

Mathematica Compendium

Miguel Antonio Méndez Hernández

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1 Changig between Basis

Given a vektor $\vec{x} = \lambda_1 \vec{v}_1 + \dots + \lambda_n \vec{v}_n$ with $\vec{v}_1, \dots, \vec{v}_n$ being a the basis vector, we can write \vec{x} in term of its components $(a, b, c, \dots)^T$.

Now imagine having another basis $\vec{b}_1, \dots, \vec{b}_n$ and a the same vector \vec{x} with all of its components. The key here is that we want to know how our original vector is described in with other basis.

So our vector \vec{x} is described:

$$\lambda_1 \vec{v}_1 + \dots + \lambda_n \vec{v}_n = \mu_1 \vec{b}_1 + \dots + \mu_n \vec{b}_n$$

in the corresponding bases.

Now lets write then a matrix vector multiplication

$$(\vec{v}_1, \dots, \vec{v}_n) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = (\vec{b}_1, \dots, \vec{b}_n) \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$$

Now lets use the following notation for the Basis and the coefficient vectors.

$$\boxed{P_B(\vec{x})_B = P_C(\vec{x})_C}$$

Here P_X is the matrix of the generic basis X vectors and $(\vec{x})_X$ the coefficients of the vector \vec{x} described by that basis.

Now that conversion of basis is just a matter of solving for our desired vector by multiplying by the inverse of the corresponding basis matrix.

Example:

Lets find \vec{x} in terms of tha basis C

$$P_C^{-1} P_B(\vec{x})_B = (\vec{x})_C$$

Now let us understan what really is happening in the step where we multiply by the inverse matrix.

$$\begin{aligned} P_C^{-1} P_B &= P_C^{-1}(\vec{b}_1, \dots, \vec{b}_n) \\ &= (P_C^{-1} \vec{b}_1, \dots, P_C^{-1} \vec{b}_n) \\ &= ((\vec{b}_1)_C, \dots, (\vec{b}_n)_C) \end{aligned}$$

Now it is clear that the inverse is just taking our basis vector and tranforming them in to basis vector with respect to C .

For the sake of completness here some importan Theoremes and definitions:

1.1 Theoremes and definitions concerning the change of basis

In this section we will use a slightly different notation. Here K_X is the **coefficient** vector with respect to a basis also called the coordinates of the vector. And φ_X is the P_X that is the matrix of the basis vectors with respect to X .

I Let V be a K -vector space with a basis $B = (v_1, \dots, v_n)$. Then there exists exactly one isomorphism

$$\varphi_B : K^n \rightarrow V$$

such that

$$\varphi_B(e_i) = v_i, \quad \text{for } 1 \leq i \leq n.$$

II The isomorphism φ_B from the previous Theorem is called the *coordinate mapping*, and for $v \in V$, we define

$$K_B(\vec{v}) := \varphi_B^{-1}(v) \in K^n$$

as the *coordinates of v with respect to B* . Or like above $P_B^{-1} P_C(\vec{x})_C = (\vec{x})_B$

This is the previously discussed $\vec{v} = P_B^{-1} \vec{v}$ that takes a vector from the other basis and translates its into our basis.

III We define the *Traslation* of a vector to another basis as:

$$T_B^A = \varphi_B^{-1} \circ \varphi_A$$

With φ_A begin the matrix of basis vectors of the new coordinate system and φ_B^{-1} the inverse of the starting basis Or $P_B^{-1} \circ P_C$.

Another way of finding this transition matrix is to write our basis vectors as a linear combination of the basis vector of the other base and then use Gauß-Jordan to find the coefficients. the result of the corresponding matrices are going to make our transition matrix.

$$\begin{array}{ccc} & V & \\ \varphi_A \nearrow & & \nwarrow \varphi_B \\ K^n & \xrightarrow{T_B^A = \varphi_B^{-1} \circ \varphi_A} & K^n \end{array}$$

IV Let $v \in V$ be arbitrary, with $K_A(v) = (x_1, \dots, x_n)^T$ and $K_B(v) = (y_1, \dots, y_n)^T$. Then the following holds:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = T_B^A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

If the coordinates of v with respect to A are known, then the matrix T_B^A can be used to compute the coordinates of v with respect to B .

$$T_B^A = B^{-1}A$$

V Let V and W be finitely generated K -vector spaces with bases A and B , respectively, and let $f \in \text{Hom}(V, W)$. Then we have

$$M_B^A(f) = \varphi_B^{-1} \circ f \circ \varphi_A$$

In plain english the matrix that represents f with respect to the bases A and B is obtained by "coordinatizing" V (via φ_A), applying f , and then transforming the result into coordinates of W (via φ_B^{-1}).

- First, φ_A turns a vector $v \in V$ into its coordinate vector in K^n . This is just the matrix of the base of K^n .
- Then f acts on the original vector v . This us our linear transformation matrix in our basis
- Finally, φ_B^{-1} translates the result $f(v)$ into a coordinate vector in K^m (according to basis B).

Thus, $M_B^A(f)$ tells you how the coordinates change through f .

VI Let V and W be finitely generated vector spaces with bases A and A' for V , and B and B' for W , respectively. Furthermore, let $f : V \rightarrow W$ be a linear map. Then the following holds:

$$M_{B'}^{A'}(f) = T_{B'}^B \cdot M_B^A(f) \cdot (T_{A'}^A)^{-1}.$$

Now here's what this means:

- $M_B^A(f)$ is the matrix representing f relative to the original bases A (for V) and B (for W).
- $M_{B'}^{A'}(f)$ is the matrix representing f relative to the new bases A' and B' .
- $T_{A'}^A$ is the change of basis matrix from A' to A .
- $T_{B'}^B$ is the change of basis matrix from B to B' .

The formula says that to itepret a linear transformation in terms of the new coordinates we have to:

1. Take a vector of the new coordinates
2. Mulipty it the matrix of its basis vectors. (Translate to our basis)
3. Apply the linear transformation
4. Inverse the change of basis by taking the inverse of the matrix in step 2.