Representation Theory of Finite Groups

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Contents

I. REPLACE

Let G be a finite group of order n, and write $G = \{g_1, \ldots, g_n\}$. Fix $g \in G$; then $gg_i = gg_j$ if and only if $i = j$. Thus there exists some $\sigma_g \in S_i$ such that $gg_i = g_{\sigma_g(i)}$ for all $i \in \{1, 2, \ldots, n\}$. In particular, $\phi : G \to S_n$ by $\phi(g) = \sigma_g$ is an embedding (injective group homomorphism). This observation is usually referred to as Cayley's Theorem.

Now let V be an n–dimensional complex vector space. We then denote $\mathrm{GL}(V)$ as the group of invertible linear operators $T : V \to V$. Now define $\psi : S_n \to \mathrm{GL}_n(V)$ by $\psi(\sigma) = T_\sigma$ where if $\{b_1, \ldots, b_n\}$ is a basis for V and $T_\sigma(b_i) = b_{\sigma(i)}$. This is an injective group homomorphism, so $\psi \circ \phi : G \to \mathrm{GL}(V)$ is an embedding of G into $\mathrm{GL}(V)$.

Definition. Let G be a finite group, and V a finite dimensional $\mathbb{C}$–vector space. A **representation** of G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. We call $\dim(V)$ the **degree** of the representation.

In particular, if V is n–dimensional, then $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$.

Example. Consider $\rho : G \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$ given by $\rho(g) = 1$ for all $g \in G$. This is called the *trivial representation*.

Example. Consider $\rho : S_n \to \mathbb{C}^\times$ given by $\rho(\sigma) = \mathrm{sgn}(\sigma)$, which is called the *sign representation*.

Example. The representation fo G afforded by Cayley's theorem is called the *regular representation* of G. The next example is a good way to understand the regular rep of G.

Example. Consider G, $X = \{x_1, \ldots, x_n\}$, and $V = \mathrm{Free}(X)$. Suppose G acts on X. Then $\rho : G \to \mathrm{GL}(V)$ given by $\rho(g)(x_i) = gx_i$. In particular, if we take $X = G$, then this is the regular representation of G

Example. Consider the 4–gon, with vertices labelled a, b, c, d. Take $X = \{a, b, c, d\}$ and the regular representation $\rho : D_4 \to \mathrm{GL}(V)$. This action has a geometric notion.

Example. Let C_n be a cyclic group of order n; let us define some $\rho : C_n \to \mathrm{GL}(V)$. Say $\rho(x) = T$ where $t \in \mathrm{GL}(V)$; then this is a representation if and only if $T^n = I$.

Definition. We say that two representations $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ are **isomorphic** if there exists an isomorphism $T : V \to W$ such that for all $g \in G$,

$$T \circ \rho(g) = \tau(g) \circ T$$

Suppose $\rho : G \to \mathrm{GL}(V)$ and $T : V \to W$ is an isomorphism. Then we can define $\tau : G \to \mathrm{GL}(W)$ by $\tau(G) = T \circ \rho(g) \circ T^{-1}$; this $\rho \cong \tau$. In other words, the representation is unique up to isomorphism under change of basis.

Example. Consider $G = \{g_1, \ldots, g_n\} = \{h_1, \ldots, h_n\}$, and fix $g \in G$. Let $gg_i = g_{\alpha(i)}$ and $gh_i = h_{\beta(i)}$ where $\alpha, \beta \in S_n$. Fix an n–dimensional vector space V with basis $\{b_1, \ldots, b_n\}$. Then two regular representations are given by

$$\rho_1 : G \to \mathrm{GL}(V), \rho(g)(b_i) = b_{\alpha(i)}$$
$$\rho_2 : G \to \mathrm{GL}(V), \rho(g)(b_i) = b_{\beta(i)}$$

Let $\gamma \in S_n$ be such that $h_{\gamma(i)} = g_i$, and define $T : V \to V$ by $T(v_i) = b_{\gamma(i)}$. Then

$$gg_i = g_{\alpha(i)} = gh_{\gamma(i)} = h_{\beta\gamma(i)} = g_{\gamma^{-1}\beta\gamma(i)}$$

so that $\alpha = \gamma^{-1}\beta\gamma$. Thus for each b_i,

$$\begin{aligned} T \circ \rho_1(g) \circ T^{-1}(b_i) &= T \circ \rho_1(g)(b_{\gamma^{-1}(i)}) \\ &= T(b_{\alpha\gamma^{-1}(i)})b_{\gamma\alpha\gamma^{-1}(i)} \\ &= b_{\beta(i)} = \rho_2(g)(b_i) \end{aligned}$$

so that $T \circ \rho_1(g) \circ T^{-1} = \rho_2(g)$.
Note: conjugate elements have the same cycle type.

Subrepresentations

What should a subrepresentation of $\rho : G \to \mathrm{GL}(V)$ mean?

We would like a subspace $W \leq V$ such that $\tau : G \to \mathrm{GL}(W)$ is a representation given by $\tau(g)(w) = \rho(g)(w)$ for all $w \in W$. Moreover, to make this well-defined, we need W to b4 $\rho(g)$–invariant for every $g \in G$ ($\rho(g)(W) \subseteq W$).