Representation Theory of Finite Groups

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Contents

I. REPLACE

Let G be a finite group of order n, and write $G = \{g_1, \ldots, g_n\}$. Fix $g \in G$; then $gg_i = gg_j$ if and only if $i = j$. Thus there exists some $\sigma_g \in S_i$ such that $gg_i = g_{\sigma_g(i)}$ for all $i \in \{1, 2, \ldots, n\}$. In particular, $\phi : G \to S_n$ by $\phi(g) = \sigma_g$ is an embedding (injective group homomorphism). This observation is usually referred to as Cayley's Theorem.

Now let V be an n–dimensional complex vector space. We then denote $\mathrm{GL}(V)$ as the group of invertible linear operators $T : V \to V$. Now define $\psi : S_n \to \mathrm{GL}_n(V)$ by $\psi(\sigma) = T_\sigma$ where if $\{b_1, \ldots, b_n\}$ is a basis for V and $T_\sigma(b_i) = b_{\sigma(i)}$. This is an injective group homomorphism, so $\psi \circ \phi : G \to \mathrm{GL}(V)$ is an embedding of G into $\mathrm{GL}(V)$.

Definition. Let G be a finite group, and V a finite dimensional $\mathbb{C}$–vector space. A **representation** of G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. We call $\dim(V)$ the **degree** of the representation.

In particular, if V is n–dimensional, then $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$.

Example. Consider $\rho : G \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$ given by $\rho(g) = 1$ for all $g \in G$. This is called the *trivial representation*.

Example. Consider $\rho : S_n \to \mathbb{C}^\times$ given by $\rho(\sigma) = \mathrm{sgn}(\sigma)$, which is called the *sign representation*.

Example. The representation fo G afforded by Cayley's theorem is called the *regular representation* of G. Here is a good way to understand the regular rep of G.

Consider $X = \{v_g : g \in G\}$ be a set of symbols, and let $V = \mathrm{Free}(X)$. Then define $\rho : G \to \mathrm{GL}(V)$ given by $\rho(g)(v_h) = v_{gh}$ for all $g, h \in G$.

Example. Consider G, $X = \{x_1, \ldots, x_n\}$, and $V = \mathrm{Free}(X)$. Suppose G acts on X. Then $\rho : G \to \mathrm{GL}(V)$ given by $\rho(g)(x_i) = gx_i$.