REPLACE

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I. Graph Colourings

1 List Colourings

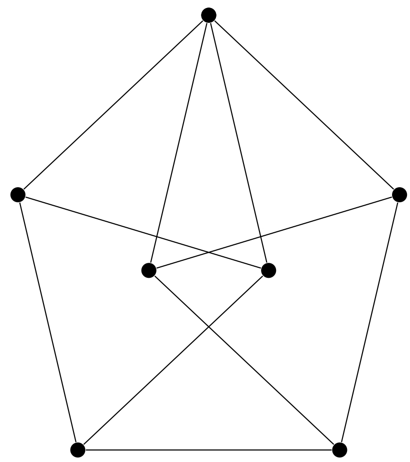
Recall that a colouring of a graph G is an assignment to each $v \in V(G)$ an element $c(v)$ of some set C called "colors" such that if v and v' are neighbours, then $c(v) \neq c(v')$. Then the **chromatic number** $\chi(G)$ is the smallest cardinality $|C|$ such that there exists a colouring of G from C.

There are some basic upper bounds on the chromatic number of a graph:

1. $\chi(G) \leq |V(G)|$, by colouring every vertex distinctly
2. $\chi(G) \leq \Delta(G) + 1$, by randomly colouring the graph based on colours not used on the neighbours

Note that these upper bounds are in fact tight; for example, the complete graph is tight for both, and an odd cycle is tight for (2).

There are some graphs for which the chromatic number is not known: consider the graph given by $V(G) = \mathbb{R}^2$ where vertices are adjacent if they have euclidean distance 1. This graph is not 3−colorable, by taking for example the subgraph



Recently there was a construction showing that the graph is not 4−colourable, and there is an easy upper bound of 7, so that $5 \leq \chi(G) \leq 7$.

We also define the notion of a list colouring:

Definition. A list assignment is an assignment of a set $L(v)$ of colors to each vertex v. Then a graph is k−list-colorable if you can always colour $V(G)$ whenever every vertex has a list of size at least k.

Note that $\chi(G) \leq \chi_\ell(G)$ since asssigning an identical list of size k is a valid list assignmnet and yields a standard coloring. In many cases list colorings can be hard to determine, but in some cases the exact value is known. Consider the complete bipartite graph $K_{k,q}$ where $q \geq k$. We then have the following classification:

1.1 Proposition. *$\chi_\ell(K_{k,q}) \leq k$ if and only if $q < k^k$, and $\chi_\ell(K_{k,q}) = k+1$ if and only if $q \geq k^k$.*

Proof Note that $\chi_\ell(K_{k,q}) \leq k+1$ always works by taking arbitrary colors on the k–side, and on the q–side, since the lists have size k, there is always a distinct color.

Now $q < k^k$. Try to color the k vertices such that two vertices have the same color. If this works, then for every list of size k on the q–side, there are only $k-1$ disallowed colours, so we may choose a valid color from the corresponding list. Otherwise, every vertex on the k–side has a distinct color; this is forced precisely when all the lists are disjoint. But then since $q < k^k$, there must be some selection of colors from the lists on the k–side such that the set of colors is distinct from every list on the q–side, and we may choose colors from the q–side without issue.

Otherwise if $q \geq k^k$, consider lists given by disjoint sets on the k–side, and then for every possible assignment of colors on the k–side, give a corresponding list for some vertex of the q–side that contains a list with those colors. Since $q \geq k^k$, we will exhaust all possibilities, so there is no valid coloring from those lists. ■

Recall that a planar graph G is one for which there exists an embedding of G into the plane such that each edge is a disjoint curve. Note that it suffices to consider edges which are polygonal curves, which consist of a finite number of straight line segments; in fact we can also do it with straight line segments (requiring that the graph is simple).

1.2 Theorem. (Thomassen) *If G is planar, then $(G) \leq 5$.*

In fact, we prove a stronger statement. We call an "almost-triangulation" a planar drawing in which every face except possibly the infinite face is a triangle. We prove this: let w be a given almost-triangulation with lists of available colour $L(v)$ assigned to every vertex v such that

1. $|L(v)| = 5$ for all vertices that are not on the infinite face,
2. two neighbouring vertices of the infinite face, a and b are colored distinctly,
3. and all other vertices of the infinite face have lists of 3 colours.

Then this almost-triangulation has a proper list colouring with respect to the given lists.

This implies the theorem since any planar drawing can be made an almost-triangulation by adding edges, and 5-element lists can be reduced to lists of the size above.

Proof We consider two cases in an induction proof.

1. There is a "long diagonal" connecting two of the vertices of the infinite face (that is not an edge of the infinite face).
2. There is no long diagonal.

The induction is on the number of vertices. When $n = 1, 2$ it is trivial, and when $n = 3$ it is a 3-cycle and it is certainly fine.

Now for the induction step, we have the two cases.

1. Cut the graph along the long diagonal to get G_1, G_2. Without loss of generality, G_1 is exactly as described in the statement, so it can be properly list coloured from the given lists. Then give the endpoints of the copied long diagonal in G_2 so that the endpoint colours are fixed; and by induction, colour it as well. Since the endpoints have the same colouring, we can put the two coloured graphs back together to obtain a proper list colouring of G.

2. Let $u \in V(G)$ be the neighbour of a on the infinite face different from b. Consider the neighbourhood of u, $N(u) = \{a, w, v_1, v_2, \ldots, v_k\}$ where w is on the infinite face different from a. We have $|L(w)| = 3$ and $|L(v_i)| = 5$ for all $i = 1, \ldots, k$ since there is no long diagonal. Choose two different colours γ and Δ in $L(u) \setminus \{\alpha\}$; they certainly exist since $|L(u)| = 3$. Delete γ and δ from all the lists of vertices in $\{v_1, \ldots, v_k\}$, and then by induction we can list colour $G \setminus \{u\}$ from the modified lists. This can be extended to a list colouring of G since u shares no colour in its list with any $\{v_1, \ldots, v_k\}$, and at least one of δ or γ will not be used in w. ■

n−connected means if you remove any n vertices, the graph remains connected

Take K_4, and have lists with colours $1, 2, 3, 4$ (or any graph which is uniquely 4−colorable). Inscribe a triangle in each face with lists $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$. Always align so that the degree 3 vertex is adjacent to the $1, 2$ and $1, 3$.

1.3 Theorem. (Grötsch) *If G is planar with girth at least 4, then $\chi(G) \leq 3$ and $\chi_\ell \leq 4$.*

If G is planar with n vertices and e edges, then $e \leq 3n - 6$ so that $\delta \leq 5$. If G is planar with n vertices and e edges with girth 4, then $e \leq 2n - 4$ so $\delta \leq 3$. This gives an easy proof of the list colouring value.

1.4 Theorem. *Let G be planar with girth at least 5. Then $\chi_\ell(G) \leq 3$.*

Proof Suppose G is a planar graph with girth at least 5 such that

1. There are at most 6 pre-colored vertices on the outer face which form a path or a cycle (edges need not be on the outer face),
2. there are some vertices with $|L(u)| = 2$ on the outer face boundary, and
3. There are no edges joining vertices with $|L(v)| < 3$ except for those in (1)

We will prove by induction on $|V(G)|$. Assume that G is a minimal counterexample. Then

1. $|V(G)| \leq 3$
2. G is connected
3. Outer face bounded by a cycle
4. No cut vertex in the graph (G is 2-connected); outer cycle has C
5. C has no chord
6. No separating cycle with at most 6 vertices
7. Pre-coloured path/cycle is a non-empty path (can just remove an edge)
8. No path of length 2 inside C except (see paper)
9. No path of length 3 inside C except starting at a list-2-vertex
10. The precolored path P and the outer cycle C has $|V(C)| \leq |V(P)| + 2$.

We will allow some precolored vertices which form a path or cycle with at most 6 vertices (edges can be chords), and some vertices with $|L(u)| = 2$, all on the outer face boundary. Except for edges in this path/cycle, there are no other edges joining vertices with $|L(u)| < 3$. All other vertices have at least 3 available colors. ■

1.5 Theorem. (Grötsch) *If G is planar with girth at least 4, then $\chi(G) \leq 3$.*

Proof If there is no 4-cycle, we are done by the previous theorem. If G contains no 4-cycle, we may simply add a 4-cycle artifically by adding edges.

Note that we may even precolor a 4-cycle or 5-cycle. Then that coloring can be extended to G. Suppose G is a minimal counterexample. First note that there is no separating 4 or 5 cycle: otherwise, one can colour the interior and exterior of the cycle. Thus assume the precolored cycle is on the boundary. If there is another separating 4 or 5 cycle inside. Then colour the outer face by induction, then the inner face.

Let C be a 4-cycle in G, and C is facial. If C is pre-colored, we have a problem: we can assume $C \neq C_0$, for if not, delete an edge in C_0 and refer to the original case. In this case, we may ... ■

1.6 Proposition. *The following are equivalent:*

(i) $\chi(G) \leq 3$

(ii) There exists an orientation of G such that all cycles are balanced modulo 3

(iii) There exists an orientation of G such that all closed walks are balanced modulo 3

Proof $(iii \Rightarrow ii)$ is immediate.

To see $(ii \Rightarrow iii)$, we can simply take the orientation from (ii). If a closed walk is not a cycle, it has a repeated vertex, and we can verify that the walk is balanced on each component.

For $(i \Rightarrow ii)$, we must simply orient the edges such that $0- > 1$, $1- > 2$, $2- > 3$

For $(iii \Rightarrow i)$, colour some vertex 0. Then for any other vertex, take a path connecting the vertices and walk along the path by adding one for every forward traversal, and subtract one for each backwards traversal, modulo 3. If there are multiple paths, then the multiple paths would form a walk which is balanced modulo 3, so the lengths must be the same. ■

Definition. A **cut** in a graph. Partition the vertex set into two pieces. Then a cut is the set of edges between the two vertex sets. A **minimal cut** is a cut containing no other cuts.

Note that a cut is minimal if and only if each side of the cut is connected. If G is planar, then the dual graph is formed as follows: each face becomes a vertex, and the vertices are joined by an edge if the corresponding faces are adjacent. The number of edges is unchanged, and the number of vertices and faces swaps.

Given an orientation on the original graph, we can pass the orientation to the dual graph by setting the orientation anticlockwise relative to the intersection. Let $E \subseteq E(G)$. Then E is a minimal cut in G if and only if E^* is a cycle, and E is a cycle in G if and only if E^* is a minimal cut in G^*.

Assume G is planar. If G is 4-edge-connected, then each cut has at least 4 edges, G^* has girth at least 4, then $\chi(G^*) \leq 3$, then the following equivalent things hold:

(i) G^* has an orientation so that all cycles are balanced modulo 3

(ii) Ghas an orientation such that all cuts are balanced modulo 3

(iii) G has an orientation such that $d^+(v) \equiv d^-(v)$ modulo 3

[Tutte] If G is 4-edge-connected, then there exists an orientation on G such that all degrees are balanced modulo 3. Currently proven for 6-edge-connected. If G is 4-edge connected, then there exists an orientation on G and a flow 1 or 2 on each edge such that at each vertex the inflow equals the outflow. This is equivalent to the conjecture by reversing the orientation for all edges which have flow 2, or by simply placing flow 1 on every edge in the graph.

In fact, one can remove the modular condition. Assume each edge has a flow 1 or 2 or 3 or 4, and assume that each inflow is equivalen to the out flow modulo 5.

1.7 Proposition. *If G is planar and 4-edge-onnected, then there exists an orientation such that G is balanced modulo 3.*

1.8 Proposition. *If G is cubic and 3-edge-connected, there exists an orientation which is balanced modulo 3 if and only if G is bipartite.*

Does there exists an orientation on G such that G is balanced modulo k? Or such that each vertex v has out degree $p(v)$ modulo k?

If the second holds for every p and k is odd, then the first holds. Let v be a vertex with degree $d(v)$; we want that $d^+(v) \cong d^-(v)$, in other words that $2d^+(v) \cong d(v) \pmod{k}$, s

$$\frac{k-1}{2} \cdot 2d^+(v) \cong \frac{k-1}{2} d(v) \Rightarrow d^+(v) \cong \frac{-(k-1)}{2} d(v)$$

Suppose $k = 2$. Here's a necessary condition: then $|E(G)| = \sum_{v \in V(G)} d^+(v) \cong \sum_{v \in V(G)} p(v)$, modulo 2. In fact, if G is connected and $\sum_{v \in V(G)} p(v) \cong |E(G)$, then such an orientation exists. Do do this, fix any orientation. If there is a vertex which does not satisfy the requirements, by parity, there must be some other vertex which does not satisfy the requirements. Take a path connecting the vertices and flip all the edges, repeating until the graph is balanced. [Jaeger] If G is 1000-edge-connected, then there exists an orientation on G balanced modulo 3. This has been proven in the affirmative for 8-edge-connected, then 6-edge-connected. It is enough to prove this for 5. [Jaeger] If G is $(2k-2)$–edge-connected, then there exists an orientation on G that is balanced modulo k if k is odd. It has been shown that if there is a $(2k^2+2)$–edge-connected graph, then there exists an orientation on G with any out degrees modulo k, also true for k is even. If G is $(3k-3)$–edge connected, then the same holds, but only for k odd.

Suppose G is 4-edge-connected: then there exists an orientation of G balanced modulo 4. This is equivalent to the 3-flow conjecture. Given an orientation balanced modulo 3, by a previous exercise, we can also balance each vertex modulo k for any k.

If $k = 5$, the statement says that G is 8-edge-connected implies G is balanced modulo 5.

Let G be 2-edge-connected, then there exists an orientation on G with flow values on $\{1, 2, 3, 4\}$ such that the inflow and the outflow are equal for all $v \in V(G)$. It suffices to verify this for cubic 3-connected graphs. Note that for cubic graphs, the edge and vertex connectivity are the same. k connected means there are k internally vertex disjoint paths, and k–edge-connected means there are k internally edge disjoint paths.

Example. Assume the 5-flow-conjecture holds for G cubic 3-connected. Then prove that it holds for G 2-edge-connected. There's a couple cases: if there is a vertex of degree 2 with edges going to the same vertex, simply add the same flow value going in and out. If there is a vertex of degree 2 with edges going to distinct edges, simply merge the edges, apply induction, and then apply the flow assigned to that edge to both pieces.

If there is a vertex with degree large, remove two of the edges so as not to create a bridge, and apply the same argument. What happens if we have all vertex of degree 3? We need to deal with the case where G is 2-edge connected. Isolate the pair of edges e_1 and e_2. First close the loops, and then multiply the flows or perhaps re-orient so that the edges agree.

Now assume G is cubic and 3-edge-connected. Then take the graph and replace every edge by 3 edges to get some G' that is 9-edge-connected. Therefore, by the result above (Jaeger with $k = 5$), it has an orientation that is balanced modulo 5. Then replace each triple of edges with the oriented net sum of the number of edges.

Example. K_8 is 7-edge-connected nad has no orientation balanced modulo 5.

Let's consider factors modulo k. A d-factor is a spanning subgraph of G such that every vertex of the subgraph has degree d.

> ***1.9 Theorem.*** *Let G be bipartite with bipartition $V(G) = A \cup B$ with $V(G) = \{v_1, \ldots, v_n\}$. For every v_i, let d_i be a natural number. We want a spanning subgraph of $H \subseteq G$ such that $d_H(v_i) \cong d_i \pmod{k}$ where k is odd. Then H exists if G is $(3k-3)$–edge connected and $\sum_{v_i \in A} d_i \cong \sum_{v_i \in B} d_i \pmod{k}$.*

Proof Apply the $(3k-3)$ result, and assign the function $p(v_i) = d_i$ for $v_i \in A$ and $p(v_j) = d(v_j) - d_j$ for $v_j \in B$. Certainly $\sum_{v_i \in V(G)} p_i \cong |E(G)|$ modulo k by the modular summation condition on the d_i. Then we simply take all $A- > B$ edges. ■