

Representation Theory of Finite Groups

Alex Rutar*
University of Waterloo

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**arutar@uwaterloo.ca*

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I. Introduction

Let G be a finite group of order n , and write $G = \{g_1, \dots, g_n\}$. Fix $g \in G$; then $gg_i = gg_j$ if and only if $i = j$. Thus there exists some $\sigma_g \in S_n$ such that $gg_i = g_{\sigma_g(i)}$ for all $i \in \{1, 2, \dots, n\}$. In particular, $\phi : G \rightarrow S_n$ by $\phi(g) = \sigma_g$ is an embedding (injective group homomorphism). This observation is usually referred to as Cayley's Theorem.

Now let V be an n -dimensional complex vector space. We then denote $\text{GL}(V)$ as the group of invertible linear operators $T : V \rightarrow V$. Now define $\psi : S_n \rightarrow \text{GL}(V)$ by $\psi(\sigma) = T_\sigma$ where if $\{b_1, \dots, b_n\}$ is a basis for V and $T_\sigma(b_i) = b_{\sigma(i)}$. This is an injective group homomorphism, so $\psi \circ \phi : G \rightarrow \text{GL}(V)$ is an embedding of G into $\text{GL}(V)$.

Definition. Let G be a finite group, and V a finite dimensional \mathbb{C} -vector space. A **representation** of G is a group homomorphism $\rho : G \rightarrow \text{GL}(V)$. We call $\dim(V)$ the **degree** of the representation.

In particular, if V is n -dimensional, then $\text{GL}(V) \cong \text{GL}_n(\mathbb{C})$.

Example. 1. Consider $\rho : G \rightarrow \text{GL}(\mathbb{C}) \cong \mathbb{C}^\times$ given by $\rho(g) = 1$ for all $g \in G$. This is called the *trivial representation*.

2. Consider $\rho : S_n \rightarrow \mathbb{C}^\times$ given by $\rho(\sigma) = \text{sgn}(\sigma)$, which is called the *sign representation*.

3. The representation of G afforded by Cayley's theorem is called the *regular representation* of G . The next example is a good way to understand the regular rep of G .

4. Consider G , $X = \{x_1, \dots, x_n\}$, and $V = \text{Free}(X)$. Suppose G acts on X . Then $\rho : G \rightarrow \text{GL}(V)$ given by $\rho(g)(x_i) = gx_i$. In particular, if we take $X = G$, then this is the regular representation of G .

5. Consider the 4-gon, with vertices labelled a, b, c, d . Take $X = \{a, b, c, d\}$ and the regular representation $\rho : D_4 \rightarrow \text{GL}(V)$. This action has a geometric notion.

6. Let C_n be a cyclic group of order n ; let us define some $\rho : C_n \rightarrow \text{GL}(V)$. Say $\rho(x) = T$ where $t \in \text{GL}(V)$; then this is a representation if and only if $T^n = I$.

Definition. We say that two representations $\rho : G \rightarrow \text{GL}(V)$ and $\tau : G \rightarrow \text{GL}(W)$ are **isomorphic** if there exists an isomorphism $T : V \rightarrow W$ such that for all $g \in G$,

$$T \circ \rho(g) = \tau(g) \circ T$$

Suppose $\rho : G \rightarrow \text{GL}(V)$ and $T : V \rightarrow W$ is an isomorphism. Then we can define $\tau : G \rightarrow \text{GL}(W)$ by $\tau(g) = T \circ \rho(g) \circ T^{-1}$; this $\rho \cong \tau$. In other words, the representation is unique up to isomorphism under change of basis.

Example. Consider $G = \{g_1, \dots, g_n\} = \{h_1, \dots, h_n\}$, and fix $g \in G$. Let $gg_i = g_{\alpha(i)}$ and $gh_i = h_{\beta(i)}$ where $\alpha, \beta \in S_n$. Fix an n -dimensional vector space V with basis $\{b_1, \dots, b_n\}$. Then two regular representations are given by

$$\rho_1 : G \rightarrow \text{GL}(V), \rho_1(g)(b_i) = b_{\alpha(i)}$$

$$\rho_2 : G \rightarrow \text{GL}(V), \rho_2(g)(b_i) = b_{\beta(i)}$$

Let $\gamma \in S_n$ be such that $h_{\gamma(i)} = g_i$, and define $T : V \rightarrow V$ by $T(b_i) = b_{\gamma(i)}$. Then

$$gg_i = g_{\alpha(i)} = gh_{\gamma(i)} = h_{\beta\gamma(i)} = g_{\gamma^{-1}\beta\gamma(i)}$$

so that $\alpha = \gamma^{-1}\beta\gamma$. Thus for each b_i ,

$$\begin{aligned} T \circ \rho_1(g) \circ T^{-1}(b_i) &= T \circ \rho_1(g)(b_{\gamma^{-1}(i)}) \\ &= T(b_{\alpha\gamma^{-1}(i)})b_{\gamma\alpha\gamma^{-1}(i)} \\ &= b_{\beta(i)} = \rho_2(g)(b_i) \end{aligned}$$

so that $T \circ \rho_1(g) \circ T^{-1} = \rho_2(g)$.

Note: conjugate elements have the same cycle type.

SUBREPRESENTATIONS

What should a subrepresentation of $\rho : G \rightarrow \text{GL}(V)$ mean?

We would like a subspace $W \leq V$ such that $\tau : G \rightarrow \text{GL}(W)$ is a representation given by $\tau(g)(w) = \rho(g)(w)$ for all $w \in W$. Moreover, to make this well-defined, we need W to be $\rho(g)$ -invariant for every $g \in G$ ($\rho(g)(W) \subseteq W$).

Suppose $T : V \rightarrow V$ is a linear operator, and $W \leq V$ is a T -invariant subspace; i.e. $T(W) \subseteq W$. In particular, the restriction operator $T_W : W \rightarrow W$ is well-defined.

Definition. Let $\rho : G \rightarrow \text{GL}(V)$ be a representation. A subspace $W \subseteq V$ is said to be **G -stable** if W is $\rho(g)$ -invariant for all $g \in G$. A **subrepresentation** of ρ is a representation $\rho_W : G \rightarrow \text{GL}(W)$ where for all $g \in G$ and $w \in W$, $\rho_W(g)(w) = \rho(g)(w)$ where W is a G -stable subspace of V .

Example. Suppose $\rho : G \rightarrow \text{GL}(V)$ be the regular representation. Take $W = \text{span}\{\sum_{g \in G} v_g\}$, which is clearly G -stable, and $\rho_W : G \rightarrow \text{GL}(W)$ is isomorphic to the trivial representation.

Similarly, let $\rho : S_n \rightarrow \text{GL}(V)$ be the regular representation, $W = \text{span}\{\sum_{\sigma \in S_n} \text{sgn}(\sigma)v_\sigma\}$; this is isomorphic to the sign representation.

0.1 Theorem. Let $\rho : G \rightarrow \text{GL}(V)$ be a representation, $W \leq V$ G -stable. Then there exists a G -stable subspace W' such that $V = W \oplus W'$.

PROOF Take any inner product $\langle x, y \rangle$ on V . Then for any $x, y \in V$, define

$$\langle x, y \rangle^* = \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle$$

This is also an inner product. Let $x, y \in V$ and let $h \in G$. Then

$$\begin{aligned} \langle \rho(h)(x), \rho(h)(y) \rangle^* &= \sum_{g \in G} \langle \rho(g)\rho(h)(x), \rho(g)\rho(h)(y) \rangle \\ &= \sum_{g \in G} \langle \rho(gh)(x), \rho(gh)(y) \rangle \\ &= \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle \end{aligned}$$

Thus every $\rho(h)$ is unitary with respect to $\langle \cdot, \cdot \rangle^*$. Let $W \leq V$ be G -stable, and take $W' = W^\perp$ with respect to $\langle \cdot, \cdot \rangle^*$. Then $V = W \oplus W'$. Let's see that W^\perp is G -stable. Let $x \in W^\perp$, $w \in W$,

and $g \in G$, so that

$$\begin{aligned} \langle \rho(g)(x), w \rangle^* &= \langle x, \rho(g)^*(w)^* \rangle = \langle x, \rho(g)^{-1}(w) \rangle^* \\ &= \langle x, \underbrace{\rho(g^{-1})(w)}_{\in W} \rangle^* \\ &= 0 \end{aligned}$$

and $\rho(g)(W^\perp) \subseteq W^\perp$ as required. \blacksquare

Definition. Let $\rho : G \rightarrow \text{GL}(V)$ be a representation, and $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ where each W_i is G -stable. For each i , let $\rho_i = \rho|_{W_i}$. For each $v = \sum w_i \in V$, we have $\rho(g)(v) = \sum \rho(g)(w_i) = \sum \rho_i(g)(w_i)$. In this case, we write

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k$$

and call ρ a direct sum of the ρ_i 's.

The previous definition is written as an internal direct sum of V . Externally, given vector spaces W_1, \dots, W_k and representations $\rho_i : G \rightarrow \text{GL}(W_i)$, we can define

$$(\rho_1 \oplus \cdots \oplus \rho_k) : G \rightarrow \text{GL}(W_1 \oplus \cdots \oplus W_k)$$

by $(\rho_1 \oplus \cdots \oplus \rho_k)(g)(w_1, \dots, w_k) = (\rho_1(g)(w_1), \dots, \rho_k(g)(w_k))$. If $\rho_i : G \rightarrow \text{GL}(W_i)$ is a subrepresentation for $\rho : G \rightarrow \text{GL}(V)$, we often say “ W_i is a subrepresentation of V ”.

Definition. Let $\rho : G \rightarrow \text{GL}(V)$ be a representation. We say ρ is **irreducible** if $V \neq \{0\}$ and the only G -stable subspaces of V are $\{0\}$ and V .

Clearly,

0.2 Theorem. Every representation $\rho : G \rightarrow \text{GL}(V)$ can be written as a direct sum of irreducible sub-representations.

Example. Let $\rho : S_3 \rightarrow \text{GL}(\mathbb{C}^3)$ be the permutation representation with respect to the standard basis $\{e_1, e_2, e_3\}$. Consider $W_1 = \text{span}\{e_1 + e_2 + e_3\}$ and $W_2 = \text{span}\{e_1 - e_2, e_2 - e_3\}$. Is W_2 irreducible?

More generally, if $V = W_1 \oplus \cdots \oplus W_k$ and $\dim W_i = 1$ and $\deg(\rho_i) = 1$,

$$\rho(gh)(\sum w_i) = \sum \rho_i(gh)(w_i) = \sum \rho_i(g)\rho_i(h)(w_i) = \sum \rho_i(h)\rho_i(g)(w_i)$$

so that $\rho(gh) = \rho(hg)$. In our example, this does not happen, since $\rho(g) \neq I$ when $g \neq 1$ and S_3 is not abelian.

Example. Let $\rho : S_3 \rightarrow \text{GL}(V)$ be the regular representation. Let $W_1 = \text{span}\{\sum_{\sigma \in S_3} v_\sigma\}$ and $W_2 = \text{span}\{\sum_{\sigma \in S_3} \text{sgn}(\sigma)v_\sigma\}$, and Now let's focus on W_3 . A basis for W_3 is given by

$$\begin{aligned} e_1 &= v_e - v_{(123)} & e_2 &= v_e - v_{(123)} \\ e_3 &= v_{(12)} - v_{(13)} & e_4 &= v_{(12)} - v_{(23)} \end{aligned}$$

Recall that $S_3 = \langle (12), (123) \rangle$; suffices to show stability with respect to generators.

$$\begin{aligned} \rho(12) : e_1 &\mapsto e_4, e_2 \mapsto e_3, e_3 \mapsto e_2, e_4 \mapsto e_1 \\ \rho(123) : e_1 &\mapsto e_2 - e_1, e_2 \mapsto -e_1, e_3 \mapsto e_4 - e_3, e_4 \mapsto -e_3 \end{aligned}$$

Let $U_1 = \text{span}\{e_1 - e_4, e_2 + e_3 - e_1\}$

1 TENSOR PRODUCTS

Let $\rho : G \rightarrow \text{GL}(V)$ and $\tau : G \rightarrow \text{GL}(W)$ be representations. We define the representation $\rho \otimes \tau : G \rightarrow \text{GL}(V \otimes W)$

$$(\rho \otimes \tau)(g)(v \otimes w) = \rho(g)(v) \otimes \tau(g)(w)$$

2 CHARACTER THEORY

We define the character of ρ by $\rho : G \rightarrow \mathbb{C}$ as $\chi(G) = \text{Tr}(\rho(g))$.

Remark. If we choose a basis β for V , then define $A(g) = [\rho(g)]_\beta$ and $\chi(G)$ is given by the sum of the diagonal entries of $A(g)$. Furthermore, if $A, B \in M_n(\mathbb{C})$, then $\text{Tr}(AB) = \text{Tr}(BA)$.

The remark implies a number of facts:

- (i) $\rho \cong \tau$, then $\text{Tr}(\rho(g)) = \text{Tr}(\tau(g))$.
- (ii) $\text{Tr}(T)$ is the sum of eigenvalues of T
- (iii) $\chi(1) = \dim(V)$.

2.1 Proposition. *For every $g \in G$ the eigenvalues of $\rho(g)$ have modulus 1. In particular, $\chi(g^{-1}) = \overline{\chi(g)}$.*

PROOF Set $n = |G|$; then $\rho(g)^n = \rho(g^n) = I$ so that $\lambda^n - 1 = 0$ for any eigenvalue λ , so $|\lambda| = 1$. Furthermore,

$$\overline{\chi(g)} = \overline{\sum \lambda_i} = \sum \overline{\lambda_i} = \sum \lambda_i^{-1} = \chi(g^{-1})$$

proving the second component. ■

2.2 Proposition. *Let $\rho : G \rightarrow \text{GL}(V)$ and $\tau : G \rightarrow \text{GL}(W)$. Then $\chi_{\rho \oplus \tau} = \chi_\rho + \chi_\tau$ and $\chi_{\rho \otimes \tau} = \chi_\rho \cdot \chi_\tau$.*

PROOF Let $\beta_1 = \{v_1, \dots, v_n\}$ be a basis for V and $\beta_2 = \{w_1, \dots, w_m\}$ a basis for W .

Then a basis for $V \oplus W$ is given by $\beta = \{(v_1, 0), \dots, (v_n, 0), (0, w_1), \dots, (0, w_m)\}$. In particular,

$$[(\rho \oplus \tau)(g)]_\beta = \begin{pmatrix} [\rho(g)]_{\beta_1} & \\ & [\tau(g)]_{\beta_2} \end{pmatrix}$$

and the trace result follows.

A basis for $V \otimes W$ is given by $\gamma = \{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ in lexicographic order. Fix $g \in G$, and set $A = [\rho(g)]_{\beta_1}$, $B = [\tau(g)]_{\beta_2}$. Fix $v_i \otimes w_j \in \gamma$. Then

$$\begin{aligned} (\rho \otimes \tau)(g)(v_i \otimes w_j) &= \rho(g)(v_i) \otimes \tau(g)(w_j) \\ &= (a_{1i}v_1 + \dots + a_{ni}v_n) \otimes (b_{1j}w_1 + \dots + b_{mj}w_m) \\ &= \dots + a_{ii}b_{jj} \cdot (v_i \otimes w_j) + \dots \\ &= \text{Tr}([\rho \otimes \tau](g))_\delta = \sum_{i,j} a_{ii}b_{jj} = \text{Tr}(A)\text{Tr}(B) = \chi_\rho(g) \cdot \chi_\tau(g) \end{aligned} \quad \blacksquare$$

Example. Suppose $\rho : S_n \rightarrow \text{GL}(\mathbb{C}^n)$ is the permutation representation with respect to $\{e_1, \dots, e_n\}$. Then $\chi(\sigma) = |\{e_i : \rho(\sigma)(e_i) = e_i\}| = |\text{Fix}(\sigma)|$, which is the number of indices i fixed by σ . Since S_n acts transitively on $\{1, \dots, n\}$, there is exactly 1 orbit, so by Burnside's lemma,

$$n! = |S_n| = \sum_{\sigma \in S_n} \chi(\sigma)$$

Example. Let $\rho : G \rightarrow \text{GL}(V)$ be the regular representation. Note that if $g \neq 1$, then for all $h \in G$, $gh \neq h$. In particular, this means that $\chi(g) = 0$ if $g \neq 1$, and $\chi(1) = |G|$ (the dimension of V).

Example. Let $\rho : S_3 \rightarrow \text{GL}(V)$ be the regular representation. Recall that $V = W_1 \oplus W_2 \oplus U_1 \oplus U_2$ where W_1 is the trivial representation, W_2 is the sign representation, and U_1, U_2 are isomorphic. Let $S_3 = \langle (12), (123) \rangle$; then we have

x_1	1	1
x_2	-1	1
x_3	a	b
x_4	a	b

In particular, $\chi(12) = 1 - 1 + 2a = 0$ and $\chi(123) = 1 + 1 + 2b = 0$, so $b = -1$.

Example. Let $\rho : G \rightarrow \text{GL}(V)$ be a representation. In particular, $\rho(ghg^{-1}) = \rho(g)\rho(h)\rho(g)$ so that $\text{Tr} \rho(ghg^{-1}) = \text{Tr} \rho(h)$ so $\chi(ghg^{-1}) = \chi(h)$; in other words, that characters are constant on conjugacy classes.

2.3 Lemma. (Schur) Let $\rho : G \rightarrow \text{GL}(V)$ and $\tau : G \rightarrow \text{GL}(W)$ be irreducible representations, and suppose $T : V \rightarrow W$ is linear such that for all $g \in G$, $\tau(g) \circ T = T \circ \rho(g)$. Then either $T = 0$ or T is an isomorphism and $\rho \cong \tau$. Moreover, if $V = W$ and $\rho = \tau$, then T is a scalar multiple of the identity.

PROOF Assume $T \neq 0$.

Let's first see that T is injective, and let $v \in \ker(T)$. Then for any $g \in G$, $T(\rho(g)(v)) = \tau(g)(T(v)) = 0$, so $\rho(g)(v) \in \ker(T)$. Thus $\ker(T)$ is G -stable (with respect to ρ). Since ρ is irreducible and $T \neq 0$, $\ker(T) = \{0\}$.

We also have that T is surjective. Let $v \in \text{Im}(T)$ and say $v = T(X)$ with $x \in V$. Then for $g \in G$, $\tau(g)(v) = \tau(g)(T(x)) = T(\rho(g)(x)) \in \text{Im}(T)$ so $\text{Im}(T)$ is G -stable, and again by irreducibility of τ , $\text{Im}(T) = W$. Thus T is an isomorphism.

Now let $\lambda \in \mathbb{C}$ be an eigenvalue of T and consider $T' = T - \lambda I$. Now, note that for $g \in G$, $\rho(g)T' = T'\rho(g)$, but T' has non-trivial kernel, so in fact $T' = 0$. ■

2.4 Corollary. Let $\rho : G \rightarrow \text{GL}(V)$ and $\tau : G \rightarrow \text{GL}(W)$ be irreducible, and $T : V \rightarrow W$ linear. Consider

$$T' = \frac{1}{|G|} \sum_{g \in G} \tau(g)^{-1} T \rho(g)$$

Then

- (i) If $T' \neq 0$, then $\rho \cong \tau$ via T' .
- (ii) If $V = W$, $\rho = \tau$, then $T' = \text{Tr}(T)/\dim(V) \cdot I$.

PROOF Clearly $T' : V \rightarrow W$ is linear, and for any $h \in G$,

$$\begin{aligned}\tau(h)T' &= \tau(h) \frac{1}{|H|} \sum_{g \in G} \tau(g^{-1})T\rho(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \tau(hg^{-1})T\rho(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \tau(g^{-1})T(\rho(g)) \\ &= \frac{1}{|G|} \sum_{g \in G} \tau(g^{-1})T\rho(g)\rho(h) \\ &= T'\rho(h)\end{aligned}$$

If $V = W$ and $\rho = T$, then $\text{Tr}(T') = \frac{1}{|G|} \text{Tr}(T) \cdot |G| = \text{Tr}(T) = \alpha \dim(V)$, so $\alpha = \text{Tr}(T)/\dim(V)$. ■

Let $\rho : G \rightarrow \text{GL}(V)$ and $\tau : G \rightarrow \text{GL}(W)$ be irreducible representations, and $T : V \rightarrow W$ linear. Let β be a basis for V and γ a basis for W . Then for $g \in G$, let $[\rho(g)]_\beta = (a_{ij}(g))$, $[\tau(g)]_\gamma = (b_{kl}(g))$, $[T]_\beta^\gamma = (x_{ki})$, and $[T']_\beta^\gamma = (x'_{ki})$.

By matrix multiplication, $x'_{ki} = \frac{1}{|G|} \sum_g \sum_{j,l} b_{kl}(g^{-1})x_{lj}a_{ji}(g)$. If $\rho \not\cong \tau$, then $T' = 0$, so by viewing the RHS as a polynomial in the x_{ij} , we have

$$\frac{1}{|G|} \sum_g b_{kl}(g^{-1})a_{ji}(g) = 0$$

But now if $\rho = \tau$, then $T' = \lambda I$ where $\lambda = \text{Tr}(T)/\dim(V)$ so that

$$\frac{1}{|G|} \sum_g \sum_{j,l} a_{kl}(g^{-1})x_{lj}a_{ji}(g) = \lambda \delta_{ki} = \frac{1}{\dim(V)} \sum_{j,l} \delta_{ki} \delta_{jl} x_{lj}$$

Then by equating coefficients of x_{lj} , we have

$$\frac{1}{|G|} \sum_g a_{kl}(g^{-1})a_{ji}(g) = \frac{1}{\dim(V)} \delta_{ki} \delta_{jl}$$

Remark. If G is a finite group, then consider the vector space of all functions $\phi : G \rightarrow \mathbb{C}$. For any ϕ, ψ in this vector space, $\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_g \phi(g) \overline{\psi(g)}$ defines an inner product. Then if χ_1, χ_2 are characters of G , then

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_g \chi_1(g) \overline{\chi_2(g)}$$

We thus have:

2.5 Theorem. If χ is a character of an irreducible representation, then $\langle \chi, \chi \rangle = 1$, and if χ_1 and χ_2 correspond to non-isomorphic representations, then $\langle \chi_1, \chi_2 \rangle = 0$.

PROOF Say $[\rho(g)]_\beta = (a_{ij}(g))$ where ρ is an irreducible representation with character χ . Then

$$\begin{aligned} \langle \chi, \chi \rangle &= \frac{1}{|G|} \sum_g \chi(g) \chi(g^{-1}) = \frac{1}{|G|} \sum_g \chi(g^{-1}) \chi(g) \\ &= \frac{1}{|G|} \sum_g \sum_{i,j} a_{ii}(g^{-1}) a_{jj}(g) = \sum_{i,j} \left(\frac{1}{|G|} \sum_g a_{ii}(g^{-1}) a_{jj}(g) \right) \\ &= \sum_{i,j} \left(\frac{1}{|G|} \sum_g a_{ii}(g^{-1}) a_{ii}(g) \right) \\ &= \sum_i \frac{1}{\dim(V)} = 1 \end{aligned}$$

To see the second part,

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_g \chi_1(g) \chi_2(g^{-1}) = \frac{1}{|G|} \sum_g \sum_{ij} a_{ii}(g) a_{jj}(g^{-1}) = \sum_{i,j} 0 = 0 \quad \blacksquare$$

If χ is a character corresponding to an irreducible representation, we say χ is irreducible. If ρ and τ are isomorphic representations, we say χ_ρ and χ_τ are isomorphic (in fact $\chi_\rho = \chi_\tau$).

2.6 Corollary. Let $\rho : G \rightarrow \text{GL}(V)$ be a representation with character χ . Say $V = W_1 \oplus \cdots \oplus W_k$ is an irreducible decomposition of V . If $\tau : G \rightarrow \text{GL}(W)$ is an irreducible representations with character ϕ , then the number of W_i isomorphic to W (i.e. $\rho_i \cong \tau$) is $\langle \chi, \phi \rangle$.

PROOF Write $\chi = n_1 \chi_1 + \cdots + n_l \chi_l$, where the χ_i are pairwise non-isomorphic. Then $\langle \chi, \chi_i \rangle = n_i$. ■

Let $\tau : G \rightarrow \text{GL}(V)$ be irreducible, and let τ have character ϕ . Then

$$\langle \chi, \phi \rangle = \sum_{i=1}^k \langle \chi_i, \phi \rangle$$

Now, $\langle \chi_i, \phi \rangle = 1$ if and only if $\rho_i \cong \tau$, so that $\langle \chi, \phi \rangle$ counts the number of times in which τ appears in the irreducible decomposition of ρ .

2.7 Corollary. If two representations of G have the same character, then they are isomorphic.

PROOF They have the same irreducible decomposition. ■

2.8 Corollary. If $\rho : G \rightarrow \text{GL}(V)$ is a representation and χ is a character, then $\langle \chi, \chi \rangle \in \mathbb{N}$ and $\langle \chi, \chi \rangle = 1$ if and only if χ is irreducible.

PROOF If χ_1, \dots, χ_k are irreducible, write $\chi = n_1 \chi_1 + \cdots + n_k \chi_k$ so that $\langle \chi, \chi \rangle = n_1^2 + \cdots + n_k^2 \in \mathbb{N}$. ■

2.9 Proposition. Every irreducible representation of G occurs as a subgroup for the regular representation of G , with multiplicity equal to its degree.

PROOF Let χ be an irreducible character of G . Then

$$\langle \chi, \chi_{\text{reg}} \rangle = \frac{1}{|G|} \sum_g \chi(g) \overline{\chi_{\text{reg}}(g)} = \frac{1}{|G|} \chi(1) \overline{\chi_{\text{reg}}(1)} = \frac{1}{|G|} \deg(\chi) \quad \blacksquare$$

2.10 Corollary. Let χ_1, \dots, χ_k be the distinct irreducible characters of G , with $\deg(\chi_i) = n_i$. Then $\sum n_i^2 = |G|$ for $g \neq 1$, $\sum_{i=1}^k n_i \chi_i(g) = 0$

PROOF Recall that $\chi_{\text{reg}} = n_1 \chi_1 + \dots + n_k \chi_k$. Then $\chi_{\text{reg}}(1) = |G| = n_1^2 + \dots + n_k^2$, and evaluation at $g \neq 1$ gives the desired result. \blacksquare

Definition. Let G be a group. A function $f : G \rightarrow \mathbb{C}$ is called a class function if f is constant on each conjugacy class, i.e. for all $a, b \in G$, $f(bab^{-1}) = f(a)$.

2.11 Proposition. Let $\rho : G \rightarrow \text{GL}(V)$ be a representation. Then

$$\rho_f = \sum_g f(g) \rho(g)$$

is a linear operator on V . If ρ is irreducible of degree n , then $\rho_f = \lambda I$, where $\lambda = \frac{|G|}{n} \langle f, \bar{\chi} \rangle$ where χ is the character of ρ .

PROOF Note that

$$\begin{aligned} \rho_f \circ \rho(h) &= \sum_g f(g) \rho(g) \rho(h) = \sum_g f(g) \rho(gh) \\ &= \sum_g f(hgh^{-1}) \rho(hg) \\ &= \sum_g f(g) \rho(h) \rho(g) = \rho(h) \circ \rho_f \end{aligned}$$

so by Schur, $\rho_f = \lambda I$ where $\lambda = \text{Tr}(\rho_f)/n$. However, $\text{Tr}(\rho_f) = \text{Tr}(\sum_g f(g) \rho(g)) = \sum_g f(g) \chi(g) = |G| \langle f, \bar{\chi} \rangle$. \blacksquare

Recall that

- $\langle \chi, \chi \rangle = 1$ if and only if χ is irreducible
- If χ_ρ and χ_τ are irreducible then $\langle \chi_\rho, \chi_\tau \rangle = 0$ if $\rho \not\cong \tau$, and 1 otherwise.
- If χ' is an irreducible subrepresentation of χ , then $\langle \chi, \chi' \rangle$ is the multiplicity of χ' in χ .
- $|G| = n_1^2 + \dots + n_k^2$ where n_i is the multiplicity of χ_i as an irreducible subrepresentation of the regular representation.
- Every irreducible character is a character of some subrepresentation of the regular rep?
- ... every irreducible representation is a subrepresentation of the regular rep?

and

$$\rho_f = \sum_g f(g)\rho(g) = \lambda I$$

where $\lambda = |G|/\dim(V) \cdot \langle f, \bar{\chi} \rangle$.

2.12 Proposition. *Let G be a group. The irreducible characters of G form an orthonormal basis for the vector space of class functions on G .*

PROOF Let $\beta = \{\chi_1, \dots, \chi_k\}$ be the irreducible characters of G . We know that β is orthonormal, and hence linearly independent. Let $W = \text{span}(\beta)$. To show $W = V$ where V is the space of class functions, we prove that $W^\perp = \{0\}$. Let $f \in W^\perp$, and suppose $\rho : G \rightarrow \text{GL}(V)$ is irreducible. By A2, $\bar{\chi}_1, \dots, \bar{\chi}_k$ are all irreducible characters of G . Thus $\rho_f = 0$. By considering irreducible decompositions, $\rho_f = 0$ for all representations $\rho : G \rightarrow \text{GL}(V)$. In particular, when ρ is the regular representation,

$$0 = \rho_f(v_1) = \sum_g f(g)\rho(g)(v_1) = \sum_g f(g)v_g$$

so by independence of $\{v_g : g \in G\}$, $f(g) = 0$ for all $g \in G$. ■

2.13 Corollary. *The number of irreducible characters of G is equal to the number of conjugacy classes of G .*

PROOF Let C_1, \dots, C_k be the conjugacy classes. Then a basis for $V_{\text{class}} = \{\phi_1, \dots, \phi_k\}$ where each ϕ_i is the indicator for C_i . Since bases must have the same size, the result follows. ■

2.14 Proposition. *Let G be a group, $g \in G$, and O_g the conjugacy class of g . Let χ_1, \dots, χ_k be the irreducible characters of G . Then*

1. $\sum_{i=1}^k |\chi_i(g)|^2 = |G|/|O_g|$
2. If h is not conjugate to g , then $\sum_{i=1}^k \chi_i(g)\overline{\chi_i(h)} = 0$.

PROOF Define $\phi : G \rightarrow \mathbb{C}$ where $\phi(x)$ is the indicator function for O_g . Write $\phi = \sum_{i=1}^k \lambda_i \chi_i$ where

$$\lambda_i = \langle \phi, \chi_i \rangle = \frac{1}{|G|} \sum_x \phi(x)\overline{\chi_i(x)} = \frac{|O_g|\overline{\chi_i(g)}}{|G|}$$

Therefore,

$$\phi(x) = \frac{|O_g|}{|G|} \sum_{i=1}^k \overline{\chi_i(g)} \chi_i(x)$$

Then the result follows by evaluating ϕ at g and h . ■

Example. Let's compute the character table of S_3 . There are 2 degree 1 representations, and 3 irreducible characters since there are three conjugacy classes (cycle types). In particular, $|S_3| = 6 = 1^2 + 1^2 + n_3^2$, so $n_3 = 2$.

	ϵ (12) (123)
$(\text{triv})\chi_1$	1 1 1
$(\text{sgn})\chi_2$	1 -1 1
χ_3	2 a b

Note that the columns must be orthogonal, so by the previous proposition, we have $a = 0$ and $b = -1$.

Let χ_1, \dots, χ_k be the irreducible characters of G . Then $\sum_{g \in G} \chi_i(g) = |G|$ and $\sum_{i=1}^k |\chi_i(g)|^2 = |G|/|O_g|$.

Let G be abelian. By A1, G has $|G|$ representations of degree 1, and $[G : [G, G]] = |G|$. Since G has $|G|$ conjugacy classes, these are all of the irreducible representations of G . Suppose G is a group whose irreducible representations are all degree one. Since $n_1^2 + \dots + n_k^2 = |G|$, then $k = |G|$.

2.15 Proposition. *Let H be an abelian subgroup of G . Then any irreducible representation of G has degree at most $[G : H]$.*

PROOF Let $\rho : G \rightarrow \text{GL}(V)$ be an irreducible representation of G . Consider the restriction $\tilde{\rho} : H \rightarrow \text{GL}(V)$. Let $W \leq V$ be an irreducible subrepresentation of $\tilde{\rho}$. Since H is abelian, $\dim W = 1$. Suppose $W = \text{span}\{x\}$, and let $W' = \{\rho(g)(x) : g \in G\}$ so that W' is G -stable, and in fact $W' = V$ since ρ is irreducible.

Take $g \in G$ and $h \in H$, so $\rho(gh) = \rho(g)\rho(h)(x) = \rho(g)(\alpha x) = \alpha\rho(g)(x)$. Say g_1, \dots, g_m are coset representatives of H in G . Then $V = W' = \text{span}\{\rho(g_i)(x) : 1 \leq i \leq m\}$, then $\dim(V) \leq m = [G : H]$. ■

Example. Consider D_4 . Then the number of degree 1 representations is $[D_4 : \langle r^2 \rangle] = 4$. Since there are 5 conjugacy classes, we know that there are 5 irreducible representations, so that $n_5^2 = 8$. Let's make the character table:

D_4	1	r	r^2	s	rs
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	-1	1	-1	1
χ_5	2	a	b	c	d

But then by column orthogonality, we have $a = 0$, $b = -2$, $c = 0$, $d = 0$.

Example. Consider S_4 . Then $[S_4 : A_4] = 2$ so there are two degree 1 representations (the trivial and the sign), and the conjugacy classes are given by 1, (12), (12)(34), (123), (1234), so there are 5 irreducible representations. Since $24^2 = 1^2 + 1^2 + n_3^2 + n_4^2 + n_5^2$, we have $22 = n_3^2 + n_4^2 + n_5^2$, which forces $n_3 = 2$ and $n_4 = n_5 = 3$. Now we have

D_4	1	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	1	1	-1	-1
χ_4	3	-1	1	-1	1
χ_5	3	a	b	c	d

Note that $K = \{1, (12)(34), (13)(24), (14)(23)\} \trianglelefteq S_4$ and $H = \{1, (12), (13), (123), (132), (23)\}$, so $S_4 = KH$. Let ρ be an irreducible representation of H of degree 2:

S_3	1	(12)	(123)
α_1	1	1	1
α_2	1	-1	1
α_3	2	0	-1

Then $\rho : S_4 \rightarrow \text{GL}(V)$ by $\rho(kh) := \rho(h)$ is an irreducible representation of S_4 since $K \trianglelefteq S_4$.

3 INDUCED REPRESENTATIONS

Given a subgroup $H \leq G$ and a representation $\rho : H \rightarrow \text{GL}(V)$, construct a representation of G . Let $H \leq G$ and $\rho : H \rightarrow \text{GL}(V)$ a representation. Say the cosets of H in G are g_1H, \dots, g_mH . For each i , let $g_iV = \{g_iv : v \in V\}$ be an isomorphic copy of G , and let $W = \bigoplus_{i=1}^m g_iV$ so that every $w \in W$ can be uniquely written as $w = g_1v_1 + \dots + g_mv_m$, where $m = [G : H]$. Fix $g \in G$; then there exists $\pi \in S_m$ such that for every i , $gg_i = g_{\pi(i)}h_i$, $h_i \in H$. We then define $\text{Ind}_H^G(\rho) : G \rightarrow \text{GL}(W)$ by

$$\text{Ind}_H^G(\rho)(g)\left(\sum g_iw_i\right) = \sum g_{\pi(i)}\rho(h_i)v_i$$

Example. Let $\{1\} \leq G$ and suppose $\rho : \{1\} \rightarrow \text{GL}(\mathbb{C})$ is the trivial representation. Then $G = \{g_1, \dots, g_n\}$. Then for $g \in G$, $gg_i1 \in G$ and

$$\text{Ind}(\rho)(s)\left(\sum_{i=1}^n g_i\alpha_i\right) = \sum ggi\rho(1)(\alpha_i) = \sum ggi\alpha_i$$

so that $\text{Ind}(\rho)$ is isomorphic to the regular representation.

Example. Consider $\langle r \rangle \leq D_n$, and let $\rho : \langle r \rangle \rightarrow \text{GL}(\mathbb{C})$ be given by $\rho(r)(1) = \zeta_n$. Let the coset representatives be given by ϵ and s .

- (i) Let $r \in D_n$ so $r\epsilon = \epsilon r$ and $rs = sr^{n-1}$. Fix $W = \epsilon\mathbb{C} \oplus s\mathbb{C}$. Then $\text{Ind}(\rho) : D_n \rightarrow \text{GL}(W)$ is given by $\text{Ind}(\rho)(r)(\epsilon\alpha_1 + s\alpha_2) = \epsilon\zeta_n\alpha_1 + 1 + s\zeta_n^{n-1}\alpha_2$.
- (ii) Let $s \in D_n$. Then $s\epsilon = s\epsilon$ and $ss = \epsilon\epsilon$. Then $\text{Ind}(\rho)(s)(\epsilon\alpha_1 + s\alpha_2) = s\rho(\epsilon)(\alpha_1) + \epsilon\rho(\epsilon)(\alpha_2) = s\alpha_1 + \epsilon\alpha_2$.

Take the basis $\beta = \{\epsilon, s\}$ for W , so we have

$$[\text{Ind}(\rho)(r)]_\beta = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{n-1} \end{pmatrix} \quad [\text{Ind}(\rho)(s)]_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$