Mathematica Compendium

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Contents

1 Propositional Logic

Propositional logic is also called Boolean logic as it works on 0 and 1. In propositional logic, we use symbolic variables to represent the logic, and we can use any symbol for a representing a proposition, such *A, B, C, P, Q, R*, etc. Propositions can be either *true* or *false*, but it cannot be both.

1.1 Logic Operators and Truth Tables

NOT ($\neg$, $\sim$)

A	$\neg A$
0	1
1	0

AND ($\wedge$)

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

OR ($\vee$)

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

IMPLIES ($\implies$)

A	B	$A => B$
0	0	1
0	1	1
1	0	0
1	1	1

IFF ($\iff$)

A	B	$A <=> B$
0	0	1
0	1	0
1	0	0
1	1	1

XOR ($\oplus$)

A	B	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

NOR ($\downarrow$)

A	B	$A \downarrow B$
0	0	1
0	1	0
1	0	0
1	1	0

NAND ($\uparrow$)

A	B	$A \uparrow B$
0	0	1
0	1	1
1	0	1
1	1	0

1.2 Tautology and Contradiction

- *Tautology*: A logical formula that is always true.

- *Contradiction*: A formula that is always false.

1.3 Logical Equivalences

Commutative Laws
$$p \wedge q \Leftrightarrow q \wedge p \qquad p \vee q \Leftrightarrow q \vee p$$

Associative Laws
$$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r) \qquad (p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$$

Distributive Laws
$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) \qquad p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

Identity Laws
$$p \wedge \mathrm{T} \Leftrightarrow p \qquad p \vee \mathrm{F} \Leftrightarrow p$$

Negation Laws
$$p \vee \sim p \Leftrightarrow \mathrm{T} \qquad p \wedge \sim p \Leftrightarrow \mathrm{F}$$

Double Negation Law
$$\sim (\sim p) \Leftrightarrow p$$

Idempotent Laws
$$p \wedge p \Leftrightarrow p \qquad p \vee p \Leftrightarrow p$$

Universal Bound Laws
$$p \vee \mathrm{T} \Leftrightarrow \mathrm{T} \qquad p \wedge \mathrm{F} \Leftrightarrow \mathrm{F}$$

De Morgan's Laws
$$\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q) \qquad \sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$$

Absorption Laws
$$p \vee (p \wedge q) \Leftrightarrow p \qquad p \wedge (p \vee q) \Leftrightarrow p$$

Conditional Laws
$$(p \Rightarrow q) \Leftrightarrow (\sim p \vee q) \qquad \sim (p \Rightarrow q) \Leftrightarrow (p \wedge \sim q)$$

Complement Law
$$p \vee \neg p \Leftrightarrow \mathrm{T} \qquad p \wedge \neg p \Leftrightarrow \mathrm{F}$$

Biconditional
$$p \Leftrightarrow q \Leftrightarrow (p \Rightarrow q) \wedge (q \Rightarrow p)$$

Transitivity
$$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$$

Indirect Proof (Contrapositive)
$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$

Disjunctive Syllogism (Disjunctive Exclusion)
$$p \nabla q \equiv (p \vee q) \wedge \neg p \Rightarrow q$$
$$p \nabla q \equiv (p \wedge q) \vee \neg (p \wedge q)$$

1.4 Truth Tables

Truth tables are a fundamental tool in logic that systematically show the truth value (true or false) of a compound statement for every possible combination of the truth values of its individual component statements.

Essentially, they lay out all the scenarios and the resulting truth of the overall logical expression. This helps determine if an argument is valid, if statements are logically equivalent, or the circumstances under which a complex statement is true or false.

Example:

p	q	$\neg p$	$\neg q$	$\neg p \Rightarrow q$	$(\neg p \Rightarrow q) \wedge \neg p$	$[(\neg p \Rightarrow q) \wedge \neg p] \Rightarrow q$
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	T	T
F	F	T	T	F	F	T

1.4.1 Filling a truth table

To fill a truth table for a logical expression with truth values (True or False), you follow a specific order for the input variables. This order ensures that all possible combinations of truth values for the variables are covered.

1.4.2 General Procedure:

1. **List all possible combinations of truth values for the input variables**: If you have n variables, the number of rows in the truth table will be 2^n. Each variable can be either True (T) or False (F).

2. **Order of the input variables**:

   - Start by filling in the truth values for the first variable. It alternates between True and False every 2^{n-1} rows.
   - Then for the second variable, it alternates every 2^{n-2} rows, and so on.
   - In short: the first variable alternates every other row, the second variable every two rows, the third every four rows, and so on.

1.4.3 Example with 3 Variables (A, B, and C)

For 3 variables, there are $2^3 = 8$ possible combinations of truth values. The truth values are filled in the following order:

A	B	C	Expression Result
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

1.4.4 The pattern for filling the input truth values

- The first column (A) alternates every 4 rows: 'T, T, F, F, T, T, F, F'.
- The second column (B) alternates every 2 rows: 'T, T, F, F, T, T, F, F'.
- The third column (C) alternates every row: 'T, F, T, F, T, F, T, F'.

This ensures that all combinations of A, B, and C are covered, and you can then evaluate the logical expression for each combination.

1.4.5 Truth Table for the Expression $(A \wedge B) \vee C$

A	B	C	$(A \wedge B) \vee C$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

1.5 Disjunctive Normal Form (DNF)

Disjunctive Normal Form (DNF) is a standard way of writing a logical expression as a disjunction (OR) of conjunctions (ANDs). A DNF expression consists of a series of conjunctions of literals, where each conjunction is connected by disjunctions.

To find the DNF in a truth table take the rows of the final result where there are true statements and bind the propositions that generated it with an AND inside parenthesis. Repeat it with each of "true" rows and connect all parenthesis with OR's

Example of DNF:
    Consider the logical expression:

$$(A \land B) \lor (\neg A \land C) \lor (B \land \neg C)$$

This is in DNF because it is a disjunction (OR) of conjunctions (ANDs) of literals.

1.6 Conjunctive Normal Form (CNF)

Conjunctive Normal Form (CNF) is a standard way of writing a logical expression as a conjunction (AND) of disjunctions (ORs). A CNF expression consists of a series of disjunctions of literals, where each disjunction is connected by conjunctions.

To find the CNF proceed just as the DNF but with the "false rows and instead of ANDs inside the parenthesis use OR and connect the terms with OR. Also add a negation before each parenthesis.

Example of CNF:
    Consider the logical expression:

$$\neg(A \lor B) \land \neg(\neg A \lor C) \land \neg(B \lor \neg C)$$

This is in CNF because it is a conjunction (AND) of disjunctions (ORs) of literals.

1.7 Karnaugh Maps

Karnaugh Maps (K-Maps) are a graphical method used to simplify Boolean expressions. The main goal of a K-map is to group adjacent cells that contain 1's in order to simplify the expression. A K-map helps identify common terms, allowing the Boolean expression to be reduced to its simplest form.

1.7.1 Karnaugh Map for Two Variables

Consider the Boolean expression $(A \lor (B \land \neg A \land \neg B))$.
    We first construct a K-map for two variables, A and B. The truth table for this expression gives the following values:

A	B	$(A \lor (B \land \neg A \land \neg B))$
0	0	0
0	1	1
1	0	1
1	1	1

The corresponding K-map is:

AB	00	01	11	10
Value	0	1	1	1

Here, we group the ones together to simplify the Boolean expression. The simplified expression is:

$$A \lor B$$

1.7.2 Karnaugh Map for Three Variables

Now, let's consider the expression $\neg C$. This expression only depends on one variable, but for illustration, we will use a 3-variable K-map with variables A, B, and C.

The truth table for $\neg C$ is as follows:

A	B	C	$\neg C$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

The corresponding K-map for three variables A, B, and C is:

$AB\backslash C$	0	1
00	1	0
01	1	0
11	1	0
10	1	0

We see that the ones are grouped in a column, leading to the simplified Boolean expression:

$$\neg C$$

1.7.3 Solving a Karnaugh Map (K-Map)

To solve a Karnaugh Map (K-Map) and simplify a Boolean expression, follow these steps:

1. **Determine the Number of Variables:**
   Decide how many variables are in the Boolean function. This determines the size of the K-Map:

   - 2 variables: 2×2
   - 3 variables: 2×4
   - 4 variables: 4×4
   - etc.

2. **Fill in the K-Map:**
   Place 1's in the cells that correspond to the minterms (where the function outputs 1). You may also include don't-care conditions (usually denoted as X).

3. **Group the 1's:**
   Form groups (called *implicants*) of 1's. The groups must follow these rules:

   - Each group must contain $1, 2, 4, 8, \ldots$ (powers of 2) 1's.
   - Groups must be rectangular (e.g., 1×2, 2×2).
   - Groups can wrap around the edges of the K-Map.
   - Try to form the largest groups possible to simplify the expression.
   - Each 1 should be included in at least one group.

4. **Write the Simplified Expression:**
   For each group:

   - Identify the variables that are constant (either always 0 or always 1) across the group.
   - Write a product term (AND) using only the constant variables.
   - Combine all product terms with OR operations to get the final simplified SOP (Sum of Products) expression.

   Example:
   Given minterms $F(A, B, C) = \sum m(1, 3, 5, 7)$:

K-Map:

AB \C	0	1
00	0	1
01	0	1
11	0	1
10	0	1

Simplified expression:

$$F = B \vee C$$

1.8 Mathematical Quantifiers with Negations and Examples

Universal Quantifier: $\forall$ Means **"for all"** or **"for every"**.

- **Example:** $\forall x \in \mathbb{R},\ x^2 \geq 0$
  (For all real numbers, the square is greater than or equal to zero.)
- **Negation:** $\neg(\forall x)\, P(x) \equiv (\exists x)\, \neg P(x)$
  ("Not all" is the same as "There exists one that does not".)
- **Negated Example:** $\exists x \in \mathbb{R},\ x^2 < 0$
  (There exists a real number whose square is less than zero — this is false.)

Existential Quantifier: $\exists$ Means **"there exists at least one"**.

- **Example:** $\exists x \in \mathbb{N},\ x > 10$
  (There exists a natural number greater than 10.)
- **Negation:** $\neg(\exists x)\, P(x) \equiv (\forall x)\, \neg P(x)$
  ("There does not exist" is the same as "For all, not".)
- **Negated Example:** $\forall x \in \mathbb{N},\ x \leq 10$
  (All natural numbers are less than or equal to 10 — this is false.)

Unique Existential Quantifier: $\exists!$ Means **"there exists exactly one"**.

- **Example:** $\exists! x \in \mathbb{R},\ x + 5 = 0$
  (There exists exactly one real number such that $x + 5 = 0$.)
- **Negation:** "Not exactly one" means:

$$\neg(\exists! x)\, P(x) \equiv (\forall x)\, \neg P(x)\ \vee\ (\exists x_1 \neq x_2)\, P(x_1) \wedge P(x_2)$$

  (Either no such x exists, or more than one does.)
- **Negated Example:** $\exists x_1 \neq x_2 \in \mathbb{R},\ x_1^2 = 4 \wedge x_2^2 = 4$
  (There are multiple solutions to $x^2 = 4$.)

1.9 Common Symbols Used in Mathematical Expressions

- $>$ (greater than)

- $<$ (less than)

- $\geq$ (greater than or equal to)

- $\leq$ (less than or equal to)

- $=$ (equals)

- $\neq$ (not equal)

- $\in$ (element of a set)

- $\notin$ (not an element of)

- $\subset$ (proper subset)

- $\subseteq$ (subset)

- $\supset$ (proper superset)

- $\supseteq$ (superset)

- $\wedge$ (logical AND)

- $\vee$ (logical OR)

- $\Rightarrow$ (implies)

- $\Leftrightarrow$ (if and only if)

2 Set Theory

Set theory is a foundational branch of mathematics that studies sets, which are simply collections of objects. At its core, it deals with the fundamental concepts of membership (whether an object belongs to a set), equality (when two sets are the same), and relationships between sets (like subsets, intersections, and unions).

It might seem simple, but set theory provides the basic language and tools to define and reason about almost all mathematical objects, from numbers and functions to more complex structures. It helps us understand the concept of infinity, organize mathematical ideas logically, and resolve paradoxes that arise from dealing with collections.

In essence, set theory provides the building blocks upon which much of modern mathematics is constructed.

2.1 Basics

To be concise, a set is a collection of mathematical objects that can also be other sets.

Here is a list of common symbols for set theory.

1. **Empty set ($\emptyset$)**: The set that contains no elements. It is the unique set with zero elements.

2. **Example set with two elements**: A set that contains exactly two elements, such as $\{1, 2\}$.

3. $A \subseteq B$: Set A is a subset of B. This means every element of A is also in B.

4. $A \subseteq B$ **or** $A = B$: This notation already includes the possibility that A equals B since a set is always a subset of itself.

5. $A \cup B$: The union of sets A and B. It includes all elements that are in A, in B, or in both.

6. $A \cap B$: The intersection of sets A and B. It includes only the elements that are in both sets.

7. $A \setminus B$: The difference of sets. Elements in A that are not in B.

8. A^c **or** $\overline{A}$: The complement of set A. All elements not in A, relative to a universal set.

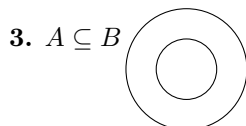9. $|A|$: The cardinality of set A, which is the number of elements in the set.
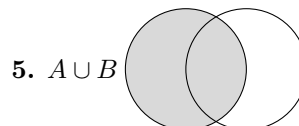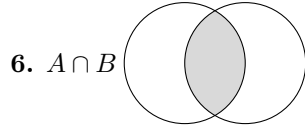
2.1.1 Visuals

1. $\emptyset$

1. Empty Set

2. $\{1, 2\}$
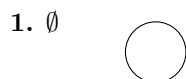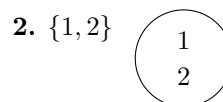
2. Set with Two Elements

3. $A \subseteq B$

3. A is a subset of B

5. $A \cup B$

5. Union of A and B

6. $A \cap B$

6. Intersection of A and B

7. $A \setminus B$

7. A minus B

8. A^c
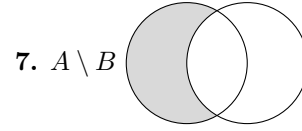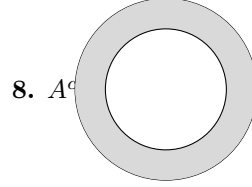
8. Complement of A

2.2 Axioms of Set Theory (Zermelo Fraenkel)

This are Zermelo Fraenkel axioms of set theory including *The Axiom of Choice.*

1. **Axiom of Extensionality:** $\forall A, B : A = B \Rightarrow (\forall C : C \in A \Leftrightarrow C \in B)$

2. **Empty-Set Axiom:** $\exists \emptyset : \forall X : X \notin \emptyset$

3. **Axiom of Pairing:** $\forall A, B : \exists C : \forall D : D \in C \Leftrightarrow (D = A \vee D = B)$

4. **Axiom of Union:** $\forall A : \exists B : \forall C : C \in B \Leftrightarrow (\exists D : C \in D \wedge D \in A)$

5. **Axiom of Infinity:** $\exists N : \emptyset \in N \wedge (\forall x : x \in N \Rightarrow x \cup \{x\} \in N)$

6. **Axiom Schema of Specification:** $\forall A : \exists B : \forall C : C \in B \Leftrightarrow (C \in A \wedge P(C))$

7. **Axiom Schema of Replacement:** $\forall A : \exists B : \forall y : y \in B \Rightarrow \exists x \in A : y = F(x)$

8. **Powerset Axiom:** $\forall A : \exists B : \forall C : C \subseteq A \Rightarrow C \in B$

9. **Foundation Axiom:** $\forall A \neq \emptyset : \exists B \in A : A \cap B = \emptyset$

10. **Axiom of Choice:** $\forall X : ([\forall A \in X : A \neq \emptyset] \wedge [\forall B, C \in X : B \neq C \Rightarrow B \cap C = \emptyset])$
    $\Rightarrow \exists Y : \forall I \in X : \exists! J \in Y : J \in I$

2.3 The Cartesian Product

The Cartesian product of two sets A and B, written $A \times B$, is the set of all ordered pairs in which the first element belongs to A and the second belongs to B:

$$A \times B = \{(a, b) : a \in A, \wedge\ b \in B\}.$$

Example:

Table 1: Cartesian Product of $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$

$A \times B$	$b \in B$		
	4	5	6
$a \in A$			
1	$(1, 4)$	$(1, 5)$	$(1, 6)$
2	$(2, 4)$	$(2, 5)$	$(2, 6)$
3	$(3, 4)$	$(3, 5)$	$(3, 6)$

The general cartesian product of n sets can be written as:

$$X_{i=1}^{n+1} A_i = (X_{i=1}^{n} A_i) \times A_{n+1} \quad \text{with} \quad X_{i=1}^{1} A_i = A_1$$

When $A_i = M$ for all i :

$$M^n := M \times M \times \cdots \times M = X_{i=1}^{n} M \quad \text{with} \quad M^1 = M$$

2.4 Laws of Set Algebra

Let X be the universal set and $A, B, C \subseteq X$.

1. $\emptyset \subseteq A$

2. $A \subseteq B \iff A \cap B = A \iff A \cup B = B \iff X \setminus B \subseteq X \setminus A \iff B \subseteq A$

3. $A \cup B = B \cup A$ (Commutative Law)

4. $A \cap B = B \cap A$ (Commutative Law)

5. $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative Law)

6. $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative Law)

7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive Law)

8. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Law)

9. $A \cup A = A$ and $A \cap A = A$ (Idempotent Law)

10. $A \setminus B = A \cap (X \setminus B) = A \cap \overline{B}$

11. $B = \overline{A} \iff (A \cup B = X \land A \cap B = \emptyset)$ (Disjoint Partition of X)

12. $\overline{A} \cap \overline{B} = \overline{A \cup B}$ (De Morgan's Law)

13. $\overline{A} \cup \overline{B} = \overline{A \cap B}$ (De Morgan's Law)

14. $\overline{\overline{A}} = A$ (Double Negation)

2.4.1 Proof of De Morgans's Law for sets and logic

The complement of $A \cup B$ is $\overline{(A \cup B)}$, and Law (11) on disjoint decomposition states:

$$B = \overline{A} \iff (A \cup B = X) \land (A \cap B = \emptyset)$$

So define $\overline{C} := A \cup B$ and $D := \overline{A} \cap \overline{B}$, and use Law (11) to show the disjoint decomposition:

$$D = C \iff A \cap B = A \cup B$$

a) To show:
$$D \cup C = X \iff (\overline{A} \cap \overline{B}) \cup (A \cup B) = X$$

$$
\begin{aligned}
(\overline{A} \cap \overline{B}) \cup (A \cup B) &= (\overline{A} \cup A \cup B) \cap (\overline{B} \cup A \cup B) \quad \text{(Law (8))} \\
&= (X \cup B) \cap (X \cup A) \\
&= X \cap X \\
&= X
\end{aligned}
$$

b) To show:
$$D \cap C = \emptyset \iff (\overline{A} \cap \overline{B}) \cap (A \cup B) = \emptyset$$

$$
\begin{aligned}
(\overline{A} \cap \overline{B}) \cap (A \cup B) &= (A \cap B \cap A) \cup (A \cap B \cap B) \quad \text{(Law (7))} \\
&= (\overline{A} \cap A \cap \overline{B}) \cup (\overline{A} \cap \overline{B} \cap B) \\
&= (\emptyset \cap \overline{B}) \cup (\overline{A} \cap \emptyset) \\
&= \emptyset \cup \emptyset \\
&= \emptyset
\end{aligned}
$$

2.5 Indexed Sets

Let X be a set, and $A_i \subseteq X$ for all $i \in J$, where J is the index set.

a) If $J = \{1, 2, \ldots, n\}$:

$$\bigcup_{i=1}^{n} A_i := A_1 \cup A_2 \cup \cdots \cup A_n = \{x \mid \exists i \in J \ (x \in A_i)\}$$

$$\bigcap_{i=1}^{n} A_i := A_1 \cap A_2 \cap \cdots \cap A_n = \{x \mid \forall i \in J \ (x \in A_i)\}$$

$$X_{i=1}^{n} A_i = \{(a_1, \ldots, a_n) \mid a_i \in A_i\}$$

b) If J is any set:

$$\bigcup_{i \in J} A_i := \{x \mid \exists i \in J \ (x \in A_i)\}$$

$$\bigcap_{i \in J} A_i := \{x \mid \forall i \in J \ (x \in A_i)\}$$

c) If J is any set, then $(A_i)_{i \in J}$ are pairwise disjoint if and only if:

$$\forall i_1, i_2 \in J, \ i_1 \neq i_2 \Rightarrow A_{i_1} \cap A_{i_2} = \emptyset$$

d) If J is any set, then $(A_i)_{i \in J}$ forms a (disjoint) decomposition of X if and only if:

$$(A_i)_{i \in J} \text{ are pairwise disjoint and } \bigcup_{i \in J} A_i = X$$

2.5.1 More Partitions Laws

Let $A_i, B_j \subseteq X$ for $i \in I$ and $j \in J$. Then the following holds:

a) De Morgan's Laws:

$$\overline{\bigcap_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i} \quad \text{and} \quad \overline{\bigcup_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i}$$

b)

$$\bigcap_{i \in I} A_i \cup \bigcap_{j \in J} B_j = \bigcap_{i,j} (A_i \cup B_j) \quad \text{with} \quad \bigcap_{i,j} = \bigcap_{(i,j) \in I \times J}$$

c)

$$\bigcup_{i \in I} A_i \cap \bigcup_{j \in J} B_j = \bigcup_{i,j} (A_i \cap B_j) \quad \text{with} \quad \bigcup_{i,j} = \bigcup_{(i,j) \in I \times J}$$

Here, $I = \{1, 2, 3, \ldots, n\}$ and $J = \{1, 2, 3, \ldots, m\}$. Then:

$$I \times J = \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq m, i, j \in \mathbb{N}\}$$

$$= \{(1, 1), (1, 2), \ldots, (1, m), (2, 1), (2, 2), \ldots, (2, m), \ldots, (n, 1), (n, 2), \ldots, (n, m)\}$$

2.6 Cardinality

The cardinality of a set is the number of elements in that set.

Let A and B be finite sets with $|A| = n$, $|B| = m$, and let X be the finite universal set. Then the following holds:

a)
$$A = (A \cap B) \cup (A \setminus B), \quad |A| = |A \cap B| + |A \setminus B|$$

b)
$$|A| = |X \setminus A| = |X| - |A|$$

$$A \setminus B = A \cap (X \setminus B), \quad |A \setminus B| = |A| - |A \cap B|$$

c)
$$|A \times B| = |A| \cdot |B| = n \cdot m$$

d) **Inclusion-Exclusion Formula for Two Disjoint Sets**
$$|A \cup B| = |A| + |B| = n + m$$

e) **Inclusion-Exclusion Formula for Two Non-Disjoint Sets** Let $|A \cap B| = k$, then:
$$|A \cup B| = |A| + |B| - |A \cap B| = n + m - k \quad \text{(since we do not count the intersection twice)}$$

f) **Inclusion-Exclusion Formula for Three Non-Disjoint Sets**
$$|A \cup B \cup C| = |(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$$
$$= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)|$$
$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

2.6.1 General Formula for the cardinality of the union of sets

$$\left| \bigcup_{i=1}^{n} M_i \right| = \sum_{I \subseteq \{1,\dots,n\} \cdot I \neq \emptyset} (-1)^{|I|-1} \left| \bigcap_{i \in I} M_i \right|$$

2.7 The Power Set

The Power Set of a set is the set of all subsets of a given set
$$P(x) := \{M : M \subset X\}$$

Its cardinality is 2^n with n being the number of elements in the original set X.

2.8 Family of Subsets

Let X be a non empty set. A subset $\mathscr{F}$ of the power set of X is called a set system of X

2.9 Partition

Let X be a non empty set. A subset $\mathscr{F}$ of the power set of X is called a partition if:

I) $M \neq \emptyset \; \forall M \in \mathscr{F}$
II) $\bigcap \mathscr{F} = X$
III) $M_1 \bigcap M_2 \neq \emptyset \implies M_1 = M_2 \; \forall \; M_1, M_2 \in \mathscr{F}$

- Every equivalence relation corresponds to a partition:
$$J : \{R : R \text{ is an equivalence relation on } X\} \to \{F : F \text{ is a partition of } X\}$$

where $J(R) := X/R$ is a bijection.

- If F is a partition of X, then we can define an equivalence relation R_F by:
$$R_F := \{(x,y) \in X \times X : \exists M \in F \text{ such that } x, y \in M\}$$

Then R_F is an equivalence relation on X.

2.10 Family of Subsets Operations

Let $\mathscr{F}$ be a system of sets (a family of subsets) on the set X. We define:

$$\bigcup_{M \in F} M := \bigcup F := \{x \in X : \text{there exists } M \in F \text{ such that } x \in M\},$$

$$\bigcap_{M \in F} M := \bigcap F := \{x \in X : x \in M \text{ for all } M \in F\}.$$

3 Relations, Maps and Functions

A relation in mathematics is a connection or relationship between elements of two sets. It's formally defined as a subset of the Cartesian product of the sets.

For example, if we have sets A and B, a relation R from A to B consists of ordered pairs (a, b) where $a \in A$ and $b \in B$, such that a is related to b according to some rule or property.

Common types of relations include:

- Functions (special relations where each input has exactly one output)

- Equivalence relations (reflexive, symmetric, and transitive)

- Partial orders (reflexive, antisymmetric, and transitive)

Relations can be represented using diagrams, matrices, or sets of ordered pairs, and they're fundamental to many areas of mathematics including algebra, calculus, and discrete mathematics.

3.1 Types of relations

Let A be a set and X be a relation on A.

- **Reflexive:** $\forall a \in A : (a, a) \in X$ (or written as $a \sim a$)

- **Irreflexive:** $\forall a \in A : (a, a) \notin X$

- **Symmetric:** $\forall a, b \in A : (a, b) \in X \Rightarrow (b, a) \in X$ (or written as $(a \sim b) \Rightarrow (b \sim a)$)

- **Antisymmetric:** $\forall a, b \in A : (a, b) \in X$ and $(b, a) \in X \Rightarrow a = b$

- **Transitive:** $\forall a, b, c \in A : (a, b) \in X$ and $(b, c) \in X \Rightarrow (a, c) \in X$ (or written as $(a \sim b)$ and $(b \sim c) \Rightarrow (a \sim c)$)

- **Total:** $\forall a, b \in A : a \neq b \Rightarrow (a, b) \in X$ or $(b, a) \in X$

3.2 Equivalence relation

An equivalence relation is a relation that is symmetric, transitive and reflexive.
Example:
$$R := \{(a, b) \in \mathbb{N} \times \mathbb{N} : a = b\}$$

3.3 The Graph

$$\text{graph}(f) := \{(x, f(x)) : x \in X\}$$

3.4 The identity

$$\text{id}(f) := \ idx := (x, x) : x \in X$$

3.5 Image and Domain

3.5.1 Image

The image (or range) of a relation R from set A to set B is the set of all elements in B that are related to at least one element in A. Formally, if $R \subseteq A \times B$ is a relation, then the image of R is defined as:

$$\text{Im}(R) = \{b \in B \mid \exists a \in A \text{ such that } (a, b) \in R\}$$

In other words, the image consists of all the "output" values that appear in the ordered pairs of the relation. For example, if $R = \{(1, 4), (2, 5), (3, 4), (2, 6)\}$, then $\text{Im}(R) = \{4, 5, 6\}$.

3.5.2 Domain

The domain of a relation R from set A to set B is the set of all elements in A that are related to at least one element in B. Formally, if $R \subseteq A \times B$ is a relation, then the domain of R is defined as:

$$\text{Dom}(R) = \{a \in A \mid \exists b \in B \text{ such that } (a, b) \in R\}$$

In other words, the domain consists of all the "input" values that appear in the ordered pairs of the relation. For example, if $R = \{(1, 4), (2, 5), (3, 4), (2, 6)\}$, then $\text{Dom}(R) = \{1, 2, 3\}$.

3.6 Equivalence Class

Let $\sim$ be an equivalence relation on a set A. For an element $x \in A$, the **equivalence class** of x, denoted by $[x]$, is the set of all elements in A that are equivalent to x. Formally, it is defined as:

$$[x] := \{y \in A \mid x \sim y\} \subseteq A$$

In other words, the equivalence class of x contains all elements y in A such that x is related to y under the equivalence relation $\sim$.

3.7 Quotient Space

Let $\sim$ be an equivalence relation on a set A. The **quotient space** of A by $\sim$, denoted by $A/\sim$ (or sometimes A/R), is the set of all distinct equivalence classes of elements in A. Formally, it is defined as:

$$A/\sim := \{[x] \mid x \in A\}$$

The quotient space $A/\sim$ is a partition of the original set A into disjoint equivalence classes. Each element of the quotient space is an equivalence class $[x]$, which itself is a subset of A.

3.8 Definition of a Map

A map (or function) from a set A to a set B, denoted as $f : A \to B$, is a relation that associates each element of the set A with exactly one element of the set B.

Formally, a function $f : A \to B$ is a subset of $A \times B$ such that for every $a \in A$, there exists exactly one $b \in B$ where $(a, b) \in f$. We typically write $f(a) = b$ to indicate that f maps the element a to the element b.

Example: Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. A possible function $f : A \to B$ could be defined as:

$$f(1) = x, \quad f(2) = y, \quad f(3) = z$$

This function can also be represented as the set of ordered pairs $\{(1, x), (2, y), (3, z)\}$.

3.9 Composition of Maps

The composition of two functions is the operation of applying one function to the result of another function. If we have functions $f : A \to B$ and $g : B \to C$, then the composition of g and f, denoted as $g \circ f$ (read as "g composed with f"), is a function from A to C defined by:

$$(g \circ f)(a) = g(f(a)) \quad \text{for all } a \in A$$

The composition applies f first, then applies g to the result. Note that the codomain of f must match the domain of g for the composition to be defined.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x + 3$. Then:

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 3$$

$$(f \circ g)(x) = f(g(x)) = f(x + 3) = (x + 3)^2 = x^2 + 6x + 9$$

Note that $g \circ f \neq f \circ g$ in general, which shows that function composition is not commutative.

3.10 Types of Functions

3.10.1 Injective Functions

An injective function (also called a one-to-one function) is a function that maps distinct elements from the domain to distinct elements in the codomain.

Formally, a function $f : A \to B$ is injective if for all $a_1, a_2 \in A$:

$$a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$$

Equivalently, using the contrapositive:

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$$

3.10.2 Surjective Functions

A surjective function (also called an onto function) is a function whose image equals its codomain, meaning that every element in the codomain has at least one preimage in the domain.

Formally, a function $f : A \to B$ is surjective if:

$$\forall b \in B, \exists a \in A \text{ such that } f(a) = b$$

3.10.3 Bijective Functions

A bijective function (also called a one-to-one correspondence) is a function that is both injective and surjective. In other words, every element in the codomain is mapped to by exactly one element in the domain.

Formally, a function $f : A \to B$ is bijective if it is both:

- Injective: $\forall a_1, a_2 \in A, a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

- Surjective: $\forall b \in B, \exists a \in A \text{ such that } f(a) = b$

Bijective functions establish a perfect pairing between elements of the domain and codomain, where each element in the domain corresponds to exactly one element in the codomain, and vice versa. A bijection allows us to define an inverse function $f^{-1} : B \to A$.

3.11 Propositions on Images and Preimages under Set Operations

Let $f : X \to Y$ be a function.

3.11.1 Union and Cut Sets

i) For subsets $A_1, A_2 \subseteq X$, we have:

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2) \quad \text{and} \quad f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

ii) For subsets $B_1, B_2 \subseteq Y$, we have:

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2) \quad \text{and} \quad f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

Proof. We prove the second part of 1 and the first part of 2; the remaining statements are left as exercises.

1. Let $y \in f(A_1 \cap A_2)$. Then there exists $x \in A_1 \cap A_2$ such that $f(x) = y$. Since $x \in A_1$ and $x \in A_2$, it follows that $y \in f(A_1)$ and $y \in f(A_2)$, hence $y \in f(A_1) \cap f(A_2)$. Therefore, every element of $f(A_1 \cap A_2)$ is also an element of $f(A_1) \cap f(A_2)$, so:

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

2. Let $x \in f^{-1}(B_1 \cup B_2)$. Then $f(x) \in B_1 \cup B_2$, which means $f(x) \in B_1$ or $f(x) \in B_2$. Thus, $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, which implies:

$$x \in f^{-1}(B_1) \cup f^{-1}(B_2)$$

Hence, both sets contain the same elements and are therefore equal:

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$$

3.11.2 Union and Cut of the whole Domain and Range

Let $f : X \to Y$ be a function.

i) Let $\mathcal{F}$ be a collection of subsets of X. Then:

$$f\left(\bigcup_{A \in \mathcal{F}} A\right) = \bigcup_{A \in \mathcal{F}} f(A) \quad \text{and} \quad f\left(\bigcap_{A \in \mathcal{F}} A\right) \subseteq \bigcap_{A \in \mathcal{F}} f(A)$$

ii) Let $\mathcal{G}$ be a collection of subsets of Y. Then:

$$f^{-1}\left(\bigcup_{B \in \mathcal{G}} B\right) = \bigcup_{B \in \mathcal{G}} f^{-1}(B) \quad \text{and} \quad f^{-1}\left(\bigcap_{B \in \mathcal{G}} B\right) = \bigcap_{B \in \mathcal{G}} f^{-1}(B)$$

Proof (partial). We show the first statement of part (ii); the rest follows analogously.
Let $x \in f^{-1}\left(\bigcup_{B \in \mathcal{G}} B\right)$. Then:

$$f(x) \in \bigcup_{B \in \mathcal{G}} B \quad \Leftrightarrow \quad \exists B \in \mathcal{G} \text{ such that } f(x) \in B \quad \Leftrightarrow \quad \exists B \in \mathcal{G} \text{ such that } x \in f^{-1}(B)$$

Hence:

$$x \in \bigcup_{B \in \mathcal{G}} f^{-1}(B)$$

It follows that:

$$f^{-1}\left(\bigcup_{B \in \mathcal{G}} B\right) = \bigcup_{B \in \mathcal{G}} f^{-1}(B)$$

3.12 Inverse of a Function

The inverse of a function $f : A \to B$ is a function $f^{-1} : B \to A$ that reverses the operation of f. That is, if f maps an element $a \in A$ to an element $b \in B$, then the inverse function f^{-1} maps b back to a. Formally, a function $f : A \to B$ has an inverse $f^{-1} : B \to A$ if and only if f is bijective (both injective and surjective). The inverse function satisfies the following properties:

$$f^{-1}(f(a)) = a \quad \text{for all } a \in A$$

$$f(f^{-1}(b)) = b \quad \text{for all } b \in B$$

In other words, composing a function with its inverse yields the identity function. That is:

$$f^{-1} \circ f = id_A \quad \text{and} \quad f \circ f^{-1} = id_B$$

where id_A and id_B are the identity functions on sets A and B, respectively.

3.12.1 Steps to Find the Inverse of a Function

To find the inverse of a function $f(x)$, follow these steps:

1. Replace $f(x)$ with y: $y = f(x)$

2. Interchange the variables x and y: $x = f(y)$

3. Solve for y in terms of x: $y = f^{-1}(x)$

4. Verify that the resulting function is indeed the inverse by checking that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$

3.12.2 Example: Finding the Inverse of $f(x) = 2x + 3$

Let's apply the steps above to find the inverse of $f(x) = 2x + 3$.

Step 1: Replace $f(x)$ with y.

$$y = 2x + 3$$

Step 2: Interchange the variables x and y.

$$x = 2y + 3$$

Step 3: Solve for y in terms of x.

$$x = 2y + 3 \tag{1}$$
$$x - 3 = 2y \tag{2}$$
$$\frac{x - 3}{2} = y \tag{3}$$

So, the inverse function is:

$$f^{-1}(x) = \frac{x - 3}{2}$$

Step 4: Verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

Let's verify $f^{-1}(f(x)) = x$:

$$f^{-1}(f(x)) = f^{-1}(2x + 3) \tag{4}$$
$$= \frac{(2x + 3) - 3}{2} \tag{5}$$
$$= \frac{2x}{2} \tag{6}$$
$$= x \tag{7}$$

And let's verify $f(f^{-1}(x)) = x$:

$$f(f^{-1}(x)) = f\left(\frac{x - 3}{2}\right) \tag{8}$$
$$= 2\left(\frac{x - 3}{2}\right) + 3 \tag{9}$$
$$= (x - 3) + 3 \tag{10}$$
$$= x \tag{11}$$

Since both compositions yield the identity function, $f^{-1}(x) = \frac{x-3}{2}$ is indeed the inverse of $f(x) = 2x + 3$.

3.12.3 Properties of the Inverse Function

Let $f : X \to Y$ be a function.

- Assume $x \sim y$ if $f(x) = f(y)$ so is $\sim$ an equivalence relation.

- Consider the quotient set $X_f := X/\sim$, where $\sim$ is the equivalence relation defined by $x \sim x' \iff f(x) = f(x')$. Let $q_f : X \to X_f$ be the canonical projection defined by $q_f(x) = [x]$, and let $\iota_f : f(X) \to Y$ be the inclusion map, $y \mapsto y$. Then the function

$$\hat{f} : X_f \to f(X), \quad [x] \mapsto f(x)$$

is a bijection, and the original map f can be written as the composition

$$f = \iota_f \circ \hat{f} \circ q_f.$$

Here, q_f is surjective, $\hat{f}$ is bijective, and ι_f is injective. This yields the following commutative diagram:

3.13   Transformations of a Function

Transformations modify the appearance of a function's graph without altering its basic shape. Here, we examine how different algebraic changes to a function $f(x)$ affect its graph:

- **Vertical Translation:** $f(x) + a$

  - Shifts the graph **upward** if $a > 0$, and **downward** if $a < 0$.
  - Each point on the graph moves vertically by a units.

- **Horizontal Translation:** $f(x + a)$

  - Shifts the graph **left** if $a > 0$, and **right** if $a < 0$.
  - This is opposite of what might be expected: adding to x shifts the graph in the negative direction.

- **Vertical Scaling (Stretch/Compression):** $af(x)$

  - If $|a| > 1$: the graph is **stretched** vertically (taller and narrower).
  - If $0 < |a| < 1$: the graph is **compressed** vertically (shorter and wider).
  - If $a < 0$: includes a reflection across the **x-axis**.

- **Horizontal Scaling (Stretch/Compression):** $f(ax)$

  - If $|a| > 1$: the graph is **compressed** horizontally (narrower).
  - If $0 < |a| < 1$: the graph is **stretched** horizontally (wider).
  - If $a < 0$: includes a reflection across the **y-axis**.

- **Reflection across the x-axis:** $-f(x)$

  - Flips the graph upside-down over the x-axis.
  - Each point (x, y) becomes $(x, -y)$.

- **Reflection across the y-axis:** $f(-x)$

  - Flips the graph left-to-right over the y-axis.
  - Each point (x, y) becomes $(-x, y)$.

4 Topological Vocabulary

In this section, we introduce essential vocabulary used in topology. Each term is accompanied by a brief explanation and its formal mathematical definition.

- **Open Set**
  A subset $U \subseteq X$ of a topological space is called *open* if for every point $x \in U$, there exists an $\varepsilon > 0$ such that the open ball $B_\varepsilon(x) \subseteq U$.
  Intuitively, an open set contains none of its boundary points and every point has some "wiggle room" around it.

- **Closed Set**
  A subset $A \subseteq X$ is called *closed* if its complement $X \setminus A$ is open. Equivalently, A contains all its limit points.
  That is, A is closed if it includes its boundary.

- **Interior Point**
  A point $x \in A$ is an *interior point* of $A \subseteq X$ if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$.
  The set of all interior points of A is called the *interior* of A, denoted $\text{int}(A)$.

- **Boundary Point**
  A point $x \in X$ is a *boundary point* of a set $A \subseteq X$ if every open ball around x contains both points in A and in $X \setminus A$.
  The set of all boundary points is called the *boundary* of A, denoted ∂A.

- **Accumulation Point / Limit Point**
  A point $x \in X$ is an *accumulation point* of a set $A \subseteq X$ if every open ball $B_\varepsilon(x)$ contains a point of $A \setminus \{x\}$.
  In other words, points of A cluster arbitrarily close to x, even if $x \notin A$.

- **Isolated Point**
  A point $x \in A$ is an *isolated point* if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \{x\}$.
  That is, x stands alone in A without other points of A nearby.

- **Compact Set**
  A set $K \subseteq X$ is *compact* if every open cover of K has a finite subcover.
  In $\mathbb{R}^n$, this is equivalent to K being closed and bounded (by the Heine–Borel theorem).

- **Dense Set**
  A subset $D \subseteq X$ is *dense* in X if every point $x \in X$ is either in D or is a limit point of D.
  Equivalently, the closure of D is X, i.e., $\overline{D} = X$.

- **Open Ball** $(B_\varepsilon(x))$
  For a metric space (X, d), the *open ball* centered at $x \in X$ with radius $\varepsilon > 0$ is defined as:

$$B_\varepsilon(x) := \{y \in X \mid d(x,y) < \varepsilon\}$$

It represents the set of all points within distance ε from x, excluding the boundary.

5 Mathematical Proofs

In this section I will provide with some examples of different types of proofs.

5.1 Proof by Direct Argument

For any integer n, if n is even, then n^2 is even.

Proof. We will prove this theorem by direct argument.

Assume n is an even integer. Then we can write $n = 2k$ for some integer k.

Now, we compute n^2:

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

This shows that n^2 is even, as it can be expressed as $2m$ where $m = 2k^2$ is an integer. Therefore, we conclude that if n is even, then n^2 is even. □

5.2 Proof by Contradiction

If n is an integer such that n^2 is even, then n is even.

Proof. We will prove this theorem by contradiction. Assume that n is an integer such that n^2 is even, but n is odd. Then we can write $n = 2k + 1$ for some integer k.

Now, we compute n^2:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

This shows that n^2 is odd, which contradicts our assumption that n^2 is even. Therefore, our assumption that n is odd must be false, and thus n must be even. □

5.3 Proof by Induction

For all $n \in \mathbb{N}$, the sum of the first n positive integers is given by:

$$S(n) = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Proof. We will prove this theorem by induction on n.

Base Case: For $n = 1$:

$$S(1) = 1 = \frac{1(1+1)}{2}$$

The base case holds.

Inductive Step: Assume that the statement holds for some $n = k$, i.e., assume that:

$$S(k) = 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$$

We need to show that the statement holds for $n = k + 1$:

$$S(k + 1) = S(k) + (k + 1)$$

By the inductive hypothesis, we have:

$$S(k + 1) = \frac{k(k+1)}{2} + (k + 1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Thus, the statement holds for $n = k + 1$. By the principle of mathematical induction, the statement holds for all $n \in \mathbb{N}$. □

5.4 Proof by Exhaustion

The only integer solutions to the equation $x^2 + y^2 = 1$ are $(0, 1), (1, 0), (0, -1), (-1, 0)$.

Proof. We will prove this theorem by exhaustion. We will check all possible integer values of x and y such that $x^2 + y^2 = 1$.

The possible integer values for x and y are $-1, 0, 1$. We will check each case:
- If $x = 0$: - Then $y^2 = 1$ gives $y = 1$ or $y = -1$. - Solutions: $(0, 1), (0, -1)$. - If $x = 1$: - Then $y^2 = 0$ gives $y = 0$. - Solution: $(1, 0)$. - If $x = -1$: - Then $y^2 = 0$ gives $y = 0$. - Solution: $(-1, 0)$.

Thus, the only integer solutions to the equation are:

$$(0, 1), (1, 0), (0, -1), (-1, 0)$$

$\square$

5.5 Proof by Cases

For any integer n, n^2 is even if and only if n is even.

Proof. We will prove this theorem by cases.

Case 1: Assume n is even. Then we can write $n = 2k$ for some integer k.

Now, we compute n^2:
$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

This shows that n^2 is even.

Case 2: Assume n is odd. Then we can write $n = 2k + 1$ for some integer k.

Now, we compute n^2:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

This shows that n^2 is odd.

Since both cases have been considered, we conclude that n^2 is even if and only if n is even. $\square$

5.6 Proof by Construction

There exists an irrational number x such that x^2 is rational.

Proof. We will construct an irrational number x such that x^2 is rational.

Let $x = \sqrt{2}$. We know that $\sqrt{2}$ is irrational. Now, we compute x^2:

$$x^2 = (\sqrt{2})^2 = 2$$

Since 2 is a rational number, we have constructed an irrational number $x = \sqrt{2}$ such that $x^2 = 2$ is rational. Therefore, the theorem is proved. $\square$

5.7 Proof by Counterexample

The statement "All prime numbers are odd" is false.

Proof. To prove this theorem, we will provide a counterexample.

The number 2 is a prime number, as its only divisors are 1 and 2. However, 2 is even, which contradicts the statement that all prime numbers are odd.

Therefore, the statement "All prime numbers are odd" is false. $\square$

5.8   Proof by Contrapositive

If n is an integer such that n^2 is odd, then n is odd.

Proof. We will prove this theorem by contrapositive. The contrapositive of the statement is: If n is an integer such that n is even, then n^2 is even.

Assume n is even. Then we can write $n = 2k$ for some integer k.

Now, we compute n^2:
$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

This shows that n^2 is even.

Since the contrapositive statement is true, the original statement "If n^2 is odd, then n is odd" is also true.   □

5.9   Proof by Reduction to Absurdity

The square root of 2 is irrational.

Proof. We will prove this theorem by reduction to absurdity. Assume that $\sqrt{2}$ is rational. Then we can write:
$$\sqrt{2} = \frac{p}{q}$$

where p and q are integers with no common factors (i.e., the fraction is in simplest form).

Squaring both sides gives:
$$2 = \frac{p^2}{q^2}$$

Rearranging gives:
$$p^2 = 2q^2$$

This implies that p^2 is even, and therefore p must be even (since the square of an odd number is odd). Let $p = 2k$ for some integer k. Substituting this back into the equation gives:
$$(2k)^2 = 2q^2$$
$$4k^2 = 2q^2$$
$$2k^2 = q^2$$

This implies that q^2 is even, and therefore q must also be even. Since both p and q are even, they have a common factor of 2, which contradicts our assumption that p and q have no common factors. Therefore, our assumption that $\sqrt{2}$ is rational must be false, and thus $\sqrt{2}$ is irrational.   □

5.10   Proof by Analogy

The set of rational numbers is dense in the set of real numbers.

Proof. We will prove this theorem by analogy.

Consider the set of rational numbers $\mathbb{Q}$ and the set of real numbers $\mathbb{R}$. The density of $\mathbb{Q}$ in $\mathbb{R}$ means that between any two real numbers, there exists a rational number.

For example, between the real numbers 1 and 2, we can find the rational number $\frac{3}{2} = 1.5$. Similarly, between any two real numbers a and b (where $a < b$), we can find a rational number $r = \frac{a+b}{2}$.

This shows that the set of rational numbers is dense in the set of real numbers. Therefore, the theorem is proved by analogy.   □

6 Means and Proofs

In math there are a lot of means. In this section I will show some of them with the correspoinding proofs.

- **Arithmetic Mean:** The arithmetic mean of n numbers $x_1, x_2, \ldots, x_n$ is given by:

$$A = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

- **Geometric Mean:** The geometric mean of n numbers $x_1, x_2, \ldots, x_n$ is given by:

$$G = \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$$

- **Harmonic Mean:** The harmonic mean of n numbers $x_1, x_2, \ldots, x_n$ is given by:

$$H = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

- **Quadratic Mean:** The quadratic mean (or root mean square) of n numbers $x_1, x_2, \ldots, x_n$ is given by:

$$Q = \sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}}$$

6.1 Proof of the Arithmetic Mean-Geometric Mean Inequality

Let $a_1, a_2, \ldots, a_n > 0$. We will prove by induction that:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Base Case: $n = 2$

We want to prove:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$$

Let $a_1, a_2 > 0$. Then by the identity

$$\left(\frac{a_1 - a_2}{2} \right)^2 \geq 0,$$

we get

$$\frac{a_1^2 - 2a_1 a_2 + a_2^2}{4} \geq 0 \Rightarrow a_1^2 + a_2^2 \geq 2a_1 a_2.$$

So,

$$(a_1 + a_2)^2 \geq 4a_1 a_2 \Rightarrow \left(\frac{a_1 + a_2}{2} \right)^2 \geq a_1 a_2,$$

and taking square roots gives the desired result:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}.$$

For $n \geq 2$

$$A_{n+1} := \left(\sum_{i=1}^{n+1} a_i \right)/(n+1) = \frac{a_1 + a_2 + \cdots + a_n + a_{n+1}}{n+1}$$

$$G_{n+1} := \sqrt[n+1]{a_1 a_2 \cdots a_n a_{n+1}} = \sqrt[n+1]{(a_1 a_2 \cdots a_n) a_{n+1}}$$

$$A_{n-1}^{n+1} = \left(\frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1} \right)^{n-1} = A_{n+1}^{n-1}$$

$$G_{n+1}^{n+1} := \left(\sqrt[n+1]{a_1 a_2 \cdots a_n a_{n+1}} \right)^{n+1} = (a_1 a_2 \cdots a_n) a_{n+1} = \sqrt[n]{(a_1 a_2 \cdots a_n)^n} a_{n+1}^{n+1} = G_n^n a_{n+1}^{n+1}$$

Then

$$G_{n+1}^{n+1} A_{n+1}^{n-1} = G_n^n a_{n+1}^{n+1} A_{n+1}^{n-1} \leq A_n^n A_{n+1}^{n-1}$$

This comes from

$$G_n \leq A_n$$

$$G_n^n \leq A_n^n$$

$$G_n^n a_{n+1} \leq A_n^n a_{n+1}$$

$$G_n^n a_{n+1}^{n+1} A_{n+1}^{n-1} \leq A_n^n A_{n+1}^{n-1} = \left(A_n^n \left(a_{n+1} A_{n+1} \right)^{n-1} \right)^{\frac{n}{n}}$$

$$\leq A_n^n \left(\frac{a_{n+1} + A_{n+1} + \dots A_{n+1}}{n} \right)^n$$

$$A_n^n \left(\frac{a_{n+1} + (n-1)A_{n+1}}{n} \right)^n$$

$$\left(A_n \frac{a_{n+1} + (n-1)A_{n+1}}{n} \right)^n$$

Note that

$$\left(A_n \frac{a_{n+1} + (n-1)A_{n+1}}{n} \right) \rightarrow \left(\sqrt{A_n \frac{a_{n+1} + (n-1)A_{n+1}}{n}} \right)^2$$

$$\leq \left(\frac{A_n + \frac{a_{n+1} + (n-1)A_{n+1}}{n}}{2} \right)^{2n}$$

Now with power of n we have

$$\left(A_n + \frac{a_{n+1} + (n-1)A_{n+1}}{n} \right)^n \leq \left(\frac{A_n + \frac{a_{n+1} + (n-1)A_{n+1}}{n}}{2} \right)^{2n}$$

$$= \left(\frac{A_n}{2} + \frac{a_{n+1} + (n-1)A_{n+1}}{2n} \right)^{2n}$$

$$= \left(\frac{A_n n}{2n} + \frac{a_{n+1} + (n-1)A_{n+1}}{2n} \right)^{2n}$$

$$= \left(\frac{A_n n + a_{n+1} + (n-1)A_{n+1}}{2n} \right)^{2n}$$

$$= \left(\frac{A_{n+1}(n+1) + (n-1)A_{n+1}}{2n} \right)^{2n}$$

$$= \left(\frac{2n A_{n+1}}{2n} \right)^{2n} = (A_{n+1})^{2n}$$

Now we have prooven that $G_{n+1}^{n+1} A_{n+1}^{n-1} \leq (A_{n+1})^{2n}$ by dividing both sides by A_{n+1}^{n-1} we get

$$G_{n+1}^{n+1} \leq A_{n+1}^{n+1}$$

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6.2 Proof of the Harmonic Mean Geometric Mean Inequality

Let $a_1, a_2, \ldots, a_n > 0$. We will prove the inequality.
We know that $G_n \leq A_n$

$$G_n \leq A_n$$

$$\sqrt[n]{x_1 \cdot x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}$$

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$$

This concludes the proof of the harmonic mean-geometric mean inequality.
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7 The Natural Numbers

In this section we will take a look at the natural numbers, which are the numbers we use for counting. The natural numbers are defined as follows: This not going not be a deep dive just a look at the axioms and the basic construction of the natural numbers. The natural numbers are defined as follows:

We will now define the set of natural numbers, $\mathbb{N}$, via the following 9 axioms. These axioms are known as the **Peano Axioms**. The first 4 axioms define equality on the set $\mathbb{N}$.

Axiom 1: For every $x \in \mathbb{N}$, we have $x = x$. (Reflexivity)

Axiom 2: For every $x, y \in \mathbb{N}$, if $x = y$ then $y = x$. (Symmetry)

Axiom 3: For every $x, y, z \in \mathbb{N}$, if $x = y$ and $y = z$ then $x = z$. (Transitivity)

Axiom 4: For all x, y, if $x \in \mathbb{N}$ and $x = y$, then $y \in \mathbb{N}$. (Closure of Equality)

The remaining 5 axioms define the structure of $\mathbb{N}$:

Axiom 5: $0 \in \mathbb{N}$

Axiom 6: If $x \in \mathbb{N}$, then the successor $S(x) \in \mathbb{N}$.

Axiom 7: There is no $x \in \mathbb{N}$ such that $S(x) = 0$.

Axiom 8: For all $x, y \in \mathbb{N}$, if $S(x) = S(y)$, then $x = y$.

Axiom 9: Let $P(x)$ be a statement about the natural number x. If:

- $P(0)$ is true, and
- for all $n \in \mathbb{N}$, if $P(n)$ is true, then $P(S(n))$ is also true,

then $P(x)$ is true for all $x \in \mathbb{N}$. (Mathematical Induction)

As shorthand, we denote:

$$S(0) = 1, \quad S(S(0)) = 2, \quad S(S(S(0))) = 3, \quad \text{and so on.}$$

7.1 Arithmetic on $\mathbb{N}$

We have defined the set $\mathbb{N}$, but we still have no way of working with its elements. In this section, we will formalize, using only the above 9 axioms, the two basic operations on natural numbers: addition and multiplication.

7.1.1 Addition

We define addition recursively:

$$a + 0 = a$$
$$a + S(b) = S(a + b)$$

for all $a, b \in \mathbb{N}$.

    Example: Show that $1 + 2 = 3$:

$$
\begin{aligned}
1 + 2 &= 1 + S(1) \\
&= S(1 + 1) \\
&= S(1 + S(0)) \\
&= S(S(1 + 0)) \\
&= S(S(1)) \\
&= S(2) \\
&= 3
\end{aligned}
$$

7.1.2 Multiplication

We define multiplication recursively in terms of repeated addition:

$$a \cdot 0 = 0$$
$$a \cdot S(b) = a + (a \cdot b)$$

for all $a, b \in \mathbb{N}$.

    Example: Show that $2 \cdot 2 = 4$:

$$
\begin{aligned}
2 \cdot 2 &= 2 \cdot S(1) \\
&= 2 + (2 \cdot 1) \\
&= 2 + (2 \cdot S(0)) \\
&= 2 + (2 + (2 \cdot 0)) \\
&= 2 + (2 + 0) \\
&= 2 + 2 \\
&= 2 + S(1) \\
&= S(2 + 1) \\
&= S(2 + S(0)) \\
&= S(S(2 + 0)) \\
&= S(S(2)) \\
&= S(3) \\
&= 4
\end{aligned}
$$

7.2 Ordering the Natural Numbers

For any $a, b \in \mathbb{N}$, we say $a \leq b$ if there exists some $s \in \mathbb{N}$ such that $a + s = b$. If such $s \neq 0$, then $a < b$.

8 The Archimedean Principle

The Archimedean Principle is a fundamental property of the real numbers $\mathbb{R}$, and it essentially states that the natural numbers are unbounded in $\mathbb{R}$.

8.1 Statement

For any real number $x \in \mathbb{R}$, there exists a natural number $n \in \mathbb{N}$ such that $n > x$.

In other words, no matter how large a real number you choose, there is always a natural number that is larger. Similarly, for any positive real number $\epsilon > 0$, there exists a natural number $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

8.2 Proof

We prove the Archimedean Principle by contradiction.

Assume that there exists some real number $x \in \mathbb{R}$ such that $n \leq x$ for all $n \in \mathbb{N}$. That is, x is an upper bound for the set $\mathbb{N} \subset \mathbb{R}$.

Let $S = \sup(\mathbb{N})$, the least upper bound of $\mathbb{N}$. Then $S - 1 < \sup(\mathbb{N})$, so $S - 1$ is not an upper bound of $\mathbb{N}$. Hence, there exists $n_0 \in \mathbb{N}$ such that:

$$n_0 > S - 1 \Rightarrow n_0 + 1 > S$$

But $n_0 + 1 \in \mathbb{N}$, which contradicts the assumption that S is an upper bound of $\mathbb{N}$. Therefore, our assumption must be false, and the theorem is proven.

$\square$

8.3 Equivalent Formulations

The Archimedean Principle is often stated in different but equivalent ways:

- For any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

- For any $a, b \in \mathbb{R}$ with $a > 0$, there exists $n \in \mathbb{N}$ such that $na > b$.

8.4 Applications

1. **Density of Rational Numbers:** The Archimedean Principle helps in proving that between any two real numbers, there exists a rational number.

2. **Limits and Infinitesimals:** It ensures that sequences like $\left\{ \frac{1}{n} \right\}$ converge to 0, foundational in real analysis and calculus.

3. **Bounding Functions:** It is used in analysis to show that functions do not grow faster than natural numbers in certain contexts.

4. **Non-Existence of Infinitely Small Numbers:** The principle implies that real numbers do not contain infinitesimals (nonzero numbers smaller than all $\frac{1}{n}$), distinguishing $\mathbb{R}$ from non-standard number systems.

9 Fundamental Theorem of Arithmetic

- The Fundamental Theorem of Arithmetic says that every integer greater than 1 can be factored uniquely into a product of primes.

- Euclid's lemma says that if a prime divides a product of two numbers, it must divide at least one of the numbers.

- The least common multiple $[a, b]$ of nonzero integers a and b is the smallest positive integer divisible by both a and b.

Theorem. (Fundamental Theorem of Arithmetic) Every integer greater than 1 can be written in the form

$$p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$$

where $n_i \geq 0$ and the p_i are distinct primes. The factorization is unique, except possibly for the order of the factors.

Example.

$$4312 = 2 \cdot 2156 = 2 \cdot 2 \cdot 1078 = 2 \cdot 2 \cdot 2 \cdot 539 = 2 \cdot 2 \cdot 2 \cdot 7 \cdot 77 = 2 \cdot 2 \cdot 2 \cdot 7 \cdot 7 \cdot 11$$

That is,

$$4312 = 2^3 \cdot 7^2 \cdot 11$$

9.1 Lemmas

Lemma. If $m \mid pq$ and $\gcd(m, p) = 1$, then $m \mid q$.

Proof. Write $1 = \gcd(m, p) = am + bp$ for some $a, b \in \mathbb{Z}$. Then

$$q = amq + bpq$$

Since $m \mid amq$ and $m \mid bpq$ (because $m \mid pq$), we conclude $m \mid q$.

Lemma. If p is prime and $p \mid a_1 a_2 \cdots a_n$, then $p \mid a_i$ for some i.

Proof. (Case $n = 2$): Suppose $p \mid a_1 a_2$, and $p \nmid a_1$. Then $\gcd(p, a_1) = 1$, and by the previous lemma, $p \mid a_2$.

For general $n > 2$: Assume the result is true for $n - 1$. Suppose $p \mid a_1 a_2 \cdots a_n$. Group as $(a_1 a_2 \cdots a_{n-1}) a_n$.

By the $n = 2$ case, either $p \mid a_n$ or $p \mid a_1 a_2 \cdots a_{n-1}$, and by induction, $p \mid a_i$ for some i.

9.2 Proof of the Fundamental Theorem

Existence: Use induction on $n > 1$. Base case: $n = 2$ is prime.

Inductive step: If n is prime, done. Otherwise $n = ab$, with $1 < a, b < n$. By induction, both a and b factor into primes, so n does too.

Uniqueness: Suppose:

$$p_1^{m_1} \cdots p_j^{m_j} = q_1^{n_1} \cdots q_k^{n_k}$$

with all p_i and q_i distinct primes.

Since p_1 divides the LHS, it divides the RHS. So $p_1 \mid q_i^{n_i}$ for some i, hence $p_1 = q_i$. Reorder so $p_1 = q_1$. Then:

If $m_1 > n_1$, divide both sides by $q_1^{n_1}$:

$$p_1^{m_1 - n_1} \cdots p_j^{m_j} = q_2^{n_2} \cdots q_k^{n_k}$$

But then p_1 divides LHS but not RHS, contradiction. So $m_1 = n_1$. Cancel and repeat.

Eventually, all p_i match with some q_i, and the exponents are equal. So the factorizations are the same up to order.

9.3   Least Common Multiple

The least common multiple of a and b, denoted $[a, b]$, is the smallest positive integer divisible by both.

Example:
$$[6, 4] = 12, \quad [33, 15] = 165$$

Fact:
$$[a, b] \cdot \gcd(a, b) = ab$$

Let:
$$a = p_1 \cdots p_l q_1 \cdots q_m, \quad b = q_1 \cdots q_m r_1 \cdots r_n$$

Then:

$$\gcd(a, b) = q_1 \cdots q_m$$
$$[a, b] = p_1 \cdots p_l q_1 \cdots q_m r_1 \cdots r_n$$
$$ab = p_1 \cdots p_l q_1^2 \cdots q_m^2 r_1 \cdots r_n$$

So:
$$[a, b] \cdot \gcd(a, b) = ab$$

Example:
$$\gcd(36, 90) = 18, \quad [36, 90] = 180, \quad 36 \cdot 90 = 32400 = 18 \cdot 180$$

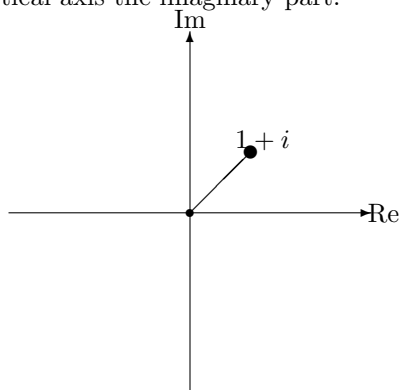# 10   Complex Numbers

What are Complex Numbers?

A complex number is a number of the form:

$$z = a + bi,$$

where $a, b \in \mathbb{R}$, and i is the imaginary unit defined by $i^2 = -1$. The set of all complex numbers is denoted by $\mathbb{C}$.

The Complex Plane

Complex numbers can be represented graphically in the **complex plane**, where the horizontal axis represents the real part and the vertical axis the imaginary part.

The point $1 + i$ is located at (1,1), showing 1 unit on the real axis and 1 unit on the imaginary axis.

Operations in Cartesian Coordinates

Let $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers.

- **Addition:**
$$z_1 + z_2 = (a + c) + (b + d)i$$

- **Multiplication:**
$$z_1 \cdot z_2 = (ac - bd) + (ad + bc)i$$

- **Quotient:**
$$\frac{z_1}{z_2} = \frac{(a + bi)}{(c + di)} \frac{(c - di)}{(c - di)} = \frac{(a + bi)(c - di)}{c^2 + d^2} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

Polar Coordinates

A complex number can also be expressed in polar form as:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta},$$

where:

$$r = |z| = \sqrt{a^2 + b^2} \quad \text{(modulus)}$$

$$\theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right) \quad \text{(argument)}$$

Important: The value of θ depends on the quadrant where the complex number lies:

- Quadrant I: $a > 0, b > 0$ — use $\tan^{-1}(b/a)$

- Quadrant II: $a < 0, b > 0$ — add π to $\tan^{-1}(b/a)$

- Quadrant III: $a < 0, b < 0$ — add π to $\tan^{-1}(b/a)$

- Quadrant IV: $a > 0, b < 0$ — use $\tan^{-1}(b/a)$

Multiplication and Division in Polar Coordinates

Given:
$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2},$$

- **Multiplication:**
$$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

- **Division:**
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

This polar form is especially useful in simplifying powers and roots of complex numbers using De Moivre's Theorem.

Exponentiation and Roots (De Moivre's Theorem)

Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ be a complex number in polar form.

Exponentiation

To raise z to the power $n \in \mathbb{N}$, we use De Moivre's Theorem:
$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)) = r^n e^{in\theta}$$

Roots of Complex Numbers

To find the nth roots of a complex number $z = re^{i\theta}$, we use the formula:
$$z^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \quad k = 0, 1, \ldots, n-1$$

This yields n distinct roots, each separated by an angle of $\frac{2\pi}{n}$ in the complex plane.

Example: Solve $z^4 = 1 + \sqrt{3}i$

Step 1: Convert RHS to polar form.
Let $w = 1 + \sqrt{3}i$. Real part: $a = 1$, Imaginary part: $b = \sqrt{3}$

$$r = |w| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = 2$$

$$\theta = \arg(w) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

So,
$$w = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$$

Step 2: Solve $z^4 = w \Rightarrow z = w^{1/4}$
Using the root formula:
$$z_k = 2^{1/4}\left(\cos\left(\frac{\pi + 2k\pi}{12}\right) + i\sin\left(\frac{\pi + 2k\pi}{12}\right)\right), \quad k = 0, 1, 2, 3$$

So the four roots are:
$$z_0 = 2^{1/4}\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right)$$
$$z_1 = 2^{1/4}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right)$$
$$z_2 = 2^{1/4}\left(\cos\left(\frac{9\pi}{12}\right) + i\sin\left(\frac{9\pi}{12}\right)\right) = 2^{1/4}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$
$$z_3 = 2^{1/4}\left(\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right)$$

These represent the four complex 4th roots of $1 + \sqrt{3}i$, equally spaced around the circle of radius $2^{1/4}$ in the complex plane.