Representation Theory of Finite Groups

Alex Rutar*
University of Waterloo

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*arutar@uwaterloo.ca
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I. Introduction

Let G be a finite group of order n, and write $G = \{g_1, \ldots, g_n\}$. Fix $g \in G$; then $gg_i = gg_j$ if and only if $i = j$. Thus there exists some $\sigma_g \in S_i$ such that $gg_i = g_{\sigma_g(i)}$ for all $i \in \{1, 2, \ldots, n\}$. In particular, $\phi : G \to S_n$ by $\phi(g) = \sigma_g$ is an embedding (injective group homomorphism). This observation is usually referred to as Cayley's Theorem.

Now let V be an n–dimensional complex vector space. We then denote $\mathrm{GL}(V)$ as the group of invertible linear operators $T : V \to V$. Now define $\psi : S_n \to \mathrm{GL}_n(V)$ by $\psi(\sigma) = T_\sigma$ where if $\{b_1, \ldots, b_n\}$ is a basis for V and $T_\sigma(b_i) = b_{\sigma(i)}$. This is an injective group homomorphism, so $\psi \circ \phi : G \to \mathrm{GL}(V)$ is an embedding of G into $\mathrm{GL}(V)$.

Definition. Let G be a finite group, and V a finite dimensional $\mathbb{C}$–vector space. A **representation** of G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. We call $\dim(V)$ the **degree** of the representation.

In particular, if V is n–dimensional, then $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$.

Example.

1. Consider $\rho : G \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$ given by $\rho(g) = 1$ for all $g \in G$. This is called the *trivial representation*.
2. Consider $\rho : S_n \to \mathbb{C}^\times$ given by $\rho(\sigma) = \mathrm{sgn}(\sigma)$, which is called the *sign representation*.
3. The representation fo G afforded by Cayley's theorem is called the *regular representation* of G. The next example is a good way to understand the regular rep of G.
4. Consider G, $X = \{x_1, \ldots, x_n\}$, and $V = \mathrm{Free}(X)$. Suppose G acts on X. Then $\rho : G \to \mathrm{GL}(V)$ given by $\rho(g)(x_i) = gx_i$. In particular, if we take $X = G$, then this is the regular representation of G
5. Consider the 4–gon, with vertices labelled a, b, c, d. Take $X = \{a, b, c, d\}$ and the regular representation $\rho : D_4 \to \mathrm{GL}(V)$. This action has a geometric notion.
6. Let C_n be a cyclic group of order n; let us define some $\rho : C_n \to \mathrm{GL}(V)$. Say $\rho(x) = T$ where $t \in \mathrm{GL}(V)$; then this is a representation if and only if $T^n = I$.

Definition. We say that two representations $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ are **isomorphic** if there exists an isomorphism $T : V \to W$ such that for all $g \in G$,

$$T \circ \rho(g) = \tau(g) \circ T$$

Suppose $\rho : G \to \mathrm{GL}(V)$ and $T : V \to W$ is an isomorphism. Then we can define $\tau : G \to \mathrm{GL}(W)$ by $\tau(G) = T \circ \rho(g) \circ T^{-1}$; this $\rho \cong \tau$. In other words, the representation is unique up to isomorphism under change of basis.

Example. Consider $G = \{g_1, \ldots, g_n\} = \{h_1, \ldots, h_n\}$, and fix $g \in G$. Let $gg_i = g_{\alpha(i)}$ and $gh_i = h_{\beta(i)}$ where $\alpha, \beta \in S_n$. Fix an n–dimensional vector space V with basis $\{b_1, \ldots, b_n\}$. Then two regular representations are given by

$$\rho_1 : G \to \mathrm{GL}(V), \rho(g)(b_i) = b_{\alpha(i)}$$
$$\rho_2 : G \to \mathrm{GL}(V), \rho(g)(b_i) = b_{\beta(i)}$$

Let $\gamma \in S_n$ be such that $h_{\gamma(i)} = g_i$, and define $T : V \to V$ by $T(v_i) = b_{\gamma(i)}$. Then

$$gg_i = g_{\alpha(i)} = gh_{\gamma(i)} = h_{\beta\gamma(i)} = g_{\gamma^{-1}\beta\gamma(i)}$$

so that $\alpha = \gamma^{-1}\beta\gamma$. Thus for each b_i,

$$\begin{aligned} T \circ \rho_1(g) \circ T^{-1}(b_i) &= T \circ \rho_1(g)(b_{\gamma^{-1}(i)}) \\ &= T(b_{\alpha\gamma^{-1}(i)})b_{\gamma\alpha\gamma^{-1}(i)} \\ &= b_{\beta(i)} = \rho_2(g)(b_i) \end{aligned}$$

so that $T \circ \rho_1(g) \circ T^{-1} = \rho_2(g)$.
Note: conjugate elements have the same cycle type.

Subrepresentations

What should a subrepresentation of $\rho : G \to \mathrm{GL}(V)$ mean?

We would like a subspace $W \leq V$ such that $\tau : G \to \mathrm{GL}(W)$ is a representation given by $\tau(g)(w) = \rho(g)(w)$ for all $w \in W$. Moreover, to make this well-defined, we need W to b4 $\rho(g)$–invariant for every $g \in G$ ($\rho(g)(W) \subseteq W$).

Suppose $T : V \to V$ is a linear operator, and $W \leq V$ is a T–invariant subspace; i.e. $T(W) \subseteq W$. In particular, the restriction operator $T_W : W \to W$ is well-defined.

Definition. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. A subspqce $W \subseteq V$ is said to be ***G*–stable** if W is $\rho(g)$–invariant for all $g \in G$. A **subrepresentation** of ρ is a representation $\rho_W : G \to \mathrm{GL}(W)$ where for all $g \in G$ and $w \in W$, $\rho_W(g)(w) = \rho(g)(w)$ where W is a G–stable subspace of V.

Example. Suppose $\rho : G \to \mathrm{GL}(V)$ be the regular representation. Take $W = \mathrm{span}\{\sum_{g\in G} v_g\}$, which is clearly G–stable, and $\rho_W : G \to \mathrm{GL}(W)$ is isomorphic to the trivial representation.

Similarly, let $\rho : S_n \to \mathrm{GL}(V)$ be the regular representation, $W = \mathrm{span}\{\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)v_\sigma\}$; this is isomorphic to the sign representation.

0.1 Theorem. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation, $W \leq V$ G–stable. Then there exists a G–stable subspace W' such that $V = W \oplus W'$.*

Proof Take any inner product $\langle x, y\rangle$ on V. Then for any $x, y \in V$, define

$$\langle x, y\rangle^* = \sum_{g\in G} \langle \rho(g)(x), \rho(g)(y)\rangle$$

This is also an inner product. Let $x, y \in V$ and let $h \in G$. Then

$$\begin{aligned} \langle \rho(h)(x), \rho(h)(y)\rangle^* &= \sum_{g\in G} \langle \rho(g)\rho(h)(x), \rho(g)\rho(h)(y)\rangle \\ &= \sum_{g\in G} \langle \rho(gh)(x), \rho(gh)(y)\rangle \\ &= \sum_{g\in G} \langle \rho(g)(x), \rho(g)(y)\rangle \end{aligned}$$

Thus every $\rho(h)$ is unitary with respect to $\langle\cdot,\cdot\rangle^*$. Let $W \leq V$ be G–stable, and take $W' = W^\perp$ with respect to $\langle\cdot,\cdot\rangle^*$. Then $V = W \oplus W'$. Let's see that $W^\perp$ is G–stable. Let $x \in W^\perp$, $w \in W$, and $g \in G$, so that

$$\begin{aligned}\langle \rho(g)(x), w\rangle^* &= \langle x, \rho(g)^*(w)^*\rangle = \langle x, \rho(g)^{-1}(w)\rangle^* \\ &= \langle x, \underbrace{\rho(g^{-1})(w)}_{\in W}\rangle^* \\ &= 0\end{aligned}$$

and $\rho(g)(W^\perp) \subseteq W^\perp$ as required. ■

Definition. Let $\rho : G \to \mathrm{GL}(V)$ be a representation, and $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ where each W_i is G–stable. For each i, let $\rho_i = \rho_{w_i}$. For each $v = \sum w_i \in V$, we have $\rho(g)(v) = \sum \rho(g)(w_i) = \rho_i(g)(w_i)$. In this case, we write

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k$$

and call ρ a direct sum of the ρ_i's.

The previous definition is written as an internal direct sum of V. Externally, given vector spaces $W_1, \ldots, W_k$ and representations $\rho_i : G \to \mathrm{GL}(W_i)$, we can define

$$(\rho_1 \oplus \cdots \oplus \rho_k) : G \to \mathrm{GL}(W_1 \oplus \cdots \oplus W_k)$$

by $(\rho_1 \oplus \cdots \oplus \rho_k)(g)(w_1, \ldots, w_k) = (\rho_1(g)(w_1), \ldots, \rho_k(g)(w_k))$. If $\rho_i : G \to \mathrm{GL}(W_i)$ is a subrepresentation fo $\rho : G \to \mathrm{GL}(V)$, we often say "$W_i$ is a subrepresentation of V".

Definition. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. We say ρ is **irreducible** if $V \neq \{0\}$ and the only G–stable subspaces of V are $\{0\}$ and V.

Clearly,

> ***0.2 Theorem.*** *Every representation $\rho : G \to \mathrm{GL}(V)$ can be written as a direct sum of irreducible sub-representations.*

Example. Let $\rho : S_3 \to \mathrm{GL}(\mathbb{C}^3)$ be the permutation representation with respect to the standard basis $\{e_1, e_2, e_3\}$. Consider $W_1 = \mathrm{span}\{e_1 + e_2 + e_3\}$ and $W_2 = \mathrm{span}\{e_1 - e_2, e_2 - e_3\}$. Is W_2 irreducible?

More generally, if $V = W_1 \oplus \cdots \oplus W_k$ and $\dim W_i = 1$ and $\deg(\rho_i) = 1$,

$$\rho(gh)(\sum w_i) = \sum \rho_i(gh)(w_i) = \sum \rho_i(g)\rho_i(h)(w_i) = \sum \rho_i(h)\rho_i(g)(w_i)$$

so that $\rho(gh) = \rho(hg)$. In the our example, this does not happen, since $\rho(g) \neq I$ when $g \neq 1$ and S_3 is not abelian.

Example. Let $\rho : S_3 \to \mathrm{GL}(V)$ be the regular representation. Let $W_1 = \mathrm{span}\{\sum_{\sigma \in S_3} v_\sigma\}$ and $W_2 = \mathrm{span}\{\sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) v_\sigma\}$, and

$$W_3 = \sum \alpha_\sigma v_\sigma | \ \alpha \begin{matrix} +\alpha_{(123)} + \alpha_{(1,3,2)} \\ = 0 \\ \alpha_{(12)} + \alpha_{(13)} + \alpha_{(23)} \\ = 0 \end{matrix} \ \epsilon$$

Now let's focus on W_3. A basis for W_3 is given by

$$e_1 = v_\epsilon - v_{(123)} \qquad e_2 = v_\epsilon - v_{(123)}$$
$$e_3 = v_{(12)} - v_{(13)} \qquad e_4 = v_{(12)} - v_{(23)}$$

Recall that $S_3 = \langle (12), (123) \rangle$; suffices to show stability with respect to generators.

$$\rho(12) : e_1 \mapsto e_4, e_2 \mapsto e_3, e_3 \mapsto e_2, e_4 \mapsto e_1$$
$$\rho(123) : e_1 \mapsto e_2 - e_1, e_2 \mapsto -e_1, e_3 \mapsto e_4 - e_3, e_4 \mapsto -e_3$$

Let $U_1 = \operatorname{span}\{e_1 - e_4, e_2 + e_3 - e_1\}$

1 Tensor Products

Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be representations. We define the representation $\rho \otimes \tau : G \to \mathrm{GL}(V \otimes W)$

$$(\rho \otimes \tau)(g)(v \otimes w) = \rho(g)(v) \otimes \tau(g)(w)$$

2 Character Theory

We define the character of ρ by $\rho : G \to \mathbb{C}$ as $\chi(G) = (\rho(g))$.

Remark. If we choose a basis β for V, then define $A(g) = [\rho(g)]_\beta$ and $\chi(G)$ is given by the sum of the diagonal entries of $A(g)$. Furthermore, if $A, B \in M_n(\mathbb{C})$, then $(AB) = (BA)$.

The remark implies a number of facts:

(i) $\rho \cong \tau$, then $(\rho(g)) = (\tau(g))$.
(ii) (T) is the sum of eigenvalues of T
(iii) $\chi(1) = \dim(V)$.

2.1 Proposition. *For every $g \in G$ the eigenvalues of $\rho(g)$ have modulus 1. In particular, $\chi(g^{-1}) = \overline{\chi(g)}$.*

Proof Set $n = |G|$; then $\rho(g)^n = \rho(g^n) = I$ so that $\lambda^n - 1 = 0$ for any eigenvalue λ, so $|\lambda| = 1$. Furthermore,

$$\overline{\chi(g)} = \overline{\sum \lambda_i} = \sum \overline{\lambda_i} = \sum \lambda_i^{-1} = \chi(g^{-1})$$

proving the second component. ∎

2.2 Proposition. *Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$. Then $\chi_{\rho\oplus\tau} = \chi_\rho + \chi_\tau$ and $\chi_{\rho\otimes\tau} = \chi_\rho \cdot \chi_\tau$.*

Proof Let $\beta_1 = \{v_1, \ldots, v_n\}$ be a basis for V and $\beta_2 = \{w_1, \ldots, w_m\}$ a basis for W.

Then a basis for $V \oplus W$ is given by $\beta = \{(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)\}$. In particular,

$$[(\rho \oplus \tau)(g)]_\beta = \begin{pmatrix} [\rho(g)]_{\beta_1} & \\ & [\tau(g)]_{\beta_2} \end{pmatrix}$$

and the trace result follows.

A basis for $V \otimes W$ is given by $\gamma = \{v_i \otimes w_j : 1 \le i \le n, 1 \le j \le m\}$ in lexicographic order. Fix $g \in G$, and set $A = [\rho(g)]_{\beta_1}$, $B = [\rho(g)]_{\beta_2}$. Fix $v_i \otimes w_j \in \gamma$. Then

$$\begin{aligned}(\rho \otimes \tau)(g)(v_i \otimes w_j) &= \rho(g)(v_i) \otimes \tau(g)(w_j) \\ &= (a_{1i}v_1 + \cdots + a_{ni}v_n) \otimes (b_{1j}w_1 + \cdots + b_{mj}v_m) \\ &= \cdots + a_{ii}b_{jj} \cdot (v_i \otimes w_j) + \cdots \\ &= ([\rho \otimes \tau)(g)]_\delta) = \sum_{i,j} a_{ii}b_{jj} = (A)() = \chi_\rho(g) \cdot \chi_\tau(g)\end{aligned}$$

■

Example. Suppose $\rho : S_n \to \mathrm{GL}(\mathbb{C}^n)$ is the permutation representation with respect to $\{e_1, \ldots, e_n\}$. Then $\chi(\sigma) = |\{e_i : \rho(\sigma)(e_i) = e_i\}| = |\mathrm{Fix}(\sigma)|$, which is the number of indices i fixed by σ. Since S_n acts transitively on $\{1, \ldots, n\}$, there is exactly 1 orbit, so by Burnside's lemma,

$$n! = |S_n| = \sum_{\sigma \in S_n} \chi(\sigma)$$

Example. Let $\rho : G \to \mathrm{GL}(V)$ be the regular representation. Note that if $g \neq 1$, then for all $h \in G$, $gh \neq h$. In particular, this means that $\chi(g) = 0$ if $g \neq 1$, and $\chi(1) = |G|$ (the dimension of V).

Example. Let $\rho : S_3 \to \mathrm{GL}(V)$ be the regular representation. Recall that $V = W_1 \oplus W_2 \oplus V_1 \oplus V_2$. Let $S_3 = \langle (12), (123) \rangle$.