Functional Analysis

Alex Rutar*
University of Waterloo

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*arutar@uwaterloo.ca
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Contents

I. Fundamentals of Functional Analysis

1 Basic Elements of Functional Analysis

Throughout, we denote by $\mathbb{F}$ either the field $\mathbb{R}$ or the field $\mathbb{C}$.

Banach Spaces

Definition. Let X be a vector space over $\mathbb{F}$. A **norm** is a functional $\|\cdot\| : X \to \mathbb{R}$ such that it is

- *(non-negative)* $\|x\| \geq 0$ for any $x \in X$
- *(non-degenerate)* $\|x\| = 0$ if and only if $x = 0$
- *(subadditivity)* $\|x+y\| \leq \|x\| + \|y\|$ for $x, y \in X$
- *($|\cdot|$−homogeneity)* $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{F}$, $x \in X$.

We call the pair $(X, \|\cdot\|)$ a **normed vector space**. Furthermore, we say that $(X, \|\cdot\|)$ is a **Banach space** provided that X is complete with respect to the metric $\rho(x,y) = \|x-y\|$.

Example. (i) $(\mathbb{F}, |\cdot|)$ is a Banach space.

(ii) $(\mathbb{F}^b, \|\cdot\|_p)$, $x = (x_j)_{j=1}^n$,

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^n |x_j|^p\right)^{1/p} & 1 \leq p < \infty \\ \max_{j=1,\dots,n} |x_j| & p = \infty \end{cases}$$

(iii) Consider the space

$$L_p^{\mathbb{F}} = \left\{ f : [0,1] \to \mathbb{F} \;\middle|\; f \text{ is Lebesgue measurable}, \left(\int_0^1 |f|^p\right)^{1/p} < \infty \right\} \Big/ \sim_{\text{a.e.}}$$

where $1 \leq p < \infty$.

(iv) $L_\infty^{\mathbb{F}}[0,1]$, $\|f\|_\infty = \operatorname{ess\,sup}_{t \in [0,1]} |f(t)|$.

(v) Let (X, d) be a metric space. Then

$$C_b^{\mathbb{F}}(x) = \{ f : X \to \mathbb{F} \mid f \text{ is continuous and bounded} \}, \quad \|f\|_\infty = \sup_{x \in X} |f(x)|$$

is a Banach space.

Here is a more interesting example:

Example. Let (X,d) be a metric space. We define the space of Lipschitz functions

$$\mathrm{Lip}^{\mathbb{F}}(X,d) = \left\{ f : X \to \mathbb{F} \;\middle|\; f \text{ is bounded}, L(f) = \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|f(x)-f(y)|}{d(x,y)} < \infty \right\}$$

We note that for $f : X \to \mathbb{F}$ that

$$f \in \mathrm{Lip}^{\mathbb{F}}(X,d) \Leftrightarrow \text{there is } L \geq 0 \text{ s.t. } |f(x)-f(x)| \leq Ld(x,y) \text{ for all } x,y \in X \tag{1.1}$$

It is easy to verify that $L(f) = \min\{L \geq 0 : (1.1) \text{ holds for } f\}$. It is an easy exercise to see that $\mathrm{Lip}^{\mathbb{F}}$ is a vector space, and that $L : \mathrm{Lip}^{F}(X,d) \to \mathbb{R}$ is a **semi-norm** (non-negative, subadditive, $|\cdot|$–homogeneous). However, we do not have non-degeneracy (for example, constants are taken to 0). We define the Lipschitz norm

$$\|f\|_{\mathrm{Lip}} = \|f\|_\infty + L(f)$$

1.1 Proposition. $(\mathrm{Lip}^{\mathbb{F}}(X,d), \|\cdot\|_{\mathrm{Lip}})$ *is a Banach space.*

Proof Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $(\mathrm{Lip}^{\mathbb{F}}(X,d), \|\cdot\|_{\mathrm{Lip}})$. Since $\|\cdot\|_\infty \leq \|\cdot\|_{\mathrm{Lip}}$ on $\mathrm{Lip}^{F}(X,d)$, we see that $(f_n)_{n=1}^\infty$ is uniformly Cauchy (and bounded), and hence there is $f = \lim_{n\to\infty} f_n$ in $C_b^{\mathbb{F}}(X)$, where the limit is taken with respect to $\|\cdot\|_\infty$, since $(C_b^{\mathbb{F}}(X), \|\cdot\|_\infty)$ is a Banach space. If $x,y \in X$, then

$$\begin{aligned} |f(x)-f(x)| &= \lim_{n\to\infty} |f_n(x)-f_n(y)| \leq \sup_{n\in\mathbb{N}} |f_n(x)-f_n(y)| \\ &\leq \sup_{n\in\mathbb{N}} L(f_n) d(x,y) \leq \sup_{n\in\mathbb{N}} \|f_n\|_{\mathrm{Lip}}\, d(x,y) \end{aligned}$$

Since Cauchy sequences are bounded, we see that $|f(x)-f(y)| \leq Ld(x,y)$, where $L = \sup_{n\in\mathbb{N}} \|f_n\|_{\mathrm{Lip}} < \infty$. Thus by (1.1), $f \in \mathrm{Lip}^{\mathbb{F}}(X,d)$. Exercise: one may verify that $\|f - f_n\|_{\mathrm{Lip}} \to 0$. ■

Another collection of basic examples are given by the sequence spaces. We can define

$$\ell_1^{\mathbb{F}} = \left\{ x = (x_j)_{j=1}^\infty \in \mathbb{F}^{\mathbb{N}} \;\middle|\; \|x\|_1 = \sum_{j=1}^\infty |x_j| < \infty \right\}$$

It is easy to see that $(\ell_1, \|\cdot\|_1)$ is a normed vector space.

For $1 < p < \infty$, and write

$$\ell_p^{\mathbb{F}} = \left\{ x \in \mathbb{F}^{\mathbb{N}} \;\middle|\; \|x\|_p = \left(\sum_{j=1}^\infty |x_j|^p \right)^{1/p} < \infty \right\}$$

Note that $0 \in \ell_p$, $\alpha \in \mathbb{F}$, $\alpha x \in \ell_p$ if $x \in \ell_p$. Let $q = p/(p-1)$ so that $1/p + 1/q = 1$. Then q is called the **conjugate index**. We have

1.2 Proposition. (Young's Inequality) *If $a,b \geq 0$ in $\mathbb{R}$, then $ab \leq a^p/p + b^q/q$, with equality only if $a^p = b^q$.*

and

1.3 Proposition. (Hölder's Inequality) *If $x \in \ell_p$ and $y \in \ell_q$, then $xy = (x_i y_i)_{i=1}^{\infty} \in \ell_1$, with*

$$\sum_{i=1}^{\infty} |x_i y_i| \le \|x\|_p \|y\|_q$$

with equality exactly when $\operatorname{sgn}(x_i y_i) = \operatorname{sgn}(x_k y_k)$ *for all $j,k \in \mathbb{N}$ where $x_i y_i \neq 0 \neq x_k y_k$, and $|x|^p = (|x_j|^p)_{j=1}^{\infty}$ and $|y|^q$ are linearly dependent in ℓ_1.*

and finally

1.4 Proposition. (Minkowski's Inequality) *If $x, y \in \ell_p$, then $\|x+y\|_p \le \|x\|_p + \|y\|_p$ with equality exactly when one of x or y is a non-negative scalar combination of the other.*

Review of Topology

Let X denote a non-empty set, and $\mathcal{P}(X)$ denote the power set of X.

Definition. A **topology** on a set X is a set τ of subsets of X such that

(i) $\emptyset, X \in \tau$

(ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.

(iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \le i \le n$, then $\bigcap_{i=1}^{n} U_i \in \tau$.

The sets $U \in \tau$ are called the **open sets** in X, and sets of the form $X \setminus U$ for some open set U are caled the **closed sets** in X. The pair (X, τ) is called a **topological space**.

The metric topology on a metric space (X, d) is the topology

$$\tau_d = \{ U \subseteq X \mid \text{ for each } x_0 \in U, \text{ there is } \delta = \delta(x_0) \text{ s.t. } B_\delta(x_0) \subseteq U \}$$

Example. (i) Given two metrics d, ρ on X, we say that $d \sim \rho$ if and only if there are $c, C > 0$ such that

$$c d(x,y) \le \rho(x,y) \le C d(x,y) \text{ for any } x, y \in X$$

Note that $d \sim \rho$ implies that $\tau_d = \tau_\rho$, but the reverse implication is not true. An example of this are the metrics on $X = \mathbb{R}$ given by $d(x,y)$ and $\rho(x,y) = \frac{|x-y|}{1+|x-y|}$. Then $d \not\sim \rho$ but $\tau_d = \tau_\rho$.

(ii) "Sorgenfry line" Set $X = \mathbb{R}$, and consider

$$\sigma = \{ V \subseteq \mathbb{R} \mid \text{ for any } s \in V, \text{ there is } \delta = \delta(s) > 0 \text{ s.t. } [s, s+\delta) \subseteq V \}$$

It is an exercise to verify that $\tau_{|\cdot|} \subsetneq \sigma$. We say that σ is **finer** than $\tau_{|\cdot|}$.

(iii) Relative topology: let (X, τ) be a topological space, and $\emptyset \neq A \subseteq X$. Then we can define a topology $\tau|_A = \{U \cap A : U \in \tau\}$.

Definition. Let (X, τ) and (Y, σ) be topological spaces, and $f : X \to Y$. We say that f is $(\tau - \sigma -)$**continuous** at x_0 in X if,

- given $V \in \sigma$ such that $f(x_0) \in V$, then there exists $U \in \tau$ such that $x_0 \in U$ and $f(U) \subseteq V$.

We say that f is $(\tau - \sigma -)$continuous if it is continuous at each x_0 in X.

Space of bounded continuous functions into a normed space

Let $(Y, \|\cdot\|)$ denote a normed space. We let $\tau_{\|\cdot\|}$ denote the topology given by the metric $\rho(x,y) = \left\|x-y\right\|$. Let (X,τ) denote any topological space. Then we write

$$C_b^Y(X) = \left\{ f : X \to Y \;\middle|\; f \text{ is bounded and } \tau - \tau_{\|\cdot\|} - \text{continuous} \right\}$$

With pointwise operations, we see that $C_b^Y(X)$ is a vector space. We also define for $f \in C_b^Y(X)$, $\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}$, making $(C_b^Y(X), \|\cdot\|_\infty)$ a normed vector space.

1.5 Theorem. *If $(Y, \|\cdot\|)$ is a Banach space, then $(C_b^Y(X), \|\cdot\|_\infty)$ is a Banach space.*

Proof Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $(C_b^Y(X), \|\cdot\|_\infty)$. Then for any $x \in X$, we have that $(f_n(x))_{n=1}^\infty$ is Cauchy in $(Y, \|\cdot\|)$ since $\|f_n(x) - f_m(x)\| \leq \|f_n - f_m\|_\infty$, and hence admis a limit $f(x)$. In particular, $x \mapsto f(x)$ defines a function from X to Y. We shall fix $x_0 \in X$ and show that f is continuous at x_0. Given $\epsilon > 0$, we let

- n_1 be so $n, m \geq n_1$ so that $\|f_n - f_m\|_\infty < \epsilon/4$.
- n_2 be so $n \geq n_2$ so that $\|f_n(x_0) - f(x_0)\| < \epsilon/4$.
- $N = \max\{n_1, n_2\}$.
- $U \in \tau$, $x_0 \in U$ such that $f_N(U) \subseteq B_{\epsilon/4}(f(x_0)) \subset Y$.

Then for $x \in U$, we let n_x be so $n_x \geq n_1$ and $n \geq n_x$, so that $\|f_n(x) - f(x)\| < \epsilon/4$. We then have

$$\begin{aligned}\|f(x) - f(x_0)\| &\leq \left\|f(x) - f_{n_x}(x)\right\| + \left\|f_{n_x}(x) - f_N(x)\right\| + \|f_N(x) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| \\ &< \frac{\epsilon}{4} + \left\|f_{n_x} - f_N\right\|_\infty + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon\end{aligned}$$

in other words that $f(U) \subseteq B_\epsilon(f(x_0))$.

Now let us check that $\|f\|_\infty < \infty$. Since $|\|f_n\|_\infty - \|f_m\|_\infty| \leq \|f_n - f_m\|_\infty$, so $(\|f_n\|_\infty)_{n=1}^\infty \subseteq \mathbb{R}$ is Cauchy, hence bounded. If $x \in X$, then

$$\|f(x)\| = \lim_{n\to\infty} \|f_n(x)\| \leq \sup_{n\in\mathbb{N}} \|f_n(x)\| \leq \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$$

so $\|f\|_\infty = \sup_{x\in X} \|f(x)\| < \infty$.

Notice that if ϵ, n_1 are as above, and further x_0, N are as above, we have for $n \geq n_1$

$$\|f_n(x_0) - f(x_0)\| \leq \|f_n(x_0) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| < \frac{\epsilon}{2}$$

so $\|f_n - f\|_\infty = \sup_{x_0\in X} \|f_n(x_0) - f(x_0)\| \leq \epsilon/2 < \epsilon$. This is uniform since n_1 is chosen uniformly in X. ■

1.6 Corollary. *$(C_b^{\mathbb{F}}(X), \|\cdot\|_\infty)$ is a Banach space.*

Let's first note the following general priniple: let (X,d), (Y,ρ) be metric spaces, where (X,d) is complete. If $\psi : X \to Y$ is a $(d-\rho-)$isometry, then $(\psi(X), \rho|_{\psi(X)})$ is a complete metric space.

Example. (i) Let T be a non-empty set and let

$$\ell_\infty(T) = \left\{ x = (x_t)_{t\in T} \in \mathbb{F}^T \;\middle|\; \|x\|_\infty \right\} < \infty$$

With pointwise operations, $(\ell_\infty, \|\cdot\|_\infty)$ is a normed space. In fact, it is a Banach space. Let us note that

$$f \mapsto (f(t))_{t\in T} : C_b(T, \mathcal{P}(T)) \to \ell_\infty(T)$$

is a surjective linear isometry, and the result follows.

(ii) Let $c = \{ x \in \ell_\infty \mid \lim_{n\to\infty} x_n \text{ exists} \}$. Then $(c, \|\cdot\|_\infty)$ is a Banach space. Consider the topological space given by $\omega = \mathbb{N} \cup \{\infty\}$, with topology

$$\tau_\omega = \mathcal{P}(\mathbb{N}) \cup \bigcup_{n\in\mathbb{N}} \{k \in \mathbb{N} : k \geq n\}$$

The map $f \mapsto (f(n))_{n=1}^\infty : C_b(\omega) \to c$ is a linear surjective isometry.

(iii) $c_0 = \left\{ x \in \mathbb{F}^{\mathbb{N}} \;\middle|\; \lim_{n\to\infty} x_n = 0 \right\} \subseteq c \subseteq \ell_\infty$.

1.7 Lemma. *If $x_0 \in X$ where (X, τ) is a topological space, then*

$$\mathcal{I}(x_0) = \{ f \in C_b(x) \mid f(x_0) = 0 \}$$

is closed, hence complete, subspace of $C_b(X)$.

Proof If $(f_n)_{n=1}^\infty \subseteq \mathcal{I}(x_0)$ and $f = \lim_{n\to\infty} f_n$ with respect to $\|\cdot\|_\infty$ in $C_b(X)$, then $f(x_0) = \lim_{n\to\infty} f_n(x_0) = 0$. Thus $f \in \mathcal{I}(x_0)$, and closed subsets of complete spaces are themselves complete. ∎

Now, $f \mapsto (f(n))_{n=1}^\infty : \mathcal{I}(\infty) \to c_0$ is a (linear) surjective isometry.

(iv) Consider the Sorgenfty line $(\mathbb{R}, \sigma)$: verify that

$$c_b(\mathbb{R}, \sigma) = \left\{ f : \mathbb{R} \to \mathbb{F} \;\middle|\; f \text{ is bounded and } \lim_{t\to t_0^+} f(t) = f(t_0) \text{ for } t \in \mathbb{R} \right\}$$

2 Linear operators and linear functionals

Let X, Y be vector spaces. We let $\mathcal{L}(X,Y) = \{ S : X \to Y \mid S \text{ is linear} \}$; this is itself a vector space with pointwise operations. Let $(X, \|\cdot\|)$ be a normed space. We denote

$$\begin{aligned} D(X) &= \{x \in X : \|x\| < 1\} \\ S(X) &= \{x \in X : \|x\| = 1\} \\ B(X) &= \{x \in X : \|x\| \leq 1\} \end{aligned}$$

(Yes, this notation is confusion. No, I didn't choose it.)

2.1 Proposition. *If X, Y are normed spaces and $S \in \mathcal{L}(X,Y)$, then the following are equivalent:*

(i) S is continuous

(ii) S is continuous at some $x_0 \in X$

(iii) $\|S\| = \sup_{x\in D(X)} \|Sx\| < \infty$.

Moreover, in this case, we have

$$\begin{aligned} \|S\| &= \min\{L > 0 : \|Sx\| \leq L\|x\| \text{ for } x \in X\} \\ &= \sup_{x\in S(X)} \|Sx\| = \sup_{x\in B(X)} \|Sx\| \end{aligned}$$

Proof $(i \Rightarrow ii)$ Obvious

$(ii \Rightarrow iii)$ Note that

$$Sx_0 + D(Y) = \{Sx_0 + y : t \in D(Y)\} = \{y \in Y : \left\|Sx_0 - y'\right\| < 1\}$$

is a neighbourhood of Sx_0. By the definition of metric continuity, there is $\delta > 0$ such that

$$x_0 + \delta D(X) = \{x_0 + \delta x : x \in D(x)\} = \{x' \in X : \left\|x_0 - x'\right\| < \delta\}$$

such that

$$Sx_0 + \delta S(D(X)) = S(x_0 + \delta D(x) \subseteq Sx_0 + D(Y)$$

which implies that $\delta S(D(X)) \subseteq D(Y)$ and $S(D(X)) \subseteq D(Y)/\delta$, in other words that $\|Sx\| \leq 1/\delta$ for $x \in D(X)$.

$(iii \Rightarrow i)$ If $x \in X$ and $\epsilon > 0$, then

$$\|Sx\| = (\|x\| + \epsilon)\left\|S\left(\frac{1}{\|x\| + \epsilon}\|x\|\right)\right\| \leq (\|x\| + \epsilon)\|S\|$$

Then, letting $\epsilon \to 0^+$, we see that

$$\|Sx\| \leq \|x\|\|S\| = \|S\|\|X\|$$

If $x, x' \in X$, then $\|Sx - S'x\| \leq \|S\|\|x - x'\|$ is S is Lipschitz, hence continuous.

To complete the proof, the content of (iii) implies (i) tellus us that the Lipschitz constant $L(S) \leq \|S\|$. Furthermore, if $\|x\| = 1$, the preceding proof gives us that $\|S\|_{S(X)}$.

Conversely,

$$\|S\| = \sup_{x \in D(X) \setminus \{0\}} \|Sx\| = \sup_{x \in D(X) \setminus \{0\}} \|x\|\left\|S\left(\frac{1}{\|x\|}x\right)\right\| \leq \sup_{x \in S(X)} \|Sx\|$$

The remaining equivalence is obvious. ■

We now let $\mathcal{B}(X, Y) = \{\, S \in \mathcal{L}(X, Y) \mid S \text{ is bounded} \,\}$. We will see that $\|\cdot\|$, above, defines a norm on $B(X, Y)$.

2.2 Theorem. *If X, Y are normed spaces, then $(\mathcal{B}(X, Y), \|\cdot\|)$ is a normed space. Furthermore, if Y is a Banach spaces, then so to is $(\mathcal{B}(X, Y), \|\cdot\|)$.*

Proof Define

$$\Gamma : \mathcal{B}(X, Y) \to C_b^Y(B(X))$$

given by $\Gamma(S) = S|_{B(X)}$. Then, by definition, Γ is linear, with

$$\|\Gamma(S)\|_\infty = \sup_{x \in B(X)} \|Sx\| = \|S\|$$

Thus $\|\cdot\|$ is a norm: if $S, T \in \mathcal{B}(X, Y)$, $\alpha \in \mathbb{F}$,

$$\|S + T\| = \|\Gamma(S + T)\|_\infty = \|\Gamma(S) + \Gamma(T)\|_\infty \leq \|\Gamma(S)\|_\infty + \|\Gamma(T)\|_\infty = \|S\| + \|T\|$$
$$\|\alpha S\| = \|\Gamma(\alpha S)\|_\infty = |\alpha|\|\Gamma(S)\|_\infty = |\alpha|\|S\|.$$

Furthermore, $\Gamma : \mathcal{B}(X,Y) \to C_b^Y(\mathcal{B}(X))$ is an isometry.

Now suppose that Y is a Banach space. We will show that $\Gamma(\mathcal{B}(X,Y))$ is closed in $C_b^Y(B(X))$, and hence $B(X,Y) = \Gamma^{-1}(\Gamma(\mathcal{B}(X,Y)))$ is complete. Let $(S_n)_{n=1}^{\infty} \subset \mathcal{B}(X,Y)$ be $\|\cdot\|$–Cauchy. Then $(\Gamma(S_n))_{n=1}^{\infty}$ is $\|\cdot\|_\infty$–Cauchy in $C_b^Y(B(X))$, and hence there is $f \in C_b^Y(B(X))$ such that $\lim_{n\to\infty} \|\Gamma(S_n) - f\|_\infty = 0$. Then we let $S : X \to Y$ be given by

$$Sx = \begin{cases} \|x\| f\left(\frac{x}{\|x\|}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

If $x, x' \in X$ and $\alpha \in \mathbb{F}$ are all such that $x, x', x + \alpha x' \neq 0$, then

$$\begin{aligned} S(x+\alpha x') &= \left\|x+\alpha x'\right\| f\left(\frac{1}{x+\alpha x'}(x+\alpha x')\right) \\ &= \left\|x+\alpha x'\right\| \lim_{n\to\infty} S_n\left(\frac{1}{x+\alpha x'}(x+\alpha x')\right) \\ &= \lim_{n\to\infty}(S_n x + \alpha S_n x') = \lim_{n\to\infty}\left[\|x\| S_n\left(\frac{1}{\|x\|}x\right) + \alpha \left\|x'\right\| S_n\left(\frac{1}{\|x\|}x'\right)\right] \\ &= \|x\| f\left(\frac{x}{\|x\|}\right) + \alpha \left\|x'\right\| f\left(\frac{x'}{\|x\|}\right) \\ &= Sx + \alpha S x' \end{aligned}$$

The above computation is easily performed if any of x, x', $x+\alpha x'$ are 0. Hence $S \in \mathcal{L}(X,Y)$. We se that S is continuous (say, at a point on $S(X)$), so $S \in \mathcal{B}(X,Y)$. Finally, as $S|_{B(X)} = f = \lim_{n\to\infty} S_n|_{B(X)}$ (with respect to the uniform norm), we have

$$\|S - S_n\| = \sup_{x\in B(X)} \|(S-S_n)x\| = \|f - \Gamma(S_n)\|_\infty$$

goes to 0 as n goes to infinity. ∎

Definition. Given a vector space X, let $X' = \mathcal{L}(X,\mathbb{F})$ denote the **algebraic dual**. If further X is a normed space, we let $X^* = \mathcal{B}(X,\mathbb{F})$ denote the (continuous) dual.

2.3 Corollary. *If X is a normed spaces, then X^* is always a Banach space.*

2.4 Theorem. *Let for $x \in \ell_1$, $f_x : c_0 \to \mathbb{F}$ be given by $f_x(y) = \sum_{j=1}^{\infty} x_j y_j$. Then $f_x \in c_0^*$ with $\|f_x\| = \|x\|_1$. Furthermore, every element of c_0^* arises as above.*

Proof If $x \in \ell_1$ and $y \in c_0 \subseteq \ell_\infty$, then

$$\sum_{j=1}^{\infty} |x_j y_j| \leq \sum_{j=1}^{\infty} |x_j| \left\|y\right\|_\infty = \|x\|_1 \left\|y\right\|_\infty < \infty$$

so $f_x(y) = \sum_{j=1}^{\infty} x_j y_j$ is well-defined. It is obvious that f_x is linear: $f_x(y+\alpha y') = f_x(y) + \alpha f(y')$ for $y, yl \in c_0$ and $\alpha \in \mathbb{F}$. Also, $\|f_x\| \leq \|x\|_1$. We let $y^n = (\overline{\operatorname{sgn} x}, \dots, \overline{\operatorname{sgn} x_n}, 0, 0, \dots) \in c_0$, with $\left\|y^n\right\| = 1$. Then

$$\|f_x\| \geq |f_x(y^n)| = \sum_{j=1}^{n} x_i \overline{\operatorname{sgn} x_i} = \sum_{j=1}^{n} |x_i|$$

so that $\|f_x\| \geq \|x\|_1$, and hence equality holds.

Now let $f \in c_0^*$, and write $e_n = (0, \ldots, 0, 1, 0, 0, \ldots) \in c_0$, and let $x_n = f(e_n)$. Then, let $y \in c_0$ and $y^n = (y_1, \ldots, y_n, 0, 0, \ldots)$ and we have

$$\left\|y - y^n\right\|_\infty = \sup_{j \geq n+1} |y_j|$$

which goes to 0 as n goes to infinity. Then since f is continuous, we have

$$f(y) = \lim_{n\to\infty} f(y^n) = \lim_{n\to\infty} \sum_{j=1}^{n} y_j x_j = \sum_{j=1}^{\infty} x_j y_j = f_x(y)$$

We use sequence $(y^n)_{n=1}^\infty$ as in $y^n \in c_0$, to see that

$$\sum_{j=1}^{n} |x_i| = |f(y^n)| \leq \|f\| < \infty$$

so $x \in \ell_1$. Thus $f = f_x$, as desired. ∎

2.5 Corollary. *$\ell_1 \cong c^*$ isometrically isomorphically.*

Proof For $y \in c$, let $L(y) = \lim_{n\to\infty} y_n$. Given $y \in c$, let $y^n = (y_1, \ldots, y_n, L(y), L(y), \ldots) \in c$. Notice that $\left\|y - y^n\right\|_\infty \to 0$ similarly as above.

We let $1 = (1, 1, \ldots)$, and $1_n = (0, \ldots, 0, 1, 1, \ldots)$. If $m < n$, then $1_n - 1_m \in c_0$, so

$$|f(1_n) - f(1_m)| = |f_x(1_n - 1_m)| \leq \sum_{j=m+1}^{n} |x_j|$$

so that $(f(1_n))_{n=1}^\infty$ is Cauchy in $\mathbb{F}$. Let $x_0 = \lim_{n\to\infty} f(1_n)$. Let $\tilde{x} = (x_0, x_1, \ldots) \in \ell_1$. Then letting $x_j = f(e_j)$, we see that

$$f(y) = \lim_{n\to\infty} f(y^n) = \sum_{j=1}^{\infty} x_j y_j + x_0 L(y)$$ ∎

Similarly as above, we may show that $\|f\| = \|\tilde{x}\|_1$.

Remark. We write $c_0^* \cong \ell_1$ isometrically.

2.6 Corollary. *$(\ell_1, \|\cdot\|_1)$ is complete.*

3 Axiom of Choice and the Hahn-Banach Theorem

Definition. Let S be a non-empty set. A **partial ordering** is a binary relation $\leq$ on S which satisfies for $s, t, n \in S$,

(i) *(reflexivity)* $s \leq s$
(ii) *(transitivity)* $s \leq t$, $t \leq u$ implies $s \leq u$
(iii) *(anti-symmetry)* $s \leq t$, $t \leq s$ implies $s = t$

We call the pair $(S,\leq)$ a **partially ordered set**. We say that $(S,\leq)$ is **totally ordered** if, given $s,t \in S$, at least one of $s \leq t$ or $t \leq s$ holds. We say that $(S,\leq)$ is **well-ordered** if given any $\emptyset \neq S_0 \subseteq S$, there is some $s_0 \in S_0$ such that $s_0 \leq s$ for $s \in S_0$. A **chain** in a poset $(S,\leq)$ is any $\emptyset \neq C \subseteq S$ such that $(S, \leq|_C)$ is totally ordered.

Example. (i) $X \neq \emptyset$, $(\mathcal{P}(X),\subseteq)$ is a poset
(ii) $(\mathbb{R},\leq)$ is a totally ordered set
(iii) $(\mathbb{N},\leq)$, $(\omega = \mathbb{N}\cup\{\infty\},\leq)$, are well-ordered sets.

3.1 Theorem. *The following are equivalent:*

(i) (Axiom of Choice 1): For any $x \neq \emptyset$, there is a function $\gamma : \mathcal{P}(X)\setminus\{\emptyset\} \to X$ such that $\gamma(A) \in A$ for each $A \in \mathcal{P})X_{\setminus}\{\emptyset\}$.

(ii) (Axiom of Choice 2): Given any $\{A_\lambda\}_{\lambda\in\Lambda}$ where $A_\lambda \neq \emptyset$ for each λ,

$$\prod_{\lambda\in\Lambda} A_\lambda = \{(a_\lambda)_{\lambda\in\Lambda} : a_\lambda \in A_\lambda \textit{ for each } \lambda\} \neq \emptyset$$

(iii) (Zorn's Lemma): In a poset $(S,\leq)$, if each chain $C \subseteq S$ admits an upper bound in S, then $(S,\leq)$ admis a maximal element.

(iv) (Well-ordering principle): Any $S \neq \emptyset$ admits a well-ordering

Proof Exercise. ■

Definition. Let X be a vector space (over k). A subset $S \subseteq X$ is called

- **linearly independent** if for any distinct $x_1,\dots,x_n \in S$, the equation $0 = \alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ where $\alpha_i \in k$ implies $\alpha_1 = \cdots = \alpha_n = 0$.
- **spanning** if each $x \in X$ admits $x_i \in S$ and $\alpha_i \in k$ such that $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$.
- **Hamel basis** if it is both linearly independent and spanning

3.2 Proposition. *Any vector space X admits a Hamel basis.*

Proof Let $\mathcal{L} = \{L \subseteq X : L \text{ is linearly independent}\}$. Then $(\mathcal{L},\subseteq)$ is a poset. Verify that for any chain $\mathcal{C} \subseteq \mathcal{L}$, that $U = \bigcup_{L\in\mathcal{C}} L \in \mathcal{L}$ and is an upper bound for $\mathcal{C}$. Apply Zorn to find a maximal element M in $(\mathcal{L},\subseteq)$. Verify that M is spanning for X. ■

3.3 Corollary. *If X is an infinite dimensional normed space, then there exists $f \in X' \setminus X^*$.*

Proof Our assumption provides $\{e_n\}_{n=1}^\infty$ which is linearly independent. By normalizing each element, we may and will suppose that each $\|e_n\| = 1$. Let

$$\operatorname{span}\{e_n\}_{n=1}^\infty = \left\{ \sum_{j=1}^m \alpha_j e_{n_j} : m \in \mathbb{N}, \alpha_i \in \mathbb{F}, n_1 < \cdots < n_m \right\}$$

and let B be any linearly independent set containing $\{e_n\}_{n=1}^\infty$. Define $f : X = \operatorname{span} B \to \mathbb{F}$ be given for $x = \sum_{b\in B\setminus\{e_n\}_{n=1}^\infty} \alpha_b b + \sum_{j=1}^n \alpha_j e_{n_j}$ by $f(x) = \sum_{j=1}^m \alpha_j n_j$. The point is that $f(e_n) = n$ and $f(e) = 0$ for any other $e \in B$. Notice that

$$\|f\| = \sup_{x\in B(X)} |f(x)| \geq \sup_{n\in\mathbb{N}} |f(e_n)| = \sup_{n\in\mathbb{N}} n = \infty$$ ■

Definition. Let X be a $\mathbb{R}-$vector space. A **sublinear functional** is any $\rho : X \to \mathbb{R}$ such that it satisies

- *(non-negative homogenity)* $\rho(tx) = t\rho(x)$ for $t \geq 0$, $x \in X$.
- *(subadditivity)* $\rho(x+y) \leq \rho(x) + \rho(y)$ for $x, y \in X$.

3.4 Theorem. (Hahn-Banach) *Let X be a $\mathbb{R}-$vector space, $\rho : X \to \mathbb{R}$ a sublinear functional, $Y \subseteq X$ a subspace and $f \in Y'$ such that $f \leq \rho|_Y$. Then there exists $F \in X'$ such that $F|_Y = f$ and $F \leq \rho$ on X.*

Proof We first do this for extensions by a single point $x \in X \setminus Y$. We wish to find $c \in \mathbb{R}$ such that

$$f(y) + \alpha c \leq \rho(y + \alpha x)$$

for $y \in Y$ and $\alpha \in \mathbb{R}$. In this case, we let $F : \operatorname{span} Y \cup \{x\} \to \mathbb{R}$ be given by $F(y+\alpha x) = f(y)+\alpha c$, and we have that F is linear and satisfies $F \leq \rho$ on $\operatorname{span} Y \cup \{s\}$. To do this, let y_+, y_- in Y and observe that $f(y_+) + f(y_-) = f(y_+ + y_-) \leq \rho(y_+ + y_-) \leq \rho(y_+ + x) + \rho(y_- - x)$ so that $f(y_-) - \rho(y_- - x) \leq \rho(y_+ + x) - f(y_+)$. It thus follows that

$$\sup\{f(y) - \rho(y - x) : y \in Y\} \leq \in \{\rho(y + x) - f(y) : y \in Y\}$$

so we may find $c \in \mathbb{R}$ for which

$$\sup\{f(y) - \rho(y - x) : y \in Y\} \leq c \leq \inf\{\rho(y + x) - f(y) : y \in Y\}$$

If $t > 0$, then for $y \in Y$,

$$c \leq \rho\left(\frac{1}{t}y + x\right) - f\left(\frac{1}{t}y\right) \Rightarrow tc \leq \rho(y + tx) - f(y) \Rightarrow f(y) + tc \leq \rho(y + tx)$$

and if $s > 0$, then for $y \in Y$,

$$f\left(\frac{1}{s}y\right) - \rho\left(\frac{1}{s}y - x\right) \leq c \Rightarrow sc \leq f(y) - \rho(y + sx) \Rightarrow f(y) - sc \leq \rho(y - sx)$$

Clearly, $f(y) + 0 \leq \rho(y + 0x)$. Hence, we have our desired inequality.

We now use Zorn's lemma to lift this result to the whole space. Consider the set of "p-extensions" of f,

$$\mathcal{E} = \left\{ (\mathcal{M}, \psi) \;\middle|\; Y \subseteq \mathcal{M} \subseteq X, \mathcal{M} \text{ is a subspace}, \psi \in \mathcal{M}', \psi|_Y = f, \psi \leq P|_{\mathcal{M}} \right\}$$

Define a partial order on $\mathcal{E}$ by

$$(\mathcal{M}, \psi) \leq (\mathcal{N}, \phi) \text{ iff } \mathcal{M} \subseteq \mathcal{N}, \phi|_{\mathcal{M}} = \psi$$

Suppose $\mathcal{C} \subseteq \mathcal{E}$ is a chain with respect to $\leq$. We let

- $\mathcal{U} = \bigcup_{(\mathcal{M},\varphi)} \mathcal{M}$ which is a subspace, since $\mathcal{C}$ is a chain.
- and define $\phi : \mathcal{U} \to \mathbb{R}$ by $\phi(x) = \psi(x)$ whenever $x \in \mathcal{M}$, which is again well-defined since C is a chain.

Furthermore, we see that $\phi \in U'$, since if $x, y \in \mathcal{U}$, get $x \in \mathcal{M}$, $y \in \mathcal{N}$ for some $(\mathcal{M}, \psi) \leq (\mathcal{N}, \psi') \in \mathcal{C}$. Then $\phi(x+y) = \psi'(x+y) = \psi'(x) + \psi'(y) = \phi(x) + \phi(y)$, etc. Likewise, $\psi \leq p|_{\mathcal{U}}$. Thus by Zorn's lemma, $\mathcal{E}$ admits a maximal element $\mathcal{M}, F$ Then $\mathcal{M} = X$, for if not, then we would find $x \in X \setminus \mathcal{M}$ and we apply step one to $\operatorname{span} \mathcal{M} \cup \{x\}$ to get F', a strictly larger element violating maximality. ∎

Trivially, any $\mathbb{C}$−vector siace is a $\mathbb{R}$−vector space.

3.5 Lemma. *Let X be a $\mathbb{C}$−vector space.*

(i) If $f \in X'_{\mathbb{R}}$ into $\mathbb{R}$, then define $f_{\mathbb{C}}$ given by $f_{\mathbb{C}}(x) = f(x) - if(ix)$ defines an element of $X' = X'_{\mathbb{C}}$.

(ii) If $g \in X'$, then $f = \operatorname{Re} g$ in $X'_{\mathbb{R}}$ satisfies $g = f_{\mathbb{C}}$.

(iii) If X is a normed $\mathbb{C}$−vector space, then for $f \in X'_{\mathbb{R}}$,

$$f \in X^*_{\mathbb{R}} \text{ if and only if} f_{\mathbb{C}} \in X^* = X^*_{\mathbb{C}} \text{ with } \|f\| = \|f_{\mathbb{C}}\|$$

Proof (i) and (ii) are straightforward exercises; let's see (iii). We let fr $x \in X$, $z = \operatorname{sgn} f_{\mathbb{C}}(x)$. Then

$$\mathbb{R} \ni |f_{\mathbb{C}}(x)| = \overline{z} f_{\mathbb{C}}(x) = f_{\mathbb{C}}(\overline{z}x) = \operatorname{Re} f_{\mathbb{C}}(\overline{z}x) = f(\overline{z}x) = |f(\overline{z}x)|$$
$$\leq \|f\|\|\overline{z}x\| = \|f\||\overline{z}|\|x\| = \|f\|\|x\|$$

so we see that $\|f_{\mathbb{C}}\| \leq \|f\|$. Conversely,

$$|f(x)| = |\operatorname{Re} f_{\mathbb{C}}(x)| \leq |f_{\mathbb{C}}(x)| \leq \|f_{\mathbb{C}}\|\|x\| \text{ so that } \|f\| \leq \|f_{\mathbb{C}}\|$$

■

3.6 Corollary. *If X is a normed space, $Y \subseteq X$ a subspace and $f \in Y^*$, then there exists $F \in X^*$ such that $F|_Y = f$ and $\|F\| = \|f\|$.*

Proof Define $\rho : X \to \mathbb{R}$ be given by $p(x) = \|f\| \cdot \|x\|$, so p is sublinear and $\operatorname{Re} f \leq p|_Y$. Apply Hahn-banach to to this data and get $\tilde{F} \in X^*_{\mathbb{R}}$ such that $\tilde{F}|_Y = \operatorname{Re} f$ and $\tilde{F} \leq p$, and let $F = \tilde{F}_{\mathbb{C}}$. ■

3.7 Corollary. *If X is a normed space, $x \in C$, then there is $f \in X^*$ such that*

$$\|x\| = f(x) = |f(x)| \text{ and } \|f\| = 1$$

Proof Let $f_0 : \mathbb{F}x \to \mathbb{F}$ be given by $f_0(\alpha x) = \alpha\|x\|$. If $x \neq 0$, then

$$\|f_0\| = \sup_{\|\alpha x\| \leq 1} |f_0(\alpha x)| = \sup_{\|\alpha x\| \leq 1} |\alpha|\|x\| = 1$$

and apply the previous corollary. If $x = 0$, this is trivial. ■

3.8 Theorem. *Let X be a normed space and X^{**} denote the bidual. For $x \in X$, define $\hat{x} : X^* \to \mathbb{F}$ by $\hat{x}(f) = f(x)$. Then $\hat{x} \in X^{**}$ with $\|\hat{x}\| = \|x\|$, so that $x\hat{x} : X \to X^{**}$ is a linear isometry.*

Proof Notice that $|\hat{x}(f)| = |f(x) \leq \|f\|\|x\|$ so $\|\hat{x}\| \leq \|x\|$. The last corollary provides for $x \in X$ an $f_x \in S(X^*)$ with $|f_x(x)| = \|x\|$. Then $\|\hat{x}\| \leq |\hat{x}(f_x)| = \|x\|$. Hence $\|\hat{x}\| = \|x\|$. Clearly $x \mapsto \hat{x}$ is linear. ■

Remark. Since X^{**}, being a dual space, is complete, we have that $\hat{X} = \{\hat{x} : x \in X\}$ satisfies that its closure $\overline{\hat{X}} \subseteq X^{**}$ is complete. Hence $\overline{\hat{X}}$ is a Banach space containing a dense copy of X. Often, we will simply write $\overline{\hat{X}} = \overline{X}$ and call it the **completion** of X.

Geometric Hahn-Banach

If $A, B \subset X$ with $A \cap B = \emptyset$ (and other suitable assumptions), we will find a $\mathbb{R}-$hyperplane between A and B.

Definition. In a vector space, a **hyperplane** is any set of the form $x_0 + \ker f$ with $x_0 \in X$ and $f \in X'$. Then a $\mathbb{R}-$**hyperplane** is any set of the form $x_0 + \ker \operatorname{Re} f$.

> ***3.9 Proposition.*** *Let X be a normed space.*
> *(i) If $f \in X^* \setminus \{0\}$, then $\underline{\ker f}$ is closed and nowhere dense.*
> *(ii) if $f \in X' \setminus X^*$, then $\overline{\ker f} = X$.*
> *Thus a hyperplane in X is either closed and nowhere dense, or it is dense.*

Proof To see (i), $\ker f = f^{-1}(\{0\})$ is a closed set since f is continuous. Furthermore, if $Y \subsetneq X$ is a proper closed subspace, then it is nowhere dense. If not, then there would exist $y_0 \in T$ and $\delta > 0$ such that $y_0 + \delta D(X) \subseteq Y$. But then $D(X) \subseteq \frac{1}{\delta}(Y - y_0) = Y$, so $X = \operatorname{span} D(X) \subseteq Y$, a contradiction.

To see (ii), suppose that $\ker f$ is not dense in X. Then there would be $x_0 \in X$ and $\delta > 0$ such that $(x_0 + \delta D(X)) \cap \ker f = \emptyset$, so

$$0 \notin f(x_0 + \delta D(X)) = f(x_0) + \delta f(D(X)) \Longrightarrow \frac{1}{\delta} f(x_0) \notin -f(D(X)) = f(D(X)) \tag{3.1}$$

But then $\|f\| \leq \frac{1}{\delta} f(x_0)$, for if $\|f\| > \frac{1}{\delta} f(x_0)$, there would be $x \in D(X)$ such that $|f(x)| > \frac{1}{\delta}|f(x_0)|$. Thus

$$\left|\frac{f(x_0)}{\delta f(x)}\right| < 1 \Longrightarrow \frac{f(x_0)}{\delta f(x)} = \frac{1}{\delta} f(x)$$

contradicting the statement in (3.1). ■

Definition. Let $\emptyset \neq A \subseteq X$. We say that A is

- **convex** if for $a, b \in A$ and $0 < \lambda < 1$, $(1-\lambda)a + \lambda b \in A$.
- **absorbing** at $a_0 \in A$ if for any $x \in X$, there is $\epsilon(a_0, x) > 0$ such that $a_0 + tx \in A$ for $0 \leq t < \epsilon$.

For example, if X is a normed space, then any open set is absorbing around any of its points.

> ***3.10 Lemma. (Minkowski Functional)*** *Let $A \subset X$ be a convex set containing 0 and absorbing at 0. Define $p : X \to \mathbb{R}$ by $p(x) = \inf\{t > 0 : x \in tA\}$. Then p is a sublinear functional. Moreover, we have that*
> *(i) $\{x \in X : p(x) < 1\} \subseteq A \subseteq \{x \in X : p(x) \leq 1\}$; and*
> *(ii) if X is normed and A is a neighbourhood of 0, then there is $N > 0$ such that $p(x) \leq N\|x\|$ for $x \in X$.*

Proof First note, for any $x \in X$, if A is absorbing at 0, there is $s > 0$ such that $sx \in A$, so $x \in \frac{1}{s}A$ and hence $0 \leq p(x) < \infty$.

Let's see non-negative homogeneity. Clearly $p(0) = 0$. If $s > 0$ and $x \in X$, then

$$p(sx) = \inf\{t > 0 : sx \in tA\} = \inf\left\{t > 0 : x \in \frac{t}{s}A\right\} = s \cdot \inf\left\{\frac{t}{s} > 0 : x \in \frac{t}{s}\right\} = sp(x)$$

We also have subadditivity. First, note that if $s,t>0$ and $a,b\in A$, then

$$sa+tb=(s+t)\left(\frac{s}{s+t}a+\frac{s}{s+t}b\right)\in(s+t)A\Longrightarrow sA+tA\subseteq(s+t)A$$

by convexity, and also $(s+t)A=\{(s+t)a:a\in A\}\subseteq\{sa+tb:a,b\in A\}=sA+tA$. Thus $sA+tA=(s+t)A$. Now for $x,y\in X$, we have

$$\begin{aligned}p(x)+p(y)&=\inf\{s>0:x\in sA\}+\inf\{t>0:y\in tA\}\\&=\inf\{s+t:s>0,t>0,x\in sA,y\in tA\}\\&\geq\inf\{s+t:s>0,t>0,x+y\in sA+tA=(s+t)A\}\\&=\inf\{r>0:x+y\in rA\}=p(x+y)\end{aligned}$$

so that p is a sublinear functional. Then

(i) If $p(x)<1$, then there is $0<t<1$ so $x\in tA$; i.e. $\frac{1}{t}x\in A$ and $x=(1-t)=+t\frac{1}{t}x\in A$. The second inclusion is obvious.

(ii) The assumptions provide $\delta>0$ so $\delta D(X)\subseteq A$. Then for $x\in X$ and $\epsilon>0$,

$$x\in(\|x\|+\epsilon)D(X)=\frac{\|x\|+\epsilon}{\delta}\delta D(X)\subseteq\frac{\|x\|+\epsilon}{\delta}A$$

so $p(x)\leq\frac{\|x\|+\epsilon}{\delta}$ so $p(x)\leq\frac{1}{\delta}\|x\|$; the result follows with $N=1/\delta$. ■

3.11 Theorem. (Hyperplane Separation) *Let X be an $\mathbb{F}$ −vector space, $A,B\subset X$ be convex with $A\cap B=\emptyset$ and A absorbing at some a_0. Then there are $f\in X'$ and $\alpha\in\mathbb{R}$ such that*

$$\operatorname{Re}f(a)\geq\alpha\geq\operatorname{Re}f(b)$$

for $a\in A$ and $b\in B$. Moreover, if X is normed, then

- *If A is a neighbourhood of a_0, we have $f\in X^*$; and*
- *if A is absorbing around each of its points (for example if A is open), then we have $\operatorname{Re}f(a)>\alpha\geq\operatorname{Re}f(b)$.*

Proof We first re-centre at 0. Let $A-B=\{a-b:a\in A,b\in B\}$. Then it is easy to verify that

(i) $A-B$ is absorbing at any a_0-b, $b\in B$

(ii) $A-B$ is convex

(iii) if X is normed and A a neighbourhood of a_0, then $A-B$ is a neighbourhood of each a_0-b, $b\in B$; and if A is absorbing around any of its points (resp. open), then A_B is absorbing around any of its points (resp. open).

Let $x_0=a_0-b_0$ for some $b_0\in V$, and set $C=x_0-(A-B)$, so we have $0=x_0-x_0\in C$. Then by the above points, C is absorbing at 0, convex, and if X is normed and A a neighbourhood of a_0, then C is a neighbourhood of 0; and if A is absorbing at any of its points (resp. A is open), then C is absorbing at each of its points (resp. open).

Let p be the Minkowski functional of C. Notice that since $A\cap B=\emptyset$, $0\notin A-B$ so $x_0\notin C$. Thus by (i) of the lemma, $p(x_0)>1$.

Let us find f and α. Let $f_0:\mathbb{R}x_0\to\mathbb{R}$, by $f_0(sx)=sp(x_0)$. Hence f_0 is linear and $f_0\leq p|_{Rx_0}$, so by Hahn-Banach, get $f\in X'_{\mathbb{R}}$ such that $f\leq p$ on X. If $a\in A$ and $b\in B$, then

$x_0-(a-b) \in C$, so by (i) of the lemma, since $p(x_0) \geq 1$, we have $f(x_0-(a-b)) \leq p(x_0-(a-b)) \leq 1$. Thus $f(x_0)+f(b) \leq 1+f(a)$ so in fact $f(b) \leq f(a)$. Thus there exists some $\alpha \in \mathbb{R}$ such that

$$\sup\{f(b) : b \in B\} \leq \alpha \leq \inf\{f(a) : a \in A\}$$

If $\mathbb{F} = \mathbb{R}$, we are done; otherwise, we shall replace f by $f_{\mathbb{C}}$

For the remainder of the proof, we suppose X is a normed space, and A is a neighbourhood of a_0. Then part (ii) of the lemma provides $N > 0$ so that $p(x) \leq N\|x\|$. Then for $x \in X$, $f(x) \leq p(x) \leq N\|x\|$ and $-f(x) = p(-x) \leq N\|-x\| = N\|x\|$ so $|f(x)| \leq N\|x\|$, in other words that $\|f\| \leq N$ and $f \in X^*$. If A is absorbing around any of its points, then $f(a) > \alpha$ for any $a \in A$. Indeed, suppose $f(a) = \alpha$. Then there would be $t > 0$ so $a + t(-x_0) \in A$. But then $\alpha \leq f(a - tx_0) = f(a) - tf(x_0) < \alpha$, a contradiction. ■

Definition. If $\emptyset \neq S \subset X$, then its **convex hull** is given by

$$(S) = \{\sum_{i=1}^{n} \lambda_j x_j : n \in \mathbb{N}, x_1, \ldots, x_n \in S \text{ and } \lambda_1, \ldots, \lambda_n \geq 0 \text{ with } \sum_{j=1}^{n} \lambda_j = 1\}$$

One can verify that (S) is in fact convex, and is the smallest convex set containing S, i.e.

$$(S) = \bigcap\{C : S \subseteq C \subseteq X, C \text{ convex}\}$$

If X is normed, we let (S) denote the **closed convex hull**, i.e. the closure of the convex hull.

Definition. A **half-space** of X is any set of the form $H = \{x \in X : \operatorname{Re} f(x) \leq \alpha\}$ for some $f \in X'$, $\alpha \in \mathbb{R}$.

If X is normed, then the last proposition shows H is closed if and only if f is bounded.

3.12 Theorem. *If X is a normed vector space and $\emptyset \neq S \subset X$, then $(S) = \cap\{H : S \subseteq H \subset X, H \text{ a closed half space}\}$.*

Proof It is immediate that $(S) \subseteq \cap\{H : S \subseteq H \subset X, H \text{ a closed half-space}\}$. Thus suppose $x_0 \notin (S)$. Then there is $\delta > 0$ such that $(x_0 + \delta D(X)) \cap (S) = \emptyset$. Since $x_0 + \delta D(X)$ is open and convex, hyperplace separation gives provides $f \in X^*$ and $\alpha \in \mathbb{R}$ so $\operatorname{Re} f(a) > \alpha \geq \operatorname{Re} f(b)$ for $a \in x_0 + \delta D(X)$ and $b \in (S)$. Then $S \subset H = \{y \in X : \operatorname{Re} f(x) \leq \alpha\}$ but $x_0 \notin H$. ■

4 Some Applications of Baire Category Theorem

4.1 Theorem. (Baire Category I) *If (X,d) is a complete metric space and $\{U_n\}_{n=1}^{\infty}$ is a countable collection of dense, open subsets, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.*

Definition. Let (X,d) be a metric space. A subset $F \subset X$ is **nowhere dense** if $X \setminus F$ is dense in X; equivalently, $\overline{F}$ contains no non-trivlal open subsets. We say that a subset $M \subseteq X$ is **meagre** (1st category) if $M = \bigcup_{n=1}^{\infty} F_n$ and each F_n is nowehere dense; and a set is **non-meagre** (2nd category) otherwise.

4.2 Theorem. (Baire Category II) *Let (X,d) be a complete metric space. Then a non-empty open $U \subseteq X$ is non-meagre.*

Proof Suppose not, so $U = \bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} \overline{F}_n$, each F_n (hence $\overline{F_n}$) nowhere dense. Then each $V_n = X \setminus \overline{F_n}$ is open and dense, and hence by BCT I, $G = \bigcap_{n=1}^{\infty} V_n$ is dense in X, and hence $U \cap G \neq \emptyset$, violating assumption ■

4.3 Theorem. (Banach-Steinhaus) *Let X, Y be normed spaces, $U \subseteq X$ be non-meagre, and $\mathcal{F} \subset \mathcal{B}(X,Y)$ be such that for each $x \in U$, $\sup\{\|Tx\| : T \in \mathcal{F}\} < \infty$ (pointwise bounded). Then $\mathcal{F}$ is uniformly bounded, i.e. $\sup\{\|T\| : T \in \mathcal{F}\} < \infty$.*

Proof Let for each $n \in \mathbb{N}$

$$F_n = \bigcap_{T \in \mathcal{F}} T^{-1}(nB(Y)) = \{x \in X : \|Tx\| \leq n \text{ for all } T \in \mathcal{F}\}$$

so each F_n is closed and, by the pointwise boundedness assumption, $U \subseteq \bigcup_{n=1}^{\infty} F_n$. By assumption of non-meagreness of U, at least one F_{n_0} admis an interior point: there is $x_0 \in F_{n_0}$ and $\delta > 0$ such that $x_0 + \delta D(X) \subseteq F_{n_0}$. Then if $x \in D(X)$, we have

$$Tx = \frac{1}{\delta}\left[T\left(x_0 + \frac{\delta}{2}x\right) - T\left(x_0 - \frac{\delta}{2}x\right)\right]$$

so $\|Tx\| \leq \frac{2}{\delta} n_0$, in other words

$$\|T\| = \sup_{x \in D(x)} \|Tx\| \leq \frac{2n_0}{\delta} < \infty$$

where the bound is independent of T. ■

4.4 Theorem. (Open Mapping) *Let X, Y be Banach spaces, and $T \in B(X,Y)$ surjective. Then T is an open map; i.e. $T(U)$ is open in Y whenver U is open in X.*

Remark. Given $x \in X$ and $\alpha \in \mathbb{F} \setminus \{0\}$, non-empty $A \subset X$, we have that $\overline{x + \alpha A} = x + \alpha \overline{A}$. Indeed, note that for $(a_k)_{k=1}^{\infty} \subset A$, we have

$$a_k \to a \in \overline{A} \text{ if and only if } x + \alpha a_k \to x + \alpha a \in x + \alpha \overline{A}$$

4.5 Lemma. *With the assumptions as above, we have that if $\overline{T(D(X)} \supset rB(Y)$ for some $r > 0$, then $T(D(X)) \supseteq rD(Y)$.*

Proof Let $z \in rD(Y)$ and let $0 < \delta < 1$ be so $\|z\| < r(1-\delta) < r$. Set $y = z/(1-\delta)$ so $\|y\| < r/(1-\delta)$. It suffices to show that $y \in \frac{1}{1-\delta} T(D(X))$. To begin, let $A = T(D(X)) \cap rB(Y)$, so $\overline{A} = rB(Y)$. Indeed, if $y \in rB(Y) \subseteq \overline{T(D(X))}$, then there is $(y_k)_{k=1}^{\infty} \subset \overline{T(D(X))}$, so $y = \lim y_k$. But then there is $x_k \in D(X)$ so each $\|y_k - T(x_k)\| < 1/k$ so $y = \lim T(x_k)$ with each $x_k \in D(X)$.

Now we inductively build a sequence $(y_n)_{n=1}^{\infty}$ as follows.

- Since $y \in rD(Y) \subseteq \overline{A}$, there is $y_1 \in A \cap (y + \delta r D(Y))$
- $y \in y_1 + \delta r(D(Y)) \subseteq \overline{y_1 + \delta A} = y_1 + \delta \overline{A}$, so there is $y_2 \in (y_1 + \delta A) \cap (y + \delta^2 r D(Y))$
- $y \in y_n + \delta^n r D(Y) \subseteq \overline{y_n + \delta^n A}$, so there is $y_{n+1} \in (y_n + \delta^n A) \cap (y + \delta^{n+1} r D(Y))$

By construction, $y_{n+1} - y_n \in \delta^n A$, so $\|y_{n+1} - y_n\| \leq \delta^n r$ and there is $x_n \in \delta^n D(X)$ such that $y_{n+1} - y_n = Tx_n$. Likewise, $y_1 \in A \subseteq T(D(X))$ so $y = T(x_0)$ for some $x_0 \in D(X)$. Notice that each $y_n \in y + \delta^n r(Y)$, so $\|y_n - y\| \leq \delta^n r \to 0$. Since X is complete, we let $x = \sum_{n=0}^{\infty} x_n$, and by construction

$$\|x\| \leq \sum_{n=0}^{\infty} \|x_n\| < \sum_{n=0}^{\infty} \delta^n = \frac{1}{1-\delta}$$

Then by linearity and continuity of T, we have

$$Tx = \sum_{n=0}^{\infty} Tx_n = y_1 + \sum_{n=1}^{\infty}(y_{n+1} - y_n) = y_N + \sum_{n=N}^{\infty}(y_{n+1} - y_n) \to y$$

so that indeed $T(x) = y$, as required. ■

Remark. So far, we've only used completeness of X and continuity and linearity of T.

We now proceed with the proof of the open mapping theorem.

Proof It suffices to see that $T(D(X))$ contains a neighbourhood of 0 in Y. Indeed, if $\emptyset \neq U \subseteq X$ is open, $x \in U$, then there is $\delta > 0$ such that $x + \delta D(X) \subseteq U$, so $U - x \supseteq \delta D(X)$. If $T(D(X)) \supseteq rD(Y)$, then $T(U - x) \supseteq \delta T(D(X)) \supseteq r\delta D(Y)$ so that $Tx + r\delta D(Y) \subseteq T(U)$. In other words, $T(U)$ is a neighbourhood of any of its points, and thus open.

Now write $X = \bigcup_{n=1}^{\infty} nD(X)$, and we assume that $T(X) = Y$. Hence $Y = \bigcup_{n=1}^{\infty} nT(D(X))$, so $Y = \bigcup_{n=1}^{\infty} n\overline{T(D(X))}$. But Y is complete, so by Baire category theorem, there is some n so that $n\overline{T(D(X))}$ has non-empty interior. Since $nT(D(X))$ is convex and symmetric, and hence $n\overline{T(D(X))}$ is convex and symmetric as well. Thus if $y \in D(Y)$, then $y_0 \pm \epsilon \in y_0 + \epsilon D(Y)$ so

$$\epsilon y = \frac{1}{2}[y_0 + \epsilon y - (y_0 - \epsilon y)] \in n\overline{T(D(X))}$$

and $\frac{\epsilon}{n} y \in \overline{T(D(X))}$, i.e. $\frac{\epsilon}{n} D(Y) \subseteq \overline{T(D(X))}$. Thus applying the main lemma, $\frac{\epsilon}{n} D(Y) \subseteq T(D(X))$. ■

4.6 Theorem. (Inverse Mapping) *If X, Y are Banach spaces and $T \in \mathcal{B}(X, Y)$ is invertible, $T^{-1} \in \mathcal{B}(Y, X)$*

Proof Direct application of the open mapping theorem. ■

Let X, Y be normed spaces. Then we define for $(x, y) \in X \oplus Y$, and we let $\|(x, y)\|_1 = \|x\| + \|y\|$. It is easy to check that $\|\cdot\|_1$ is a norm on $X \oplus Y$, and if X, Y are Banach, then so is $(X \oplus Y, \|\cdot\|_1)$. In this case, we write $X \oplus_1 Y$.

4.7 Theorem. (Closed Graph) *Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Then $T \in \mathcal{B}(X, Y)$ if and only if $\Gamma(T) = \{(x, Tx) : x \in X\}$ is closed in $X \oplus_1 Y$.*

Proof Let $T \in \mathcal{B}(X, Y)$. If $(x_n) \to x$ in X, then $Tx_n \to Tx$ in Y. Thus if $(x, y) \in \overline{\Gamma(T)}$, then $(x, y) = \lim(x_n, Tx_n)$ where $(x_n, Tx_n) \in \Gamma(T)$. But then

$$\|y - Tx\| \leq \|y - Tx_n\| + \|Tx_n - Tx\| \leq \|x - x_n\| + \|y - Tx_n\| + \|Tx_n - tx\| = \|(x - y) - (x_n, Tx_n)\|_1$$

so in fact $y = Tx$ so $(x, y) = (x, Tx)$.

Conversely, if $\Gamma(T)$ is closed in $X \oplus_1 Y$, then $\Gamma(T)$ is a Banach space. Define $S : \Gamma(T) \to X$ by $S(x, Tx) = x$. Notice that S is linear, and

$$\|S(x, Tx)\| = \|x\| \le \|(x, Tx)\|_1$$

so $\|S\| \le 1$, so S is bounded. It is also clear that S is bijective, with $S^{-1} : X \to \Gamma(T)$ given by $S^{-1}(x) = (x, Tx)$. Thus the inverse mapping theorem gives that S^{-1} is also bounded. Hence for any $x \in X$,

$$\|Tx\| \le \|(x, Tx)\|_1 = \left\|S^{-1}x\right\| \le \|x\| \left\|S^{-1}\right\|$$

so that T is in fact bounded. ■

4.8 Theorem. (Closed graph test) *Given normed spaces and $T \in \mathcal{L}(X, Y)$, we have that $\Gamma(T)$ is closed in $X \oplus_1 Y$ if and only if whenever $x_n \to 0$ for which we may assume that Tx_n converges in Y, say $y = \lim Tx_n$, then $y = 0$ too.*

Proof We have $(x_n, Tx_n) \to (x, z) \in \overline{\Gamma(T)}$ if and only if $(x_n - x, T(x_n - x)) \to (x, z) - (x, Tx) = (0, z - Tx)$. Set $y = z - Tx$. We have $(x, z) \in \Gamma(T)$ if and only if $z = Tx$ if and only if $y = 0$. ■