University of Glasgow

Pre-course: Constitutive Modelling of Soft Tissues

Lecture 3

Strain Energy Functions

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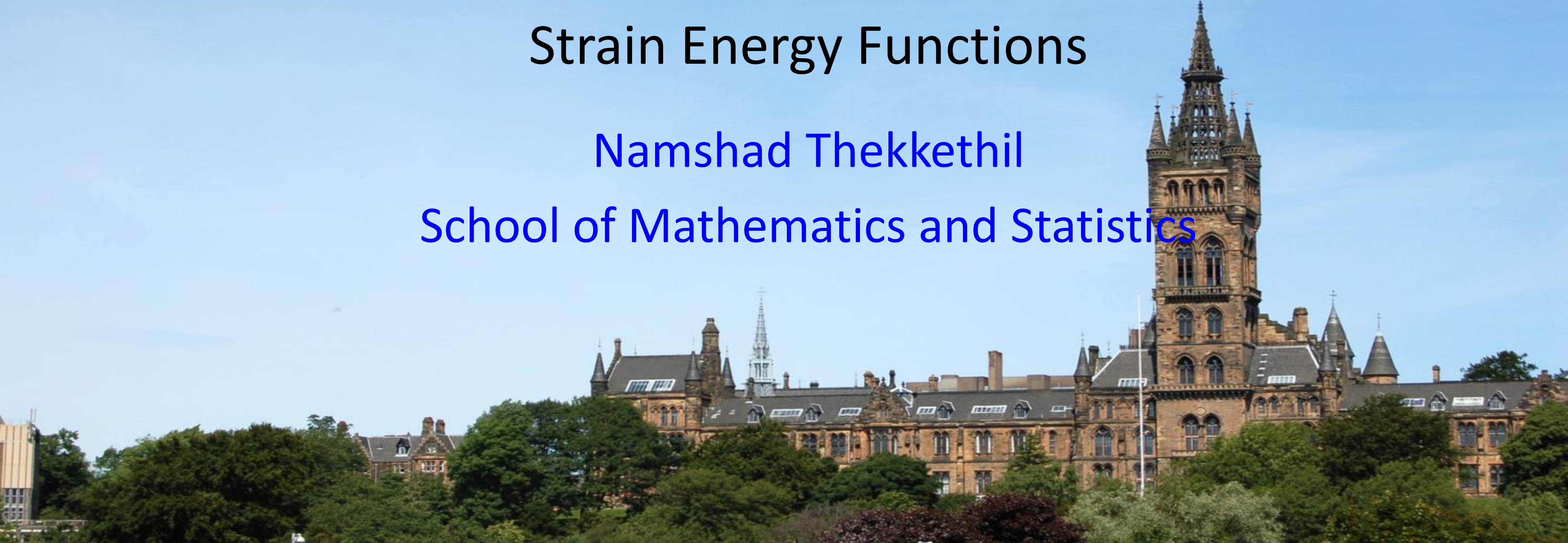


Overview

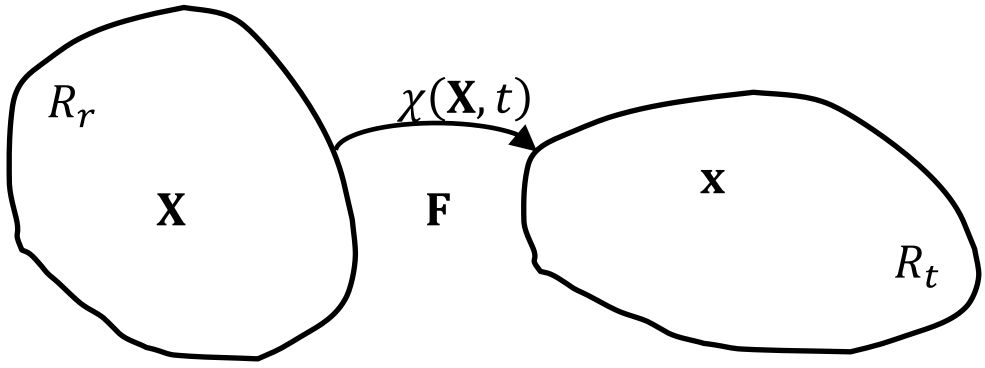
- Summary of balance equations
- Constitutive Equations – Strain Energy Function
- Objectivity
- Isotropy
- Stress and Strain Tensors
- Linear Elasticity
- Nonlinear Elasticity

Summary of Balance Equations

Mass balance: $$\int_{R_t} \rho dv = \int_{R_r} \rho_r dV$$

Linear Momentum balance: $$\int_{R_t} \left[\rho \frac{\partial \mathbf{v}}{\partial t} - \operatorname{div} \boldsymbol{\sigma} - \rho \mathbf{b}\right] dv = 0$$

Angular Momentum balance: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$



$\boldsymbol{\sigma}$ -> Cauchy Stress
$\mathbf{v}$ -> Velocity
$\mathbf{b}$ -> Body force

Unknowns: $\rho, \mathbf{v}\ (3\ components), \boldsymbol{\sigma}\ (9\ components)$ → $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ → $\rho, \mathbf{v}\ (3\ components), \boldsymbol{\sigma}\ (6\ components)$

13 Unknowns

10 Unknowns

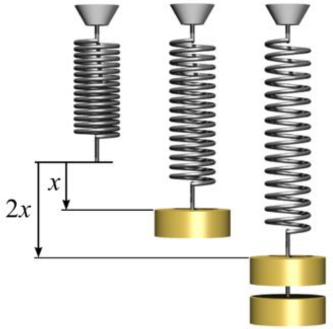
1+3 = 4 Equations

Constitutive Equations

Stress-strain relationship

$$\boldsymbol{\sigma} = f(\mathbf{x}, \mathbf{v}, \mathbf{F}, \mathbf{L}, \dots)$$

$$F = k\,\Delta x$$



Homogeneous Elastic material: $\Rightarrow$ $\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F})$

No pre-stress: $\Rightarrow$ $\mathbf{g}(\mathbf{I}) = \mathbf{0}$

Hyperelastic or ***Green elastic*** materials: Stress-strain relationship from strain energy function

Energy Balance Equation

Total rate of working = Rate of work due to body force + Rate of work due to surface force

$$P(t) = \int_{R_t} \rho \mathbf{b}.\mathbf{v} dv + \int_{S_t} (\boldsymbol{\sigma}\mathbf{n}).\mathbf{v} da$$

Simplification, See slide no. 22

Energy balance: $P(R_t) = \frac{d}{dt}\int_{R_t} {}^1\!/_2\, \rho \mathbf{v}.\mathbf{v} dv + \int_{R_t} \mathrm{tr}(\boldsymbol{\sigma}\mathbf{L}) dv$

Rate of Kinetic energy

Rate of stored Elastic energy

The rate of change of the total stored elastic energy…

For hyperelastic material, we have

Rate of Stored elastic energy **per unit reference volume**

Rate of stored Elastic energy $\Rightarrow \displaystyle\int_{R_t} \mathrm{tr}(\boldsymbol{\sigma}\mathbf{L})dv$

$$\int_{R_t} \mathrm{tr}(\boldsymbol{\sigma}\mathbf{L})dv = \int_{R_r} \mathrm{tr}(\boldsymbol{\sigma}\mathbf{L})J dV = \int_{R_r} \mathrm{tr}(J\boldsymbol{\sigma}\mathbf{L})dV$$

$dv = JdV$

W: Stored elastic energy (strain energy) per unit reference volume $\Rightarrow$ $\mathrm{tr}(J\boldsymbol{\sigma}\mathbf{L}) = \dfrac{\partial}{\partial t}W(\mathbf{F})$

$$\int_{R_r} \mathrm{tr}(J\boldsymbol{\sigma}\mathbf{L})dV = \int_{R_r} \frac{\partial}{\partial t}W(\mathbf{F})dV$$

Constitutive equations $\Rightarrow$ Expression for W

The rate of change of the total stored elastic energy…

Since $W(\mathbf{F})$ is only a function of $\mathbf{F}$, we have

Index notation

$$\frac{\partial W}{\partial F_{ij}}\frac{\partial F_{ij}}{\partial t} = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial W}{\partial F_{ij}}\frac{\partial F_{ij}}{\partial t}$$

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial F_{ij}}\frac{\partial F_{ij}}{\partial t}$$

Derivative of scalar w.r.t tensor

$$\frac{\partial W}{\partial F_{ij}} = \left(\frac{\partial W}{\partial \mathbf{F}}\right)_{ji}$$

$$\frac{\partial \mathbf{F}}{\partial t} = \dot{\mathbf{F}} = \mathbf{LF} \Rightarrow \frac{\partial F_{ij}}{\partial t} = (\mathbf{LF})_{ij}$$

$$L_{ij} = \frac{\partial v_i}{\partial x_j}$$

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial F_{ij}}\frac{\partial F_{ij}}{\partial t} \Rightarrow \frac{\partial W}{\partial t} = \left(\frac{\partial W}{\partial \mathbf{F}}\right)_{ji}(\mathbf{LF})_{ij} \Rightarrow \frac{\partial W}{\partial t} = \mathrm{tr}\left(\frac{\partial W}{\partial \mathbf{F}}\mathbf{LF}\right) \Rightarrow \frac{\partial W}{\partial t} = \mathrm{tr}\left(\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}\mathbf{L}\right)$$

Trace

$$\mathrm{tr}(\mathbf{AB}) = A_{ji}B_{ij}$$

$$\mathrm{tr}(\mathbf{ABC}) = \mathrm{tr}(\mathbf{CAB})$$

The rate of change of the total stored elastic energy

$$\mathrm{tr}(J\boldsymbol{\sigma}\mathbf{L}) = \frac{\partial}{\partial t}W(\mathbf{F}) \quad \Rightarrow \quad \mathrm{tr}(J\boldsymbol{\sigma}\mathbf{L}) = \mathrm{tr}\left(\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}\mathbf{L}\right)$$

$$\frac{\partial W}{\partial t} = \mathrm{tr}\left(\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}\mathbf{L}\right)$$

$$J\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} \quad \Rightarrow \quad \boxed{\boldsymbol{\sigma} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}}$$

Stress tensor

Cauchy stress tensor as a function of F

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}) = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}$$

Nominal/Engineering stress

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} \qquad S_{ij} = \frac{\partial W}{\partial F_{ji}}$$

First Piola Kirchhoff's stress

$$\mathbf{P} = \mathbf{S}^T$$

Stress in the reference configuration

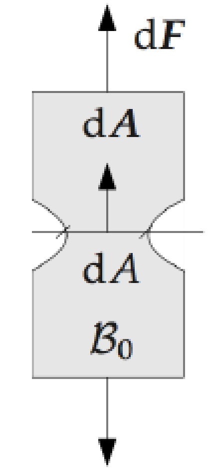
Force per unit reference area

S is not symmetric, but **FS** is symmetric

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma} = \mathbf{h}(\mathbf{F})$$

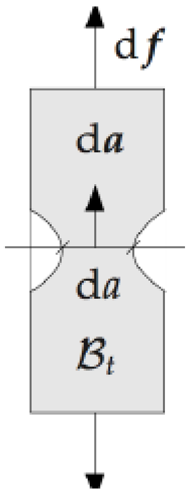
$$(\mathbf{FS})^T = \mathbf{FS}$$

$$\boldsymbol{\sigma} = J^{-1}\mathbf{FS}$$



Reference Configuration

First PK stress **P**



Current/ deformed Configuration

Cauchy stress $\boldsymbol{\sigma}$

Conjugate stress and strain tensors

Rate of total elastic stored energy per unit reference volume

$$\mathrm{tr}(J\boldsymbol{\sigma}\mathbf{L}) \xrightarrow[\boldsymbol{\sigma} = J^{-1}\mathbf{FS}]{} \mathrm{tr}(J\boldsymbol{\sigma}\mathbf{L}) = \mathrm{tr}(\mathbf{FSL}) = \mathrm{tr}(\mathbf{SLF}) \xrightarrow[\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}]{} \boxed{\mathrm{tr}(J\sigma\mathbf{L}) = \mathrm{tr}(\mathbf{S}\dot{\mathbf{F}})}$$

S and **F** are conjugate pairs

We define

$$\mathbf{E} = {}^{1}/_{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) \Longrightarrow \dot{\mathbf{E}} = {}^{1}/_{2}\left(\mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F}\right)$$

Green strain tensor

$$d\mathbf{x}.\,d\mathbf{x} - d\mathbf{X}.\,d\mathbf{X} = d\mathbf{X}.\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)d\mathbf{X}$$

$$\mathrm{tr}(\mathbf{S}\dot{\mathbf{F}}) = \mathrm{tr}(\mathbf{S}\mathbf{F}^{-T}\mathbf{F}^T\dot{\mathbf{F}}) \xrightarrow[\mathbf{T} = \mathbf{S}\mathbf{F}^{-T}]{} \mathrm{tr}(\mathbf{S}\dot{\mathbf{F}}) = \mathrm{tr}(\mathbf{T}\mathbf{F}^T\dot{\mathbf{F}}) \Longrightarrow \boxed{\mathrm{tr}(\mathbf{S}\dot{\mathbf{F}}) = \mathrm{tr}(\mathbf{T}\dot{\mathbf{E}})}$$

2nd Piola Kirchoff's stress tensor

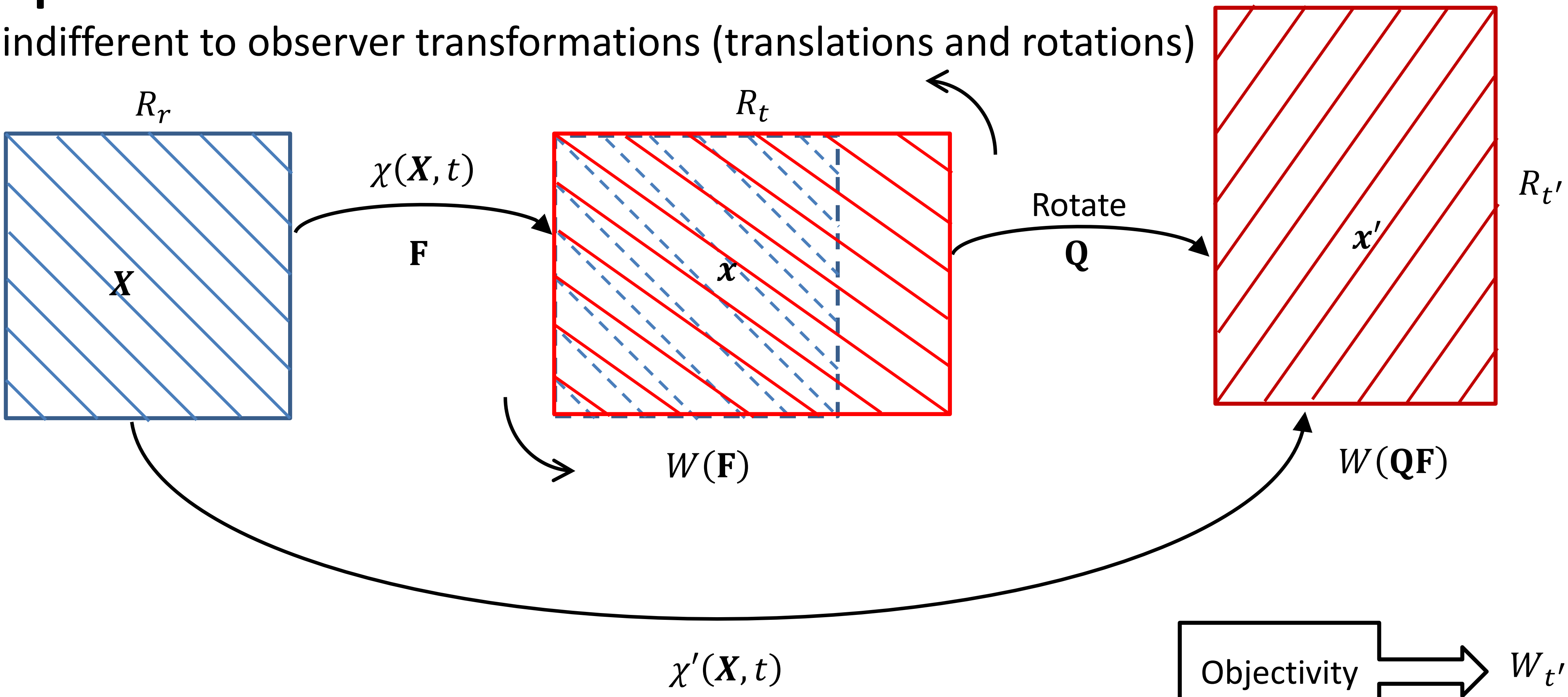
T and **E** are conjugate pairs

More conjugate stress/strain pairs: See slide no. 23

Objectivity

Principle of material frame-indifference.

W is indifferent to observer transformations (translations and rotations)



Objectivity $\Rightarrow$ $W_{t'} = W_t$

$$\mathbf{F}' = \frac{\partial \chi'(\boldsymbol{X}, t)}{\partial \boldsymbol{X}} = \frac{\partial \chi'(\boldsymbol{X}, t)}{\partial \chi(\boldsymbol{X}, t)} \frac{\partial \chi(\boldsymbol{X}, t)}{\partial \boldsymbol{X}} = \mathbf{QF}$$

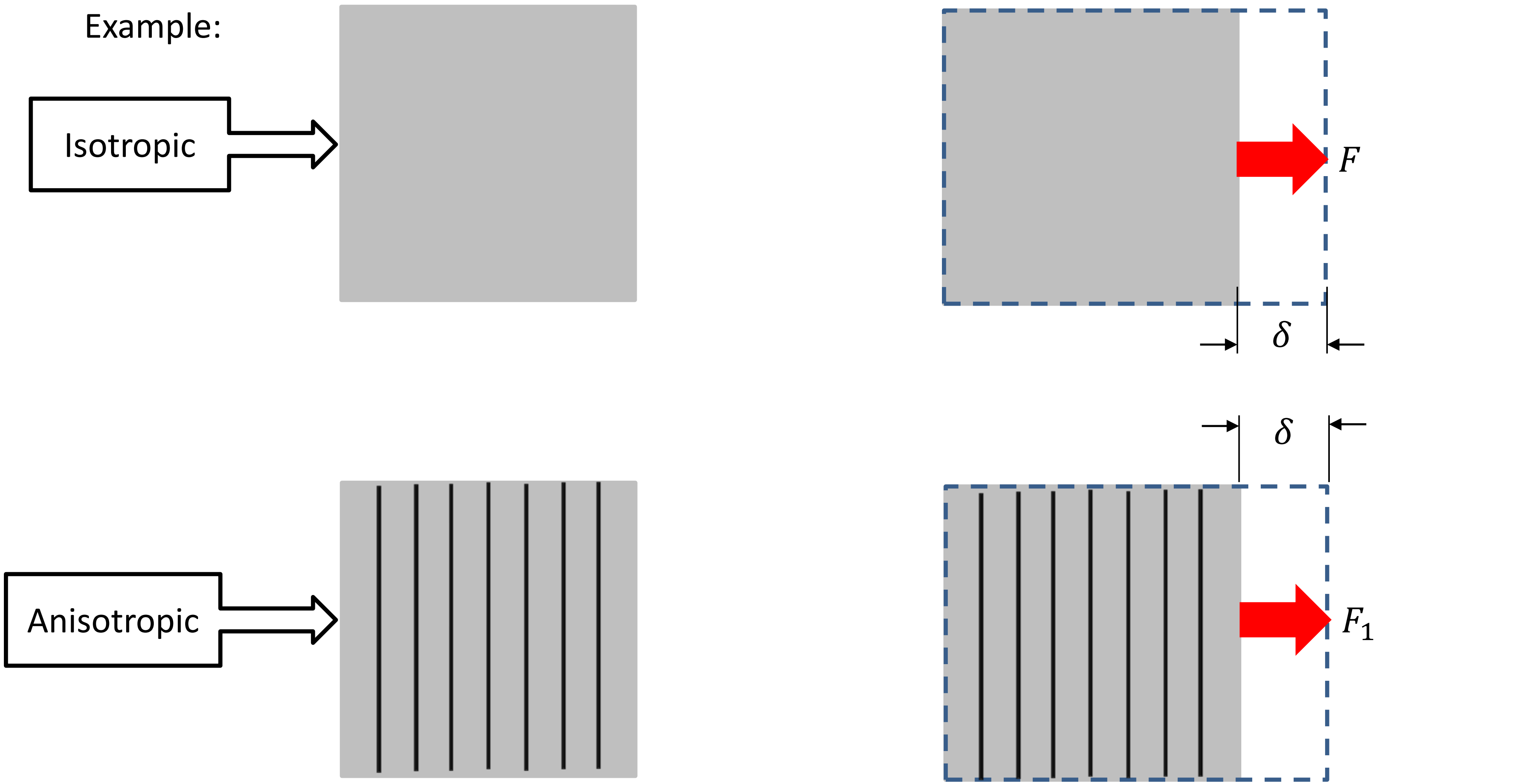
$$\boxed{W(\mathbf{QF}) = W(\mathbf{F})}$$

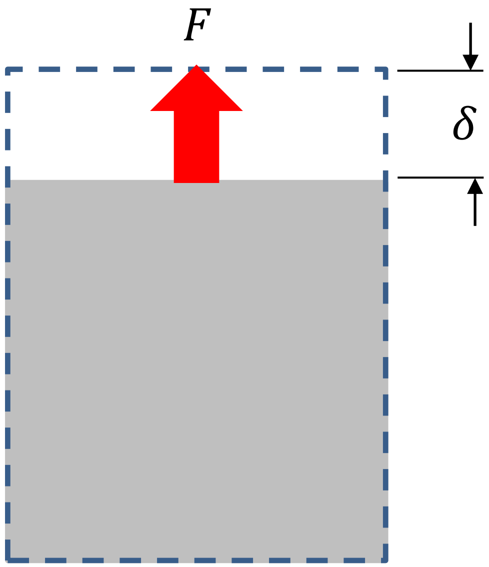
Isotropy…

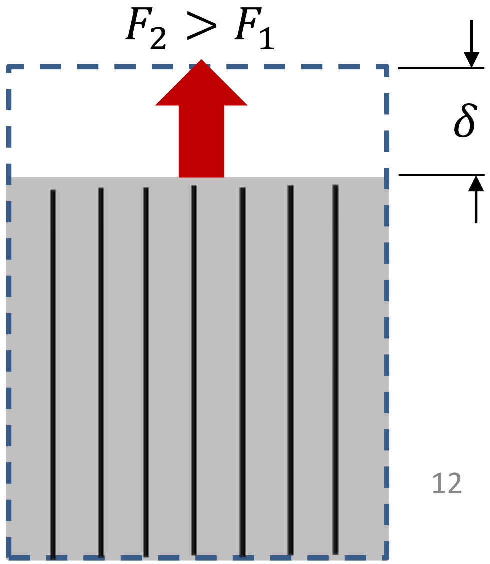
Isotropy -> Material Properties same in all directions

Anisotropy -> Material Properties not necessarily same in all directions

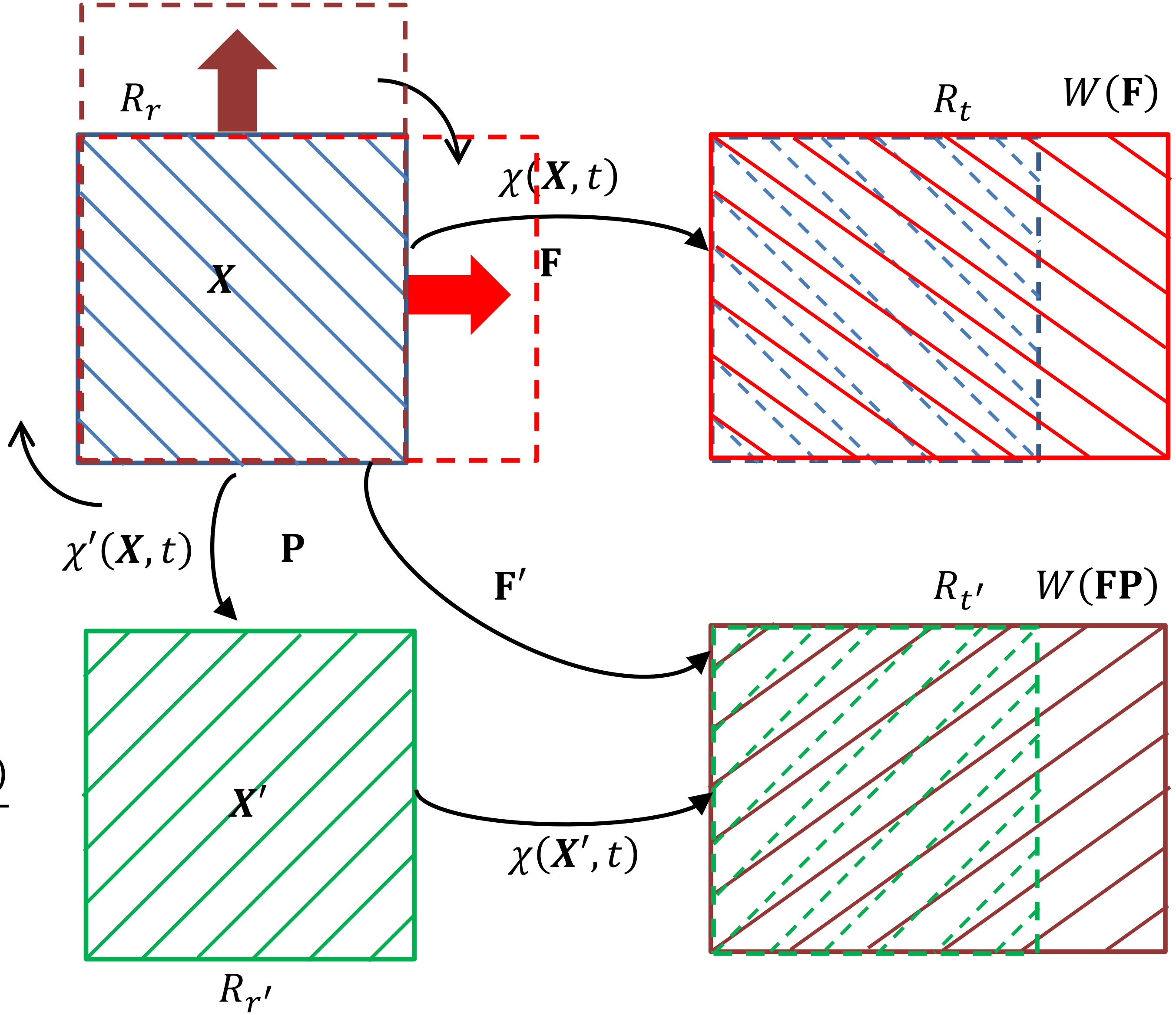
Example:







Isotropy



$$\mathbf{P} = \frac{\partial \chi(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}$$

$$\mathbf{F} = \frac{\partial \chi(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}$$

$$\mathbf{F}' = \frac{\partial \chi(\boldsymbol{X}', t)}{\partial \boldsymbol{X}} = \frac{\partial \chi(\boldsymbol{X}', t)}{\partial \boldsymbol{X}'} \frac{\partial \chi'(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}$$

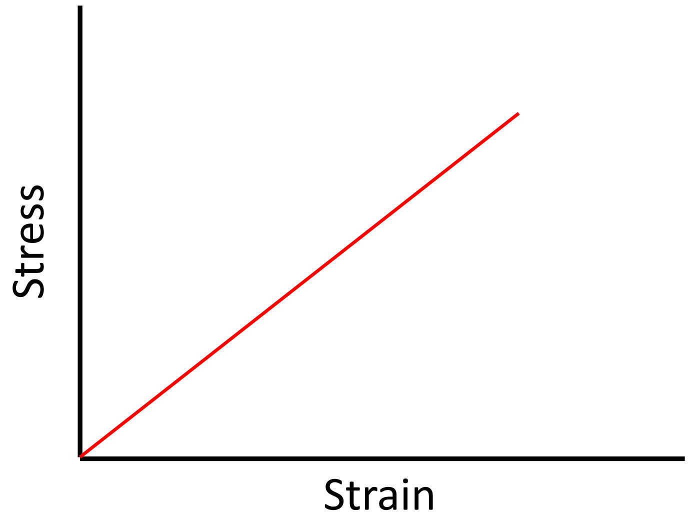
$$\mathbf{F}' = \mathbf{FP}$$

Isotropic $\Rightarrow$ $W_{t'} = W_t$

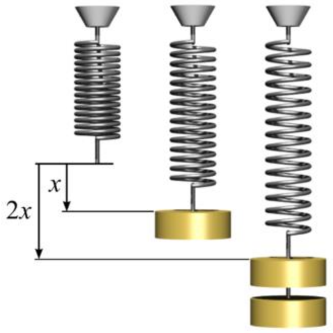
$$\boxed{W(\mathbf{FP}) = W(\mathbf{F})}$$

Types of deformation

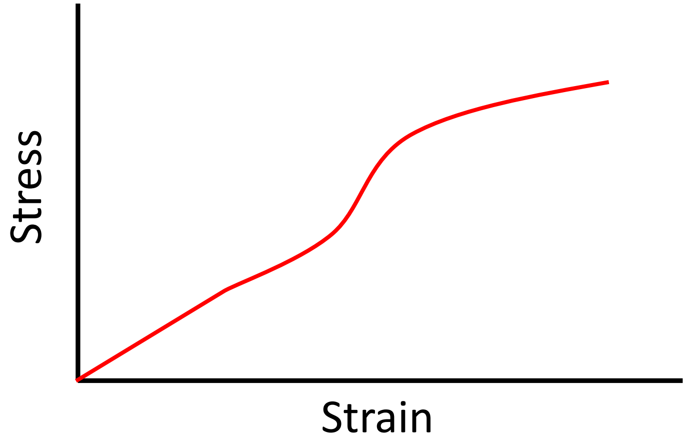
Linear Elastic -> Small deformation



$$F = k\,\Delta x$$



Nonlinear Elastic -> Large deformation



Linear Elasticity

Displacement: $\mathbf{u} = \mathbf{x} - \mathbf{X}$ $\Rightarrow$ $\mathbf{F} = \mathrm{Grad}\,\mathbf{x} = \mathbf{I} + \mathbf{H}$

$\mathbf{H} = \mathrm{Grad}\,\mathbf{u}$

Linear Elasticity -> Small deformation $\Rightarrow$ $\varepsilon \equiv \sqrt{\mathbf{H}.\mathbf{H}} = \sqrt{\mathbf{H}_{ij}^{2}} \ll 1$

Infinitesimal strain tensor

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{T}\right)$$

Polar decomposition, Objectivity & Isotropy $\Rightarrow$ $\boldsymbol{\sigma} = \dfrac{\partial \mathbf{g}}{\partial \mathbf{U}}\mathbf{e} + \mathcal{O}\left(\varepsilon^{2}\right)$ $\Rightarrow$ $\boldsymbol{\sigma} = \mathbf{c}\mathbf{e}$ $\Rightarrow$ $\sigma_{ij} = c_{ijkh} e_{kh}$

$\mathbf{F} = \mathbf{R}\mathbf{U}$

$\varepsilon \ll 1$

$$c_{ijkh} = \frac{\partial g_{ij}}{\partial U_{kh}}(\mathbf{I})$$

$$\mathbf{c} = \frac{\partial \mathbf{g}}{\partial \mathbf{U}}(\mathbf{I}) = \frac{\partial^{2} W}{\partial \mathbf{U} \partial \mathbf{U}}(\mathbf{I})$$

See derivation in slide no. 24-26

Elasticity tensor (4th Order)

$$\boxed{\boldsymbol{\sigma} = \lambda(\mathrm{tr}\,\mathbf{e})\mathbf{I} + 2\mu\mathbf{e}} \Rightarrow \boxed{W = \frac{1}{2}\left[\lambda(\mathrm{tr}\,\mathbf{e})^{2} + 2\mu\mathbf{e}\cdot\mathbf{e}\right]}$$

Hook's law

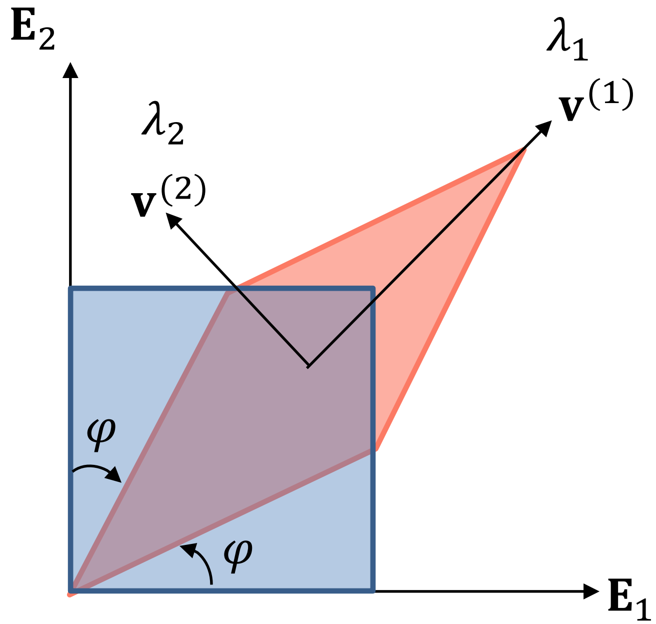
λ, μ -> Lame' moduli

Nonlinear Elasticity

For isotropic materials

$$W(\mathbf{F}) = W(\mathbf{FQ}) \Rightarrow W(\mathbf{F}) = W(\mathbf{V})$$

Objectivity $\quad \mathbf{F} = \mathbf{VR}$



$\gamma = \tan\varphi$

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}$$

$\lambda_1, \lambda_2, \lambda_3$ -> Eigen values (principal stretches) of $\mathbf{V}$

$\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}$-> Eigen vectors of $\mathbf{V}$

$$W(\mathbf{V}) = W'(\lambda_1, \lambda_2, \lambda_3)$$

i_1, i_2, i_3 -> principal invariants of $\mathbf{V}$

$$W'(\lambda_1, \lambda_2, \lambda_3) = W''(i_1, i_2, i_3)$$

$i_1 = \lambda_1 + \lambda_2 + \lambda_3 \qquad i_2 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 \qquad i_3 = \lambda_1\lambda_2\lambda_3$

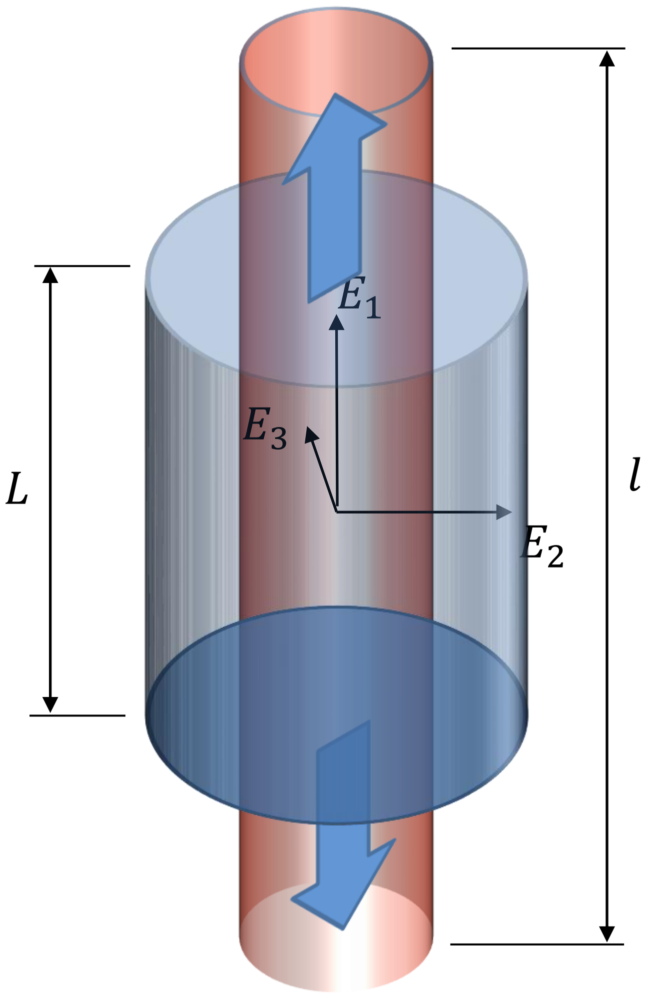
$$W'(\lambda_1, \lambda_2, \lambda_3) = W'''(I_1, I_2, I_3)$$

$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \qquad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2 \qquad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$

I_1, I_2, I_3 −> principal invariants of $\mathbf{C} = \mathbf{F}^T\mathbf{F}$

W as a function of the principal stretches

Simple elongation of a circular cylinder



$$\lambda_1 = \lambda = \frac{l}{L} = \frac{dx_1}{dX_1} \qquad \lambda_2 = \frac{dx_2}{dX_2} \qquad \lambda_3 = \frac{dx_3}{dX_3}$$

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \Rightarrow \mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$W = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 - 2\log J\right) + \frac{1}{2}\kappa(J-1)^2$$

μ, κ -> Material properties, always > 0

$$J = \det \mathbf{F} = \lambda_1 \lambda_2 \lambda_3$$

$$J\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} = \mu\left(\lambda_i^2 - 1\right) + \kappa J(J-1)$$

See derivation in slide no. 27-28

$\lambda_1 = \lambda$

$\sigma_1 = \sigma$

$$\Rightarrow J\sigma = \mu\left(\lambda^2 - 1\right) + \kappa J(J-1)$$

$\lambda_2 = \lambda_3$

$\sigma_2 = \sigma_3 = 0$

$$\Rightarrow 0 = \mu\left(\lambda_2^2 - 1\right) + \kappa J(J-1)$$

W as a function of the principal invariants I_1, I_2, and I_3…

$$\bar{W}(I_1, I_2, I_3)$$

$$I_1 = \mathrm{tr}(\mathbf{C})$$

$$I_2 = \frac{1}{2}\left[I_1^2 - \mathrm{tr}\left(\mathbf{C}^2\right)\right] \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F}$$

$$I_3 = \det \mathbf{C}$$

Nominal stress

$$\mathbf{S} = \frac{\partial \bar{W}}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{F}} + \frac{\partial \bar{W}}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{F}} + \frac{\partial \bar{W}}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{F}}$$

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^T$$

$$\frac{\partial I_2}{\partial \mathbf{F}} = \frac{\partial\left[\frac{1}{2}\left(I_1^2 - \mathrm{tr}\left(\mathbf{C}^2\right)\right)\right]}{\partial \mathbf{F}} = 2I_1\mathbf{F}^T - \frac{1}{2}\frac{\partial}{\partial \mathbf{F}}\mathrm{tr}\left(\mathbf{C}^2\right)$$

$$\frac{\partial I_3}{\partial \mathbf{F}} = \frac{\partial}{\partial \mathbf{F}}(\det \mathbf{F})^2 = 2I_3\mathbf{F}^{-1}$$

See derivation in slide no. 29

W as a function of the principal invariants I_1, I_2, and I_3

$$\bar{W}(I_1, I_2, I_3)$$

$$I_1 = \mathrm{tr}(\mathbf{C})$$

$$I_2 = \frac{1}{2}\left[I_1^2 - \mathrm{tr}\left(\mathbf{C}^2\right)\right] \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} \qquad I_1(\mathbf{F}), I_2(\mathbf{F}), I_3(\mathbf{F})$$

$$I_3 = \det\mathbf{C} = (\det\mathbf{F})^2$$

Nominal stress $\mathbf{S} = \frac{\partial \bar{W}}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{F}} + \frac{\partial \bar{W}}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{F}} + \frac{\partial \bar{W}}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{F}}$ $\Rightarrow$ $\mathbf{S} = 2\frac{\partial \bar{W}}{\partial I_1}\mathbf{F}^T + 2\frac{\partial \bar{W}}{\partial I_2}\left(2I_1\mathbf{F}^T - 2\mathbf{F}^T\mathbf{F}\mathbf{F}^T\right) + 2I_3\frac{\partial \bar{W}}{\partial I_3}\mathbf{F}^{-1}$

Cauchy stress

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{S} \quad \Rightarrow \quad \boldsymbol{\sigma} = 2I_3^{-1/2}\left(\frac{\partial \bar{W}}{\partial I_1} + I_1\frac{\partial \bar{W}}{\partial I_2}\right)\mathbf{B} - 2I_3^{-1/2}\frac{\partial \bar{W}}{\partial I_2}\mathbf{B}^2 + 2I_3^{1/2}\frac{\partial \bar{W}}{\partial I_3}\mathbf{I}$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T$$

W as a function of the principal invariants I_1, I_2, and I_3

neo-Hookean

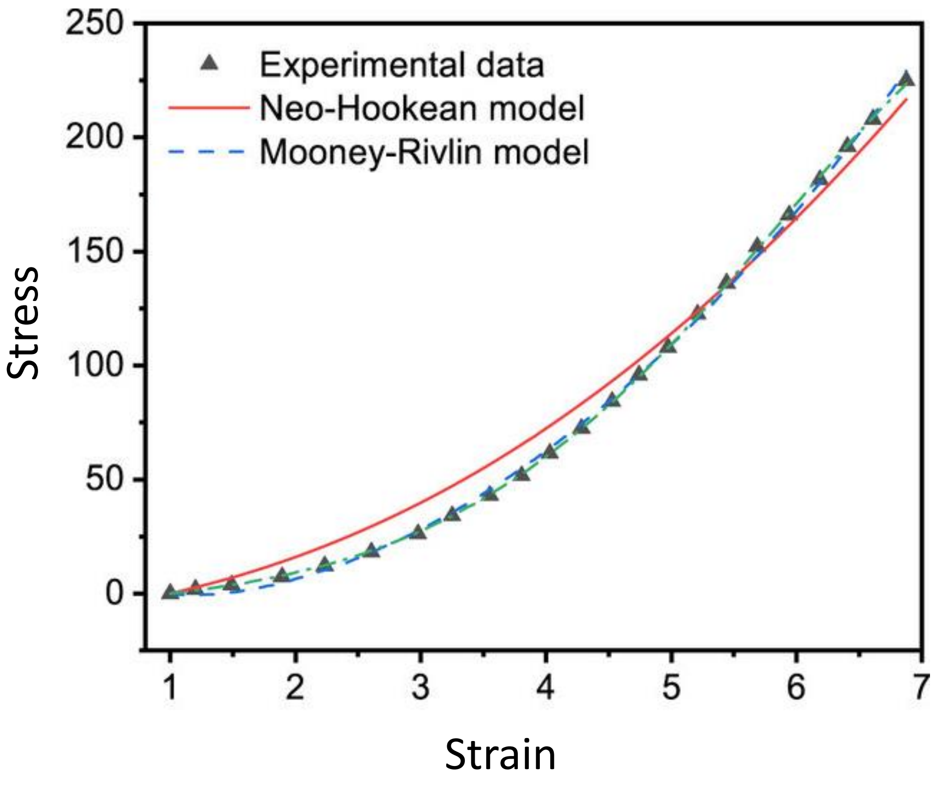
$$W = \frac{1}{2}\mu(I_1 - 3)$$

$$\mu > 0$$

Mooney-Rivlin

$$W = \frac{1}{2}\mu_1(I_1 - 3) - \frac{1}{2}\mu_2(I_2 - 3)$$

$$\mu_1 \geq 0, \mu_2 \leq 0, \mu_1 - \mu_2 = \mu > 0$$



Dong, X., & Duan, Z. (2022). Comparative study on the sealing performance of packer rubber based on elastic and hyperelastic analyses using various constitutive models. *Materials Research Express*, *9*(7), 075301.

Extra Notes

Energy Balance Equation

$$P(t) = \int_{R_t} \rho\mathbf{b}.\mathbf{v}dv + \int_{S_t} (\boldsymbol{\sigma}\mathbf{n}).\mathbf{v}da = \int_{R_t} \rho\mathbf{b}.\mathbf{v}dv + \int_{S_t} (\boldsymbol{\sigma}\mathbf{v}).\mathbf{n}da = \int_{R_t} [\rho\mathbf{b}.\mathbf{v} + \mathrm{div}(\boldsymbol{\sigma}\mathbf{v})]dv$$

(**σ** is symmetric) (Divergence theorem)

$$= \int_{R_t} [\rho\mathbf{b}.\mathbf{v} + \mathrm{div}(\boldsymbol{\sigma}).\mathbf{v} + \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{L})]dv = \int_{R_t} \left[\left(\rho\mathbf{b} + \mathrm{div}(\boldsymbol{\sigma})\right).\mathbf{v} + \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D})\right]dv = \int_{R_t} [\rho\dot{\mathbf{v}}.\mathbf{v} + \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D})]dv$$

$(\left(\sigma_{ij}v_j\right)_{,i} = \sigma_{ij,i}v_j + \sigma_{ij}v_{j,i} = \sigma_{ij,i}v_j + \sigma_{ij}L_{ji})$ $(\sigma_{ij}L_{ji} = \sigma_{ij}\left(D_{ji} + W_{ji}\right) = \sigma_{ij}D_{ji})$ Using momentum balance

$$P(R_t) = \frac{d}{dt}\int_{R_t} {}^{1}\!/_{2}\,\rho\mathbf{v}.\mathbf{v}dv + \int_{R_t} \mathrm{tr}(\boldsymbol{\sigma}\mathbf{L})dv$$

Conjugate stress and strain tensors

Stored energy rate $\Rightarrow$ $$\frac{\partial W}{\partial t} = \mathrm{tr}\left(\mathbf{T}^{(2)}\dot{\mathbf{E}}^{(2)}\right) = \mathrm{tr}\left[{}^{1}\!/_{2}\left(\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}\right)\dot{\mathbf{E}}^{(1)}\right]$$

$$\mathbf{T}^{(1)} = {}^{1}\!/_{2}\left(\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}\right) \quad \Rightarrow \quad \mathrm{tr}\left(\mathbf{T}^{(2)}\dot{\mathbf{E}}^{(2)}\right) = \mathrm{tr}\left[\mathbf{T}^{(1)}\dot{\mathbf{E}}^{(1)}\right]$$

Another conjugate pair
$\mathbf{T}^{(1)}$ and $\mathbf{E}^{(1)}$

$$\frac{\partial W}{\partial t} = J\mathrm{tr}(\sigma\mathbf{D}) = \mathrm{tr}\left(\mathbf{S}\dot{\mathbf{F}}\right) = \mathrm{tr}\left(\mathbf{T}^{(2)}\dot{\mathbf{E}}^{(2)}\right) = \mathrm{tr}\left(\mathbf{T}^{(1)}\dot{\mathbf{E}}^{(1)}\right)$$

In general

$$\frac{\partial W}{\partial t} = \mathrm{tr}\left(\mathbf{T}^{(m)}\dot{\mathbf{E}}^{(m)}\right) \qquad \mathbf{E}^{(m)} = \frac{\mathbf{U}^{m} - \mathbf{I}}{m}$$

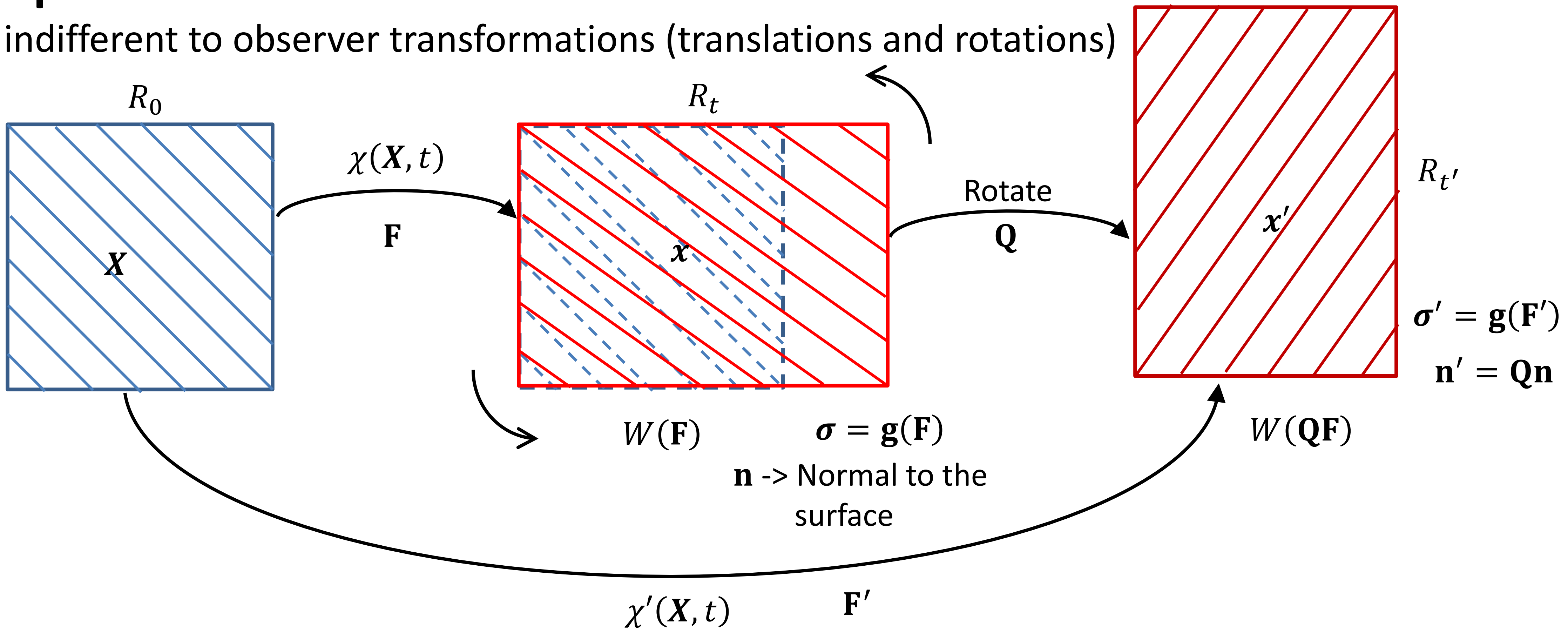
$W^{(m)}$ as a function of the general strain tensor $\mathbf{E}^{(m)}$

$$W^{(m)}\left(\mathbf{E}^{(m)}\right) = W^{(1)}\left(\mathbf{E}^{(1)}\right) = W^{(2)}\left(\mathbf{E}^{(2)}\right) \quad \Rightarrow \quad \mathbf{T}^{(m)} = \frac{\partial W}{\partial \mathbf{E}^{(m)}}$$

Objectivity

Principle of material frame-indifference.

W is indifferent to observer transformations (translations and rotations)



Traction vector on R_t: $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$

Traction vector on $R_{t'}$: $\mathbf{t}' = \boldsymbol{\sigma}'\mathbf{n}' = \boldsymbol{\sigma}'\mathbf{Q}\mathbf{n}$

$$\xrightarrow{\mathbf{t}' = \mathbf{Q}\mathbf{t}} \boldsymbol{\sigma}'\mathbf{Q}\mathbf{n} = \mathbf{Q}\boldsymbol{\sigma}\mathbf{n} \Rightarrow \boldsymbol{\sigma}'\mathbf{Q} = \mathbf{Q}\boldsymbol{\sigma} \xrightarrow{\mathbf{Q}^{-1} = \mathbf{Q}^T} \boldsymbol{\sigma}' = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T$$

$$\mathbf{g}(\mathbf{F}') = \mathbf{Q}\mathbf{g}(\mathbf{F})\mathbf{Q}^T$$

Polar decomposition

Polar decomposition theorem

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

$\mathbf{U}, \mathbf{V}$ -> Positive definite symmetric tensor
$\mathbf{R}$ -> Proper orthogonal tensor

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$$

$$\mathbf{V}^2 = \mathbf{F}\mathbf{F}^T$$

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$$

Isotropy: $\mathbf{g}(\mathbf{Q}\mathbf{F}) = \mathbf{Q}\mathbf{g}(\mathbf{F})\mathbf{Q}^T \xrightarrow[\mathbf{Q} = \mathbf{R}]{} \mathbf{g}(\mathbf{R}\mathbf{F}) = \mathbf{R}\mathbf{g}(\mathbf{F})\mathbf{R}^T \Rightarrow \mathbf{g}(\mathbf{R}\mathbf{U}) = \mathbf{R}\mathbf{g}(\mathbf{U})\mathbf{R}^T \Rightarrow \mathbf{g}(\mathbf{F}) = \mathbf{R}\mathbf{g}(\mathbf{U})\mathbf{R}^T$

Linear Elasticity

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left[(\mathbf{I} + \mathbf{H})^T(\mathbf{I} + \mathbf{H}) - \mathbf{I}\right] = \frac{1}{2}\left[\mathbf{H} + \mathbf{H}^T + \mathcal{O}\left(\varepsilon^2\right)\right] \Rightarrow \mathbf{E} = \mathbf{e} + \mathcal{O}\left(\varepsilon^2\right)$$

Infinitesimal strain tensor $\Rightarrow$ $\mathbf{e} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right)$

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} = (\mathbf{I} + \mathbf{H})^T(\mathbf{I} + \mathbf{H}) = \mathbf{I} + 2\mathbf{e} + \mathcal{O}\left(\varepsilon^2\right) \Rightarrow \mathbf{U} = \mathbf{I} + \mathbf{e} + \mathcal{O}\left(\varepsilon^2\right) \Rightarrow \mathbf{U}^{-1} = \mathbf{I} + \mathbf{e} + \mathcal{O}\left(\varepsilon^2\right)$$

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = [\mathbf{I} + \mathbf{H}]\left[\mathbf{I} - \mathbf{e} + \mathcal{O}\left(\varepsilon^2\right)\right] = \mathbf{I} + \frac{1}{2}\left(\mathbf{H} - \mathbf{H}^T\right) + \mathcal{O}\left(\varepsilon^2\right) \Rightarrow \mathbf{R} = \mathbf{I} + \boldsymbol{w} + \mathcal{O}\left(\varepsilon^2\right)$$

$$\mathbf{w} = \frac{1}{2}\left(\mathbf{H} - \mathbf{H}^T\right)$$

$$\mathbf{g}(\mathbf{U}) = \mathbf{g}(\mathbf{I} + \mathbf{e}) = \mathbf{g}(\mathbf{I}) + \frac{\partial \mathbf{g}}{\partial \mathbf{U}}(\mathbf{I})\mathbf{e} + \mathcal{O}\left(\varepsilon^2\right)$$

$$\boldsymbol{\sigma} = \mathbf{R}\mathbf{g}(\mathbf{U})\mathbf{R}^T = \left[\mathbf{I} + \mathbf{w} + \mathcal{O}\left(\varepsilon^2\right)\right]\left[\mathbf{g}(\mathbf{I}) + \frac{\partial \mathbf{g}}{\partial \mathbf{U}}\mathbf{e} + \mathcal{O}\left(\varepsilon^2\right)\right]\left[\mathbf{I} - \mathbf{w} + \mathcal{O}\left(\varepsilon^2\right)\right]$$

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{I}) + \mathbf{w}\mathbf{g}(\mathbf{I}) - \mathbf{g}(\mathbf{I})\mathbf{w} + \frac{\partial \mathbf{g}}{\partial \mathbf{U}}\mathbf{e} + \mathcal{O}\left(\varepsilon^2\right) \Rightarrow \boldsymbol{\sigma} = \frac{\partial \mathbf{g}}{\partial \mathbf{U}}\mathbf{e} + \mathcal{O}\left(\varepsilon^2\right)$$

No pre-stress: $\mathbf{g}(\mathbf{I}) = \mathbf{0}$

W as a function of the principal stretches…

$$W(\lambda_1, \lambda_2, \lambda_3) \qquad \frac{\partial W}{\partial t} = \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial t}$$

We have

$$\left.\begin{aligned} \frac{\partial W}{\partial t} &= J\mathrm{tr}(\boldsymbol{\sigma}\mathbf{D}) \\ \boldsymbol{\sigma} &= \sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)} \end{aligned}\right\} \Rightarrow \frac{\partial W}{\partial t} = J\mathrm{tr}\left(\sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}\, \mathbf{D}\right) = J \sum_{i=1}^{3} \sigma_i D_{ii}$$

σ_i -> Principal stresses

D_{ii} -> Normal component of $\mathbf{D}$ referred to the axes $\mathbf{v}^{(i)}$

$$D_{ii} = \mathbf{v}^{(i)}.\left(\mathbf{D}\mathbf{v}^{(i)}\right)$$

$$\mathbf{D} = \frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}}\right)\mathbf{R}^T \Rightarrow \mathbf{D} = \mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-1}\right)\mathbf{R}^T$$

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)} \Rightarrow \mathbf{D} = \mathbf{R} \sum_{i=1}^{3} \left(\dot{\lambda}_i \lambda_i^{-1} \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}\right) \mathbf{R}^T \Rightarrow \mathrm{D} = \sum_{i=1}^{3} \left(\dot{\lambda}_i \lambda_i^{-1} \mathbf{R}\mathbf{u}^{(i)} \otimes \mathbf{R}\mathbf{u}^{(i)}\right)$$

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)} \qquad \mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)} \Rightarrow \mathbf{D} = \sum_{i=1}^{3} \left(\dot{\lambda}_i \lambda_i^{-1} \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}\right) \Rightarrow D_{ii} = \dot{\lambda}_i \lambda_i^{-1}$$

W as a function of the principal stretches

$$D_{ii} = \dot{\lambda}_i \lambda_i^{-1} \Rightarrow \frac{\partial W}{\partial t} = \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i} \dot{\lambda}_i = \sum_{i=1}^{3} J \sigma_i \dot{\lambda}_i \lambda_i^{-1} \Rightarrow \frac{\partial W}{\partial \lambda_i} = J \sigma_i \lambda_i^{-1} \Rightarrow \sigma_i = J^{-1} \lambda_i \frac{\partial W}{\partial \lambda_i}$$

$$J = \lambda_1 \lambda_2 \lambda_3$$

Cauchy stress

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} J^{-1} \lambda_i \frac{\partial W}{\partial \lambda_i} \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}$$

Nominal stress

$$\mathbf{S} = J \mathbf{F}^{-1} \boldsymbol{\sigma} \Rightarrow \mathbf{S} = J \mathbf{U}^{-1} \mathbf{R}^T \sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)} \Rightarrow \mathbf{S} = J \mathbf{U}^{-1} \sum_{i=1}^{3} \sigma_i \mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)} \Rightarrow \mathbf{S} = \sum_{i=1}^{3} J \sigma_i \lambda_i^{-1} \mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)}$$

$$\mathbf{F}^{-1} = \mathbf{U}^{-1} \mathbf{R}^T \qquad \mathbf{R}^T \mathbf{v}^{(i)} = \mathbf{u}^{(i)} \qquad \mathbf{U}^{-1} \mathbf{u}^{(i)} = \lambda_i^{-1} \mathbf{v}^{(i)}$$

W as a function of the principal invariants I_1, I_2, and I_3

$$\frac{\partial I_1}{\partial \mathbf{F}} = \frac{\partial \mathrm{tr}\left(\mathbf{F}^T\mathbf{F}\right)}{\partial \mathbf{F}} \qquad \mathrm{tr}\left(\mathbf{F}^T\mathbf{F}\right) = \left(\mathbf{F}^T\right)_{i\beta}(\mathbf{F})_{\beta i} \qquad \mathrm{tr}\left(\mathbf{F}^T\mathbf{F}\right) = F_{\beta i}F_{\beta i} = F_{\beta i}^2$$

$$\frac{\partial I_1}{\partial \mathbf{F}} = \frac{\partial \mathrm{tr}\left(\mathbf{F}^T\mathbf{F}\right)}{\partial \mathbf{F}} \Rightarrow \left(\frac{\partial I_1}{\partial \mathbf{F}}\right)_{k\alpha} = \frac{\partial F_{\beta i}^{\mathbf{2}}}{\partial F_{\alpha k}} = 2F_{\beta i}\frac{\partial F_{\beta i}}{\partial F_{\alpha k}} \Rightarrow \left(\frac{\partial I_1}{\partial \mathbf{F}}\right)_{k\alpha} = 2F_{\alpha k} \Rightarrow \frac{\partial I_1}{\partial \boldsymbol{F}} = 2\mathbf{F}^T$$

$$\frac{\partial I_2}{\partial \mathbf{F}} = \frac{\partial \left[\frac{1}{2}\left(I_1^2 - \mathrm{tr}\left(\mathbf{C}^2\right)\right)\right]}{\partial \mathbf{F}}$$

$$\left[\frac{\partial}{\partial \mathbf{F}}\mathrm{tr}\left(\mathbf{C}^2\right)\right]_{k\alpha} = \frac{\partial\left(C_{\beta i}^2\right)}{\partial F_{\alpha k}} = 2C_{\beta i}\frac{\partial\left(C_{\beta i}\right)}{\partial F_{\alpha k}} = 2C_{\beta i}\frac{\partial\left(F_{\beta p}^T F_{pi}\right)}{\partial F_{\alpha k}} = 2C_{\beta i}\left[F_{p\beta}\frac{\partial\left(F_{pi}\right)}{\partial F_{\alpha k}} F_{pi}\frac{\partial\left(F_{p\beta}\right)}{\partial F_{\alpha k}}\right]$$

$$= 2C_{\beta k}\left[F_{\alpha\beta}\right] + 2C_{ki}\left[F_{\alpha i}\right] = 4\mathbf{F}^T\mathbf{F}\mathbf{F}^T$$

$$\frac{\partial I_2}{\partial \mathbf{F}} = \frac{\partial \left[\frac{1}{2}\left(I_1^2 - \mathrm{tr}\left(\mathbf{C}^2\right)\right)\right]}{\partial \mathbf{F}} = 2I_1\mathbf{F}^T - \frac{1}{2}\frac{\partial}{\partial \mathbf{F}}\mathrm{tr}\left(\mathbf{C}^2\right) \qquad \frac{\partial I_3}{\partial \mathbf{F}} = \frac{\partial}{\partial \mathbf{F}}(\det \mathbf{F})^2 = 2I_3\mathbf{F}^{-1}$$