FEM

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## FEM

The main constitutes of a finite element method for the solution of a boundary-value problem are

* The variational or weak statement of the problem;
* The approximate solution of the variatinal equations through the use of "finite element functions."

Strong form of the two-point boundary-value problem :

$$
(S)
\begin{cases}
\textrm{Given $\mathscr{l}:\bar{\Omega} \rightarrow \mathbb{R}$ and constants $\mathscr{q}$ and $\mathscr{h}$, find $u:\bar{\Omega} \rightarrow \mathbb{R}$, such that} \\
u\_{,\,xx}(x)+\mathscr{l}(x)=0 \quad \textrm{on}\,\, \Omega \\
u(1)=\mathscr{q} \\
-u\_{,\,x}(0)=\mathscr{h}
\end{cases}
$$

The exact solution of is trivial to obtain, namely:

### Weak form

*Trial solutions*:

where  *functions*, writen as , satisfy

*Weighting functions*, or *variations*:

Weak form (W):

$$
(W)
\begin{cases}
\textrm{Given $\mathscr{l}$, $\mathscr{q}$ and $\mathscr{h}$ as before, find $u \in \delta$ such that for all $w \in \mathscr{V}$} \\
u(1) = \mathscr{q} \\
w(1) = 0 \\
\int\_{0}^1 w\_{,x} u\_{,x} \textrm{d}x = \int\_0^1 w \mathscr{l} \textrm{d}x +w(0)\mathscr{h}
\end{cases}
$$

Formulations of this type are often called *virtual work*, or *virtual displacement, principles* in mechanics. The 's are the *virtual displacements*.

### Proposition

1. Let be a solution of (). Then is also a solution of ().
2. Let be a solution of (). Then is also a solution of ().

Both () and () possess unique solutions, thus the strong and weak solutions are the same.

### Remarks

The boundary contion is not explicitly mentioned in the statement of (), but is implied by the satisfaction of the variational equation. Boundary conditions of this type are referred to as *natural boundary conditions*. On the other hand, trial solutons are explicitly required to satisfy the boundary condition . Boundary conditions of this type are called *essential boundary conditions*

*Finite element methods* are to obtain approximate solutions to the original boundary-value problem in terms of approximating and by convenient, finite-dimensional collections of functions.

Let

The weak form can be rewritten as

### Galerkin's aproximation method

Finite-dimensional approximation functions of $\mathscr\delta$ and are denoted by and , respectively. The superscript refers to the association of and with a *mesh*, or *discretization*, of the domain , which is parameterized by a characteristic length scale . Thus, and are subsets of and , respectively:

$$
\delta^h \subset \delta \quad \textrm{(i.e., if } u^h \in \delta^h \textrm{, then } u^h \in \delta \textrm{)} \\
\mathscr{V}^h \subset \mathscr{V} \quad \textrm{(i.e., if } w^h \in \mathscr{V}^h \textrm{, then } w^h \in \mathscr{V} \textrm{)}
$$

which indicate:

$$
u^h (1) = \mathscr{q} \\
w^h (0) = 0
$$

To each member , we construct a function by

where is a gaven function satiffying the ssential boundary condition

Thus, satisfies the requisite boundary condition

is bla