REPLACE

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Contents

I. REPLACE

1. For $a, b, k \in \mathbb{N}$,

$$\binom{a+b}{k} = \sum_{j=1}^{k} \binom{a}{j} \cdot \binom{b}{k-j} \tag{0.1}$$

We prove this with a bijection:

$$\mathcal{B}(a+b, k) \leftharpoondown \bigcup_{j=0}^{k} \mathcal{B}(a, j) \times \mathcal{B}(b, k-j)$$

given by $S \mapsto (S \cap \{1, \ldots, a\}, (S \cap \{a+1, \ldots, a+b\})^{(-a)}$ and $(P, Q) \mapsto P \cup Q^{(a)}$, where $\mathcal{B}(n, i)$ is the set of $i-$element subsets of $\{1, 2, \ldots, n\}$ and for $C \subseteq \mathbb{Z}$ and $q \in \mathbb{Z}$, $C^{(q)} = \{c+q : c \in C\}$. Note that the equation in fact gives the polynomial identity

$$\binom{x+y}{k} = \sum_{j=0}^{k} \binom{x}{j}\binom{y}{k-j}$$

in $\mathbb{Q}[x, y]$. We denote the falling factorial $(x)_i = x(x-1)(x-2)\cdots(x-i+1)$, which has degree i for each $i \in \mathbb{N}$. In particular, $(x)_i = i!\binom{x}{i}$, so multiplying our identity by $k!$, we get

$$(x+y)_k = \sum_{j=0}^{k} \binom{k}{j}(x)_j(y)_{k-j}$$

Compare this with the standard binomial theorem

$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}$$

These are called sequences of binomial type.

2. Here's another identity. For $n \geq 0$ and $s, t \geq 1$,

$$\binom{n+s+t-1}{s+t-1} = \sum_{k=0}^{n} \binom{k+s-1}{s-1}\binom{n-k+t-1}{t-1}$$

Let $\mathcal{M}(m, r)$ denote a multiset of size m with elements of r types, so that $|\mathcal{M}(m, r)| = \binom{m+r-1}{r-1}$. Let's define a bijection

$$\mathcal{M}(n, s+t) \rightleftharpoons \bigcup_{k=1}^{n} \mathcal{M}(k, s) \times \mathcal{M}(n-k, t) \tag{0.2}$$

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$\mu = (m_1, \ldots, m_{s+t}) \mapsto ((m_1, \ldots, m_s), (m_{s+1}, \ldots, m_{s+t}))$ and $(\nu, \theta) \mapsto \nu\theta$. Note that if f, g are polynomials of degree d and e respectively, then $\sum_{k=0}^n f(k)g(n-k)$ is a polynomial in n of degree $d + e - 1$.

Is there some way to understand (0.2)? It is unclear, with our known techniques, that this corresponds to a polynomial identity since there is a variable n in the exponent. However, we can use generating functions. Define

$$\sum_{n=0}^{\infty} \binom{n+s+t-1}{s+t-1} z^n = \sum_{n=0}^{\infty} |\mathcal{M}(n, s+t)| z^n = \sum_{(m_1, \ldots, m_{s+t})} z^{m_1 + \cdots + m_{s+t}}$$

$$= \left(\sum_{m=0}^{\infty} z^m \right)^{s+t}$$

$$= \frac{1}{(1-z)^{s+t}} = \frac{1}{(1-z)^s} \frac{1}{(1-z)^t}$$

$$= \sum_{k=0}^{\infty} \binom{k+s-1}{s-1} z^k \sum_{\ell=0}^{\infty} \binom{\ell+t-1}{t-1} z^\ell$$

$$= \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} \binom{k+s-1}{s-1} \binom{n-k+t-1}{t-1} \right)$$

Similarly, (0.1) is equivalent to saying $(1+z)^{a+b} = (1+z)^a (1+z)^b$. Note that $(1+z)^n = \sum_{k=0}^n \binom{n}{k} z^k = \sum_{k=0}^{\infty} \binom{n}{k} z^k$ for $n \in \mathbb{N}$.

Can we substitute $\frac{1}{(1-q)^t} = (1+z)^n$ where $z = -q$ and $n = -t$?

3. Consider

$$(x_1 + x_2)^n = \sum_{i=0}^n \binom{n}{i} x_1^i x_2^{n-i}$$

and

$$(x_1 + x_2)^n = \sum_{f: N_n \to \{1,2\}} \prod_{j=1}^n x_{f(j)}$$

More generally, we can consider

$$(x_1 + \cdots + x_k)^n = \sum_{f: N_n \to N_k} \prod_{j \in N_n} x_{f(j)}$$

If we set all $x_1 = \cdots = x_k = 1$, then k^n gives the number of functions from N_n to N_k. If we set $x_i = q^i$ for all $i \in N_k$, then we get

$$\left(\frac{q - q^{k+1}}{1 - q} \right)^n = (q + q^2 + \cdots + q^k)^n = \sum_{f: N_n \to N_k} q^{f(1) + \cdots + f(k)}$$

Collect all the terms in $(x_1 + \cdots + x_k)^n$ that produce the same monomial. Given a multiset μ with $m_1 + \cdots + m_k = n$, write $x_1^{m_1} \cdots x_k^{m_k} = \underline{x}^\mu$. Then

$$(x_1 + \cdots + x_k)^n = \frac{n!}{m_1! \cdots m_k!} \underline{x}^\mu = \sum_{\mu \in \mathcal{M}(n,t)} \binom{n}{\mu} \underline{x}^\mu$$

4. How can we interpret

$$P_n(q) = \prod_{i=1}^{n}(1 + q + q^2 + \cdots + q^{i-1})$$

In general, if we set $q = 1$, we see that $P_n(1) = n!$. We might hope that there is some weight function on permutations $w : \mathcal{S}_n \to \mathbb{N}$ such that $P_n(q) = \sum_{\sigma \in \mathcal{S}_n} q^{w(\sigma)}$. Recall the bijection $I_n : \mathcal{S}_n \to \mathcal{Q}_n$ from chapter 1. Let's find some weight function $v : \mathcal{Q}_n \to \mathbb{N}$ such that $\sum_{\rho \in \mathcal{Q}_n} x^{v(\rho)} = P_n(q)$, then "pull back" the definition of $v : \mathcal{Q}_n \to \mathbb{N}$ to get a definition for $\omega : \mathcal{S}_n \to \mathbb{N}$. Note that $\sum_{h \in N_r} q^{h-1} = 1 + q + \cdots + q^{r-1}$. Thus

$$\sum_{\rho=(h_1,\ldots,h_n)\in\mathcal{Q}_n} q^{(h_1-1)+(h_2-1)+\cdots+(h_n-1)} = \prod_{i=1}^{n}(1 + q + \cdots + q^{i-1}) = P_n(q)$$

so we can define $v(\rho) = |\rho| - n$ and $\sum_{q \in \mathcal{Q}_n} q^{|\rho|-n} = P_n(q)$.

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