1 Introduction

Recommended Problems

P1.1

Evaluate each of the following expressions for the complex number $z = \frac{1}{2}e^{j\pi/4}$.

(a) $Re\{z\}$

(b) $Im\{z\}$

(c) $|z|$

(d) $\sphericalangle z$

(e) z^* (* denotes complex conjugation)

(f) $z + z^*$

P1.2

Let z be an arbitrary complex number.

(a) Show that

$$Re\{z\} = \frac{z + z^*}{2}$$

(b) Show that

$$jIm\{z\} = \frac{z - z^*}{2}$$

P1.3

Using Euler's formula, $e^{j\theta} = \cos\theta + j\sin\theta$, derive the following relations:

(a) $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

(b) $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

P1.4

(a) Let $z = re^{j\theta}$. Express in polar form (i.e., determine the magnitude and angle for) the following functions of z:

(i) z^*

(ii) z^2

(iii) jz

(iv) zz^*

(v) $\dfrac{z}{z^*}$

(vi) $\dfrac{1}{z}$

(b) Plot in the complex plane the vectors corresponding to your answers to Problem P1.4a(i)–(vi) for $r = \frac{2}{3}$, $\theta = \pi/6$.

P1.5

Show that

$$(1 - e^{j\alpha}) = 2 \sin\left(\frac{\alpha}{2}\right) e^{j[(\alpha - \pi)/2]}$$

P1.6

For $x(t)$ indicated in Figure P1.6, sketch the following:

(a) $x(-t)$

(b) $x(t + 2)$

(c) $x(2t + 2)$

(d) $x(1 - 3t)$

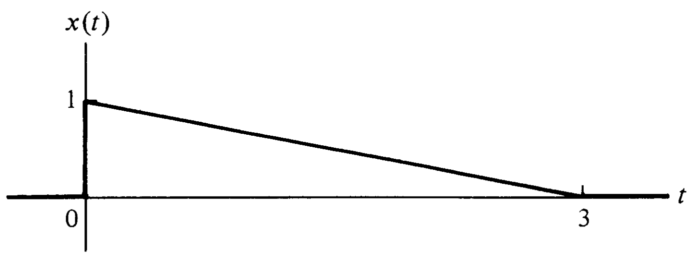

Figure P1.6

P1.7

Evaluate the following definite integrals:

(a) $\int_0^a e^{-2t}\,dt$

(b) $\int_2^\infty e^{-3t}\,dt$

Resource: Signals and Systems
Professor Alan V. Oppenheim

1 Introduction

Solutions to Recommended Problems

S1.1

(a) Using Euler's formula,

$$e^{j\pi/4} = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$

Since $z = \frac{1}{2}e^{j\pi/4}$,

$$Re\{z\} = \frac{1}{2}Re\left\{\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right\} = \frac{\sqrt{2}}{4}$$

(b) Similarly,

$$Im\{z\} = \frac{1}{2}Im\left\{\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right\} = \frac{\sqrt{2}}{4}$$

(c) The magnitude of z is the product of the magnitudes of $\frac{1}{2}$ and $e^{j\pi/4}$. However, $|\frac{1}{2}| = \frac{1}{2}$, while $|e^{j\theta}| = 1$ for all θ. Thus,

$$|z| = |\tfrac{1}{2}e^{j\pi/4}| = |\tfrac{1}{2}|\,|e^{j\pi/4}| = \tfrac{1}{2}\cdot 1 = \tfrac{1}{2}$$

(d) The argument of z is the sum of the arguments of $\frac{1}{2}$ and $e^{j\pi/4}$. Since $\sphericalangle\frac{1}{2} = 0$ and $\sphericalangle e^{j\theta} = \theta$ for all θ,

$$\sphericalangle z = \sphericalangle\left(\frac{1}{2}e^{j\pi/4}\right) = \sphericalangle\frac{1}{2} + \sphericalangle e^{j\pi/4} = 0 + \frac{\pi}{4} = \frac{\pi}{4}$$

(e) The complex conjugate of z is the product of the complex conjugates of $\frac{1}{2}$ and $e^{j\pi/4}$. Since $\frac{1}{2}^* = \frac{1}{2}$ and $(e^{j\theta})^* = e^{-j\theta}$ for all θ,

$$z^* = (\tfrac{1}{2}e^{j\pi/4})^* = \tfrac{1}{2}^*(e^{j\pi/4})^* = \tfrac{1}{2}e^{-j\pi/4}$$

(f) $z + z^*$ is given by

$$z + z^* = \frac{1}{2}e^{j\pi/4} + \frac{1}{2}e^{-j\pi/4} = \frac{e^{j\pi/4} + e^{-j\pi/4}}{2} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Alternatively,

$$Re\{z\} = \frac{z + z^*}{2}, \qquad \text{or} \qquad z + z^* = 2Re\{z\} = 2\frac{\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$$

S1.2

(a) Express z as $z = \sigma + j\Omega$, where $Re\{z\} = \sigma$ and $Im\{z\} = \Omega$. Recall that z^* is the complex conjugate of z, or $z^* = \sigma - j\Omega$. Then

$$\frac{z + z^*}{2} = \frac{(\sigma + j\Omega) + (\sigma - j\Omega)}{2} = \frac{2\sigma + 0}{2} = \sigma$$

(b) Similarly,

$$\frac{z - z^*}{2} = \frac{(\sigma + j\Omega) - (\sigma - j\Omega)}{2} = \frac{0 + 2j\Omega}{2} = j\Omega$$

S1.3

(a) Euler's relation states that $e^{j\theta} = \cos\theta + j\sin\theta$. Therefore, $e^{-j\theta} = \cos(-\theta) + j\sin(-\theta)$. But, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. Thus, $e^{-j\theta} = \cos\theta - j\sin\theta$. Substituting,

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \frac{(\cos\theta + j\sin\theta) + (\cos\theta - j\sin\theta)}{2} = \frac{2\cos\theta}{2} = \cos\theta$$

(b) Similarly,

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{(\cos\theta + j\sin\theta) - (\cos\theta - j\sin\theta)}{2j} = \frac{2j\sin\theta}{2j} = \sin\theta$$

S1.4

(a) (i) We first find the complex conjugate of $z = re^{j\theta}$. From Euler's relation, $re^{j\theta} = r\cos\theta + jr\sin\theta = z$. Thus,

$$z^* = r\cos\theta - jr\sin\theta = r\cos\theta + jr(-\sin\theta)$$

But $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$. Thus,

$$z^* = r\cos(-\theta) + jr\sin(-\theta) = re^{-j\theta}$$

(ii) $z^2 = (re^{j\theta})^2 = r^2(e^{j\theta})^2 = r^2e^{j2\theta}$

(iii) $jz = e^{j\pi/2}re^{j\theta} = re^{j[\theta+(\pi/2)]}$

(iv) $zz^* = (re^{j\theta})(re^{-j\theta}) = r^2e^{j(\theta-\theta)} = r^2 \cdot 1$

(v) $\dfrac{z}{z^*} = \dfrac{re^{j\theta}}{re^{-j\theta}} = e^{j(\theta+\theta)} = e^{j2\theta}$

(vi) $\dfrac{1}{z} = \dfrac{1}{re^{j\theta}} = \dfrac{1}{r}e^{-j\theta}$

(b) From part (a), we directly plot the result in Figure S1.4-1, noting that for $z = re^{j\theta}$, r is the radial distance to the origin and θ is the angle counterclockwise subtended by the vector with the positive real axis.

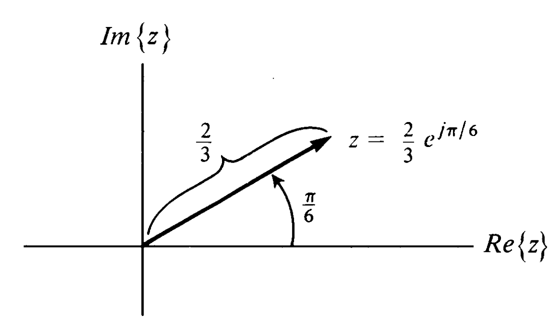

Figure S1.4-1

(i)

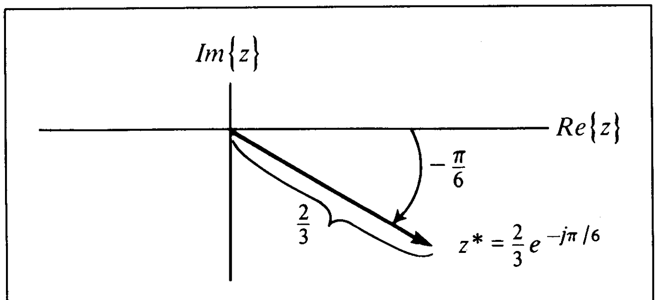

Figure S1.4-2

(ii)

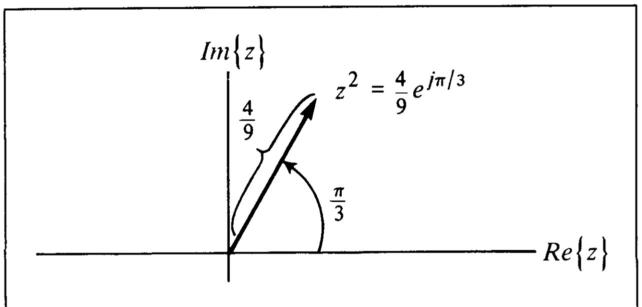

Figure S1.4-3

(iii)

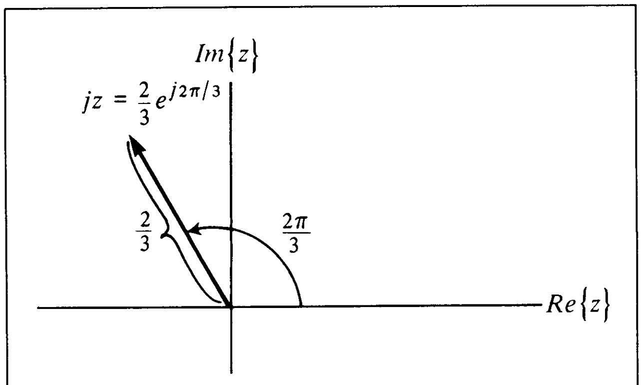

Figure S1.4-4

(iv)

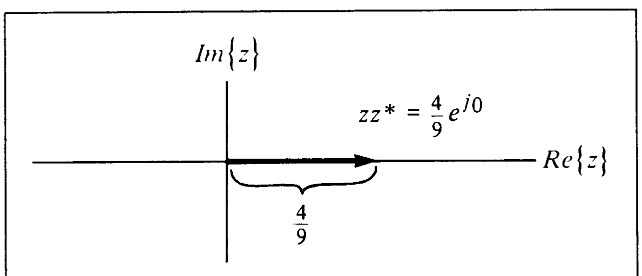

Figure S1.4-5

(v)

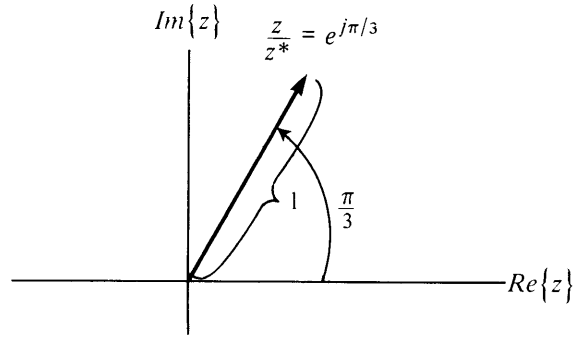

Figure S1.4-6

(vi)

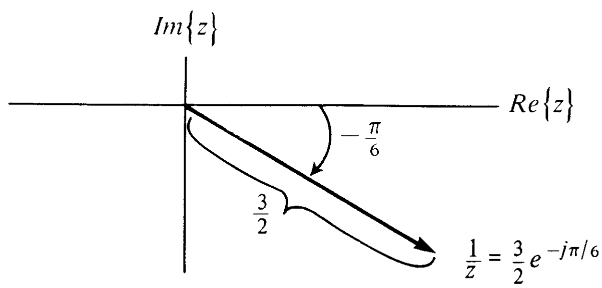

Figure S1.4-7

S1.5

This problem shows a useful manipulation. Multiply by $e^{+j\alpha/2}e^{-j\alpha/2} = 1$, yielding

$$e^{+j\alpha/2}e^{-j\alpha/2}(1 - e^{j\alpha}) = e^{j\alpha/2}(e^{-j\alpha/2} - e^{j\alpha/2})$$

Now we note that $2j \sin(-x) = -2j \sin x = e^{-x} - e^{x}$. Therefore,

$$1 - e^{j\alpha} = e^{j\alpha/2}\left(-2j \sin \frac{\alpha}{2}\right)$$

Finally, we convert $-j$ to complex exponential notation, $-j = e^{-j\pi/2}$. Thus,

$$1 - e^{j\alpha} = e^{j\alpha/2}\left(2e^{-j\pi/2} \sin \frac{\alpha}{2}\right) = 2 \sin \frac{\alpha}{2}\, e^{j[(\alpha-\pi)/2]}$$

S1.6

There are three things a linear scaling of the form $x(at + b)$ can do: (i) reverse direction $\Rightarrow$ a is negative; (ii) stretch or compress the time axis $\Rightarrow$ $|a| \neq 1$; (iii) time shifting $\Rightarrow$ $b \neq 0$.

(a) This is just a time reversal.

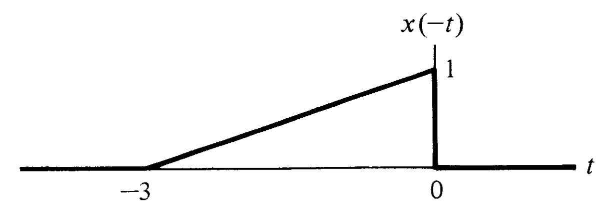

Figure S1.6-1

Note: Amplitude remains the same. Also, reversal occurs about $t = 0$.

(b) This is a shift in time. At $t = -2$, the vertical portion occurs.

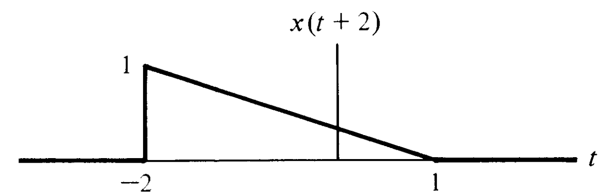

Figure S1.6-2

(c) A scaling by a factor of 2 occurs as well as a time shift.

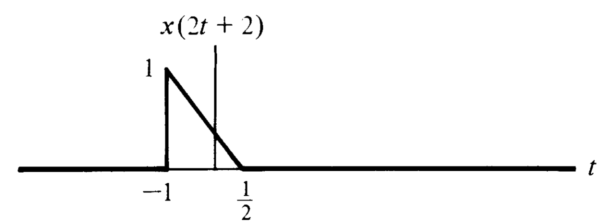

Figure S1.6-3

Note: $a > 1$ induces a compression.

(d) All three effects are combined in this linear scaling.

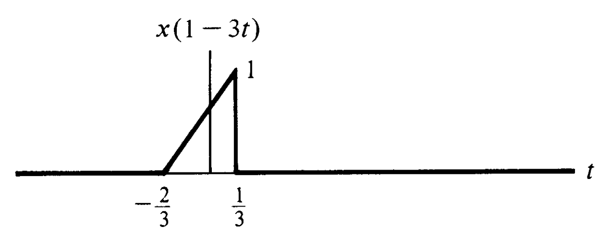

Figure S1.6-4

S1.7

This should be a review of calculus.

(a) $$\int_0^a e^{-2t}\,dt = -\tfrac{1}{2}e^{-2t}\Big|_0^a = -\tfrac{1}{2}e^{-2a} - [-\tfrac{1}{2}e^{-2(0)}]$$
$$= \tfrac{1}{2} - \tfrac{1}{2}e^{-2a}$$

(b) $$\int_2^\infty e^{-3t}\,dt = -\tfrac{1}{3}e^{-3t}\Big|_2^\infty = \lim_{t\to\infty}(-\tfrac{1}{3}e^{-3t}) + \tfrac{1}{3}e^{-3(2)}$$

Therefore,

$$\int_2^\infty e^{-3t}\,dt = 0 + \tfrac{1}{3}e^{-6} = \tfrac{1}{3}e^{-6}$$