

# PMATH 465

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# I. Fundamentals of Manifolds

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## 1 INTRODUCTION TO TOPOLOGY

### BASIC CONSTRUCTIONS

**Definition.** A **topology** on a set  $X$  is a set  $\tau$  of subsets of  $X$  such that

- (i)  $\emptyset \in \tau$  and  $X \in \tau$
- (ii) If  $U_\alpha \in \tau$  for all  $\alpha \in A$ , then  $\bigcup_{\alpha \in A} U_\alpha \in \tau$ .
- (iii) If  $n \in \mathbb{N}$  and  $U_i \in \tau$  for each  $1 \leq i \leq n$ , then  $\bigcap_{i=1}^n U_i \in \tau$ .

The sets  $U \in \tau$  are called the **open sets** in  $X$ , and sets of the form  $X \setminus U$  for some open set  $U$  are called the **closed sets** in  $X$ .

**Definition.** When  $X$  is a topological space and  $A \subseteq X$ , the **interior** of  $A$  (denoted  $A^\circ$ ) is the union of all open sets contained in  $A$ . Similarly, we define the **closure** of  $A$  (denoted  $\bar{A}$ ) as the intersection of all closed sets containing  $A$ . Then the **boundary** of  $A$ , denoted by  $\partial A$ , is the set  $\partial A = \bar{A} \setminus A^\circ$ .

*Example.* Let  $X$  be any set. The **discrete topology** on  $X$  is the topology  $\tau = \mathcal{P}(X)$ , and the **trivial topology** on  $X$  is the topology  $\tau = \{\emptyset, X\}$ .

**Definition.** A **basis** for a topology on a set  $X$  is a set  $\mathcal{B}$  of subsets of  $X$

- (i)  $\bigcup_{B \in \mathcal{B}} B = X$
- (ii) for all  $a \in X$  and  $U, V \in \mathcal{B}$  such that  $a \in U \cap V$ , then there exists  $W \in \mathcal{B}$  with  $a \in W \subseteq U \cap V$ .

When  $\mathcal{B}$  is a basis for a topology on  $X$ , the topology on  $X$  **generated** by  $\mathcal{B}$  is the set  $\tau$  of subsets of  $X$  such that for  $W \subseteq X$ ,  $W \in \tau$  if and only if for all  $a \in W$ , there exists  $U \in \mathcal{B}$  such that  $a \in U \subseteq W$ .

Note that  $\tau$ , as above, is a topology on  $X$  since

- (i)  $\emptyset \in \tau$  vacuously and  $X \in \tau$  obviously.
- (ii) If  $A_k \in \tau$  for all  $k \in K$  (where  $K$  is any set of indices), then given  $a \in \bigcup_{k \in K} A_k$ , we can choose  $\ell \in K$  so that  $a \in A_\ell$ . Then since  $A_\ell \in \tau$ , we can choose  $U_\ell \in \mathcal{B}$  so that  $a \in U_\ell \subseteq A_\ell$ . Thus  $a \in U_\ell \subseteq A_\ell \subseteq \bigcup_{k \in K} A_k$ .
- (iii) By induction, it suffices to prove that if  $A, B \in \tau$ , then  $A \cap B \in \tau$ . Suppose  $A, B \in \tau$ , and let  $a \in A \cap B$ . Since  $A \in \tau$ , we can choose  $U \in \mathcal{B}$  so that  $a \in U \subseteq A$ . Since  $B \in \tau$ , we can choose  $V \in \mathcal{B}$  so that  $a \in V \subseteq B$ . Then we have  $a \in U \cap V$ . Since  $\mathcal{B}$  is a basis, we can choose  $W \in \mathcal{B}$  with  $a \in W \subseteq U \cap V$ , so  $a \in W \subseteq U \cap V \subseteq A \cap B$ .

Note that when  $\tau$  is the topology on  $X$  generated by the basis  $\mathcal{B}$ , for  $A \subseteq X$ ,  $A \in \tau$  if and only if there exists some  $S \subseteq \mathcal{B}$  such that  $A = \bigcup_{s \in S} s$ . In this sense, the topology  $\tau$  on  $X$  generated by the basis  $\mathcal{B}$  is the coarsest topology which contains  $\mathcal{B}$ .

**Definition. (Subspace Topology)** When  $Y$  is a topological space and  $X \subseteq Y$  is a subset of  $Y$ , we define the **subspace topology** on  $X$  to be the topology for which a set  $U \subseteq X$  is open if and only if  $U = X \cap V$  for some open set  $V$ .

If  $\mathcal{C}$  is a basis for the topology on  $Y$ , then  $\mathcal{B} = \{X \cap V \mid V \in \mathcal{C}\}$  is a basis for the subspace topology on  $X$ .

**Definition. (Disjoint Union Topology)** If  $X$  and  $Y$  are topological spaces with  $X \cap Y = \emptyset$ , then the **disjoint union topology** on  $X \cup Y$  is the topology in which a subset  $U \subseteq X \cup Y$  is open in  $X \cup Y$  if and only if  $U \cap X$  is open in  $X$  and  $U \cap Y$  is open in  $Y$ .

**Definition. (Product Topology)** If  $X$  and  $Y$  are topological spaces, the **product topology** on  $X \times Y$  is the topology generated by the basis

$$\mathcal{B} = \{ U \times V \mid U \in \mathcal{C}, V \in \mathcal{D} \}$$

where  $\mathcal{C}$  and  $\mathcal{D}$  are bases for the topologies on  $X, Y$  respectively.

**Definition. (Infinite Product Topology)** We define the infinite product to be

$$\prod_{k \in K} \left\{ f : K \rightarrow \bigcup_{k \in K} X_k \mid f(k) \in X_k \text{ for all } k \in K \right\}$$

There are two standard topologies on  $X$ . The first is the **box topology**,

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \mid U_k \text{ is open in } X_k \right\}$$

and the **product topology**

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \mid \begin{array}{l} U_k \text{ is open in } X_k \\ U_k = X_k \text{ for all but finitely many indices } k \end{array} \right\}$$

*Example. (Metric Topology)*  $\mathbb{R}^n$  has a standard **inner product**, and for  $u, v \in \mathbb{R}^n$ ,  $\langle u, v \rangle = u \cdot v = V^T u = \sum_{i=1}^n u_i v_i$ . This gives the standard norm on  $\mathbb{R}^n$  for  $u \in \mathbb{R}^n$ ,  $\|u\| = \sqrt{\langle u, u \rangle}$ . This gives the standard metric on  $\mathbb{R}^n$ : for  $a, b \in \mathbb{R}^n$ ,  $d(a, b) = \|b - a\|$ .

Given a metric on a set  $Y$ , we obtain (by restriction) an induced metric on any subset  $X \subseteq Y$ . Given a metric space  $X$ , we define the **metric topology** on  $X$  to be the topology which is generated by the set of open balls

$$B(a, r) = \{ x \in X \mid d(a, x) < r \}$$

where  $x \in X, r > 0$ .

## MAPS ON TOPOLOGICAL SPACES

**Definition.** When  $X$  and  $Y$  are topological spaces and  $f : X \rightarrow Y$ , we say that  $f$  is **continuous** when it has the property that  $f^{-1}(V)$  is open in  $X$  for every open set  $V$  in  $Y$ . We say that  $f : X \rightarrow Y$  is a **homeomorphism** when  $f$  is bijective and both  $f$  and  $f^{-1}$  are continuous. Then  $X, Y$  are **homeomorphic** if there exists a homeomorphism  $f : X \rightarrow Y$ .

**1.1 Theorem. (Glueing Lemma)** Let  $X$  and  $Y$  be topological spaces, and let  $f : X \rightarrow Y$  be a function. Suppose either

(i)  $X = \bigcup_{k \in K} A_k$  where each  $A_k$  is open in  $X$ , or

(ii)  $X = \bigcup_{k=1}^n A_k$  where each  $A_k$  is closed in  $X$

and each restriction map  $f_k : A_k \rightarrow Y$  is continuous, then  $f$  is continuous.

PROOF Exercise. ■

**Definition.** A topological space  $X$  is **compact** when it has the property that for every set  $\mathcal{S}$  of open subsets of  $X$  with  $X = \bigcup_{U \in \mathcal{S}} U$ , there exists a finite subset  $\mathcal{F} \subseteq \mathcal{S}$  such that  $X = \bigcup_{F \in \mathcal{F}} F$ .

Note that when  $X \subseteq Y$  is a subspace,  $X$  is compact if and only if  $X$  has the property that for every set  $\mathcal{T}$  with  $X \subseteq \bigcup_{T \in \mathcal{T}} T$ , there exists a finite subset  $\mathcal{G} \subseteq \mathcal{T}$  such that  $X \subseteq \bigcup_{G \in \mathcal{G}} G$ .

**Definition.** A topological space  $X$  is **connected** when there do not exist non-empty disjoint open sets  $U, V \subseteq X$  such that  $X = U \cup V$ .

Note that if  $Y$  is a metric space and  $X \subseteq Y$  is a subspace, then  $X$  is connected if and only if there do not exist open sets  $U, V \subseteq Y$  such that

$$X \cap U \neq \emptyset, X \cap V \neq \emptyset, U \cap V = \emptyset, \text{ and } X \subseteq U \cup V$$

**Definition.** A topological space  $X$  is called **path connected** when it has the property that for all  $a, b \in X$ , there exists a continuous map  $\alpha : [0, 1] \rightarrow X$  with  $\alpha(0) = a$  and  $\alpha(1) = b$ .

It is easy to see that if  $X$  is path connected, then  $X$  is connected.

**Definition.** Let  $X$  be a topological space. If we define a relation  $\sim$  on  $X$  by taking  $a \sim b$  if and only if there exists a connected subspace  $A \subseteq X$  with  $a \in A$  and  $b \in A$ .

It is clear that this is an equivalence relation. Note that when  $X$  is a topological space, its connected components are connected, and each connected subspace of  $X$  is contained in one of its connected components.

**Definition.** Let  $X$  be a topological space. Define a relation  $\approx$  on  $X$  by  $a \approx b$  if and only if there exists a continuous map  $\alpha : [0, 1] \rightarrow X$  with  $\alpha(0) = a$  and  $\alpha(1) = b$ . Such a map  $\alpha$  is called a **continuous path**.

One can show that if  $X$  is **locally path connected** (which means that  $X$  has a basis for its topology which consists of path connected sets), then the path components of  $X$  are equal to the connected components of  $X$ , and that these components are open.

## QUOTIENT TOPOLOGY

**Definition. (Quotient Topology)** Let  $X$  be a topological space and let  $\sim$  be an equivalence relation on  $X$ . The set of equivalence classes is denoted  $X/\sim$ , and  $X/\sim$  is called the **quotient** of  $X$  by  $\sim$ . The map  $\pi : X \rightarrow X/\sim$  given by  $\pi(a) = [a]$  is called the **natural projection map** or **quotient map**. We define the **quotient topology** on  $X/\sim$  by stipulating that for  $W \subseteq X/\sim$ ,  $W$  is open in  $X/\sim$  if and only if  $\pi^{-1}(W)$  is open in  $X$ .

When a group  $G$  acts on a topological space  $X$ , we define an equivalence relation  $\sim$  on  $X$  by  $a \sim b$  if and only if  $b = g \cdot a$  for some  $g \in G$ . The equivalence classes are orbits. In this context, we also write  $X/\sim$  as  $X/G$ .

When  $X, Y$  are any topological spaces and  $\pi : X \rightarrow Y$  is surjective, we can define an equivalence relation  $\sim$  on  $X$  by  $a \sim b$  if and only if  $\pi(a) = \pi(b)$ . We then have a natural bijection from  $Y$  to  $X/\sim$  in which  $y \in Y$  corresponds to the fibre  $\pi^{-1}(y) \in X/\sim$ .

If  $Y$  has the topology such that for  $W \subseteq Y$ ,  $W$  is open in  $Y$  if and only if  $\pi^{-1}(W)$  is open in  $X$ . In this case, we also use the terminology “quotient map” for  $\pi$ .

**Remark.** Let  $X$  be a topological space and let  $\sim$  be an equivalence relation on  $X$ . Let  $Y$  be any set. If  $f : X \rightarrow Y$  is constant on the equivalence classes, then  $f$  induces a well-defined map  $\bar{f} : X/\sim \rightarrow Y$  given by define  $\bar{f}([a]) = f(a)$ .

*Example.* Define an equivalence class on  $[0, 1] \subseteq \mathbb{R}$  by  $s \sim t$  if and only if  $s = t$  or  $\{s, t\} = \{0, 1\}$ . Then  $[0, 1]/\sim \cong \mathbb{S}^1$ . Define  $f : [0, 1] \rightarrow \mathbb{S}^1$  by  $f(t) = e^{i2\pi t}$ . Note that  $f(0) = f(1)$ , so  $f$  induces a continuous map  $\bar{f} : [0, 1]/\sim \rightarrow \mathbb{S}^1$ . The inverse map can be constructed as follows. We define  $g : \mathbb{S}^1 \rightarrow [0, 1]/\sim$  by

$$g(x, y) = \begin{cases} \left[ \frac{1}{2\pi} \cos^{-1} x \right] & : y \geq 0 \\ \left[ 1 - \frac{1}{2\pi} \cos^{-1} x \right] & : y \leq 0 \end{cases}$$

Then  $g$  is continuous by the Glueing lemma.

In particular, the same proof shows that  $\mathbb{R}/\mathbb{Z}$  is homeomorphic to  $\mathbb{S}^1$ .

*Example.* The projective space  $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$  can be defined in several ways.  $\mathbb{P}^n$  is the set of all 1-dimensional vector subspaces of  $\mathbb{R}^{n+1}$ , or  $\mathbb{P}^n = \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times$ , or  $\mathbb{P}^n = \mathbb{S}^n / \pm 1$  where  $\mathbb{S}^n = \{u \in \mathbb{R}^{n+1} : |u| = 1\}$ .

Let us show that  $\mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times$  is homeomorphic to  $\mathbb{S}^n / \pm 1$ . Define  $f : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{S}^n$  by  $f(x) = x/|x|$ , and  $g = \pi \circ f$ . Then  $g$  is given by  $g(x) = \{\pm x/|x|\}$ . Note that for  $t \in \mathbb{R}^\times$ ,

$$g(tx) = \left[ \frac{t}{|t|} \cdot \frac{x}{|x|} \right] = \left[ \frac{x}{|x|} \right]$$

since  $t/|t| = \pm 1$ . Thus  $g$  induces a continuous map  $\bar{g}$  on the quotient. We construct the inverse map in a similar way.

**Definition.** Let  $X$  be a topological space. Then

- $X$  is **T1** when for all  $a, b \in X$  there exists an open set  $U$  in  $X$  with  $a \in U$  and  $b \notin U$
- $X$  is **T2** or **Hausdorff** when for all  $a, b \in X$ , there exist disjoint open sets  $U, V \subseteq X$  with  $a \in U$  and  $b \in V$
- $X$  is **T3** or **regular** when  $X$  is T1 and for every  $a \in X$  and every closed set  $B \subseteq X$  with  $a \notin B$ , there exist open sets  $U, V \subseteq X$  with  $a \in U$ ,  $B \subseteq V$ .
- $X$  is **T4** or **normal** when  $X$  is T1 and for all disjoint closed sets  $A, B \subseteq X$  there exist disjoint open sets  $U, V \subseteq X$  with  $A \subseteq U$  and  $B \subseteq V$ .

**Definition.** Let  $X$  be a topological space.

- $X$  is **first countable** when for every  $a \in X$ , there exists a countable set  $B_a$  of open sets in  $X$  which contain  $a$  such that for every open set  $W$  in  $X$  with  $a \in W$ , there exists  $U \in B_a$  with  $a \in U \subseteq W$ .
- $X$  is **second countable** when there exists a countable basis for the topology on  $X$ .

*Example.* (i)  $X$  is T1 if and only if every 1-point subset of  $X$  is closed in  $X$

(ii) Every compact Hausdorff space is regular.

(iii) Every second countable regular space is normal.

(iv) Every metric space is normal.

(v) If  $X$  is second countable, then every open cover admits a countable subcover.

(vi) Every second countable space  $X$  contains a countable dense subset.

**1.2 Lemma. (Urysohn)** If  $X$  is normal and  $A, B \subseteq X$  are disjoint and closed, then there is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(A) = \{0\}$  and  $f(B) = \{1\}$ .

**1.3 Theorem. (Tietze Extension)** If  $X$  is normal and  $f : A \rightarrow \mathbb{R}$  is continuous for some  $A \subseteq X$  closed, then there exists a continuous map  $F : X \rightarrow \mathbb{R}$  such that  $F|_A = f$  and  $\sup_{a \in A} |f(a)| = \sup_{x \in X} |F(x)|$ .



**1.4 Theorem. (Urysohn's Metrization)** If  $X$  is second countable and regular, then  $X$  is metrizable.

**Definition.** An  $n$ -dimensional topological manifold is a Hausdorff, second countable topological space  $M$  which is **locally homeomorphic** to  $\mathbb{R}^n$ , meaning for every  $p \in M$ , there exists an open set  $U \subseteq M$  with  $p \in U$  and an open set  $V \subseteq \mathbb{R}^n$  and a homeomorphism  $\phi : U \subseteq M \rightarrow V \subseteq \mathbb{R}^n$ . Such a homeomorphism  $\phi$  is called a **(local) coordinate chart** or **chart** on  $M$  at  $p$ . The domain  $U$  of a chart  $\phi : U \subseteq M \rightarrow \phi(U) \subseteq \mathbb{R}^n$  is called a (local) **coordinate neighbourhood** at  $p$ . Note that we can choose a set of charts

$$\mathcal{A} = \{\phi_k : U_k \subseteq M \rightarrow \phi_k(U_k) : k \in K\}$$

where  $K$  is any non-empty set such that  $M = \bigcup_{k \in K} U_k$ . Such a set of charts is called an **atlas** for  $M$ .

**Definition.** Two charts are called  $\phi : U \rightarrow \phi(U)$  and  $\psi : V \rightarrow \psi(V)$  are called **(smoothly) compatible** when either  $U \cap V = \emptyset$  or  $\phi^{-1} \circ \psi$  and  $\psi \circ \phi^{-1}$  are smooth (meaning partial derivatives of all orders exist). We say that an atlas is **smooth** if every pair of charts is compatible.

Note that a smooth atlas  $\mathcal{A}$  on  $M$  can be extended to a unique maximal smooth atlas  $\mathcal{M}$  on  $M$  by adding to  $\mathcal{A}$  every possible homeomorphism  $\psi : U \subseteq M \rightarrow \psi(U) \subseteq \mathbb{R}^n$  which is compatible with all of the existing charts (since if  $\psi$  and  $\chi$  are both compatible with every chart  $\phi \in \mathcal{A}$ , then  $\psi$  and  $\chi$  will be compatible with each other). The maps  $\psi \circ \phi^{-1}$  are called **transition maps** or **change of coordinate maps**. A maximal smooth atlas  $\mathcal{M}$  on  $M$  is called a **smooth structure** on  $M$ .

**Definition.** An  $n$ -dimensional **smooth (or  $C^\infty$ ) manifold** is an  $n$ -dimensional topological manifold with a smooth structure.

**Remark.** A topological manifold can have different smooth structures. For example, take  $\mathcal{A} = \{\phi\}$  where  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is the identity map, and  $\mathcal{B} = \{\psi\}$  where  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is a homeomorphism given by  $\psi(x) = x^3$ , since  $\sqrt[3]{x}$  is not smooth at the origin.

What if we tried  $\mathcal{B} = \{\psi\}$  where  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is a homeomorphism which is not  $C^\infty$ ? This is trivially a smooth atlas.

Typically, a manifold is given with a standard smooth structure.

**Remark.** We can give a smooth manifold  $M$  an (at most countable) atlas of charts all of which are of one of the forms

- $\phi : U \subseteq M \rightarrow B(0, 1)$
- $\phi : U \subseteq M \rightarrow (0, 1)^n$
- $\phi : U \subseteq M \rightarrow \mathbb{R}^n$

Note that the maximal atlas  $\mathcal{M}$  is determined from any subset  $\mathcal{A} \subset \mathcal{M}$  such that the domains of the charts in  $\mathcal{A}$  cover  $M$ .

**Definition.** Let  $M$  be an  $m$ -dimensional smooth manifold and  $N$  be an  $n$ -dimensional smooth manifold and let  $f : M \rightarrow N$  be a function. Then we say  $f$  is **smooth** at  $p$  when for some (hence for any) chart  $\phi$  on  $M$  at  $p$  and for some (hence any) chart  $\psi$  on  $N$  at  $f(p)$ , the map  $\psi^{-1} \circ f \circ \phi$  is smooth at  $x = \phi(p)$ , and  $f$  is **smooth** if  $f$  is smooth at every  $p \in M$ . We say that  $f$  is a **diffeomorphism** when  $f$  is invertible and both  $f$  and  $f^{-1}$  are smooth. We say that  $M$  and  $N$  are **diffeomorphic**, and write  $M \cong N$ , when there exists a diffeomorphism  $f : M \rightarrow N$ .

*Remark.* It is conceivable that a topological manifold  $M$  could be both of dimension  $n$  and of dimension  $m$  with  $n \neq m$ . To do this, we would need to have a homeomorphism from an open set in  $\mathbb{R}^n$  to an open set in  $\mathbb{R}^m$ . In fact, this cannot happen by invariance of domain, proven using tools from algebraic topology.

When  $M$  is smooth, it is easy to see that this cannot happen. If  $\psi \circ \phi^{-1}$  and  $\phi \circ \psi^{-1}$  were smooth inverses, then the matrices  $D(\psi \circ \phi^{-1})(\phi(p))$  and  $D(\phi \circ \psi^{-1})(\psi(p))$  would be inverse matrices. But then a product of a matrix in  $M_{m \times n}(\mathbb{R})$  and in  $M_{n \times m}(\mathbb{R})$  cannot be inverses when  $m \neq n$ .

*Remark.* Manifolds are sometimes constructed using quotient constructions. These quotients can be given by polygons with pairs of edges identified up to orientation.

There are other kinds of manifolds (other than  $C^\infty$  manifolds); for example, one can define  $C^k$  manifolds, or analytic  $C^\omega$  manifold has an atlas in which the transition maps are analytic.

- Example.*
1.  $\mathbb{R}^n$  is a smooth  $n$ -dimensional manifold. It can be given an atlas consisting of 1 chart, the identity map.
  2. Any  $n$ -dimensional vector space over  $\mathbb{R}$  is a smooth  $n$ -dimensional manifold. It can be given an atlas with one chart. If  $\{u_1, \dots, u_n\}$  is a basis for  $V$ , then one can define  $\phi : V \rightarrow \mathbb{R}^n$  by  $\phi(\sum t^i u_i) = (t^1, \dots, t^n) = t \in \mathbb{R}^n$ .
  3. Every open subset of a smooth  $n$ -dimensional manifold is also a smooth  $n$ -dimensional manifold.
  4.  $M_{n \times m}(\mathbb{R})$  is an  $n \cdot m$ -dimensional manifold with pointwise  $\mathbb{R}^{nm}$  structure.
  5.  $\{A \in M_{n \times m}(\mathbb{R}) : \text{rank}(A) = \min\{n, m\}\}$  is a smooth manifold with one chart, since it is an open submanifold of  $M_{n \times m}$ . Suppose  $n > m$ ; then take all  $n \times n$  submatrices which have non-zero determinant (open by continuity of  $\det$ ), and maximal rank means that  $A$  is contained in one of these open subsets.
  6. The disjoint union of countably many  $n$ -dimensional smooth manifolds.
  7. The cartesian product of finitely many smooth manifolds is a smooth manifold. Let  $\dim(M_k) = n_k$ , the  $\dim(M_1 \times \dots \times M_\ell) = \sum_{k=1}^\ell n_k$ . If  $\phi_k : U_k \subseteq M_k \rightarrow \phi_k(U_k) \subseteq \mathbb{R}^{n_k}$  is a chart on  $M_k$ , then  $\chi_k : \prod_{k=1}^\ell U_k \rightarrow \prod_{k=1}^\ell \mathbb{R}^{n_k}$  given by  $\chi_k(p_1, \dots, p_\ell) = (\phi_1(p), \dots, \phi_\ell(p))$  is a chart in  $M_1 \times \dots \times M_\ell$ .
  8. One can show that  $\mathbb{S}^n$  is a smooth  $n$ -dimensional manifold.

*Remark.* For  $A \in M_{n \times m}(\mathbb{R})$ , we denote the entry in the  $k^{\text{th}}$  row and  $\ell^{\text{th}}$  column by  $A_\ell^k$ .

*Example.*  $\mathbb{S}^n$  is an example of an  $n$ -dimensional smooth manifold. It can, for example, be given a smooth atlas which contains  $2(n+1)$  charts as follows. For  $1 \leq k \leq n+1$ , let

$$\begin{aligned} U_k &= \{x \in \mathbb{S}^n : x^k > 0\} \\ \phi_k : U_k &\rightarrow B(0, 1) \subseteq \mathbb{R}^n \\ \phi_k(x) &= (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^{n+1}) \\ \phi_k^{-1}(t^1, \dots, t^n) &= \left(t^1, \dots, t^{k-1}, \sqrt{1 - \sum_{i=1}^{k-1} (t^i)^2}, t^k, \dots, t^n\right) \end{aligned}$$

and the corresponding opposite charts for  $x^k < 0$ . Note that  $\mathbb{S}^n$  is a metric space. It has 2 standard metrics: either the one inherited from  $\mathbb{R}^n$ , or the arclength distance  $d_s(u, v) = \cos^{-1}(u \cdot v)$ .

We can also give  $\mathbb{S}^n$  an atlas which only uses 2 charts, by stereographic projection from a north pole and a south pole.

This stereographic projection also shows that the rational points on the sphere are dense in  $\mathbb{S}^n$ , via the map

$$\phi(x) = \alpha \left( \frac{1}{1-x^{n+1}} \text{right} \right) = \left( \frac{x^1}{1-x^{n+1}}, \dots, \frac{x^n}{1-x^{n+1}} \right)$$

One can also find  $\phi^{-1}$  and verify that they are both rational functions. In particular,  $\phi^{-1}(\mathbb{Q}^n) \subseteq \mathbb{S}^n$  is dense.

*Example.* The projective space  $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$  is commonly defined in at least 3 ways:

$$\begin{aligned} \mathbb{P}^n &= \{1\text{-dimensional subspaces of } \mathbb{R}^{n+1}\} \\ \mathbb{P}^n &= \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times = \{[x] : 0 \neq x \in \mathbb{R}^{n+1}, [x] = \{tx : t \in \mathbb{R}^\times\}\} \\ \mathbb{P}^n &= \mathbb{S}^n / \pm 1 \end{aligned}$$

We can give  $\mathbb{P}^n$  a smooth atlas with  $n+1$  charts as follows: for  $1 \leq k \leq n+1$ , set

$$\begin{aligned} U_k &= \{[x] \in \mathbb{P}^n : x^k \neq 0\} \\ \phi_k : U_k &\rightarrow \mathbb{R}^n, \phi_k([x]) = \left( \frac{x^1}{x^k}, \dots, \frac{x^{k-1}}{x^k}, \frac{x^{k+1}}{x^k}, \dots, \frac{x^{n+1}}{x^k} \right) \end{aligned}$$

with  $\phi_k^{-1}(t_1, \dots, t^n) = [(t_1, \dots, t^{k-1}, 1, t^k, \dots, t^n)]$ .

### EXAMPLES OF SMOOTH MAPS

- The inclusion  $f : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$  given by  $f(x) = x$
- The quotient map  $f : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$
- The exponential map  $f : \mathbb{R} \rightarrow \mathbb{S}^1$  given by  $f(t) = e^{i2\pi t}$ , or more generally  $f : \mathbb{R}^n \rightarrow \mathbb{T}^n$  given by  $f(t^1, \dots, t^n) = (e^{2\pi i t^1}, \dots, e^{2\pi i t^n})$
- The determinant map  $f : M_n(\mathbb{R}) \rightarrow \mathbb{R}$  given by  $f(A) = \det(A)$  is smooth
- For  $A \in M_n(\mathbb{R})$ , left and right multiplication by  $A$ , the transpose map, and the inverse map  $f(A) = A^{-1}$  are smooth.

### PARTITIONS OF UNITY

**1.5 Lemma.** *Every open cover of a manifold has an (at most) countable subcover.*

**PROOF** Let  $\mathcal{S}$  be any open cover of  $M$ , and let  $\mathcal{B}$  be a countable basis for the topology on  $M$ . For each  $p \in M$ , choose  $U_p \in \mathcal{S}$  with  $p \in U_p$ , then choose  $B_p \in \mathcal{B}$  with  $p \in B_p \subseteq U_p$ . Then  $\{B_p : p \in M\} \subseteq \mathcal{B}$  is an open cover of  $M$ , and it is a subset of  $\mathcal{B}$ , so it is (at most) countable; but then  $\{U_p : p \in M\}$  gives an at most countable subcover of  $\mathcal{S}$ . ■

As a result, every manifold has a countable basis  $\mathcal{B}$  such that for each  $B \in \mathcal{B}$ , there is a chart  $\phi : U \rightarrow \phi(U)$  on  $M$  with  $\phi(U) = B(0, 2)$  and  $\phi(B) = B(0, 1)$ .

**1.6 Lemma.** *Let  $M$  be a manifold, and let  $\mathcal{S}$  be any open cover of  $M$ . Then there exists an at most countable open cover  $\mathcal{B}$  of  $M$  such that*

1. *for each  $B \in \mathcal{B}$  there is a chart  $\phi_B : C_B \rightarrow \phi_B(C_B) = B(0, 1)$  with  $B \subseteq C_B \subseteq U_B \subseteq \mathcal{S}$  for some  $U_B \in \mathcal{S}$  and  $\phi_B(B) = B(0, 1)$ .*

2.  $\{C_B : B \in \mathcal{B}\}$  is locally finite, meaning that every point in  $M$  has an open neighbourhood which only intersects with finitely many of the sets  $C_B$ ,  $B \in \mathcal{B}$  (and hence also the sets  $\overline{B}$ ,  $B \in \mathcal{B}$ ).

PROOF Choose a countable set  $\mathcal{V} = \{V_1, V_2, \dots\}$  of regular coordinate balls which cover  $M$  with charts  $\phi_i : W_i \rightarrow \phi_i(W_i) = B(0, 2)$  such that  $V_i = \phi_i^{-1}(B(0, 1))$ . We use the sets  $V_i$  to construct a strongly ascending chain of compact sets  $K_i$  in  $M$  with  $K_i \subseteq H_{i+1}^{-1}$  for each  $i$ , and  $M = \bigcup_{i=1}^{\infty} K_i$  as follows:

- Let  $K_1 = \overline{V_1}$ ; since  $K_1$  is compact, we can choose  $\ell_1 \in \mathbb{N}$  so that  $K_1 \subseteq V_1 \cup \dots \cup V_{\ell_1}$ .
- Then we let  $K_2 = \overline{V_1 \cup \dots \cup V_{\ell_1}}$ . Since  $K_2$  is compact, we can choose  $\ell_2 > \ell_1$  so that  $K_2 \subseteq V_1 \cup \dots \cup V_{\ell_2}$ , and set  $K_3 = \overline{V_1 \cup \dots \cup V_{\ell_2}}$ .

Repeat the above process to obtain  $K_1 \subseteq K_2^\circ \subseteq K_2 \subseteq K_3^\circ \subseteq \dots$  with  $\bigcup_{i=1}^k K_i = M$ . For each  $m \in \mathbb{N}$ , note that  $K_{m+1} \setminus K_m^\circ$  is compact and contained in the open set  $K_{m+2} \setminus K_{m-1}$  (with  $K_0 = \emptyset$ ). For each  $p \in K_{m+1} \setminus K_m^\circ$ , choose  $U_p \in \mathcal{S}$  with  $p \in U_p$  and then choose a regular coordinate ball  $B_p$  and a chart  $\phi_p : C_p \subseteq M \rightarrow \phi_p(C_p) = B(0, 2) \subseteq \mathbb{R}^n$  with  $\phi_p(B_p) = B(0, 1)$  and  $C_p \subseteq U_p \cap (K_{m+2}^\circ \setminus K_{m-1})$ . The coordinate balls  $B_p$ ,  $p \in K_{m+1} \setminus K_m^\circ$  cover the compact set  $K_{m+1} \setminus K_m^\circ$ , so we can choose a finite set  $\mathcal{B}_m$  of such regular coordinate balls  $B_p$  so that  $K_{m+1} \setminus K_m^\circ \subseteq \bigcup \mathcal{B}_m \subseteq K_{m+2}^\circ \setminus K_{m-1}$ .

Now, the set  $\mathcal{B} = \bigcup_{m=1}^{\infty} \mathcal{B}_m$  is a countable set of such regular coordinate balls. Note that for each  $B \in \mathcal{B}$ , we have chart  $\phi_B : C_B \rightarrow \phi_B(C_B) = B(0, 2)$  and the set  $\{C_B : B \in \mathcal{B}\}$  is locally finite since every point in  $M$  is contained in one of the sets  $K_{m+2}^\circ \setminus K_{m-1}$  and each of these sets only intersects with the coordinate balls from the finite sets  $\mathcal{B}_l$  with  $m-2 \leq l \leq m+2$ . ■

**1.7 Theorem. (Partitions of Unity)** Let  $M$  be a smooth manifold, and let  $\mathcal{S}$  be any open cover of  $M$ . There exists a set  $\{\psi_u : u \in \mathcal{S}\}$  of smooth maps  $\psi_u : M \rightarrow \mathbb{R}$  such that

1.  $\psi_u(M) \subseteq [0, 1]$  for each  $u \in \mathcal{S}$
2.  $\text{supp}(\psi_u) \subseteq U$  for each  $U \in \mathcal{S}$
3.  $\{\text{supp}(\psi_u) : u \in \mathcal{S}\}$  is locally finite: every point in  $M$  contains an open neighbourhood which only intersects finitely many of the sets  $\text{supp}(\psi_u)$ ,  $u \in \mathcal{S}$
4.  $\sum_{u \in \mathcal{S}} \psi_u = 1$

Such a set of functions  $\{\psi_u : u \in \mathcal{S}\}$  is called a (smooth) **partition of unity** on  $M$  for  $\mathcal{S}$  (or **subordinate** to  $\mathcal{S}$ ).

PROOF Let  $\mathcal{B}$  be a countable set of regular coordinate balls as in the previous lemma. Recall that the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(t) = \begin{cases} e^{1/t} & : t < 0 \\ 0 & : t \geq 0 \end{cases}$$

is smooth, so the function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  given by  $g(x) = f(|x|^2 - 1)$  is smooth with  $g(x) > 0$  for  $|x| < 1$  and  $g(x) = 0$  for  $|x| \geq 1$ . For each  $B \in \mathcal{B}$ , we define a smooth bump function  $\sigma_B : M \rightarrow \mathbb{R}$  by

$$\sigma_B(p) = \begin{cases} g(\phi_B(p)) & : p \in B \\ 0 & : p \notin B \end{cases}$$

where  $\phi_B : C_B \subseteq M \rightarrow \phi_B(C_B) = B(0, 2)$  with  $\phi_B(B) = B(0, 1)$  as in the previous lemma. Note that  $\sigma(B)$  is smooth with  $\sigma_B(p) > 0$  for  $p \in B$  and  $\sigma_B(p) = 0$  for  $p \notin B$ . Now for each  $B \in \mathcal{B}$ ,

define  $\tau'_B : M \rightarrow \mathbb{R}$  by

$$\tau_B = \frac{\sigma_B}{\sum_{c \in \mathcal{B}} \sigma_c}$$

Note that  $\sum_{c \in \mathcal{B}} \sigma_c$  is well-defined by the local finiteness of  $\mathcal{B}$  and  $\sum_{c \in \mathcal{B}} \sigma_c(p) > 0$ . Furthermore, note that  $\tau_B(p) > 0$  for all  $p \in B$ , and  $\tau_B(p) = 0$  for all  $p \notin B$ , and  $\sum_{B \in \mathcal{B}} \tau_B = 1$ . Then define  $\rho_V : M \rightarrow \mathbb{R}$  by  $\rho_V = \sum_{B \in \mathcal{B}_V} \tau_B$ . ■

## 2 IMMERSIONS, EMBEDDING, SUBMANIFOLDS

**2.1 Theorem. (Inverse Function Theorem)** Let  $U \subseteq \mathbb{R}^n$  be open,  $p \in U$ , and  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  be smooth and suppose  $Df(p)$  is invertible. Then  $f$  is a local diffeomorphism.

**2.2 Corollary.** Let  $n < m$  and  $U \subseteq \mathbb{R}^n$  be open, and let  $p \in U$ , and  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  be smooth and suppose  $Df(p)$  has rank  $n$ . Then the range of  $f$  is locally equal to the graph of a smooth function. Such a map  $f$  is called a local **immersion** at  $p$ .

**PROOF** Since  $Df(p)$  is an  $m \times n$  matrix of rank  $n$ , some  $n$  rows of  $Df(p)$  form an invertible submatrix. Reorder the variables in  $\mathbb{R}^m$  (if necessary) so that the top  $n$  rows form an invertible matrix. Write elements in  $U \subseteq \mathbb{R}^n$  as  $t$  and write elements of  $\mathbb{R}^m$  as  $(x, y)$ . Also write  $(x, y) = f(t) = (u(t), v(t))$  so

$$Df = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix}$$

with  $\frac{\partial u}{\partial t}(p)$  invertible. Then by the inverse function theorem,  $u(t)$  is a local diffeomorphism. Say  $u : U_0 \subseteq U \rightarrow V_0 \subseteq \mathbb{R}^n$  is the diffeomorphism, and let  $g : V_0 \rightarrow U_0$  be its inverse. Then the range of  $f$  is locally equal to the graph of the function  $y = v(g(x)) =: h(x)$ . If  $(x, y) \in \Gamma(f)$  with  $(x, y) = f(t) = (u(t), v(t))$ , then since  $x = u(t)$  we have  $t = g(x)$  so  $y = v(t) = v(g(x)) = h(x)$ . If  $(x, y) \in \Gamma(h)$ , then  $y = h(x) = v(g(x))$  and we can choose  $t = g(x)$  to get  $x = u(t)$  and  $y = v(g(x)) = v(t)$  so that  $(x, y) = (u(t), v(t)) = f(t)$ . ■

**2.3 Theorem. (Implicit Function)** Let  $n < m$ ,  $U \subseteq \mathbb{R}^m$  be open,  $p \in U$ , and  $f : U \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$  be smooth. Suppose  $Df(p)$  has rank  $n$  and let  $q = f(p)$ . Then the level set  $f^{-1}(q)$  is locally equal to a graph of a smooth function.

**2.4 Theorem.** Let  $U \subseteq \mathbb{R}^n$  be open with  $p \in U$ , let  $f : U \rightarrow \mathbb{R}^m$  be smooth with  $f(p) = q$  and suppose that  $Df$  has constant rank  $r$  in  $U$ . Then the level set (or fibre)  $f^{-1}(q)$  is locally equal to the graph of a smooth function (with  $n - r$  independent variables and  $r$  dependent variables).

**PROOF** Since  $Df$  is an  $m \times n$  matrix of rank  $r$ , there is some  $r \times r$  submatrix of  $Df(p)$  which is invertible; without loss of generality, it is the upper left submatrix. Write elements in  $\mathbb{R}^n$  as  $(x, y)$  with  $x \in \mathbb{R}^r$  and  $y \in \mathbb{R}^{n-r}$  and write elements in  $\mathbb{R}^m$  as  $(u, v)$  with  $u \in \mathbb{R}^r$  and  $v \in \mathbb{R}^{m-r}$ , with say  $p = (a, b)$  and  $q = f(p) = (c, d)$ . Then we have  $(u, v) = f(x, y) = (u(x, y), v(x, y))$  so that

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

with  $\frac{\partial u}{\partial x}(p) = \frac{\partial u}{\partial x}(a, b)$  being an invertible  $r \times r$  matrix. Define  $F : U \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^m$  by  $F(x, y) = (u(x, y), y)$ . Then

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ 0 & I \end{pmatrix}$$

so that  $DF(p)$  is invertible. By the IVT,  $F$  is a local diffeomorphism, say  $F : U_0 \subseteq U \subseteq \mathbb{R}^m \rightarrow V_0 \subseteq \mathbb{R}^m$  is a diffeomorphism with  $U_0$  an open rectangular box. Let  $G : V_0 \rightarrow U_0$  denote the smooth inverse of  $F$ . Note that  $G$  is of the form  $G(u, y) = (g(u, y), y)$  for some smooth function  $g : V_0 \rightarrow \mathbb{R}^r$ . We claim that  $f^{-1}(q) = f^{-1}(c, d)$  is locally equal to the graph of  $x = g(c, y)$ . First, note that

$$(u, y) = F(G(u, y)) = F(g(u, y), y) = (u(g(u, y), y), y)$$

so that, in particular,  $u(g(u, y), y) = u$  and so

$$f(G(u, y)) = (u(g(u, y), y), v(g(u, y), y)) = (u, h(u, y))$$

where  $h(u, y) = v(g(u, y), y)$ . Thus

$$Df(x, y) \cdot DG(u, y) = D(f \circ G)(u, y) = \begin{pmatrix} I & 0 \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial y} \end{pmatrix}$$

Since  $Df$  has constant rank  $r$  and  $DG$  is invertible, the matrix on the right is of rank  $r$  for all  $(u, v) \in V_0$ . Thus it follows that  $\frac{\partial h}{\partial y} = 0$  for all  $u, b$ , so that  $h(u, y)$  is independent of  $y$  and  $h(u, y) = h(u, b)$  for all  $y$ ; let  $k(u) = h(u, b)$ . Let us calculate  $k(c)$ . We have

$$\begin{aligned} f(a, b) = (c, d) &\implies (u(a, b), v(a, b)) = (c, d) \\ &\implies u(a, b) = c \\ &\implies F(a, b) = (u(a, b), b) = (c, b) \\ &\implies (a, b) = G(c, b) \\ &\implies (c, d) = f(a, b) = f(G(c, b)) = (c, h(c, b)) = (c, k(c)) \\ &\implies k(c) = d \end{aligned}$$

Finally, let us show that  $f^{-1}(c, d)$  is (locally) the graph of  $x = g(c, y)$ . We have

$$\begin{aligned} (x, y) = f^{-1}(c, d) &\implies f(x, y) = (c, d) \\ &\implies u(x, y) = c \text{ and } v(x, y) = d \\ &\implies F(x, y) = (u(x, y), y) = (c, y) \\ &\implies (x, y) = G(c, y) = (g(c, y), y) \\ &\implies x = g(c, y) \end{aligned}$$

We thus have

$$\begin{aligned} x = g(c, y) &\implies G(c, y) = (g(c, y), y) = (x, y) \\ &\implies f(x, y) = f(G(c, y)) = (c, h(c, y)) = (c, k(c)) = (c, d) \end{aligned}$$

as required. ■

**Definition.** When  $N$  and  $M$  are smooth manifolds and  $f : N \rightarrow M$  is a smooth map, we say that  $f$  has **rank  $r$**  at  $p \in N$  when for some (hence for every) chart  $\phi$  on  $N$  at  $p$  and for some (hence every) chart  $\psi$  on  $M$  at  $f(p)$ , the matrix  $D(\psi f \phi^{-1})(\phi(p))$  has rank  $r$ .

**2.5 Corollary.** Let  $N$  and  $M$  be smooth manifolds, with  $p \in N$ . Let  $f : N \rightarrow M$  be smooth with  $f(p) = q \in M$ . Suppose  $f$  has constant rank  $r$  in an open neighbourhood of  $p$ . Then there exists a chart  $\phi$  on  $N$  at  $p$  and a chart  $\psi$  on  $M$  at  $q = f(p)$  such that  $\phi(p) = 0$  and  $\psi(q) = 0$  and

$$(\psi \circ f \circ \phi^{-1})(x^1, \dots, x^r, \dots, x^n) = (x^1, \dots, x^r, 0, \dots, 0)$$

where  $n = \dim(N)$  and  $m = \dim(M)$ .

**PROOF** Choose any chart  $\phi_0$  on  $N$  at  $p$  and any chart  $\psi_0$  on  $M$  at  $q$  with  $\phi_0(p) = 0$  and  $\psi_0(q) = 0$ . Then  $D(\psi_0 f \phi_0^{-1})$  has constant rank  $r$  near 0. Let  $\phi_1$  and  $\psi_1$  be linear permutation maps so that the upper left  $r \times r$  submatrix of  $D(\psi_1 \psi_0 f \phi_0^{-1} \phi_1^{-1})(0)$ . Say  $f_1 = \psi_1 \psi_0 f \phi_0^{-1} \phi_1^{-1}$ . Let  $F, G, f_1$  be as in the proof of the rank theorem (for the function  $f_1$ ). Let us verify that for the charts  $\phi = F \phi_1 \phi_0$  and  $\psi = H \psi_1 \psi_0$  where  $H(u, v) = (u, v - k(u))$  we have  $(\psi f \phi^{-1})(u, y) = (u, 0)$ . ■

**2.6 Corollary.** When  $f : M \rightarrow N$  is a smooth map of smooth manifolds with constant rank  $r$  in  $M$ , for  $q \in \text{im } f$ , the level set (fibre)  $f^{-1}(q)$  can be given charts (obtained from canonical charts) to make it a smooth  $(\dim M - r)$ -dimensional manifold.

**Definition.** Let  $N$  and  $M$  be smooth manifolds (of dimensions  $m$  and  $n$ ). A smooth map  $f : N \rightarrow M$  is called a (smooth) **immersion** when  $f$  has rank  $n$  in  $N$ . An **immersed submanifold** of  $M$  is the image of an immersion  $f : N \rightarrow M$  or the image of an injective immersion  $f : N \rightarrow M$ .

Note that when  $f : N \rightarrow M$  is injective, we can give the image  $f(N)$  a smooth atlas which makes  $f : N \rightarrow f(N)$  a diffeomorphism. When we do this, the resulting topology on  $f(N) \subseteq M$  does not necessarily agree with the subspace topology of  $M$ .

**Definition.** An **embedded submanifold** of  $M$  is a subset  $N \subseteq M$  which is a smooth manifold such that the inclusion map  $f : N \rightarrow M$  (given by  $f(p) = p$ ) is an immersion such that the topology in the previous remark agrees with the subspace topology.

When  $f : M \rightarrow N$  is a smooth map of smooth manifolds of constant rank  $r$  and  $q \in \text{im } f$ , the level set  $f^{-1}(q)$  is an embedded submanifold of  $M$ .

**Remark.** When  $N \subseteq M$  is an embedded submanifold,

- If  $f : M \rightarrow K$  is smooth, then the restriction  $f : N \rightarrow K$  is smooth
- If  $f : K \rightarrow M$  is smooth and  $f(K) \subseteq N$ , then  $f : K \rightarrow N$  is smooth

**Example.**  ${}_n(\mathbb{R})$  is a smooth manifold. Recall that  $\text{GL}_n(\mathbb{R})$  is a smooth  $n^2$ -dimensional manifold, since it is open in the  $n^2$ -dimensional vector space  $M_n(\mathbb{R})$ . We have  ${}_n(\mathbb{R}) = f^{-1}(\{1\})$  where  $f$  is the determinant evaluation map. Then for fixed  $\ell$ ,  $\det X = \sum_{i=1}^n (-1)^{i+\ell} X_{\ell}^i \deg X_{\ell}^{(i)}$ , where  $X_{\ell}^{(i)}$  is the matrix obtained from  $X$  by removing row  $i$  and column  $j$ . We have

$$Df = \left( \mathbb{P} f x_1^1, \dots, \frac{\partial f}{\partial x_n^n} \right) \in M_{1 \times n^2}(\mathbb{R})$$

with  $\frac{\partial f}{\partial x_{\ell}^k} = (-1)^{k+\ell} \det X_{\ell}^{(k)}$ , so that  $Df = 0$  if and only if  $\det X = 0$ . Thus  $f$  has constant rank 1, so  ${}_n(\mathbb{R}) = f^{-1}(1)$  is an embedded submanifold of  $M_n(\mathbb{R})$  of dimension.

### 3 TANGENT VECTORS

**Definition.** A vector  $u$  in  $\mathbb{R}^n$  at a point  $a \in \mathbb{R}^n$  is an ordered pair  $(a, u)$ .

**Definition.** Let  $M$  be a smooth manifold and let  $p \in M$ . A **tangent vector** on  $M$  at  $p$  is a set of vectors  $X = \{\phi_*x : \phi \text{ is a chart on } M \text{ at } p\}$ , where  $\phi_*x$  is a vector in  $\mathbb{R}^n$  at the point  $x = \phi(p)$  such that when  $\phi$  and  $\psi$  are two charts on  $M$  at  $p$ , we have  $\psi_*X = D(\psi\phi^{-1})(\phi(p))\phi_*X$ .

The set of all tangent vectors on  $M$  at  $p$  is denoted by  $T_pM$ . Note that  $T_pM$  is an  $n$ -dimensional vector space. When  $I \subseteq \mathbb{R}$  is an open interval,  $s \in I$ , and  $\alpha : I \subseteq \mathbb{R} \rightarrow M$  is a smooth map with  $\alpha(s) = p$ , we define  $\alpha'(s)$  to be the tangent vector  $\alpha'(s) \in T_pM$  given by  $\phi_*\alpha'(s) = \beta'(s)$  where  $\beta(t) = \phi(\alpha(t))$ . Note that, by the chain rule, we do have  $\phi_*\alpha'(s) = D(\psi\phi^{-1})\phi_*\alpha'(s)$ .

When  $\phi$  is a chart on  $M$  at  $p$ , we often write

$$x = x(p) = \phi(p) = (\phi^1(p), \dots, \phi^n(p)) = (x^1(p), \dots, x^n(p))$$

(so each  $x^k = \phi^k$  is a function  $x^k, \phi^k : U \subseteq M \rightarrow \mathbb{R}$ ). When  $\psi$  is another chart and we write  $y = \psi(p)$ , we often write  $y = y(x) = (\psi\phi^{-1})(x) = (y^1(x), \dots, y^n(x))$  and we write

$$\frac{\partial y}{\partial x} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix}$$

With this notation, if  $u = \phi_*X$  and  $v = \psi_*X$ , then  $v = D(\psi\phi^{-1})u = \frac{\partial y}{\partial x}u$ , so  $V^k = \sum_{i=1}^n \frac{\partial y^k}{\partial x^i} u^i$ .

**Definition.** Let  $f : M \rightarrow N$  be a smooth map of smooth manifolds with  $p \in M$ . We define the **induced map** or the **pushforward**  $f_*$  or the **differential**  $df$  to be the map  $f_* = df : T_pM \rightarrow T_{f(p)}N$  given as follows. Given  $X \in T_pM$ , choose  $\alpha : (-\epsilon, \epsilon) \rightarrow M$  smooth with  $\alpha(0) = p$ ,  $\alpha'(0) = X$ , when we let  $\beta(t) = f(\alpha(t))$  and define  $df(x) = f_*(x) = \beta'(0)$ . Given a chart  $\phi$  on  $M$  at  $p$  and  $\psi$  on  $N$  at  $f(p)$ , if  $u = \phi_*X$  and  $V = \psi_*(f_*X)$ , then verify that  $v = D(\psi f \phi^{-1})(\phi(p))u$ .

1. When  $\phi$  is a chart on  $M$  at  $p$  and  $\psi$  is a chart on  $N$  at  $f(p)$ ,  $\psi_*f_*X = D(\psi f \phi^{-1})_{\phi(p)}\phi_*X$
2. The map  $df = f_*$  is linear
3. If  $g : L \rightarrow M$  and  $f : M \rightarrow N$  are smooth, then  $(f \circ g)_* = f_* \circ g_*$ .
4. When  $\iota : M \rightarrow M$  is the identity map,  $d\iota : T_pM \rightarrow T_pM$  is the identity map
5. If  $f : M \rightarrow N$  is a diffeomorphism, then  $f_* : T_pM \rightarrow T_pM$  is an isomorphism.
6. For  $f : M \rightarrow N$  smooth,  $f$  is of rank  $r$  at  $p$  if and only if  $f_*$  is of rank  $r$  at  $p$ .

When  $U \subseteq \mathbb{R}^n$  is open,  $U$  is a manifold with atlas  $\{\emptyset\}$  where  $\phi$  is the identity map. In this case, we identify  $X \in T_pU$  with  $\phi_*X \in \mathbb{R}^n$ . With this convention,  $\phi_*X$  is equal to  $\phi_*X$  where the second  $\phi_*$  is the pushforward. When  $N \subseteq M$  is a submanifold (immersed or embedded), the inclusion map  $\iota : N \rightarrow M$  is an injective immersion. Thus, the map  $\iota_* : T_pN \rightarrow T_pM$ . In this situation, we identify  $T_pN$  with the subspace  $\iota_*(T_pN) \subseteq T_pM$ .

Let  $X$  be the vector on  $\mathbb{S}^2$  at  $p$  with  $\phi_*X = (1, 0)$ . Let  $\iota : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  be the inclusion map. We have  $\phi^{-1}(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$  with  $u = \phi_*X = (1, 0)$ . Then  $\iota_*X = D(\psi\phi^{-1})_{\phi(p)}\phi_*X$  where  $\psi$  is the identity on  $\mathbb{R}^3$ .

#### TANGENT VECTORS AS DIFFERENTIAL OPERATORS

Recall that a vector  $u \in \mathbb{R}^n$  at a point  $a \in \mathbb{R}^n$  acts as a differential operator on smooth maps  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  by directional derivative. Choose any smooth map  $\alpha : (-\epsilon, \epsilon) \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$  with



$\alpha(0) = a$  and  $\alpha'(0) = u$ , and define  $u(f) = u_a(f) = D_u f(a) = \beta'(0)$  where  $\beta(t) = f(\alpha(t))$ . Since  $\beta(t) = f(\alpha(t))$ , we have  $\beta'(t) = Df(\alpha(t)) \cdot \alpha'(t)$  so

$$\begin{aligned} u(f) &= D_u f(a) = \beta'(0) = Df(a) \cdot u \\ &= \left( \frac{\partial f}{\partial x^1}(a), \dots, \frac{\partial f}{\partial x^n}(a) \right) \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a) \cdot u^i \end{aligned}$$

or as a differential operator,  $u = \sum_{i=1}^n u^i \frac{\partial}{\partial x^i}$ .

**Definition.** When  $M$  is a smooth manifold,  $p \in M$ , and  $X \in T_p M$ ,  $X$  acts as a differential operator on a smooth function  $f : M \rightarrow \mathbb{R}$  as follows: choose a smooth map  $\alpha : (-\epsilon, \epsilon) \subseteq \mathbb{R} \rightarrow M$  with  $\alpha(0) = p$  and  $\alpha'(0) = X$ , and define  $X(f) = X_p(f) = \beta'(0)$  where  $\beta(t) = f(\alpha(t))$ .

When  $\phi$  is a chart on  $M$  at  $p$ , then

$$\begin{aligned} X(f) &= (\phi_* X)(f \circ \phi^{-1}) = D_{\phi_* X}(f \circ \phi^{-1})(\phi(p)) \\ &= D(f \circ \phi^{-1})(\phi(p)) \cdot (\phi_* X) \\ &= \sum_{i=1}^n \frac{\partial f \circ \phi^{-1}}{\partial x^i}(\phi(p)) \cdot u^i \end{aligned}$$

where  $u = \phi_* X \in \mathbb{R}^n$ . So when  $u = \phi_* X \in \mathbb{R}^n$ ,  $X$  acts as the differential operator  $X = \sum_{i=1}^n u^i \frac{\partial}{\partial x^i} \Big|_p$  where  $\frac{\partial}{\partial x^i} \Big|_p(f) = \frac{\partial f \circ \phi^{-1}}{\partial x^i}(\phi(p))$ . With this notation,

$$T_p M = \text{span} \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

If  $\phi$  and  $\psi$  are two charts at  $p$  on  $M$ , then  $T_p M$  has two representations as differential operators. Let us determine how  $\frac{\partial}{\partial x^k}$  and  $\frac{\partial}{\partial y^\ell}$  are related. When  $X \in T_p M$ ,  $u = \phi_* X \in \mathbb{R}^n$  and  $v = \psi_* X \in \mathbb{R}^n$ , we have  $V = D(\psi \circ \phi^{-1})(\phi(p)) \cdot u = \left( \frac{\partial y}{\partial x} \right)(\phi(p)) \cdot u$ . When  $u = \frac{\partial}{\partial x^j}$ ,

$$v = \left( \frac{\partial y}{\partial x} \right) \cdot e_k$$

so  $v = \left( \frac{\partial y}{\partial x} \right)_k^\ell = \frac{\partial y^\ell}{\partial x^k}$  so that

$$v = \sum_{i=1}^n \frac{\partial y^i}{\partial x^k} \frac{\partial}{\partial y^i}$$

**Definition.** A **derivation** on  $M$  at  $p$  is a linear map  $L : C^\infty(M) \rightarrow \mathbb{R}$  or  $L : C_p^\infty(M) \rightarrow \mathbb{R}$  where  $C_p^\infty(M)$  is the vector space of **germs** of smooth functions on  $M$  at  $p$ , which satisfies the product rule at  $p$ :

$$L(fg) = L(f) \cdot g(p) + f(p) \cdot L(g)$$

Every  $X \in T_p M$  gives a derivation on  $M$  at  $p$ . Moreover, it can be shown that every derivation on  $M$  at  $p$  is of this form. Thus allows us to give an alternate definition for  $T_p M$  as the space of derivations on  $M$  at  $p$ .

**Definition.** Let  $TM$  be the disjoint union of all the tangent spaces. A **vector field** on  $M$  is a function  $X : M \rightarrow TM$  such that  $X(p) \in T_p M$ .

Given a chart  $\phi : U \rightarrow \phi(U)$  on  $M$ , the restriction of  $X$  to  $U$  determines and is determined by the vector field  $\phi_* X$  on  $\phi(U) \subseteq \mathbb{R}^n$  by  $(\phi_* X)(\phi(p)) = \phi_*(X(p))$ , or  $(\phi_* X)(x) = \phi_*(X(\phi^{-1}(x))) \in \mathbb{R}^n$ . We say that  $X$  is **smooth** at  $p$  when for some chart  $\phi$  on  $M$  at  $p$ , the vector field  $\phi_* X$  is smooth at  $\phi(p)$ . When  $X$  is a smooth vector field on  $M$ ,  $X$  acts as a differential operator  $X : C^\infty(M) \rightarrow C^\infty(M)$  by  $X(f)(p) = X_p(f)$ .