PMATH 465

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Contents

I. Fundamentals of Manifolds

1 Introduction to Topology

Basic Constructions

Definition. A **topology** on a set X is a set τ of subsets of X such that
(i) $\emptyset \in \tau$ and $X \in \tau$
(ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
(iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \leq i \leq n$, then $\bigcap_{i=1}^{n} U_i \in \tau$.
The sets $U \in \tau$ are called the **open sets** in X, and sets of the form $X \setminus U$ for some open set U are caled the **closed sets** in X.

Definition. When X is a topological space and $A \subseteq X$, the **interior** of A (denoted A°) is the union of all open sets contained in A. Similarly, we define the **closure** of A (denoted $\overline{A}$) as the intersction of all closed sets containing A. Then the **boundary** of A, denoted by ∂A, is the set $\partial A = \overline{A} \setminus A^\circ$.

Example. Let X be any set. The **discrete topology** on X is the topology $\tau = \mathcal{P}(X)$, and the **trivial topology** on X is the topology $\tau = \{\emptyset, X\}$.

Definition. A **basis** for a topology on a set X is a set $\mathcal{V}$ of subsets of X
(i) $\bigcup_{B \in \mathcal{B}} b = X$
(ii) for all $a \in X$ and $U, V \in \mathcal{B}$ such that $a \in U \cap V$, then there exists $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$.
When $\mathcal{B}$ is a basis for a topology on X, the topology on X **generated** by $\mathcal{B}$ is the set τ of subsets of X such that for $W \subseteq X$, $W \in \tau$ if and only if for all $a \in W$, there exists $U \in \mathcal{B}$ such that $a \in U \subseteq W$.

Note that τ, as above, is a topology on X since
(i) $\emptyset \in \tau$ vacuously and $X \in \tau$ obviously.
(ii) If $A_k \in \tau$ for all $k \in K$ (where K is any set of indices), then given $a \in \bigcup_{x \in K} A_k$, we can choose $\ell \in K$ so that $a \in A_\ell$. Then since $A_\ell \in \tau$, we can choose $U_\ell \in \mathcal{B}$ so that $a \in U_\ell \subseteq A_\ell$. Thus $a \in U_\ell \subseteq A_\ell \subseteq \bigcup_{k \in K} A_k$.
(iii) By induction, it suffices to prove that if $A, B \in \tau$, then $A \cap B \in \tau$. Suppose $A, B \in \tau$, and let $a \in A \cap B$. Since $A \in \tau$, we cna choose $U \in \mathcal{B}$ so that $a \in U \subseteq A$. Since $B \in \tau$, we can choose $V \in \mathcal{B}$ so that $a \in V \subseteq B$. Then we have $a \in U \cap V$. Since $\mathcal{B}$ is a basis, we can chose $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$, so $a \in W \subseteq U \cap V \subseteq A \cap B$.
Note that when τ is the topology on X generated by the basis $\mathcal{B}$, for $A \subseteq X$, $A \in \tau$ if and only if there exists some $S \subseteq \mathcal{B}$ such that $A = \bigcup_{s \in S} s$. In this sense, the topology τ on X generated by the basis $\mathcal{B}$ is the coarsest topology which contains $\mathcal{B}$.

Definition. (Subspace Topology) When Y is a topological space and $X \subseteq Y$ is a subset of Y, we define the **subspace topology** on X to be the topology for which as set $U \subseteq X$ is open if and only if $U = X \cap V$ for some open set V.

If $\mathcal{C}$ is a basis for the topology on Y, then $\mathcal{B} = \{\, X \cap V \mid V \in \mathcal{C} \,\}$ is a basis for the subspace topology on X.

Definition. (Disjoint Union Topology) If X and Y are topological spaces with $X \cap Y = \emptyset$, then the **disjoint union topology** on $X \cup Y$ is the topology in which a subset $U \subseteq X \cup Y$ is open in $X \cup Y$ if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y.

Definition. (Product Topology) If X and Y are topological spaces, the **product topology** on $X \times Y$ is the topology generted by the basis

$$\mathcal{B} = \{ U \times V \mid U \in \mathcal{C}, V \in \mathcal{D} \}$$

where $\mathcal{C}$ and $\mathcal{D}$ are bases for the topologies on X, Y respectively.

Definition. (Infinite Product Topology) We define the infinite product to be

$$\prod_{k \in K} \left\{ f : K \to \bigcup_{k \in K} X_k \;\middle|\; f(k) \in X_k \text{ for all } k \in K \right\}$$

There are two standard topologies on X. The first is the **box topology**,

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \;\middle|\; U_k \text{ is open in } X_k \right\}$$

and the **product topology**

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \;\middle|\; \begin{array}{l} U_k \text{ is open in } X_k \\ U_k = X_k \text{ for all but finitely many indices } k \end{array} \right\}$$

Example. (Metric Topology) $\mathbb{R}^n$ has a standard **inner product**, and for $u, v \in \mathbb{R}^n$, $\langle u, v \rangle = u \cdot v = V^T u = \sum_{i=1}^n u_i v_i$. This gives the standard norm on $\mathbb{R}^n$ for $u \in \mathbb{R}^n$, $\|u\| = \sqrt{\langle u, v \rangle}$. This gives the standard metric on $\mathbb{R}^n$: for $a, \in \mathbb{R}^n$, $d(a, b) = \|b - a\|$.

Given a metric on a set Y, we obtain (by restriction) an induced metric on any subset $X \subseteq Y$. Given a metric space X, we define the **metric topology** on X to be the topology which is generated by the set of open balls

$$B(a, r) = \{ x \in X \mid d(a, x) < r \}$$

where $x \in X$, $r > 0$.

Maps on Topological Spaces

Definition. When X and Y are topological spaces and $f : X \to Y$, we say that f is **continuous** when it has the property that $f^{-1}(V)$ is open in X for every open set V in Y. We say that $f : X \to Y$ is a **homeomorphism** when f is bijective and both f and f^{-1} are continuous. Then X, Y are **homeomorphic** if there exists a homeomorphism $f : X \to Y$.

> ***1.1 Theorem. (Glueing Lemma)*** *Let X and Y be topological spaces, and let $f : X \to Y$ be a function. Suppose either*
> (i) $X = \bigcup_{k \in K} A_k$ *where each* A_k *is open in* X, *or*
> (ii) $X = \bigcup_{k=1}^n A_k$ *where each* A_k *is closed in* X
> *and each restriction map* $f_k : A_k \to Y$ *is continuous, then* f *is continuous.*

Proof Exercise. ■

Definition. A topological space X is **compact** when it has the property that for every set $\mathcal{S}$ of open subsets of X with $X = \bigcup_{U\in\mathcal{S}} U$, there exists a finite subset $\mathcal{F} \subseteq \mathcal{S}$ such that $X = \bigcup_{F\in\mathcal{F}} F$.

Note that when $X \subseteq Y$ is a subspace, X is compact if and only if X has the property that for every set $\mathcal{T}$ with $X \subseteq \bigcup_{T\in\mathcal{T}} T$, there exists a finite subset $\mathcal{G} \subseteq \mathcal{T}$ uch that $X \subseteq \bigcup_{G\in\mathcal{G}} G$.

Definition. A topological space X is **connected** when there do not exist non-empty disjoint open sets $U, V \in X$ such that $X = U \cup V$.

Note that if Y is a metric space and $X \subseteq Y$ is a subspsace, then X if connected if and only if there do not exist open sets $U, V \in Y$ such that

$$X \cap U \neq \emptyset, X \cap V \neq \emptyset, U \cap V = \emptyset, \text{ and } X \subseteq U \cap V$$

Definition. A topological space X is called **path connected** when it has the property that for all $a, b \in X$, there exists a continuous map $\alpha : [0,1] \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$.

It is easy to see that if X is path connected, then X is connected.

Definition. Let X be a topological space. If we define a relation $\sim$ on C by taking $a \sim b$ if and only if there exists a connected subspsace $A \subseteq X$ with $a \in A$ and $b \in B$.

It is clear that this is an equivalence relation. Note that when X is a topological space, its connected components are connected, and each connected subspace of X is contained in one of its connected components.

Definition. Let X be a topological space. Define a relation $\approx$ on X by $a \approx b$ if and only if there exists a continuous map $\alpha : [0,1] \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$. Such a map α is called a **continuous path**.

One can show that if X is **locally path connected** (which means that X has a basis for its topology which consists of path connected sets), then the path components of X are equal to the connected components of X, and that these components are open.

Quotient Topology

Definition. (Quotient Topology) Let X be a topological space and let $\sim$ be an equivalence relation on X. The set of equivalence classes is denoted $X/\sim$, and $X/\sim$ is called the **quotient** of X by $\sim$. The map $\pi : X \to X/\sim$ given by $\pi(a) = [a]$ is called the natural **projection map** or **quotient map**. We define the **quotient topology** on $X/\sim$ by stipulating that for $W \subseteq X/\sim$, W is open in $X/\sim$ if and only if $\pi^{-1}(W)$ is open in X.

When a group G acts on a topological space X, we define an equivalence relation $\sim$ on X by $a \sim b$ if and only if $b = g \cdot a$ for some $g \in G$. The equivalence classes are orbits. In this context, we also write $X/\sim$ as X/G.

When X, Y are any toplogical spaces and $\pi : X \to Y$ is surjective, we can define an equivalence relation X by $a \sim b$ if and only if $\pi(a) = \pi(b)$. We then have a natural bijection from Y to $X/\sim$ in which $y \in Y$ corresponds to the fibre $\pi^{-1}(y) \in X/\sim$.

If Y has the topology such that for $W \subseteq Y$, W is open in Y if and only if $q^{-1}(W)$ is open in X. In this case, we also use the terminology "quotient map" for π.

Remark. Let X be a topological space and let $\sim$ be an equivalence relation on X. Let Y be any set. If $f : X \to Y$ is constant on the equivalence classes, then f induces a well-defined map $\overline{f} : X/\sim \to Y$ given by define $\overline{f}([a]) = f(a)$.

Example. Define an equivalence class on $[0,1] \subseteq \mathbb{R}$ by $s \sim t$ if and only if $s = t$ or $\{s,t\} = \{0,1\}$. Then $[0,1]\big/\sim \;\cong \mathbb{S}^1$. Define $f : [0,1] \to \S^1$ by $f(t) = e^{i2\pi t}$. Note that $f(0) = f(1)$, so f induces a continuous map $\overline{f} : [0,1]\big/\sim \;\to \mathbb{S}^1$. The inverse map can be constructed as follows. We define $g : \mathbb{S}^1 \to [0,1]\big/\sim$ by

$$g(x,y) = \begin{cases} \left[\frac{1}{2\pi}\cos^{-1} x\right] & : y \geq 0 \\ \left[1 - \frac{1}{2\pi}\cos^{-1} x\right] & : y \leq 0 \end{cases}$$

Then g is continuous by the Glueing lemma.

In particular, the same proof shows that $\mathbb{R}/\mathbb{Z}$ is homeomorphic to $\mathbb{S}^1$.

Example. The projective space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ can be defined in several ways. $\mathbb{P}^n$ is the set of all 1−dimensional vector subspaces of $\mathbb{R}^{n+1}$, or $\mathbb{P}^n = \mathbb{R}^{n+1}\setminus\{0\}\big/\mathbb{R}^\times$, or $\mathbb{P}^n = \mathbb{S}^n\big/\pm 1$ where $\mathbb{S}^n = \{u \in \mathbb{R}^{n+1} : |u| = 1\}$.

Let us show that $\mathbb{R}^{n+1}\setminus\{0\}\big/\mathbb{R}^\times$ is homeomorphic to $\mathbb{S}^n\big/\pm 1$. Define $f : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{S}^n$ by $f(x) = x/|x|$, and $g = \pi \circ f$. Then g is given by $g(x) = \{\pm x/|x|\}$. Note that for $t \in \mathbb{R}^\times$,

$$g(tx) = \left[\frac{t}{|t|} \cdot \frac{x}{|x|}\right] = \left[\frac{x}{|x|}\right]$$

since $t/|t| = \pm 1$. Thus g induces a continuous map $\overline{g}$ on the quotient. We construct the inverse map in a similar way.

Definition. Let X be a topological space. Then

- X is **T1** when for all $a, b \in X$ there exists an open set U in X with $a \in U$ and $b \notin U$
- X is **T2** or **Hausdorff** when for all $a, b \in X$, there exist disjoint open sets $U, V \subseteq X$ with $a \in U$ and $v \in B$
- X is **T3** or **regular** when X is T1 and for every $a \in X$ and every closed set $B \subseteq X$ with $a \notin B$, there exist open sets $U, V \subseteq X$ with $a \in U$, $B \subseteq V$.
- X is **T4** or **normal** when X is T1 and for all disjoint closed sets $A, B \subseteq X$ there exist disjoint open sets $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$.

Definition. Let X be a topological space.

- X is **first countable** when for every $a \in X$, there exists a countable set B_a of open sets in X which contain a such that for every open set W in X with $a \in W$, there exists $U \in \mathcal{B}_a$ with $a \in U \subseteq W$.
- X is **second countable** when there exists a countable basis for the topology on X.

Example. (i) X is T1 if and only if every 1-point subset of X is closed in X

(ii) Every compact Hausdorff space is regular.

(iii) Every second countable regular space is normal.

(iv) Every metric space is normal.

(v) If X is second countable, then every open cover admits a countable subcover.

(vi) Every secound countable space X contains a countable dense subset.

1.2 Lemma. (Urysohn) *If X is normal and $A, B \subseteq X$ are disjoint and closed, then there is a countinuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

1.3 Theorem. (Tietze Extension) *If X is normal and $f : A \to \mathbb{R}$ is continuous for some $A \subseteq X$ closed, then there exists a continuous map $F : X \to \mathbb{R}$ such that $F|_A = f$ and* $\sup_{a \in A} |f(a)| = \sup_{x \in X} |F(x)|$.

1.4 Theorem. (Urysohn's Metrization) *If X is second countable and regular, then X is metrizable.*

Definition. An **n-dimensional topological manifold** is a Hausdorff, second countable topological space M which is **locally homeomorphic** to $\mathbb{R}^n$, meaning for every $p \in M$, there exists an open set $U \subseteq M$ with $p \in U$ and an open set $V \subseteq \mathbb{R}^n$ and a homeomorphism $\phi : U \subseteq M \to V \subseteq \mathbb{R}^n$. Such a homomorphism ϕ is called a **(local) coordinate chart** or **chart** on M at p. The domain U of a chart $\phi : U \subseteq M \to \phi(U) \subseteq \mathbb{R}^n$ is called a (local) **coordinate neighbourhood** at p. Note that we can choose a set of charts

$$\mathcal{A} = \{\phi_k : U_k \subseteq M \to \phi_k(U_k) : k \in K\}$$

where K is any non-empty set such that $M = \bigcup_{k \in K} U_k$. Such a set of charts is called an **atlas** for M.

Definition. Two charts are called $\phi : U \to \phi(U)$ and $\psi : V \to \psi(V)$ are called **(smoothly) compatible** when either $U \cap V = \emptyset$ or $\phi^{-1} \circ \psi$ and $\psi \circ \phi^{-1}$ are smooth (meaning partial derivatives of all orders exist). We say that an atlas is **smooth** if every pair of charts is compatible.

Note that a smooth atlas $\mathcal{A}$ on M can be extend to a unique maximal smooth atlas $\mathcal{M}$ on M by adding to $\mathcal{A}$ every possible homeomorphism $\psi : U \subseteq M \to \phi(U) \subseteq \mathbb{R}^n$ which is compatible with all of the existing charts (since if ψ and χ are both compatible with every chart $\psi \in \mathcal{A}$, then ψ and χ will be compatible with each other). The maps $\psi \circ \phi^{-1}$ are called **transition maps** or **change of coordinate maps**. A maximal smooth atlas $\mathcal{M}$ on M is called a **smooth structure** on M.

Definition. An n−**dimensional smooth (or C^∞) manifold** is an n−dimensional topological manifold with a smooth structure.

Remark. A topological manifold can have different smooth structures. For example, take $\mathcal{A} = \{\phi\}$ where $\phi : \mathbb{R} \to \mathbb{R}$ is the identity map, and $\mathcal{B} = \{\psi\}$ where $\psi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism given by $\psi(x)x^3$, since $\sqrt[3]{x}$ is not smooth at the origin.

What if we tried $\mathcal{B} = \{\psi\}$ where $\psi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism which is not C^∞? This is trivially a smooth atlas.

Typically, a manifold is given with a standard smooth structure.

Remark. We can give a smooth manifold M an (at most countable) atlas of charts all of which are of one of the forms

- $\phi : U \subseteq M \to B(0,1)$
- $\phi : U \subseteq M \to (0,1)^n$
- $\phi : U \subseteq M \to \mathbb{R}^n$

Note that the maximal atlas $\mathcal{M}$ is determined from any subset $\mathcal{A} \subset \mathcal{M}$ such that the domains of the charts in $\mathcal{A}$ cover M.

Definition. Let M be an m−dimensional smooth manifold and N be an n−dimensional smooth manifold and let $f : M \to N$ be a function. Then we say f is smooth **smooth** at p when for some (hence for any) chart at ϕ on M at p and for some (hence any) chart ψ on N at $f(p)$, the map $\phi^{-1} \circ f \circ \psi$ is smooth at $x = \phi(p)$, and f is **smooth** if f is smooth at ever $p \in M$. We say that f is a **diffeomorphism** when f is invertible and both f and f^{-1} are smooth. We say that M and N are **diffeomorphic**, and write $M \cong N$, when there exists a diffeomorphism $f : M \to N$.

Remark. If is conceivable that a topological manifold M could be both of dimension n and of dimension m with $n \neq m$. To do this, we would need to have a homeomorphism from an open set in $\mathbb{R}^n$ to an open set in $\mathbb{R}^m$. In fact, this cannot happen by invariance of domain, proven using tools from algebraic topology.

When M is smooth, it is easy to see that this cannot happen. If $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ were smooth inverses, then the matrices $D(\psi \circ \phi^{-1})(\phi(p))$ and $D(\phi \circ \psi^{-1})(\psi(p))$ would be inverse matrices. But then a product of a matrix in $M_{m\times n}(\mathbb{R})$ and in $M_{n\times m}(\mathbb{R})$ cannot be inverses when $m \neq n$.

Remark. Manifolds are sometimes constructed using quotient constructions. These quotients can be given by polygons with pairs of edges identified up to orientation.

There are other kinds of manifolds (other than C^∞ manifolds); for example, one can define C^k manifolds, or analytic C^ω manifold has an atlas in which the transition maps are analytic.

Example.
1. $\mathbb{R}^n$ is a smooth n–dimensional manifold. It can be given an atlas consisting of 1 chart, the identity map.
2. Any n–dimensional vector space over $\mathbb{R}$ is a smooth n–dimensional manifold. It can be given an atlas with one chart. If $\{u_1, \dots, u_n\}$ is a basis for V, then one can define $\phi : V \to \mathbb{R}^n$ by $\phi(\sum t^i u_i) = (t^1, \dots, t^n) = t \in \mathbb{R}^n$.
3. Every open subset of a smooth n–dimensional manifold is also a smooth n–dimensional manifl.d
4. $M_{n\times m}(\mathbb{R})$ is an $n \cdot m$–dimensional manifold with pointwise $\mathbb{R}^{nm}$ structure.
5. $\{A \in M_{n\times m}(\mathbb{R}) : \operatorname{rank}(A) = \min\{n,m\}\}$ is a smooth manifold with one chart, since it is an open submanifold of $M_{n\times m}$. Suppose $n > m$; then take all $n \times n$ submatrices which have non-zero determinant (open by continuity of det), and maximal rank means that A is contained in one of these open subsets.
6. The disjoint union of countably many n–dimensional smooth manifolds.
7. The cartesian product of finitely many smooth manifolds is a smooth manifold. Let $\dim(M_k) = n_k$; the $\dim(M_1 \times \cdots \times M_\ell) = \sum_{k=1}^{\ell} n_k$. If $\phi_k : U_k \subseteq M_k \to \phi_k(U_k) \subseteq \mathbb{R}^{n_k}$ is a chart on M_k, then $\chi_k : \prod_{k=1}^{\ell} U_k \to \prod_{k=1}^{\ell} \mathbb{R}^{n_k}$ given by $\chi_k(p_1, \dots, p_\ell) = (\phi_1(p), \dots, \phi_\ell(p)$ is a chart in $M_1 \times \cdots \times M_\ell$.
8. One can show that $\mathbb{S}^n$ is a smooth n–dimensional manifold.

Remark. For $A \in M_{n\times m}(\mathbb{R})$, we denote the entry in the k^{th} row and ℓ^{th} column by A^k_ℓ.

Example. $\mathbb{S}^n$ is an example of an n–dimensional smooth manifold. It can, for example, be given a smooth atlas which contains $2(n+1)$ charts as follows. For $1 \le k \le n+1$, let

$$U_k = \{x \in \mathbb{S}^n : x^k > 0\}$$
$$\phi_k : U_k \to B(0,1) \subseteq \mathbb{R}^n$$
$$\phi_k(x) = (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^{n+1})$$
$$\phi_k^{-1}(t^1, \dots, t^n) = \left(t_1, \dots, t^{k-1}, \sqrt{1 - \sum (t^i)^2}, t^k, \dots, t^n\right)$$

and the corresponding opposite charts for $x^k < 0$. Note that $\mathbb{S}^n$ is a metric space. It has 2 standard metrics: eithre the one inherited from $\mathbb{R}^n$, or the arclength distance $d_s(U,v) = \cos^{-1}(u \cdot v)$.

We can also given $\mathbb{S}^n$ an atlas which only uses 2 charts, by stereographic projection from a north pole and a south pole.

This stereographic projection also shows that the rational points on the sphere are dense in $\mathbb{S}^n$, via the map

$$\phi(x) = \alpha\left(\frac{1}{1-x^{n+1}} right\right) = \left(\frac{x^1}{1-x^{n+1}}, \dots, \frac{x^n}{1-x^{n+1}}\right)$$

One can also find ϕ^{-1} and verify that they are both rational functions. In particular, $\phi^{-1}(\mathbb{Q}^n) \subseteq \mathbb{S}^n$ is dense.

Example. The projective space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ is commonly defined in at least 3 ways:

$$\begin{aligned}
\mathbb{P}^n &= \{\text{1-dimensional subspaces of } \mathbb{R}^{n+1}\} \\
\mathbb{P}^n &= \mathbb{R}^{n+1} \setminus \{0\} \big/ \mathbb{R}^\times = \{[x] : 0 \neq x \in \mathbb{R}^{n+1}\}, [x] = \{tx : t \in \mathbb{R}^\times\} \\
\mathbb{P}^n &= \mathbb{S}^n \big/ \pm 1
\end{aligned}$$

We can given $\mathbb{P}^n$ a smooth atlas with $n+1$ charts as follows: for $1 \leq k \leq n+1$, set

$$\begin{aligned}
&U_k = \{[x] \in \mathbb{P}^n : x^k \neq 0\} \\
&\phi_k : U_k \to \mathbb{R}^n, \phi_k([x]) = \left(\frac{x^1}{x^k}, \dots, \frac{x^{k-1}}{x^{k-1}}, \frac{x^{k+1}}{x^{k+1}}, \dots, x^{n+1}x^k\right)
\end{aligned}$$

with $\phi_k^{-1}(t_1, \dots, t^n) = [(t_1, \dots, t^{k-1}, 1, t^k, \dots, t^n)]$.

Examples of Smooth Maps

- The inclusion $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ given by $f(x) = x$
- The quotient map $f : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$
- The exponential map $f : \mathbb{R} \to \mathbb{S}^1$ given by $f(t) = e^{i2\pi t}$, or more generally $f : \mathbb{R}^n \to \mathbb{T}^n$ given by $f(t^1, \dots, t^n) = (e^{2\pi i t^1}, \dots, e^{2\pi i t^n})$
- The determinant map $f : M_n(\mathbb{R}) \to \mathbb{R}$ given by $f(A) = \det(A)$ is smooth
- For $A \in M_n(\mathbb{R})$, left and right multiplication by A, the transpose map, and the inverse map $f(A) = A^{-1}$ are smooth.

Partitions of Unity

1.5 Lemma. *Every open cover of a manifold has an (at most) countable subcover.*

Proof Let $\mathcal{S}$ be any open cover of M, and let $\mathcal{B}$ be a countable basis for the topology on M. For each $p \in M$, choose $U_p \in \mathcal{S}$ with $p \in U_p$, then choose $B_p \in \mathcal{B}$ with $p \in B_p \subseteq U_p$. Then $\{B_p : p \in M\} \subseteq B$ is an open cover of M, and it is a subset of $\mathcal{B}$, so it is (at most) countable; but then $\{U_p : p \in M\}$ gives an at most countable subcover of $\mathcal{S}$. ∎

As a result, every manifold has a countable basis $\mathcal{B}$ such that for each $B \in \mathcal{B}$, there is a chart $\phi : U \to \phi(U)$ on M with $\phi(U) = B(0,2)$ and $\phi(B) = B(0,1)$.

1.6 Lemma. *Let M be a manifold, and let $\mathcal{S}$ be any open cover of M. Then there exists an at most countable open cover $\mathcal{B}$ of M such that*

1. *for each $B \in \mathcal{B}$ there is a chart $\phi_B : C_B \to \phi_B(C_B) = B(0,1)$ with $B \subseteq C_B \subseteq U_B \subseteq S$ for some $U_B \in S$ and $\phi_B(B) = B(0,1)$.*

2. *$\{C_B : B \in \mathcal{B}\}$ is locally finite, meaning that every point in M has an open neighbourhood which only intersects with finitely many of the sets C_B, $B \in \mathcal{B}$ (and hence also the sets $\overline{B}$, $B \in \mathcal{B}$).*

Proof Choose a countable set $\mathcal{V} = \{V_1, V_2, \dots\}$ of regular coordinate balls which cover M with charts $\phi_i : W_i \to \phi_i(W_i) = B(0,2)$ such that $V_i = \phi_i^{-1}(B(0,1))$. We use the sets V_i to construct a strongly ascending chain of compact sets K_i in M with $K_i \subseteq H_{i+1}^{-1}$ for each i, and $M = \bigcup_{i=1}^{\infty} K_i$ as follows:

- Let $K_i = \overline{V_1}$; since K_1 is compact, we can choose $\ell_1 \in \mathbb{N}$ so that $K_1 \subseteq V_1 \cup \cdots \cup V_{\ell_1}$.
- Then we let $K_2 = \overline{V_1 \cup \cdots \cup V_{\ell_1}}$. Since K_2 is compact, we can choose $\ell_2 > \ell_1$ so that $K_2 \subseteq V_1 \cup \cdots \cup V_{\ell_2}$, and set $K_3 = \overline{V_1 \cup \cdots V_{\ell_2}}$.

Repeat the above process to obtain $K_1 \subseteq K_2^\circ \subseteq K_2 \subseteq K_3^\circ \subseteq \cdots$ with $\bigcup_{i=1}^{k} K_i = M$. For each $m \in \mathbb{N}$, note that $K_{m+1} \setminus K_m^\circ$ is compact and contained in the open set $K_{m+2} \setminus K_{m-1}$ (with $K_0 = \emptyset$). For each $p \in K_{m+1} \setminus K_m^\circ$, choose $U_p \in \mathcal{S}$ with $p \in U_p$ and then choose a regular coordinate ball B_p and a chart $\phi_p : C_p \subseteq M \to \phi_p(C_p) = B(0,2) \subseteq \mathbb{R}^n$ with $\phi_p(B_p) = B(0,1)$ and $C_p \subseteq U_p \cap (K_{m+2}^\circ \setminus K_{m-1})$. The coordinate balls B_p, $p \in K_{m+1} \setminus K_m^\circ$ cover the compact set $K_{m+1} \setminus K_m^\circ$, so we can choose a *finite* set $\mathcal{B}_m$ of such regular coordinate balls B_p so that $K_{m+1} \setminus K_m^\circ \subseteq \cup \mathcal{B}_m \subseteq K_{m+2}^\circ \setminus K_{m-1}$.

Now, the set $\mathcal{B} = \bigcup_{m=1}^{\infty} \mathcal{B}_m$ is a countable set of such regular coordinate balls. Note that for each $B \in \mathcal{B}$, we have chart $\phi_B : C_B \to \phi_B(C_B) = B(0,2)$ and the set $\{C_B : B \in \mathcal{B}\}$ is locally finite since every point in M is contained in one of the sets $K_{m+2}^\circ \setminus K_{m-1}$ and each of these sets only intersects with the coordinate balls from the finite sets $\mathcal{B}_l$ with $m-2 \le l \le m+2$.■

1.7 Theorem. (Partitions of Unity) *Let M be a smooth manifold, and let $\mathcal{S}$ be any open cover of M. There exists a set $\{\psi_u : u \in \mathcal{S}\}$ of smooth maps $\psi_u : M \to \mathbb{R}$ such that*

1. *$\psi_u(M) \subseteq [0,1]$ for each $u \in \mathcal{S}$*
2. *$\operatorname{supp}(\psi_u) \subseteq U$ for ech $U \in \mathcal{S}$*
3. *$\{\operatorname{supp}(\psi_u) : u \in \mathcal{S}\}$ is locally finite: every point in M contains an open neighbourhood whicl only intersects finitely many of the sets $\operatorname{supp}(\psi_n)$, $u \in \mathcal{S}$*
4. *$\sum_{u \in \mathcal{S}} \psi_u = 1$*

Such a set of functions $\{\psi_u : u \in \mathcal{S}\}$ is called a (smooth) **partition of unity** on M for $\mathcal{S}$ (or **subordinate** to $\mathcal{S}$).

Proof Let $\mathcal{B}$ be a countable set of regular coordinate balls as in the previous lemma. Recall that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} e^{1/t} & : t < 0 \\ 0 & : t \ge 0 \end{cases}$$

is smooth, so the function $g : \mathbb{R}^n \to \mathbb{R}$ given by $g(x) = f(|x|^2 - 1)$ is smooth with $g(x) > 0$ for $|x| < 1$ and $g(x) = 0$ for $|x| \ge 1$. For each $B \in \mathcal{B}$, we define a smooth bump function $\sigma_B : M \to \mathbb{R}$ by

$$\sigma_B(p) = \begin{cases} g(\phi_B(p)) & : p \in B \\ 0 & : p \notin B \end{cases}$$

where $\phi_B : C_B \subseteq M \to \phi_B(C_B) = B(9,2)$ with $\phi_B(B) = B(0,1)$ as in the previous lemma. Note that $\sigma(B)$ is smooth with $\sigma_B(p) > 0$ for $p \in B$ and $\sigma_B(p) = 0$ for $p \notin B$. Now for each $B \in \mathcal{B}$, define $\tau_B' : M \to \mathbb{R}$ by

$$\tau_B = \frac{\sigma_B}{c \in \mathcal{B}\sigma_c}$$

Note that $\sum_{c\in\mathcal{B}} \sigma_c$ is well-defined by the local finiteness of $\mathcal{B}$ and $\sum_{c\in\mathcal{B}} \sigma_c(p) > 0$. Furthermore, note that $\tau_B(p) > 0$ for all $p \in B$, and $\tau_B(p) = 0$ for all $p \notin B$, and $\sum_{B\in\mathcal{B}} \tau_B = 1$. Then define $\rho_V : M \to \mathbb{R}$ by $\rho_V = \sum_{B\in\mathcal{B}_V} \tau_B$. ■

2 Immersions, Embedding, Submanifolds

2.1 Theorem. (Inverse Function Theorem) *Let $U \subseteq \mathbb{R}^n$ be open, $p \in U$, and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be smooth and suppose $Df(p)$ is invertible. Then f is a local diffeomorphism.*

2.2 Corollary. *Let $n < m$ and $U \subseteq \mathbb{R}^n$ be open, and let $p \in U$, and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be smooth and suppose $Df(p)$ has rank n. Then the range of f is locally equal to the graph of a smooth function. Such a map f is called a local* ***immersion*** *at p.*

Proof Since $Df(p)$ is an $m \times n$ matrix of rank n, some n rows of $Df(p)$ form an invertible submatrix. Reorder the variables in $\mathbb{R}^m$ (if necessary) so that the top n rows form an invertible matrix. Write elements in $U \subseteq \mathbb{R}^n$ as t and write elements of $\mathbb{R}^m$ as (x,y). Also write $(x,y) = f(t) = (u(t), v(t))$ so

$$Df = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix}$$

with $\frac{\partial u}{\partial t}(p)$ invertible. Then by the inverse function theorem, $u(t)$ is a local diffeomorphism. Say $u : U_0 \subseteq U \to V_0 \subseteq \mathbb{R}^n$ is the diffeomorphism, and let $g : V_0 \to U_0$ be its inverse. Then the range of f is locally equal to the graph of the function $y = v(g(x)) =: h(x)$. If $(x,y) \in \Gamma(f)$ with $(x,y) = f(t) = (u(t), v(t))$, then since $x = u(t)$ we have $t = g(x)$ so $y = v(t) = v(g(x)) = k(x)$. If $(x,y) \in \Gamma(k)$, then $y = k(x) = v(g(x))$ and we can choose $t = g(x)$ to get $x = u(t)$ and $y = v(g(x)) = v(t)$ so that $(x,y) = (u(x), v(t)) = f(t)$. ■

2.3 Theorem. (Implicit Function) *Let $n < m$, $U \subseteq \mathbb{R}^m$ be open, $p \in U$, and $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be smooth. Suppose $Df(p)$ has rank n and let $q = f(p)$. Then the level set $f^{-1}(q)$ is locally equal to a graph of a smooth function.*

2.4 Theorem. *Let $U \subseteq \mathbb{R}^n$ be open with $p \in U$, let $f : U \to \mathbb{R}^m$ be smooth with $f(p) = q$; and suppose that Df has constant rank r in U. Then the level set (or fibre) $f^{-1}(q)$ is locally equal to the graph of a smooth function (with $n - r$ independent variables and r dependent variables).*

Proof Since Df is an $m \times n$ matrix of rank r, there is some $r \times r$ submatrix of $Df(p)$ which is invertible; without loss of generality, it is the upper left submatrix. Write elements in $\mathbb{R}^n$ as (x,y) with $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^{n-r}$ and write elements in $\mathbb{R}^m$ as (u,v) with $u \in \mathbb{R}^r$ and $v \in \mathbb{R}^{m-r}$, with say $p = (a,b)$ and $q = f(p) = (c,d)$. Then we have $(u,v) = f(x,y) = \big(u(x,y), v(x,y)\big)$ so that

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

with $\frac{\partial u}{\partial x}(p) = \frac{\partial u}{\partial x}(a,b)$ being an invertible $r \times r$ matrix. Define $F : U \subseteq \mathbb{R}^m \to \mathbb{R}^m$ by $F(x,y) = (u(x,y), y)$. Then

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ 0 & I \end{pmatrix}$$

so that $DF(p)$ is inertible. By the IVT, F is a local diffeomorphism, say $F : U_0 \subseteq U \subseteq \mathbb{R}^n \to V_0 \subseteq \mathbb{R}^n$ is a diffeomorphism with U_0 an open rectangular box. Let $G : V_0 \to U_0$ denote the smooth inverse of F. Note that G is of the form $G(u,y) = (g(u,y), y)$ for some smooth function $g : V_0 \to \mathbb{R}^r$. We claim that $f^{-1}(q) = f^{-1}(c,d)$ is locally equal to the graph of $x = g(c,y)$. FIrst, note that

$$(u,y) = F(G(u,y)) = F(g(u,y),y) = \Big(u(g(u,y),y),y\Big)$$

so that, in particular, $u(g(u,y),y) = u$ and so

$$f(G(u,y)) = (u(g(u,y),y), v(g(u,y),y)) = (u, h(u,y))$$

where $h(u,y) = v(g(u,y),y)$. Thus

$$Df(x,y) \cdot DG(u,y) = D(f \circ G)(u,y) = \begin{pmatrix} I & 0 \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial y} \end{pmatrix}$$

Since Df has constant rank r and DG is invertible, the matrix on the right is of rank r for all $(u,v) \in V_0$. Thus it follows that $\frac{\partial h}{\partial y} = 0$ for all u, b, so that $h(u,y)$ is independent of y and $h(u,y) = h(u,b)$ for all y; let $k(u) = h(u,b)$. Let us calculate $k(c)$. We have

$$\begin{aligned} f(a,b) = (c,d) &\implies (u(a,b), v(a,b)) = (c,d) \\ &\implies u(a,b) = c \\ &\implies F(a,b) = (u(a,b),b) = (c,b) \\ &\implies (a,b) = G(c,b) \\ &\implies (c,d) = f(a,b) = f(G(c,b)) = (c, h(c,b)) = (c, k(c)) \\ &\implies k(c) = d \end{aligned}$$

Finally, let us show that $f^{-1}(c,d)$ is (locally) the graph of $x = g(c,y)$. We have

$$\begin{aligned} (x,y) = f^{-1}(c,d) &\implies f(x,y) = (c,d) \\ &\implies u(x,y) = c \text{ and } v(x,y) = d \\ &\implies F(x,y) = (u(x,y),y) = (c,y) \\ &\implies (x,y) = G(c,y) = (g(c,y),y) \\ &\implies x = g(c,y) \end{aligned}$$

We thus have

$$\begin{aligned} x = g(c,y) &\implies G(c,y) = (g(c,y),y) = (x,y) \\ &\implies f(x,y) = f(G(x,y)) = (c, h(c,y)) = (c, k(c)) = (c,d) \end{aligned}$$

as required. ■

Definition. When N and M are smooth manifolds and $f : N \to M$ is a smooth map, we say that f has **rank r** at $p \in N$ when for some (hence for every) chart ϕ on N at p and for some (hence every) chart ψ on M at $f(p)$, the matrix $D(\psi f \phi^{-1})(\phi(p))$ has rank r.

2.5 Corollary. *Let N and M be smooth manifolds, with $p \in N$. Let $f : N \to M$ be smooth with $f(p) = q \in M$. Suppose f has constant rank r in an open neighbourhood of p. Then there exists a chart ϕ on N at p and a chart ψ on M at $q = f(p)$ such that $\phi(p) = 0$ and $\psi(q) = 0$ and*

$$(\psi \circ f \circ \phi^{-1})(x^1, \ldots, x^r, \ldots, x^n) = (x^1, \ldots, x^r, 0, \ldots, 0)$$

where $n = \dim(N)$ and $m = \dim(M)$.

Proof Choose any chart ϕ_0 on N at p and any chart ψ_0 on M at q with $\phi_0(p) = 0$ and $\psi_0(q) = 0$. Then $D(\psi_0 f \phi_0^{-1})$ has constant rank r near 0. Let ϕ_1 and ψ_1 be linear permutation maps so that the upper left $r \times r$ submatrix of $D(\psi, \psi_0, f \phi_0^{-1} \phi_1^{-1})(0)$. Say $f_1 = \psi_1 \psi_0 f \phi_0^{-1} \phi_1^{-1}$. Let F, G, f_1 be as in the proof of the rank theorem (for the function f_1). Let us verify that for the charts $\phi = F \phi_1 \phi_0$ and $\psi = H \psi_1 \psi_0$ where $H(u,v) = (u, v - k(u))$ we have $(\psi f \phi^{-1})(u, y) = (u, 0)$.