REPLACE

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Contents

I. REPLACE

1. For $a,b,k \in \mathbb{N}$,
$$\binom{a+b}{k} = \sum_{j=1}^{k} \binom{a}{j} \cdot \binom{b}{k-j} \tag{0.1}$$
We prove this with a bijection:
$$\mathcal{B}(a+b,k) \leftrightarrows \bigcup_{j=0}^{k} \mathcal{B}(a,j) \times \mathcal{B}(b,k-j)$$
given by $S \mapsto (S \cap \{1,\ldots,a\}, (S \cap \{a+1,\ldots,a+b\})^{(-a)}$ and $(P,Q) \mapsto P \cup Q^{(a)}$, where $\mathcal{B}(n,i)$ is the set of i–element subsets of $\{1,2,\ldots,n\}$ and for $C \subseteq \mathbb{Z}$ and $q \in \mathbb{Z}$, $C^{(q)} = \{c+q : c \in C\}$. Note that the equation in fact gives the polynomial identity
$$\binom{x+y}{k} = \sum_{j=0}^{k} \binom{x}{j}\binom{y}{k-j}$$
in $\mathbb{Q}[x,y]$. We denote the falling factorial $(x)_i = x(x-1)(x-2)\cdots(x-i+1)$, which has degree i for each $i \in \mathbb{N}$. In particular, $(x)_i = i!\binom{x}{i}$, so multiplying our identity by $k!$, we get
$$(x+y)_k = \sum_{j=0}^{k} \binom{k}{j} (x)_j (y)_{k-j}$$
Compare this with the standard binomial theorem
$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}$$
These are called sequences of binomial type.

2. Here's another identity. For $n \geq 0$ and $s,t \geq 1$,
$$\binom{n+s+t-1}{s+t-1} = \sum_{k=0}^{n} \binom{k+s-1}{s-1}\binom{n-k+t-1}{t-1}$$
Let $\mathcal{M}(m,r)$ denote a multiset of size m with elements of r types, so that $|\mathcal{M}(m,r)| = \binom{m+r-1}{r-1}$. Let's define a bijection
$$\mathcal{M}(n,s+t) \rightleftharpoons \bigcup_{k=1}^{n} \mathcal{M}(k,s) \times \mathcal{M}(n-k,t) \tag{0.2}$$

$\mu = (m_1,\ldots,m_{s+t}) \mapsto ((m_1,\ldots,m_s),(m_{s+1},\ldots,m_{s+t}))$ and $(\nu,\theta) \mapsto \nu\theta$. Note that if f,g are polynomials of degree d and e respectively, then $\sum_{k=0}^{n} f(k)g(n-k)$ is a polynomial in n of degree $d+e-1$.

Is there some way to understand (0.2)? It is unclear, with our known techniques, that this corresponds to a polynomial identity since there is a variable n in the exponent. However, we can use generating functions. Define

$$\begin{aligned}
\sum_{n=0}^{\infty}\binom{n+s+t-1}{s+t-1}z^n = \sum_{n=0}^{\infty}|\mathcal{M}(n,s+t)|z^n &= \sum_{(m_1,\ldots,m_{s+t})} z^{m_1+\cdots+m_{s+t}} \\
&= \left(\sum_{m=0}^{\infty} z^m\right)^{s+t} \\
&= \frac{1}{(1-z)^{s+t}} = \frac{1}{(1-z)^s}\frac{1}{(1-z)^t} \\
&= \sum_{k=0}^{\infty}\binom{k+s-1}{s-1}z^k \sum_{\ell=0}^{\infty}\binom{\ell+t-1}{t-1}z^\ell \\
&= \sum_{n=0}^{\infty} z^n\left(\sum_{k=0}^{n}\binom{k+s-1}{s-1}\binom{n-k+t-1}{t-1}\right)
\end{aligned}$$

Similarly, (0.1) is equivalent to saying $(1+z)^{a+b} = (1+z)^a(1+z)^b$. Note that $(1+z)^n = \sum_{k=0}^{n}\binom{n}{k}z^k = \sum_{k=0}^{\infty}\binom{n}{k}z^k$ for $n \in \mathbb{N}$.

Can we substitute $\frac{1}{(1-q)^t} = (1+z)^n$ where $z=-q$ and $n=-t$?

3. Consider

$$(x_1+x_2)^n = \sum_{i=0}^{n}\binom{n}{i}x_1^i x_2^{n-i}$$

and

$$(x_1+x_2)^n = \sum_{f:N_n\to\{1,2\}}\prod_{j=1}^{n} x_{f(j)}$$

More generally, we can consider

$$(x_1+\cdots+x_k)^n = \sum_{f:N_n\to N_k}\prod_{j\in N_n} x_{f(j)}$$

If we set all $x_1=\cdots=x_k=1$, then k^n gives the number of functions from N_n to N_k. If we set $x_i = q^i$ for all $i \in N_k$, then we get

$$\left(\frac{q-q^{k+1}}{1-q}\right)^n = (q+q^2+\cdots+q^k)^n = \sum_{f:N_n\to N_k} q^{f(1)+\cdots+f(k)}$$

Collect all the terms in $(x_1+\cdots+x_k)^n$ that produce the same monomial. Given a multiset μ with $m_1+\cdots+m_k=n$, write $x_1^{m_1}\cdots x_k^{m_k} = \underline{x}^\mu$. Then

$$(x_1+\cdots+x_k)^n = \frac{n!}{m_1!\cdots m_k!}\underline{x}^\mu = \sum_{\mu\in\mathcal{M}(n,t)}\binom{n}{\mu}\underline{x}^\mu$$

4. How can we interpret

$$P_n(q) = \prod_{i=1}^{n}(1+q+q^2+\cdots+q^{i-1})$$

In general, if we set $q=1$, we see that $P_n(1)=n!$. We might hope that there is some weight function on permutations $w:\mathcal{S}_n\to\mathbb{N}$ such that $P_n(q)=\sum_{\sigma\in\mathcal{S}_n} q^{w(\sigma)}$. Recall the bijection $I_n:\mathcal{S}_n\to\mathcal{Q}_n$ from chapter 1. Let's find some weight function $\nu:\mathcal{Q}_n\to\mathbb{N}$ such that $\sum_{\rho\in\mathcal{Q}_n} x^{\nu(\rho)}=P_n(q)$, then "pull back" the definition of $\nu:\mathcal{Q}_n\to\mathbb{N}$ to get a definition for $\omega:\mathcal{S}_n\to\mathbb{N}$. Note that $\sum_{h\in N_r} q^{h-1}=1+q+\cdots+q^{r-1}$. Thus

$$\sum_{\rho=(h_1,\ldots,h_n)\in\mathcal{Q}_n} q^{(h_1-1)+(h_2-1)+\cdots+(h_n-1)} = \prod_{i=1}^{n}(1+q+\cdots+q^{i-1}) = P_n(q)$$

so we can define $\nu(\rho)=|\rho|-n$ and $\sum_{q\in\mathcal{Q}_n} q^{|\rho|-n}=P_n(q)$. We also have

$$\sum_{\rho\in\mathcal{Q}_n} q^{(h_1-1)+\cdots+(h_n-1)} = (1+q+\cdots+q^{n-1})(1+q+\cdots+q^{n-2})\cdots(1+q)(1)$$

For notation, define $[m]_q=1+q+\cdots+q^{m-1}=\frac{1-q^m}{1-q}$. Then $[m]_q!=[m]_q[m-1]_q\cdots[1]_q$.

	1	q	q^2	q^3	q^4	
$q[3]_q$	0	1	1	1		
$[2]_q[3]_q$	1	2	2	1		
$-q[2]_q[3]_q$	0	−1	−2	−2	−1	
$q^2[2]_q[3]_q$	0	0	1	2	2	1
$[6]_q$	1	1	1	1	1	1

so that $[6]_q=(1-q+q^2)[2]_q[3]_q$. An **inversion** in $\sigma=a_1\ldots a_n\in\mathcal{S}_n$ is a pair (i,j) of indices $1\le i<j\le n$ with $a_i>a)j$. Define $\mathrm{Inv}(\sigma)$ as the set of inversions of σ, and $\mathrm{inv}(\sigma)=|\mathrm{Inv}(\sigma)|$. Notice that if $\sigma=a_1\ldots a_n\mapsto\rho=(h_1,\ldots,h_n)$, then for each $1\le i\le n$, h_i-1 is the number of inversions of σ with i in the first coordinate. Recall

$$\begin{aligned}\mathcal{S}_n &\leftrightarrows \mathcal{B}(n,k)\times\mathcal{S}_k\times\mathcal{S}_{n-k}\\ \sigma=a_1\ldots a_n &\leftrightarrow (A,\beta,\gamma)\\ \mathrm{inv}(\sigma) &= w(A)+\mathrm{inv}(\beta)+\mathrm{inv}(\gamma)\end{aligned}$$

Assuming such a weight fuction $w(A)$ exists, then

$$\begin{aligned}[n]!_q=\sum_{\sigma\in\mathcal{S}_n} q^{\mathrm{inv}(\sigma)} &= \sum_{(A,\beta,\gamma)} q^{w(A)+\mathrm{inv}(\beta)+\mathrm{inv}(\gamma)}\\ &= [k]!_q\cdot[n-k]!_q\cdot\sum_{A\in\mathcal{B}(n,k)} q^{w(A)}\end{aligned}$$

so that

$$\sum_{A\in\mathcal{B}(n,k)} q^{w(A)} = \frac{[n]!_q}{[k]!_q\cdot[n-k]!_q} = \begin{bmatrix} n\\ k\end{bmatrix}_q$$

$$\sum_{S\in\mathcal{B}(n,k)} q^{\mathrm{sum}(S)} = q^{\binom{k+1}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q$$

0.1 Theorem. *Let V be an n–dimensional vector space over a finite field $\mathbb{F}_q$. Then for $0 \le k \le n$, the number of k–dimensional subspaces of V is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

0.2 Lemma. *Let $L : V \to W$ be a linear transformation that is surjective. Then* $\dim V = \dim W + \dim(\ker L)$. *So if this is over a finite field $\mathbb{R}_q$, every $w \in W$ is the image of exactly $q^{\dim(\ker(L)}$ vectors $v \in V$.*

For every $w \in W$, is the image of exactly q^k vectors in V. The number of ordered bases of V is $q^{\binom{n}{2}}(q-1)^n[n]!_q$.

0.3 Theorem. *Let V be an n–dimensional vector space over a finite field $\mathbb{F}_q$. For $0 \le k \le n$, the number of k–dimensional subspaces of V is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

Proof Let $\mathrm{OB}(V)$ be the set of ordered bases of V, and let $G(V,k)$ be the set of k–dimensional subspaces of V. Define a function

$$\mathrm{OB}(V) \to \bigcup_{U\in G(V,k)} (\{U\}\times \mathrm{OB}(U)\times \mathrm{OB}(V/U))$$

as follows. Given $(v_1,\ldots,v_n)$ an ordered basis of V, let $U = \mathrm{span}_{\mathbb{F}_q}\{v_1,\ldots,v_k\}$. Then $(v_1,\ldots,v_k) \in \mathrm{OB}(U)$ and $(v_{k+1}+U,\ldots,v_n+U) \in \mathrm{OB}(V/U)$. Consider the map $L : V \to V/U$ given by $L(v) = v+U$, so that every $v+U$ in V/U is the image of q^k vectors in V. Thus $(v_{k-1}+U,\ldots,v_n+U)$ is the image of $q^{k(n-k)}$ sequences $(z_{k+1},\ldots,z_n)$ of vectors in V. Thus the function $(v_1,\ldots,v_n) \mapsto (U,(v_1,\ldots,v_k),(v_{k+1}+U,\ldots,v_n+U))$ is surjective and hits everything on the RHS $q^{k(n-k)}$ times. But then counting both sides,

$$\begin{aligned} q^{\binom{n}{2}}(q-1)^n[n]!_q &= \sum_{U\in G(V,k)} 1\cdot q^{\binom{k}{2}}(q-1)^k[k]!_q\cdot q^{\binom{n-k}{2}}(q-1)^{n-k}[n-k]!_q\cdot q^{k(n-k)} \\ q^{\binom{n}{2}}[n]!_q &= |G(V,k)|\cdot[k]!_q\cdot[n-k]!_q q^{\binom{k}{2}+\binom{n-k}{2}+k(n-k)} \\ [n]!_q &= |G(V,k)|\cdot[k]!_q\cdot[n-k]!_q \end{aligned}$$

giving our desired result. ■

A **set partition** π of a set V is a collection of subsets $\pi = \{B_1,\ldots,B_k\}$ of V such that
- Each B_i is not empty
- $B_i \cap B_j = \emptyset$ if $i \ne j$
- $B_1 \cup \cdots \cup B_k = V$

Let $\Pi(n,k)$ be the set of set partitions of N_n with k blocks, and set $S(n,k) = |\Pi(n,k)|$. Certainly $S(0,0) = 1$ for the empty set partition. If $n \ge 1$, then $S(n,0) = 0$, $S(n,n) = 1$, and $S(n,1) = 1$. We can also define a recurrence relation. Let $\Pi'(n,k)$ be those $\pi \in \Pi(n,k)$ in which $\{n\}$ is a block, and $\Pi''(n,k)$ is the set of π in which n is in a block of size at least 2. Note that $\Pi'(n,k) \leftrightarrows \Pi(n-1,k-1)$ by removing or adding the independent element.

Furthemore, the function which removes the element n from a block in $\Pi''(n,k)$ is a surjective function onto $\Pi(n-1,k)$ which hits every element of $\Pi(n-1,k)$ k times. Thus combing these observations, $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$. Thus we can compute

$S(n,k)$	0	1	2	3	4	5	6
0	1	X	X	X	X	X	X
1	0	1	X	X	X	X	X
2	0	1	1	X	X	X	X
3	0	1	3	1	X	X	X
4	0	1	7	6	1	X	X
5	0	1	15	25	10	1	X
6	0	1	31		1		

From homework 2, we have that

$$x^n = \sum_{k=0}^{n} k! S(n,k) \binom{n}{k}$$

Invert this using Binomial Inversion.

0.4 Theorem. (Binomial Inversion) *Let $a_0, a_1, \ldots$ be a sequence.*

Proof For $h \in \mathbb{N}$, let $b_h = \sum_{i=0}^{h} \binom{h}{i} a_i$. Let $A(t) = \sum_{i=0}^{\infty} a_i t^i$ and $B(t) = \sum_{h=0}^{\infty} b_h t^h$. Then

$$\begin{aligned}
B(t) &= \sum_{h=0}^{\infty} t^h \sum_{i=0}^{h} \binom{h}{i} a_i \\
&= \sum_{i=0}^{\infty} a_i t^i \sum_{h=i}^{\infty} \binom{h}{h-i} t^{h-i} \\
&= \sum_{i=0}^{\infty} a_i t^i \sum_{j=0}^{\infty} \binom{i+j}{j} t^j = \sum_{i=0}^{\infty} \frac{a_i t^i}{(1-t)^{i+1})} \\
&= \frac{1}{1-t} \sum_{i=0}^{\infty} a_i \left(\frac{t}{1-t}\right)^i = \frac{1}{1-t} A\left(\frac{t}{1-t}\right)
\end{aligned}$$

Let $z = t/(1-t)$, so that $t = z/(1+z)$. Thus

$$B\left(\frac{z}{1+z}\right) = (1+z)A(z)$$

so that

$$\begin{aligned}
\sum_{i=0}^{\infty} a_i z^i = \frac{1}{1+z} B\left(\frac{z}{1+z}\right) &= \sum_{h=0}^{\infty} b_h \frac{z^h}{(1+z)^{h+1}} \\
&= \sum_{h=0}^{\infty} b_h \sum_{j=0}^{\infty} \binom{j+h}{h} z^h (-z)^j \\
&= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{n-j} b_j
\end{aligned}$$

Thus for all $m \in \mathbb{N}$, $a_m = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} b_j$. ■

In particular, applying this to Stirling numbers of the second kind, for all $n \in \mathbb{N}$ in $\mathbb{R}[x]$, we have

$$x^n = \sum_{k=0}^{n} S(n,k)\binom{x}{k}k!$$

Let $b_i = i^n$ for $i = 0, 1, 2, \ldots$. If $k > n$ or $k > i$, then $S(n,k)\binom{i}{k} = 0$; thus,

$$\begin{aligned} i^n &= \sum_{k=0}^{n} S(,nk)\binom{i}{k}k! = \sum_{k=0}^{\min(n,i)} S(n,k)\binom{i}{k}k! = \sum_{k=0}^{i} S(n,k)\binom{i}{k}k! \\ &= \sum_{k=0}^{i}\binom{i}{k}a_k \end{aligned}$$

where $a_k = k!S(n,k)$ for all $k \in \mathbb{N}$. But then apply binomial inversion to get

$$a_k = \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}b_j$$

Suppose $m^n = \sum_{k=0}^{n} S(n,k)\binom{m}{k}k!$. Then $[m]_q^n = \sum_{k=0}^{\infty} S[n,k]_q \begin{bmatrix} m \\ k \end{bmatrix}_q [k]!_q$, where $S[n,k]_q = \sum_{\pi\in\Pi(n,k)} q^{w(\pi)}$. Is there some function $w : \Pi(n,k) \to \mathbb{N}$ that makes this work?

II. Power Series Identities

(i) $\dfrac{1}{(1-z)^h} = \displaystyle\sum_{k=0}^{\infty} \binom{k+h-1}{h-1} z^k$

(ii) Let $a_0, a_1, \dots$ be a sequence, and $b_h = \sum_{i=0}^{h} \binom{h}{i} a_i$. Then $a_m = \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} b_i$.

(iii) General Binomial Series. For $k \in \mathbb{N}$, let $\binom{y}{k} = \frac{y(y-1)\cdots(y-k+1)}{k!} \in \mathbb{Q}[y]$. Then we define

$$(1+x)^y = \sum_{k=0}^{\infty} \binom{y}{k} x^k$$

which is a power series in x. Each coefficient of $[x^n]$ is in $\mathbb{Q}[y]$. Then by Vandermone convolution,

$$\begin{aligned}
(1+x)^y (1+x)^z &= \sum_{i=0}^{\infty} \binom{y}{i} x^i \sum_{j=0}^{\infty} \binom{z}{j} x^j \\
&= \sum_{n=0}^{\infty} x^n \left(\sum_{i=0}^{n} \binom{y}{i} \binom{z}{n-i} \right) \\
&= \sum_{n=0}^{\infty} \binom{y+z}{n} x^n = (1+x)^{y+z}
\end{aligned}$$

Furthermore, if $y = -p < 0$ is an integer, then

$$\begin{aligned}
(1+x)^{-p} &= \sum_{k=0}^{\infty} \binom{-p}{k} x^k \\
&= \sum_{k=0}^{\infty} \binom{k+p-1}{p-1}
\end{aligned}$$

For $\alpha \in \mathbb{C}$, $f(x) = (1+x)^{\alpha}$ is analytic for $|x| < 1$. In particular, by Taylor's theorem,

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} c_k x^k$$

where $c_k = \frac{1}{k!} \frac{d^k}{dx^k} (1+x)^{\alpha} |_{x=0}$.