# Representation Theory of Finite Groups

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# Contents

# I. Introduction

Let $G$ be a finite group of order $n$, and write $G = \{g_1, \dots, g_n\}$. Fix $g \in G$; then $gg_i = gg_j$ if and only if $i = j$. Thus there exists some $\sigma_g \in S_i$ such that $gg_i = g_{\sigma_g(i)}$ for all $i \in \{1, 2, \dots, n\}$. In particular, $\phi : G \to S_n$ by $\phi(g) = \sigma_g$ is an embedding (injective group homomorphism). This observation is usually referred to as Cayley's Theorem.

Now let $V$ be an $n$–dimensional complex vector space. We then denote $\mathrm{GL}(V)$ as the group of invertible linear operators $T : V \to V$. Now define $\psi : S_n \to \mathrm{GL}_n(V)$ by $\psi(\sigma) = T_\sigma$ where if $\{b_1, \dots, b_n\}$ is a basis for $V$ and $T_\sigma(b_i) = b_{\sigma(i)}$. This is an injective group homomorphism, so $\psi \circ \phi : G \to \mathrm{GL}(V)$ is an embedding of $G$ into $\mathrm{GL}(V)$.

***Definition.*** Let $G$ be a finite group, and $V$ a finite dimensional $\mathbb{C}$–vector space. A **representation** of $G$ is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. We call $\dim(V)$ the **degree** of the representation.

In particular, if $V$ is $n$–dimensional, then $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$.

*Example.*

1. Consider $\rho : G \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$ given by $\rho(g) = 1$ for all $g \in G$. This is called the *trivial representation*.
2. Consider $\rho : S_n \to \mathbb{C}^\times$ given by $\rho(\sigma) = \mathrm{sgn}(\sigma)$, which is called the *sign representation*.
3. The representation fo $G$ afforded by Cayley's theorem is called the *regular representation* of $G$. The next example is a good way to understand the regular rep of $G$.
4. Consider $G$, $X = \{x_1, \dots, x_n\}$, and $V = \mathrm{Free}(X)$. Suppose $G$ acts on $X$. Then $\rho : G \to \mathrm{GL}(V)$ given by $\rho(g)(x_i) = gx_i$. In particular, if we take $X = G$, then this is the regular representation of $G$
5. Consider the 4–gon, with vertices labelled $a, b, c, d$. Take $X = \{a, b, c, d\}$ and the regular representation $\rho : D_4 \to \mathrm{GL}(V)$. This action has a geometric notion.
6. Let $C_n$ be a cyclic group of order $n$; let us define some $\rho : C_n \to \mathrm{GL}(V)$. Say $\rho(x) = T$ where $t \in \mathrm{GL}(V)$; then this is a representation if and only if $T^n = I$.

***Definition.*** We say that two representations $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ are **isomorphic** if there exists an isomorphism $T : V \to W$ such that for all $g \in G$,

$$T \circ \rho(g) = \tau(g) \circ T$$

Suppose $\rho : G \to \mathrm{GL}(V)$ and $T : V \to W$ is an isomorphism. Then we can define $\tau : G \to \mathrm{GL}(W)$ by $\tau(G) = T \circ \rho(g) \circ T^{-1}$; this $\rho \cong \tau$. In other words, the representation is unique up to isomorphism under change of basis.

*Example.* Consider $G = \{g_1, \dots, g_n\} = \{h_1, \dots, h_n\}$, and fix $g \in G$. Let $gg_i = g_{\alpha(i)}$ and $gh_i = h_{\beta(i)}$ where $\alpha, \beta \in S_n$. Fix an $n$–dimensional vector space $V$ with basis $\{b_1, \dots, b_n\}$. Then two regular representations are given by

$$\rho_1 : G \to \mathrm{GL}(V), \rho(g)(b_i) = b_{\alpha(i)}$$
$$\rho_2 : G \to \mathrm{GL}(V), \rho(g)(b_i) = b_{\beta(i)}$$

Let $\gamma \in S_n$ be such that $h_{\gamma(i)} = g_i$, and define $T : V \to V$ by $T(v_i) = b_{\gamma(i)}$. Then

$$gg_i = g_{\alpha(i)} = gh_{\gamma(i)} = h_{\beta\gamma(i)} = g_{\gamma^{-1}\beta\gamma(i)}$$

so that $\alpha = \gamma^{-1}\beta\gamma$. Thus for each $b_i$,

$$\begin{aligned} T \circ \rho_1(g) \circ T^{-1}(b_i) &= T \circ \rho_1(g)(b_{\gamma^{-1}(i)}) \\ &= T(b_{\alpha\gamma^{-1}(i)}) b_{\gamma\alpha\gamma^{-1}(i)} \\ &= b_{\beta(i)} = \rho_2(g)(b_i) \end{aligned}$$

so that $T \circ \rho_1(g) \circ T^{-1} = \rho_2(g)$.

Note: conjugate elements have the same cycle type.

### Subrepresentations

What should a subrepresentation of $\rho : G \to \mathrm{GL}(V)$ mean?

We would like a subspace $W \leq V$ such that $\tau : G \to \mathrm{GL}(W)$ is a representation given by $\tau(g)(w) = \rho(g)(w)$ for all $w \in W$. Moreover, to make this well-defined, we need $W$ to b4 $\rho(g)$–invariant for every $g \in G$ ($\rho(g)(W) \subseteq W$).

Suppose $T : V \to V$ is a linear operator, and $W \leq V$ is a $T$–invariant subspace; i.e. $T(W) \subseteq W$. In particular, the restriction operator $T_W : W \to W$ is well-defined.

***Definition.*** Let $\rho : G \to \mathrm{GL}(V)$ be a representation. A subspqce $W \subseteq V$ is said to be ***G*–stable** if $W$ is $\rho(g)$–invariant for all $g \in G$. A **subrepresentation** of $\rho$ is a representation $\rho_W : G \to \mathrm{GL}(W)$ where for all $g \in G$ and $w \in W$, $\rho_W(g)(w) = \rho(g)(w)$ where $W$ is a $G$–stable subspace of $V$.

*Example.* Suppose $\rho : G \to \mathrm{GL}(V)$ be the regular representation. Take $W = \operatorname{span}\{\sum_{g\in G} v_g\}$, which is clearly $G$–stable, and $\rho_W : G \to \mathrm{GL}(W)$ is isomorphic to the trivial representation.

Similarly, let $\rho : S_n \to \mathrm{GL}(V)$ be the regular representation, $W = \operatorname{span}\{\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) v_\sigma\}$; this is isomorphic to the sign representation.

**0.1 Theorem.** *Let $\rho : G \to \mathrm{GL}(V)$ be a representation, $W \leq V$ $G$–stable. Then there exists a $G$–stable subspace $W'$ such that $V = W \oplus W'$.*

Proof Take any inner product $\langle x, y\rangle$ on $V$. Then for any $x, y \in V$, define

$$\langle x, y\rangle^* = \sum_{g\in G} \langle \rho(g)(x), \rho(g)(y)\rangle$$

This is also an inner product. Let $x, y \in V$ and let $h \in G$. Then

$$\begin{aligned} \langle \rho(h)(x), \rho(h)(y)\rangle^* &= \sum_{g\in G} \langle \rho(g)\rho(h)(x), \rho(g)\rho(h)(y)\rangle \\ &= \sum_{g\in G} \langle \rho(gh)(x), \rho(gh)(y)\rangle \\ &= \sum_{g\in G} \langle \rho(g)(x), \rho(g)(y)\rangle \end{aligned}$$

Thus every $\rho(h)$ is unitary with respect to $\langle\cdot,\cdot\rangle^*$. Let $W \leq V$ be $G$–stable, and take $W' = W^\perp$ with respect to $\langle\cdot,\cdot\rangle^*$. Then $V = W \oplus W'$. Let's see that $W^\perp$ is $G$–stable. Let $x \in W^\perp$, $w \in W$, and $g \in G$, so that

$$\begin{aligned}\langle \rho(g)(x), w\rangle^* &= \langle x, \rho(g)^*(w)^*\rangle = \langle x, \rho(g)^{-1}(w)\rangle^* \\ &= \langle x, \underbrace{\rho(g^{-1})(w)}_{\in W}\rangle^* \\ &= 0\end{aligned}$$

and $\rho(g)(W^\perp) \subseteq W^\perp$ as required. ■

***Definition.*** Let $\rho : G \to \mathrm{GL}(V)$ be a representation, and $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ where each $W_i$ is $G$–stable. For each $i$, let $\rho_i = \rho_{w_i}$. For each $v = \sum w_i \in V$, we have $\rho(g)(v) = \sum \rho(g)(w_i) = \rho_i(g)(w_i)$. In this case, we write

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k$$

and call $\rho$ a direct sum of the $\rho_i$'s.

The previous definition is written as an internal direct sum of $V$. Externally, given vector spaces $W_1, \ldots, W_k$ and representations $\rho_i : G \to \mathrm{GL}(W_i)$, we can define

$$(\rho_1 \oplus \cdots \oplus \rho_k) : G \to \mathrm{GL}(W_1 \oplus \cdots \oplus W_k)$$

by $(\rho_1 \oplus \cdots \oplus \rho_k)(g)(w_1, \ldots, w_k) = (\rho_1(g)(w_1), \ldots, \rho_k(g)(w_k))$. If $\rho_i : G \to \mathrm{GL}(W_i)$ is a subrepresentation fo $\rho : G \to \mathrm{GL}(V)$, we often say "$W_i$ is a subrepresentation of $V$".

***Definition.*** Let $\rho : G \to \mathrm{GL}(V)$ be a representation. We say $\rho$ is **irreducible** if $V \neq \{0\}$ and the only $G$–stable subspaces of $V$ are $\{0\}$ and $V$.

Clearly,

> ***0.2 Theorem.*** *Every representation $\rho : G \to \mathrm{GL}(V)$ can be written as a direct sum of irreducible sub-representations.*

*Example.* Let $\rho : S_3 \to \mathrm{GL}(\mathbb{C}^3)$ be the permutation representation with respect to the standard basis $\{e_1, e_2, e_3\}$. Consider $W_1 = \mathrm{span}\{e_1 + e_2 + e_3\}$ and $W_2 = \mathrm{span}\{e_1 - e_2, e_2 - e_3\}$. Is $W_2$ irreducible?

More generally, if $V = W_1 \oplus \cdots \oplus W_k$ and $\dim W_i = 1$ and $\deg(\rho_i) = 1$,

$$\rho(gh)(\sum w_i) = \sum \rho_i(gh)(w_i) = \sum \rho_i(g)\rho_i(h)(w_i) = \sum \rho_i(h)\rho_i(g)(w_i)$$

so that $\rho(gh) = \rho(hg)$. In the our example, this does not happen, since $\rho(g) \neq I$ when $g \neq 1$ and $S_3$ is not abelian.

*Example.* Let $\rho : S_3 \to \mathrm{GL}(V)$ be the regular representation. Let $W_1 = \mathrm{span}\{\sum_{\sigma \in S_3} v_\sigma\}$ and $W_2 = \mathrm{span}\{\sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) v_\sigma\}$, and

$$W_3 = \sum \alpha_\sigma v_\sigma | \, \alpha \begin{array}{c} +\alpha_{(123)} + \alpha_{(1,3,2)} \\ = 0 \\ \alpha_{(12)} + \alpha_{(13)} + \alpha_{(23)} \\ = 0 \end{array} \epsilon$$

Now let's focus on $W_3$. A basis for $W_3$ is given by

$$e_1 = v_\epsilon - v_{(123)} \qquad e_2 = v_\epsilon - v_{(123)}$$
$$e_3 = v_{(12)} - v_{(13)} \qquad e_4 = v_{(12)} - v_{(23)}$$

Recall that $S_3 = \langle (12), (123) \rangle$; suffices to show stability with respect to generators.

$$\rho(12) : e_1 \mapsto e_4, e_2 \mapsto e_3, e_3 \mapsto e_2, e_4 \mapsto e_1$$
$$\rho(123) : e_1 \mapsto e_2 - e_1, e_2 \mapsto -e_1, e_3 \mapsto e_4 - e_3, e_4 \mapsto -e_3$$

Let $U_1 = \operatorname{span}\{e_1 - e_4, e_2 + e_3 - e_1\}$

# 1 Tensor Products

Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be representations. We define the representation $\rho \otimes \tau : G \to \mathrm{GL}(V \otimes W)$

$$(\rho \otimes \tau)(g)(v \otimes w) = \rho(g)(v) \otimes \tau(g)(w)$$

# 2 Character Theory

We define the character of $\rho$ by $\rho : G \to \mathbb{C}$ as $\chi(G) = (\rho(g))$.

*Remark.* If we choose a basis $\beta$ for $V$, then define $A(g) = [\rho(g)]_\beta$ and $\chi(G)$ is given by the sum of the diagonal entries of $A(g)$. Furthermore, if $A, B \in M_n(\mathbb{C})$, then $(AB) = (BA)$.

The remark implies a number of facts:

(i) $\rho \cong \tau$, then $(\rho(g)) = (\tau(g))$.
(ii) $(T)$ is the sum of eigenvalues of $T$
(iii) $\chi(1) = \dim(V)$.

***2.1 Proposition.*** *For every $g \in G$ the eigenvalues of $\rho(g)$ have modulus 1. In particular, $\chi(g^{-1}) = \overline{\chi(g)}$.*

Proof Set $n = |G|$; then $\rho(g)^n = \rho(g^n) = I$ so that $\lambda^n - 1 = 0$ for any eigenvalue $\lambda$, so $|\lambda| = 1$. Furthermore,

$$\overline{\chi(g)} = \overline{\sum \lambda_i} = \sum \overline{\lambda_i} = \sum \lambda_i^{-1} = \chi(g^{-1})$$

proving the second component. ∎

***2.2 Proposition.*** *Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$. Then $\chi_{\rho\oplus\tau} = \chi_\rho + \chi_\tau$ and $\chi_{\rho\otimes\tau} = \chi_\rho \cdot \chi_\tau$.*

Proof Let $\beta_1 = \{v_1, \ldots, v_n\}$ be a basis for $V$ and $\beta_2 = \{w_1, \ldots, w_m\}$ a basis for $W$.

Then a basis for $V \oplus W$ is given by $\beta = \{(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)\}$. In particular,

$$[(\rho \oplus \tau)(g)]_\beta = \begin{pmatrix} [\rho(g)]_{\beta_1} & \\ & [\tau(g)]_{\beta_2} \end{pmatrix}$$

and the trace result follows.

A basis for $V \otimes W$ is given by $\gamma = \{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ in lexicographic order. Fix $g \in G$, and set $A = [\rho(g)]_{\beta_1}$, $B = [\rho(g)]_{\beta_2}$. Fix $v_i \otimes w_j \in \gamma$. Then

$$\begin{aligned}(\rho \otimes \tau)(g)(v_i \otimes w_j) &= \rho(g)(v_i) \otimes \tau(g)(w_j)\\ &= (a_{1i}v_1 + \cdots + a_{ni}v_n) \otimes (b_{1j}w_1 + \cdots + b_{mj}v_m)\\ &= \cdots + a_{ii}b_{jj} \cdot (v_i \otimes w_j) + \cdots\\ &= ([\rho \otimes \tau)(g)]_\delta) = \sum_{i,j} a_{ii}b_{jj} = (A)() = \chi_\rho(g) \cdot \chi_\tau(g)\end{aligned}$$ ■

*Example.* Suppose $\rho : S_n \to \mathrm{GL}(\mathbb{C}^n)$ is the permutation representation with respect to $\{e_1, \ldots, e_n\}$. Then $\chi(\sigma) = |\{e_i : \rho(\sigma)(e_i) = e_i\}| = |\mathrm{Fix}(\sigma)|$, which is the number of indices $i$ fixed by $\sigma$. Since $S_n$ acts transitively on $\{1, \ldots, n\}$, there is exactly 1 orbit, so by Burnside's lemma,

$$n! = |S_n| = \sum_{\sigma \in S_n} \chi(\sigma)$$

*Example.* Let $\rho : G \to \mathrm{GL}(V)$ be the regular representation. Note that if $g \neq 1$, then for all $h \in G$, $gh \neq h$. In particular, this means that $\chi(g) = 0$ if $g \neq 1$, and $\chi(1) = |G|$ (the dimension of $V$).

*Example.* Let $\rho : S_3 \to \mathrm{GL}(V)$ be the regular representation. Recall that $V = W_1 \oplus W_2 \oplus U_1 \oplus U_2$ where $W_1$ is the trivial representation, $W_2$ is the sign representation, and $U_1, U_2$ are isomorphic. Let $S_3 = \langle (12), (123) \rangle$; then we have

| $x_1$ | 1 | 1 |
|---|---|---|
| $x_2$ | −1 | 1 |
| $x_3$ | $a$ | $b$ |
| $x_4$ | $a$ | $b$ |

In particular, $\chi(12) = 1 - 1 + 2a = 0$ and $\chi(123) = 1 + 1 + 2b = 0$, so $b = -1$.

*Example.* Let $\rho : G \to \mathrm{GL}(V)$ be a representation. In particular, $\rho(ghg^{-1}) = \rho(g)\rho(h)\rho(g)$ so that $\rho(ghg^{-1}) = \rho(h)$ so $\chi(ghg^{-1}) = (h)$; in other words, that characters are constant on conjugacy classes.

**2.3 Lemma. (Schur)** *Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be irreducible representations, and suppose $T : V \to W$ is linear such that for all $g \in G$, $\tau(g) \circ T = T \circ \rho(g)$. Then either $T = 0$ or $T$ is an isomorphism and $\rho \cong \tau$. Moreover, if $V = W$ and $\rho = \tau$, then $T$ is a scalar multiple of the identity.*

Proof Assume $T \neq 0$.

Let's first see that $T$ is injective, and let $v \in \ker(T)$. Then for any $g \in G$, $T(\rho(g)(v)) = \tau(g)(T(v)) = 0$, so $\rho(g)(v) \in \ker(T)$. Thus $\ker(T)$ is $G$−stable (with respect to $\rho$). Since $\rho$ is irreducible and $T \neq 0$, $\ker(T) = \{0\}$.

We also have that $T$ is surjective. Let $v \in \mathrm{Im}(T)$ and say $v = T(X)$ with $x \in V$. Then for $g \in G$, $\tau(g)(v) = \tau(g)(T(x)) = T(\rho(g)(x)) \in \mathrm{Im}(T)$ so $\mathrm{Im}(T)$ is $G$−stable, and again by irreducibility of $\tau$, $\mathrm{Im}(T) = W$. Thus $T$ is an isomorphism.

Now let $\lambda \in \mathbb{C}$ be an eigenvalue of $T$ and consider $T' = T - \lambda I$. Now, note that for $g \in G$, $\rho(g)T' = T'\rho(g)$, but $T'$ has non-trivial kernel, so in fact $T' = 0$. ■

***2.4 Corollary.*** *Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be irreducible, and $T : V \to W$ linear. Consider*

$$T' = \frac{1}{|G|} = \sum_{g \in G} \tau(g)^{-1} T \rho(g)$$

*Then*

*(i) If $T' \neq 0$, then $\rho \cong \tau$ via $T'$.*

*(ii) If $V = W$, $\rho = \tau$, then $T' = (T)/\dim(V) \cdot I$.*

Proof Clearly $T' : V \to W$ is linear, and for any $h \in G$,

$$\begin{aligned}
\tau(h)T' &= \tau(h)\frac{1}{|H|}\sum_{g \in G} \tau(g^{-1})T\rho(g) \\
&= \frac{1}{|G|}\sum_{g \in G} \tau(hg^{-1})T\rho(g) \\
&= \frac{1}{|G|}\sum_{g \in G} \tau(g^{-1})T(\rho(gh)) \\
&= \frac{1}{|G|}\sum_{g \in G} \tau(g^{-1})T\rho(g)\rho(h) \\
&= T'\rho(h)
\end{aligned}$$

If $V = W$ and $\rho = T$, then $(T') = \frac{1}{|G|}(T) \cdot |G| = (T) = \alpha \dim(V)$, so $\alpha = (T)/\dim(V)$. ■

*Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be irreducible representations, and $T : V \to W$ linear. Let $\beta$ be a basis for $V$ and $\gamma$ a basis for $W$. Then for $g \in G$, let $[\rho(g)]_\beta = (a_{ij}(g))$, $[\tau(g)]_\gamma = (b_{kl}(g))$, $[T]_\beta^\gamma = (X_{ki})$, and $[T']_\beta^\gamma = (x'_{ki})$.*

*By matrix multiplication, $x'_{ki} = \frac{1}{|G|}\sum_g \sum_{j,l} b_{kl}(g^{-1})x_{lj}a_{ji}$. If $\rho \ncong \tau$, then $T' = 0$, so by viewing the RHS as a polynomial in the $x_{ij}$, we have*

$$\frac{1}{|G|}\sum_g b_{kl}(g^{-1})a_{ji}(g) = 0$$

*But now it $\rho = \tau$, then $T' = \lambda I$ where $\lambda = (T)/\dim(B)$ so that*

$$\frac{1}{|G|}\sum_g \sum_{j,l} a_{kl}(g^{-1})x_{lj}a_{ji}(g) = \lambda\delta_{ki} = \frac{1}{\dim(V)}\sum_{j,l}\delta_{ki}\delta_{jl}x_{lj}$$

*Then by equating coefficients of $x_{lj}$, we have*

$$\frac{1}{|G|}\sum_g a_{kl}(g^{-1})a_{ji}(g) = \frac{1}{\dim(V)}\delta_{ki}\delta_{jl}$$

*Remark.* If $G$ is a finite group, the consider the vector space of all functions $\phi : G \to \mathbb{C}$. For any $\phi, \psi$ in this vector space, $\langle \phi, \psi \rangle = \frac{1}{|G|}\sum_g \phi(g)\overline{\psi(g)}$ defines an inner product. Then if $\chi_1, \chi_2$ are characters of $G$, then

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|}\sum_g \chi_1(g)\chi_2(g^{-1})$$

We thus have:

**2.5 Theorem.** *If $\chi$ is a character of an irreducible representation, then $\langle \chi, \chi = 1$, and if $\chi_1$ and $\chi_2$ correspond to non-isomorphic representations, then $\langle \chi_1, \chi_2 \rangle = 0$.*

Proof Say $[\rho(g)]_\beta = (a_{ij}(g))$ where $\rho$ is an irreducible representation with character $\chi$. Then

$$\begin{aligned}
\langle \chi, \chi \rangle &= \frac{1}{|G|} \sum_g \chi(g)\chi(g^{-1}) = \frac{1}{|G|} \sum_g \chi(g^{-1})\chi(g) \\
&= \frac{1}{|G|} \sum_g \sum_{i,j} a_{ii}(g^{-1}) a_{jj}(g) = \sum_{i,j} \left( \frac{1}{|G|} \sum_g a_{ii}(g^{-1}) a_{jj}(g) \right) \\
&= \sum_{i,j} \left( \frac{1}{|G|} \sum_g a_{ii}(g^{-1}) a_{ii}(g) \right. \\
&= \sum_i \frac{1}{\dim(V)} = 1
\end{aligned}$$

To see the second part,

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_g \chi_1(g)\chi_2(g^{-1}) = \frac{1}{|G|} \sum_g \sum_{ij} a_{ii}(g) a_{jj}(g^{-1}) = \sum_{i,j} 0 = 0 \qquad \blacksquare$$

*If $\chi$ is a character corresponding to an irreducible representation, we say $\chi$ is irreducible. If $\rho$ and $\tau$ are isomorphic representations, we say $\chi_\rho$ and $\chi_\tau$ are isomorphic (in fact $\chi_\rho = \chi_\tau$).*

**2.6 Corollary.** *Let $\rho : G \to \mathrm{GL}(V)$ be a representation with character $\chi$. Say $V = W_1 \oplus \cdots \oplus W_k$ is an irreducible decomposition of $V$. If $\tau : G \to \mathrm{GL}(W)$ is an irreducible representations with character $\phi$, then the number of $W_i$ isomorphic to $W$ (i.e. $\rho_i \cong \tau$) is $\langle \chi, \phi \rangle$.*

Proof Write $\chi = n_1\chi_1 + \cdots + n_l\chi_l$, where the $\chi_i$ are pairwise non-isomorphic. Then $\langle \chi, \chi_i \rangle = n_i$. $\blacksquare$