Functional Analysis

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I. Fundamentals of Functional Analysis

1 Basic Elements of Functional Analysis

Throughout, we denote by $\mathbb{F}$ either the field $\mathbb{R}$ or the field $\mathbb{C}$.

Banach Spaces

Definition. Let X be a vector space over $\mathbb{F}$. A **norm** is a functional $\|\cdot\| : X \to \mathbb{R}$ such that it is

- *(non-negative)* $\|x\| \geq 0$ for any $x \in X$
- *(non-degenerate)* $\|x\| = 0$ if and only if $x = 0$
- *(subadditivity)* $\|x+y\| \leq \|x\| + \|y\|$ for $x, y \in X$
- *($|\cdot|$−homogeneity)* $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{F}$, $x \in X$.

We call the pair $(X, \|\cdot\|)$ a **normed vector space**. Furthermore, we say that $(X, \|\cdot\|)$ is a **Banach space** provided that X is complete with respect to the metric $\rho(x,y) = \|x-y\|$.

Example. (i) $(\mathbb{F}, |\cdot|)$ is a Banach space.

(ii) $(\mathbb{F}^b, \|\cdot\|_p)$, $x = (x_j)_{j=1}^n$,

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^n |x_j|^p\right)^{1/p} & 1 \leq p < \infty \\ \max_{j=1,\ldots,n} |x_j| & p = \infty \end{cases}$$

(iii) Consider the space

$$L_p^{\mathbb{F}} = \left\{ f : [0,1] \to \mathbb{F} \;\middle|\; f \text{ is Lebesgue measurable}, \left(\int_0^1 |f|^p\right)^{1/p} < \infty \right\} \Big/ \sim_{\text{a.e.}}$$

where $1 \leq p < \infty$.

(iv) $L_\infty^{\mathbb{F}}[0,1]$, $\|f\|_\infty = \operatorname{ess\,sup}_{t \in [0,1]} |f(t)|$.

(v) Let (X, d) be a metric space. Then

$$C_b^{\mathbb{F}}(x) = \{ f : X \to \mathbb{F} \mid f \text{ is continuous and bounded} \}, \quad \|f\|_\infty = \sup_{x \in X} |f(x)|$$

is a Banach space.

Here is a more interesting example:

Example. Let (X,d) be a metric space. We define the space of Lipschitz functions

$$\mathrm{Lip}^{\mathbb{F}}(X,d) = \left\{ f : X \to \mathbb{F} \;\middle|\; f \text{ is bounded}, L(f) = \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|f(x)-f(y)|}{d(x,y)} < \infty \right\}$$

We note that for $f : X \to \mathbb{F}$ that

$$f \in \mathrm{Lip}^{\mathbb{F}}(X,d) \Leftrightarrow \text{there is } L \geq 0 \text{ s.t. } |f(x)-f(x)| \leq Ld(x,y) \text{ for all } x,y \in X \tag{1.1}$$

It is easy to verify that $L(f) = \min\{L \geq 0 : (1.1) \text{ holds for } f\}$. It is an easy exercise to see that $\mathrm{Lip}^{\mathbb{F}}$ is a vector space, and that $L : \mathrm{Lip}^F(X,d) \to \mathbb{R}$ is a **semi-norm** (non-negative, subadditive, $|\cdot|$–homogeneous). However, we do not have non-degeneracy (for example, constants are taken to 0). We define the Lipschitz norm

$$\|f\|_{\mathrm{Lip}} = \|f\|_\infty + L(f)$$

1.1 Proposition. $(\mathrm{Lip}^{\mathbb{F}}(X,d), \|\cdot\|_{\mathrm{Lip}})$ *is a Banach space.*

Proof Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $(\mathrm{Lip}^{\mathbb{F}}(X,d), \|\cdot\|_{\mathrm{Lip}})$. Since $\|\cdot\|_\infty \leq \|\cdot\|_{\mathrm{Lip}}$ on $\mathrm{Lip}^F(X,d)$, we see that $(f_n)_{n=1}^\infty$ is uniformly Cauchy (and bounded), and hence there is $f = \lim_{n\to\infty} f_n$ in $C_b^{\mathbb{F}}(X)$, where the limit is taken with respect to $\|\cdot\|_\infty$, since $(C_b^{\mathbb{F}}(X), \|\cdot\|_\infty)$ is a Banach space. If $x,y \in X$, then

$$\begin{aligned} |f(x)-f(x)| &= \lim_{n\to\infty} |f_n(x)-f_n(y)| \leq \sup_{n\in\mathbb{N}} |f_n(x)-f_n(y)| \\ &\leq \sup_{n\in\mathbb{N}} L(f_n) d(x,y) \leq \sup_{n\in\mathbb{N}} \|f_n\|_{\mathrm{Lip}}\, d(x,y) \end{aligned}$$

Since Cauchy sequences are bounded, we see that $|f(x)-f(y)| \leq Ld(x,y)$, where $L = \sup_{n\in\mathbb{N}} \|f_n\|_{\mathrm{Lip}} < \infty$. Thus by (1.1), $f \in \mathrm{Lip}^{\mathbb{F}}(X,d)$. Exercise: one may verify that $\|f - f_n\|_{\mathrm{Lip}} \to 0$. ■

Another collection of basic examples are given by the sequence spaces. We can define

$$\ell_1^{\mathbb{F}} = \left\{ x = (x_j)_{j=1}^\infty \in \mathbb{F}^{\mathbb{N}} \;\middle|\; \|x\|_1 = \sum_{j=1}^\infty |x_j| < \infty \right\}$$

It is easy to see that $(\ell_1, \|\cdot\|_1)$ is a normed vector space.

For $1 < p < \infty$, and write

$$\ell_p^{\mathbb{F}} = \left\{ x \in \mathbb{F}^{\mathbb{N}} \;\middle|\; \|x\|_p = \left(\sum_{j=1}^\infty |x_j|^p \right)^{1/p} < \infty \right\}$$

Note that $0 \in \ell_p$, $\alpha \in \mathbb{F}$, $\alpha x \in \ell_p$ if $x \in \ell_p$. Let $q = p/(p-1)$ so that $1/p + 1/q = 1$. Then q is called the **conjugate index**. We have

1.2 Proposition. (Young's Inequality) *If $a,b \geq 0$ in $\mathbb{R}$, then $ab \leq a^p/p + b^q/q$, with equality only if $a^p = b^q$.*

and

1.3 Proposition. (Hölder's Inequality) *If $x \in \ell_p$ and $y \in \ell_q$, then $xy = (x_i y_i)_{i=1}^{\infty} \in \ell_1$, with*

$$\left| \sum_{i=1}^{\infty} x_i y_i \right| \leq \|x\|_p \left\|y\right\|_q$$

with equality exactly when $\operatorname{sgn}(x_i y_i) = \operatorname{sgn}(x_k y_k)$ for all $j,k \in \mathbb{N}$ where $x_i y_i \neq 0 \neq x_k y_k$, and $|x|^p = (|x_j|^p)_{j=1}^{\infty}$ and $|y|^q$ are linearly dependent in ℓ_1.

and finally

1.4 Proposition. (Minkowski's Inequality) *If $x, y \in \ell_p$, then $\left\|x+y\right\|_p \leq \|x\|_p + \left\|y\right\|_p$ with equality exactly when one of x or y is a non-negative scalar combination of the other.*

Review of Topology

Let X denote a non-empty set, and $\mathcal{P}(X)$ denote the power set of X.

Definition. A **topology** on a set X is a set τ of subsets of X such that
(i) $\emptyset, X \in \tau$
(ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
(iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \leq i \leq n$, then $\bigcap_{i=1}^{n} U_i \in \tau$.

The sets $U \in \tau$ are called the **open sets** in X, and sets of the form $X \setminus U$ for some open set U are caled the **closed sets** in X. The pair (X, τ) is called a **topological space**.

The metric topology on a metric space (X, d) is the topology

$$\tau_d = \{\, U \subseteq X \mid \text{ for each } x_0 \in U, \text{ there is } \delta = \delta(x_0) \text{ s.t. } B_\delta(x_0) \subseteq U \,\}$$

Example. (i) Given two metrics d, ρ on X, we say that $d \sim \rho$ if and only if there are $c, C > 0$ such that

$$c d(x,y) \leq \rho(x,y) \leq C d(x,y) \text{ for any } x, y \in X$$

Note that $d \sim \rho$ implies that $\tau_d = \tau_\rho$, but the reverse implication is not true. An example of this are the metrics on $X = \mathbb{R}$ given by $d(x,y)$ and $\rho(x,y) = \frac{|x-y|}{1+|x-y|}$. Then $d \nsim \rho$ but $\tau_d = \tau_\rho$.

(ii) "Sorgenfry line" Set $X = \mathbb{R}$, and consider

$$\sigma = \{\, V \subseteq \mathbb{R} \mid \text{ for any } s \in V, \text{ there is } \delta = \delta(s) > 0 \text{ s.t. } [s, s+\delta) \subseteq V \,\}$$

It is an exercise to verify that $\tau_{|\cdot|} \subsetneq \sigma$. We say that σ is **finer** than $\tau_{|\cdot|}$.

(iii) Relative topology: let (X, τ) be a topological space, and $\emptyset \neq A \subseteq X$. Then we can define a topology $\tau|_A = \{U \cap A : U \in \tau\}$.

Definition. Let (X, τ) and (Y, σ) be topological spaces, and $f : X \to Y$. We say that f is $(\tau-\sigma-)$**continuous** at x_0 in X if,

- given $V \in \sigma$ such that $f(x_0) \in V$, then there exists $U \in \tau$ such that $x_0 \in U$ and $f(U) \subseteq V$.

We say that f is $(\tau-\sigma-)$continuous if it is continuous at each x_0 in X.

Space of bounded continuous functions into a normed space

Let $(Y, \|\cdot\|)$ denote a normed space. We let $\tau_{\|\cdot\|}$ denote the topology given by the metric $\rho(x,y) = \left\|x-y\right\|$. Let (X,τ) denote any topological space. Then we write

$$C_b^Y(X) = \left\{ f : X \to Y \;\middle|\; f \text{ is bounded and } \tau - \tau_{\|\cdot\|} - \text{continuous} \right\}$$

With pointwise operations, we see that $C_b^Y(X)$ is a vector space. We also define for $f \in C_b^Y(X)$, $\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}$, making $(C_b^Y(X), \|\cdot\|_\infty)$ a normed vector space.

1.5 Theorem. *If $(Y, \|\cdot\|)$ is a Banach space, then $(C_b^Y(X), \|\cdot\|_\infty)$ is a Banach space.*

Proof Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $(C_b^Y(X), \|\cdot\|_\infty)$. Then for any $x \in X$, we have that $(f_n(x))_{n=1}^\infty$ is Cauchy in $(Y, \|\cdot\|)$ since $\|f_n(x) - f_m(x)\| \leq \|f_n - f_m\|_\infty$, and hence admis a limit $f(x)$. In particular, $x \mapsto f(x)$ defines a function from X to Y. We shall fix $x_0 \in X$ and show that f is continuous at x_0. Given $\epsilon > 0$, we let

- n_1 be so $n, m \geq n_1$ so that $\|f_n - f_m\|_\infty < \epsilon/4$.
- n_2 be so $n \geq n_2$ so that $\|f_n(x_0) - f(x_0)\| < \epsilon/4$.
- $N = \max\{n_1, n_2\}$.
- $U \in \tau$, $x_0 \in U$ such that $f_N(U) \subseteq B_{\epsilon/4}(f(x_0)) \subset Y$.

Then for $x \in U$, we let n_x be so $n_x \geq n_1$ and $n \geq n_x$, so that $\|f_n(x) - f(x)\| < \epsilon/4$. We then have

$$\begin{aligned}\|f(x) - f(x_0)\| &\leq \left\|f(x) - f_{n_x}(x)\right\| + \left\|f_{n_x}(x) - f_N(x)\right\| + \|f_N(x) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| \\ &< \frac{\epsilon}{4} + \left\|f_{n_x} - f_N\right\|_\infty + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon\end{aligned}$$

in other words that $f(U) \subseteq B_\epsilon(f(x_0))$.

Now let us check that $\|f\|_\infty < \infty$. Since $|\|f_n\|_\infty - \|f_m\|_\infty| \leq \|f_n - f_m\|_\infty$, so $(\|f_n\|_\infty)_{n=1}^\infty \subseteq \mathbb{R}$ is Cauchy, hence bounded. If $x \in X$, then

$$\|f(x)\| = \lim_{n\to\infty} \|f_n(x)\| \leq \sup_{n\in\mathbb{N}} \|f_n(x)\| \leq \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$$

so $\|f\|_\infty = \sup_{x\in X} \|f(x)\| < \infty$.

Notice that if ϵ, n_1 are as above, and further x_0, N are as above, we have for $n \geq n_1$

$$\|f_n(x_0) - f(x_0)\| \leq \|f_n(x_0) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| < \frac{\epsilon}{2}$$

so $\|f_n - f\|_\infty = \sup_{x_0\in X} \|f_n(x_0) - f(x_0)\| \leq \epsilon/2 < \epsilon$. This is uniform since n_1 is chosen uniformly in X. ■

1.6 Corollary. *$(C_b^{\mathbb{F}}(X), \|\cdot\|_\infty)$ is a Banach space.*

Let's first note the following general priniple: let (X,d), (Y,ρ) be metric spaces, where (X,d) is complete. If $\psi : X \to Y$ is a $(d-\rho-)$isometry, then $(\psi(X), \rho|_{\psi(X)})$ is a complete metric space.

Example. (i) Let T be a non-empty set and let

$$\ell_\infty(T) = \left\{ x = (x_t)_{t\in T} \in \mathbb{F}^T \;\middle|\; \|x\|_\infty \right\} < \infty$$

With pointwise operations, $(\ell_\infty, \|\cdot\|_\infty)$ is a normed space. In fact, it is a Banach space. Let us note that

$$f \mapsto (f(t))_{t\in T} : C_b(T, \mathcal{P}(T)) \to \ell_\infty(T)$$

is a surjective linear isometry, and the result follows.

(ii) Let $c = \{ x \in \ell_\infty \mid \lim_{n\to\infty} x_n \text{ exists} \}$. Then $(c, \|\cdot\|_\infty)$ is a Banach space. Consider the topological space given by $\omega = \mathbb{N} \cup \{\infty\}$, with topology

$$\tau_\omega = \mathcal{P}(\mathbb{N}) \cup \bigcup_{n\in\mathbb{N}} \{k \in \mathbb{N} : k \geq n\}$$

The map $f \mapsto (f(n))_{n=1}^\infty : C_b(\omega) \to c$ is a linear surjective isometry.

(iii) $c_0 = \left\{ x \in \mathbb{F}^{\mathbb{N}} \,\middle|\, \lim_{n\to\infty} x_n = 0 \right\} \subseteq c \subseteq \ell_\infty$.

1.7 Lemma. *If $x_0 \in X$ where (X, τ) is a topological space, then*

$$\mathcal{I}(x_0) = \{ f \in C_b(x) \mid f(x_0) = 0 \}$$

is closed, hence complete, subspace of $C_b(X)$.

Proof If $(f_n)_{n=1}^\infty \subseteq \mathcal{I}(x_0)$ and $f = \lim_{n\to\infty} f_n$ with respect to $\|\cdot\|_\infty$ in $C_b(X)$, then $f(x_0) = \lim_{n\to\infty} f_n(x_0) = 0$. Thus $f \in \mathcal{I}(x_0)$, and closed subsets of complete spaces are themselves complete. ■

Now, $f \mapsto (f(n))_{n=1}^\infty : \mathcal{I}(\infty) \to c_0$ is a (linear) surjective isometry.

(iv) Consider the Sorgenfty line $(\mathbb{R}, \sigma)$: verify that

$$c_b(\mathbb{R}, \sigma) = \left\{ f : \mathbb{R} \to \mathbb{F} \,\middle|\, f \text{ is bounded and } \lim_{t\to t_0^+} f(t) = f(t_0) \text{ for } t \in \mathbb{R} \right\}$$

2 Linear operators and linear functionals

Let X, Y be vector spaces. We let $\mathcal{L}(X,Y) = \{ S : X \to Y \mid S \text{ is linear} \}$; this is itself a vector space with pointwise operations. Let $(X, \|\cdot\|)$ be a normed space. We denote

$$\begin{aligned} D(X) &= \{x \in X : \|x\| < 1\} \\ S(X) &= \{x \in X : \|x\| = 1\} \\ B(X) &= \{x \in X : \|x\| \leq 1\} \end{aligned}$$

(Yes, this notation is confusion. No, I didn't choose it.)

2.1 Proposition. *If X, Y are normed spaces and $S \in \mathcal{L}(X,Y)$, then the following are equivalent:*

(i) S is continuous

(ii) S is continuous at some $x_0 \in X$

(iii) $\|S\| = \sup_{x\in D(X)} \|Sx\| < \infty$.

Moreover, in this case, we have

$$\begin{aligned} \|S\| &= \min\{L > 0 : \|Sx\| \leq L\|x\| \text{ for } x \in X\} \\ &= \sup_{x\in S(X)} \|Sx\| = \sup_{x\in B(X)} \|Sx\| \end{aligned}$$

Proof $(i \Rightarrow ii)$ Obvious

$(ii \Rightarrow iii)$ Note that

$$Sx_0 + D(Y) = \{Sx_0 + y : t \in D(Y)\} = \{y \in Y : \left\|Sx_0 - y'\right\| < 1\}$$

is a neighbourhood of Sx_0. By the definition of metric continuity, there is $\delta > 0$ such that

$$x_0 + \delta D(X) = \{x_0 + \delta x : x \in D(x)\} = \{x' \in X : \left\|x_0 - x'\right\| < \delta\}$$

such that

$$Sx_0 + \delta S(D(X)) = S(x_0 + \delta D(x) \subseteq Sx_0 + D(Y)$$

which implies that $\delta S(D(X)) \subseteq D(Y)$ and $S(D(X)) \subseteq D(Y)/\delta$, in other words that $\|Sx\| \leq 1/\delta$ for $x \in D(X)$.

$(iii \Rightarrow i)$ If $x \in X$ and $\epsilon > 0$, then

$$\|Sx\| = (\|x\| + \epsilon)\left\|S\left(\frac{1}{\|x\| + \epsilon}\|x\|\right)\right\| \leq (\|x\| + \epsilon)\|S\|$$

Then, letting $\epsilon \to 0^+$, we see that

$$\|Sx\| \leq \|x\|\|S\| = \|S\|\|X\|$$

If $x, x' \in X$, then $\|Sx - S'x\| \leq \|S\|\|x - x'\|$ is S is Lipschitz, hence continuous.

To complete the proof, the content of (iii) implies (i) tellus us that the Lipschitz constant $L(S) \leq \|S\|$. Furthermore, if $\|x\| = 1$, the preceding proof gives us that $\|S\|_{S(X)}$.

Conversely,

$$\|S\| = \sup_{x \in D(X)\setminus\{0\}} \|Sx\| = \sup_{x \in D(X)\setminus\{0\}} \|x\|\left\|S\left(\frac{1}{\|x\|}x\right)\right\| \leq \sup_{x \in S(X)} \|Sx\|$$

The remaining equivalence is obvious. ■

We now let $\mathcal{B}(X, Y) = \{\, S \in \mathcal{L}(X, Y) \mid S \text{ is bounded} \,\}$. We will see that $\|\cdot\|$, above, defines a norm on $B(X, Y)$.