PMATH 465

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Contents

I. Fundamentals of Manifolds

1 Introduction to Topology

Definition. A **topology** on a set X is a set τ of subsets of X such that

(i) $\emptyset \in \tau$ and $X \in \tau$

(ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.

(iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \leq i \leq n$, then $\bigcap_{i=1}^{n} U_i \in \tau$.

The sets $U \in \tau$ are called the **open sets** in X, and sets of the form $X \setminus U$ for some open set U are caled the **closed sets** in X.

Definition. When X is a topological space and $A \subseteq X$, the **interior** of A (denoted A°) is the union of all open sets contained in A. Similarly, we define the **closure** of A (denoted $\overline{A}$) as the intersction of all closed sets containing A. Then the **boundary** of A, denoted by ∂A, is the set $\partial A = \overline{A} \setminus A^\circ$.

Example. Let X be any set. The **discrete topology** on X is the topology $\tau = \mathcal{P}(X)$, and the **trivial topology** on X is the topology $\tau = \{\emptyset, X\}$.

Definition. A **basis** for a topology on a set X is a set $\mathcal{V}$ of subsets of X

(i) $\bigcup_{B \in \mathcal{B}} b = X$

(ii) for all $a \in X$ and $U, V \in \mathcal{B}$ such that $a \in U \cap V$, then there exists $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$.

When $\mathcal{B}$ is a basis for a topology on X, the topology on X **generated** by $\mathcal{B}$ is the set τ of subsets of X such that for $W \subseteq X$, $W \in \tau$ if and only if for all $a \in W$, there exists $U \in \mathcal{B}$ such that $a \in U \subseteq W$.

Note that τ, as above, is a topology on X since

(i) $\emptyset \in \tau$ vacuously and $X \in \tau$ obviously.

(ii) If $A_k \imath \tau$ for all $k \in K$ (where K is any set of indices), then given $a \in \bigcup_{x \in K} A_k$, we can choose $\ell \in K$ so that $a \in A_\ell$. Then since $A_\ell \in \tau$, we can choose $U_\ell \in \mathcal{B}$ so that $a \in U_\ell \subseteq A_\ell$. Thus $a \in U_\ell \subseteq A_\ell \subseteq \bigcup_{k \in K} A_k$.

(iii) By induction, it suffices to prove that if $A, B \in \tau$, then $A \cap B \in \tau$. Suppose $A, B \in \tau$, and let $a \in A \cap B$. Since $A \in \tau$, we cna choose $U \in \mathcal{B}$ so that $a \in U \subseteq A$. Since $B \in \tau$, we can choose $V \in \mathcal{B}$ so that $a \in V \subseteq B$. Then we have $a \in U \cap V$. Since $\mathcal{B}$ is a basis, we can chose $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$, so $a \in W \subseteq U \cap V \subseteq A \cap B$.

Note that when τ is the topology on X generated by the basis $\mathcal{B}$, for $A \subseteq X$, $A \in \tau$ if and only if there exists some $S \subseteq \mathcal{B}$ such that $A = \bigcup_{s \in S} s$. In this sense, the topology τ on X generated by the basis $\mathcal{B}$ is the coarsest topology which contains $\mathcal{B}$.

Definition. (Subspace Topology) When Y is a topological space and $X \subseteq Y$ is a subset of Y, we define the **subspace topology** on X to be the topology for which as set $U \subseteq X$ is open if and only if $U = X \cap V$ for some open set V.

If $\mathcal{C}$ is a basis for the topology on Y, then $\mathcal{B} = \{\, X \cap V \mid V \in \mathcal{C} \,\}$ is a basis for the subspace topology on X.

Definition. (Disjoint Union Topology) If X and Y are topological spaces with $X \cap Y = \emptyset$, then the **disjoint union topology** on $X \cup Y$ is the topology in which a subset $U \subseteq X \cup Y$ is open in $X \cup Y$ if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y.

Definition. (Product Topology) If X and Y are topological spaces, the **product topology** on $X \times Y$ is the topology generted by the basis

$$\mathcal{B} = \{\, U \times V \mid U \in \mathcal{C}, V \in \mathcal{D} \,\}$$

where $\mathcal{C}$ and $\mathcal{D}$ are bases for the topologies on X, Y respectively.

Definition. (Infinite Product Topology) We define the infinite product to be

$$\prod_{k \in K} \left\{ f : K \to \bigcup_{k \in K} X_k \,\middle|\, f(k) \in X_k \text{ for all } k \in K \right\}$$

There are two standard topologies on X. The first is the **box topology**,

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \,\middle|\, U_k \text{ is open in } X_k \right\}$$

and the **product topology**

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \,\middle|\, \begin{array}{l} U_k \text{ is open in } X_k \\ U_k = X_k \text{ for all but finitely many indices } k \end{array} \right\}$$

Metric Topology

$\mathbb{R}^n$ has a standard **inner product**, and for $u, v \in \mathbb{R}^n$, $uv = u \cdot v = V^T u = \sum_{i=1}^n u_i v_i$. This gives the standard norm on $\mathbb{R}^n$ for $u \in \mathbb{R}^n$, $\|u\| = \sqrt{uv}$. This gives the standard metric on $\mathbb{R}^n$: for $a, \in \mathbb{R}^n$, $d(a,b) = \|b - a\|$.

Given a metric on a set Y, we obtain (by restriction) an induced metric on any subset $X \subseteq Y$. Given a metric space X, we define the **metric topology** on X to be the topology which is generated by the set of open balls

$$B(a,r) = \{\, x \in X \mid d(a,x) < r \,\}$$

where $x \in X$, $r > 0$.

Definition. When X and Y are topological spaces and $f : X \to Y$, we say that f is **continuous** when it has the property that $f^{-1}(V)$ is open in X for every open set V in Y.