Introduction to Galois Theory

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I. Structure of Finite Groups

1 Group Quotients

Universal Property of Quotients

Let $H \trianglelefteq G$ be a normal subgroup of G, and let $\pi : G \to G/H$ be the natural projection map. This map has the following universal property:

1.1 Theorem. (Universal Property of Quotients) *Let $\phi : G \to G'$ be a homomorphism. If $H \subset \ker(\phi)$, there is a unique homomorphism $\overline{\phi} : G/H \to G'$ so that $\phi = \overline{\phi} \circ \pi$.*
In particular, $\ker(\overline{\phi}) = \ker(\phi)/H$ and $\operatorname{im}(\overline{\phi}) = \operatorname{im}(\phi)$.

One can rephrase this universal property as follows. Suppose $\phi : G \to G'$ is a homomorphism of groups and $H \trianglelefteq G$ is a normal subgroup. If $H \leq \ker(\phi)$, then ϕ induces a homomorphism $\overline{\phi} : G/H \to G'$ given by $xH \mapsto \phi(x)$ such that $\ker(\overline{\phi}) = \ker(\phi)/H$, $\operatorname{im}(\overline{\phi}) = \operatorname{im}(\phi)$.

Proof Define $\overline{\phi}(xH) = \phi(x)$. Then $\overline{\phi} \circ \pi(g) = \overline{\phi}(gH) = \phi(g)$, so $\overline{\phi} \circ \pi = \phi$. This map is well-defined: suppose $xH = yH$. Then $y^{-1}x \in H$, so $\phi(y^{-1}x) = 0$ since $H \leq \ker(\phi)$. Thus

$$\overline{\phi}(xH) = \phi(x) = \phi(yy^{-1}x) = \phi(y)\phi(y^{-1}x) = \phi(y) = \overline{\phi}(yH)$$

so $\overline{\phi}$ is well-defined.

To see that $\overline{\phi}$ is unique, let ψ satisfy the universal property as well, so $\psi \circ \pi = \phi$. In particular, $\phi(h) = \psi \circ \pi(g) = \psi(gN)$, so $\psi(gN) = \overline{\phi}(gN)$ so $\overline{\phi}$ is unique.

$\overline{\phi}$ is a homomorphism since ϕ is:

$$\overline{\phi}((aH)(bH)) = \overline{\phi}((ab)H) = \phi(ab) = \phi(a)\phi(b) = \overline{\phi}(aH)\overline{\phi}(bH)$$

Finally,

$$xH \in \ker(\overline{\phi}) \iff \overline{\phi}(xH) = 0 \iff \phi(x) = 0 \iff x \in \ker(\phi) \qquad \blacksquare$$

1.2 Corollary. (First Isomorphism) *Suppose $\phi : G \to H$ is a surjective homomorphism. Then $G/\ker(\phi) \cong H$.*

Proof Take $H = \ker(\phi)$, so $\overline{\phi} : G/\ker(\phi) \to H$ is surjective since $\operatorname{im}(\overline{\phi}) = \operatorname{im}(\phi) = H$ and injective since $\ker(\overline{\phi}) = \ker(\phi)/\ker(\phi) = \{1\}$. $\blacksquare$

Correspondence Theorem

1.3 Theorem. *Let $\phi : G \to G'$ be a homomorphism of groups. ϕ induces two maps on the set of subgroups Γ and Γ' of G and G' respectively:*

$$\phi_* : \Gamma \to \Gamma' \textit{ given by } \phi_*(H) = \phi(H)$$
$$\phi^* : \Gamma' \to \Gamma \textit{ given by } \phi^*(H') = \phi^{-1}(H')$$

Then $\phi_* \circ \phi^*(H') = H' \cap \operatorname{im}(\phi)$ and $\phi^* \circ \phi_*(H) = \langle H, \ker(\phi) \rangle$.

Recall that $H' \cap \operatorname{im}(\phi)$ is the largest subgroup of H' contained in $\operatorname{im}(\phi)$, and $\langle H, \ker(\phi)\rangle$ is the smallest group containing H and $\ker(\phi)$.

1.4 Corollary. *Let G be a group and $N \trianglelefteq G$. Then the quotient map $\pi : G \to G/N$ is a bijection from the set of subgroups of G containing N to the set of subgroups of G/N.*

Proof Recall that π is a group homomorphism, and $\ker(\phi) = N$ and $\operatorname{im}(\phi) = G/N$. Then $\pi_* \circ \pi^*(H') = H' \cap \operatorname{im}(\pi) = H'$ and $\pi^* \circ \pi_*(H) = \langle H, \ker(\pi)\rangle = H$ so π is a bijection. ■

2 Group Actions

Definition. We say that a group **G acts on a set X** if there is a map $G \times X \to X$ satisfying $g(hx) = (gh)x$ and $1x = x$.

Equivalently, an action of G on X is a map $g \mapsto \pi_g$, which assigns to each $g \in G$ a permutation $\pi_G \in S_X$ which respects the operation of G; that is to say, if $g, h \in G$, then $\pi_{gh} = \pi_g \circ \pi_h$. In other words, an action of G on X is a homomorphism $\pi : G \to S_X$.

The action is often written in multiplicative form: we say $\pi_g(a) = b$ and can write $g \cdot a = b$, with $a, b \in X$ and $g \in G$.

Example. The most classic example of a group action is the action of G on itself by conjugation. For each $g \in G$, define the map $\phi_g : G \to G$ given by $\phi_g(x) = gxg^{-1}$. Since ϕ_g is an automorphism, it is certainly a permutation, and for any $g, h \in G$,

$$\phi_{gh}(x) = (gh)x(gh)^{-1} = g(hgh^{-1})g^{-1} = \phi_g \circ \phi_h(x)$$

Definition. Let π be an action of G on X.

1. The **kernel** of the action is the kernel of π as a homomorphism $G \to S_X$; in other words, the set $\{g \in G : g \cdot a = a \text{ for all } a \in X\}$.
2. The action is **faithful** if the kernel is $\{1\}$ (equivalently, if π is injective).
3. Given $a \in X$, the **orbit** of a is the set $G \cdot a = \{g \cdot a : g \in G\}$

If G acts faithfully on X, then G is isomorphic to a subgroup of S_X with isomorphism given by π.

2.1 Proposition. *Let G act on X. The orbits of the action partition X.*

Proof The orbits clearly cover X since $a \in G \cdot x$ for any $a \in X$. Suppose $G \cdot a$ and $G \cdot b$ are orbits. Either they or disjoint, or $x \in G \cdot a \cap G \cdot b$. Thus get g, h so that $x = g \cdot a = h \cdot b$. But

$$(g^{-1}h) \cdot b = g^{-1} \cdot (h \cdot b) = g^{-1} \cdot (g \cdot a) = (g^{-1}g) \cdot a = 1 \cdot 1 = a$$

so $a \in G \cdot b$. Thus $G \cdot a \subseteq G \cdot b$; the reverse inclusion follows identically, so $G \cdot a = G \cdot b$. ■

Definition. An action of G on X is **transitive** if it has only one orbit, X.

Definition. Let π be an action of G on X. Given $a \in X$, the **stabilizer** of a is the set $G_a = \{g \in G : g \cdot a = a\}$.

2.2 Proposition. (Orbit-Stabilizer) *Suppose G acts on X. For every $a \in X$,*

(i) $G_a \leq G$

(ii) $|G \cdot a| = [G : G_a]$

Hence if G is finite, then every orbit has size dividing G.

Proof 1. It suffices to show that G_a is closed under multiplication and inverses. Let $g, h \in G_a$. Then $(gh)\cdot a = g\cdot(h\cdot a) = g\cdot a = a$, so $gh \in G_a$. Similarly, $g^{-1}\cdot a = g^{-1}\cdot(g\cdot a) = (g^{-1}g)\cdot a = 1\cdot a = 1$.

2. Let g, h be arbitrary. Then

$$\begin{aligned} g\cdot a = h\cdot a &\iff h^{-1}\cdot(g\cdot a) = h^{-1}\cdot(h\cdot a) \\ &\iff (h^{-1}g)\cdot a = a \\ &\iff h^{-1}g \in G_a \\ &\iff hG_a = gG_a \end{aligned}$$

so that $g\cdot a$ depends only on gG_a. Thus the number of distinct values of $g\cdot a$ equals the number of left cosets of G_a. ∎

Conjugation and the Class Equation

Recall the action of G on itself by conjugation: the maps ϕ_g are given by $\phi_g(x) = gxg^{-1}$.

Definition. The **conjugacy class** of an element $a \in A$ is the set $G\cdot a = \{gag^{-1} : g \in G\} := \operatorname{conj}(a)$.

By general properties of group actions, G is partitioned by its conjugacy classes, and $|\operatorname{conj}(g)| = [G : G_a]$. In particular, when G is finite, $|\operatorname{conj}(a)| \,\big|\, |G|$ for any $g \in G$. Furthermore, the stabilizer G_a satisfies

$$G_a = \{g \in G : g\cdot a = a\} = \{g \in G : gag^{-1} = g\} = \{g \in G : ga = ag\} = C_G(a)$$

which is the centralizer of a in G. We thus have that $|\operatorname{conj}(g)| = [G : C_G(g)]$.

What happens when $\operatorname{conj}(g) = \{g\}$? In this case, we say that g is **central** (and otherwise call the conjugacy classes **non-central**). In this special case,

$$\begin{aligned} |\operatorname{conj}(g)| = 1 &\iff [G : C_G(g)] = 1 \\ &\iff G = C_G(g) \\ &\iff ga = ag \forall a \in G \\ &\iff g \in Z(G) \end{aligned}$$

Thus G is the disjoint union of $Z(G)$ and its non-central conjugacy classes. In particular, if $a_1, \ldots, a_m$ are representatives of the non-central conjugacy classes, we have

$$|G| = |Z(G)| + \sum_{i=1}^{m} |\operatorname{conj}(a_i)| = |Z(G) + \sum_{i=1}^{m} [G : C_G(a_i)]$$

Conjugation Action on Subgroups

Let G be a group, $P, Q \le G$ be subgroups. Let $\mathcal{K}$ denote the set of conjugates of P in G.

2.3 Proposition. *For any $A \in \mathcal{K}$, $A \le G$. If $A, B \in \mathcal{K}$, then $|A| = |B|$.*

In other words, $\mathcal{K}$ is composed of subgroups of G conjugate to P, all of which have the same size as P.

Proof If $a, b \in hPh^{-1}$, then $a = hp_1h^{-1}$, $b = hp_2h^{-1}$ so $ab = h(p_1p_2)h^{-1} \in hPh^{-1}$. Similarly, $a^{-1} = (hp_1h^{-1})^{-1} = hp_1^{-1}h^{-1} \in hPh^{-1}$ as well.

To see that $|A| = |B|$, since A, B are conjugate, get x so $B = xAx^{-1}$. The map $\alpha : A \to B$ given by $a \mapsto xax^{-1}$ is a bijection. It is injective, since if $xa_1x^{-1} = xa_2x^{-1}$ then $a_1 = a_2$; and it is surjective, since if $b \in B$, get $a \in A$ so $xax^{-1} = b$. ■

Given this setup, Q acts on $\mathcal{K}$ by conjugation: for $g \in Q$ and $hPh^{-1} \in \mathcal{K}$, we define $g \cdot hPh^{-1} = g(hPh^{-1})g^{-1} = (gh)P(gh)^{-1} \in \mathcal{K}$.

The orbits are equivalence classes of conjugates of P, where $h_1Ph_1^{-1} \sim h_2Ph_2^{-1}$ if they are conjugate by some element of Q.

Recall that $N_G(H) = \{g \in G : gHg^{-1} = H\}$; note that $N_G(H)$ is the largest subgroup of G containg H as a normal subgroup. Then the stabilizers are given by $Q_{P_i} = \{q \in Q : qP_iq^{-1} = P_i\} = N_G(P_i) \cap Q$.

3 Structure of Finitely Generated Abelian Groups

4 Sylow Theorems

Lagrange's theorem, that says that the order of any subgroup of a group G must divide its order. From the previous section, for finite abelian G, if $m \,\Big|\, |G|$ is any factor, then G has a subgroup of order m. This does not necessarily hold for groups which are not abelian.

4.1 Proposition. *There exists a group G and $m \,\Big|\, |G|$ so there is no subgroup of G with order m.*

Proof Take $G = A_4$, so $|G| = 12$. I claim that H has no group of order 6. For contradiction, suppose $H \leq G$ and $|H| = 6$. Let $a \in G$ such that $|a| = 3$; there are 8 such elements. Consider the cosets H, aH, a^2H. Since $[G : H] = 2$, there are 3 cases:

- $aH = H$, so $a \in H$
- $aH = a^2H$, so $H = aH$ and $a \in H$
- $a^2H = H$ so $H = aH$ and $a \in H$, since $a^3 = 1$.

Thus all 8 elements of order 3 are in H, contradiction. ■

While in general these subgroups do not exist, a partial converse is given by the First Sylow Theorem.

Sylow p–groups

Definition. Let p be a prime. We say that a group G is a **p-group** if $|G| = p^k$, $k \in \mathbb{N}$. If $H \leq G$ is a p-group, we say that H is a **p-subgroup**. If $|H| = p^k \| |G|$ with k maximal, then we say that G is a **Sylow p-subgroup of G**.

Before we prove the First Sylow Theorem, let's recall Cauchy's Theorem. Some standard proofs resort to the class equation; here, I will present a different alternative approach.

4.2 Theorem. (Cauchy) *Let G be a finite group and let $p \,\Big|\, |G|$ be prime. If r is the number of solutions to the equation $x^p = 1$, then $p | r$.*

Proof Let $|G| = n$, $p|n$ prime, and define

$$S = \{(a_1, a_2, \ldots, a_p) : a_i \in G, a_1 a_2 \cdots a_p = 1\}$$

and note that $|S| = n^{p-1}$. Define $\sim$ on S by $a \sim b$ if a and b are cyclic permutations of each other.

If all components of a p–tuple are equal, then its equivalence class has 1 member. Otherwise, its equivalence class has p members.

If r denotes the number of solutions to $x^p = 1$, then r is equal to the number of equivalence classes with exactly 1 member. Let s denote the number of equivalence classes with p members; then, $r + ps = n^{p-1}$ and since $p|n$, $p|r$ as well. ■

4.3 Corollary. *If $p \,\big|\, |G|$ is prime, then there exists $H \leq G$ with $|H| = p$.*

Proof By Cauchy's Theorem, there is at least one non-trivial solution to the equation $x^p = 1$. Let g be such an element; then $H = \langle g \rangle \leq G$ has order p. ■

In a sense, Cauchy's Theorem provides a partial converse to Lagrange's Theorem. However, the First Sylow Theorem is a strengthening of this claim. In particular, Cauchy's Theorem follows as an easy corollary.

4.4 Theorem. (First Sylow) *Let G be a finite group and let p be a prime dividing its order. Then G contains a Sylow p–subgroup.*

Proof The proof follows by induction on $|G|$. If $|G| = 2$, then G is its own Sylow 2-subgroup. If $|G| \geq 2$ is finite, let $p \,\big|\, |G|$, and say $|G| = p^n m$ where $p \nmid m$.

Case 1: $p \,\big|\, |Z(G)|$. By Cauchy, there exists $a \in Z(G)$ so that $o(a) = p$. Since $\langle a \rangle \subseteq Z(G)$, $\langle a \rangle \trianglelefteq G$. If $n = 1$, we are done; otherwise, by induction, $G/\langle a \rangle$ has a Sylow p–subgroup $\overline{H}$. By correspondence, $\overline{H} = H/\langle a \rangle$ for some $H \leq G$. Thus, $p^{n-1} = |H|/p$, so $|H| = p^n$ and H is a Sylow p–subgroup of G.

Case 2: $p \nmid |Z(G)|$. By the Class equation, there is some a_i so that $p \nmid [G : C_G(a_i)] = |G|/|C_G(a_i)|$. Thus $p^n \,\big|\, |C_G(a_i)|$ where a_i is non-central. Since $a_i \notin Z(G)$, $|C_G(a_i)| < |G|$. By induction, $C_G(a_i)$ has a Sylow p–subgroup, which is also a Sylow p–subgroup of G. ■

Structure of Sylow p–subgroups

Let G be a group and suppose $H \leq G$.

4.5 Lemma. *Suppose $p \,\big|\, |G|$, P is a Sylow p–subgroup of G, and Q is a p–subgroup of G. Then $Q \cap N_G(P) = Q \cap P$.*

Proof Since $P \subseteq N_G(P)$, $P \cap Q \subseteq N_G(P) \cap Q$. For notation, set $N = N_G(P)$ and $H = N_G(P) \cap Q$. It remains to show $H \subseteq P \cap Q$.

Write $|P| = p^n$ and $|H| = p^m$. Since $P \trianglelefteq N$, $HP \leq N$. Thus

$$|HP| = \frac{|H| \cdot |P|}{|H \cap P|} = p^k, k \leq n$$

As well, $P \subseteq HP$ so $n \leq k$, and $P = HP$. Thus $H \subseteq HP = P$. ■

4.6 Lemma. *Let G, p, P, Q be as in the previous lemma, and let $\mathcal{K}$ denote the set of conjugates of P in G. Let Q act on $\mathcal{K}$ by conjugation, so the orbits have representatives $P = P_1, P_2, \dots, P_r$. Then, $|\mathcal{K}| = \sum_{i=1}^r [Q : Q \cap P_i]$.*

Proof By the Orbit-Stabilizer lemma,

$$\begin{aligned}|\mathcal{K}| = \sum_{i=1}^r |Q \cdot P_i| = \sum_{i=1}^r [Q : Q_{P_i}] \\ = \sum_{i=1}^r [Q : N_G(P_i) \cap Q] \\ = \sum_{i=1}^r [Q : P_i \cap Q]\end{aligned}$$

where the last line follows from the previous lemma. ∎

4.7 Theorem. (Second Sylow) *If P and Q are Sylow p–subgroups of G, then there exists $g \in G$ so that $P = gQg^{-1}$.*

Since the conjugation action preserves the order of groups, the Sylow p–subgroups of G are precisely the equivalence class of any Sylow p–subgroup of G.

Proof Let $\mathcal{K}$ be the set of conjugates of P in G, and let P act on $\mathcal{K}$ by conjugation. Recall that for $P_i, P_j \in \mathcal{K}$, $|P_i| = |P_j|$.

Let $P = P_1, P_2, \dots, P_r$ be orbit represntatives. Then by the Lemma above,

$$|\mathcal{K}| = \sum_{i=1}^r [P : P \cap P_i] = 1 + \sum_{i=2}^r [P : P_i \cap P] \equiv 1 \pmod p$$

since $p \,\Big|\, [P : P_i \cap P]$: this follows since $P_i \cap P \lneq P$ and $|P| = p^n$.

Now let Q act on K by conjugation. Reindexing if necessary, let the orbits have representatives $P = P_1, P_2, \dots, P_s$. If $Q \neq P_i$ for $i = 1, 2, \dots, s$, then by the same argument as above, $|\mathcal{K}| = \sum_{i=1}^s [Q : P_i \cap Q] \equiv 0 \pmod p$, a contradiction. Thus $Q = P_i$ and so Q is a conjugate of P. ∎

Now Sylow's third theorem follows easily:

4.8 Theorem. (Third Sylow) *Let $p \,\Big|\, |G|$ be prime, $|G| = p^n m$ with $\gcd(p, m) = 1$, and n_p denote the number of Sylow p–subgroups of G. Then if P is any Sylow p–subgroup of G,*

1. *$n_p \equiv 1 \pmod p$*
2. *$n_p = [G : N_G(P)]$*

In particular, $n_p | m$, and $n_p = 1$ if and only if $N_G(P) = G$; in other words, that P is a normal subgroup of G.

Proof Let P be a Sylow p–subgroup of G and let $\mathcal{K}$ be the set of conjugates of P in G. From the proof of Sylow's second theorem, $n_p = |\mathcal{K}| \equiv 1 \pmod p$.

Now let G act on K by conjugation so $\mathcal{K} = G \cdot P$. By the Orbit-Stabilizer theorem, $|G| = |G_P| \cdot |G \cdot P|$. Since $G_P = N_G(P) \cap G = N_G(P)$, $p^n m = |N_G(P)| \cdot n_p$. Thus $n_p | p^n m$, and since $n_p \not\equiv 0 \pmod p$, $n_p | m$. ∎

Remark. $\operatorname{disc} f(x)$ is not a square in F iff $\operatorname{Gal} f(x) \not\subseteq A_2$ iff $\operatorname{Gal} f(x) = S_2$ iff $f(x)$ is irreducible.

Example. Prove that there is no simple group of order 56.

Note that $56 = 2^3 \cdot 7$. Since $n_7 \equiv 1 \pmod 7$ and $n_7 | 8$, we have $n_7 \in \{1, 8\}$. If $n_7 = 1$, then G has a normal Sylow 7–subgroup. By Lagrange, distinct Sylow 7-subgroups intersect trivially. Thus there are $8 \cdot 6 = 48$ elements of order 7 in G. This forces $n_2 = 1$. In either case, G is not simple.

Remark. If $p \neq q$ are prime, $p, q \mid |G|$. Then if H_p, H_q are $p-$ and $q-$subgroups, then $H_p \cap H_q = \{1\}$. Similarly, if $|G| = pm$ and H, K are Sylow $p-$subgroups, then $H = K$ or $H \cap K = \{1\}$.

Example. If $|G| = pq$, where p, q prime, $p < q$, $p \nmid q - 1$. Then G is cyclic.

Since $n_p \equiv 1 \pmod p$ and $n_p \mid |q$. We cannot have $n_p = q$, so G has a normal Sylow $p-$subgroup H_p. Since $p < q$, $q \nmid p - 1$, so $n_q = 1$ and G has a normal Sylow $q-$subgroup H_q, say H_q. Since $H_p \cap H_q = \{1\}$, $G \cong H_p \times H_q \cong \mathbb{Z}_{pq}$ since p, q are coprime.

Example. If $|G| = 30$, then G has a subgroup isomorphic to $\mathbb{Z}_{15}$. Since $n_5 \equiv 1 \pmod 5$ and $n_5 | 6$, $n_5 \in \{1, 6\}$. Similarly, $n_3 \equiv 1 \pmod 3$, and $n_3 | 10$, so $n_3 \in \{1, 10\}$. By counting elements, at least one must be normal. Let H_3, H_5 be Sylow subgroups. Since $3 \nmid 5 - 1$, $Z_{15} \cong H_3 H_5 \leq G$ by the previous example.

Example. If $|G| = 60$, $n_5 > 1$, then G is simple. Since $|G| = 60$, $n_5 \equiv 1 \pmod 5$ and $n_5 | 12$, we must have $n_5 = 6$ (accounting for 25 elements). Suppose $N \trianglelefteq G$.

Case 1: $5 \mid |H|$. Then H contains a Sylow 5–subgroup of G. Since H is normal, H contains all conjugate other Sylow 5-subgroups, so $|H| \geq 25$ and $|H| = 30$. By the previous example, $n_5 = 1$ since $\mathbb{Z}_{15}$ has only 1 Sylow 5-subgroup.

Case 2: $|H| \in \{2, 3, 4, 6, 12\}$. If $|H| = 12$, H has a normal Sylow 2- or 3-subgroup, which is normal in G. Call it K. If $|H| = 6$, then H has a normal Sylow 3-subgroup which is normal in G. Call it K. By replacing H with K if necessary, we may assume $|H| \in \{2, 3, 4\}$. Consider $\overline{G} = G/H$. Then $|\overline{G}| = \{15, 20, 30\}$. In any case, $\overline{G}$ has a normal Sylow 5-subgroup; call it $\overline{P}$. By correspondence, $\overline{P} = P/H$. P is a normal subgroup of G, so P is a proper, non-trivial normal subgroup of G. As well, $|P| = |\overline{P}| \cdot |H| = 5$, so $5 \mid |H|$ and $5 \mid |P|$. This contradicts Case 1.

Example. A_5 is simple since $|A_5| = 60$ and $\langle (12345) \rangle$, $\langle (13245) \rangle$ are distinct Sylow 5-subgroups.

II. Fields

5 Irreducible Polynomials

Definition. Let R be an integral domain. We say $f(x) \in R[x]$ is **irreducible** over R if f is a non-unit, non-irreducible, and whenever $f(x) = g(x)h(x)$, then either g is a unit or h is a unit. Otherwise, f is **reducible**.

Remark. A canonical way to construct new fields as follows. Suppose F be a field and I an ideal of $F[x]$. Since $F[x]$ is a PID ($F[x]$ has a division algorithm), then $I = \langle p(x) \rangle$, $p(x) \in F[x]$. Moreover, I is maximal if and only if $p(x)$ is irreducible. Thus $F[x]/I$ is a field if and only if $p(x)$ is irreducible.

5.1 Proposition. *Let F be a field. If $f(x) \in F[x]$, $\deg f(x) > 1$ and $f(x)$ has a root in F, then $f(x)$ is reducible over F. In particular, if $\deg f(x) \in \{2,3\}$, then $f(x)$ is irreducible over F if and only if f has no roots in F.*

Proof By the division algorithm, $f(x) = (x-a)q(x) + r(x)$ where $\deg r(x) \leq 1$. Then $f(x) = 0 + r = r$, so $f(x) = (x-a)q(x) + f(a)$, so $(x-a) \mid f(x)$ if and only if $f(a) = 0$. From this, the first claim follows immediately.

For the second claim, if $g(x)|f(x)$, then either $\deg g = \deg f$, $\deg g = 2$, or $\deg g = 1$. If every divisor has the same degree as f, then f is irreducible; otherwise, f has a factor of degree 1 and the claim follows by the initial observation. ■

5.2 Lemma. (Gauss' Lemma) *Let R be a UFD with field of fractions F. Let $p(x) \in R[x]$. If $p(x) = A(x)B(x)$ with $A(x), B(x)$ non-constant in $F[x]$, then there exists $r \in F^\times$ such that $a(x) = rA(x), b(x) = r^{-1}B(x) \in R[x]$.*

Proof PMATH 347. ■

Remark. Gauss' Lemma states that if $p(x) \in R[x]$ is reducible over F, then $p(x)$ is reducible over R. In particular, if $p(x)$ is irreducible over $\mathbb{Z}$, then $p(x)$ is irreducible over $\mathbb{Q}$ as well.

Let R be an integral domain and I a proper ideal. If $p(x) \in R[x]$ with coefficients a_i, then $\overline{p}(x) \in (R/I)[x]$ with coefficients $a_i + I$. The map $p(x) \mapsto \overline{p}(x)$ is a ring homomorphism.

5.3 Proposition. *Let I be a proper ideal of an integral domain R, and $p(x) \in R[x]$ non-constant and monic. If $\overline{p}(x)$ cannot be factored in $(R/I)[x]$ into polynomials of lesser degree, then $p(x)$ is irreducible in* $\mathrm{Frac}(R)[x]$.

Proof Suppose $p(x)$ is reducible over $\mathrm{Frac}(R)$; by Gauss' Lemma, write $p(x) = f(x)g(x)$ is a non-trivial factorization over $R[x]$ with $\deg f, \deg g < \deg p$. Without loss of generality, $f(x)$ and $g(x)$ are also monic. Thus, in $(R/I)[x]$, $\overline{p}(x) = \overline{f}(x) = \overline{g}(x)$. Since $I \subsetneq R$, $1 \notin I$, so $\deg \overline{f} = \deg f$, $\deg \overline{g} = \deg g$, $\deg \overline{p} = \deg p$ and $\overline{f} = \overline{g}\overline{h}$ is a non-trivial factorization. ■

5.4 Corollary. *Let $f(x) \in \mathbb{Z}[x]$, $\deg f(x) \geq 1$. Let $p \in \mathbb{Z}$ be a prime. If $\overline{f}(x) \in \mathbb{Z}_p[x]$ such that $\deg f(x) = \deg \overline{f}(x)$ and $\overline{f}(x)$ is irreducible over $\mathbb{Z}_p$, then $f(x)$ is irreducible over $\mathbb{Q}$.*

Proof Take $R = \mathbb{Z}$, $I = (p)$ in the previous lemma. ■

5.5 Proposition. (Eisenstein's Criterion) *Let R be an integral domain and P a prime ideal of R. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. If $a_i \in P$ and $a_0 \notin P^2$, then $f(x)$ is irreducible over R.*

Proof Suppose $f(x)$ is reducible over R. Since $f(x)$ is monic, $f(x) = g(x)h(x)$, where $g(x), h(x) \in R[x]$ with $\deg g(x), \deg h(x) < \deg f(x)$. Therefore,

$$\begin{aligned} \overline{f}(x) &= \overline{g}(x)\overline{h}(x) \\ &= x^n \in (R/P)[x] \end{aligned}$$

Since P is prime, R/P is an integral domain. Thus $\overline{g}(0) = \overline{h}(0) = 0$ and $g(0), h(0) \in P$, so $a_0 = g(0)h(0) \in P^2$. ■

Example. 1. $f(x,y) = x^2 + y^2 - 1 \in \mathbb{Q}[x,y]$ is irreducible. Let $g(y) = y^2 + (x^2 - 1)$, and take $P = \langle x+1 \rangle$. Since $x+1$ is irreducible, P is a prime ideal of $\mathbb{Q}[x]$. Moreover, $x^2 - 1 \in P$ but $(x+1)^2 \notin P^2$, so by Eisenstein, $f(x,y)$ is irreducible.

2. Suppose $f(x) = x^n - d$, where d is not a perfect square. Then f is irreducible over $\mathbb{Q}$ by Eisenstein.

3. $f(x) = x^3 + 2x + 16$. Consider modulo 3, $\overline{f}(x) = x^3 + 2x + 1$, which is irreducible by checking $0, 1, 2$ as roots.

4. $f(x) = x^4 + 5x^3 + 6x^2 - 1$. Then $\overline{f} = x^4 + x^3 + 1 \in \mathbb{Z}_2[x]$ is irreducible by checking roots and the unique irreducible quadriatic $x^2 + x + 1$.

5. Let p be a prime, and $f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 = (x^p - 1)/(x-1)$, so

$$f(x+1) = \frac{(x+1)^p - 1}{x} = x^{p-1} + \binom{p}{p-1} x^{p-2} + \cdots + \binom{p}{2} x + \binom{p}{1}$$

Since $f(x)$ is irreducible if and only if $f(x+a)$ is irreducible, $f(x)$ is irreducible by Eisenstein.

6 Field Extensions

6.1 Proposition. *The polynomial ring $F[x]$ has a division algorithm (i.e. it is a Euclidean domain). Thus $F[x]$ is a PID.*

Proof PMATH 347. ■

Definition. Let K be a field. $F \subseteq K$ is a **subfield** of K if F is a field under the same operations. A **field extension** of F is a field K which contains an isomorphic copy of F as a subfield. In this case, we write K/F. We say $F_1/F_2/\cdots/F_n$ is a **tower of fields** if each F_i/F_{i+1} is a field extension.

Remark. Suppose $f(x) \in F[x]$ is irreducible. Then $K = F[x]\big/\langle f(x)\rangle$ contains F in the following natural way: define $\phi : F \to K$ by $\phi(x) = x + \langle f(x)\rangle$. It follows that ϕ is injective: if $\phi(x) = \phi(y)$, then $x - y \in \langle f(x)\rangle$. Since $x - y \in F$ but $\langle f(x)\rangle \neq F[x]$, we must have $x - y = 0$ so $x = y$.

If $\mathrm{char}(F) = p > 0$, then there is a natural injection $\mathbb{Z}_p \to F$: consider the map $\phi : \mathbb{Z} \to F$ given by $n \mapsto n \cdot 1_F$; apply the first isomorphism theorem.

Definition. Let $\alpha_1, \ldots, \alpha_n \in K$. The **field extension of F generated by $\alpha_1, \ldots, \alpha_n$** is

$$F(\alpha_1, \ldots, \alpha_n) = \left\{ \frac{f(\alpha_1, \ldots, \alpha_n)}{g(\alpha_1, \ldots, \alpha_n)} : f, g \in F[x_1, \ldots, x_n], g(\alpha_1, \ldots, \alpha_n) \neq 0 \right\}$$

Remark. Note that $K/F(\alpha_1, \ldots, \alpha_n)/F$.

prop:f-ext

6.2 Proposition. *Suppose K/F, $\alpha \in K$. If α is a root of some non-zero $f(x) \in F[x]$, which is irreducible over F, then $F(\alpha) \cong F[x]\big/\langle f(x)\rangle$. Moreover, if $\deg f(x) = n$, then $F(\alpha) =$* $\mathrm{span}_F\{1, \alpha, \ldots, \alpha^{n-1}\}$.

Proof Let $\alpha \in K$ be a root of $f(x) \in F[x]$ with $\deg f(x) = n$. Consider the map

$$\phi : F[x] \to F(\alpha), \qquad \phi(g(x)) = g(\alpha)$$

One can verify that this is a ring homomorphism. Set $I = \ker(\phi)$: since $F[x]$ is a PID, $I = \langle g(x)\rangle$; since $f(x) \in I$, $f(x) = g(x)h(x)$ for some $h(x) \in F[x]$. Since I is a proper ideal, g is not a unit, so by irreducibility of f, h is a unit and $\langle g(x)\rangle = \langle f(x)\rangle$. Thus by the first isomorphism theorem, $F[x]\big/\langle f(x)\rangle \cong \phi(F[x])$ via $h(x) + \langle f(x)\rangle \mapsto h(\alpha)$.

By definition, $\phi(F[x]) \subseteq F(\alpha)$. Since $\phi(F[x])$ is a field (up to isomorphism) which contains $\alpha = \phi(x)$ and F, $F(\alpha) \subseteq \phi(F[x])$, so equality holds.

Finally, by the division algorithm,

$$F[x]\big/\langle f(x)\rangle = \left\{ c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_0 + \langle f(x)\rangle, c_i \in F \right\}$$

Thus $F(\alpha) = \{c_{n-1}\alpha^{n-1} + \cdots + c_a\alpha + c_0 : c_i \in F\} = \mathrm{span}_F\{1, \alpha, \ldots, \alpha^{n-1}\}$. ■

Remark. Suppose $g \in F[x]$ such that $g(\alpha) = 0$. Since $F[x]$ is an integral domain, g must have an irreducible factor f with $f(\alpha) = 0$. In particular,

1. If $h(x) \in F[x]$, $h(\alpha) = 0$ then $h(x) \in \langle f(x)\rangle$ and $f(x) \mid h(x)$.
2. $\langle f(x)\rangle$ contains a unique, monic, irreducible polynomial. If $g(x) \in \langle f(x)\rangle$ is irreducible, then $g(x) = uf(x)$.

Definition. Let K/F be an extension and $\alpha \in K$ a root of a nonzero polynomial in $F[x]$. Then, there exists a unique monic irreducible $f(x) \in F[x]$ such that $f(\alpha) = 0$. We call $f(x)$ the **minimal polynomial** of α over F. If $\deg f(x) = n$, then n is the **degree of α over F**.

6.3 Proposition. *Let K/F and $\alpha \in K$ with minimal polynomial $f(x) \in F[x]$, with $\deg_F(\alpha) = n$. Then $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a basis for K/F.*

Proof That it spans follows from the previous proposition (Proposition 6.2). If the set is linearly dependent, then the coefficients in the dependence relation would give a polynomial g with $g(\alpha) = 0$ and $\deg g \leq n - 1$, a contradiction. ■

6.4 Corollary. *Let $\alpha, \beta \in K$ have the same minimal polynomial $f(x) \in F[x]$. Then $F(\alpha) \cong F(\beta)$.*

Proof This is immediate since $F(\alpha) \cong F[x]\big/\langle f(x) \rangle \cong F(\beta)$. ■

Finite Extensions

Definition. We say that K/F is a **finite extension** if K is a finite dimensional F−vector space. We call $\dim_F K$ the **degree** of K/F and denote this dimension by $[K:F]$.

6.5 Theorem. *If K/E and E/F are extensions, then $[K:F] = [K:E][E:F]$.*

Proof Let $\{v_1, \dots, v_n\}$ be a basis for K/E and $\{w_1, \dots, w_m\}$ a basis for E/F. Let's show $\{w_i v_j : i \in [n], j \in [m]\}$ is a basis for K/F. Suppose $\sum_{i,j} c_{ij} v_i w_j = 0$. Then $\sum_i \left(\sum_j c_{ij} w_j\right) v_i = 0$; since the v_i are linearly independent, for each i, $\sum_j c_{ij} w_j = 0$ is linearly independent. It is clear that this sets spans, so it is indeed a basis. ■

Definition. Let K/F be an extension. We say $\alpha \in K$ is **algebraic over *F*** if it is the root of a non-zero polynomial. Otherwise, we say α is **transcendental over *F***. We say K/F is algebraic if every $\alpha \in K$ is algebraic over F. Otherwise, we say K/F is transcendental.

Remark. If $\alpha \in K$ is algebraic over F, then α has a minimal polynomial in $F[x]$.

6.6 Theorem. *If K/F is finite, then K/F is algebraic.*

Proof Suppose $[K:F] = n < \infty$, and let $\alpha \in K$. Consider $\alpha, \alpha^2, \dots, \alpha^{n+1}$. If $\alpha^i = \alpha^j$ for some $i \neq j$ then α is a root of $f(x) = x^j - x^i$. Otherwise, since $\{\alpha, \alpha^2, \dots, \alpha^{n+1}\}$ is linearly dependent over F, there is some dependence relation and α is a root of $f(x) = c_{n+1}x^{n+1} + \cdots + c_1 x \neq 0$. ■

Definition. We say that K is a **finitely generated** extension of F if there exists $\alpha_1, \dots, \alpha_n \in K$ such that $K = F(\alpha_1, \dots, \alpha_n)$.

6.7 Proposition. *If K is a finitely generated and algebraic extension of F, then K/F is finite.*

Proof Suppose K/F is algebraic, where $K = F(\alpha_1, \dots, \alpha_n)$, $\alpha_i \in K$. If $n = 1$, then $[F(\alpha_1):F] = \deg_F(\alpha_1) < \infty$.
Assume the result for n and consider $K = F(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$. Then

$$[F(\alpha_1, \dots, \alpha_n, \alpha_{n+1})] = [F(\alpha_1, \dots, \alpha_n)(\alpha_{n+1}) : F(\alpha_1, \dots, \alpha_n)] \cdot [F(\alpha_1, \dots, \alpha_n) : F] < \infty$$

by the tower theorem. ■

6.8 Proposition. *If K/E and E/F are both algebraic, then K/F is algebraic.*

Proof Let $\alpha \in K$. Since K/E is algebraic, α has a minimal polynomial in E:

$$p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \in E[x]$$

Thus α is algebraic over $F(c_0, c_1, \dots, c_{n-1})$. Note that $[F(c_{n-1}, \dots, c_1, c_0)(\alpha) : F(c_{n-1}, \dots, c_1, c_0)] < \infty$. Since $F(c_{n-1}, \dots, c_1, c_0) \subseteq E$, $F(c_{n-1}, \dots, c_1, c_0)/F$ is algebraic and finitely generated, so $[F(c_{n-1}, \dots, c_1, c_0) : F] < \infty$. By the tower theorem, $[F(c_{n-1}, \dots, c_1, c_0, \alpha) : F] < \infty$, so α is algebraic over F. ■

6.9 Proposition. *Let K/F be a extension. The set of elements of K which are algebraic over F form a subfield of K.*

Proof Let L denote the elements algebraic over F. If $\alpha, \beta \in L$, then $\alpha, \beta, \alpha - \beta, \alpha\beta, \beta^{-1} \in F(\alpha, \beta)$ and $[F(\alpha, \beta) : F] < \infty$ and since finite implies algebraic, these elements are all algebraic. ■

Splitting Fields

Definition. Let $f(x) \in F[x]$ be non-constant. We say $f(x)$ **splits** in an extension K of F if it factors completely into linear factors over K.

6.10 Theorem. (Kronecker) *Let $f(x) \in F[x]$ be non-constant. Then there exists an extension K of F such that $f(x)$ has a root in K.*

Proof Let $f(x) \in F[x]$ be non-constant; since $F[x]$ is a UFD, let $p|f$ where p is irreducible. Let $K = F[t]/(p(t))$, so $t + (p(t))$ is a root of $p(x)$, which is also a root of $f(x)$. ■

6.11 Corollary. *Let $f(x) \in F[x]$ be non-constant. There exists an extension K of F such that $f(x)$ splits over K.*

Proof Repeated application of Kronecker. ■

Definition. Let $f(x) \in F[x]$ be non-constant. A minimal extension K of F with the property that f splits over K is called a **splitting field** for f.

If $f(x) \in F[x]$, there is an extension K/F such that $f(x)$ splits over K. But then a splitting field for $f(x)$ over F is $F(\alpha_1, \dots, \alpha_n)$ where the α_i are the roots of f.

Example. Find a splitting field for $f(x) = x^4 + x^2 - 6$ over $\mathbb{Q}$. Over $\mathbb{C}$, $f(x) = (x + \sqrt{3}i)(x - \sqrt{3}i)(x - \sqrt{2})(x + \sqrt{2})$. Thus a splitting field for $f(x)$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{2}, \sqrt{3}i)$.

6.12 Lemma. *Let F, F' be fields. If $\phi : F \to F'$ is an isomorphism, then the natural map $\tilde{\phi} : F[x] \to F'[x]$ is an isomorphism.*

Proof It's long but easy. ■

We'll just write $\widetilde{\varphi} \equiv \varphi$.

lem:iso-ext ***6.13 Lemma. (Isomorphism Extension)*** *Let F, F' be fields, $\phi : F \to F'$ be an isomorphism. Let $f(x) \in F[x]$ be irreducible, α a root of $f(x)$ in an extension of F. β is a root of $\phi(f(x))$ in some extension of F'. Then there exists an isomorphism $\psi : F(\alpha) \to F'(\beta)$ such that $\psi|_F = \phi$ and $\psi(\alpha) = \beta$.*

Proof The following diagram commutes:

$$\begin{tikzcd}
F(\alpha) \arrow[rrr, dashed, "\psi"] \arrow[d, "\rho_1"', "\wr"] & & & F'(\beta) \\
& F \arrow[ul, hook] \arrow[r, "\phi"] & F' \arrow[ur, hook] & \\
F[x]/\langle f(x)\rangle \arrow[rrr, "\sim", "\sigma : \overline{g(x)} \mapsto \overline{\phi(g(x))}"'] & & & F'[x]/\langle \phi(f(x))\rangle \arrow[uuu, "\rho_2"', "\wr"]
\end{tikzcd}$$

where ψ exists by composing maps. If $a \in F$, then

$$\psi(a) = \rho_2 \circ \sigma \circ \rho_1(a) = \rho_2 \circ \sigma(\overline{a}) = \rho_2(\overline{\phi(a)}) = \phi(a) = a$$

As well, we verify that

$$\psi(\alpha) = \rho_2 \circ \sigma \circ \rho_1(\alpha) = \rho_2 \circ \sigma(\overline{x}) = \rho_2(\overline{\phi(x)}) = \rho_2(\overline{x}) = \beta$$ ■

6.14 Corollary. *Let F be a field, $f(x) \in F[x]$ non-constant. Let K be a splitting field for $f(x)$ over F. If F' is a field and $\phi : F \to F'$ is an isomorphism, then for any K' splitting field for $\phi(f(x))$ over F', there is an isomorphism $\psi : K \to K'$ such that $\psi|_F = \phi$.*

Proof Repeatedly apply the isomorphism extension lemma (Lemma 6.13) to the roots of f. ■

6.15 Corollary. *Let $f(x) \in F[x]$ be non-constant. If K and K' are splitting fields for $f(x)$ over F, then $K \cong K'$.*

Proof Take $\phi = \mathrm{id}$ in the previous corollary. ■

Algebraic Closure

Definition. A field $\overline{F}$ is an **algebraic closure** of a field F if

- $\overline{F}/F$ is algebraic
- Every non-constant polynomial in $F[x]$ splits over $\overline{F}$.

A field F is **algebraically closed** if every non-constant polynomial $f(x) \in F[x]$ has a root in F.

Example. $\mathbb{C}$ is an algebraic closure for $\mathbb{R}$, but not for $\mathbb{Q}$.

6.16 Proposition. *If $\overline{F}$ is an algebraic closure for F, then $\overline{F}$ is algebraically closed.*

Proof Let $\overline{F}$ be an algebraic closer for F. Let $f(x) \in \overline{F}(x)$ be non-constant; by Kronecker, $f(x)$ has a root α in some extension of $\overline{F}$. Since $\overline{F}(\alpha)/\overline{F}$ is algebraic and $\overline{F}/F$ is algebraic, $\overline{F}(\alpha)/F$ is algebraic. Thus α is the root of some non-zero polynomial $p(x) \in F[x]$. Now, $p(x)$ splits over $\overline{F}$ so $\alpha \in \overline{F}$ and $\overline{F}$ is algebraically closed. ■

6.17 Theorem. *For every field F, there exists an algebraically closed field containing F.*

Proof Exercise. ■

6.18 Theorem. *Let K be an algebraically closed field which contains F. The collection of elements in K which are algebraic over F is an algebraic closure.*

Proof Let $L = \{\alpha \in K : \alpha \text{ is algebraic over} F\}$. We claim that L is an algebraic closure for F. By construction, L/F is algebraic. Let $f(x) \in F[x]$, $\deg f(x) \geq 1$. Since $f(x)$ splits over K, $f(x) = u(x-\alpha_1)\cdots(x-\alpha_n)$. Since $u \in F$, $\alpha_i \in K$. But, $f(\alpha_i) = 0$ for $i = 1, \ldots, n$ and so $\alpha_i \in L$ and $f(x)$ splits over L. ■

7 Examples of Field Extensions

Cyclotomic Extensions

What is the splitting field of $f(x) = x^n - 1$?

Definition. We call the roots of $x^n - 1$ (in $\mathbb{C}$) the $\boldsymbol{n}^{\textbf{th}}$ **roots of unity**.

If $\zeta_n = e^{2\pi i/n}$, they are $1, \zeta_n, \zeta_n^2, \cdots, \zeta_n^{n-1}$. Thus, the splitting field over $\mathbb{Q}$ is $\mathbb{Q}(\zeta_n)$. What is $[\mathbb{Q}(\zeta_n) : \mathbb{Q}]$? When $n = p$ is prime, $x^p - 1 = (x-1)(1 + x + x^2 + \cdots + x^{p-1})$. Since $\Phi_p(x) = x^{p-1} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$ (from before), so $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p - 1$.

Example. Since $\zeta_5 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\mathbb{Q}(\zeta_6) = \mathbb{Q}(i\sqrt{3})$ so $\deg(x^2 + 3) = 2$.

Note that the n^{th} roots of unity form a finite cyclic subgroup of $\mathbb{C}$; in fact, they are the only finite cyclic subgroups of $\mathbb{C}$. A generator of this group is called a **primitive $\boldsymbol{n}^{\textbf{th}}$ root of unity**, which happens precisely for ζ_n^k where $\gcd(k, n) = 1$. Thus there are $\phi(n)$ primitive n^{th} roots of unity.

Definition. The $\boldsymbol{n}^{\textbf{th}}$ **cyclotomic polynomial** is

$$\Phi_n(x) = \prod_{k \in (\mathbb{Z}_n)^\times} (x - \zeta_n^k)$$

7.1 Theorem. *$\Phi_n(x)$ is the minimal polynmial for ζ_n, and $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \phi(n)$.*

Proof Note that ζ_n is a root of $x^n - 1$, so ζ_n is algebraic over $\mathbb{Q}$. By Gauss' lemma, let $f(x) \in \mathbb{Z}[x]$ be the minimal polynomial of ζ_n over $\mathbb{Q}$ so that $f(x) \mid (x^n - 1)$ over $\mathbb{Z}[x]$. Recall that

$$x^n - 1 = \prod_{j \in \mathbb{Z}_n} (x - \zeta_n^j)$$

If $j \notin (\mathbb{Z}_n)^\times$, then ζ_n^j satisfies $x^{\frac{n}{\gcd(n,j)}} - 1$ but ζ_n does not, so ζ and ζ_n^j are not conjugates. Thus the only possible conjugates for ζ_n are the ζ_n^j where $j \in (\mathbb{Z}_n)^\times$; it suffices to show that these are precisely the conjugates. In particular, let's show that if $\theta = \zeta_n^t$ and p is prime with $p \nmid n$, then θ^p is conjugate to θ. With this, the result follows: if j is coprime to n, write $j = p_1^{e_1} \cdots p_m^{e_m}$ with $p_i \nmid n$ and repeatedly apply the above result to ζ_n for each p_i, e_i times.

Thus let's prove the claim. Write $x^n - 1 = f(x)g(x)$ with $f, g \in \mathbb{Z}[x]$; since θ^p is a root of $x^n - 1$, either it is a root of $f(x)$ - in which case we're done - or it is a root of $g(x)$. Suppose $g(\theta^p) = 0$, so θ is a root of $g(x^p) \in \mathbb{Z}[x]$ so $f(x) \mid g(x^p)$ over $\mathbb{Z}[x]$. Modulo p, $\overline{f}(x) \mid \overline{g}(x^p) = \overline{g}(x)^p$ in $\mathbb{Z}_p[x]$. Since $\mathbb{Z}_p[x]$ is a UFD, let $s(x)$ be an irreducible factor of $f(x)$ so that $s|\overline{f}$ and thus $s|\overline{g}$. But then $x^n - \overline{1} = \overline{f}\,\overline{g}$, so $s^2 \mid (x^n - 1)$ and $s \mid \overline{n}x^{n-1}$. Since n is coprime to p, this implies $s = cx$ for some $c \in \mathbb{Z}_p$. But then $cx \mid x^n - \overline{1}$, a contradiction. ∎

Finite Fields

Definition. Let F be a field of characteristic p. Then the map $\phi : F \to F$ given by $x \mapsto x^p$ is called the **Frobenius map**.

7.2 Proposition. *The Frobenius map is an injective ring homomorphism.*

Proof We have that $\phi(xy) = x^p y^p = (xy)^p$, and

$$\phi(x+y) = (x+y)^p = \sum_{i=0}^{p} x^i y^{p-i} \binom{p}{i} = x^p + y^p$$

since $p \mid \binom{p}{i}$ for all $1 \le i \le p-1$. Injectivity is immediate since $\phi(1) = 1$ and the only ideals of F are $\{0\}$ and $\{F\}$, forcing $\ker(\phi) = \{0\}$. ∎

7.3 Corollary. *If F is a finite field, the Frobenius map is an automorphism.*

7.4 Proposition. *Suppose F is finite. Then*

1. $F^\times = \langle \alpha \rangle$ *is a cyclic group.*
2. $|F| = p^n$.
3. $|F| = p^n$ *if and only if F is the splitting field for* $x^{p^n} - x$ *over* $\mathbb{Z}_p$.
4. *Finite fields of a fixed size are unique up to isomorphism.*

Proof

1. Write $F^\times \cong C_{n_1} \times \cdots \times C_{n_k}$ where $n_1 | n_2 | \cdots | n_k$. Then each C_{n_i} has a subgroup $D_i \cong C_{n_k}$; but then every $x \in D_1 \times \cdots \times D_k$ satisfies $x^{n_k} = 1$. Since there are ${n_k}^k$ such elements and $x^{n_k} = 1$ has at most n_k roots, this forces $k = 1$ and $F^\times$ is cyclic.
2. Recall that $F/\mathbb{Z}_p$ where $p = \operatorname{char} F$. Thus $[F : \mathbb{Z}_p] = n < \infty$ so that $F = \mathbb{Z}_p(\alpha)$ and $|F| = p^n$.
3. Suppose $|F| = p^n$; by Lagrange, every $a \in F^\times$ satisfies $x^{p^n - 1} - 1$ so that every $a \in F$ satisfies $x^{p^n} - x$, so $x^{p^n} - x$ splits over F. Take $f(x) = x^{p^n} - x$, so that $f'(x) = -1$ and f is separable. Thus, any splitting field F must have at least p^n elemenets, so $|F|$ is minimal and F is a splitting field of $x^{p^n} - x$.
   Conversely, suppose F is the splitting field of $x^{p^n} - x$ over $\mathbb{Z}_p$. Consider $K = \{\alpha \in F : f(\alpha) = 0\}$, so that $K \le F$. In particular, F splits in K, forcing $K = F$. Thus, $|F| = |K| \le p^n$ since f can have at most p^n roots. However, as above, $f(x)$ is separable, so $|F| = |K| = p^n$.
4. Splitting fields are unique up to isomorphism. ∎

Since the splitting field is unique, for any prime p and $n \in \mathbb{N}$, there exists a unique field of order p^n (up to isomorphism). We denote the field $\mathbb{F}_{p^n}$.

7.5 Theorem. *If E is a subfield of* $\mathbb{F}_{p^n}$, *then* $E \cong \mathbb{F}_{p^r}$, *where* $r|n$. *Moreover, if* $r|n$, *then* $\mathbb{F}_{p^n}$ *has a unique subfield of order* p^r.

Proof Let E be a subfield of $\mathbb{F}_{p^n}$, so $n = [\mathbb{F}_{p^n} : \mathbb{F}_p] = [\mathbb{F}_{p^n} : E][E : \mathbb{F}_p]$. Set $r = [E : \mathbb{F}_p]$, $r|n$, and $|E| = p^r$.

Conversely, suppose $r|n$, and consider $\mathbb{F}_{p^n} = \{\alpha \in \overline{\mathbb{F}_p} : \alpha^{p^n} - \alpha = 0\}$. Since $r|n$, write $p^n - 1 = (p^r - 1)(p^{n-r} + p^{n-2r} + \cdots + p^r + 1)$. From before,

$$\begin{aligned} E &= \{\alpha \in \overline{\mathbb{F}_p} : \alpha^{p^r} - \alpha = 0\} \\ &= \{\alpha \in \overline{\mathbb{F}_p} : \alpha^{p^r - 1} - 1 = 0\} \cup \{0\} \\ &\subseteq \mathbb{F}_{p^n} \end{aligned}$$

Moreover, $|E| = p^r$. If K is any other subfield and $|K| = p^r$, then for any $0 \ne \alpha \in K$, $\alpha^{p^r - 1} = 1$ since $K^\times$ is cyclic, and $K \subseteq E$. ∎

III. Galois Theory

TODO

- talk about maps $\sigma : K \hookrightarrow k^a$ (algebraic closure of k).
- full proof of algebraic closure
- isomorphism extension lemma in terms of emebeddings
- use lower case k for base field to distinguish.
- Use universal property of simple field extensions

8 Galois Groups

Let $f(x) \in F[x]$ be non-constant, and $\alpha_1, \ldots, \alpha_n$ be the roots of $f(x)$ in its splitting field. Our goal is to study these roots by permuting them using automorphisms of K.

Definition. Let K/F. Recall that $\mathrm{Aut}(K)$ is the group of automorphisms of K. We define $\mathrm{Gal}(K/F) = \{\phi \in \mathrm{Aut}(K) : \phi|_F = \mathrm{id}\} \leq \mathrm{Aut}(K)$.

8.1 Lemma. *Let K/F. If $\alpha \in K$ is a root of $f(x) \in F[x]$ and $\phi \in \mathrm{Gal}(K/F)$, then $\phi(\alpha)$ is also a root of $f(x)$.*

Proof Note that $0 = \phi(f(\alpha)) = f(\phi(\alpha))$ since ϕ fixes the coefficients of f. ■

8.2 Corollary. *If $\alpha \in K$ is algebraic over F and $\phi \in \mathrm{Gal}(K/F)$, then $\phi(\alpha)$ is algebraic over F and has the same minimal polynomial in $F[x]$.*

Example. Compute $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q})$. If $\phi \in \mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q})$, then $\phi(\sqrt{2}) = \pm\sqrt{2}$ and $\phi(\sqrt{3}) = \pm\sqrt{3}$. Thus the automorphisms are given by.

$$\phi_1 = \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \end{cases} \qquad \phi_2 = \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \end{cases}$$

$$\phi_3 = \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \end{cases} \qquad \phi_4 = \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \end{cases}$$

and $G = \{\phi_1, \phi_2, \phi_3, \phi_4\}$. Since $|\phi_i| = 2$ for all i, G is abelian, so $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Example. Consider $G = \mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$. If $\phi \in G$, then $\phi(\sqrt[3]{2}) \in \{\sqrt[3]{2}, \sqrt[3]{2}\zeta_3, \sqrt[3]{2}\zeta_3^2\}$, so $\phi(\sqrt[3]{2}) = \sqrt[3]{2}$. Thus $\phi = \mathrm{id}$ and $G = \{\mathrm{id}\}$.

Let F be a field, $f(x) \in F[x]$, $\deg f(x) = n \geq 1$. Let K be the splitting field for $f(x)$ over F, so the roots of $f(x)$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let $G = \mathrm{Gal}(K/F)$, so for any $\phi \in G$, $\phi(\alpha_i) = \alpha_j$. In particular, for any $\phi \in \mathrm{Gal}(K/F)$, $\phi(\alpha_i) = \alpha_{\pi(i)}$ for some $\pi \in S_n$. Thus the map $\mathrm{Gal}(K/F) \to S_n$ given by $\phi \mapsto \pi$ is injective.

Remark. If $f(x) \in F[x]$, K the splitting field for $f(x)$, then we write $\mathrm{Gal}(K/F) = \mathrm{Gal}(f(x))$.

Example. Consider $f(x) = (x^2 - 2)(x^2 - 3) \in \mathbb{Q}[x]$. Then $\mathrm{Gal}(f(x)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $\alpha_1 = \sqrt{2}$, $\alpha_2 = -\sqrt{2}$, $\alpha_3 = \sqrt{3}$, $\alpha_4 = -\sqrt{3}$, so $\mathrm{Gal}(f(x)) = \{\epsilon, (34), (12), (12)(34)\}$.

Example. $\mathrm{Gal}(x^2+1) \cong \mathbb{Z}_2$ over $\mathbb{Q}[x]$, but $\mathrm{Gal}(x^2+1) = \{1\}$ over $\mathbb{Z}_2[x]$.

8.3 Corollary. *Let F be a field, $f(x) \in F[x]$ irreducible, K the splitting field for $f(x)$ over F. Then for any roots $\alpha, \beta \in K$ of $f(x)$, there exists $\phi \in \mathrm{Gal}(K/F)$ such that $\phi(\alpha) = \beta$.*

Proof By the isomorphism extension lemma (Lemma 6.13), $\mathrm{id} : F \to F$ extents to an automorphism $\phi : F(\alpha) \to F(\beta)$ such that $\alpha \mapsto \beta$, which extends to an isomorphism $K \to K$. ∎

Definition. A subgroup H of S_n is **transitive** if for all $i, j \in \{1, 2, \dots, n\}$, there exists $\pi \in H$ such that $\pi(i) = j$.

8.4 Corollary. *Let $f(x) \in F[x]$, $\deg f(x) = n \geq 1$, $f(x)$ separable and irreducible. Then $\mathrm{Gal}(f(x))$ is isomorphic to a transitive subgroup of S_n.*

Example. Compute $G = \mathrm{Gal}(x^3 - 2)$ over $\mathbb{Q}[x]$. Since $f(x) = x^3 - 2$ is irreducible, $f(x)$ is also separable. Then G is isomorphic to a transitive subgroup of S_3. Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of $f(x)$, and $x = \{\alpha_1, \alpha_2, \alpha_3\}$. Then G acts on X via $\phi \cdot \alpha_i = \phi(\alpha_i)$. By Orbit-Stabilizer, $|G| = |G \cdot \alpha| \cdot |\mathrm{Stab}(\alpha_1)|$. By transitivity, $|G \cdot \alpha| = 3$, so $3 \,\big|\, |G|$ and $G \cong A_3$ or S_3.

Consider G as a subgroup of S_3 relative to the order $\alpha_1 = \sqrt[3]{2}$, $\alpha_2 = \alpha_1 \zeta_3$, $\alpha_3 = \alpha_1 \zeta_3^2$. Note that $x^3 - 2$ is irreducible over $\mathbb{Q}(\zeta_3)$ since $x^3 - 2$ has no roots in $\mathbb{Q}(\zeta_3)$. Thus by the isomorphism extension lemma, there exists $\phi \in G$ such that the following diagram commutes:

$$
\begin{tikzcd}
\mathbb{Q}(\zeta_3, \alpha_1) \arrow[r, "\phi:\phi(\alpha_1)=\alpha_1"] & \mathbb{Q}(\zeta_3, \alpha_1) \\
\mathbb{Q}(\zeta_3) \arrow[u, hook] \arrow[r, "\zeta_3 \mapsto \zeta_3^2"] & \mathbb{Q}(\zeta_3) \arrow[u, hook] \\
\mathbb{Q} \arrow[u, hook] \arrow[r, "\mathrm{id}"] & \mathbb{Q} \arrow[u, hook]
\end{tikzcd}
$$

Thus $\phi(\alpha_1) = \alpha_1$, $\phi(\alpha_2) = \alpha_3$ and $\phi(\alpha_3) = \alpha_2$. Hence $\phi \sim (23) \in G$ is an element of order 2, so $G \cong S_3$.

Remark. When computing $G = \mathrm{Gal}(K/F)$, it is useful to know $|G|$.

Definition. Suppose K/F and E/F are field extensions. Any homomorphism $\phi : K \to E$ which fixes F is called an ***F*−map** from K to E.

Remark. If $\phi : K \to E$ is a F−map, since K is a field, ϕ is automatically injective. Furthermore, for any $\alpha \in F$, $v \in K$, $\phi(\alpha v) = \alpha \phi(v)$, so ϕ is F−linear.

If $\phi : K \to K$ and $[K : F] < \infty$, then ϕ is surjective and $\phi : K \to K$ is an F−map if and only if $\phi \in \mathrm{Gal}(K/F)$.

8.5 Lemma. *Let K/F, E/F, $[K : E] < \infty$. The number of distinct F−maps $\phi : K \to E$ is at most $[K : F]$.*

Proof We proceed inductively on the number of generators of K/F. If $K = F(\alpha_1)$ and $\phi : K \to E$ is an F–map, then α_1 and $\phi(\alpha_1)$ have the same minimal polynomial over F. Thus there are at most $[F(\alpha_1) : F] = [K : F]$ options $\phi(\alpha_1)$, so there are at most $[K : F]$ many such F–maps.

Now assume $K = F(\alpha_1, \dots, \alpha_n)$, and let $L = F(\alpha_1, \dots, \alpha_{n-1})$. Let $\phi : K \to E$ be an F–map, so $\phi|_L : L \to E$ is an F–map. By induction, the number of possible $\phi|_L$ is at most $[L : F]$. Since ϕ is completely determined by $\phi|_L$ and $\phi(\alpha_n)$, there are at most $[L : F][L(\alpha_n) : L] = [K : F]$ possibilities for ϕ. ■

Remark. How can it happen that $|\mathrm{Gal}(K/F)| < [K : F]$? It could be that the extension is not normal; i.e. the extension has conjugates not contained in the extension.

It can also happen that there are repeated roots: consider $G = \mathrm{Gal}(\mathbb{Z}_2(t)/\mathbb{Z}_2(t^2))$, so $[\mathbb{Z}_2(t) : \mathbb{Z}_2(t^2)] = 2$. Then $t \mapsto x^2 - t^2 \in \mathbb{Z}(t^2)[x]$, so $(x - t)^2 \in \mathbb{Z}(t)[x]$. Thus if $\phi \in G$, then $\phi(t) = t$, so $\phi = \mathrm{id}$ and $G = \{1\}$.

9 Separable and Normal Extensions

Definition. We say $\alpha \in K$ is **separable** if α is algebraic over F and its minimal polynomial is separable (over F). We say K/F is **separable** if K/F is algebraic and all elements of K are separable over F. A field F is **perfect** if every algebraic extension of F is separable.

Remark. Suppose $f(x) \in F[x]$ is irreducible. Then $f(x)$ is separable if and only if $f'(x) \neq 0$.

9.1 Proposition. *Let $f(x) \in F[x]$ be irreducible.*

1. *If $\mathrm{char}(F) = 0$, then $f(x)$ is separable.*
2. *If $\mathrm{char}(F) = p > 0$ then $f(x)$ is not separable if and only if $f(x) = g(x^p)$ for some $g(x) \in F[x]$.*

Proof Immediate from the preceding remark. ■

9.2 Corollary.
1. *If $\mathrm{char}(F) = 0$, then F is perfect.*
2. *If $\mathrm{char}(F) = p$, then F is perfect if and only if $\phi(x) = x^p$ is an automorphism.*

Proof (1) is clear, so we prove (2). In characteristic p, ϕ is always injective.

First suppose $\phi(x) = x^p$ is also surjective. Suppose there exists $f(x) \in F[x]$ irreducible but not separable. Thus $f(x) = g(x^p)$, and write

$$\begin{aligned} f(x) &= a_n x^{pm_n} + \cdots + a_1 x^{pm_1} + a_0 \\ &= b_n^p x^{pm_n} + \cdots + b_1^p x^{pm_1} + b_0^p \\ &= (b_n x^{m_n} + \cdots + b_x x^{m_1} + b_0)^p \end{aligned}$$

Conversely, suppose x^p is not an automorphism; in particular, x^p is not surjective. Let $\alpha \notin \mathrm{im}(\phi)$. But then $f(x) = x^p - \alpha$ is irreducible, but if K is the splitting field for F, then r is a root so $r^p = \alpha$ and $(x - r)^p = x^p - \alpha$ and f is not separable. ■

Remark. Since the Frobenius map is an isomorphism when F is a finite field, every finite field is perfect.

thm:gal-size

9.3 Theorem. *Let $f(x) \in F[x]$ be non-constant and separable, and K the splitting field for $f(x)$ over F. Then $|\mathrm{Gal}(K/F)| = [K : F]$.*

Proof We proceed by induction on $[K : F]$. If $[K : F] = 1$, this is obvious.

Otherwise, let $[K : F] = n > 1$. Let $p(x) \in F[x]$ be an irreducible factor of $f(x)$, so $p(x)$ is also separable over F. Say the roots of $p(x)$ are $\alpha_1, \alpha_2, \dots, \alpha_m$ where $m = \deg p(x)$; suppose $\alpha_1 \notin F$ and let $E = F(\alpha_1)$. Then $K/E/F$ is a tower of fields with $[K : E] = \frac{n}{m} < n$. Furthermore, K is the splitting field for $f(x)$ over E, so by induction, $|\mathrm{Gal}(K/E)| = [K : E] = \frac{n}{m}$.

Since $p(x) \in F[x]$ is irreducible, for all j, get $\phi_j \in \mathrm{Gal}(K/F)$ such that $\phi_j(\alpha_1) = \alpha_j$; note that $\phi_1, \dots, \phi_m$ are distinct in $\mathrm{Gal}(K/F)$. Moreover, $\phi_j^{-1} \circ \phi_i(\alpha_1) \neq \alpha_1 \in E$. Thus $\phi_j^{-1} \circ \phi_i \notin \mathrm{Gal}(K/E)$, so $\phi_i \mathrm{Gal}(K/E) \neq \phi_j \mathrm{Gal}(K/E)$. Thus $|\mathrm{Gal}(K/F)/\mathrm{Gal}(K/E)| \geq m$. Thus $|\mathrm{Gal}(K/F)| \geq m|\mathrm{Gal}(K/E)| = n$, and we're done. ■

Definition. We say an extension K/F is **simple** if there exists $\alpha \in K$ such that $K = F(\alpha)$. We say α is a **primitive element** for K/F.

thm:prim-el

9.4 Theorem. (Primitive Element) *If K/F is finite and separable, then K/F is simple.*

Proof Suppose K/F is finite and separable.

First suppose F is finite, so that K is also finite and $K^\times = \langle \alpha \rangle$ for some $\alpha \in K$. Thus, $K = F(\alpha)$.

Otherwise, F is infinite, and write $K = F(\pi_1, \dots, \pi_n)$ for some $\pi_i \in K$. It suffices to prove the result for $n = 2$; say, $K = F(\alpha, \beta)$. Let p, q be the minimal polynomial of α and β respectively. Let L be the splitting field for $p(x)q(x)$ over K, and let $\alpha = \alpha_1, \dots, \alpha_n$ and $\beta = \beta_1, \dots, \beta_k$ the distinct conjugates in L of α and β (since K/F is separable). Let

$$S = \left\{ \frac{\alpha_i - \alpha_1}{\beta_1 - \beta_j} : 1 < i \leq n, 1 < j \leq m \right\}$$

Since S is finite and F is infinite, get $u \in S \setminus F$ so that $\gamma := \alpha + u\beta \neq \alpha_i + u\beta_j$ for any $i, j \neq 1$. Certainly $F(\gamma) \subseteq F(\alpha, \beta)$. Let $h(x)$ be the minimal polynomial for β over $F(\gamma)$. Since $q(x) \in F(\gamma)[x]$ and $q(\beta) = 0$, $h(x)|q(x)$. As well, $h(x)|p(\gamma - ux)$ since $p(\gamma - u\beta) = 0$; but the only shared root is β by choice of u, $\deg h = 1$ and $\beta \in F(\gamma)$. ■

9.5 Corollary. *If F is perfect and $[K : F] < \infty$, then K/F is simple.*

TODO: move def'n of conjugates somewhere more logical.

Definition. Let $[K : F] < \infty$. We say K/F is **normal** if K is the splitting field of some non-constant $f(x) \in F[x]$ over F. Suppose $\alpha \in K$ has minimal polynomial $p(x) \in F[x]$. The roots of $p(x)$ in its splitting field are called the ***F*–conjugates** (or just **conjugates** when the base field is clear) of α.

Remark. If $\phi : K \to E$ is an F–map and α has minimal polynomial $p(x) \in F[x]$, then $p(\phi(\alpha)) = \phi(p(\alpha)) = \phi(0) = 0$, so that $\phi(\alpha)$ is also a conjugate of $p(x)$ in a splitting field L/F.

thm:char-norm

9.6 Theorem. (Characterization of Normal Extensions) *Let $[K : F] < \infty$. The following are equivalent:*

1. *K/F is normal.*
2. *For every L/K, if ϕ is an F–map from L to L, then $\phi|_K \in \mathrm{Gal}(K/F)$.*

3. *If $\alpha \in K$, then all of the F−conjugates of α are in K.*
4. *If $\alpha \in K$, then its minimal polynomial splits over K.*

Proof (1 ⇒ 2) If K/F is normal, then K is the splitting field of some $f(x) \in F[x]$. Let $\phi : L \to L$ be an F−map. Write $K = F(\alpha_1, \dots, \alpha_n)$ where α_i are the roots of $f(x)$ in K. It suffices to show that $\phi|_K(K) \subseteq K$. For each i, there exists j such that $\phi|_K(\alpha_i) = \phi(\alpha_i) = \alpha_j \in K$. Since each $x \in K$ is a F−linear combination of the α_i, it follows that $\phi(x) \in K$, and the result follows.

(2 ⇒ 3) Let $\alpha \in K$ with minimal polynomial $f(x) \in F[x]$. Since $[K : F] < \infty$, $K = F(\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in K$. For each i, let h_i be the minimal polynomial for α_i over F. Let $p(x) = f(x)h_1(x)h_2(x)\cdots h_n(x)$ and L be the splitting field of $p(x)$ over F. Such a choice is necessary to ensure $L/K/F$. Let $\beta \in L$ be a root of $f(x)$, and get $\phi \in \mathrm{Gal}(L/F)$ such that $\phi(\alpha) = \beta$. By assumption, $\phi|_K \in \mathrm{Gal}(K/F)$, so $\beta = \phi(\alpha) \in K$, as required.

(3 ⇒ 4) Immediate.

(4 ⇒ 1) Since $[K : F] < \infty$, $K = F(\alpha_1, \dots, \alpha_n)$ for $\alpha_i \in K$. Let $h_i(x)$ be the minimal polynomial for α_i over F, and set $f(x) = h_1(x)\cdots h_n(x)$. Then the splitting field for $f(x)$ over F is $F(\alpha_1, \dots, \alpha_n) = K$. ∎

Example. $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal. $\mathbb{F}_{p^n}/\mathbb{F}_p$ is normal, since it is the splitting field of $x^{p^n} - x$. $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is normal with $\Phi_n(x)$. $\mathbb{Z}_p(t)/\mathbb{Z}_p(t^n)$ is normal with $x^p - t^p$.

10 Galois Extensions and the Fundamental Theorem

Definition. We say that K/F is **Galois** if K/F is normal and separable.

Remark. If F is perfect and K/F is finite, then K/F is Galois if and only if K/F is normal.

Definition. Let K be a field and $G \leq \mathrm{Aut}(K)$. Then the **fixed field** of G is

$$\mathrm{Fix}(G) = \left\{ a \in K : \phi(a) = a \text{ for all } \phi \in G \right\}$$

Remark. Certainly $\mathrm{Fix}(\mathrm{Gal}(K/F)) \supseteq F$ by definition.

thm:char-gal

10.1 Theorem. (Characterization of Galois Extensions) *The following are equivalent:*

1. *K is the splitting field of a non-constant separable $f(x) \in F[x]$ over F.*
2. $|\mathrm{Gal}(K/F)| = [K : F]$
3. $\mathrm{Fix}(\mathrm{Gal}(K/F)) = F$
4. *K/F is Galois*

Proof (1 ⇒ 2) This is Theorem 9.3.

(2 ⇒ 3) Assume $|\mathrm{Gal}(K/F)| = [K : F]$ and set $E = \mathrm{Fix}(\mathrm{Gal}(K/F))$ so that $K/E/F$ is a tower of fields. Moreover, $\mathrm{Gal}(K/E) \leq \mathrm{Gal}(K/F)$ is a subgroup so $[K : F] = |\mathrm{Gal}(K/F)| \geq |\mathrm{Gal}(K/E)|$. Let $a \in E$ and $\phi \in \mathrm{Gal}(K/F)$. Then $\phi(a) = a$ by the definition of E, so $\mathrm{Gal}(K/E) = \mathrm{Gal}(K/F)$. Thus

$$[K : F] = |\mathrm{Gal}(K/F)| = |\mathrm{Gal}(K/E)| \leq [K : E] \leq [K : F]$$

so equality holds and $[E : F] = 1$ by the tower theorem.

(3 ⇒ 4) Assume $\mathrm{Fix}(\mathrm{Gal}(K/F)) = F$. Let $\alpha \in K$ with minimal polynomial $p(x) \in F[x]$; we must show $p(x)$ that splits over K with no repeated roots. Let $G = \mathrm{Gal}(K/F)$ and $\{\alpha_1, \dots, \alpha_n\} = \{\phi(\alpha) : \phi \in G\} \subseteq K$. Without loss of generality, $\alpha = \alpha_1$, and consider $h(x) = (x-\alpha_1)\cdots(x-\alpha_n) \in K[x]$. Then if $\phi \in G$, $\phi(h(x)) = h(x) \in (\mathrm{Fix}\, G)[x] = F[x]$ since ϕ acts by permutation on the α_i. Thus $h(x)$ splits over K with no repeated roots, and in fact $h(x) = p(x)$ since every root of $h(x)$ is a F−conjugate of α, and thus a root of $p(x)$.

(4 ⇒ 1) Since K/F is finite, $K = F(\alpha_1, \dots, \alpha_n)$, $\alpha_i \in K$. For each i, let $q_i(x) \in F[x]$ be its minimal polynomial. Say $p_1(x), \dots, p_m(x)$ is a list of distinct $q_i(x)$. Then $f(x) = p_1(x) \cdots p_m(x)$, and since K/F is normal, its splitting field over F is K, and by A6, $f(x)$ is separable. ■

Example. Consider $\alpha = \sqrt{2+\sqrt{3}} \in \mathbb{C}$, with minimal polyomial $x^4 - 4x^2 + 1$. Since $\mathbb{Q}$ is perfect, we only need to check normality, and $f(x)$ has roots $\pm\sqrt{2\pm\sqrt{3}}$. The $\mathbb{Q}$−conjugates of α are $\pm\alpha, \pm\beta$ where $\beta = \sqrt{2-\sqrt{3}}$. Since $\alpha\beta = 1$, $\beta = \alpha^{-1}$. Thus $\pm\alpha$, $\pm\beta \in \mathbb{Q}(\alpha)$ and $\mathbb{Q}(\alpha)/\mathbb{Q}$ is normal.

	α	$-\alpha$	β	$-\beta$	S_4
ϕ_1	α	$-\alpha$	β	$-\beta$	ϵ
ϕ_2	$-\alpha$	α	$-\beta$	β	(12)(34)
ϕ_3	β	$-\beta$	α	$-\alpha$	(13)(24)
ϕ_3	$-\beta$	β	$-\alpha$	α	(14)(23)

so $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

thm:artin

10.2 Theorem. (Artin) *Let K be a field, H a finite subgroup of* $\mathrm{Aut}(K)$*. Let $F = \mathrm{Fix}\, H$. Then*

1. *K/F is Galois*
2. $\mathrm{Gal}(K/F) = H$
3. $|H| = [K : F]$

Proof Let $\alpha \in K$ and $\sigma_1, \dots, \sigma_r \in H$ with r maximal such that the $\sigma_i(\alpha)$ are distinct. If $\tau \in G$ is arbitrary, then $(\tau \circ \sigma_i(\alpha))$ differs from $(\sigma_i(\alpha))$ only by a permutation: by maximality of r, $\tau \circ \sigma_i(\alpha) = \sigma_j(\alpha)$ for every i and some j. Injectivity of τ shows that it is indeed a permutation. Thus taking $\tau = \sigma_1^{-1}$ if necessary, we may assume that $\sigma_1(\alpha) = \alpha$ and α is a root of the polynomial

$$f(x) = \prod_{i=1}^{r} (x - \sigma_i(\alpha))$$

and for any $\tau \in G$, $\tau(f) = f$. Thus $f(x) \in (\mathrm{Fix}\, H)[x] = F[x]$. Since the $\sigma_i(\alpha)$ are distinct, f is separable.

Since $\alpha \in K$ was arbitrary and $r \le |H|$, we see that every $\alpha \in K$ is the root of a separable polynomial with degree at most $|H|$ and coefficients in F, and the polynomial splits in K. Thus K/F and since the minimal polynomial of each $\alpha \in F$ splits completely in K, K/F is normal by Theorem 9.6. In particular, by the primitive element theorem (Theorem 9.4), $K = F(\alpha)$ where the degree of α is at most $|H|$, so that $[K : F] \le |H|$.

Note that $H \subseteq \mathrm{Gal}(K/F)$ and $|H| \le |\mathrm{Gal}(K/F)| \le [K : F]$; we have shown that $[K : F] \le |H|$, so we're done. ■

The Fundamental Theorem of Galois Theory

We adopt the following notation for the rest of this section. Suppose K/F: then $\mathcal{E} = \{E : F \subseteq E \subseteq K\}$ is the set of intermediate subfields of K/F, and $\mathcal{H}$ is the set of subgroups of $\mathrm{Gal}(K/F)$. We then define the **Galois correspondence** by

$$\begin{aligned} \mathcal{E} &\longleftrightarrow \mathcal{H} \\ E &\longmapsto \mathrm{Gal}(K/E) \\ \operatorname{Fix} H &\longleftarrow\!\shortmid H \end{aligned}$$

Note that if $E_1 \subseteq E_2$ in $\mathcal{E}$, then $\mathrm{Gal}(K/E_1) \supseteq \mathrm{Gal}(K/E_2)$. Similarly, if $H_1 \subseteq H_2$ in $\mathcal{H}$, then $\operatorname{Fix} H_1 \supseteq \operatorname{Fix} H_2$. Thus the Galois correspondence is inclusion reversing.

thm:ftgt

10.3 Theorem. (Fundamental Theorem of Galois Theory) *Let K/F be a finite Galois extension. The Galois correspondences give an inclusion-reversing bijection (antitone Galois connection) between $\mathcal{E}$ and $\mathcal{H}$:*

1. *If $E \in \mathcal{E}$, then $\operatorname{Fix}(\mathrm{Gal}(K/E)) = E$. In particular, K/E is Galois.*
2. *If $H \in \mathcal{H}$, then $\mathrm{Gal}(K/\operatorname{Fix}(H)) = H$.*

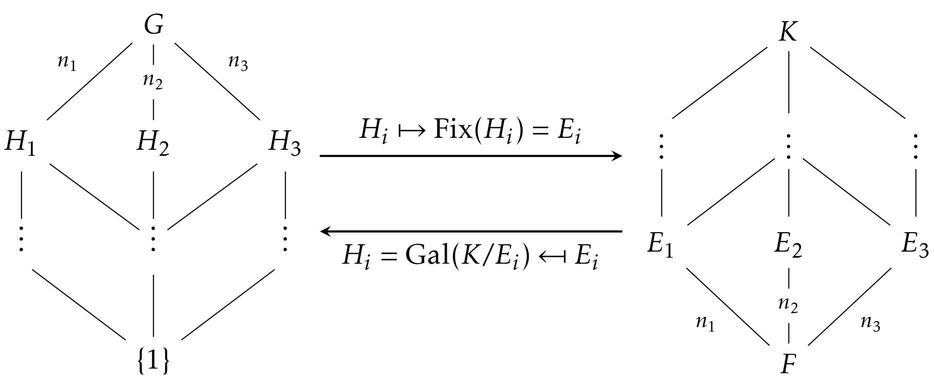
Proof
1. K/F is normal and separable, so K/E is also normal and separable so that K/E is Galois. Thus the result follows by Theorem 10.1.
2. This is a direct application of Theorem 10.2. ■

10.4 Corollary. *Suppose K/F is finite Galois. If $H_1 \subseteq H_2$ in $\mathcal{H}$, then $[H_2 : H_1] = [\operatorname{Fix} H_1 : \operatorname{Fix} H_2]$.*

Proof We have

$$\begin{aligned} [\operatorname{Fix} H_1 : \operatorname{Fix} H_2] &= \frac{[K : \operatorname{Fix} H_2]}{[K : \operatorname{Fix} H_1]} \\ &= \frac{|\mathrm{Gal}(K/\operatorname{Fix} H_2)|}{|\mathrm{Gal}(K/\operatorname{Fix} H_1)|} \\ &= \frac{|H_2|}{|H_1|} = [H_2 : H_1] \end{aligned}$$ ■

To summarize the previous results, perhaps the easiest way to visualize it is with a digram. On the left, we have the subgroup lattice of $G = \mathrm{Gal}(K/F)$, and on the right, we have the intermediate fields of K/F.



Example. Consider $G = \mathrm{Gal}(x^3 - 2)$ and set $\alpha = \sqrt[3]{2}$. Since $\mathbb{Q}$ is perfect and $x^3 - 2$ is irreducible, then $x^3 - 2$ is separable, so $\mathbb{Q}(\alpha, \zeta_3)$ is the splitting field for $x^3 - 2$ over $\mathbb{Q}$. Then $|G| = [\mathbb{Q}(\alpha, \zeta_3) : \mathbb{Q}] = 6$ and since $G \leq S_3$, $G \cong S_3$.

prop:int-conj

10.5 Proposition. *Let E be an intermediate subfield of K/F. For any $\phi \in \mathrm{Gal}(K/F)$, $\phi\,\mathrm{Gal}(K/E)\phi^{-1} = \mathrm{Gal}(K/\phi(E))$.*

Proof For any $\psi \in \mathrm{Aut}(K)$,

$$\begin{aligned} \psi \in \mathrm{Gal}(K/E) &\iff \psi(\alpha) = \alpha \text{ for all } \alpha \in E \\ &\iff \psi \circ \phi^{-1} \circ \phi(\alpha) = \phi^{-1} \circ \phi(\alpha) \text{ for all } \alpha \in E \\ &\iff \psi \circ \phi^{-1}(\beta) = \phi^{-1}(\beta) \text{ for all } \beta \in \phi(E) \\ &\iff \phi \circ \psi \circ \phi^{-1}(\beta) = \beta \text{ for all } \beta \in \phi(E) \\ &\iff \phi \circ \psi \circ \phi^{-1} \in \mathrm{Gal}(K/\phi(E)) \end{aligned}$$ ■

Definition. Let $K/E/F$ and $H \leq \mathrm{Gal}(K/F)$. We say E is **invariant** under H if $\phi(E) = E$ for all $\phi \in H$.

10.6 Proposition. *Suppose K/F is finite and Galois. If E is an intermediate subfield of K/F, then the following are equivalent:*
1. *E/F is Galois*
2. *E is $\mathrm{Gal}(K/F)$–invariant*
3. $\mathrm{Gal}(K/E) \trianglelefteq \mathrm{Gal}(K/F)$

Proof $(2 \Leftrightarrow 3)$ This is straightfoward in light of Proposition 10.5.

$(1 \Rightarrow 2)$ Suppose E/F is Galois and take $\phi \in \mathrm{Gal}(K/F)$. Since E/F is Galois, $\phi|_E \in \mathrm{Gal}(E/F)$; thus, $\phi|_E(E) = \phi(E) = E$.

$(2 \Rightarrow 1)$ Suppose E is G–invariant where $G = \mathrm{Gal}(K/F)$. By A7, E/F is separable. To show normality, we show that E is closed under conjugation. Let $\alpha \in E$ with minimal polynomial $f(x) \in F[x]$. Since K/F is normal, $f(x)$ splits over K. Let $\beta \in K$ be a F–conjugate of α. Since $f(x) \in F[x]$ is irreducible, there exists $\phi \in G$ such that $\phi(\alpha) = \beta$ so that $\beta = \phi(\alpha) \in \phi(E) = E$. ■

10.7 Proposition. *Let $K/E/F$, K/F finite and Galois. If E/F is Galois, then* $\mathrm{Gal}(E/F) \cong \mathrm{Gal}(K/F)/\mathrm{Gal}(K/E)$.

Proof Consider the map $\psi : \mathrm{Gal}(K/F) \to \mathrm{Gal}(E/F)$ given by $\psi(\phi) = \phi|_E$. Then $\ker\psi = \mathrm{Gal}(K/E)$ and the result follows by the first isomorphism theorem. ■

11 Galois Group Computations

Example. (Cyclotomic Galois Group) Let's compute $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. Note that $\mathbb{Q}(\zeta_n)$ is the splitting field for the separable polynomial $\Phi_n(x)$ over $\mathbb{Q}$ so that $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is Galois. To see that $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathbb{Z}_n^\times$, one can realize that the map $\psi : \mathbb{Z}_n^\times \to G$ by $\psi(k) = \{\zeta_n \mapsto \zeta_n^k\}$ is an isomorphism.

Example. (Finite Field Galois Group) We can also compute $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. Since $\mathbb{F}_{p^n}$ is the splitting field of $x^{p^n}-x$ over $\mathbb{F}_p$, $\mathbb{F}_{p^n}/\mathbb{F}_p$ is Galois with index n. Consider the Frobenius map $\phi:\mathbb{F}_{p^n}\to\mathbb{F}_{p^n}$ such that $\phi(a)=a^p$; by Fermat, $\phi\in\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. Let $j=|\phi|$, so $j\leq n$. Furthermore, since ϕ is an automorphism, every element of $\mathbb{F}_{p^n}$ is a root of $x^{p^j}-x$, which is only possible if $j\geq n$. Thus equiality holds and $G=\langle\phi\rangle$.

We now turn towards computing the Galois groups of arbitrary splitting fields of cubic and quadratic polynomials. To do this, we need to introduce some new machinery.

Definition. Let $f(x)\in F[x]$ be non-constant with splitting field K. Say $f(x)=u(x-\alpha_1)\cdots(x-\alpha_n)\in K[x]$. We say

$$\operatorname{disc} f(x)=\prod_{i<j}(\alpha_i-\alpha_j)^2$$

is the **discriminant** of $f(x)$.

Remark. (i) $\operatorname{disc}(f(x))\neq 0$ if and only if $f(x)$ is separable.

(ii) If $f(x)=x^2+bx+c$, then $\operatorname{disc} f(x)=b^2-4c$.

11.1 Lemma. *Suppose $f(x)\in F[x]$ is non-constant. Then* $\operatorname{disc} f(x)\in F$.

Proof If $f(x)$ is not separable, this is obvious, so suppose $f(x)$ is separable. For all $\phi\in\mathrm{Gal}(f(x))$, $\phi(\operatorname{disc} f(x))=\operatorname{disc} f(x)$, so $\operatorname{disc} f(x)\in\mathrm{Fix}(\mathrm{Gal}(f(x)))=F$. ■

prop:disc-an

11.2 Proposition. *Suppose* $\operatorname{char} F\neq 2$, *$f(x)$ separable with degree $n\geq 2$. Set $G=\mathrm{Gal}\, f(x)$ and $d=\prod_{i<j}(\alpha_i-\alpha_j)$.*

If $\phi\in G\subseteq S_n$, then $\phi(d)=\pm d$. Moreover, $\phi(d)=d$ if and only if $\phi\in A_n$. In particular, $\mathrm{Gal}(K/F(d))=G\cap A_n$ and $G\subseteq A_n$ if and only if $d\in\mathrm{Fix}(G)=F$.

Proof Let $\phi\in G$, so $d,\phi(d)$ are roots of $x^2-d^2\in F[x]$; thus, $\phi(d)=\pm d$. Observe that S_n acts on $X=\{d,-d\}$ by

$$\sigma\cdot\prod_{i<j}(\alpha_i-\alpha_j)=\prod_{i<j}(\alpha_{\sigma(i)}-\alpha_{\sigma(j)})$$

Moreover, $\epsilon\cdot d=d$ and $((n)(n-1))\cdot d=-d$, so the action is transitive. By Orbit-Stabilizer, $n!=|S_n|=|\mathrm{Stab}(d)|\cdot|S_n\cdot d|=|\mathrm{Stab}(d)|\cdot 2$, so $\mathrm{Stab}(d)=A_n$ since A_n is the only index 2 subgroup of S_n. ■

For the remainder of this section, we will assume that $\operatorname{char} F\neq 2,3$.

Galois Groups from Cubic Splitting Fields

We first treat the case where $f(x)$ is cubic. If $f(x)\in F[x]$ is irreducible and separable, then $\mathrm{Gal}\, f(x)\cong S_3$ or A_3. Suppose $g(x)=x^3+\alpha x^2+\beta x+\gamma\in F[x]$ irreducible and separable and consider $f(x)=g(x-\alpha/3)=x^3+bx+c\in F[x]$. Note that $f(x)$ is still irreducible and separable; in particular, $\mathrm{Gal}\, f(x)=\mathrm{Gal}\, g(x)$. Such a cubic is called a **depressed cubic**. One can compute $\operatorname{disc} f(x)=-4b^3-27c^2$. Then by applying Proposition 11.2, we see that

$$\mathrm{Gal}\, f(x)=\begin{cases}A_3 & :\operatorname{disc} f(x)=d^2, d\in F\\ S_3 & :\text{otherwise}\end{cases}$$

Galois Groups from Quartic Splitting Fields

Suppose $f(x) = x^4 + \alpha x^3 + \beta x^2 + \gamma x + \delta \in F[x]$; as before, we take $g(x) = f(x - \alpha/4) = x^4 + bx^2 + cx + d$, and $\mathrm{Gal}(f(x)) = \mathrm{Gal}(g(x))$. If $G = \mathrm{Gal}\, f(x)$, then G is a transitive subgroup of S_4 with $4 \mid |G|$. Thus, the possible options are S_4, A_4, D_4, V, C_4, where $V = \{\epsilon, (12)(34), (13)(24), (14)(23)\}$.

Let the roots of $f(x)$ be given by $\alpha_1, \ldots, \alpha_4$. Let $K = F(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and set

$$u = \alpha_1\alpha_2 + \alpha_3\alpha_4$$
$$v = \alpha_1\alpha_3 + \alpha_2\alpha_4$$
$$w = \alpha_1\alpha_4 + \alpha_2\alpha_3$$

We define the **resolvent cubic** of $f(x)$

$$\mathrm{Res}\, f(x) = (x-u)(x-v)(x-w) = x^3 - bx^2 - 4dx + 4bd - c^2 \in F[x]$$

where the coefficients may be evaluated by the reader.

Let $L = F(u, v, w)$, so that $K/L/F$. Since K/F is Galois, K/L is Galois, and $\mathrm{Gal}(\mathrm{Res}\, f(x)) = \mathrm{Gal}(L/F)$. Since $\mathrm{Gal}(K/L) = G \cap V$ and L/F is Galois, $\mathrm{Gal}(K/L) \trianglelefteq \mathrm{Gal}(K/F)$, and $\mathrm{Gal}(L/F) = G/G \cap V$. Let $m = |\mathrm{Gal}(\mathrm{Res}\, f(x))|$.

G	S_4	A_4	D_4	V	C_4
$G \cap V$	V	V	V	V	C_2
$G/(G \cap V)$	S_3	C_3	C_2	$\{1\}$	C_2
m	6	3	2	1	2

Note that G is uniquely determined when $m \in \{1, 3, 6\}$, so let's examine the case $m = 2$. Since $\deg(\mathrm{Res}\, f(x)) = 3$ and $m = 2$, exactly one of u, v, or w is in F. Without loss of generality, assume $u \in F$. Either option for G has a 4-cycle which fixes u, so $\sigma = (1324) \in G$ and $\sigma^2 = (12)(34) \in G$. Consider

$$(x - \alpha_1\alpha_2)(x - \alpha_3\alpha_4) = x^2 - ux + d$$
$$(x - (\alpha_1 + \alpha_2))(x - (\alpha_3 + \alpha_4)) = x^2 + (b - u)$$

Let's see that $G = \langle\sigma\rangle \cong C_4$ if and only if both of these polynomials split over L.

($\Rightarrow$) Suppose $G = \langle\sigma\rangle$. Then $\mathrm{Gal}(K/L) = G \cap V = \langle\sigma^2\rangle$, so $\alpha_1\alpha_2$, $\alpha_3\alpha_4$,$\alpha_1 + \alpha_2$,$\alpha_3 + \alpha_4 \in \mathrm{Fix}\langle\sigma^2\rangle = L$.

($\Leftarrow$) Conversely, suppose $\alpha_1\alpha_2$, $\alpha_3\alpha_4$,$\alpha_1 + \alpha_2$,$\alpha_3 + \alpha_4 \in L$. Then $\alpha_1\alpha_2 \in L(\alpha_1)$ that $\alpha_1, \alpha_2 \in L(\alpha_1)$. Then since $v - w = (\alpha_1 - \alpha_2)(\alpha_3 - \alpha_4) \in L$, so $\alpha_3 - \alpha_4 \in L(\alpha_1)$ as well, so that $\alpha_3, \alpha_4 \in L(\alpha_1)$.

Now, $K = F(\alpha_1, \ldots, \alpha_4) = L(\alpha_1)$, and $[K : L] = [L(\alpha_1) : L] = |\mathrm{Gal}(K/L)|$. The polynomial $p(x) = x^2 - (\alpha_1 + \alpha_2)x + \alpha_1\alpha_2 \in L[x]$ has $p(\alpha_1) = 0$ so that $[K : L] \le 2$. Thus $[K : F] \le 4$, which forces $G = C_4$. TODO: why is $[L : F] \le 2$?

Example. Consider $f(x) = x^4 - 2x - 2$. Then $\mathrm{Res}\, f(x) = x^3 + 8x - 4$ has no rational roots, and is irreducible. Now, $\mathrm{disc}(\mathrm{Res}\, f(x)) = -4 \cdot (8^3) - 27 \cdot 4^2 < 0$ is not a square in $\mathbb{Q}$, so $\mathrm{Gal}(\mathrm{Res}\, f(x)) = S_3$. Thus $\mathrm{Gal}\, f(x) \cong S_4$.

Example. Consider $g(x) = x^4 + 5x + 5$, irreducible by Eisenstein, so $\mathrm{Res}\, g(x) = x^3 - 20x - 25 = (x-5)(x^2 + 5x + 5)$. Thus $\mathrm{Gal}\,\mathrm{Res}\, g(x) = \mathbb{Z}_2$, and $m = 2$. We let $u = 5 \in \mathbb{Q}$. Consider $x^2 - 5x - 5$ and $x^2 - 5$. The roots of $x^2 + 5x + 5$ are $\frac{-5 \pm \sqrt{5}}{2}$, so $L = \mathbb{Q}(\sqrt{5})$. The roots of $x^2 - 5$ are also in L. Thus $\mathrm{Gal}\, f(x) = \mathbb{Z}_4$.

12 Solvability and Radical Extensions

Throughout this section, we assume that $\operatorname{char} F = 0$.

Definition. A group G is **solvable** if there exists a chain of subgroups $G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$ such that G_i/G_{i+1} is abelian.

Example. Any abelian solvable is abelian. We have $S_4 \supseteq A_4 \supseteq V \supseteq \{1\}$, so S_4 is solvable. If G is simple, then G is solvable if and only if G is abelian. For example, A_5 is simple and non-abelian, and thus not solvable.

12.1 Proposition. *If G is solvable and $N \leq G$, then N is solvable; if $N \trianglelefteq G$, then G/N is solvable.*

Proof Since G is solvable, get $G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$. Then

- Consider the sequence $N = G_0 \cap N \trianglerighteq G_1 \cap N \trianglerighteq \cdots \trianglerighteq G_n \cap N = \{1\}$, since normality is preserved under intersection. Furthermore,

$$N \cap G_i \big/ N \cap G_{i+1} \cong (N \cap G_i)G_{i+1} \big/ G_{i+1} \subseteq G_i \big/ G_{i+1}$$

  is abelian.
- Consider the sequence $G/N = G_0/N \trianglerighteq G_1/N \trianglerighteq \cdots \trianglerighteq G_n/N = \{1\}$ and use the third isomorphism theorem. TODO: finish this, something is weird: N is not a normal subgroup of G_i, use correspondence theorem for normal subgroups. ■

12.2 Proposition. *Let $N \trianglelefteq G$; then N is solvable if and only if N and G/N are solvable.*

Proof The forward direction is done; conversely, suppose N and G/N are solvable. Let

$$N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_m = \{1\}$$
$$G/N = G_0/N \supseteq G_1/N \supseteq \cdots \supseteq G_l/N = \{N\}$$

By the third isomorphism theorem, $G_i/N \big/ G_{i+1}/N \cong G_i/G_{i+1}$, so $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq N$. TODO: fix this. ■

Remark. Let G be finite, solvable. By refining the chain as much as possible, we may assume $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$ with G_i/G_{i+1}, and no $H_i \leq G$ with $G_i \supsetneq H_i \supseteq G_{i+1}$ normal. That is to say, G_i/G_{i+1} is abelian and simple, so $\left|G_i/G_{i+1}\right|$ prime.

Definition. We say K/F is a **simple radical extension** if $K = F(\alpha)$ for some $\alpha \in K$ such that $\alpha^n \in F$ for some $n \in \mathbb{N}$. A **radical tower** over F is a tower $K_m/K_{m-1}/\cdots/K_1/F$ such that K_1/F and K_{i+1}/K_i are each simple radical extensions. We say K/F is **radical** if there exists a radical tower over F starting at K. We say $f(x) \in F[x]$ is **solvable by radicals** over F if its splitting field is contained in a radical extension of F.

Example. Consider $f(x) = x^4 - 4x^2 + 2$. Then $\mathbb{Q}(\sqrt{2+\sqrt{2}}) \supseteq \mathbb{Q}(\sqrt{2}) \supseteq \mathbb{Q}$ is solvable by radicals over $\mathbb{Q}$.

Definition. We say an extension K/F is **cyclic** if K/F is finite and Galois, and $\operatorname{Gal}(K/F)$ is cyclic.

prop:prim-cy

12.3 Proposition. *If F contains a primitive n^{th} root of unity and $K = F(\alpha)$ with $\alpha^n \in F$, then K/F is cyclic.*

Proof Consider $f(x) = x^n - \alpha^n \in F[x]$. Let $\zeta \in F$ be a primitive n root of unity. The roots of $f(x)$ in K are $\alpha\zeta^i$ for $i \in \{0,1,\dots,n-1\}$. Thus K is the splitting field for $f(x)$ over F, so K/F is Galois. For each $\phi \in \mathrm{Gal}(K/F)$, there exists a unique $0 \le i \le n-1$ such that $\phi(\alpha) = \alpha\zeta^i$. Write $i = \Gamma(\phi)$, and it is straightforward to verify that $\Gamma : \mathrm{Gal}(K/F) \to \mathbb{Z}_n$ is an injective homomorphis. Thus $\mathrm{Gal}(K/F)$ is isomorphic to a cyclic subgroup of Z_n, and thus cyclic. ■

TODO: finish all the proofs in this section.

Definition. We say $\{\sigma_1,\dots,\sigma_n\} \subseteq \mathrm{Aut}\, K$ is **linearly dependent** over K if there exists $a_i \in L$, not all zero, such that $a_1\sigma_1(\alpha)+\cdots+a_n\sigma_n(\alpha) = 0$ for all $\alpha \in K$. Otherwise, we say $\{\sigma_1,\dots,\sigma_n\}$ is **linearly independent**.

12.4 Lemma. *Let $[K : F] < \infty$. Then any finite subset of* $\mathrm{Gal}(K/F)$ *is linearly independent over K.*

Proof Suppose not; it suffices to prove the result for $\mathrm{Gal}(K/F)$. Let $\{\sigma_1,\dots,\sigma_r\}$ be a minimal linearly dependent subset of $\mathrm{Gal}(K/F)$ and let

$$a_1\sigma_1 + \cdots + a_r\sigma_r = 0$$

be a non-trivial dependence relation; note that each $a_i \in K^\times$ by minimality. Certainly, $r > 1$.

Let $\beta \in K$ be such that $\sigma_1(\beta) \neq \sigma_2(\beta)$. We then have for any $\alpha \in K$ that

$$a_1\sigma_1(\alpha)\sigma_1(\beta) + a_2\sigma_2(\alpha)\sigma_2(\beta) + \cdots + a_r\sigma_r(\alpha)\sigma_r(\beta) = 0 \qquad (12.1)$$ {eq:1}

$$a_1\sigma_1(\alpha)\sigma_1(\beta) + a_2\sigma_2(\alpha)\sigma_1(\beta) + \cdots + a_r\sigma_r(\alpha)\sigma_1(\beta) = 0 \qquad (12.2)$$ {eq:2}

where Eq. (12.1) follows since $\sigma_i(\alpha\beta) = \sigma_i(\alpha)\sigma_i(\beta)$. Subtracting Eq. (12.1) and Eq. (12.2), we get

$$a_2\sigma_2(\alpha)[\sigma_2(\beta) - \sigma_1(\beta)] + \cdots + a_r\sigma_r(\alpha)[\sigma_r(\beta) - \sigma_1(\beta)] = 0$$

which is a dependence relation on $\{\sigma_2,\dots,\sigma_r\}$, contradicting minimality. ■

We now provide a converse to Proposition 12.3. TODO: maybe merge the theorems?

12.5 Proposition. *Let F be a field which contains a primitive n^{th} root of unity. If K/F is cyclic with $[K : F] = n$, then K/F is simple radical.*

Proof Suppose $\zeta \in F$ is a primitive n^{th} root of unity and K/F is cyclic of degree n. Let $G = \mathrm{Gal}(K/F) = \langle\sigma\rangle$, $|G| = n$ for some $\sigma \in G$. For $\alpha \in K$, define

$$g(\alpha) := \alpha + \zeta\sigma(\alpha) + \zeta^2\sigma^2(\alpha) + \cdots + \zeta^{n-1}\sigma^{n-1}(\alpha)$$

Note that $\zeta\sigma(g(\alpha)) = g(\alpha)$ so that $\sigma(g(\alpha)) = \zeta^{-1}g(\alpha)$. In particular,

$$\sigma(g(\alpha)^n) = \sigma(g(\alpha))^n = \left(\zeta^{-1}g(\alpha)\right)^n = g(\alpha)^n$$

Thus for all $\alpha \in K$, since $G = \langle\sigma\rangle$, $g(\alpha)^n \in \mathrm{Fix}\, G = F$. Moreover, since G is linearly independent over K, there exists $\alpha \in K$ such that $g(\alpha) \neq 0$. Furthermore, $\sigma^i(g(\alpha)) = \zeta^{-i}g(\alpha) \neq g(\alpha)$ for any $1 \le i \le n-1$; thus $g(\alpha) \notin \mathrm{Fix}\, H$ for any $\{1\} \neq H \le G$. Thus by the fundamental theorem of galois theory (Theorem 10.3), $g(\alpha) \notin E$ for any $F \subseteq E \subsetneq K$, so $F(g(\alpha)) = K$. ■

12.6 Proposition. *Let $K/E/F$, E/F Galois, K/E radical. Then there exists L/K such that L/F is Galois and L/E is radical such that* $\mathrm{Gal}(L/E)$ *is solvable.*

Proof We prove the result when K/E is simple radical; the more general case follows by induction. Suppose $K = E(\alpha)$ where $\alpha^n = \beta \in E$. Also suppose $G = \mathrm{Gal}(E/F) = \{\sigma_1, \dots, \sigma_r\}$. Consider

$$f(x) = \Phi_n \prod_{i=1}^{r} (x^n - \sigma_i(\beta)) \in (\mathrm{Fix}\, G)[x] = F[x]$$

and let L be the splitting field for $f(x)$ over K; let's show that L has the desired properties.

- L/F is Galois. First note that L is the splitting field for $f(x)$ over E. Since E/F is Galois, E is the splitting field of some separable polynomial $h(x) \in F[x]$. Then L is the splitting field for $h(x)f(x)$, and since $\mathrm{char}\, F = 0$ so that F is perfect, L/F is Galois.
- L/E is radical. Let ζ be a root of $\Phi_n(x)$ in L. We extend each $\sigma_i \in G$ to a $\sigma_i^* \in \mathrm{Gal}(L/F)$. Thus, the roots of $f(x)$ are of the form $\zeta^i \sigma_i^*(\alpha)$, so $L = E(\zeta, \sigma_1^*(\alpha), \dots, \sigma_r^*(\alpha))$.

  Let $E_0 = E(\zeta)$ and for $1 \le i \le r$, $E_i = E(\zeta, \sigma_1^*(\alpha), \dots, \sigma_i^*(\alpha))$ so $E_r = L$. Note that $\zeta^n = 1 \in E$ and $\sigma_i^*(\alpha)^n = \sigma_i^*(\alpha^n) = \sigma_i^*(\beta) = \sigma_i(\beta) \in E$. Thus,

  $$E \subseteq E_0 \subseteq E_1 \subseteq \cdots \subseteq E_r = L$$

  is a radical tower, so that L/E is radical.
- $\mathrm{Gal}(L/E)$ is solvable. Let $G_i = \mathrm{Gal}(L/E_i)$, so by the fundamental theorem of galois theory,

  $$\{1\} = G_r \le G_{r-1} \le \cdots \le G_2 \le G_1 \le G_0 \le G'$$

  where $G_0 = \mathrm{Gal}(L/E(\zeta))$. Moreover, $G_0 \le G' := \mathrm{Gal}(L/E)$. First, $G_0 = \mathrm{Gal}(L/E(\zeta)) \trianglelefteq \mathrm{Gal}(L/E)$ since $E(\zeta)/E$ is Galois (splitting field of $\Phi_n(x)$ over E). Furthermore, $G'/G_0 \cong \mathrm{Gal}(E(\zeta)/E)$ is abelian in the same way that $\mathbb{Q}(\zeta)/\mathbb{Q}$ is abelian.

  Now, $\mathrm{Gal}(L/E_{i+1}) \trianglelefteq \mathrm{Gal}(L/E_i)$ since E_{i+1}/E_i is Galois (E_{i+1}/E_i is simple radical with $\zeta \in E_i$ and $\sigma_{i+1}^*(\alpha)^n \in E_i$. By the proposition, E_{i+1}/E_i is cyclc. Also, $G_i/G_{i+1} \cong \mathrm{Gal}(E_{i+1}/E_i)$ is cyclic (correspondence between simple radical and cyclic). ■

12.7 Corollary. *Take $E = F$. If K/F is radical, then there exists L/K such that L/F is radical and Galois with* $\mathrm{Gal}(L/F)$ *is solvable.*

12.8 Theorem. (Galois) *Let $f(x) \in F[x]$. Then $f(x)$ is solvable over F if and only if* $\mathrm{Gal}\, f(x)$ *is solvable.*

Proof ($\Rightarrow$) Reading
($\Leftarrow$) Suppose $f(x)$ is solvable by radicals over F. Say $f(x) = p_1(x)^{i_1} \cdots p_l(x)^{i_l}$ where the p_i are distinct and irreducible. By replacing $f(x)$ with $p_1(x) \cdots p_l(x)$, we may assume $f(x)$ is separable. Let E be the splitting field of $f(x)$ over F. Then E/F is Galois. Moreover, $E \subseteq K$, K/F is radical. Then by the proposition, there exists L/K such that L/F is Galois and radical. Since E/F is Galois, $\mathrm{Gal}(L/E) \trianglelefteq \mathrm{Gal}(L/F$. Thsn $\mathrm{Gal}(E/F) \cong \mathrm{Gal}(L/F) / \mathrm{Gal}(L/E)$. ■

Example. If $1 \leq \deg(x) < 5$, then $f(x)$ is solvable by raicals. Let $g(x)$ be the product of distinct factors of $f(x)$. Then $\mathrm{Gal}(g(x)) \leq S_4$ since $g(x)$ is separable, and S_4 is solvable.

Remark. Note that $S_n = \langle (12), (123\cdots n)\rangle$. If p is prime, then $S_p = \langle \tau, \sigma \rangle$ where τ is any transposition and σ is any $p-$cycle.

> ***12.9 Lemma.*** *Let $f(x) \in \mathbb{Q}[x]$ be irreducible with prime degree p. If $f(x)$ has exactly 2 non-real roots, then* $\mathrm{Gal}\, f(x) = S_p$.

Proof Let α be a root of $f(x)$, then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg f(x) = p$. Thus $p \bigm| [K : \mathbb{Q}]$ where k is the splitting field of $f(x)$ over $\mathbb{Q}$. Thus there exists $\sigma \in \mathrm{Gal}\, f(x)$, $|\sigma| = p$. Without loss of generality, $\sigma = (123\cdots p)$. Moreover, $\phi : \mathbb{C} \to \mathbb{C}$ by $\phi(z) = \overline{z}$ is a $\mathbb{Q}-$map. By the normality theorem, $\phi|_K \in \mathrm{Gal}\, f(x)$. Since $f(x)$ has only 2 non-real roots, $\phi|_K = (ij)$. Thus $\mathrm{Gal}\, f(x) = S_p$. ■

Example. Consider $f(x) = x^5 + 2x^3 - 24x - 2$, irreducible by Eisenstein. By IVT, $f(x)$ has at least 3 real roots. Computing the sum of squares of roots as $\sum \alpha_i^2 = (\sum \alpha_i)^2 - 2\sum_{i<j} \alpha_i \alpha_j = -4$, one sees that not all rots of $f(x)$ are real. Since non-real roots of $f(x)$ appear in conjugate pairs, $f(x)$ has exactly 2 non-real roots. By the lemma, $\mathrm{Gal}\, f(x) = S_5$, S_5 is not solvable, so $f(x)$ is not solvable by radicals.

Exam questions!

1. Minimal polynomials / field extensions
2. show K/F Galois, compute $\mathrm{Gal}(K/F)$
3. Answer questions about $\mathrm{Gal}(f(x))$ (probably quartic)
4. questions similar to assignment questions, times 3
5. 2 proofs from lecture, from the second half (post midterm)
6. new proof, and an assignment proof
7. solvability by radicals
8. give example / DNE (10 parts)