

Functional Analysis

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Contents

| | | |
|------------------|---|---|
| Chapter I | Fundamentals of Functional Analysis | |
| 1 | Basic Elements of Functional Analysis | 1 |

I. Fundamentals of Functional Analysis

1 BASIC ELEMENTS OF FUNCTIONAL ANALYSIS

Throughout, we denote by \mathbb{F} either the field \mathbb{R} or the field \mathbb{C} .

BANACH SPACES

Definition. Let X be a vector space over \mathbb{F} . A **norm** is a functional $\|\cdot\| : X \rightarrow \mathbb{R}$ such that it is

- (*non-negative*) $\|x\| \geq 0$ for any $x \in X$
- (*non-degenerate*) $\|x\| = 0$ if and only if $x = 0$
- (*subadditivity*) $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in X$
- (*$|\cdot|$ -homogeneity*) $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{F}$, $x \in X$.

We call the pair $(X, \|\cdot\|)$ a **normed vector space**. Furthermore, we say that $(X, \|\cdot\|)$ is a **Banach space** provided that X is complete with respect to the metric $\rho(x, y) = \|x - y\|$.

Example. (i) $(\mathbb{F}, |\cdot|)$ is a Banach space.

(ii) $(\mathbb{F}^n, \|\cdot\|_p)$, $x = (x_j)_{j=1}^n$,

$$\|x\|_p = \begin{cases} \left(\sum_{j=1}^n |x_j|^p \right)^{1/p} & 1 \leq p < \infty \\ \max_{j=1, \dots, n} |x_j| & p = \infty \end{cases}$$

(iii) Consider the space

$$L_p^{\mathbb{F}} = \left\{ f : [0, 1] \rightarrow \mathbb{F} \mid f \text{ is Lebesgue measurable, } \left(\int_0^1 |f|^p \right)^{1/p} < \infty \right\} \Big/ \sim_{\text{a.e.}}$$

where $1 \leq p < \infty$.

(iv) $L_{\infty}^{\mathbb{F}}[0, 1]$, $\|f\|_{\infty} = \text{ess sup}_{t \in [0, 1]} |f(t)|$.

(v) Let (X, d) be a metric space. Then

$$C_b^{\mathbb{F}}(X) = \{ f : X \rightarrow \mathbb{F} \mid f \text{ is continuous and bounded } \}, \quad \|f\|_{\infty} = \sup_{x \in X} |f(x)|$$

is a Banach space.

Here is a more interesting example:

Example. Let (X, d) be a metric space. We define the space of Lipschitz functions

$$\text{Lip}^{\mathbb{F}}(X, d) = \left\{ f : X \rightarrow \mathbb{F} \left| f \text{ is bounded, } L(f) = \sup_{\substack{x, y \in X \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x, y)} < \infty \right. \right\}$$

We note that for $f : X \rightarrow \mathbb{F}$ that

$$f \in \text{Lip}^{\mathbb{F}}(X, d) \Leftrightarrow \text{there is } L \geq 0 \text{ s.t. } |f(x) - f(y)| \leq Ld(x, y) \text{ for all } x, y \in X \quad (1.1)$$

It is easy to verify that $L(f) = \min\{L \geq 0 : (1.1) \text{ holds for } f\}$. It is an easy exercise to see that $\text{Lip}^{\mathbb{F}}$ is a vector space, and that $L : \text{Lip}^{\mathbb{F}}(X, d) \rightarrow \mathbb{R}$ is a **semi-norm** (non-negative, subadditive, $|\cdot|$ -homogeneous). However, we do not have non-degeneracy (for example, constants are taken to 0). We define the Lipschitz norm

$$\|f\|_{\text{Lip}} = \|f\|_{\infty} + L(f)$$

1.1 Proposition. $(\text{Lip}^{\mathbb{F}}(X, d), \|\cdot\|_{\text{Lip}})$ is a Banach space.

PROOF Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $(\text{Lip}^{\mathbb{F}}(X, d), \|\cdot\|_{\text{Lip}})$. Since $\|\cdot\|_{\infty} \leq \|\cdot\|_{\text{Lip}}$ on $\text{Lip}^{\mathbb{F}}(X, d)$, we see that $(f_n)_{n=1}^{\infty}$ is uniformly Cauchy (and bounded), and hence there is $f = \lim_{n \rightarrow \infty} f_n$ in $C_b^{\mathbb{F}}(X)$, where the limit is taken with respect to $\|\cdot\|_{\infty}$, since $(C_b^{\mathbb{F}}(X), \|\cdot\|_{\infty})$ is a Banach space. If $x, y \in X$, then

$$\begin{aligned} |f(x) - f(y)| &= \lim_{n \rightarrow \infty} |f_n(x) - f_n(y)| \leq \sup_{n \in \mathbb{N}} |f_n(x) - f_n(y)| \\ &\leq \sup_{n \in \mathbb{N}} L(f_n) d(x, y) \leq \sup_{n \in \mathbb{N}} \|f_n\|_{\text{Lip}} d(x, y) \end{aligned}$$

Since Cauchy sequences are bounded, we see that $|f(x) - f(y)| \leq Ld(x, y)$, where $L = \sup_{n \in \mathbb{N}} \|f_n\|_{\text{Lip}} < \infty$. Thus by (1.1), $f \in \text{Lip}^{\mathbb{F}}(X, d)$. Exercise: one may verify that $\|f - f_n\|_{\text{Lip}} \rightarrow 0$. \blacksquare

Another collection of basic examples are given by the sequence spaces. We can define

$$\ell_1^{\mathbb{F}} = \left\{ x = (x_j)_{j=1}^{\infty} \in \mathbb{F}^{\mathbb{N}} \left| \|x\|_1 = \sum_{j=1}^{\infty} |x_j| < \infty \right. \right\}$$

It is easy to see that $(\ell_1, \|\cdot\|_1)$ is a normed vector space.

For $1 < p < \infty$, and write

$$\ell_p^{\mathbb{F}} = \left\{ x \in \mathbb{F}^{\mathbb{N}} \left| \|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} < \infty \right. \right\}$$

Note that $0 \in \ell_p$, $\alpha \in \mathbb{F}$, $\alpha x \in \ell_p$ if $x \in \ell_p$. Let $q = p/(p-1)$ so that $1/p + 1/q = 1$. Then q is called the **conjugate index**. We have

1.2 Proposition. (Young's Inequality) If $a, b \geq 0$ in \mathbb{R} , then $ab \leq a^p/p + b^q/q$, with equality only if $a^p = b^q$.

and

1.3 Proposition. (Hölder's Inequality) If $x \in \ell_p$ and $y \in \ell_q$, then $xy = (x_i y_i)_{i=1}^\infty \in \ell_1$, with

$$\left| \sum_{i=1}^\infty x_i y_i \right| \leq \|x\|_p \|y\|_q$$

with equality exactly when $\text{sgn}(x_i y_i) = \text{sgn}(x_k y_k)$ for all $j, k \in \mathbb{N}$ where $x_i y_i \neq 0 \neq x_k y_k$, and $|x|^p = (|x_j|^p)_{j=1}^\infty$ and $|y|^q$ are linearly dependent in ℓ_1 .

and finally

1.4 Proposition. (Minkowski's Inequality) If $x, y \in \ell_p$, then $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ with equality exactly when one of x or y is a non-negative scalar combination of the other.

REVIEW OF TOPOLOGY

Let X denote a non-empty set, and $\mathcal{P}(X)$ denote the power set of X .

Definition. A **topology** on a set X is a set τ of subsets of X such that

- (i) $\emptyset, X \in \tau$
- (ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
- (iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \leq i \leq n$, then $\bigcap_{i=1}^n U_i \in \tau$.

The sets $U \in \tau$ are called the **open sets** in X , and sets of the form $X \setminus U$ for some open set U are called the **closed sets** in X . The pair (X, τ) is called a **topological space**.

The metric topology on a metric space (X, d) is the topology

$$\tau_d = \{ U \subseteq X \mid \text{for each } x_0 \in U, \text{ there is } \delta = \delta(x_0) \text{ s.t. } B_\delta(x_0) \subseteq U \}$$

Example. (i) Given two metrics d, ρ on X , we say that $d \sim \rho$ if and only if there are $c, C > 0$ such that

$$cd(x, y) \leq \rho(x, y) \leq Cd(x, y) \text{ for any } x, y \in X$$

Note that $d \sim \rho$ implies that $\tau_d = \tau_\rho$, but the reverse implication is not true. An example of this are the metrics on $X = \mathbb{R}$ given by $d(x, y)$ and $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$. Then $d \sim \rho$ but $\tau_d = \tau_\rho$.

(ii) "Sorgenfrey line" Set $X = \mathbb{R}$, and consider

$$\sigma = \{ V \subseteq \mathbb{R} \mid \text{for any } s \in V, \text{ there is } \delta = \delta(s) > 0 \text{ s.t. } [s, s + \delta) \subseteq V \}$$

It is an exercise to verify that $\tau_{|\cdot|} \subsetneq \sigma$. We say that σ is **finer** than $\tau_{|\cdot|}$.

(iii) Relative topology: let (X, τ) be a topological space, and $\emptyset \neq A \subseteq X$. Then we can define a topology $\tau|_A = \{U \cap A : U \in \tau\}$.

Definition. Let (X, τ) and (Y, σ) be topological spaces, and $f : X \rightarrow Y$. We say that f is $(\tau - \sigma)$ -**continuous** at x_0 in X if,

- given $V \in \sigma$ such that $f(x_0) \in V$, then there exists $U \in \tau$ such that $x_0 \in U$ and $f(U) \subseteq V$.

We say that f is $(\tau - \sigma)$ -continuous if it is continuous at each x_0 in X .

SPACE OF BOUNDED CONTINUOUS FUNCTIONS INTO A NORMED SPACE

Let $(Y, \|\cdot\|)$ denote a normed space. We let $\tau_{\|\cdot\|}$ denote the topology given by the metric $\rho(x, y) = \|x - y\|$. Let (X, τ) denote any topological space. Then we write

$$C_b^Y(X) = \left\{ f : X \rightarrow Y \mid f \text{ is bounded and } \tau - \tau_{\|\cdot\|} \text{-continuous} \right\}$$

With pointwise operations, we see that $C_b^Y(X)$ is a vector space. We also define for $f \in C_b^Y(X)$, $\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}$, making $(C_b^Y(X), \|\cdot\|_\infty)$ a normed vector space.

1.5 Theorem. *If $(Y, \|\cdot\|)$ is a Banach space, then $(C_b^Y(X), \|\cdot\|_\infty)$ is a Banach space.*

PROOF Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $(C_b^Y(X), \|\cdot\|_\infty)$. Then for any $x \in X$, we have that $(f_n(x))_{n=1}^\infty$ is Cauchy in $(Y, \|\cdot\|)$ since $\|f_n(x) - f_m(x)\| \leq \|f_n - f_m\|_\infty$, and hence admits a limit $f(x)$. In particular, $x \mapsto f(x)$ defines a function from X to Y . We shall fix $x_0 \in X$ and show that f is continuous at x_0 . Given $\epsilon > 0$, we let

- n_1 be so $n, m \geq n_1$ so that $\|f_n - f_m\|_\infty < \epsilon/4$.
- n_2 be so $n \geq n_2$ so that $\|f_n(x_0) - f(x_0)\| < \epsilon/4$.
- $N = \max\{n_1, n_2\}$.
- $U \in \tau$, $x_0 \in U$ such that $f_N(U) \subseteq B_{\epsilon/4}(f(x_0)) \subset Y$.

Then for $x \in U$, we let n_x be so $n_x \geq n_1$ and $n \geq n_x$, so that $\|f_n(x) - f(x)\| < \epsilon/4$. We then have

$$\begin{aligned} \|f(x) - f(x_0)\| &\leq \|f(x) - f_{n_x}(x)\| + \|f_{n_x}(x) - f_N(x)\| + \|f_N(x) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| \\ &< \frac{\epsilon}{4} + \|f_{n_x} - f_N\|_\infty + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon \end{aligned}$$

in other words that $f(U) \subseteq B_\epsilon(f(x_0))$.

Now let us check that $\|f\|_\infty < \infty$. Since $|\|f_n\|_\infty - \|f_m\|_\infty| \leq \|f_n - f_m\|_\infty$, so $(\|f_n\|_\infty)_{n=1}^\infty \subseteq \mathbb{R}$ is Cauchy, hence bounded. If $x \in X$, then

$$\|f(x)\| = \lim_{n \rightarrow \infty} \|f_n(x)\| \leq \sup_{n \in \mathbb{N}} \|f_n(x)\| \leq \sup_{n \in \mathbb{N}} \|f_n\|_\infty < \infty$$

so $\|f\|_\infty = \sup_{x \in X} \|f(x)\| < \infty$.

Notice that if ϵ, n_1 are as above, and further x_0, N are as above, we have for $n \geq n_1$

$$\|f_n(x_0) - f(x_0)\| \leq \|f_n(x_0) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| < \frac{\epsilon}{2}$$

so $\|f_n - f\|_\infty = \sup_{x_0 \in X} \|f_n(x_0) - f(x_0)\| \leq \epsilon/2 < \epsilon$. This is uniform since n_1 is chosen uniformly in X . ■

1.6 Corollary. $(C_b^{\mathbb{F}}(X), \|\cdot\|_\infty)$ is a Banach space.