alpha = 1/\|T^{-1}\|$, then $\|Tx\| \geq \alpha\|x\|$ for all $x \in X$.

$\Leftarrow$) $Tx = 0 \Rightarrow x = 0$ since $\|Tx\| \geq \alpha\|x\|$. So T is injective. We claim that $R(T)$ is closed in X. In fact, for all $y \in \overline{R(T)}$, there exists $\{y_n\}_{n=1}^{\infty} \subset R(T)$ such that $y_n \to y$ $(n \to \infty)$. Then there is $\{x_n\}_{n=1}^{\infty}$ such that $Tx_n = y_n$ for each $n \in \mathbb{N}_+$. So $\|y_n - y_m\| = \|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \geq \alpha\|x_n - x_m\| \geq 0$ for each $n, m \in \mathbb{N}_+$, and hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. So $x_n \to x$ as $n \to \infty$ for some $x \in X$. Hence, $y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = T\lim_{n\to\infty} x_n = Tx \in R(T)$, i.e., $R(T) = \overline{R(T)} = X \Rightarrow T$ is surjective, and hence it's a bijection. Then T is invertible by Theorem 28.4. ☐

Definition 28.6. *Suppose X, Y are normed linear spaces, then $(X \times Y, \|\bullet\|)$ is defined by $(x,y) + (x',y') = (x+x', y+y')$, $\alpha(x,y) = (\alpha x, \alpha y)$, and $\|(x,y)\| = \|x\| + \|y\|$ for all $x, x' \in X$, $y, y' \in Y$, and $\alpha \in \mathbb{F}$.*

The following theorem is trivial.

Theorem 28.7. *$X \times Y$ is a Banach space if X, Y are Banach spaces.*

Theorem 28.8. ***Closed Graph Theorem***

Let X and Y be Banach spaces, then a linear operator $T : X \to Y$ is bounded iff the graph of T, $G(T) := \{(x,Tx) : x \in X\}$, is closed in $X \times Y$.

Proof. $\Rightarrow$) For each $(x,y) \in \overline{G(T)}$, there exists $\{(x_n,y_n)\}_{n=1}^{\infty} \subset G(T)$ such that $(x_n,y_n) \to (x,y)$ $(n \to \infty)$ where $y_n = Tx_n$ for each $n \in \mathbb{N}_+$. Then $\|(x_n - x, y_n - y)\| \to 0$ as $n \to \infty \Rightarrow x_n \to x$ $(n \to \infty)$ in X and $y_n \to y$ $(n \to \infty)$ in Y. Hence, $y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = T\lim_{n\to\infty} x_n = Tx \Rightarrow (x,y) = (x,Tx) \in G(T)$.

$\Leftarrow$) Since $G(T)$ is closed in $X \times Y$, we see that $G(T)$ is a Banach subspace in $X \times Y$. Define $P : G(T) \to X$ as $(x,Tx) \mapsto x$ for each $(x,Tx) \in G(T)$. Clearly, P

is surjective and linear. If $P(x, Tx) = 0$, then $x = 0 \Rightarrow Tx = 0 \Rightarrow (x, Tx) = (0, 0)$. Hence, P is injective.

Now, $\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|$ for each $x \in X \Rightarrow \|P\| \leq 1$. By Theorem 28.4, $P^{-1} \in \mathscr{B}(X, G(T))$, it follows that $\|Tx\| \leq \|x\| + \|Tx\| = \|(x, Tx)\| = \|P^{-1}x\| \leq \|P^{-1}\|\|x\|$. $\square$

29. Hahn-Banach Theorem: 6/13/2018

Definition 29.1. *Suppose X is a real linear space and $p : X \to \mathbb{R}$ is a functional. If p is subadditive (i.e., $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$) and absolutely homogeneous (i.e., $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{R}$ and $x \in X$), then p is **sublinear**.*

Theorem 29.2. ***Hahn-Banach Theorem I***

Let M be a linear subspace of a real linear space X, and let $p : X \to \mathbb{R}$ be sublinear on X, then for each linear functional f on M with $f(x) \leq p(x)$ for all $x \in M$, there exists a linear functional g on X such that $g(x) = f(x)$ for all $x \in M$ and $g(x) \leq p(x)$ for all $x \in X$.

Proof. Suppose $M \subsetneqq X$. Select $z \in X \setminus M$, then $z \neq 0$. Set $M_z = \{w = x + \alpha z (\in X) : x \in M, \alpha \in \mathbb{R}\}$. For each $w \in M_z$, there are unique $x \in M$ and $\alpha \in \mathbb{R}$ such that $w = x + \alpha z$ since $w = x + \alpha z = x' + \alpha' z$ implies $\alpha' = \alpha$ (otherwise $z = (x - x')/(\alpha' - \alpha) \in M$ yields a contradiction) and $x' = x$.

We only prove the case where $M_z = X$ here. For $M_z \neq X$, just repeat the following procedure with the help of Zorn's Lemma.

For all $x, y \in M$, $f(x) + f(y) = f(x+y) \leq p(x+y) \leq p(x-z) + p(z+y) \Rightarrow f(x) - p(x-z) \leq p(z+y) - f(y)$. So there exists $t \in \mathbb{R}$ such that $\sup_{x \in M}(f(x) - p(x-z)) \leq t \leq \inf_{y \in M}(p(z+y) - f(y))$.

Define a functional $g : M_z \to \mathbb{R}$ by $g(w) = f(x) + \alpha t$ for each $w = x + \alpha z \in M_z$. Clearly, g is linear and $g(x) = f(x)$ when $x \in M$. For $w = x + \alpha z$ with $\alpha > 0$, $p(z + y) - f(y) \geq t$ for all $y \in M$. Let $y = x/\alpha$, then $p(z + x/\alpha) - f(x/\alpha) \geq t \Rightarrow f(x) + \alpha t \leq p(x + \alpha z)$, i.e., $g(w) \leq p(w)$. For $w = x + \alpha z$ with $\alpha < 0$, $f(x) - p(x - z) \leq t$ for all $x \in M$. Then $f(-x/\alpha) - p(-x/\alpha - z) \leq t \Rightarrow f(x) + \alpha t \leq p(x + \alpha z)$, i.e., $g(w) \leq p(w)$. Thus, $g(w) \leq p(w)$ holds for all $w \in M_z$. $\square$

Theorem 29.3. ***Hahn-Banach Theorem II***

Let M be a linear subspace of a complex linear space X, and let $p : X \to \mathbb{R}$ be sublinear on X, then for each complex-valued linear functional f on M with $|f(x)| \leq p(x)$ for all $x \in M$, there exists a complex-valued linear functional g on X such that $g(x) = f(x)$ for all $x \in M$ and $|g(x)| \leq p(x)$ for all $x \in X$.

Corollary 29.4. *Let M be a subspace of $(X, \| \bullet \|)$ over $\mathbb{C}$. If $f \in M^*$, then there exists $g \in X^*$ such that $g(x) = f(x)$ for each $x \in M$ and $\|g\| = \|f\|$.*

Proof. Set $p(x) = \|f\|\|x\|$ for all $x \in M$, then $|f(x)| \leq p(x)$ whenever $x \in M$.

By Theorem 29.3, there exists $g \in X^*$ such that $g(x) = f(x)$ for all $x \in M$ and $|g(x)| \le p(x) = \|f\|\|x\|$ for all $x \in X$, then $\|g\| \le \|f\|$.

Also,

$$\|g\| = \sup_{\substack{\|x\|=1 \\ x\in X}} |g(x)| \ge \sup_{\substack{\|x\|=1 \\ x\in M}} |g(x)| = \sup_{\substack{\|x\|=1 \\ x\in M}} |f(x)| = \|f\|. \qquad \square$$

Remark 29.5. *The analogous version for X over $\mathbb{R}$ of Corollary 29.4 also holds.*

Corollary 29.6. *If $x_0 \in X$ and $x_0 \neq 0$, then there exists $f \in X^*$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$.*

Proof. Let $M = \{\alpha x_0 : \alpha \in \mathbb{F}\}$ and let f_0 be defined by $f_0(x) = \alpha\|x_0\|$ for each $x = \alpha x_0 \in M$. f is well-defined since α is unique for each $x \in M$. Clearly, $f_0 \in M^*$ and $|f_0(x)| = |\alpha|\|x_0\| = \|\alpha x_0\| = \|x\|$ for each $x \in M \Rightarrow \|f_0\| = 1$. By Corollary 29.4, there exists $f \in X^*$ such that $\|f\| = \|f_0\| = 1$ and $f(x) = f_0(x)$ for all $x \in M$. In particular, $f(x_0) = f_0(x_0) = \|x_0\|$. $\square$

Corollary 29.7. *If $X^* = \{0\}$, then $X = \{0\}$.*

Corollary 29.8. *Let X be a real or complex normed linear space and suppose that $f(x) = 0$ for all $f \in X^*$, then $x = 0$.*

Theorem 29.9. $(C[0,1])^* = V[0,1]$ *in the sense of isometrical isomorphism.*

Remark 29.10. *Suppose X is a normed linear space. Fix $x \in X$, then for each $f \in X^*$, define $F_x \in X^{**}$ by $F_x(f) = f(x)$. There actually exists a linear mapping $J : X \to X^{**}$ such that for all $x \in X$, $x \mapsto J(x) = F_x$ and $\|Jx\| = \|x\|$. Here, J is called the* ***canonical mapping*** *from X to X^{**}. It can be proved that J is an isometrical isomorphism from X to $J(X) \subset X^{**}$, i.e., $X = J(X) \subset X^{**}$ in the sense of isometrical isomorphism.*

30. Weak and Strong Convergence: 6/15/2018

If $u_0 \in L^2[a,b]$ satisfying $\int_a^b u_0(t)\mathrm{d}t = \min_{u\in L^2[a,b]} \int_a^b u(t)\mathrm{d}t$, does there exist $\{u_n\}_{n=1}^\infty \subset L^2[a,b]$ such that $\int_a^b u_n(t)\mathrm{d}t \to \int_a^b u_0(t)\mathrm{d}t$ as $n \to \infty$?

Set $f(u) = \int_a^b u(t)\mathrm{d}t$ for all $u \in L^2[a,b]$, then $f \in (L^2[a,b])^*$ since $|f(u)| \le (b-a)^{1/2}\|u\|$ for all $u \in L^2[a,b]$. Then if the desired sequence exists, $\lim_{n\to\infty} f(u_n) = f(u_0) = \min_{u\in L^2[a,b]} f(u)$.

Also, if $u_n \to u_0$ $(n \to \infty)$, then as $n \to \infty$, $u_n \to u_0 \Rightarrow \|u_n - u_0\| \to 0 \Rightarrow \|f(u_n) - f(u_0)\| \le (b-a)^{1/2}\|u_n - u_0\| \to 0$.

Definition 30.1. *Let $\{u_n\}_{n=1}^\infty \subset (X, \|\bullet\|)$ and $u_0 \in (X, \|\bullet\|)$, then $\{u_n\}_{n=1}^\infty$* ***weakly converges*** *to u_0 as $n \to \infty$ if for each $f \in X^*$, $f(u_n) \to f(u)$ $(n \to \infty)$, denoted $u_n \rightharpoonup u_0$ $(n \to \infty)$ or $u_n \xrightarrow{w} u_0$ $(n \to \infty)$. Also, $\{u_n\}_{n=1}^\infty$* ***strongly converges*** *to u_0 as $n \to \infty$ if $\|u_n - u_0\| \to 0$ $(n \to \infty)$, denoted $u_n \to u_0$ $(n \to \infty)$.*

Remark 30.2. *Clearly, if* $u_n \to u_0$ $(n \to \infty)$*, the* $u_n \rightharpoonup u_0$ $(n \to \infty)$ *since for each* $f \in X^*$, $0 \le |f(u_n) - f(u_0)| = |f(u_n - u_0)| \le \|f\|\|u_n - u_0\| \to 0$ $(n \to \infty)$.

But the converse does not hold in general. For instance, in ℓ^p*, for each* $f \in (\ell^p)^*$ *where* $p > 1$*, there exists* $\{a_n\}_{n=1}^\infty \subset \ell^q$ *such that* $f(x) = \sum_{n=1}^\infty a_n x_n$ *for all* $x = \{x_n\}_{n=1}^\infty \in \ell^p$ *where* $1/p + 1/q = 1$*. Let* $\{e_n\}_{n=1}^\infty \subset \ell^p$ *be the canonical orthonormal basis, then* $\|e_n - e_m\| = 2^{1/p} \nrightarrow 0$ $(n, m \to \infty$ *with* $n \neq m)$*, which implies that* $\{e_n\}_{n=1}^\infty$ *does not strongly converge in* ℓ^p*. But for each* $f \in (\ell^p)^*$, $f(e_n) = a_n \to 0 = f(0)$ *as* $n \to \infty$ *since* $\sum_{n=1}^\infty |a_n|^q < \infty$*, i.e.,* $e_n \rightharpoonup 0$ $(n \to \infty)$ *in* ℓ^p.

Hence, we've shown that

$$\{\textit{Weak Convergence}\} \supset \{\textit{Strong Convergence}\}.$$

Remark 30.3. *In* ℓ^1*, weak convergence is equivalent to strong convergence.*

Theorem 30.4. *If* $\{u_n\}_{n=1}^\infty$ *converges weakly in* $(X, \|\bullet\|)$ *as* $n \to \infty$*, then*

(i) *the limit is unique; and*

(ii) $\{u_n\}_{n=1}^\infty$ *is bounded in* $(X, \|\bullet\|)$.

Proof. (i) Suppose that $u_n \rightharpoonup u_0$ and $u_n \rightharpoonup u_0'$ as $n \to \infty$, that for each $f \in X^*$, $f(u_n) \to f(u_0)$ and $f(u_n) \to f(u_0')$ $(n \to \infty)$, which implies that $f(u_0) = f(u_0') \Rightarrow f(u_0 - u_0') = 0$ for all $x \in X^*$. Then $u_0 - u_0' = 0 \Leftrightarrow u_0 = u_0'$ by Corollary 29.8.

(ii) Without loss of generality, assume that $u_n \rightharpoonup u$ $(n \to \infty)$ with $u_n \neq u$ for each $n \in \mathbb{N}_+$, then there exists $f_n \in X^*$ such that $f_n(u_n - u) = \|u_n - u\|$ and $\|f_n\| = 1$ for all $n \in \mathbb{N}_+$ by Corollary 29.6. For each $f \in X^*$ and $n \in \mathbb{N}_+$, define $F_n(f) = f(u_n - u)$, then $\{F_n\}_{n=1}^\infty \subset X^{**}$. Note that X^{**} is complete by Corollary 25.5. We have $F_n(f) \to 0$ $(n \to \infty)$ for all $f \in X^*$, which, by Corollary 27.1, yields that there exists $M > 0$ such that $\|F_n\| \le M$ for all $n \in \mathbb{N}_+$. Now $\|u_n - u\| = f_n(u_n - u) = F_n(f_n) \le \|f_n\|\|F_n\| = \|F_n\| \le M \Rightarrow \|u_n\| \le M + \|u\|$ for all $n \in \mathbb{N}_+$. □

Remark 30.5. *If* $u_n \rightharpoonup u$ $(n \to \infty)$*, then* $f(u_n) \to f(u)$ $(n \to \infty)$ *for each* $f \in X^*$*. So* $|f(u_n - u)| \le \|f\|\|u_n - u\|$.

Remark 30.6. *If* $f(x) = 0$ *for each* $x \in X$*, then* $f = 0$.

If $f(x) = 0$ *for each* $f \in X^*$*, then* $x = 0$ *by Corollary 29.8.*

If $F(f) = 0$ *for all* $F \in X^{**}$*, then* $f = 0$ *by Corollary 29.8.*

Definition 30.7. *Suppose* $\{f_n\}_{n=1}^\infty \subset X^*$ *and* $f \in X^*$*, then consider the following types of convergence, where (i) is called* ***strong convergence*** *of* $\{f_n\}_{n=1}^\infty$ *and (ii) is called* ***weak* convergence*** *of* $\{f_n\}_{n=1}^\infty$*:*

(i) $\|f_n - f\| \to 0$ *as* $n \to \infty$*, denoted* $f_n \to f$ *as* $n \to \infty$*;*

(ii) $|f_n(x) - f(x)| \to 0$ *as* $n \to \infty$ *for all* $x \in X$*, denoted* $f_n \xrightarrow{w^*} f$ *as* $n \to \infty$*;*

(iii) $|F(f_n) - F(f)| \to 0$ *as* $n \to \infty$ *for all* $F \in X^{**}$.

Remark 30.8. *Compare weak* convergence with weak convergence (see Definition 30.1),that is, if for all* $f \in X^*$, $|f(x_n) - f(x)| \to 0$ $(n \to \infty)$*, then* $x_n \rightharpoonup x$ $(n \to \infty)$.

Remark 30.9. *Let $\{T_n\}_{n=1}^{\infty} \subset \mathscr{B}(X,Y)$, $T \in \mathscr{B}(X,Y)$, and $x \in X$, then consider the following types of convergence (where (iii) is actually not necessary):*

(i) $\|T_n x - Tx\| \to 0$ *as* $n \to \infty$*;*
(ii) $\|T_n - T\| \to 0$ *as* $n \to \infty$*;*
(iii) $|F(T_n) - F(T)| \to 0$ *as* $n \to \infty$ *for all* $F \in (\mathscr{B}(X,Y))^*$*; and*
(iv) $|f(T_n x) - f(Tx)| \to 0$ *as* $n \to \infty$ *for all* $f \in Y^*$.

Theorem 30.10. *In a Hilbert space $\mathcal{H}$, if $\{u_n\}_{n=1}^{\infty} \subset \mathcal{H}$, $u \in \mathcal{H}$, then $u_n \rightharpoonup u$ $(n \to \infty) \Leftrightarrow$ for each $z \in \mathcal{H}$, $\langle u_n, z\rangle \to \langle u, z\rangle$ $(n \to \infty)$.*

Proof. $\Rightarrow$) For each $z \in \mathcal{H}$, $f(x) = \langle x, z\rangle$ $(x \in \mathcal{H})$ is an element of $\mathcal{H}^*$ by (b) of Corollary 25.7. Then as $n \to \infty$, $f(u_n) \to f(u) \Leftrightarrow \langle u_n, z\rangle \to \langle u, z\rangle$.

$\Leftarrow$) For each $f \in \mathcal{H}^*$, by Theorem 25.6, there exists $x_f \in \mathcal{H}$ such that $f(x) = \langle x, x_f\rangle$ for all $x \in \mathcal{H}$. Since $\langle u_n, x_f\rangle \to \langle u, x_f\rangle$ as $n \to \infty$, it follows that $f(u_n) \to f(u)$ as $n \to \infty$. Thus, $u_n \rightharpoonup u$ as $n \to \infty$. □

31. Adjoint Operator: 6/20/2018

Theorem 31.1. *Suppose $\mathcal{H}, \mathcal{W}$ are Hilbert spaces, then for each $T \in \mathscr{B}(\mathcal{H}, \mathcal{W})$, there exists unique $T^* \in \mathscr{B}(\mathcal{W}, \mathcal{H})$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in \mathcal{H}$ and $y \in \mathcal{W}$.*

Proof. Fix $y \in \mathcal{W}$, define a functional $f : \mathcal{H} \to \mathbb{F}$ by $f(x) = \langle Tx, y\rangle$ for all $x \in \mathcal{H}$. It's clear that f is linear. Since $|f(x)| = |\langle Tx, y\rangle| \le \|Tx\|\|y\| \le (\|T\|\|y\|)\|x\|$ for all $x \in \mathcal{H}$, we see that $f \in \mathcal{H}^*$. By Theorem 25.6, there is a unique $x_f \in \mathcal{H}$ such that $f(x) = \langle x, x_f\rangle$ for all $x \in \mathcal{H}$. Define a mapping $T^* : \mathcal{W} \to \mathcal{H}$ by $T^*y = x_f$ for all $y \in \mathcal{W}$. Note that T^* is well-defined since x_f is uniquely determined by y, then $\langle Tx, y\rangle = \langle x, T^*y\rangle$ whenever $x \in \mathcal{H}, y \in \mathcal{W}$.

Now it's suffices to show T^* is linear, bounded, and unique.

For all $x \in \mathcal{H}$, $y_1, y_2 \in \mathcal{W}$ and $\lambda, \mu \in \mathbb{F}$, $\langle x, T^*(\lambda y_1 + \mu y_2)\rangle = \langle Tx, \lambda y_1 + \mu y_2\rangle = \bar{\lambda}\langle Tx, y_1\rangle + \bar{\mu}\langle Tx, y_2\rangle = \bar{\lambda}\langle x, T^*y_1\rangle + \bar{\mu}\langle x, T^*y_2\rangle = \langle x, \lambda T^*y_1 + \mu T^*y_2\rangle$. So $T^*(\lambda y_1 + \mu y_2) = \lambda T^*y_1 + \mu T^*y_2$, which indicates that T^* is linear.

Boundedness of T^* can be derived through two routes here. One is that, by Theorem 25.6, it follows $\|T^*y\| = \|x_f\| = \|f\| \le \|T\|\|y\|$ for all $y \in \mathcal{W}$. So $\|T^*\| \le \|T\|$ and then T^* is bounded. The other one is from $\|T^*y\|^2 = \langle T^*y, T^*y\rangle = \langle T(T^*y), y\rangle \le \|T(T^*y)\|\|y\| \le \|T\|\|T^*y\|\|y\|$ for all $y \in \mathcal{H}$. So $\|T^*y\| \le \|T\|\|y\|$ for all $y \in \mathcal{W}$ and then T^* is bounded with $\|T^*\| \le \|T\|$.

Since $\langle x, T'y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in \mathcal{H}$ and $y \in \mathcal{W}$ if there exists another desired mapping $T' : \mathcal{W} \to \mathcal{H}$, then $T' = T^*$. So T^* is unique. □

Definition 31.2. *$T^*(\in \mathscr{B}(\mathcal{W}, \mathcal{H}))$ given in Theorem 31.1 is said to be the **adjoint operator** of $T(\in \mathscr{B}(\mathcal{H}, \mathcal{W}))$.*

Example 31.3. *Suppose $T \in \mathscr{B}(\ell^2)$ is defined by $Tx = (0, x_1, x_2, \cdots)$ for all $x = (x_1, x_2, \cdots) \in \ell^2$. For each $x = (x_1, x_2, \cdots), y = (y_1, y_2, \cdots) \in \ell^2$, $\langle x, T^*y\rangle = \langle Tx, y\rangle \Leftrightarrow \langle (x_1, x_2, \cdots), (z_1, z_2, \cdots)\rangle = \langle (0, x_1, x_2, \cdots), (y_1, y_2, \cdots)\rangle \Leftrightarrow x_1\bar{z}_1 + x_2\bar{z}_2 +$*

$\cdots = x_1\bar{y}_2 + x_2\bar{y}_3 + \cdots$ *where* $z := T^*y = (z_1, z_2, \cdots) \in \ell^2$. *If* $z_1 = y_2, z_2 = y_3, z_3 = y_4, \cdots$, *then the above identity holds for all* $x \in \ell^2$. *So* $T^*(y_1, y_2, \cdots) = (y_2, y_3, \cdots)$ *for all* $y = (y_1, y_2, \cdots) \in \ell^2$ *by uniqueness of* T^*.

Theorem 31.4. *Suppose* $T \in \mathscr{B}(\mathcal{H}, \mathcal{W})$, *then*

(a) $T^{**} = T$;
(b) $\|T^*\| = \|T\|$;
(c) $\|T^*T\| = \|T\|^2$.

Proof. (a) For all $x \in \mathcal{H}$ and $y \in \mathcal{W}$, $\langle y, T^{**}x\rangle = \langle T^*y, x\rangle = \overline{\langle x, T^*y\rangle} = \overline{\langle Tx, y\rangle} = \langle y, Tx\rangle$, which shows that $T^{**} = T$.

(b) It's shown in the proof of Theorem 25.6 that $\|T^*\| \leq \|T\|$, so $\|T\| = \|T^{**}\| \leq \|T^*\| \leq \|T\|$ by (a). Thus, $\|T^*\| = \|T\|$.

(c) Note that $\|T^*Tx\| \leq \|T^*\|\|Tx\| \leq \|T^*\|\|T\|\|x\| \Rightarrow \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$ by (b). Also, $\|Tx\| = \sqrt{\langle Tx, Tx\rangle} = \sqrt{\langle T^*Tx, x\rangle} \leq \sqrt{\|T^*Tx\|\|x\|} \leq \sqrt{\|T^*T\|}\|x\|$ for all $x \in \mathcal{H}$. So $\|T\|^2 \leq \|T^*T\|$ and then $\|T^*T\| = \|T\|^2$. $\square$

32. Review of the Course: 6/22/2018

33. Final Examination: 7/5/2018

34. Appendix and Supplements to the Notes

We give a more detailed definition of metric space here, compared with that in Definition 2.1, together with other related notions.

Definition 34.1. *A* ***metric space*** *is a nonempty set* S *with a function* $d : S \times S \to \mathbb{R}$ *such that the following rules hold for all* $x, y, z \in S$:

- $d(x, x) = 0$;
- $d(x, y) \geq 0$;
- *Separation:* $d(x, y) = 0 \Rightarrow x = y$;
- *Symmetry:* $d(x, y) = d(y, z)$; *and*
- *Triangle inequality:* $d(x, y) + d(y, z) \geq d(x, z)$.

A ***pseudo-metric space*** *satisfies the above except (possibly) for separation.*

A ***quasi-metric space*** *satisfies the above except (possibly) for symmetry.*

A ***semi-metric space*** *often means a space satisfying the above except (possibly) for the triangle inequality. At times though, it also refers to a pseudo-metric space, but sometimes it also refers to quasi-metric space.*

Generalized metric space *or* ***Lawvere metric space*** *satisfies the above except (possibly) for separation and symmetry.*

It is also possible to drop the $d(x, x) = 0$ *requirement, such things are called* ***partial-metric spaces*** *and come in both pseudo/semi and quasi variants.*

In relation to normed and semi-normed spaces (allowing to assign zero length to some non-zero elements), where the terminology is standard, defining $d(x, y) = \|x-y\|$

associated with a normed spaces a metric space and with a semi-normed space a metric space where $d(x,y)=0 \Rightarrow x=y$ may fail. This could be seen as support for calling such spaces ***semi-metric spaces****.*

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