

PMATH 465

Alex Rutar*
University of Waterloo

Fall 2019[†]

*arutar@uwaterloo.ca

[†]Last updated: September 23, 2019

Contents

Chapter I	Fundamentals of Manifolds	
1	Introduction to Topology	1

I. Fundamentals of Manifolds

1 INTRODUCTION TO TOPOLOGY

BASIC CONSTRUCTIONS

Definition. A **topology** on a set X is a set τ of subsets of X such that

- (i) $\emptyset \in \tau$ and $X \in \tau$
- (ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
- (iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \leq i \leq n$, then $\bigcap_{i=1}^n U_i \in \tau$.

The sets $U \in \tau$ are called the **open sets** in X , and sets of the form $X \setminus U$ for some open set U are called the **closed sets** in X .

Definition. When X is a topological space and $A \subseteq X$, the **interior** of A (denoted A°) is the union of all open sets contained in A . Similarly, we define the **closure** of A (denoted \bar{A}) as the intersection of all closed sets containing A . Then the **boundary** of A , denoted by ∂A , is the set $\partial A = \bar{A} \setminus A^\circ$.

Example. Let X be any set. The **discrete topology** on X is the topology $\tau = \mathcal{P}(X)$, and the **trivial topology** on X is the topology $\tau = \{\emptyset, X\}$.

Definition. A **basis** for a topology on a set X is a set \mathcal{B} of subsets of X

- (i) $\bigcup_{B \in \mathcal{B}} B = X$
- (ii) for all $a \in X$ and $U, V \in \mathcal{B}$ such that $a \in U \cap V$, then there exists $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$.

When \mathcal{B} is a basis for a topology on X , the topology on X **generated** by \mathcal{B} is the set τ of subsets of X such that for $W \subseteq X$, $W \in \tau$ if and only if for all $a \in W$, there exists $U \in \mathcal{B}$ such that $a \in U \subseteq W$.

Note that τ , as above, is a topology on X since

- (i) $\emptyset \in \tau$ vacuously and $X \in \tau$ obviously.
- (ii) If $A_k \in \tau$ for all $k \in K$ (where K is any set of indices), then given $a \in \bigcup_{k \in K} A_k$, we can choose $\ell \in K$ so that $a \in A_\ell$. Then since $A_\ell \in \tau$, we can choose $U_\ell \in \mathcal{B}$ so that $a \in U_\ell \subseteq A_\ell$. Thus $a \in U_\ell \subseteq A_\ell \subseteq \bigcup_{k \in K} A_k$.
- (iii) By induction, it suffices to prove that if $A, B \in \tau$, then $A \cap B \in \tau$. Suppose $A, B \in \tau$, and let $a \in A \cap B$. Since $A \in \tau$, we can choose $U \in \mathcal{B}$ so that $a \in U \subseteq A$. Since $B \in \tau$, we can choose $V \in \mathcal{B}$ so that $a \in V \subseteq B$. Then we have $a \in U \cap V$. Since \mathcal{B} is a basis, we can choose $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$, so $a \in W \subseteq U \cap V \subseteq A \cap B$.

Note that when τ is the topology on X generated by the basis \mathcal{B} , for $A \subseteq X$, $A \in \tau$ if and only if there exists some $S \subseteq \mathcal{B}$ such that $A = \bigcup_{s \in S} s$. In this sense, the topology τ on X generated by the basis \mathcal{B} is the coarsest topology which contains \mathcal{B} .

Definition. (Subspace Topology) When Y is a topological space and $X \subseteq Y$ is a subset of Y , we define the **subspace topology** on X to be the topology for which a set $U \subseteq X$ is open if and only if $U = X \cap V$ for some open set V .

If \mathcal{C} is a basis for the topology on Y , then $\mathcal{B} = \{X \cap V \mid V \in \mathcal{C}\}$ is a basis for the subspace topology on X .

Definition. (Disjoint Union Topology) If X and Y are topological spaces with $X \cap Y = \emptyset$, then the **disjoint union topology** on $X \cup Y$ is the topology in which a subset $U \subseteq X \cup Y$ is open in $X \cup Y$ if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y .

Definition. (Product Topology) If X and Y are topological spaces, the **product topology** on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} = \{ U \times V \mid U \in \mathcal{C}, V \in \mathcal{D} \}$$

where \mathcal{C} and \mathcal{D} are bases for the topologies on X, Y respectively.

Definition. (Infinite Product Topology) We define the infinite product to be

$$\prod_{k \in K} \left\{ f : K \rightarrow \bigcup_{k \in K} X_k \mid f(k) \in X_k \text{ for all } k \in K \right\}$$

There are two standard topologies on X . The first is the **box topology**,

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \mid U_k \text{ is open in } X_k \right\}$$

and the **product topology**

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \mid \begin{array}{l} U_k \text{ is open in } X_k \\ U_k = X_k \text{ for all but finitely many indices } k \end{array} \right\}$$

Example. (Metric Topology) \mathbb{R}^n has a standard **inner product**, and for $u, v \in \mathbb{R}^n$, $uv = u \cdot v = V^T u = \sum_{i=1}^n u_i v_i$. This gives the standard norm on \mathbb{R}^n for $u \in \mathbb{R}^n$, $\|u\| = \sqrt{uv}$. This gives the standard metric on \mathbb{R}^n : for $a, b \in \mathbb{R}^n$, $d(a, b) = \|b - a\|$.

Given a metric on a set Y , we obtain (by restriction) an induced metric on any subset $X \subseteq Y$. Given a metric space X , we define the **metric topology** on X to be the topology which is generated by the set of open balls

$$B(a, r) = \{ x \in X \mid d(a, x) < r \}$$

where $x \in X, r > 0$.

MAPS ON TOPOLOGICAL SPACES

Definition. When X and Y are topological spaces and $f : X \rightarrow Y$, we say that f is **continuous** when it has the property that $f^{-1}(V)$ is open in X for every open set V in Y . We say that $f : X \rightarrow Y$ is a **homeomorphism** when f is bijective and both f and f^{-1} are continuous. Then X, Y are **homeomorphic** if there exists a homeomorphism $f : X \rightarrow Y$.

1.1 Theorem. (Glueing Lemma) Let X and Y be topological spaces, and let $f : X \rightarrow Y$ be a function. Suppose either

(i) $X = \bigcup_{k \in K} A_k$ where each A_k is open in X , or

(ii) $X = \bigcup_{k=1}^n A_k$ where each A_k is closed in X

and each restriction map $f_k : A_k \rightarrow Y$ is continuous, then f is continuous.

PROOF Exercise. ■

Definition. A topological space X is **compact** when it has the property that for every set \mathcal{S} of open subsets of X with $X = \bigcup_{U \in \mathcal{S}} U$, there exists a finite subset $\mathcal{F} \subseteq \mathcal{S}$ such that $X = \bigcup_{F \in \mathcal{F}} F$.

Note that when $X \subseteq Y$ is a subspace, X is compact if and only if X has the property that for every set \mathcal{T} with $X \subseteq \bigcup_{T \in \mathcal{T}} T$, there exists a finite subset $\mathcal{G} \subseteq \mathcal{T}$ such that $X \subseteq \bigcup_{G \in \mathcal{G}} G$.

Definition. A topological space X is **connected** when there do not exist non-empty disjoint open sets $U, V \subseteq X$ such that $X = U \cup V$.

Note that if Y is a metric space and $X \subseteq Y$ is a subspace, then X is connected if and only if there do not exist open sets $U, V \subseteq Y$ such that

$$X \cap U \neq \emptyset, X \cap V \neq \emptyset, U \cap V = \emptyset, \text{ and } X \subseteq U \cup V$$

Definition. A topological space X is called **path connected** when it has the property that for all $a, b \in X$, there exists a continuous map $\alpha : [0, 1] \rightarrow X$ with $\alpha(0) = a$ and $\alpha(1) = b$.

It is easy to see that if X is path connected, then X is connected.

Definition. Let X be a topological space. If we define a relation \sim on X by taking $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a \in A$ and $b \in A$.

It is clear that this is an equivalence relation. Note that when X is a topological space, its connected components are connected, and each connected subspace of X is contained in one of its connected components.

Definition. Let X be a topological space. Define a relation \approx on X by $a \approx b$ if and only if there exists a continuous map $\alpha : [0, 1] \rightarrow X$ with $\alpha(0) = a$ and $\alpha(1) = b$. Such a map α is called a **continuous path**.

One can show that if X is **locally path connected** (which means that X has a basis for its topology which consists of path connected sets), then the path components of X are equal to the connected components of X , and that these components are open.

QUOTIENT TOPOLOGY

Definition. (Quotient Topology) Let X be a topological space and let \sim be an equivalence relation on X . The set of equivalence classes is denoted X/\sim , and X/\sim is called the **quotient** of X by \sim . The map $\pi : X \rightarrow X/\sim$ given by $\pi(a) = [a]$ is called the **natural projection map** or **quotient map**. We define the **quotient topology** on X/\sim by stipulating that for $W \subseteq X/\sim$, W is open in X/\sim if and only if $\pi^{-1}(W)$ is open in X .

When a group G acts on a topological space X , we define an equivalence relation \sim on X by $a \sim b$ if and only if $b = g \cdot a$ for some $g \in G$. The equivalence classes are orbits. In this context, we also write X/\sim as X/G .

When X, Y are any topological spaces and $\pi : X \rightarrow Y$ is surjective, we can define an equivalence relation \sim on X by $a \sim b$ if and only if $\pi(a) = \pi(b)$. We then have a natural bijection from Y to X/\sim in which $y \in Y$ corresponds to the fibre $\pi^{-1}(y) \in X/\sim$.

If Y has the topology such that for $W \subseteq Y$, W is open in Y if and only if $\pi^{-1}(W)$ is open in X . In this case, we also use the terminology “quotient map” for π .

Remark. Let X be a topological space and let \sim be an equivalence relation on X . Let Y be any set. If $f : X \rightarrow Y$ is constant on the equivalence classes, then f induces a well-defined map $\bar{f} : X/\sim \rightarrow Y$ given by define $\bar{f}([a]) = f(a)$.

Example. Define an equivalence class on $[0, 1] \subseteq \mathbb{R}$ by $s \sim t$ if and only if $s = t$ or $\{s, t\} = \{0, 1\}$. Then $[0, 1]/\sim \cong SS^1$. Define $f : [0, 1] \rightarrow S^1$ by $f(t) = e^{i2\pi t}$. Note that $f(0) = f(1)$, so f induces a continuous map $\bar{f} : [0, 1]/\sim \rightarrow SS^1$. The inverse map can be constructed as follows. We define $g : SS^1 \rightarrow [0, 1]/\sim$ by

$$g(x, y) = \begin{cases} \left[\frac{1}{2\pi} \cos^{-1} x \right] & : y \geq 0 \\ \left[1 - \frac{1}{2\pi} \cos^{-1} x \right] & : y \leq 0 \end{cases}$$

Then g is continuous by the Glueing lemma.

In particular, the same proof shows that \mathbb{R}/\mathbb{Z} is homeomorphic to SS^1 .

Example. The projective space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ can be defined in several ways. \mathbb{P}^n is the set of all 1-dimensional vector subspaces of \mathbb{R}^{n+1} , or $\mathbb{P}^n = \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times$, or $\mathbb{P}^n = SS^n / \pm 1$ where $SS^n = \{u \in \mathbb{R}^{n+1} : |u| = 1\}$.

Let us show that $\mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times$ is homeomorphic to $SS^n / \pm 1$. Define $f : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow SS^n$ by $f(x) = x/|x|$, and $g = \pi \circ f$. Then g is given by $g(x) = \{\pm x/|x|\}$. Note that for $t \in \mathbb{R}^\times$,

$$g(tx) = \left[\frac{t}{|t|} \cdot \frac{x}{|x|} \right] = \left[\frac{x}{|x|} \right]$$

since $t/|t| = \pm 1$. Thus g induces a continuous map \bar{g} on the quotient. We construct the inverse map in a similar way.

Definition. Let X be a topological space. Then

- X is **T1** when for all $a, b \in X$ there exists an open set U in X with $a \in U$ and $b \notin U$
- X is **T2** or **Hausdorff** when for all $a, b \in X$, there exist disjoint open sets $U, V \subseteq X$ with $a \in U$ and $b \in V$
- X is **T3** or **regular** when X is T1 and for every $a \in X$ and every closed set $B \subseteq X$ with $a \notin B$, there exist open sets $U, V \subseteq X$ with $a \in U$, $B \subseteq V$.
- X is **T4** or **normal** when X is T1 and for all disjoint closed sets $A, B \subseteq X$ there exist disjoint open sets $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$.

Definition. Let X be a topological space.

- X is **first countable** when for every $a \in X$, there exists a countable set B_a of open sets in X which contain a such that for every open set W in X with $a \in W$, there exists $U \in B_a$ with $a \in U \subseteq W$.
- X is **second countable** when there exists a countable basis for the topology on X .

Example. (i) X is T1 if and only if every 1-point subset of X is closed in X

(ii) Every compact Hausdorff space is regular.

(iii) Every second countable regular space is normal.

(iv) Every metric space is normal.

(v) If X is second countable, then every open cover admits a countable subcover.

(vi) Every second countable space X contains a countable dense subset.

1.2 Lemma. (Urysohn) If X is normal and $A, B \subseteq X$ are disjoint and closed, then there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

1.3 Theorem. (Tietze Extension) If X is normal and $f : A \rightarrow \mathbb{R}$ is continuous for some $A \subseteq X$ closed, then there exists a continuous map $F : X \rightarrow \mathbb{R}$ such that $F|_A = f$ and $\sup_{a \in A} |f(a)| = \sup_{x \in X} |F(x)|$.

1.4 Theorem. (Urysohn's Metrization) If X is second countable and regular, then X is metrizable.

Definition. An n -dimensional topological manifold is a Hausdorff, second countable topological space M which is **locally homeomorphic** to \mathbb{R}^n , meaning for every $p \in M$, there exists an open set $U \subseteq M$ with $p \in U$ and an open set $V \subseteq \mathbb{R}^n$ and a homeomorphism $\phi : U \subseteq M \rightarrow V \subseteq \mathbb{R}^n$. Such a homeomorphism ϕ is called a **(local) coordinate chart** or **chart** on M at p . The domain U of a chart $\phi : U \subseteq M \rightarrow \phi(U) \subseteq \mathbb{R}^n$ is called a (local) **coordinate neighbourhood** at p . Note that we can choose a set of charts

$$\mathcal{A} = \{\phi_k : U_k \subseteq M \rightarrow \phi_k(U_k) : k \in K\}$$

where K is any non-empty set such that $M = \bigcup_{k \in K} U_k$. Such a set of charts is called an **atlas** for M .

Definition. Two charts are called $\phi : U \rightarrow \phi(U)$ and $\psi : V \rightarrow \psi(V)$ are called **(smoothly) compatible** when either $U \cap V = \emptyset$ or $\phi^{-1} \circ \psi$ and $\psi \circ \phi^{-1}$ are smooth (meaning partial derivatives of all orders exist). We say that an atlas is **smooth** if every pair of charts is compatible.

Note that a smooth atlas \mathcal{A} on M can be extended to a unique maximal smooth atlas \mathcal{M} on M by adding to \mathcal{A} every possible homeomorphism $\psi : U \subseteq M \rightarrow \psi(U) \subseteq \mathbb{R}^n$ which is compatible with all of the existing charts (since if ψ and χ are both compatible with every chart $\phi \in \mathcal{A}$, then ψ and χ will be compatible with each other). The maps $\psi \circ \phi^{-1}$ are called **transition maps** or **change of coordinate maps**. A maximal smooth atlas \mathcal{M} on M is called a **smooth structure** on M .

Definition. An n -dimensional **smooth (or C^∞) manifold** is an n -dimensional topological manifold with a smooth structure.

Remark. A topological manifold can have different smooth structures. For example, take $\mathcal{A} = \{\phi\}$ where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is the identity map, and $\mathcal{B} = \{\psi\}$ where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism given by $\psi(x) = x^3$, since $\sqrt[3]{x}$ is not smooth at the origin.

What if we tried $\mathcal{B} = \{\psi\}$ where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism which is not C^∞ ? This is trivially a smooth atlas.

Typically, a manifold is given with a standard smooth structure.

Remark. We can give a smooth manifold M an (at most countable) atlas of charts all of which are of one of the forms

- $\phi : U \subseteq M \rightarrow B(0, 1)$
- $\phi : U \subseteq M \rightarrow (0, 1)^n$
- $\phi : U \subseteq M \rightarrow \mathbb{R}^n$

Note that the maximal atlas \mathcal{M} is determined from any subset \mathcal{AM} such that the domains of the charts in \mathcal{AM} cover M .

Definition. Let M be an m -dimensional smooth manifold and N be an n -dimensional smooth manifold and let $f : M \rightarrow N$ be a function. Then we say f is **smooth** at p when for some (hence for any) chart ϕ on M at p and for some (hence any) chart ψ on N at $f(p)$, the map $\psi^{-1} \circ f \circ \phi$ is smooth at $x = \phi(p)$, and f is **smooth** if f is smooth at every $p \in M$. We say that f is a **diffeomorphism** when f is invertible and both f and f^{-1} are smooth. We say that M and N are **diffeomorphic**, and write $M \cong N$, when there exists a diffeomorphism $f : M \rightarrow N$.

Remark. It is conceivable that a topological manifold M could be both of dimension n and of dimension m with $n \neq m$. To do this, we would need to have a homeomorphism from an open set in \mathbb{R}^n to an open set in \mathbb{R}^m . In fact, this cannot happen by invariance of domain, proven using tools from algebraic topology.

When M is smooth, it is easy to see that this cannot happen. If $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ were smooth inverses, then the matrices $D(\psi \circ \phi^{-1})(\phi(p))$ and $D(\phi \circ \psi^{-1})(\psi(p))$ would be inverse matrices. But then a product of a matrix in $M_{m \times n}(\mathbb{R})$ and in $M_{n \times m}(\mathbb{R})$ cannot be inverses when $m \neq n$.

Remark. Manifolds are sometimes constructed using quotient constructions. These quotients can be given by polygons with pairs of edges identified up to orientation.

There are other kinds of manifolds (other than C^∞ manifolds); for example, one can define C^k manifolds, or analytic C^ω manifold has an atlas in which the transition maps are analytic.

- Example.*
1. \mathbb{R}^n is a smooth n -dimensional manifold. It can be given an atlas consisting of 1 chart, the identity map.
 2. Any n -dimensional vector space over \mathbb{R} is a smooth n -dimensional manifold. It can be given an atlas with one chart. If $\{u_1, \dots, u_n\}$ is a basis for V , then one can define $\phi : V \rightarrow \mathbb{R}^n$ by $\phi(\sum t^i u_i) = (t^1, \dots, t^n) = t \in \mathbb{R}^n$.
 3. Every open subset of a smooth n -dimensional manifold is also a smooth n -dimensional manifold.
 4. $M_{n \times m}(\mathbb{R})$ is an $n \cdot m$ -dimensional manifold with pointwise \mathbb{R}^{nm} structure.
 5. $\{A \in M_{n \times m}(\mathbb{R}) : \text{rank}(A) = \min\{n, m\}\}$ is a smooth manifold with one chart, since it is an open submanifold of $M_{n \times m}$. Suppose $n > m$; then take all $n \times n$ submatrices which have non-zero determinant (open by continuity of \det), and maximal rank means that A is contained in one of these open subsets.
 6. The disjoint union of countably many n -dimensional smooth manifolds.
 7. The cartesian product of finitely many smooth manifolds is a smooth manifold. Let $\dim(M_k) = n_k$, the $\dim(M_1 \times \dots \times M_\ell) = \sum_{k=1}^\ell n_k$. If $\phi_k : U_k \subseteq M_k \rightarrow \phi_k(U_k) \subseteq \mathbb{R}^{n_k}$ is a chart on M_k , then $\chi_k : \prod_{k=1}^\ell U_k \rightarrow \prod_{k=1}^\ell \mathbb{R}^{n_k}$ given by $\chi_k(p_1, \dots, p_\ell) = (\phi_1(p), \dots, \phi_\ell(p))$ is a chart in $M_1 \times \dots \times M_\ell$.
 8. One can show that \mathbb{S}^n is a smooth n -dimensional manifold.

Remark. For $A \in M_{n \times m}(\mathbb{R})$, we denote the entry in the k^{th} row and ℓ^{th} column by A_ℓ^k .

Example. \mathbb{S}^n is an example of an n -dimensional smooth manifold. It can, for example, be given a smooth atlas which contains $2(n+1)$ charts as follows. For $1 \leq k \leq n+1$, let

$$\begin{aligned} U_k &= \{x \in \mathbb{S}^n : x^k > 0\} \\ \phi_k : U_k &\rightarrow B(0, 1) \subseteq \mathbb{R}^n \\ \phi_k(x) &= (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^{n+1}) \\ \phi_k^{-1}(t^1, \dots, t^n) &= \left(t^1, \dots, t^{k-1}, \sqrt{1 - \sum_{i=1}^{k-1} (t^i)^2}, t^k, \dots, t^n\right) \end{aligned}$$

and the corresponding opposite charts for $x^k < 0$. Note that \mathbb{S}^n is a metric space. It has 2 standard metrics: either the one inherited from \mathbb{R}^n , or the arclength distance $d_s(u, v) = \cos^{-1}(u \cdot v)$.

We can also give \mathbb{S}^n an atlas which only uses 2 charts, by stereographic projection from a north pole and a south pole.

This stereographic projection also shows that the rational points on the sphere are dense in \mathbb{S}^n , via the map

$$\phi(x) = \alpha\left(\frac{1}{1-x^{n+1}} \text{right}\right) = \left(\frac{x^1}{1-x^{n+1}}, \dots, \frac{x^n}{1-x^{n+1}}\right)$$

One can also find ϕ^{-1} and verify that they are both rational functions. In particular, $\phi^{-1}(\mathbb{Q}^n) \subseteq \mathbb{S}^n$ is dense.

Example. The projective space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ is commonly defined in at least 3 ways:

$$\begin{aligned}\mathbb{P}^n &= \{1\text{-dimensional subspaces of } \mathbb{R}^{n+1}\} \\ \mathbb{P}^n &= \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times = \{[x] : 0 \neq x \in \mathbb{R}^{n+1}\}, [x] = \{tx : t \in \mathbb{R}^\times\} \\ \mathbb{P}^n &= \mathbb{S}^n / \pm 1\end{aligned}$$

We can give \mathbb{P}^n a smooth atlas with $n+1$ charts as follows: for $1 \leq k \leq n+1$, set

$$\begin{aligned}U_k &= \{[x] \in \mathbb{P}^n : x^k \neq 0\} \\ \phi_k : U_k &\rightarrow \mathbb{R}^n, \phi_k([x]) = \left(\frac{x^1}{x^k}, \dots, \frac{x^{k-1}}{x^k}, \frac{x^{k+1}}{x^k}, \dots, \frac{x^{n+1}}{x^k}\right)\end{aligned}$$

with $\phi_k^{-1}(t_1, \dots, t^n) = [(t_1, \dots, t^{k-1}, 1, t^k, \dots, t^n)]$.