REPLACE

Alex Rutar[*]
University of Waterloo

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[*] *arutar@uwaterloo.ca*

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Contents

I. REPLACE

1. For $a, b, k \in \mathbb{N}$,
$$\binom{a+b}{k} = \sum_{j=1}^{k} \binom{a}{j} \cdot \binom{b}{k-j} \tag{0.1}$$
We prove this with a bijection:
$$\mathcal{B}(a+b,k) \leftrightarrows \bigcup_{j=0}^{k} \mathcal{B}(a,j) \times \mathcal{B}(b,k-j)$$
given by $S \mapsto (S \cap \{1,\ldots,a\}, (S \cap \{a+1,\ldots,a+b\})^{(-a)}$ and $(P,Q) \mapsto P \cup Q^{(a)}$, where $\mathcal{B}(n,i)$ is the set of i–element subsets of $\{1,2,\ldots,n\}$ and for $C \subseteq \mathbb{Z}$ and $q \in \mathbb{Z}$, $C^{(q)} = \{c+q : c \in C\}$. Note that the equation in fact gives the polynomial identity
$$\binom{x+y}{k} = \sum_{j=0}^{k} \binom{x}{j}\binom{y}{k-j}$$
in $\mathbb{Q}[x,y]$. We denote the falling factorial $(x)_i = x(x-1)(x-2)\cdots(x-i+1)$, which has degree i for each $i \in \mathbb{N}$. In particular, $(x)_i = i!\binom{x}{i}$, so multiplying our identity by $k!$, we get
$$(x+y)_k = \sum_{j=0}^{k} \binom{k}{j} (x)_j (y)_{k-j}$$
Compare this with the standard binomial theorem
$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}$$
These are called sequences of binomial type.

2. Here's another identity. For $n \geq 0$ and $s, t \geq 1$,
$$\binom{n+s+t-1}{s+t-1} = \sum_{k=0}^{n} \binom{k+s-1}{s-1}\binom{n-k+t-1}{t-1}$$
Let $\mathcal{M}(m,r)$ denote a multiset of size m with elements of r types, so that $|\mathcal{M}(m,r)| = \binom{m+r-1}{r-1}$. Let's define a bijection
$$\mathcal{M}(n,s+t) \rightleftharpoons \bigcup_{k=1}^{n} \mathcal{M}(k,s) \times \mathcal{M}(n-k,t) \tag{0.2}$$

$\mu = (m_1, \ldots, m_{s+t}) \mapsto ((m_1, \ldots, m_s), (m_{s+1}, \ldots, m_{s+t}))$ and $(\nu, \theta) \mapsto \nu\theta$. Note that if f, g are polynomials of degree d and e respectively, then $\sum_{k=0}^{n} f(k)g(n-k)$ is a polynomial in n of degree $d+e-1$.

Is there some way to understand (0.2)? It is unclear, with our known techniques, that this corresponds to a polynomial identity since there is a variable n in the exponent. However, we can use generating functions. Define

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{n+s+t-1}{s+t-1} z^n = \sum_{n=0}^{\infty} |\mathcal{M}(n, s+t)| z^n &= \sum_{(m_1, \ldots, m_{s+t})} z^{m_1 + \cdots + m_{s+t}} \\
&= \left(\sum_{m=0}^{\infty} z^m \right)^{s+t} \\
&= \frac{1}{(1-z)^{s+t}} = \frac{1}{(1-z)^s} \frac{1}{(1-z)^t} \\
&= \sum_{k=0}^{\infty} \binom{k+s-1}{s-1} z^k \sum_{\ell=0}^{\infty} \binom{\ell+t-1}{t-1} z^\ell \\
&= \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} \binom{k+s-1}{s-1} \binom{n-k+t-1}{t-1} \right)
\end{aligned}$$

Similarly, (0.1) is equivalent to saying $(1+z)^{a+b} = (1+z)^a (1+z)^b$. Note that $(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k = \sum_{k=0}^{\infty} \binom{n}{k} z^k$ for $n \in \mathbb{N}$.

Can we substitute $\frac{1}{(1-q)^t} = (1+z)^n$ where $z = -q$ and $n = -t$?