Fiber Bundles & Characteristic Classes

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1. Useful Stuffs Mentioned in the Lecture

Lemma 1.1. *Let ξ, η be vector bundles over B, and let $f : E(\xi) \to E(\eta)$ be a continuous function which maps each vector space $F_b(\xi)$ isomorphically onto the corresponding vector space $F_b(\eta)$. Then f is necessarily a homeomorphism. Hence ξ is isomorphic to η.*

Definition 1.2. *Suppose ξ is a vector bundle on B and η is a vector bundle on B'. A* ***bundle map*** *from η to ξ is a continuous map $F : E(\eta) \to E(\xi)$ such that for each $b' \in B'$, there exists $b \in B$ such that F restricts to a linear isomorphism $F|_{b'} : \pi_\eta^{-1}(b') \to \pi_\xi^{-1}(b)$ at $b' \in B'$.*

Thus in fact, F induces $f : B' \to B$ by $b' \mapsto b$. In other words, there holds the following commutative diagram and we say that F ***covers*** *f.*

$$\begin{array}{ccc} E(\eta) & \xrightarrow{F} & E(\xi) \\ \big\downarrow{\scriptstyle \pi_\eta} & & \big\downarrow{\scriptstyle \pi_\xi} \\ B' & \xrightarrow{f} & B \end{array}$$

2. Homework

2.1. HW #1: 7/16/2018.

Homework 2.1. *Prove that γ_n^1 defines a line bundle on $\mathbb{RP}^n$.*

Proof. Recall that $\mathbb{RP}^n$ is derived by identifying antipodal points x and $-x$ on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ where x varies on S^n. Let $[x] := \{x, -x\}$ for each $x \in S^n$, then $\pi : E = \{([x], cx) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : c \in \mathbb{R}\} \to B = \mathbb{RP}^n$ projects each $([x], cx)$ to $[x]$. So $\pi^{-1}([x]) = \{([x], cx) \in$

$\mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} : c \in \mathbb{R}\}$ which can be identified with the line through x and $-x$ or simply $\mathbb{R}$. Hence $\pi^{-1}([x])$ is always a real vector space.

Let $U \subset S^n$ be an arbitrary open set that does not contain any pair of antipodal points on the sphere. Set U_1 to be the collection of equivalent classes of all elements in U. The map $h : U_1 \times \mathbb{R} \to \pi^{-1}(U_1) = \{([x], tx) : x \in U, t \in \mathbb{R}\}$ defined by $h([x], t) = ([x], tx)$ for each $(x, t) \in U \times \mathbb{R}$ is trivially a homeomorphism. It's also obvious that the union all such U_1 covers B. So for each $[x] \in B$, $[x]$ is contained in some U_1. Then for all $[x'] \in U_1$, $t \mapsto h([x'], t) = ([x'], tx') \in \pi^{-1}([x'])$ for each $t \in \mathbb{R}$ clearly defines a linear isomorphism from $\mathbb{R}$ to $\pi^{-1}([x']) = \{([x'], cx') \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} : c \in \mathbb{R}\}$. □

2.2. **HW #2: 7/17/2018.**

Homework 2.2. *Consider γ_n^1 on $\mathbb{R}\mathrm{P}^n$. Let $f : S^n \to \mathbb{R}\mathrm{P}^n$ be defined by $x \mapsto [x]$ for each x on S^n. Then $f^*\gamma_n^1$ on S^n is trivial, i.e., it is isomorphic to the trivial line bundle $(S^n \times \mathbb{R}, S^n, \pi_1)$.*

Proof. For each $[x] \in \mathbb{R}\mathrm{P}^n$, let $r([x]) \in S^n$ be a fixed representative of $[x]$. So

$$E(\gamma_n^1) = \{([x], cr([x]) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} : c \in \mathbb{R}\}$$

and then

$$\begin{aligned} E(f^*\gamma_n^1) &= \{(b, [x], cr([x])) \in S^n \times E(\gamma_n^1) : f(b) = \pi_{\gamma_n^1}([x], cr([x])) = [x], c \in \mathbb{R}\} \\ &= \{(b, f(b), cr(f(b))) : b \in S^n, c \in \mathbb{R}\}. \end{aligned}$$

Define $h : S^n \times \mathbb{R} \to E(f^*\gamma_n^1)$ via $(b, c) \mapsto (b, f(b), cr(f(b)))$ for each $(b, c) \in S^n \times \mathbb{R}$. Note that f is a continuous surjection. Thus, h is a continuous function. For each $b \in S^n$, the restriction

$$h|_b : \pi^{-1}(b) = \{b\} \times \mathbb{R} \to \pi^{-1}_{f^*\gamma_n^1}(b) = \{(b, f(b), cr(f(b))) : c \in \mathbb{R}\}$$

maps (b, c) to $(b, f(b), cr(f(b)))$ for all $c \in \mathbb{R}$. It's trivial that $h|_b$ is a linear isomorphism and then $f^*\gamma_n^1$ is isomorphic to the line bundle by Lemma 1.1. □

Homework 2.3. *In Definition 1.2, prove that $\eta \approx f^*\xi$.*

Proof. Define $h : E(\eta) \to E(f^*\xi)$ via $e' \mapsto (\pi_\eta(e'), F(e'))$ for all $e' \in E(\eta)$. Since for each $b' \in B'$, the restriction

$$\begin{aligned} h|_{b'} : \pi_\eta^{-1}(b') \to \pi^{-1}_{f^*\xi}(b') &= \{(b', e) : e \in E(\xi), \pi_\xi(e) = f(b') = b\} \\ &= \{(b', e) : e \in \pi_\xi^{-1}(b)\} \\ &= \{b'\} \times \pi_\xi^{-1}(b) \end{aligned}$$

is defined by $e' \mapsto (\pi_\eta(e'), F(e')) = (b', F(e'))$ for each $e' \in \pi_\eta^{-1}(b')$. Since $F|_{b'} : \pi_\eta^{-1}(b') \to \pi_\xi^{-1}(b)$ is a linear isomorphism by Definition 1.2, then $h|_{b'}$ is obviously an isomorphism. Note that F is continuous. So h is a continuous map. Thus, $\eta \approx f^*\xi$ by Lemma 1.1. □

2.3. **HW #3: 7/18/2018.**

Homework 2.4. *Suppose η is a subbundle of ξ, define the quotient bundle ξ/η and show that it satisfies the local triviality condition. Prove that $\xi \approx \eta \oplus (\xi/\eta)$ when the base space B is paracompact. Does this hold in general?*

Proof. Let $E(\xi/\eta) = E(\xi)/\sim$ where $\sim$ is an equivalent relationship on $E(\xi)$ such that for all $e_1, e_2 \in E(\xi)$ $e_1 \sim e_2$ iff there exists $b \in B$ with $\pi_\xi(e_1) = \pi_\xi(e_2) = b$ and $e_1 - e_2 \in \pi_\eta^{-1}(b)$. Define the projection $\pi_{\xi/\eta}$ by $\pi_{\xi/\eta}(e + \pi_\eta^{-1}(\pi_\xi(e))) = \pi_\xi(e)$ for all $b \in B$. Then for each $b \in B$, $\pi_{\xi/\eta}^{-1}(b) = \pi_\xi^{-1}(b)/\pi_\eta^{-1}(b)$.

Assume $\text{rank}(\xi) = n, \text{rank}(\eta) = k$. For each $b \in B$, there is an open neighborhood U of b such that there exist $s_1, \cdots, s_k$ as a local basis of sections for η on U. Reduce U to be a smaller one so that $\xi|_U$ is trivial. Extend $s_1, \cdots, s_k$ to a local basis of sections $s_1, \cdots, s_k, s_{k+1}, \cdots, s_n$ of ξ on U, then $s_{k+1}, \cdots, s_n$ are representatives of the local basis of sections for ξ/η on U since $\pi_{\xi/\eta}^{-1}(b)$ is a quotient space for each $b \in B$. So that ξ/η admits local basis of sections and then local triviality holds for ξ/η.

If B is paracompact, then there is a metric on ξ. By Gram-Schmidt process, $s_1, \cdots, s_n$ can be orthonormalized such that $s_1, \cdots, s_n$ are mutually perpendicular at each $b \in U$. Note that the space spanned by $s_{k+1}(b), \cdots, s_n(b)$ are the the orthogonal complement of $\pi_\eta^{-1}(b)$ in $\pi_\xi^{-1}(b)$. Such spaces attached to each $b \in B$ is the fiber-wise orthogonal complement of η in ξ, denoted $\eta^\perp$. Then $\xi = \eta \oplus \eta^\perp$. It's obvious that $\xi/\eta \approx \eta^\perp$ since the fibers of them at the same point $b \in B$ are spanned by the same local basis of sections and the topology on $\xi/\eta, \eta^\perp$ are the same by definition. The isomorphism is given by $v \in E(\eta^\perp) \mapsto [v] \in \xi/\eta$. Hence $\eta \approx \xi/\eta$.

This property also holds in general. Since for a Whitney sum, the elements in $\pi_\eta^{-1}(b), \pi_{\xi/\eta}^{-1}(b)$ are combined through cartesian product for each $b \in B$. So $\pi_\xi^{-1}(b) \approx \pi_\eta^{-1}(b) \times \pi_{\xi/\eta}^{-1}(b)$. The general isomorphism is given by the assignment

$$(v_1, \cdots, v_k, v_{k+1}, \cdots, v_n) \mapsto (v_1, \cdots, v_k, [v_{k+1}], \cdots, [v_n]).$$ □

Homework 2.5. *Consider a general treatment for line bundles on S^1.*

(a) *Let ξ be a line bundle on S^1. Since S^1 is Hausdorff and compact we can assume that there is a metric on ξ. By local triviality, there is an open cover $\{U_\alpha\}_{\alpha \in I}$ of S^1 such that ξ has nowhere vanishing local section s_α on each U_α. This defines translation function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(1) = \{-1, 1\} \approx \mathbb{Z}_2$ since there is a metric on ξ. Then on each overlap $U_\alpha \cap U_\beta$, we have $s_\alpha = g_{\alpha\beta} s_\beta$. Show that $g_{\alpha\beta}$ defines a Čech 1-cocycle in $C^1(\{U_\alpha\}_{\alpha \in I}; \mathbb{Z}_2)$.*

(b) *Show that ξ is trivial iff the Čech 1-cocycle defined above is a coboundary. Then prove that the isomorphic classes of line bundles on S^1 are in one-to-one correspondence with $H^1(S^1, \mathbb{Z}_2)$, that is, there are exactly two isomorphic classes of line bundles on S^1, one represented by the trivial line bundle and the other by Möbius strip.*

Proof. (a) By definition of translation function, it's clear that $g_{\alpha\alpha} = 1$ and $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ where $g_{\alpha\beta}$ now denotes the image $g_{\alpha\beta}(U_\alpha \cap U_\beta)$ by abuse of notation. Define Čech 1-cochain f by $f(\alpha, \beta) = g_{\alpha\beta}$. This is indeed an element of $C^1(\{U_\alpha\}_{\alpha \in I}; \mathbb{Z}_2)$, i.e., a Čech 1-cochain, since

$f(\alpha,\beta) = f(\beta,\alpha)$. Note that $s_\alpha = g_{\alpha\beta}s_\beta = g_{\alpha\beta}g_{\beta\gamma}s_\gamma = g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}s_\alpha$ on $U_\alpha \cap U_\beta \cap U_\gamma$. So $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ and

$$(\delta f)(\alpha,\beta,\gamma) = f(\alpha,\beta)f(\beta,\gamma)f(\gamma,\alpha) = g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$$

where δ is the coboundary operator. Thus, f is a Čech 1-cocycle in $C^1(\{U_\alpha\}_{\alpha\in I};\mathbb{Z}_2)$.

(b) It's known that ξ, as a line bundle, is trivial iff there exists a nowhere vanishing section s on S^1. Hence just set $s_\alpha := s|_{U_\alpha}$ for each $\alpha \in I$. Note that $\{U_\alpha\}_{\alpha\in I}$ is an open cover of a connected space S^1. It follows that all $g_{\alpha\beta}$'s are the same and so equal to 1 by the cocycle condition of $g_{\alpha\beta}$'s. Conversely, if all $g_{\alpha\beta} = 1$, then they combine to define a global nowhere vanishing section on S^1 again because S^1 is connected.

Suppose ξ is trivial, then $g_{\alpha\beta} = 1$ for all $\alpha,\beta \in I$. Define Čech 0-cochain f' by $f'(\alpha) \equiv 1$. It's obvious that f' is indeed a 1-cochain of $C^0(\{U_\alpha\}_{\alpha\in I};\mathbb{Z}_2)$. For each $\alpha,\beta \in I$, $f(\alpha,\beta) = g_{\alpha\beta} = 1 = f'(\alpha)f'(\beta) = (\delta f')(\alpha,\beta)$. Hence, f is a coboundary. Conversely, suppose f is a coboundary, then there exists 0-cochain f' in $C^0(\{U_\alpha\}_{\alpha\in I};\mathbb{Z}_2)$ such that $f = (\delta f')$, i.e., $g_{\alpha\beta} = f(\alpha,\beta) = (\delta f')(\alpha,\beta) = f'(\alpha)f'(\beta)$ always holds. Now let $\tilde{s}_\alpha = f'(\alpha)s_\alpha$ for each $\alpha \in I$, then $\tilde{\alpha}_\alpha = f'(\alpha)s_\alpha = f'(\alpha)g_{\alpha\beta}s_\beta = f'(\alpha)g_{\alpha\beta}(f'(\beta))^{-1}\tilde{s}_\beta$. So

$$\begin{aligned}\tilde{g}_{\alpha\beta} &= f'(\alpha)g_{\alpha\beta}(f'(\beta))^{-1} = f'(\alpha)f(\alpha,\beta)(f'(\beta))^{-1} \\ &= f'(\alpha)f'(\alpha)f'(\beta)(f'(\beta))^{-1} = (f'(\alpha))^2 = 1\end{aligned}$$

holds for all $\alpha,\beta \in I$. Thus, for each $\alpha \in I$, $\tilde{s}_\alpha$ is a section on U_α. Moreover, $\tilde{g}_{\alpha\beta} = 1$ always holds. Hence, ξ is trivial.

S^1 has a good cover since S^1 is a manifold. So assume $\{U_\alpha\}_{\alpha\in I}$ is a good cover of S^1. Since the usual cohomology group is isomorphic to the Čech cohomology group for good covers, we have $H^1(\{U_\alpha\}_{\alpha\in I};\mathbb{Z}_2) \approx H^1(S^1;\mathbb{Z}_2) \approx H^1(\mathbb{R}\mathrm{P}^1;\mathbb{Z}_2)$, which is cyclic of order 2. Hence, the line bundle ξ on S^1 is trivial iff $w_1(\xi) = 0$ in $H^1(S^1;\mathbb{Z}_2)$ since f is a coboundary iff ξ is trivial. So $w_1(\xi)$ distinguishes the trivial line bundle on S^1 from other nontrivial ones. Then any other nontrivial vector bundle ξ_1 on S^1 defines the same nonzero element $a := w_1(\xi)$ in $H^1(S^1;\mathbb{Z}_2)$ as a generator.

It's obvious that all trivial line bundles are isomorphic. Suppose s_ξ, s_η are global sections of nontrivial line bundles ξ,η respectively, then s_ξ, s_η vanish somewhere on S^1. Assume that s_ξ vanishes on $p \in S^1$, then pick p out of the circle. The remaining space is contractible, and so there is a global nowhere vanishing section s on it since all vector bundles on it are trivial. Parametrize S^1 as $(0,1) \to S^1$ by $t \mapsto \mathrm{e}^{2\pi it}$ such that $p = 1$. Let $\tilde{s}(\mathrm{e}^{2\pi it}) = t(1-t)s(\mathrm{e}^{2\pi it})$ for all $t \in [0,1]$. $\tilde{s}$ is a well-defined section on S^1 and only vanishes at p. So without loss of generality, we can assume that s_ξ, s_η only vanish at p. Define $F : E(\xi) \to E(\eta)$ as

$$F|_{\mathrm{e}^{2\pi it}} : \pi_\xi^{-1}(\mathrm{e}^{2\pi it}) \to \pi_\eta^{-1}(\mathrm{e}^{2\pi it}),\ r\frac{s_\xi(\mathrm{e}^{2\pi it})}{\|s_\xi(\mathrm{e}^{2\pi it})\|} \to r\frac{s_\eta(\mathrm{e}^{2\pi it})}{\|s_\eta(\mathrm{e}^{2\pi it})\|}$$

for all $t \in (0,1)$ where $r \in \mathbb{R}$. Let $F|_1 : \pi_\xi^{-1}(1) \to \pi_\eta^{-1}(1)$ be defined as $v \to v$. It's trivial that F is a isomorphism. Hence, all nontrivial line bundles on S^1 are isomorphic and then $0, a \in H^1(S^1;\mathbb{Z}_2)$ corresponds to and only to two isomorphic classes of line bundles on S^1, that is, the one represented by trivial line bundle and the other by Möbius strip. $\square$

Homework 2.6. *Process the following steps to prove that a real vector bundle ξ is orientable iff $w_1(\xi) = 0$.*

Before stating the problem, we have to clarify some notions involved here.

Suppose b_1, b_2 are bases of a linear space V and $A : b_1 \to b_2$ is the unique linear transformation from b_1 to b_2. Then b_1 and b_2 are said to have ***the same orientation*** *(or be* ***consistently oriented****) if the A has positive determinant; otherwise they have opposite orientations. The property of having the same orientation defines an equivalence relation on the set of all ordered bases for V. If V is nonzero, there are precisely two equivalence classes determined by this relation. An* ***orientation*** *on V is an assignment of $+1$ to one equivalence class and -1 to the other.*

Given a real vector bundle $\pi : E \to B$, an ***orientation*** *of E means: for each fiber $\pi^{-1}(b)$, there is an orientation of the vector space $\pi^{-1}(b)$ and one demands that each trivialization map (which is a bundle map) $h : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is fiber-wise orientation-preserving, where $\mathbb{R}^n$ is given the standard orientation.*

A vector bundle that can be given an orientation is said to be ***orientable****.*

- (a) *Show that a line bundle ξ on S^1 is orientable iff it is trivial.*
- (b) *Define the determinant line bundle of ξ to be $\det \xi := \wedge^n \xi := \amalg_{b \in B} \wedge^n \pi^{-1}(b)$ where $n = \mathrm{rank}(\xi)$. (Recall that if V is an n-dimensional vector space, then $\dim(\wedge^k V) = \binom{n}{k}$ and so $\dim(\wedge^k V) = 1$.) Show that a vector bundle ξ is orientable iff its determinant line bundle $\det \xi$ is trivial.*
- (c) *Any continuous curve $c : S^1 \mapsto B$ defines a 1-cycle in $H^1(B, \mathbb{Z}_2)$ and hence it can be paired with $w_1(\xi)$. Compute this pairing in terms of $c^* w_1(\xi)$ to prove that ξ is orientable iff $w_1(\xi) = 0$.*

Proof. (a) $\Leftarrow$) Suppose ξ is trivial, then there is a nowhere vanishing section s on S^1. It's obvious that ξ is orientable by just assigning 1 to each $[s(b)]$.

$\Rightarrow$) By local triviality, for $b \in S^1$ there is a neighborhood U_b of b and a nowhere vanishing local section s_b on U_b. When b varies on S^1, we derive an open cover $\{U_b\}_{b \in B}$ of B. Without loss of generality, assume $\{U_b\}_{b \in B}$ are open intervals on S^1 and every point of S^1 is contained in at most two sets in the cover. Suppose the line bundle ξ is orientable, then there is a local constant function f maps $[s_b(b')]$ to $\{\pm 1\}$ for all $b' \in B$. Multiply s_b by -1 if necessary, we can assume that $f([s_b(b')]) \equiv 1$ for each $b \in B$. Since S^1 is connected, each U_b intersects another set in the cover. Hence, all s_b's can combine to produce a nowhere vanishing global section s by proper adjustment on the overlaps. So ξ is trivial.

(b) $\Leftarrow$)Assume $\det \xi$ is trivial, then there is a nowhere vanishing section ω of $\det \xi$. Suppose B is the base space of ξ. Define ε such that $\varepsilon_b(f) = \mathrm{sign}(\omega_b(f))$ for each $b \in B$. ε is clearly an orientation on ξ, i.e., $\varepsilon \in \mathrm{or}(\xi)$ and hence ξ is orientable.

$\Rightarrow$) Suppose ξ is orientable and let $s_1, \cdots, s_n$ be an ordered local basis on some neighborhood U_b of b. Then there is a local constant function ε with $\varepsilon(s_1(b), \cdots, s_n(b))$ taken values in $\{\pm 1\}$. Replacing s_1 by $-s_1$ and U_b by a smaller neighborhood if necessary, we may assume that $\varepsilon \equiv 1$ on U_b. So $s_1, \cdots, s_n$ would be a positively oriented local basis with respect to the orientation

ε. Then $\omega_b = s_1 \wedge \cdots \wedge s_n$ is a nowhere vanishing section of $\wedge^n \pi^{-1}(b)$ over U_b. Moreover, for every $b' \in B$, the form ω_b is positively oriented with respect to ε_b.

Obviously, there exists an open cover $\{U_{b_i}\}_{i\in I}$ of B with nowhere vanishing sections ω_i of $\wedge^n \xi|_{U_i}$ such that $\omega_i(b')$ is positively oriented with respect to ε_{b_i} for every $i \in I$.

Suppose B is Hausdorff and paracompact, then $\{U_{b_i}\}_{i\in I}$ admits a partition of unity $\{\psi_k\}_{k\in K}$ subordinate to $\{U_{b_i}\}_{i\in I}$. This means ψ_k's are continuous functions taking values in $[0,1]$, for all $k \in K$ $\operatorname{supp}(\psi_k) \subset U_{b_{i_k}}$ for some $i_k \in I$, and $\sum_{k\in K} \psi_k = 1$ with locally finite sum.

Let $\omega = \sum_{k\in K} \psi_k \omega_{ik}$. This is obviously an global nowhere vanishing section of ξ which is positively oriented with respect to ε. Hence, $\det \xi$ is trivial.

(c) Since ξ is orientable iff the line bundle $\wedge^n \xi$ on B is trivial, it suffices to prove that $w_1(\xi) = 0$ iff $\wedge^n \xi$ is trivial. Instead of a direct proof, we first verify two general propositions.

Suppose λ is a real line bundle on B, then λ is trivial iff the its Euler class $e(\lambda) = 0 \in H^1(B; \mathbb{Z}_2)$. Basically, let $f : S^1 \to B$ be any map, then by naturality of Euler class, one has $e(f^*\lambda) = f^*e(\lambda) \in H^1(S^1; \mathbb{Z}_2)$. It's known that all line bundles on S^1 are divided into two isomorphic classes, i.e., the trivial ones and the nontrivial ones. Moreover, the trivial ones correspond to zero Euler class in $H^1(S^1; \mathbb{Z}_2)$; the nontrivial ones correspond to the unique nonzero Euler class in $H^1(S^1; \mathbb{Z}_2)$. It follows that $f^*\lambda$ is trivial on S^1 for every f if $e(\lambda) = 0$. However, since each connected component of B is path-wise connected, λ is trivial iff $f^*\lambda$ is trivial for every map $f : S^1 \to B$. Then $e(\lambda) = 0$ yields λ is trivial. The inverse part is obvious. Hence, $\wedge^n \xi$ is trivial iff $e(\wedge^n \xi) = 0$.

The second proposition goes that $w_1(\xi) = e(\wedge^n \xi) \in H^1(B; \mathbb{Z}_2)$. In fact, let $f : B' \to B$ be a splitting map for ξ, so that $f^*\xi$ is a Whitney sum $\lambda_1 \oplus \cdots \oplus \lambda_m$ of real line bundles on B' and $f^* : H^*(X; \mathbb{Z}_2) \to H^*(B'; \mathbb{Z}_2)$ is a monomorphism. It suffices to show that $f^*e(\wedge^n \xi) = f^*w_1(\xi) \in H^1(B; \mathbb{Z}_2)$. On the one hand, $f^*e(\wedge^n \xi) = e(f^* \wedge^n \xi) = e(\wedge^n f^*\xi)$ by naturality of Euler classes and naturality of exterior powers, and since $f^*\xi = \lambda_1 \oplus \cdots \oplus \lambda_m$, we have

$$f^*e(\wedge^n \xi) = e(\wedge^m f^*\xi) = e(\lambda_1 \otimes \cdots \otimes \lambda_m) = e(\lambda_1) + \cdots + e(\lambda_m) \in H^1(B', \mathbb{Z}_2).$$

On the other hand,

$$\begin{aligned} f^*w(\xi) = w(f^*\xi) &= w(\lambda_1 \oplus \cdots \oplus \lambda_m) = w(\lambda_1) \cdots w(\lambda_m) \\ &= (1 + e(\lambda_1)) \cdots (1 + e(\lambda_m)) \in H^{\Pi}(B'; \mathbb{Z}_2) \end{aligned}$$

by naturality and the Whitney product formula for total Stiefel-Whitney classes. This implies that $f^*w_1(\xi) = e(\lambda_1) + \cdots + e(\lambda_m) = f^*e(\wedge^n \xi)$, and since f^* is a monomorphism one has $w_1(\xi) = e(\wedge^n \xi) \in H^1(B, \mathbb{Z}_2)$ as claimed. Hence, $\wedge^n \xi$ is trivial iff $w_1(\xi) = 0$. So ξ is orientable iff $w_1(\xi) = 0$. $\square$

2.4. **HW #4: 7/24/2018.**

Homework 2.7. *Show that for all odd positive integer n, $\mathbb{RP}^n$ is the boundary of some compact smooth manifold.*

Proof. In the definition of the monomial $w_1^{r_1} \cdots w_n^{r_n}$, the exponents are subject to the identity $r_1 + 2r_2 + \cdots + nr_n = n$. If all w_j's are zero for odd j, then the identity fails since it's even on the left and odd on the right. So the monomial must contain w_j for some odd j. Since $\binom{n+1}{j}$ is odd when n and j are both odd. Note that we have used the fact that $\binom{p}{q}$ is odd iff every bit of the binary expansion of q is less than or equal to the corresponding bit of the binary expansion of p. This is due to Lucas.

Thus, w_j equals zero. This immediately implies that all Whitney numbers of $\mathbb{R}\mathrm{P}^n$ is zero. Hence, $\mathbb{R}\mathrm{P}^n$ is the boundary of some compact smooth manifold whenver n is odd. $\square$

Homework 2.8. *Find a 4-dimensional smooth manifold M such that $\partial M = \mathbb{R}\mathrm{P}^3$.*

Solution. Take the complex surface $z_1^2 + z_2^2 + z_3^2 = 1$ in $\mathbb{C}^3$ and intersect it with the ball $|z_1|^2 + |z_2|^2 + |z_3|^2 \leq 1$ to get an 4-dimensional smooth manifold M whose boundary is claimed to be $\partial M = \mathbb{R}\mathrm{P}^3$ and embedded in $\mathbb{C}^3$. The boundary can be derived by intersecting the complex surface with S^5, i.e., $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$.

To verify the claim explicitly, consider the map from $\mathbb{C}^2$ to $\mathbb{C}^3$ given by

$$\begin{aligned} z_1 &= i[(z^2 + w^2) - (\bar{z}^2 + \bar{w}^2)]/2, \\ z_2 &= [(z^2 + w^2) + (\bar{z}^2 + \bar{w}^2)]/2, \\ z_3 &= zw + \bar{z}\bar{w}. \end{aligned}$$

One computes that $z_1^2 + z_2^2 + z_3^3 = (|z|^2 + |w|^2)^2$ and $|z_1|^2 + |z_2|^2 + |z_3|^2 = (|z|^2 + |w|^2)^2$. It follows that the image of S^3 in $\mathbb{C}^2$ is the complex surface $z_1^2 + z_2^2 + z_3^3 = 1$ intersected with S^5, i.e., $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. Since the map $(z, w) \mapsto (z_1, z_2, z_3)$ is $2:1$ restricted to S^3, its image is $\mathbb{R}\mathrm{P}^3$. $\square$

2.5. **HW #5: 7/25/2018.**

Homework 2.9. *Consider $E = M \times \mathbb{R}^k \xrightarrow{\pi_1} M$ with $\mathrm{d} : (f_1, \cdots, f_k)^T \mapsto (df_1, \cdots, d_k)^T$ as the trivial connection. Let $\nabla^E = \mathrm{d} + \omega : (f_1, \cdots, f_k)^T \mapsto (\mathrm{d}f_1, \cdots, \mathrm{d}f_k)^T + \omega(f_1, \cdots, f_k)^T$ where $\omega \in (\omega_{ij})_{k\times k}$ and $\omega_{ij} \in \Omega^1(M)$. Show that all connections on E is of the form $\mathrm{d} + \omega$.*

Proof. For any two connections ∇_1^E, ∇_2^E on E, $\nabla_1^E - \nabla_2^E$ is $C^\infty(M)$-linear by definition. Hence there is a unique $\omega' \in \Omega^1(M; E)$ such that $(\nabla_1^E - \nabla_2^E)(\sigma) = \omega'\sigma$ for all $\sigma \in \Gamma(E)$, that is, ∇_1^E, ∇_2^E differ by an element in $\Omega^1(M; E)$. So all connections on E is of the form $\mathrm{d} + \omega$. $\square$

3. Take-Home Exam: 7/30/2018

Question 3.1. *Give a detailed proof of the second part of Chern-Weil Theorem, that is, for any formal power series*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \in \mathbb{C}[[x]]$$

the cohomology class $[\mathrm{tr}[f(R^E)]] \in H^*(M)$ is independent of the connection ∇^E.

Proof. Suppose $\tilde{\nabla}^E$ is another connection on E and $\tilde{R}^E$ is its curvature. For any $t \in [0, 1]$, let ∇_t^E be the deformed connection on E given by $\nabla_t^E = (1-t)\nabla^E + t\tilde{\nabla}^E$. Then ∇_t^E is a connection on E such that $\nabla_0^E = \nabla^E$ and $\nabla_1^E = \tilde{\nabla}^E$. Moreover,

$$\frac{\mathrm{d}\nabla_t^E}{\mathrm{d}t} = \tilde{\nabla}^E - \nabla^E \in \Omega^1(M; \mathrm{End}(E)).$$

Denote the curvature of ∇_t^E by R_t^E with $t \in [0,1]$. We study the change of $\mathrm{tr}[f(R_t^E)]$ when t varies in $[0,1]$. Let $f'(x)$ be the power series obtained from the derivative of $f(x)$ with respect to x. By Bianchi's identity, we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}[f(R_t^E)] &= \mathrm{tr}\left[\frac{\mathrm{d}R_t^E}{\mathrm{d}t}f'(R_t^E)\right] = \mathrm{tr}\left[\frac{\mathrm{d}(\nabla_t^E)^2}{\mathrm{d}t}f'(R_t^E)\right] \\ &= \mathrm{tr}\left[\left[\nabla_t^E, \frac{\mathrm{d}\nabla_t^E}{\mathrm{d}t}\right]f'(R_t^E)\right] = \mathrm{tr}\left[\left[\nabla_t^E, \frac{\mathrm{d}\nabla_t^E}{\mathrm{d}t}f'(R_t^E)\right]\right] \\ &= \mathrm{d}\mathrm{tr}\left[\frac{\mathrm{d}\nabla_t^E}{\mathrm{d}t}f'(R_t^E)\right].\end{aligned}$$

Then we derive

$$\mathrm{tr}[f(R^E)] - \mathrm{tr}[f(\tilde{R}^E)] = -\mathrm{d}\int_0^1 \mathrm{tr}\left[\frac{\mathrm{d}\nabla_t^E}{\mathrm{d}t}f'(R_t^E)\right]\mathrm{d}t.$$

Hence, $[\mathrm{tr}[f(R^E)]] \in H^*(M)$ is independent of the connection ∇^E. ☐

Question 3.2. *For complex vector bundles E, E' on M, show that*

$$c_1(E \otimes E') = r'c_1(E) + rc_1(E')$$

where $r = \mathrm{rank}(E)$ and $r' = \mathrm{rank}(E')$. Use this to show that $c_1(\bar{E}) = -c_1(E)$.

Proof. Using the splitting principle, assume that $E = E_1 \oplus E_2 \oplus \cdots \oplus E_r$ and $E' = E_1' \oplus E_2' \oplus \cdots \oplus E_{r'}'$ are splittings of E, E' into line bundles. Then $c(E) = c(E_1 \oplus E_2 \oplus \cdots \oplus E_r) = \prod_{i=1}^r c(E_i) = \prod_{i=1}^r (1+\alpha_i)$ where $\alpha_i = c_1(E_i)$. Also, $c(E') = \prod_{i=1}^{r'}(1+\beta_i)$ where $\beta = c_1(E_i')$. Hence,

$$\begin{aligned}c(E_1 \otimes E') &= c((E_1 \oplus E_2 \oplus \cdots \oplus E_r) \otimes E') = \prod_{i=1}^r c(E_i \otimes E') \\ &= \prod_{i=1}^r c(E_i \otimes (E_1' \oplus E_2' \oplus \cdots \oplus E_{r'}')) = \prod_{i=1}^r \prod_{j=1}^{r'} c(E_i \otimes E_j') \\ &= \prod_{i=1}^r \prod_{j=1}^{r'} (1 + \alpha_i + \beta_j) = 1 + r'c_1(E) + rc_1(E') + \cdots\end{aligned}$$

which implies that $c_1(E \otimes E') = r'c_1(E) + rc_1(E')$. Since $E \otimes \bar{E}$ is a trivial line bundle, then $rc_1(E) + rc_1(\bar{E}) = c_1(E \otimes \bar{E}) = 0 \Rightarrow c_1(\bar{E}) = -c_1(E)$. ☐

Question 3.3. *Let E be a complex vector bundle of rank k on M and let $\det E = \wedge^k E$ be its top exterior product bundle, aka determinant bundle.*

(a) *If $\{U_\alpha\}$ is an open cover of M on which E is trivial and $g_{\alpha\beta}$ the corresponding transition matrices, what would be the transition functions of $\det E$? For a connection ∇^E on E, construct an induced connection $\nabla^{\det E}$ and compute its curvature in terms of that of ∇^E.*

(b) *Show that $c_1(\det E) = c_1(E)$.*

Proof. (a) By the definition of determinant bundle, the transition functions of $\wedge^k E$ are given by $j_{\alpha\beta}(x) = \det g_{\alpha\beta}(x) \in GL(1, \mathbb{C}) = \mathbb{C}^*$. Given a connection ∇^E and a local frame $e_1, \cdots, e_k$ for E, the corresponding connection matrix consisting of 1-forms ω_i^j is defined by $\nabla e_i = \omega_i^j e_j$. Since $e_1 \wedge \cdots \wedge e_k$ is a frame for $\det E$, we have

$$e_1 \wedge \cdots \wedge \nabla^E e_j \wedge \cdots \wedge e_k = e_1 \wedge \cdots \wedge \omega_j^k e_k \wedge \cdots \wedge e_k = \omega_j^k \delta_{jk} e_1 \wedge \cdots \wedge e_j \wedge \cdots \wedge e_k.$$

Then by definition of product connection, $\nabla^{\det E}(e_1 \wedge \cdots \wedge e_k) = \omega_j^j e_1 \wedge \cdots \wedge e_k$, that is, the connection matrix represents $\nabla^{\det E}$, which is the trace of the connection matrix of ∇^E. Hence immediately we deduce that $R^{\det E} = -\frac{\sqrt{-1}}{2\pi}\mathrm{tr}(R^E) = -c_1(E, \nabla^E)$, i.e., minus the first Chern form associated to ∇^E.

(b) Using the splitting principle, assume $E = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is a splitting of E into line bundles, then $\det E \approx E_1 \otimes E_2 \otimes \cdots \otimes E_k$. It follows that

$$\begin{aligned} c_1(E) = c_1(E_1 \oplus E_2 \oplus \cdots \oplus E_k) &= \sum_{i=1}^k c_1(E_i) \\ &= c_1(E_1 \otimes E_2 \otimes \cdots \otimes E_k) = c_1(\det E). \end{aligned}$$

□

Question 3.4.

(a) *Assume that M is a closed oriented manifold of dimension n (so that we can integrate differential forms on M). For any $\omega = [\omega]_{(n)} + [\omega]_{(n-1)} + \cdots \in \Omega^*(M)$ written in its components of homgeneous degree, define*

$$\int_M \omega = \int_M [\omega]_{(n)}.$$

Let K be a vector field on M and define $i_K : \Omega^k(M) \to \Omega^{k-1}(M)$ to be the contraction with respect to K. That is $i_K\omega(X_1, \cdots, X_{k-1}) = \omega(K, X_1, \cdots, X_{k-1})$. Put $\mathrm{d}_K = \mathrm{d} + i_K : \Omega^*(M) \to \Omega^*(M)$. (Restricted to the invariant forms, this is the coboundary operator that defines the equivariant cohomology.) Show that

$$\int_M \mathrm{d}_K \omega = 0.$$

(b) *Let $g = \langle \bullet, \bullet \rangle$ be a metric on TM with respect to which K is an infinitesimal isometry, meaning the Lie derivative $\mathcal{L}_K g = 0$. (This implies that the family of local diffeomorphisms generated by K will preserve the metric g). Set $\theta \in \Omega^1(M)$ to be the metric dual of K, i.e., $\theta(X) = \langle K, X \rangle$. Check that $\mathrm{d}_K\theta = |K|^2 + \mathrm{d}\theta$ is invertible in $\Omega^*(M)$ provided that K is nowhere vanishing. That is $(d_K\theta)^{-1} \in \Omega^*(M)$. In this case show that for any $\omega \in \Omega^*(M)$ which is d_K-closed,*

$$\omega = \mathrm{d}_K[(\mathrm{d}_K\theta)^{-1} \wedge \theta \wedge \omega].$$

In particular, in this case, for any $\omega \in \Omega^*(M)$, $\mathrm{d}_K\omega = 0$,

$$\int_M \omega = 0.$$

Proof. Sorry I can't work out this problem...

Integral on manifold is beyond my knowledge. ☐