Representation Theory of Finite Groups

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I. Introduction

Let G be a finite group of order n, and write $G = \{g_1, \ldots, g_n\}$. Fix $g \in G$; then $gg_i = gg_j$ if and only if $i = j$. Thus there exists some $\sigma_g \in S_i$ such that $gg_i = g_{\sigma_g(i)}$ for all $i \in \{1, 2, \ldots, n\}$. In particular, $\phi : G \to S_n$ by $\phi(g) = \sigma_g$ is an embedding (injective group homomorphism). This observation is usually referred to as Cayley's Theorem.

Now let V be an n–dimensional complex vector space. We then denote $\mathrm{GL}(V)$ as the group of invertible linear operators $T : V \to V$. Now define $\psi : S_n \to \mathrm{GL}_n(V)$ by $\psi(\sigma) = T_\sigma$ where if $\{b_1, \ldots, b_n\}$ is a basis for V and $T_\sigma(b_i) = b_{\sigma(i)}$. This is an injective group homomorphism, so $\psi \circ \phi : G \to \mathrm{GL}(V)$ is an embedding of G into $\mathrm{GL}(V)$.

Definition. Let G be a finite group, and V a finite dimensional $\mathbb{C}$–vector space. A **representation** of G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. We call $\dim(V)$ the **degree** of the representation.

In particular, if V is n–dimensional, then $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$.

Example.

1. Consider $\rho : G \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$ given by $\rho(g) = 1$ for all $g \in G$. This is called the *trivial representation*.
2. Consider $\rho : S_n \to \mathbb{C}^\times$ given by $\rho(\sigma) = \mathrm{sgn}(\sigma)$, which is called the *sign representation*.
3. The representation fo G afforded by Cayley's theorem is called the *regular representation* of G. The next example is a good way to understand the regular rep of G.
4. Consider G, $X = \{x_1, \ldots, x_n\}$, and $V = \mathrm{Free}(X)$. Suppose G acts on X. Then $\rho : G \to \mathrm{GL}(V)$ given by $\rho(g)(x_i) = gx_i$. In particular, if we take $X = G$, then this is the regular representation of G
5. Consider the 4–gon, with vertices labelled a, b, c, d. Take $X = \{a, b, c, d\}$ and the regular representation $\rho : D_4 \to \mathrm{GL}(V)$. This action has a geometric notion.
6. Let C_n be a cyclic group of order n; let us define some $\rho : C_n \to \mathrm{GL}(V)$. Say $\rho(x) = T$ where $t \in \mathrm{GL}(V)$; then this is a representation if and only if $T^n = I$.

Definition. We say that two representations $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ are **isomorphic** if there exists an isomorphism $T : V \to W$ such that for all $g \in G$,

$$T \circ \rho(g) = \tau(g) \circ T$$

Suppose $\rho : G \to \mathrm{GL}(V)$ and $T : V \to W$ is an isomorphism. Then we can define $\tau : G \to \mathrm{GL}(W)$ by $\tau(G) = T \circ \rho(g) \circ T^{-1}$; this $\rho \cong \tau$. In other words, the representation is unique up to isomorphism under change of basis.

Example. Consider $G = \{g_1, \ldots, g_n\} = \{h_1, \ldots, h_n\}$, and fix $g \in G$. Let $gg_i = g_{\alpha(i)}$ and $gh_i = h_{\beta(i)}$ where $\alpha, \beta \in S_n$. Fix an n–dimensional vector space V with basis $\{b_1, \ldots, b_n\}$. Then two regular representations are given by

$$\rho_1 : G \to \mathrm{GL}(V), \rho(g)(b_i) = b_{\alpha(i)}$$
$$\rho_2 : G \to \mathrm{GL}(V), \rho(g)(b_i) = b_{\beta(i)}$$

Let $\gamma \in S_n$ be such that $h_{\gamma(i)} = g_i$, and define $T : V \to V$ by $T(v_i) = b_{\gamma(i)}$. Then

$$gg_i = g_{\alpha(i)} = gh_{\gamma(i)} = h_{\beta\gamma(i)} = g_{\gamma^{-1}\beta\gamma(i)}$$

so that $\alpha = \gamma^{-1}\beta\gamma$. Thus for each b_i,

$$\begin{aligned} T \circ \rho_1(g) \circ T^{-1}(b_i) &= T \circ \rho_1(g)(b_{\gamma^{-1}(i)}) \\ &= T(b_{\alpha\gamma^{-1}(i)}) b_{\gamma\alpha\gamma^{-1}(i)} \\ &= b_{\beta(i)} = \rho_2(g)(b_i) \end{aligned}$$

so that $T \circ \rho_1(g) \circ T^{-1} = \rho_2(g)$.

Note: conjugate elements have the same cycle type.

Subrepresentations

What should a subrepresentation of $\rho : G \to \mathrm{GL}(V)$ mean?

We would like a subspace $W \leq V$ such that $\tau : G \to \mathrm{GL}(W)$ is a representation given by $\tau(g)(w) = \rho(g)(w)$ for all $w \in W$. Moreover, to make this well-defined, we need W to b4 $\rho(g)$–invariant for every $g \in G$ ($\rho(g)(W) \subseteq W$).

Suppose $T : V \to V$ is a linear operator, and $W \leq V$ is a T–invariant subspace; i.e. $T(W) \subseteq W$. In particular, the restriction operator $T_W : W \to W$ is well-defined.

Definition. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. A subspqce $W \subseteq V$ is said to be ***G*–stable** if W is $\rho(g)$–invariant for all $g \in G$. A **subrepresentation** of ρ is a representation $\rho_W : G \to \mathrm{GL}(W)$ where for all $g \in G$ and $w \in W$, $\rho_W(g)(w) = \rho(g)(w)$ where W is a G–stable subspace of V.

Example. Suppose $\rho : G \to \mathrm{GL}(V)$ be the regular representation. Take $W = \mathrm{span}\{\sum_{g \in G} v_g\}$, which is clearly G–stable, and $\rho_W : G \to \mathrm{GL}(W)$ is isomorphic to the trivial representation.

Similarly, let $\rho : S_n \to \mathrm{GL}(V)$ be the regular representation, $W = \mathrm{span}\{\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) v_\sigma\}$; this is isomorphic to the sign representation.

0.1 Theorem. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation, $W \leq V$ G–stable. Then there exists a G–stable subspace W' such that $V = W \oplus W'$.*

Proof Take any inner product $\langle x, y \rangle$ on V. Then for any $x, y \in V$, define

$$\langle x, y \rangle^* = \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle$$

This is also an inner product. Let $x, y \in V$ and let $h \in G$. Then

$$\begin{aligned} \langle \rho(h)(x), \rho(h)(y) \rangle^* &= \sum_{g \in G} \langle \rho(g)\rho(h)(x), \rho(g)\rho(h)(y) \rangle \\ &= \sum_{g \in G} \langle \rho(gh)(x), \rho(gh)(y) \rangle \\ &= \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle \end{aligned}$$

Thus every $\rho(h)$ is unitary with respect to $\langle \cdot, \cdot \rangle^*$. Let $W \leq V$ be G–stable, and take $W' = W^\perp$ with respect to $\langle \cdot, \cdot \rangle^*$. Then $V = W \oplus W'$. Let's see that $W^\perp$ is G–stable. Let $x \in W^\perp$, $w \in W$, and $g \in G$, so that

$$\begin{aligned}\langle \rho(g)(x), w\rangle^* &= \langle x, \rho(g)^*(w)^*\rangle = \langle x, \rho(g)^{-1}(w)\rangle^* \\ &= \langle x, \underbrace{\rho(g^{-1})(w)}_{\in W}\rangle^* \\ &= 0\end{aligned}$$

and $\rho(g)(W^\perp) \subseteq W^\perp$ as required. ∎

Definition. Let $\rho : G \to \mathrm{GL}(V)$ be a representation, and $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ where each W_i is G–stable. For each i, let $\rho_i = \rho_{w_i}$. For each $v = \sum w_i \in V$, we have $\rho(g)(v) = \sum \rho(g)(w_i) = \rho_i(g)(w_i)$. In this case, we write

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k$$

and call ρ a direct sum of the ρ_i's.

The previous definition is written as an internal direct sum of V. Externally, given vector spaces $W_1, \ldots, W_k$ and representations $\rho_i : G \to \mathrm{GL}(W_i)$, we can define

$$(\rho_1 \oplus \cdots \oplus \rho_k) : G \to \mathrm{GL}(W_1 \oplus \cdots \oplus W_k)$$

by $(\rho_1 \oplus \cdots \oplus \rho_k)(g)(w_1, \ldots, w_k) = (\rho_1(g)(w_1), \ldots, \rho_k(g)(w_k))$. If $\rho_i : G \to \mathrm{GL}(W_i)$ is a subrepresentation fo $\rho : G \to \mathrm{GL}(V)$, we often say "$W_i$ is a subrepresentation of V".

Definition. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. We say ρ is **irreducible** if $V \neq \{0\}$ and the only G–stable subspaces of V are $\{0\}$ and V.

Clearly,

> ***0.2 Theorem.*** *Every representation $\rho : G \to \mathrm{GL}(V)$ can be written as a direct sum of irreducible sub-representations.*

Example. Let $\rho : S_3 \to \mathrm{GL}(\mathbb{C}^3)$ be the permutation representation with respect to the standard basis $\{e_1, e_2, e_3\}$. Consider $W_1 = \mathrm{span}\{e_1 + e_2 + e_3\}$ and $W_2 = \mathrm{span}\{e_1 - e_2, e_2 - e_3\}$. Is W_2 irreducible?

More generally, if $V = W_1 \oplus \cdots \oplus W_k$ and $\dim W_i = 1$ and $\deg(\rho_i) = 1$,

$$\rho(gh)(\sum w_i) = \sum \rho_i(gh)(w_i) = \sum \rho_i(g)\rho_i(h)(w_i) = \sum \rho_i(h)\rho_i(g)(w_i)$$

so that $\rho(gh) = \rho(hg)$. In the our example, this does not happen, since $\rho(g) \neq I$ when $g \neq 1$ and S_3 is not abelian.

Example. Let $\rho : S_3 \to \mathrm{GL}(V)$ be the regular representation. Let $W_1 = \mathrm{span}\{\sum_{\sigma \in S_3} v_\sigma\}$ and $W_2 = \mathrm{span}\{\sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) v_\sigma\}$, and

$$W_3 = \sum \alpha_\sigma v_\sigma | \ \alpha \quad \begin{matrix} +\alpha_{(123)} + \alpha_{(1,3,2)} \\ = 0 \\ \alpha_{(12)} + \alpha_{(13)} + \alpha_{(23)} \\ = 0 \end{matrix} \quad \epsilon$$

Now let's focus on W_3. A basis for W_3 is given by

$$e_1 = v_\epsilon - v_{(123)} \qquad e_2 = v_\epsilon - v_{(123)}$$
$$e_3 = v_{(12)} - v_{(13)} \qquad e_4 = v_{(12)} - v_{(23)}$$

Recall that $S_3 = \langle (12), (123) \rangle$; suffices to show stability with respect to generators.

$$\rho(12) : e_1 \mapsto e_4, e_2 \mapsto e_3, e_3 \mapsto e_2, e_4 \mapsto e_1$$
$$\rho(123) : e_1 \mapsto e_2 - e_1, e_2 \mapsto -e_1, e_3 \mapsto e_4 - e_3, e_4 \mapsto -e_3$$

Let $U_1 = \operatorname{span}\{e_1 - e_4, e_2 + e_3 - e_1\}$

1 Tensor Products

Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be representations. We define the representation $\rho \otimes \tau : G \to \mathrm{GL}(V \otimes W)$

$$(\rho \otimes \tau)(g)(v \otimes w) = \rho(g)(v) \otimes \tau(g)(w)$$

2 Character Theory

We define the character of ρ by $\rho : G \to \mathbb{C}$ as $\chi(G) = (\rho(g))$.

Remark. If we choose a basis β for V, then define $A(g) = [\rho(g)]_\beta$ and $\chi(G)$ is given by the sum of the diagonal entries of $A(g)$. Furthermore, if $A, B \in M_n(\mathbb{C})$, then $(AB) = (BA)$.

The remark implies a number of facts:

(i) $\rho \cong \tau$, then $(\rho(g)) = (\tau(g))$.
(ii) (T) is the sum of eigenvalues of T
(iii) $\chi(1) = \dim(V)$.

2.1 Proposition. *For every $g \in G$ the eigenvalues of $\rho(g)$ have modulus 1. In particular, $\chi(g^{-1}) = \overline{\chi(g)}$.*

Proof Set $n = |G|$; then $\rho(g)^n = \rho(g^n) = I$ so that $\lambda^n - 1 = 0$ for any eigenvalue λ, so $|\lambda| = 1$. Furthermore,

$$\overline{\chi(g)} = \overline{\sum \lambda_i} = \sum \overline{\lambda_i} = \sum \lambda_i^{-1} = \chi(g^{-1})$$

proving the second component. ∎

2.2 Proposition. *Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$. Then $\chi_{\rho \oplus \tau} = \chi_\rho + \chi_\tau$ and $\chi_{\rho \otimes \tau} = \chi_\rho \cdot \chi_\tau$.*

Proof Let $\beta_1 = \{v_1, \ldots, v_n\}$ be a basis for V and $\beta_2 = \{w_1, \ldots, w_m\}$ a basis for W.

Then a basis for $V \oplus W$ is given by $\beta = \{(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)\}$. In particular,

$$[(\rho \oplus \tau)(g)]_\beta = \begin{pmatrix} [\rho(g)]_{\beta_1} & \\ & [\tau(g)]_{\beta_2} \end{pmatrix}$$

and the trace result follows.

A basis for $V \otimes W$ is given by $\gamma = \{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ in lexicographic order. Fix $g \in G$, and set $A = [\rho(g)]_{\beta_1}$, $B = [\rho(g)]_{\beta_2}$. Fix $v_i \otimes w_j \in \gamma$. Then

$$\begin{aligned}(\rho \otimes \tau)(g)(v_i \otimes w_j) &= \rho(g)(v_i) \otimes \tau(g)(w_j) \\ &= (a_{1i}v_1 + \cdots + a_{ni}v_n) \otimes (b_{1j}w_1 + \cdots + b_{mj}v_m) \\ &= \cdots + a_{ii}b_{jj} \cdot (v_i \otimes w_j) + \cdots \\ &= ([\rho \otimes \tau)(g)]_\delta) = \sum_{i,j} a_{ii}b_{jj} = (A)() = \chi_\rho(g) \cdot \chi_\tau(g)\end{aligned}$$ ■

Example. Suppose $\rho : S_n \to \mathrm{GL}(\mathbb{C}^n)$ is the permutation representation with respect to $\{e_1, \ldots, e_n\}$. Then $\chi(\sigma) = |\{e_i : \rho(\sigma)(e_i) = e_i\}| = |\mathrm{Fix}(\sigma)|$, which is the number of indices i fixed by σ. Since S_n acts transitively on $\{1, \ldots, n\}$, there is exactly 1 orbit, so by Burnside's lemma,

$$n! = |S_n| = \sum_{\sigma \in S_n} \chi(\sigma)$$

Example. Let $\rho : G \to \mathrm{GL}(V)$ be the regular representation. Note that if $g \neq 1$, then for all $h \in G$, $gh \neq h$. In particular, this means that $\chi(g) = 0$ if $g \neq 1$, and $\chi(1) = |G|$ (the dimension of V).

Example. Let $\rho : S_3 \to \mathrm{GL}(V)$ be the regular representation. Recall that $V = W_1 \oplus W_2 \oplus U_1 \oplus U_2$ where W_1 is the trivial representation, W_2 is the sign representation, and U_1, U_2 are isomorphic. Let $S_3 = \langle (12), (123) \rangle$; then we have

x_1	1	1
x_2	-1	1
x_3	a	b
x_4	a	b

In particular, $\chi(12) = 1 - 1 + 2a = 0$ and $\chi(123) = 1 + 1 + 2b = 0$, so $b = -1$.

Example. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. In particular, $\rho(ghg^{-1}) = \rho(g)\rho(h)\rho(g)$ so that $\rho(ghg^{-1}) = \rho(h)$ so $\chi(ghg^{-1}) = (h)$; in other words, that characters are constant on conjugacy classes.

> ***2.3 Lemma. (Schur)*** *Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be irreducible representations, and suppose $T : V \to W$ is linear such that for all $g \in G$, $\tau(g) \circ T = T \circ \rho(g)$. Then either $T = 0$ or T is an isomorphism and $\rho \cong \tau$. Moreover, if $V = W$ and $\rho = \tau$, then T is a scalar multiple of the identity.*

Proof Assume $T \neq 0$.

Let's first see that T is injective, and let $v \in \ker(T)$. Then for any $g \in G$, $T(\rho(g)(v)) = \tau(g)(T(v)) = 0$, so $\rho(g)(v) \in \ker(T)$. Thus $\ker(T)$ is G–stable (with respect to ρ). Since ρ is irreducible and $T \neq 0$, $\ker(T) = \{0\}$.

We also have that T is surjective. Let $v \in \mathrm{Im}(T)$ and say $v = T(X)$ with $x \in V$. Then for $g \in G$, $\tau(g)(v) = \tau(g)(T(x)) = T(\rho(g)(x)) \in \mathrm{Im}(T)$ so $\mathrm{Im}(T)$ is G–stable, and again by irreducibility of τ, $\mathrm{Im}(T) = W$. Thus T is an isomorphism.

Now let $\lambda \in \mathbb{C}$ be an eigenvalue of T and consider $T' = T - \lambda I$. Now, note that for $g \in G$, $\rho(g)T' = T'\rho(g)$, but T' has non-trivial kernel, so in fact $T' = 0$. ■

2.4 Corollary. *Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be irreducible, and $T : V \to W$ linear. Consider*

$$T' = \frac{1}{|G|} = \sum_{g \in G} \tau(g)^{-1} T \rho(g)$$

Then

(i) If $T' \neq 0$, then $\rho \cong \tau$ via T'.

(ii) If $V = W$, $\rho = \tau$, then $T' = (T)/\dim(V) \cdot I$.

Proof Clearly $T' : V \to W$ is linear, and for any $h \in G$,

$$\begin{aligned}
\tau(h)T' &= \tau(h)\frac{1}{|H|}\sum_{g \in G} \tau(g^{-1}) T \rho(g) \\
&= \frac{1}{|G|}\sum_{g \in G} \tau(hg^{-1}) T \rho(g) \\
&= \frac{1}{|G|}\sum_{g \in G} \tau(g^{-1}) T (\rho(gh)) \\
&= \frac{1}{|G|}\sum_{g \in G} \tau(g^{-1}) T \rho(g)\rho(h) \\
&= T'\rho(h)
\end{aligned}$$

If $V = W$ and $\rho = T$, then $(T') = \frac{1}{|G|}(T) \cdot |G| = (T) = \alpha \dim(V)$, so $\alpha = (T)/\dim(V)$. ■

Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be irreducible representations, and $T : V \to W$ linear. Let β be a basis for V and γ a basis for W. Then for $g \in G$, let $[\rho(g)]_\beta = (a_{ij}(g))$, $[\tau(g)]_\gamma = (b_{kl}(g))$, $[T]_\beta^\gamma = (X_{ki})$, and $[T']_\beta^\gamma = (x'_{ki})$.

By matrix multiplication, $x'_{ki} = \frac{1}{|G|}\sum_g \sum_{j,l} b_{kl}(g^{-1}) x_{lj} a_{ji}$. If $\rho \ncong \tau$, then $T' = 0$, so by viewing the RHS as a polynomial in the x_{ij}, we have

$$\frac{1}{|G|}\sum_g b_{kl}(g^{-1}) a_{ji}(g) = 0$$

But now it $\rho = \tau$, then $T' = \lambda I$ where $\lambda = (T)/\dim(B)$ so that

$$\frac{1}{|G|}\sum_g \sum_{j,l} a_{kl}(g^{-1}) x_{lj} a_{ji}(g) = \lambda \delta_{ki} = \frac{1}{\dim(V)}\sum_{j,l} \delta_{ki}\delta_{jl} x_{lj}$$

Then by equating coefficients of x_{lj}, we have

$$\frac{1}{|G|}\sum_g a_{kl}(g^{-1}) a_{ji}(g) = \frac{1}{\dim(V)}\delta_{ki}\delta_{jl}$$

Remark. If G is a finite group, the consider the vector space of all functions $\phi : G \to \mathbb{C}$. For any ϕ, ψ in this vector space, $\langle \phi, \psi \rangle = \frac{1}{|G|}\sum_g \phi(g)\overline{\psi(g)}$ defines an inner product. Then if χ_1, χ_2 are characters of G, then

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|}\sum_g \chi_1(g)\chi_2(g^{-1})$$

We thus have:

2.5 Theorem. *If χ is a character of an irreducible representation, then $\langle \chi, \chi = 1$, and if χ_1 and χ_2 correspond to non-isomorphic representations, then $\langle \chi_1, \chi_2 \rangle = 0$.*

Proof Say $[\rho(g)]_\beta = (a_{ij}(g))$ where ρ is an irreducible representation with character χ. Then

$$\begin{aligned}
\langle \chi, \chi \rangle &= \frac{1}{|G|} \sum_g \chi(g)\chi(g^{-1}) = \frac{1}{|G|} \sum_g \chi(g^{-1})\chi(g) \\
&= \frac{1}{|G|} \sum_g \sum_{i,j} a_{ii}(g^{-1}) a_{jj}(g) = \sum_{i,j} \left(\frac{1}{|G|} \sum_g a_{ii}(g^{-1}) a_{jj}(g)) \right. \\
&= \sum_{i,j} \left(\frac{1}{|G|} \sum_g a_{ii}(g^{-1}) a_{ii}(g) \right. \\
&= \sum_i \frac{1}{\dim(V)} = 1
\end{aligned}$$

To see the second part,

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_g \chi_1(g)\chi_2(g^{-1}) = \frac{1}{|G|} \sum_g \sum_{ij} a_{ii}(g) a_{jj}(g^{-1}) = \sum_{i,j} 0 = 0 \qquad \blacksquare$$

If χ is a character corresponding to an irreducible representation, we say χ is irreducible. If ρ and τ are isomorphic representations, we say χ_ρ and χ_τ are isomorphic (in fact $\chi_\rho = \chi_\tau$).

2.6 Corollary. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation with character χ. Say $V = W_1 \oplus \cdots \oplus W_k$ is an irreducible decomposition of V. If $\tau : G \to \mathrm{GL}(W)$ is an irreducible representations with character ϕ, then the number of W_i isomorphic to W (i.e. $\rho_i \cong \tau$) is $\langle \chi, \phi \rangle$.*

Proof Write $\chi = n_1\chi_1 + \cdots + n_l\chi_l$, where the χ_i are pairwise non-isomorphic. Then $\langle \chi, \chi_i \rangle = n_i$. $\blacksquare$

Let $\tau : G \to \mathrm{GL}(V)$ be irreducible, and let τ have character φ. Then

$$\langle \chi, \varphi \rangle = \sum_{i=1}^{k} \langle \chi_i, \varphi \rangle$$

Now, $\langle \chi_i, \varphi \rangle = 1$ if and only if $\rho_i \cong \tau$, so that $\langle \chi, \varphi \rangle$ counts the number of times in which τ appears in the irreducible decomposition of ρ.

2.7 Corollary. *If two representations of G have the same character, then they are isomorphic.*

Proof They have the same irreducible decomposition. $\blacksquare$

2.8 Corollary. *If $\rho : G \to \mathrm{GL}(V)$ is a representation and χ is a character, then $\langle \chi, \chi \rangle \in \mathbb{N}$ and $\langle \chi, \chi \rangle = 1$ if and only if χ is ireducible.*

Proof If $\chi_1,\ldots,\chi_k$ are irreducible, write $\chi = n_1\chi_1 + \cdots + n_k\chi_k$ so that $\langle \chi,\chi\rangle = n_1^2 + \cdots + n_k^2 \in \mathbb{N}$. ■

2.9 Proposition. *Every irreducible representation of G occurs as a subgroup fo the regular representation of G, with multiplicity equal to its degree.*

Proof Let χ be an irreducible character of G. Then

$$\langle \chi,\chi_{\text{reg}}\rangle = \frac{1}{|G|}\sum_g \chi(g)\overline{\chi_{\text{reg}}(g)} = \frac{1}{|G|}\chi(1)\overline{\chi_{\text{reg}}(1)} = \frac{1}{|G|}\deg(\chi)$$ ■

2.10 Corollary. *Let $\chi_1,\ldots,\chi_k$ be the distinct irreducible characters of G, with* $\deg(\chi_i) = n_i$. *Then $\sum n_i^2 = |G|$ for for $g \neq 1$, $\sum_{i=1}^k n_1\chi_i(g) = 0$*

Proof Recall that $\chi_{\text{reg}} = n_1\chi_1 + \cdots + n_k\chi_k$. Then $\chi_{\text{reg}}(1) = |G| = n_1^2 + \cdots + n_k^2$, and evaluation at $g \neq 1$ gives the desired result. ■

Definition. Let G be a group. A function $f : G \to \mathbb{C}$ is called a class function if f is constant on each conjugacy class, i.e. for all $a,b \in G$, $f(bab^{-1}) = f(a)$.

2.11 Proposition. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Then*

$$\rho_f = \sum_g f(g)\rho(g)$$

is a linear operator on V. If ρ is irreducible of degree n, then $\rho_f = \lambda I$, where $\lambda = \frac{|G|}{n}\langle f,\overline{x}\rangle$ where χ is the character of ρ.

Proof Note that

$$\begin{aligned}\rho_f \circ \rho(h) &= \sum_g f(g)\rho(g)\rho(h) = \sum_g f(g)\rho(gh)\\ &= \sum_g f(hgh^{-1})\rho(hg)\\ &= \sum_g f(g)\rho(h)\rho(g) = \rho(h)\circ\rho_f\end{aligned}$$

so by Schur, $\rho_f = \lambda I$ where $\lambda = (\rho_f)/n$. However, $(\rho_f) = (\sum_g f(g)\rho(g)) = \sum_g f(g)\chi(g) = |G|\langle f,\overline{x}\rangle$. ■

Recall that

- $\langle\chi,\chi\rangle = 1$ *if and only if χ is irreducible*
- *If χ_ρ and χ_τ are irreducible then $\langle\chi_\rho,\chi_\tau\rangle = 0$ if $\rho \ncong \tau$, and 1 otherwise.*
- *If χ' is an irreducible subrepresentation of χ, then $\langle\chi,\chi'\rangle$ is the multiplicity of χ' in χ.*
- $|G| = n_1^2 + \cdots + n_k^2$ *where n_i is the multiplicity of χ_i as an irreducible subrepresentation of the regular representation.*
- *Every irreducible character is a character of some subrepresentation of the regular rep?*
- *... every irreducible represetation is a subrepresentation of the regular rep?*

and

$$\rho_f = \sum_g f(g)\rho(g) = \lambda I$$

where $\lambda = |G|/\dim(V)\cdot\langle f,\overline{\chi}\rangle$.

2.12 Proposition. *Let G be a group. The irreducible characters of G form an orthonormal basis for the vector space of class functions on G.*

Proof Let $\beta = \{\chi_1,\dots,\chi_k\}$ be the irreducible characters of G. We know that β is orthonormal, and hence linearly independent. Let $W = \operatorname{span}(\beta)$. To show $W = V$ where V is the space of class functions, we prove that $W^\perp = \{0\}$. Let $f \in W^\perp$, and suppose $\rho : G \to \mathrm{GL}(V)$ is irreducible. By A2, $\overline{\chi}_1,\dots,\overline{\chi}_k$ are all irreducible characters of G. Thus $\rho_f = 0$. By considering irreducible decompositions, $\rho_f = 0$ for all representations $\rho : G \to \mathrm{GL}(V)$. In particular, when ρ is the regular representation,

$$0 = \rho_f(v_1) = \sum_g f(g)\rho(g)(v_1) = \sum_g f(g)v_g$$

so by independence of $\{v_g : g \in G\}$, $f(g) = 0$ for all $g \in G$. ∎

2.13 Corollary. *The number of irreducible characters of G is equal to the number of conjugacy classes of G.*

Proof Let $C_1,\dots,C_k$ be the conjugacy classes. Then a basis for $V_{\text{class}} = \{\phi_1,\dots,\phi_k\}$ where each ϕ_i is the indicator for C_i. Since bases must have the same size, the result follows. ∎

2.14 Proposition. *Let G be a group, $g \in G$, and O_g the conjugacy class of g. Let $\chi_1,\dots,\chi_k$ be the irreducible characters of G. Then*

1. $\sum_{i=1}^k |\chi_i(g)|^2 = |G|/|O_g|$
2. *If h is not conjugate to g, then* $\sum_{i=1}^k \chi_i(g)\overline{\chi_i(h)} = 0$.

Proof Define $\phi : G \to \mathbb{C}$ where $\phi(x)$ is the indicator function for O_g. Write $\phi = \sum_{i=1}^k \lambda_i\chi_i$ where

$$\lambda_i = \langle \phi, \chi_i\rangle = \frac{1}{|G|}\sum_x \phi(x)\overline{\chi_i(x)} = \frac{|O_g|\overline{\chi_i(g)}}{|G|}$$

Therefore,

$$\phi(x) = \frac{|O_g|}{|G|}\sum_{i=1}^k \overline{\chi_i(g)}\chi_i(x)$$

Then the result follows by evaluating ϕ at g and h. ∎

Example. Let's compute the character table of S_3. There are 2 degree 1 representations, and 3 irreducible characters since there are three conjugacy classes (cycle types). In particular, $|S_3| = 6 = 1^2 + 1^2 + n_3^2$, so $n_3 = 2$.

	ϵ	(12)	(123)
(triv)χ_1	1	1	1
(sgn)χ_2	1	-1	1
χ_3	2	a	b

Note that the columns must be orthogonal, so by the previous proposition, we have $a = 0$ and $b = -1$.

Let $\chi_1, \dots, \chi_k$ be the irreducible characters of G. Then $\sum^2_{g|\chi_i} = |G|$ and $\sum_{i=1}^k |\chi_i(g)|^2 = |G|/|O_g|$.

Let G be abelian. By A1, G has $|G|$ representations of degree 1, and $[G : [G, G] = |G|$. Since G as $|G|$ conjugacy classes, these are all of the irreducible representations of G. Suppose G is a group whose irreducible representations are all degree one. Since $n_1^2 + \cdots + n_k^2 = |G|$, then $k = |G|$.

2.15 Proposition. *Let H be an abelian subgroup of G. Then any irreducible representation of G has degree at most $[G : H]$.*

Proof Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of G. Consider the restriction $\tilde{\rho} : H \to \mathrm{GL}(V)$. Let $W \leq V$ be an irreducible subrepresentation of $\tilde{G}$. Since H is abelian, $\dim W = 1$. Suppose $W = \mathrm{span}\{x\}$, and let $W' = \{\rho(g)(x) : g \in G\}$ so that V' is G–stable, and in fact $V' = V$ since ρ is irreducible.

Take $g \in G$ and $h \in H$, so $\rho(gh) = \rho(g)\rho(h)(x) = \rho(g)(\alpha x) = \alpha\rho(g)(x)$ Say $g_1, \dots, g_m$ are coset representatives of H in G. Then $V = V' = \mathrm{span}\{\rho(g_i)(x) : 1 \leq i \leq m\}$, then $\dim(V) \leq m = [G : H]$. ■

Example. Consider D_4. Then the number of degree 1 representations is $[D_4 : \langle r^2 \rangle] = 4$. Since there are 5 conjugacy classes, we know that there are 5 irreducible representations, so that $n_5^2 = 8$. Let's make the character table:

D_4 rs	1	r	r^2	s
χ_1 1	1	1	1	1
χ_2 -1	1	-1	1	1
χ_3 -1	1	1	1	-1
χ_4 1	1	-1	1	-1
χ_5 d	2	a	b	c

But then by column orthogonality, we have $a = 0$, $b = -2$, $c = 0$, $d = 0$.

Example. Consider S_4. Then $[S_4 : A_4] = 2$ so there are two degree 1 representations (the trivial and the sign), and the conjugacy classes are given by 1, (12), (12)(34), (123), (1234), so there are 5 irreducible representations. Since $24^2 = 1^2 + 1^2 + n_3^2 + n_4^2 + n_5^2$, we have $22 = n_3^2 + n_4^2 + n_5^2$, which forces $n_3 = 2$ and $n_4 = n_5 = 3$. Now we have

D_4 (1234)	1	(12)	(12)(34)	(123)
χ_1 1	1	1	1	1
χ_2 -1	1	-1	1	1
χ_3 -1	2	1	1	-1
χ_4 1	3	-1	1	-1
χ_5 d	3	a	b	c

Note that $K = \{1, (12)(34), (13)(24), (14)(23)\} \trianglelefteq S_4$ and $H = \{1, (12), (13), (123), (132), (23)\}$, so $S_4 = KH$. Let ρ be an irreducible representation of H of degree 2:

S_3	1	(12)	(123)
α_1	1	1	$1\alpha_2$
1	-1	$1\alpha_3$	2
0	-1		

Then $\rho : S_4 \to \mathrm{GL}(V)$ by $\rho(kh) := \rho(h)$ is an irreducible representation of S_4 since $K \trianglelefteq S_4$.