PMATH 465

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Contents

I. Fundamentals of Manifolds

1 Introduction to Topology

Basic Constructions

Definition. A **topology** on a set X is a set τ of subsets of X such that
- (i) $\emptyset \in \tau$ and $X \in \tau$
- (ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
- (iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \le i \le n$, then $\bigcap_{i=1}^{n} U_i \in \tau$.

The sets $U \in \tau$ are called the **open sets** in X, and sets of the form $X \setminus U$ for some open set U are caled the **closed sets** in X.

Definition. When X is a topological space and $A \subseteq X$, the **interior** of A (denoted A°) is the union of all open sets contained in A. Similarly, we define the **closure** of A (denoted $\overline{A}$) as the intersction of all closed sets containing A. Then the **boundary** of A, denoted by ∂A, is the set $\partial A = \overline{A} \setminus A^\circ$.

Example. Let X be any set. The **discrete topology** on X is the topology $\tau = \mathcal{P}(X)$, and the **trivial topology** on X is the topology $\tau = \{\emptyset, X\}$.

Definition. A **basis** for a topology on a set X is a set $\mathcal{V}$ of subsets of X
- (i) $\bigcup_{B \in \mathcal{B}} b = X$
- (ii) for all $a \in X$ and $U, V \in \mathcal{B}$ such that $a \in U \cap V$, then there exists $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$.

When $\mathcal{B}$ is a basis for a topology on X, the topology on X **generated** by $\mathcal{B}$ is the set τ of subsets of X such that for $W \subseteq X$, $W \in \tau$ if and only if for all $a \in W$, there exists $U \in \mathcal{B}$ such that $a \in U \subseteq W$.

Note that τ, as above, is a topology on X since
- (i) $\emptyset \in \tau$ vacuously and $X \in \tau$ obviously.
- (ii) If $A_k \in \tau$ for all $k \in K$ (where K is any set of indices), then given $a \in \bigcup_{x \in K} A_k$, we can choose $\ell \in K$ so that $a \in A_\ell$. Then since $A_\ell \in \tau$, we can choose $U_\ell \in \mathcal{B}$ so that $a \in U_\ell \subseteq A_\ell$. Thus $a \in U_\ell \subseteq A_\ell \subseteq \bigcup_{k \in K} A_k$.
- (iii) By induction, it suffices to prove that if $A, B \in \tau$, then $A \cap B \in \tau$. Suppose $A, B \in \tau$, and let $a \in A \cap B$. Since $A \in \tau$, we cna choose $U \in \mathcal{B}$ so that $a \in U \subseteq A$. Since $B \in \tau$, we can choose $V \in \mathcal{B}$ so that $a \in V \subseteq B$. Then we have $a \in U \cap V$. Since $\mathcal{B}$ is a basis, we can chose $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$, so $a \in W \subseteq U \cap V \subseteq A \cap B$.

Note that when τ is the topology on X generated by the basis $\mathcal{B}$, for $A \subseteq X$, $A \in \tau$ if and only if there exists some $S \subseteq \mathcal{B}$ such that $A = \bigcup_{s \in S} s$. In this sense, the topology τ on X generated by the basis $\mathcal{B}$ is the coarsest topology which contains $\mathcal{B}$.

Definition. ***(Subspace Topology)*** When Y is a topological space and $X \subseteq Y$ is a subset of Y, we define the **subspace topology** on X to be the topology for which as set $U \subseteq X$ is open if and only if $U = X \cap V$ for some open set V.

If $\mathcal{C}$ is a basis for the topology on Y, then $\mathcal{B} = \{\, X \cap V \mid V \in \mathcal{C} \,\}$ is a basis for the subspace topology on X.

Definition. (Disjoint Union Topology) If X and Y are topological spaces with $X \cap Y = \emptyset$, then the **disjoint union topology** on $X \cup Y$ is the topology in which a subset $U \subseteq X \cup Y$ is open in $X \cup Y$ if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y.

Definition. (Product Topology) If X and Y are topological spaces, the **product topology** on $X \times Y$ is the topology generted by the basis

$$\mathcal{B} = \{\, U \times V \mid U \in \mathcal{C}, V \in \mathcal{D} \,\}$$

where $\mathcal{C}$ and $\mathcal{D}$ are bases for the topologies on X, Y respectively.

Definition. (Infinite Product Topology) We define the infinite product to be

$$\prod_{k \in K} \left\{ f : K \to \bigcup_{k \in K} X_k \,\middle|\, f(k) \in X_k \text{ for all } k \in K \right\}$$

There are two standard topologies on X. The first is the **box topology**,

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \,\middle|\, U_k \text{ is open in } X_k \right\}$$

and the **product topology**

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \,\middle|\, \begin{array}{l} U_k \text{ is open in } X_k \\ U_k = X_k \text{ for all but finitely many indices } k \end{array} \right\}$$

Example. (Metric Topology) $\mathbb{R}^n$ has a standard **inner product**, and for $u, v \in \mathbb{R}^n$, $uv = u \cdot v = V^T u = \sum_{i=1}^n u_i v_i$. This gives the standard norm on $\mathbb{R}^n$ for $u \in \mathbb{R}^n$, $\|u\| = \sqrt{uv}$. This gives the standard metric on $\mathbb{R}^n$: for $a, \in \mathbb{R}^n$, $d(a,b) = \|b - a\|$.

Given a metric on a set Y, we obtain (by restriction) an induced metric on any subset $X \subseteq Y$. Given a metric space X, we define the **metric topology** on X to be the topology which is generated by the set of open balls

$$B(a,r) = \{\, x \in X \mid d(a,x) < r \,\}$$

where $x \in X$, $r > 0$.

Maps on Topological Spaces

Definition. When X and Y are topological spaces and $f : X \to Y$, we say that f is **continuous** when it has the property that $f^{-1}(V)$ is open in X for every open set V in Y. We say that $f : X \to Y$ is a **homeomorphism** when f is bijective and both f and f^{-1} are continuous. Then X, Y are **homeomorphic** if there exists a homeomorphism $f : X \to Y$.

1.1 Theorem. (Glueing Lemma) *Let X and Y be topological spaces, and let $f : X \to Y$ be a function. Suppose either*
(i) $X = \bigcup_{k \in K} A_k$ where each A_k is open in X, or
(ii) $X = \bigcup_{k=1}^n A_k$ where each A_k is closed in X
and each restriction map $f_k : A_k \to Y$ is continuous, then f is continuous.

Proof Exercise. ∎

Definition. A topological space X is **compact** when it has the property that for every set $\mathcal{S}$ of open subsets of X with $X = \bigcup_{U \in S} U$, there exists a finite subset $\mathcal{F} \subseteq \mathcal{S}$ such that $X = \bigcup_{F \in \mathcal{F}} F$.

Note that when $X \subseteq Y$ is a subspace, X is compact if and only if X has the property that for every set $\mathcal{T}$ with $X \subseteq \bigcup_{T \in \mathcal{T}} T$, there exists a finite subset $\mathcal{G} \subseteq \mathcal{T}$ uch that $X \subseteq \bigcup_{G \in \mathcal{G}} G$.

Definition. A topological space X is **connected** when there do not exist non-empty disjoint open sets $U, V \in X$ such that $X = U \cup V$.

Note that if Y is a metric space and $X \subseteq Y$ is a subspsace, then X if connected if and only if there do not exist open sets $U, V \in Y$ such that

$$X \cap U \neq \emptyset, X \cap V \neq \emptyset, U \cap V = \emptyset, \text{ and } X \subseteq U \cap V$$

Definition. A topological space X is called **path connected** when it has the property that for all $a, b \in X$, there exists a continuous map $\alpha : [0,1] \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$.

It is easy to see that if X is path connected, then X is connected.

Definition. Let X be a topological space. If we define a relation $\sim$ on C by taking $a \sim b$ if and only if there exists a connected subspsace $A \subseteq X$ with $a \in A$ and $b \in B$.

It is clear that this is an equivalence relation. Note that when X is a topological space, its connected components are connected, and each connected subspace of X is contained in one of its connected components.

Definition. Let X be a topological space. Define a relation $\approx$ on X by $a \approx b$ if and only if there exists a continuous map $\alpha : [0,1] \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$. Such a map α is called a **continuous path**.

One can show that if X is **locally path connected** (which means that X has a basis for its topology which consists of path connected sets), then the path components of X are equal to the connected components of X, and that these components are open.

Quotient Topology

Definition. (Quotient Topology) Let X be a topological space and let $\sim$ be an equivalence relation on X. The set of equivalence classes is denoted $X/\sim$, and $X/\sim$ is called the **quotient** of X by $\sim$. The map $\pi : X \to X/\sim$ given by $\pi(a) = [a]$ is called the natural **projection map** or **quotient map**. We define the **quotient topology** on $X/\sim$ by stipulating that for $W \subseteq X/\sim$, W is open in $X/\sim$ if and only if $\pi^{-1}(W)$ is open in X.

When a group G acts on a topological space X, we define an equivalence relation $\sim$ on X by $a \sim b$ if and only if $b = g \cdot a$ for some $g \in G$. The equivalence classes are orbits. In this context, we also write $X/\sim$ as X/G.

When X, Y are any toplogical spaces and $\pi : X \to Y$ is surjective, we can define an equivalence relation X by $a \sim b$ if and only if $\pi(a) = \pi(b)$. We then have a natural bijection from Y to $X/\sim$ in which $y \in Y$ corresponds to the fibre $\pi^{-1}(y) \in X/\sim$.

If Y has the topology such that for $W \subseteq Y$, W is open in Y if and only if $q^{-1}(W)$ is open in X. In this case, we also use the terminology "quotient map" for π.

Remark. Let X be a topological space and let $\sim$ be an equivalence relation on X. Let Y be any set. If $f : X \to Y$ is constant on the equivalence classes, then f induces a well-defined map $\overline{f} : X/\sim \to Y$ given by define $\overline{f}([a]) = f(a)$.

Example. Define an equivalence class on $[0,1] \subseteq \mathbb{R}$ by $s \sim t$ if and only if $s = t$ or $\{s,t\} = \{0,1\}$. Then $[0,1]\big/\!\sim\ \cong SS^1$. Define $f : [0,1] \to \S^1$ by $f(t) = e^{i2\pi t}$. Note that $f(0) = f(1)$, so f induces a continuous map $\overline{f} : [0,1]\big/\!\sim\ \to SS^1$. The inverse map can be constructed as follows. We define $g : SS^1 \to [0,1]\big/\!\sim$ by

$$g(x,y) = \begin{cases} \left[\frac{1}{2\pi}\cos^{-1} x\right] & : y \geq 0 \\ \left[1 - \frac{1}{2\pi}\cos^{-1} x\right] & : y \leq 0 \end{cases}$$

Then g is continuous by the Glueing lemma.

In particular, the same proof shows that $\mathbb{R}/\mathbb{Z}$ is homeomorphic to SS^1.

Example. The projective space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ can be defined in several ways. $\mathbb{P}^n$ is the set of all 1−dimensional vector subspaces of $\mathbb{R}^{n+1}$, or $\mathbb{P}^n = \mathbb{R}^{n+1}\setminus\{0\}\big/\mathbb{R}^\times$, or $\mathbb{P}^n = SS^n/\pm 1$ where $SS^n = \{u \in \mathbb{R}^{n+1} : |u| = 1\}$.

Let us show that $\mathbb{R}^{n+1}\setminus\{0\}\big/\mathbb{R}^\times$ is homeomorphic to $SS^n/\pm 1$. Define $f : \mathbb{R}^{n+1}\setminus\{0\} \to SS^n$ by $f(x) = x/|x|$, and $g = \pi \circ f$. Then g is given by $g(x) = \{\pm x/|x|\}$. Note that for $t \in \mathbb{R}^\times$,

$$g(tx) = \left[\frac{t}{|t|}\cdot\frac{x}{|x|}\right] = \left[\frac{x}{|x|}\right]$$

since $t/|t| = \pm 1$. Thus g induces a continuous map $\overline{g}$ on the quotient. We construct the inverse map in a similar way.

Definition. Let X be a topological space. Then

- X is **T1** when for all $a,b \in X$ there exists an open set U in X with $a \in U$ and $b \notin U$
- X is **T2** or **Hausdorff** when for all $a,b \in X$, there exist disjoint open sets $U,V \subseteq X$ with $a \in U$ and $v \in B$
- X is **T3** or **regular** when X is T1 and for every $a \in X$ and every closed set $B \subseteq X$ with $a \notin B$, there exist open sets $U,V \subseteq X$ with $a \in U$, $B \subseteq V$.
- X is **T4** or **normal** when X is T1 and for all disjoint closed sets $A,B \subseteq X$ there exist disjoint open sets $U,V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$.

Definition. Let X be a topological space.

- X is **first countable** when for every $a \in X$, there exists a countable set B_a of open sets in X which contain a such that for every open set W in X with $a \in W$, there exists $U \in \mathcal{B}_a$ with $a \in U \subseteq W$.
- X is **second countable** when there exists a countable basis for the topology on X.

Example. (i) X is T1 if and only if every 1-point subset of X is closed in X
(ii) Every compact Hausdorff space is regular.
(iii) Every second countable regular space is normal.
(iv) Every metric space is normal.
(v) If X is second countable, then every open cover admits a countable subcover.
(vi) Every secound countable space X contains a countable dense subset.

1.2 Lemma. (Urysohn) *If X is normal and $A,B \subseteq X$ are disjoint and closed, then there is a countinuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

1.3 Theorem. (Tietze Extension) *If X is normal and $f : A \to \mathbb{R}$ is continuous for some $A \subseteq X$ closed, then there exists a continuous map $F : X \to \mathbb{R}$ such that $F|_A = f$ and $\sup_{a\in A}|f(a)| = \sup_{x\in X}|F(x)|$.*

1.4 Theorem. (Urysohn's Metrization) *If X is second countable and regular, then X is metrizable.*

Definition. An **n-dimensional topological manifold** is a Hausdorff, second countable topological space M which is **locally homeomorphic** to $\mathbb{R}^n$, meaning for every $p \in M$, there exists an open set $U \subseteq M$ with $p \in U$ and an open set $V \subseteq \mathbb{R}^n$ and a homeomorphism $\phi : U \subseteq M \to V \subseteq \mathbb{R}^n$. Such a homomorphism ϕ is called a **(local) coordinate chart** or **chart** on M at p. The domain U of a chart $\phi : U \subseteq M \to \phi(U) \subseteq \mathbb{R}^n$ is called a (local) **coordinate neighbourhood** at p. Note that we can choose a set of charts

$$\mathcal{A} = \{\phi_k : U_k \subseteq M \to \phi_k(U_k) : k \in K\}$$

where K is any non-empty set such that $M = \bigcup_{k\in K} U_k$. Such a set of charts is called an **atlas** for M.

Definition. Two charts are called $\phi : U \to \phi(U)$ and $\psi : V \to \psi(V)$ are called **(smoothly) compatible** when either $U \cap V = \emptyset$ or $\phi^{-1} \circ \psi$ and $\psi \circ \phi^{-1}$ are smooth (meaning partial derivatives of all orders exist). We say that an atlas is **smooth** if every pair of charts is compatible.

Note that a smooth atlas $\mathcal{A}$ on M can be extend to a unique maximal smooth atlas $\mathcal{M}$ on M by adding to $\mathcal{A}$ every possible homeomorphism $\psi : U \subseteq M \to \phi(U) \subseteq \mathbb{R}^n$ which is compatible with all of the existing charts (since if ψ and χ are both compatible with every chart $\psi \in \mathcal{A}$, then ψ and χ will be compatible with each other). The maps $\psi \circ \phi^{-1}$ are called **transition maps** or **change of coordinate maps**. A maximal smooth atlas $\mathcal{M}$ on M is called a **smooth structure** on M.

Definition. An n–**dimensional smooth (or C^∞) manifold** is an n–dimensional topological manifold with a smooth structure.

Remark. A topological manifold can have different smooth structures. For example, take $\mathcal{A} = \{\phi\}$ where $\phi : \mathbb{R} \to \mathbb{R}$ is the identity map, and $\mathcal{B} = \{\psi\}$ where $\psi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism given by $\psi(x)x^3$, since $\sqrt[3]{x}$ is not smooth at the origin.

What if we tried $\mathcal{B} = \{\psi\}$ where $\psi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism which is not C^∞? This is trivially a smooth atlas.

Typically, a manifold is given with a standard smooth structure.

Remark. We can give a smooth manifold M an (at most countable) atlas of charts all of which are of one of the forms

- $\phi : U \subseteq M \to B(0,1)$
- $\phi : U \subseteq M \to (0,1)^n$
- $\phi : U \subseteq M \to \mathbb{R}^n$

Note that the maximal atlas $\mathcal{M}$ is determined from any subset $\mathcal{AM}$ such that the domains of the charts in $\mathcal{A}$ cover M.

Definition. Let M be an m–dimensional smooth manifold and N be an n–dimensional smooth manifold and let $f : M \to N$ be a function. Then we say f is smooth **smooth** at p when for some (hence for any) chart at ϕ on M at p and for some (hence any) chart ψ on N at $f(p)$, the map $\phi^{-1} \circ f \circ \psi$ is smooth at $x = \phi(p)$, and f is **smooth** if f is smooth at ever $p \in M$. We say that f is a **diffeomorphism** when f is invertible and both f and f^{-1} are smooth. We say that M and N are **diffeomorphic**, and write $M \cong N$, when there exists a diffeomorphism $f : M \to N$.

Remark. If is conceivable that a topological manifold M could be both of dimension n and of dimension m with $n \neq m$. To do this, we would need to have a homeomorphism from an open set in $\mathbb{R}^n$ to an open set in $\mathbb{R}^m$. In fact, this cannot happen by invariance of domain, proven using tools from algebraic topology.

When M is smooth, it is easy to see that this cannot happen. If $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ were smooth inverses, then the matrices $D(\psi \circ \phi^{-1})(\phi(p))$ and $D(\phi \circ \psi^{-1})(\psi(p))$ would be inverse matrices. But then a product of a matrix in $M_{m\times n}(\mathbb{R})$ and in $M_{n\times m}(\mathbb{R})$ cannot be inverses when $m \neq n$.

Remark. Manifolds are sometimes constructed using quotient constructions. These quotients can be given by polygons with pairs of edges identified up to orientation.