A NOTATIONS

Table.7 shows all terms' notations in this paper.

B PROOF OF MONOTONICITY AND NON-SUBMODULARITY OF POPULARITY RATIO FUNCTION (LEMMA 2)

Proof. It is proved that the expected influence spread set function σ is monotone and submodular, we don't show the proof process here. So we only need to prove that the popularity ratio function is monotone with respect to the function $\sigma(S_i), 0 < i < T$. We can prove the monotone of the popularity ratio function. For simplicity, we denote $d^n_{t-1} + d^p_{t-1} + z$ as d_t, denote $\sigma(S_i)$ as x_i, $0 < i < T$.

$$\begin{aligned}\frac{\partial r_T}{\partial x_i} &= (r_0+1)\prod_{t=1}^{i-1}\left(1+\frac{x_t}{d_t}\right)\left[\frac{1}{d_i}\prod_{t=i+1}^{T}\left(1+\frac{x_t}{d_t}\right)\right.\\ &\quad\left.+\sum_{t=i+1}^{T}\frac{-x_t}{d_t^2}\prod_{s=i,s\neq t}^{T}\left(1+\frac{x_s}{d_s}\right)\right]\\ &= r_T\cdot\left[\frac{z}{d_{i+1}\,(d_i+x_i)}+\cdots+\frac{z}{d_T\,(d_{T-1}+\sigma(S_{T-1}))}\right.\\ &\quad\left.+\frac{1}{d_T+x_T}\right] > 0\end{aligned}$$

For any $0 < i < T$, $\frac{\partial r_T}{\partial x_i} > 0$, the popularity ratio function is monotone.

However the popularity ratio function does not satisfy submodular, and we will illustrate this property with a counter example below.

Consider the simplest case that social network has only three nodes u, v, w. No edges between nodes, the initial popularity measure of Novice item is $d^n_0 = 1$ and the initial popularity measure of Popular item is $d^p_0 = 2$, the increment of popularity measure of each round $z = 1$. In this case, the original

$$r_T(\mathcal{S}) = (r_0+1)\prod_{t=1}^{T}\left(1+\frac{\sigma(S_t)}{d^n_0+d^p_0+z\cdot t+\sum_{i=1}^{t-1}\sigma(S_i)}\right)-1$$

where T is at most 3, $r_0 = \frac{d^n_0}{d^p_0} = \frac{1}{2}$.

$$\begin{aligned}&r_T(\mathcal{S})\\ &= \frac{3}{2}*(1+\frac{\sigma(S_1)}{d^n_0+d^p_0+z})*(1+\frac{\sigma(S_2)}{d^n_1+d^p_1+z})*(1+\frac{\sigma(S_3)}{d^n_2+d^p_2+z})-1\end{aligned}$$

where $d^n_0 + d^p_0 + z = 4$, $d^n_1 + d^p_1 + z = 5 + \sigma(S_1)$, $d^n_1 + d^p_1 + z = 6 + \sigma(S_1) + \sigma(S_2)$.

$$\begin{aligned}&r_T(\mathcal{S})\\ &= \frac{3}{2}*(1+\frac{\sigma(S_1)}{4})*(1+\frac{\sigma(S_2)}{5+\sigma(S_1)})*(1+\frac{\sigma(S_3)}{6+\sigma(S_1)+\sigma(S_2)})-1\end{aligned}$$

Now the two pair set $\mathcal{S} \subset \mathcal{Q}$, $\mathcal{S} = \{(u,1)\}$, $\mathcal{Q} = \{(u,1),(v,1)\}$, and a pair $b = (w,2)$, Clearly $b \notin \mathcal{S}$.

$$r_T(\mathcal{S}) = 0.875, r_T(\mathcal{S}\cup\{b\}) = 1.1875$$
$$r_T(\mathcal{Q}) = 1.25, r_T(\mathcal{Q}\cup\{b\}) = 1.5714$$
$$r_T(\mathcal{Q}\cup\{b\}) - r_T(\mathcal{Q}) = 0.3214, r_T(\mathcal{S}\cup\{b\}) - r_T(\mathcal{S}) = 0.3125$$
$$r_T(\mathcal{Q}\cup\{b\}) - r_T(\mathcal{Q}) > r_T(\mathcal{S}\cup\{b\}) - r_T(\mathcal{S})$$

In this case, the marginal value of $\mathcal{Q}$ is larger than $\mathcal{S}$. So the set popularity ratio function is not submodular. □

C SIMPLIFICATION PROCESS FROM POPULARITY RATIO FUNCTION TO ROUND WEIGHTED INFLUENCE FUNCTION (SECTION 4.2

The first step: expanding the multiplication series of Eq.(3) and only keeping the first-order terms;

$$\begin{aligned}&\prod_{t=1}^{T}\left(1+\frac{\sigma(S_t)}{d^n_0+d^p_0+z\cdot t+\sum_{i=1}^{t-1}\sigma(S_i)}\right)\\ &= \frac{\sigma(S_1)}{d^n_0+d^p_0+z}+\frac{\sigma(S_2)}{d^n_0+d^p_0+2z+\sigma(S_1)}\\ &+\cdots+\frac{\sigma(S_T)}{d^n_0+d^p_0+T\cdot z+\sum_{i=0}^{T-1}\sigma(S_i)}\end{aligned}$$

second step: removing the $\sigma(S_1), \ldots, \sigma(S_{T-1})$ in the denominator of each term left after step (a).

$$= \frac{\sigma(S_1)}{d^n_0+d^p_0+z}+\frac{\sigma(S_2)}{d^n_0+d^p_0+2z}+\cdots+\frac{\sigma(S_T)}{d^n_0+d^p_0+T\cdot z}$$

Combining the above simplified process, it can be noted that

$$\rho_T(\mathcal{S}) = \frac{\sigma(S_1)}{d^n_0+d^p_0+z}+\frac{\sigma(S_2)}{d^n_0+d^p_0+2a}+\cdots+\frac{\sigma(S_T)}{d^n_0+d^p_0+T\cdot z}$$

is our weighted overlapping influence function, where d^n_0, d^p_0, z are our predefined parameters. Thus we can denote the weighted overlapping influence function as

$$\rho_T(\mathcal{S}) = \sum_{t=1}^{T} w_t\cdot\sigma(S_t) \tag{17}$$

D PW-RR GENERATION PROCESS

RR set R is generated by independently reverse simulating the propagation from v in round t. A (random) pair-wise RR set (R, t) is a RR set R rooted at a node picked uniformly at random from V, and t is picked uniformly at random from $[T]$.

Algorithm 4 PW-RR generation process.

Input: directed graph $G = (V, E)$, IC model, Max round T
Number of PW-RR set θ
Output: the set of PW-RR set $\mathcal{R}$

```
ℛ = ∅
for 0 < θ do
    θ = θ − 1
    Generate an RR set R for a random node v ∈ V
    choose a round t uniformly at random from [T]
    put the RR set R and round t together as R^(t)
    ℛ = ℛ ∪ {R^(t)}
end for
return ℛ
```

Table 7: Notations

Notation	Description
$G = (V, E)$	a social network G with a node set V and an edge set E
N, M	the numbers of nodes and edges in G, respectively
d_t^n	the popularity measure of Novice item at the end of round t
d_t^p	the popularity measure of Popular item at the end of round t
k	number of seeds to be selected
τ	time step index, each time step is one step in one round of IC model propagation
t	round index, each round is a promotional round
T	Total rounds
S_t	seed set of round t
$\mathcal{S} = \bigcup_{t=1}^{T} S_t \times \{t\}$	pair set, a pair (u, t) is a node u at round t
$\mathcal{S}^*$	optimal pair set
$r_T(\mathcal{S})$	the popularity ratio at the end of round T
r_0	the initial popularity ratio
$p(u, v)$	the probability of node u active v
z	total natural growth of popularity measure
$\sigma(S)$	the influence spread of seed set S
$w = (w_1, w_2 ... w_T)$	the weight vector, w_t is the weight of round t
$\rho_T(\mathcal{S})$	weighted overlapping influence spread
$\rho^{(t)}(S_t)$	weighted overlapping influence spread of round t
$R, R(v)$	one RR set, $R(v)$ is the RR set rooted at node v
$R^{(t)}, R^{(t)}(v)$	one PW-RR set, $R^{(t)} = R \times \{t\}$, $R^{(t)}(v)$ is the RR set rooted at pair (v, t)
$\mathcal{R}$	the set of pair-wised RR set
θ	the number of PW-RR sets that need to be generated
l	Error probability parameter
ε	Approximate ratio parameter
$b, Q, i, j, x_i, \alpha, \beta, OPT, \delta, LB, Pr, \omega, \Omega$	parameters in derivation

E PROOF OF MONOTONICITY AND NON-SUBMODULARITY OF POPULARITY RATIO FUNCTION (LEMMA 3)

Proof. For every $t \in [T]$ and every set $\mathcal{S}$ of pairs in $V \times [T]$, define $\rho^{(t)}(\mathcal{S}) = \sigma(S_t)$. Using the fact that the influence spread function $\sigma(S)$ is monotone and submodular with respect to S, we want to show that $\rho^{(t)}(\mathcal{S})$ is monotone and submodular with respect to $\mathcal{S}$. In fact, for every $\mathcal{S} \subseteq Q \subseteq V \times [T]$, we know that $S_t \subseteq Q_t$, and therefore $\rho^{(t)}(\mathcal{S}) = \sigma(S_t) \leq \sigma(Q_t) = \rho^{(t)}(Q)$, and thus the monotonicity holds.

Now, suppose that $\mathcal{S} \subseteq Q \subseteq V \times [T]$ and $b = (v, j) \in V \times [T] \setminus Q$. If $j \neq t$, then $\mathcal{S} \cup \{b\}$ and $\mathcal{S}$ has the same node set for round t, which means $\rho^{(t)}(\mathcal{S} \cup \{b\}) - \rho^{(t)}(\mathcal{S}) = 0$. Similarly, $\rho^{(t)}(Q \cup \{b\}) - \rho^{(t)}(Q) = 0$. Thus, $\rho^{(t)}(Q \cup \{b\}) - \rho^{(t)}(Q) \leq \rho^{(t)}(\mathcal{S} \cup \{b\}) - \rho^{(t)}(\mathcal{S})$. If $j = t$, then we have $S_t \subseteq Q_t$ and $v \in V \setminus Q_t$. By the submodularity of σ, we have $\rho^{(t)}(Q \cup \{b\}) - \rho^{(t)}(Q) = \sigma(Q_t \cup \{v\}) - \sigma(Q_t) \leq \sigma(S_t \cup \{v\}) - \sigma(S_t) = \rho^{(t)}(\mathcal{S} \cup \{b\}) - \rho^{(t)}(\mathcal{S})$. Therefore submodularity also holds.

Finally, since $\rho^{(t)}(\mathcal{S})$ is monotone and submodular with respect to $\mathcal{S}$ for every t, by the well known fact that the nonnegative weighted summation of monotone submodular functions is still monotone and submodular, we know that $\rho_T(\mathcal{S}) = \sum_{t=1}^{T} w_t \sigma(S_t) = \sum_{t=1}^{T} w_t \rho^{(t)}(\mathcal{S})$ is also monotone and submodular with respect to $\mathcal{S}$. □

F PROOF OF PROPERTY OF PW-RR (LEMMA 4)

Proof. The randomness of $Y(\mathcal{S})$ is from two aspect: (1) the root of a PW-RR set is uniformly random choose, (2) the round t of the PW-RR set is uniformly random choose.

$$
\begin{aligned}
&\mathbb{E}\left[Y(\mathcal{S})\right] \\
&= \frac{1}{T} \sum_{t=1}^{T} w_t \cdot \mathbb{E}\left(\mathbb{I}\{\mathcal{S} \cap \mathcal{R} \neq \emptyset\}\right) \\
&= \frac{1}{T} \sum_{t=1}^{T} w_t \cdot \Pr\left\{S_t \cap R^{(t)} \neq \emptyset\right\} \\
&= \frac{1}{T} \sum_{t=1}^{T} w_t \cdot \frac{1}{N} \sum_{v \in V} \Pr\left\{S_t \cap R^{(t)}(v) \neq \emptyset\right\} \\
&= \frac{1}{T} \sum_{t=1}^{T} w_t \cdot \frac{1}{N} \sum_{v \in V} ap\left(S_t, v\right) \\
&= \frac{1}{T} \sum_{t=1}^{T} w_t \cdot \frac{1}{N} \sigma\left(S_t\right)
\end{aligned}
$$

For any seed set $S_t \in V$, any node $v \in V$, the probability that the seed set S_t activates node v with probability $ap(S_t, v)$. $ap(S_t, v)$ is the probability that S_t have an intersaction with a random RR set $R(v)$ rooted from node v. i.e. $ap(S, v) = \Pr\{S \cap R(v) \neq \emptyset\}$

□

G CORRECTNESS OF PRM-IMM ALGORITHM (THEOREM 5)

We first give a general conclusion (Theorem 7) to show how the greedy solution obtained by the PRM-NodeSelection approaches the optimal solution of the weighted overlapping influence maximization problem when the $\hat{\rho}_T(\mathcal{S}, \mathcal{R})$ itself satisfies monotone submodular (It is easy to proof that $\hat{\rho}_T(\mathcal{S}, \mathcal{R})$ is submodular and nondecreasing with respect to $\mathcal{S}$).

We denote the random estimation of $\rho_T(\mathcal{S})$ as $\hat{\rho}_T(\mathcal{S}, \omega)$, where $\omega \in \Omega$ is a sample in random space Ω. $\mathcal{S}^*$ is the optimal solution of $\rho_T(\mathcal{S})$, $OPT = \rho_T(\mathcal{S}^*)$. $\hat{\mathcal{S}}^g(\omega)$ is the greedy result of $\hat{\rho}_T(\cdot, \omega)$. For $\varepsilon > 0$, we say a solution $\mathcal{S}$ is bad, if $\rho_T(\mathcal{S}) < (1/2 - \varepsilon) \cdot OPT$.

Theorem 7. *for any $\varepsilon > 0$, $\varepsilon_1 \in (0, 2\varepsilon)$, $\delta_1, \delta_2 > 0$, if:*

(a) $\Pr_{\omega \sim \Omega}\{\hat{\rho}_T(\mathcal{S}^*, \omega) \geq (1-\varepsilon_1) \cdot OPT\} \geq 1 - \delta_1$

(b) for any bad $\mathcal{S}$,
$\Pr_{\omega \sim \Omega}\left\{\hat{\rho}_T(\mathcal{S}, \omega) \geq \frac{1}{2}(1-\varepsilon_1) \cdot OPT\right\} \leq \frac{\delta_2}{T^k \cdot \binom{N}{k}}$

(c) for all $\omega \sim \Omega$, $\hat{\rho}_T(\mathcal{S}, \omega)$ is monotone and submodular with respect to $\mathcal{S}$.

So that, $\Pr_{\omega \sim \Omega}\left\{\rho_T\left(\hat{\mathcal{S}}^g(\omega)\right) \geq \left(\frac{1}{2} - \varepsilon\right) \cdot OPT\right\} \geq 1 - \delta_1 - \delta_2$

Proof. Because $\hat{\rho}_T(\mathcal{S}, \omega)$ is monotone and submodular with respect to $\mathcal{S}$, due to the property of partition matroid, we know the greedy solution returns a 1/2-approximate solution.

$$\hat{\rho}_T\left(\hat{\mathcal{S}}^g(\omega), \omega\right) \geq \frac{1}{2}\hat{\rho}_T(\mathcal{S}^*, \omega)$$

With (a), we have at least $1 - \delta_1$ probability that:

$$\hat{\rho}_T\left(\hat{\mathcal{S}}^g(\omega), \omega\right) \geq \frac{1}{2}(1-\varepsilon_1) \cdot OPT$$

In T round, the number of the k size seed set is at most $T^k \cdot \binom{N}{k}$. The probability that every bad $\mathcal{S}$ satisfy (b) is less than $\frac{\delta_2}{T^k \cdot \binom{N}{k}}$, so the probability that existing a $\mathcal{S}$ to make $\rho_T(\mathcal{S}, \omega) \geq \frac{1}{2}(1-\varepsilon_1) \cdot OPT$ is at most δ_2.

So $\Pr_{\omega \sim \Omega}\left\{\rho_T\left(\hat{\mathcal{S}}^g(\omega)\right) \geq \left(\frac{1}{2} - \varepsilon\right) \cdot OPT\right\} \geq 1 - \delta_1 - \delta_2$ □

We will use the concentration inequality to find out how much θ is sufficient to satisfy the conditions (a) and (b) in Theorem 7. For all subsequences of length θ, $\mathcal{R}[\theta]$ in the probability space Ω, each PW-RR set is also independent of each other, so we can use Chernoff bounds of independent sequences to analyze, which is more simple and intuitive[6].

Theorem 8. *For any $\varepsilon > 0$, $\varepsilon_1 \in (0, 2\varepsilon)$, $\delta_1, \delta_2 > 0$:*

$$\theta^{(1)} = \frac{2w_1 N \cdot T \cdot \ln\frac{1}{\delta_1}}{OPT \cdot \varepsilon_1^2}, \theta^{(2)} = \frac{w_1 N \cdot T \cdot \ln\left(\frac{T^k \cdot \binom{N}{k}}{\delta_2}\right)}{OPT \cdot \left(\varepsilon - \frac{1}{2}\varepsilon_1\right)^2}$$

For any fixed $\theta > \theta^{(1)}$, $\Pr_{\omega \sim \Omega}\{\hat{\rho}_T(\mathcal{S}^*, \omega) \geq (1-\varepsilon_1) \cdot OPT\} \geq 1 - \delta_1$

For any fixed $\theta > \theta^{(2)}$, any bad $\mathcal{S}$,

$$\Pr_{\omega \sim \Omega}\left\{\hat{\rho}_T(\mathcal{S}, \omega) \geq \frac{1}{2}(1-\varepsilon_1) \cdot OPT\right\} \leq \frac{\delta_2}{T^k \cdot \binom{N}{k}}$$

Proof. When $\theta > \theta^{(1)}$ Notice that:

$$\hat{\rho}_T(\mathcal{S}, \mathcal{R}) = \frac{N \cdot T}{\theta}\sum_{j=1}^{\theta} Y_j^{\mathcal{R}}(\mathcal{S})$$

$$\Pr_{\mathcal{R}_0 \sim \Omega}\{\hat{\rho}_T(\mathcal{S}^*, \mathcal{R}_0) < (1-\varepsilon_1) \cdot OPT\}$$

$$= \Pr_{\mathcal{R}_0 \sim \Omega}\left\{\frac{N \cdot T}{\theta} \cdot \sum_{j=1}^{\theta} Y_j^{\mathcal{R}_0}(\mathcal{S}^*) < (1-\varepsilon_1) \cdot OPT\right\}$$

$$= \Pr_{\mathcal{R}_0 \sim \Omega}\left\{\sum_{j=1}^{\theta} Y_j^{\mathcal{R}_0}(\mathcal{S}^*) < \frac{\theta}{N \cdot T} \cdot (1-\varepsilon_1) \cdot OPT\right\}$$

$$= \Pr_{\mathcal{R}_0 \sim \Omega}\left\{\sum_{j=1}^{\theta} Y_j^{\mathcal{R}_0}(\mathcal{S}^*) - \theta \cdot \frac{\rho_T(\mathcal{S}^*)}{N \cdot T} < \frac{\theta(1-\varepsilon_1)\, OPT}{N \cdot T} - \theta\frac{\rho_T(\mathcal{S}^*)}{N \cdot T}\right\}$$

$$= \Pr_{\mathcal{R}_0 \sim \Omega}\left\{\sum_{j=1}^{\theta} Y_j^{\mathcal{R}_0}(\mathcal{S}^*) - \theta \cdot \frac{\rho_T(\mathcal{S}^*)}{N \cdot T} < -\varepsilon_1 \cdot \left(\theta \cdot \frac{\rho_T(\mathcal{S}^*)}{N \cdot T}\right)\right\}$$

$$= \Pr_{\mathcal{R}_0 \sim \Omega}\left\{\sum_{j=1}^{\theta} \frac{Y_j^{\mathcal{R}_0}(\mathcal{S}^*)}{w_1} - \theta \cdot \frac{\rho_T(\mathcal{S}^*)}{w_1 N \cdot T} < -\varepsilon_1 \cdot \left(\theta \cdot \frac{\rho_T(\mathcal{S}^*)}{w_1 N \cdot T}\right)\right\}$$

$$\leq \exp\left(-\frac{\varepsilon_1^2}{2}\theta \cdot \frac{\rho_T(\mathcal{S}^*)}{w_1 N \cdot T}\right)$$

$$\leq \exp\left(-\frac{\varepsilon_1^2}{2} \cdot \frac{2w_1 N \cdot T \cdot \ln\frac{1}{\delta_1}}{OPT \cdot \varepsilon_1^2} \cdot \frac{\rho_T(\mathcal{S}^*)}{N \cdot T}\right) = \delta_1$$

When $\theta > \theta^{(2)}$, set $\varepsilon_2 = \varepsilon - \frac{1}{2}\varepsilon_1$, $\rho_T(\mathcal{S}) < \left(\frac{1}{2} - \varepsilon\right) \cdot OPT$

$$\Pr_{\mathcal{R}_0 \sim \Omega}\left\{\hat{\rho}_{T\theta}(\mathcal{S}, \mathcal{R}_0) \geq \frac{1}{2}(1-\varepsilon_1) \cdot OPT\right\}$$

$$= \Pr_{\mathcal{R}_0 \sim \Omega}\left\{\frac{N \cdot T}{\theta} \cdot \sum_{j=1}^{\theta} Y_j^{\mathcal{R}_0}(\mathcal{S}) \geq \frac{1}{2}(1-\varepsilon_1) \cdot OPT\right\}$$

$$= \Pr_{\mathcal{R}_0 \sim \Omega}\left\{\sum_{j=1}^{\theta} Y_j^{\mathcal{R}_0}(\mathcal{S}) - \theta \cdot \frac{\rho_T(\mathcal{S})}{N \cdot T} \geq \frac{\theta}{N \cdot T}\left[\frac{1}{2}(1-\varepsilon_1) \cdot OPT - \rho_T(\mathcal{S})\right]\right\}$$

$$/* \rho_T(\mathcal{S}) < \left(\frac{1}{2} - \varepsilon\right) \cdot OPT */$$

$$\leq \Pr_{\mathcal{R}_0 \sim \Omega}\left\{\sum_{j=1}^{\theta} Y_j^{\mathcal{R}_0}(\mathcal{S}) - \theta \cdot \frac{\rho_T(\mathcal{S})}{N \cdot T} \geq \frac{\theta}{N \cdot T} \cdot \varepsilon_2 \cdot OPT\right\}$$

$$= \Pr_{\mathcal{R}_0 \sim \Omega}\left\{\sum_{j=1}^{\theta} \frac{Y_j^{\mathcal{R}_0}(\mathcal{S})}{w_1} - \theta \cdot \frac{\rho_T(\mathcal{S})}{w_1 N \cdot T} \geq \left(\varepsilon_2 \cdot \frac{OPT}{\rho_T(\mathcal{S})}\right) \cdot \theta \cdot \frac{\rho_T(\mathcal{S})}{w_1 N \cdot T}\right\}$$

$$\leq \exp\left(-\frac{\left(\varepsilon_2 \cdot \frac{OPT}{\rho_T(\mathcal{S})}\right)^2}{2 + \frac{2}{3}\left(\varepsilon_2 \cdot \frac{OPT}{\rho_T(\mathcal{S})}\right)} \cdot \theta \cdot \frac{\rho_T(\mathcal{S})}{w_1 N \cdot T}\right)$$

$$\leq \exp\left(-\frac{\varepsilon_2^2 \cdot OPT^2}{2\rho_T(\mathcal{S}) + \frac{2}{3}\varepsilon_2 \cdot OPT} \cdot \theta \cdot \frac{1}{w_1 N \cdot T}\right)$$

$$\le \exp\left(-\frac{\varepsilon_2^2\cdot OPT^2}{2\left(\frac{1}{2}-\varepsilon\right)\cdot OPT+\frac{2}{3}\left(\varepsilon-\frac{1}{2}\varepsilon_1\right)\cdot OPT}\cdot\theta\cdot\frac{1}{w_1N\cdot T}\right)$$

$$\le \exp\left(-\frac{\left(\varepsilon-\frac{1}{2}\varepsilon_1\right)^2\cdot OPT^2}{OPT}\cdot\frac{w_1N\cdot T\cdot\ln\left(\frac{T^k\cdot\binom{N}{k}}{\delta_2}\right)}{OPT\cdot\left(\varepsilon-\frac{1}{2}\varepsilon_1\right)^2}\cdot\frac{1}{w_1N\cdot T}\right)$$

$$=\frac{T^k\cdot\binom{N}{k}}{\delta_2}$$

□

Now we discuss the setting of parameters $\varepsilon_1, \delta_1, \delta_2$. The Settings of these parameters are not unique, and the method we describe below follows the settings in the original IMM algorithm. According to Theorem 8, assuming that OPT is known, the target of these parameters is to make the output of greedy solution $\hat{\mathcal{S}}^g$ is the $1/2-\varepsilon$ approximation of optimal solution with probability at least $1-1/(2N^l)$. The high probability of $1-1/(2N^l)$ was achieved because, in the next step, we would use the same high probability of $1-1/(2N^l)$to obtain a better lower-bound estimate of OPT. Thus, the correctness of the overall algorithm would be guaranteed for an assignment with a high probability of $1-1/(N^l)$. The following corollary give the setting of the parameters.

Corollary 1. *Set* $\delta_1=\delta_2=\frac{1}{4n^l}, \varepsilon_1=\varepsilon\cdot\frac{\alpha}{\frac{1}{2}\alpha+\beta}$

$$\alpha=\sqrt{l\ln N+\ln 4}, \beta=\sqrt{\frac{1}{2}\cdot\left(\ln\binom{N}{k}+l\ln N+\ln 4+k\ln T\right)}$$

For any fixed $\theta>\frac{2N\cdot T\cdot\left[\frac{1}{2}\alpha+\beta\right]^2}{\varepsilon^2\cdot OPT}$*, if the input of PRM-NodeSelection is* $\mathcal{R}_0[\theta],\mathcal{R}_0\sim\Omega$*,the probability that PRM-NodeSelection-OINS's output* $\hat{\mathcal{S}}^g(\mathcal{R}_0[\theta])$ *is the* $(1/2-\varepsilon)$ *approximation of the optimal solution is at least* $1-\frac{1}{2N^l}$.

Theorem 9. *The probability of* $LB\le OPT$ *is at least* $1-\frac{1}{2N^l}$*, which means that the probability of* $\tilde{\theta}\ge\frac{2nt\cdot\left[\frac{1}{2}\cdot\alpha+\beta\right]^2}{\varepsilon^2\cdot OPT}$ *is at least* $1-\frac{1}{2N^l}$.

G.1 Proof of Theorem 9

Theorem 10. *For any* $i=1,2,\cdots,\lfloor\log_2 N\rfloor-1$,

(1) *if* $x_i=\frac{\sum_1^k w_t\cdot N}{2^i}>OPT$*, the probability of* $\hat{\rho}_{T\theta_i}(\mathcal{S}_i,\mathcal{R}_0[\theta_i])\ge\left(1+\varepsilon'\right)\cdot x_i$ *is at most* $\frac{1}{2N^l\log_2 N}$.

(2) *if* $x_i=\frac{\sum_1^k w_t\cdot N}{2^i}\le OPT$*, the probability of* $\hat{\rho}_{T\theta_i}(\mathcal{S}_i,\mathcal{R}_0[\theta_i])\ge\left(1+\varepsilon'\right)\cdot OPT$ *is at most* $\frac{1}{2N^l\log_2 N}$.

Proof. For any k size seed set $\mathcal{S}$.

$$\hat{\rho}_{T\theta_i}(\mathcal{S},\mathcal{R}_0[\theta_i])=\frac{N\cdot T\cdot\sum_{j=1}^{\theta_i}Y_j^{\mathcal{R}_0[\theta_i]}(\mathcal{S})}{\theta_i}$$

$$\Pr_{\mathcal{R}_0\sim\Omega}\left\{\hat{\rho}_{T\theta_i}(\mathcal{S},\mathcal{R}_0)\ge\left(1+\varepsilon'\right)\cdot x_i\right\}$$

$$=\Pr_{\mathcal{R}_0\sim\Omega}\left\{\frac{N\cdot T}{\theta_i}\cdot\sum_{j=1}^{\theta_i}Y_j^{\mathcal{R}_0[\theta_i]}(\mathcal{S})\ge\left(1+\varepsilon'\right)\cdot x_i\right\}$$

$$=\Pr_{\mathcal{R}_0\sim\Omega}\left\{\sum_{j=1}^{\theta_i}Y_j^{\mathcal{R}_0[\theta_i]}(\mathcal{S})-\frac{\theta_i\cdot\rho_T(\mathcal{S})}{N\cdot T}\ge\frac{\theta_i\left(1+\varepsilon'\right)x_i}{N\cdot T}-\frac{\theta_i\cdot\rho_T(\mathcal{S})}{N\cdot T}\right\}$$

/* because $x_i>OPT\ge\rho_T(\mathcal{S})$ */

$$\le\Pr_{\mathcal{R}_0\sim\Omega}\left\{\sum_{j=1}^{\theta_i}Y_j^{\mathcal{R}_0[\theta_i]}(\mathcal{S})-\frac{\theta_i\cdot\rho_T(\mathcal{S})}{N\cdot T}\ge\frac{\theta_i}{N\cdot T}\varepsilon'\cdot x_i\right\}$$

$$=\Pr_{\mathcal{R}_0\sim\Omega}\left\{\sum_{j=1}^{\theta_i}Y_j^{\mathcal{R}_0[\theta_i]}(\mathcal{S})-\frac{\theta_i\cdot\rho_T(\mathcal{S})}{N\cdot T}\ge\frac{\varepsilon'\cdot x_i}{\rho_T(\mathcal{S})}\cdot\frac{\theta_i\cdot\rho_T(\mathcal{S})}{N\cdot T}\right\}$$

$$=\Pr_{\mathcal{R}_0\sim\Omega}\left\{\sum_{j=1}^{\theta_i}\frac{Y_j^{\mathcal{R}_0[\theta_i]}(\mathcal{S})}{w_1}-\frac{\theta_i\cdot\rho_T(\mathcal{S})}{w_1N\cdot T}\ge\frac{\varepsilon'\cdot x_i}{\rho_T(\mathcal{S})}\cdot\frac{\theta_i\cdot\rho_T(\mathcal{S})}{w_1N\cdot T}\right\}$$

$$\le\exp\left(-\frac{\left(\frac{\varepsilon'\cdot x_i}{\rho_T(\mathcal{S})}\right)^2}{2+\frac{2}{3}\left(\frac{\varepsilon'\cdot x_i}{\rho_T(\mathcal{S})}\right)}\cdot\frac{\theta_i\cdot\rho_T(\mathcal{S})}{w_1N\cdot T}\right)\quad /*\text{ chernoff bound } */$$

$$=\exp\left(-\frac{(\varepsilon'\cdot x_i)^2}{2\rho_T(\mathcal{S})+\frac{2}{3}(\varepsilon'\cdot x_i)}\cdot\frac{\theta_i}{w_1N\cdot T}\right)$$

$$\le\exp\left(-\frac{\varepsilon'^2\cdot x_i}{2+\frac{2}{3}\varepsilon'}\cdot\frac{\theta_i}{w_1N\cdot T}\right)$$

$$\le\exp\left(-\frac{\varepsilon'^2\cdot x_i}{2+\frac{2}{3}\varepsilon'}\cdot\frac{w_1N\cdot T\left(2+\frac{2}{3}\varepsilon'\right)\left(\ln T^k+\ln\binom{N}{k}+l\ln N+\ln 2+\ln\log_2 N\right)}{\varepsilon'^2x_i}\cdot\frac{1}{w_1N\cdot T}\right)$$

$$=\frac{1}{2T^k\cdot\binom{N}{k}N^l\log_2 N}$$

For any k size set $\mathcal{S}$.

$$\Pr_{\mathcal{R}_0\sim\Omega}\left\{\hat{\rho}_{T\theta_i}(\mathcal{S},\mathcal{R}_0)\ge\left(1+\varepsilon'\right)\cdot OPT\right\}$$

$$=\Pr_{\mathcal{R}_0\sim\Omega}\left\{\frac{N\cdot T}{\theta_i}\cdot\sum_{j=1}^{\theta_i}Y_j^{\mathcal{R}_0[\theta_i]}(\mathcal{S})\ge\left(1+\varepsilon'\right)\cdot OPT\right\}$$

$$\Pr_{\mathcal{R}_0\sim\Omega}\left\{\sum_{j=1}^{\theta_i}Y_j^{\mathcal{R}_0[\theta_i]}(\mathcal{S})-\frac{\theta_i\cdot\rho_T(\mathcal{S})}{N\cdot T}\ge\frac{\theta_i}{N\cdot T}\left(1+\varepsilon'\right)\cdot OPT-\frac{\theta_i\cdot\rho_T(\mathcal{S})}{N\cdot T}\right\}$$

/*$OPT\ge\rho_T(\mathcal{S})$ */

$$\le\Pr_{\mathcal{R}_0\sim\Omega}\left\{\sum_{j=1}^{\theta_i}Y_j^{\mathcal{R}_0[\theta_i]}(\mathcal{S})-\frac{\theta_i\cdot\rho_T(\mathcal{S})}{N\cdot T}\ge\frac{\theta_i}{N\cdot T}\varepsilon'\cdot OPT\right\}$$

$$\leq \Pr_{\mathcal{R}_0\sim\Omega}\left\{\sum_{j=1}^{\theta_i} Y_j^{\mathcal{R}_0[\theta_i]}(\mathcal{S}) - \frac{\theta_i \cdot \rho_T(\mathcal{S})}{N \cdot T} \geq \frac{\varepsilon' \cdot OPT}{\rho_T(\mathcal{S})} \cdot \frac{\theta_i \cdot \rho_T(\mathcal{S})}{N \cdot T}\right\}$$

$$\leq \Pr_{\mathcal{R}_0\sim\Omega}\left\{\sum_{j=1}^{\theta_i} \frac{Y_j^{\mathcal{R}_0[\theta_i]}(\mathcal{S})}{w_1} - \frac{\theta_i \cdot \rho_T(\mathcal{S})}{w_1 N \cdot T} \geq \frac{\varepsilon' \cdot OPT}{\rho_T(\mathcal{S})} \cdot \frac{\theta_i \cdot \rho_T(\mathcal{S})}{w_1 N \cdot T}\right\}$$

$$\leq \exp\left(-\frac{\left(\frac{\varepsilon' \cdot OPT}{\rho_T(\mathcal{S})}\right)^2}{2+\frac{2}{3}\left(\frac{\varepsilon' \cdot OPT}{\rho_T(\mathcal{S})}\right)} \cdot \frac{\theta_i \cdot \rho_T(\mathcal{S})}{w_1 N \cdot T}\right)$$

$$= \exp\left(-\frac{(\varepsilon' \cdot OPT)^2}{2\rho_T(\mathcal{S})+\frac{2}{3}(\varepsilon' \cdot OPT)} \cdot \frac{\theta_i}{w_1 N \cdot T}\right)$$

$$\leq \exp\left(-\frac{\varepsilon'^2 \cdot OPT}{2+\frac{2}{3}\varepsilon'} \cdot \frac{\theta_i}{w_1 N \cdot T}\right)$$

$$\leq \exp\left(-\frac{\varepsilon'^2 \cdot x_i}{2+\frac{2}{3}\varepsilon'} \cdot \frac{\theta_i}{w_1 N \cdot T}\right)$$

$$\leq \exp\left(- \frac{\varepsilon'^2 \cdot x_i}{2+\frac{2}{3}\varepsilon'} \cdot \frac{w_1 N \cdot T\left(2+\frac{2}{3}\varepsilon'\right)\left(\ln T^k + \ln\binom{N}{k} + l\ln N + \ln 2 + \ln\log_2 N\right)}{\varepsilon'^2 x_i} \cdot \frac{1}{w_1 nT}\right)$$

$$= \frac{1}{2T^k \cdot \binom{N}{k} N^l \log_2 N}$$

With the union bound $\Pr_{\mathcal{R}_0\sim\Omega}\left\{\hat{\rho}_{T\theta_i}(\mathcal{S}_i, \mathcal{R}_0) \geq (1+\varepsilon') \cdot OPT\right\} \leq \frac{1}{2N^l \log_2 N}$

□

With Theorem 10 we know that LB is a lower bound of OPT with high probability, so θ satisfy the Corollary 1. Further we can know that the probability of $LB < OPT$ is at least $1 - 1/2N^l$.

Proof. Set $LB_i = \frac{\hat{\rho}_{T\theta_i}(\mathcal{S}_i, \mathcal{R}_0)}{(1+\varepsilon')}$. When OPT $\geq x_{\lfloor \log_2 N \rfloor - 1}$.Set $i \geq 1$ is the smallest index to make $OPT \geq x_i$.

For any $i' \leq i-1$, $OPT < x_{i'}$,

For any $i'' > i-1$, $OPT \geq x_{i''}$,

We define the event ε as: for any $i' \leq i-1$, $\hat{\rho}_{T\theta_{i'}}(\mathcal{S}_{i'}, \mathcal{R}_0) < (1+\varepsilon')\, x_{i'}$, and for any $i'' \geq i$, $\hat{\rho}_{T\theta_{i'}}(\mathcal{S}_{i'}, \mathcal{R}_0) \geq (1+\varepsilon')\, x_{i'}$

Notice that $i', i'' \geq 1$, so when $i = 1$, i' is not exist. The part of Event ε about i' is true. Event ε is the event that we expected. Because as Event ε happens, $LB = LB_i$ or $LB = 1$. So Event ε indicate that $LB \leq OPT$ so the upper bound of Event ε not happen is that:

$$\Pr_{\mathcal{R}_0\sim\Omega}\{\neg\varepsilon\} \leq \sum_{i'=1}^{i-1} \Pr_{\mathcal{R}_0\sim\Omega}\left\{\hat{\rho}_{T\theta_{i'}}(\mathcal{S}_{i'}, \mathcal{R}_0) \geq (1+\varepsilon')x_{i'}\right\} + \sum_{i''=i-1}^{\lfloor \log_2 N \rfloor - 1} \Pr_{\mathcal{R}_0\sim\Omega}\left\{\hat{\rho}_{T\theta_{i''}}(\mathcal{S}_{i''}, \mathcal{R}_0) \geq (1+\varepsilon')OPT\right\}$$

With the above, we know that $\Pr_{\mathcal{R}_0\sim\Omega}\left\{\hat{\rho}_{T\theta_{i'}}(\mathcal{S}_{i'}, \mathcal{R}_0) \geq (1+\varepsilon')x_{i'}\right\} \leq \frac{1}{2N^l \log_2 N}$. And $\Pr_{\mathcal{R}_0\sim\Omega}\left\{\hat{\rho}_{T\theta_{i''}}(\mathcal{S}_{i''}, \mathcal{R}_0) \geq (1+\varepsilon')OPT\right\} \leq \frac{1}{2N^l \log_2 N}$. So $\Pr_{\mathcal{R}_0\sim\Omega}\{\neg\varepsilon\} \leq \frac{1}{2N^l}$.

When $OPT < x_{\lfloor \log_2 n \rfloor - 1}$. $LB = 1$ with probability at least $1 - \frac{1}{2N^l}$.

□

For any $\varepsilon > 0, l > 0$, PRM-IMM Guarantees that $\hat{\mathcal{S}}^g$ is the $\frac{1}{2} - \varepsilon$ approximation of OPT with probability at least $1 - \frac{1}{N^l}$. Define the Event ε as the $LB \leq OPT$, and put the $\mathcal{R}_0'[\tilde{\theta}]$ to PRM-NodeSelection to get the seed set $\hat{\mathcal{S}}^g$ is the $\frac{1}{2} - \varepsilon$ approximation of the PRM problem.

$$\rho_T\left(\hat{\mathcal{S}}^g\left(\mathcal{R}_0'[\tilde{\theta}]\right)\right) \geq \left(\frac{1}{2} - \varepsilon\right) \cdot OPT$$

with union bound:

$$\Pr_{\mathcal{R}_0\sim\Omega\mathcal{R}_0'\sim\Omega}\{\neg\varepsilon\} \leq \Pr_{\mathcal{R}_0\sim\Omega\mathcal{R}_0'\sim\Omega}\{\text{LB} > \text{OPT}\} + \Pr_{\mathcal{R}_0\sim\Omega\mathcal{R}_0'\sim\Omega}\left\{\text{LB} \leq \text{OPT} \wedge \rho_T\left(\hat{\mathcal{S}}^g\left(\mathcal{R}_0'[\tilde{\theta}]\right)\right) < \left(\frac{1}{2} - \varepsilon\right) \cdot OPT\right\}$$

And $\Pr_{\mathcal{R}_0\sim\Omega\mathcal{R}_0'\sim\Omega}\{\text{LB} > \text{OPT}\} = \Pr_{\mathcal{R}_0\sim\Omega}\{\text{LB} > \text{OPT}\} \leq \frac{1}{2N^l}$.

Now we know that:

$$\Pr_{\mathcal{R}_0\sim\Omega\mathcal{R}_0'\sim\Omega}\left\{\text{LB} \leq \text{OPT} \wedge \rho_T\left(\hat{\mathcal{S}}^g\left(\mathcal{R}_0'[\tilde{\theta}]\right)\right) < \left(\tfrac{1}{2} - \varepsilon\right) \cdot OPT\right\}$$

When $LB \leq OPT$,$\tilde{\theta} \geq \frac{2N \cdot \left[\frac{1}{2} \cdot \alpha + \beta\right]^2}{\varepsilon^2 \cdot OPT}$, and $\tilde{\theta} \leq \frac{2N \cdot \left[\frac{1}{2} \cdot \alpha + \beta\right]^2}{\varepsilon^2}$.

$\theta_{\min} = \left\lceil \frac{2N \cdot \left[\frac{1}{2} \cdot \alpha + \beta\right]^2}{\varepsilon^2 \cdot OPT} \right\rceil$, $\theta_{\max} = \left\lfloor \frac{2N \cdot \left[\frac{1}{2} \cdot \alpha + \beta\right]^2}{\varepsilon^2} \right\rfloor$.$\tilde{\theta}$ is a integer range from θ_{min} to θ_{max}.

$$\Pr_{\mathcal{R}_0\sim\Omega\mathcal{R}_0'\sim\Omega}\left\{\text{LB} \leq \text{OPT} \wedge \rho_T\left(\hat{\mathcal{S}}^g\left(\mathcal{R}_0'[\tilde{\theta}]\right)\right) < \left(\frac{1}{2} - \varepsilon\right) \cdot OPT\right\}$$

$$\leq \Pr_{\mathcal{R}_0\sim\Omega\mathcal{R}_0'\sim\Omega}\left\{\tilde{\theta} \geq \theta_{\min} \wedge \tilde{\theta} \leq \theta_{\max} \wedge \rho_T\left(\hat{\mathcal{S}}^g\left(\mathcal{R}_0'[\tilde{\theta}]\right)\right) < \left(\frac{1}{2} - \varepsilon\right) \cdot OPT\right\}$$

$$= \sum_{\theta=\theta_{\min}}^{\theta_{\max}} \Pr_{\mathcal{R}_0\sim\Omega\mathcal{R}_0'\sim\Omega}\left\{\tilde{\theta} = \theta \wedge \rho_T\left(\hat{\mathcal{S}}^g\left(\mathcal{R}_0'[\tilde{\theta}]\right)\right) < \left(\frac{1}{2} - \varepsilon\right) \cdot OPT\right\}$$

$$= \sum_{\theta=\theta_{\min}}^{\theta_{\max}} \Pr_{\mathcal{R}_0\sim\Omega\mathcal{R}_0'\sim\Omega}\left\{\tilde{\theta} = \theta \wedge \rho_T\left(\hat{\mathcal{S}}^g\left(\mathcal{R}_0'[\theta]\right)\right) < \left(\frac{1}{2} - \varepsilon\right) \cdot OPT\right\}$$

/∗ because $\mathcal{R}_0$ is independent with $\mathcal{R}_0'$ ∗ /

$$= \sum_{\theta=\theta_{\min}}^{\theta_{\max}} \Pr_{\mathcal{R}_0\sim\Omega}\{\tilde{\theta} = \theta\} \cdot \Pr_{\mathcal{R}_0'\sim\Omega}\left\{\rho_T\left(\hat{\mathcal{S}}^g\left(\mathcal{R}_0'[\theta]\right)\right) < \left(\frac{1}{2} - \varepsilon\right) \cdot OPT\right\}$$

$$\leq \sum_{\theta=\theta_{\min}}^{\theta_{\max}} \Pr_{\mathcal{R}_0\sim\Omega}\left\{\tilde{\theta} = \theta\right\} \cdot \frac{1}{2N^l} = \frac{1}{2N^l}$$

So $\Pr_{\mathcal{R}_0\sim\Omega,\mathcal{R}_0'\sim\Omega}\{\neg\varepsilon\} \leq \frac{1}{N^l}$, which means that with probability at least $1 - \frac{1}{N^l}$, the output of PRM-IMM $\hat{\mathcal{S}}^g$ is the $\frac{1}{2} - \varepsilon$ approximation of OPT.

H TIME COMPLEXITY OF PRM-IMM

The time complexity of RPM-IMM is $O((k+l)(M+N)T\log(NT)/\varepsilon^2)$.

Proof. We use the Martingale theorem in the IMM algorithm[32] to estimate the time complexity of the PRM-IMM algorithm. Then the time complexity of the IMM algorithm is $O(\mathbb{E}[\bar{\theta}+\tilde{\theta}]\cdot(EPT+1))$, where $\mathbb{E}[\bar{\theta}+\tilde{\theta}]$ is the overall number of PW-RR sets needed to be generated.

$$\mathbb{E}[\bar{\theta}+\tilde{\theta}] \leq \frac{8\left(\lambda^*+\lambda'\right)\cdot\left(1+\varepsilon'\right)^2}{\cdot OPT}+2,$$

where

$$\lambda^* = \frac{4w_1NT\cdot(\cdot\alpha+\beta)^2}{\varepsilon^2}$$
$$\lambda' = \frac{w_1NT\cdot\left(2+\frac{2}{3}\varepsilon'\right)\cdot\left(\ln\binom{N}{k}+\ell\ln N+\ln 2+\ln\log_2 N+\ln T^k\right)}{\varepsilon'^2},$$

and α and β is defined in section 5.

Therefore, $\mathbb{E}[\bar{\theta}+\tilde{\theta}] = O\left(\frac{w_1(k+l)NT\log NT}{OPT\varepsilon^2}\right)$. And $EPT = \frac{M}{N}\cdot\mathbb{E}[\sigma(v)]$ is the expected running time of generating a PW-RR set. Because $\mathbb{E}[\sigma(v)] \leq \frac{OPT}{w_1}$, so the expected running time is:

$$O\left(\frac{(k+l)(N+M)T\log NT}{\varepsilon^2}\right)$$

□

I OBJECTIVE FUNCTION WHEN NATURAL GROWTH COUNT IS VARIABLE(PROOF OF SECTION 6.2)

We use the natural growth vector : $z = [z_1, z_2, \cdots, z_t]$ with z_t denoting the natural customer count in round t.

$$d_1^n = d_0^n + z_1\cdot\frac{d_0^n}{d_0^n+d_0^p}+\sigma(S_1)$$
$$d_1^p = d_0^p + z_1\cdot\frac{d_0^p}{d_0^n+d_0^p}$$
$$r_1 = \frac{d_0^n + z_1\cdot\frac{d_0^n}{d_0^n+d_0^p}+\sigma(S_1)}{d_0^p + z_1\cdot\frac{d_0^p}{d_0^n+d_0^p}}$$
$$r_1 = r_0 + \frac{(r_0+1)\,\sigma(S_1)}{d_0^n+d_0^p+z_1}$$
$$r_t = r_{t-1} + \frac{\sigma(S_{t-1})}{d_{t-1}^n+d_{t-1}^p+z_t}(r_{t-1}+1)$$
$$r_t+1 = r_{t-1} + \frac{\sigma(S_{t-1})}{d_{t-1}^n+d_{t-1}^p+z_t}(r_{t-1}+1)+1$$
$$r_t+1 = \left(1+\frac{\sigma(S_{t-1})}{d_{t-1}^n+d_{t-1}^p+z_t}\right)(r_{t-1}+1)$$
$$r_t+1 = \left(1+\frac{\sigma(S_{t-1})}{d_{t-1}^n+d_{t-1}^p+z_t}\right)(r_{t-1}+1)$$
$$r_T(S) = (r_0+1)\prod_{t=1}^{T}\left(1+\frac{\sigma(S_t)}{d_0^n+d_0^p+\sum_{i=1}^{t}z_i+\sum_{i=1}^{t-1}\sigma(S_i)}\right)-1$$

J UPPER AND LOWER BOUND OF OBJECTIVE FUNCTION (POPULAR ITEM PROMOTION SECTION 6.3)

The objective function $r_T(\mathcal{S})$ is not easily derived. So we can obtain its upper and lower bound as follows.

Upper bound:

$$\begin{aligned}
r_t+1 &= \frac{d_t^n+d_t^p}{d_t^p} = \frac{d_{t-1}^n+d_{t-1}^p+\sigma(S_t)+z+p_t}{d_{t-1}^p+z\cdot\frac{d_{t-1}^p}{d_{t-1}^n+d_{t-1}^p}+p_t}\\
&= \frac{d_{t-1}^n+d_{t-1}^p+\sigma(S_t)+z+p_t}{d_{t-1}^n+d_{t-1}^p+z+p_t\cdot\frac{d_{t-1}^n+d_{t-1}^p}{d_{t-1}^p}}\cdot\frac{d_{t-1}^n+d_{t-1}^p}{d_{t-1}^p}\\
&< \frac{d_{t-1}^n+d_{t-1}^p+\sigma(S_t)+z+p_t}{d_{t-1}^n+d_{t-1}^p+z+p_t}\cdot\frac{d_{t-1}^n+d_{t-1}^p}{d_{t-1}^p}\\
&= \left(1+\frac{\sigma(S_t)}{d_{t-1}^n+d_{t-1}^p+z+p_t}\right)\cdot(r_{t-1}+1)
\end{aligned}$$
$$r_t+1 < \left(1+\frac{\sigma(S_t)}{d_0^n+d_0^p+z\cdot t+\sum_1^{t-1}\sigma(S_i)+\sum_1^t p_i}\right)\cdot(r_{t-1}+1)$$

$$r_{t-1}+1 < \left(1+\frac{\sigma(S_{t-1})}{d_0^n+d_0^p+z\cdot(t-1)+\sum_1^{t-2}\sigma(S_i)+\sum_1^{t-1}p_i}\right)\cdot(r_{t-2}+1)$$
$$r_T+1 < (r_0+1)\prod_{t=1}^{T}\left(1+\frac{\sigma(S_t)}{d_0^n+d_0^p+z\cdot t+\sum_1^{t-1}\sigma(S_i)+\sum_1^t p_i}\right)$$
$$r_T < r'_T$$
$$= (r_0+1)\prod_{t=1}^{T}\left(1+\frac{\sigma(S_t)}{d_0^n+d_0^p+z\cdot t+\sum_1^{t-1}\sigma(S_i)+\sum_1^t p_i}\right)-1$$

Lower bound:

$$r_t + 1 = \frac{d_t^n + d_t^p}{d_t^p} = \frac{d_{t-1}^n + d_{t-1}^p + \sigma(S_t) + z + p_t}{d_{t-1}^p + z \cdot \frac{d_{t-1}^p}{d_{t-1}^n + d_{t-1}^p} + p_t}$$

$$= \frac{d_{t-1}^n + d_{t-1}^p + \sigma(S_t) + z + p_t}{d_{t-1}^n + d_{t-1}^p + z + p_t \cdot \frac{d_{t-1}^n + d_{t-1}^p}{d_{t-1}^p}} \cdot \frac{d_{t-1}^n + d_{t-1}^p}{d_{t-1}^p}$$

$$> \frac{d_{t-1}^n + d_{t-1}^p + \sigma(S_t) + z + p_t}{d_{t-1}^n + d_{t-1}^p + z + 2 \cdot p_t} \cdot \frac{d_{t-1}^n + d_{t-1}^p}{d_{t-1}^p}$$

$$= \left(1 + \frac{\sigma(S_t) - p_t}{d_{t-1}^n + d_{t-1}^p + z + 2 \cdot p_t}\right) \cdot (r_{t-1} + 1)$$

$$r_t + 1 > \left(1 + \frac{\sigma(S_t) - p_t}{d_0^n + d_0^p + z \cdot t + \sum_1^{t-1} \sigma(S_i) + \sum_1^t p_i + p_t}\right) \cdot (r_{t-1} + 1)$$

$$r_{t-1} + 1 > \left(1 + \frac{\sigma(S_{t-1}) - p_{t-1}}{d_0^n + d_0^p + z \cdot (t-1) + \sum_1^{t-2} \sigma(S_i) + \sum_1^{t-1} p_i + p_{t-1}}\right) \cdot (r_{t-2} + 1)$$

$$r_T > (r_0 + 1) \prod_{t=1}^{T} \left(1 + \frac{\sigma(S_t) - p_t}{d_0^n + d_0^p + z \cdot t + \sum_1^{t-1} \sigma(S_i) + \sum_1^t p_i + p_t}\right) - 1$$

$$r_T > r_T''$$

$$= (r_0 + 1) \prod_{t=1}^{T} \left(1 + \frac{\sigma(S_t) - p_t}{d_0^n + d_0^p + z \cdot t + \sum_1^{t-1} \sigma(S_i) + \sum_1^t p_i + p_t}\right) - 1$$

K UPPER AND LOWER BOUND OF OBJECTIVE FUNCTION (MULTI-ITEM PROMOTION THEOREM 6)

In the setting of multiple items with promotions, we can derive the bound of r_T as follows.

Upper bound:

$$r_t + 1 = \frac{d_t^n + \sum_{j=1}^s d_t^{p^j}}{\sum_{j=1}^s d_t^{p^j}} = \frac{d_{t-1}^n + \sum_{j=1}^s d_{t-1}^{p^j} + \sigma(S_t) + z + \sum_{j=1}^s p^j}{\sum_{j=1}^s d_{t-1}^{p^j} + z \cdot \frac{\sum_{j=1}^s d_{t-1}^{p^j}}{d_{t-1}^n + \sum_{j=1}^s d_{t-1}^{p^j}} + \sum_{j=1}^s p^j}$$

$$= \frac{d_{t-1}^n + \sum_{j=1}^s d_{t-1}^{p^j} + \sigma(S_t) + z + \sum_{j=1}^s p^j}{d_{t-1}^n + \sum_{j=1}^s d_{t-1}^{p^j} + z + \left(\sum_{j=1}^s p^j\right) \cdot \frac{d_{t-1}^n + \sum_{j=1}^s d_{t-1}^{p^j}}{\sum_{j=1}^s d_{t-1}^{p^j}}} \cdot \frac{d_{t-1}^n + \sum_{j=1}^s d_{t-1}^{p^j}}{\sum_{j=1}^s d_{t-1}^{p^j}}$$

$$< \frac{d_{t-1}^n + \sum_{j=1}^s d_{t-1}^{p^j} + \sigma(S_t) + z + \sum_{j=1}^s p^j}{d_{t-1}^n + \sum_{j=1}^s d_{t-1}^{p^j} + z + \left(\sum_{j=1}^s p^j\right)} \cdot \frac{d_{t-1}^n + \sum_{j=1}^s d_{t-1}^{p^j}}{\sum_{j=1}^s d_{t-1}^{p^j}}$$

$$= \left(1 + \frac{\sigma(S_t)}{d_{t-1}^n + \sum_{j=1}^s d_{t-1}^{p^j} + z + \sum_{j=1}^s p^j}\right) \cdot (r_{t-1} + 1)$$

$$r_t + 1 < \left(1 + \frac{\sigma(S_t)}{d_0^n + \sum_{j=1}^s p^j + z \cdot t + \sum_1^{t-1} \sigma(S_i) + \sum_1^t p_i}\right) \cdot (r_{t-1} + 1)$$

$$r_{t-1} + 1 < \left(1 + \frac{\sigma(S_{t-1})}{d_0^n + \sum_{j=1}^s p^j + z \cdot (t-1) + \sum_1^{t-2} \sigma(S_i) + \sum_1^{t-1} p_i}\right) \cdot (r_{t-2} + 1)$$

$$r_T + 1 < (r_0 + 1) \prod_{t=1}^{T} \left(1 + \frac{\sigma(S_t)}{d_0^n + \sum_{j=1}^s p^j + z \cdot t + \sum_1^{t-1} \sigma(S_i) + \sum_1^t p_i}\right)$$

$$r_T < r_T' = (r_0 + 1) \prod_{t=1}^{T} \left(1 + \frac{\sigma(S_t)}{d_0^n + \sum_{j=1}^s p^j + z \cdot t + \sum_1^{t-1} \sigma(S_i) + \sum_1^t p_i}\right) - 1$$

Lower bound:

$$r_t + 1 = \frac{d_t^n + d_t^p}{d_t^p} = \frac{d_{t-1}^n + d_{t-1}^p + \sigma(S_t) + z + p_t}{d_{t-1}^p + z \cdot \frac{d_{t-1}^p}{d_{t-1}^n + d_{t-1}^p} + \sum_{j=1}^s d_t^{p^j}}$$

$$= \frac{d_{t-1}^n + d_{t-1}^p + \sigma(S_t) + z + \sum_{j=1}^s d_t^{p^j}}{d_{t-1}^n + d_{t-1}^p + z + \sum_{j=1}^s d_t^{p^j} \cdot \frac{d_{t-1}^n + d_{t-1}^p}{d_{t-1}^p}} \cdot \frac{d_{t-1}^n + d_{t-1}^p}{d_{t-1}^p}$$

$$> \frac{d_{t-1}^n + d_{t-1}^p + \sigma(S_t) + z + \sum_{j=1}^s d_t^{p^j}}{d_{t-1}^n + d_{t-1}^p + z + 2 \cdot \sum_{j=1}^s d_t^{p^j}} \cdot \frac{d_{t-1}^n + d_{t-1}^p}{d_{t-1}^p}$$

$$= \left(1 + \frac{\sigma(S_t) - \sum_{j=1}^s d_t^{p^j}}{d_{t-1}^n + d_{t-1}^p + z + 2 \cdot \sum_{j=1}^s d_t^{p^j}}\right) \cdot (r_{t-1} + 1)$$

$$r_t + 1 > \left(1 + \frac{\sigma(S_t) - \sum_{j=1}^s d_t^{p^j}}{d_0^n + \sum_{j=1}^s p^j + z \cdot t + \sum_1^{t-1} \sigma(S_i) + \sum_1^t p_i + \sum_{j=1}^s d_t^{p^j}}\right) \cdot (r_{t-1} + 1)$$

$$r_{t-1} + 1 > \left(1 + \frac{\sigma(S_{t-1}) - p_{t-1}}{d_0^n + \sum_{j=1}^s p^j + z \cdot (t-1) + \sum_1^{t-2} \sigma(S_i) + \sum_1^{t-1} p_i + p_{t-1}}\right) \cdot (r_{t-2} + 1)$$

$$r_T > (r_0 + 1) \prod_{t=1}^{T} \left(1 + \frac{\sigma(S_t) - \sum_{j=1}^s d_t^{p^j}}{d_0^n + \sum_{j=1}^s p^j + z \cdot t + \sum_1^{t-1} \sigma(S_i) + \sum_1^t p_i + \sum_{j=1}^s d_t^{p^j}}\right) - 1$$

$$r_T > r_T'' = (r_0 + 1) \prod_{t=1}^{T} \left(1 + \frac{\sigma(S_t) - \sum_{j=1}^s d_t^{p^j}}{d_0^n + \sum_{j=1}^s p^j + z \cdot t + \sum_1^{t-1} \sigma(S_i) + \sum_1^t p_i + \sum_{j=1}^s d_t^{p^j}}\right) - 1$$