PMATH 465

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Fall 2019†

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†Last updated: September 13, 2019

Contents

I. Fundamentals of Manifolds

1 Introduction to Topology

Basic Constructions

Definition. A **topology** on a set X is a set τ of subsets of X such that

(i) $\emptyset \in \tau$ and $X \in \tau$

(ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha\in A} U_\alpha \in \tau$.

(iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \leq i \leq n$, then $\bigcap_{i=1}^n U_i \in \tau$.

The sets $U \in \tau$ are called the **open sets** in X, and sets of the form $X \setminus U$ for some open set U are caled the **closed sets** in X.

Definition. When X is a topological space and $A \subseteq X$, the **interior** of A (denoted A°) is the union of all open sets contained in A. Similarly, we define the **closure** of A (denoted $\overline{A}$) as the intersction of all closed sets containing A. Then the **boundary** of A, denoted by ∂A, is the set $\partial A = \overline{A} \setminus A^\circ$.

Example. Let X be any set. The **discrete topology** on X is the topology $\tau = \mathcal{P}(X)$, and the **trivial topology** on X is the topology $\tau = \{\emptyset, X\}$.

Definition. A **basis** for a topology on a set X is a set $\mathcal{V}$ of subsets of X

(i) $\bigcup_{B\in\mathcal{B}} b = X$

(ii) for all $a \in X$ and $U, V \in \mathcal{B}$ such that $a \in U \cap V$, then there exists $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$.

When $\mathcal{B}$ is a basis for a topology on X, the topology on X **generated** by $\mathcal{B}$ is the set τ of subsets of X such that for $W \subseteq X$, $W \in \tau$ if and only if for all $a \in W$, there exists $U \in \mathcal{B}$ such that $a \in U \subseteq W$.

Note that τ, as above, is a topology on X since

(i) $\emptyset \in \tau$ vacuously and $X \in \tau$ obviously.

(ii) If $A_k \in \tau$ for all $k \in K$ (where K is any set of indices), then given $a \in \bigcup_{x\in K} A_k$, we can choose $\ell \in K$ so that $a \in A_\ell$. Then since $A_\ell \in \tau$, we can choose $U_\ell \in \mathcal{B}$ so that $a \in U_\ell \subseteq A_\ell$. Thus $a \in U_\ell \subseteq A_\ell \subseteq \bigcup_{k\in K} A_k$.

(iii) By induction, it suffices to prove that if $A, B \in \tau$, then $A \cap B \in \tau$. Suppose $A, B \in \tau$, and let $a \in A \cap B$. Since $A \in \tau$, we cna choose $U \in \mathcal{B}$ so that $a \in U \subseteq A$. Since $B \in \tau$, we can choose $V \in \mathcal{B}$ so that $a \in V \subseteq B$. Then we have $a \in U \cap V$. Since $\mathcal{B}$ is a basis, we can chose $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$, so $a \in W \subseteq U \cap V \subseteq A \cap B$.

Note that when τ is the topology on X generated by the basis $\mathcal{B}$, for $A \subseteq X$, $A \in \tau$ if and only if there exists some $S \subseteq \mathcal{B}$ such that $A = \bigcup_{s\in S} s$. In this sense, the topology τ on X generated by the basis $\mathcal{B}$ is the coarsest topology which contains $\mathcal{B}$.

Definition. (Subspace Topology) When Y is a topological space and $X \subseteq Y$ is a subset of Y, we define the **subspace topology** on X to be the topology for which as set $U \subseteq X$ is open if and only if $U = X \cap V$ for some open set V.

If $\mathcal{C}$ is a basis for the topology on Y, then $\mathcal{B} = \{\, X \cap V \mid V \in \mathcal{C} \,\}$ is a basis for the subspace topology on X.

Definition. (Disjoint Union Topology) If X and Y are topological spaces with $X \cap Y = \emptyset$, then the **disjoint union topology** on $X \cup Y$ is the topology in which a subset $U \subseteq X \cup Y$ is open in $X \cup Y$ if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y.

Definition. (Product Topology) If X and Y are topological spaces, the **product topology** on $X \times Y$ is the topology generted by the basis

$$\mathcal{B} = \{\, U \times V \mid U \in \mathcal{C}, V \in \mathcal{D} \,\}$$

where $\mathcal{C}$ and $\mathcal{D}$ are bases for the topologies on X, Y respectively.

Definition. (Infinite Product Topology) We define the infinite product to be

$$\prod_{k \in K} \left\{ f : K \to \bigcup_{k \in K} X_k \;\middle|\; f(k) \in X_k \text{ for all } k \in K \right\}$$

There are two standard topologies on X. The first is the **box topology**,

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \;\middle|\; U_k \text{ is open in } X_k \right\}$$

and the **product topology**

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \;\middle|\; \begin{array}{l} U_k \text{ is open in } X_k \\ U_k = X_k \text{ for all but finitely many indices } k \end{array} \right\}$$

Example. (Metric Topology) $\mathbb{R}^n$ has a standard **inner product**, and for $u, v \in \mathbb{R}^n$, $uv = u \cdot v = V^T u = \sum_{i=1}^n u_i v_i$. This gives the standard norm on $\mathbb{R}^n$ for $u \in \mathbb{R}^n$, $\|u\| = \sqrt{uv}$. This gives the standard metric on $\mathbb{R}^n$: for $a, \in \mathbb{R}^n$, $d(a,b) = \|b - a\|$.

Given a metric on a set Y, we obtain (by restriction) an induced metric on any subset $X \subseteq Y$. Given a metric space X, we define the **metric topology** on X to be the topology which is generated by the set of open balls

$$B(a, r) = \{\, x \in X \mid d(a, x) < r \,\}$$

where $x \in X$, $r > 0$.

Maps on Topological Spaces

Definition. When X and Y are topological spaces and $f : X \to Y$, we say that f is **continuous** when it has the property that $f^{-1}(V)$ is open in X for every open set V in Y. We say that $f : X \to Y$ is a **homeomorphism** when f is bijective and both f and f^{-1} are continuous. Then X, Y are **homeomorphic** if there exists a homeomorphism $f : X \to Y$.

1.1 Theorem. (Glueing Lemma) *Let X and Y be topological spaces, and let $f : X \to Y$ be a function. Suppose either*
(i) $X = \bigcup_{k \in K} A_k$ where each A_k is open in X, or
(ii) $X = \bigcup_{k=1}^n A_k$ where each A_k is closed in X
and each restriction map $f_k : A_k \to Y$ is continuous, then f is continuous.

Proof Exercise. ■

Definition. A topological space X is **compact** when it has the property that for every set $\mathcal{S}$ of open subsets of X with $X = \bigcup_{U \in S} U$, there exists a finite subset $\mathcal{F} \subseteq \mathcal{S}$ such that $X = \bigcup_{F \in \mathcal{F}} F$.

Note that when $X \subseteq Y$ is a subspace, X is compact if and only if X has the property that for every set $\mathcal{T}$ with $X \subseteq \bigcup_{T \in \mathcal{T}} T$, there exists a finite subset $\mathcal{G} \subseteq \mathcal{T}$ uch that $X \subseteq \bigcup_{G \in \mathcal{G}} G$.

Definition. A topological space X is **connected** when there do not exist non-empty disjoint open sets $U, V \in X$ such that $X = U \cup V$.

Note that if Y is a metric space and $X \subseteq Y$ is a subspsace, then X if connected if and only if there do not exist open sets $U, V \in Y$ such that

$$X \cap U \neq \emptyset, X \cap V \neq \emptyset, U \cap V = \emptyset, \text{ and } X \subseteq U \cap V$$

Definition. A topological space X is called **path connected** when it has the property that for all $a, b \in X$, there exists a continuous map $\alpha : [0,1] \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$.

It is easy to see that if X is path connected, then X is connected.

Definition. Let X be a topological space. If we define a relation $\sim$ on C by taking $a \sim b$ if and only if there exists a connected subspsace $A \subseteq X$ with $a \in A$ and $b \in B$.

It is clear that this is an equivalence relation. Note that when X is a topological space, its connected components are connected, and each connected subspace of X is contained in one of its connected components.

Definition. Let X be a topological space. Define a relation $\approx$ on X by $a \approx b$ if and only if there exists a continuous map $\alpha : [0,1] \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$. Such a map α is called a **continuous path**.

One can show that if X is **locally path connected** (which means that X has a basis for its topology which consists of path connected sets), then the path components of X are equal to the connected components of X, and that these components are open.

Quotient Topology

Definition. (Quotient Topology) Let X be a topological space and let $\sim$ be an equivalence relation on X. The set of equivalence classes is denoted $X/\sim$, and $X/\sim$ is called the **quotient** of X by $\sim$. The map $\pi : X \to X/\sim$ given by $\pi(a) = [a]$ is called the natural **projection map** or **quotient map**. We define the **quotient topology** on $X/\sim$ by stipulating that for $W \subseteq X/\sim$, W is open in $X/\sim$ if and only if $\pi^{-1}(W)$ is open in X.

When a group G acts on a topological space X, we define an equivalence relation $\sim$ on X by $a \sim b$ if and only if $b = g \cdot a$ for some $g \in G$. The equivalence classes are orbits. In this context, we also write $X/\sim$ as X/G.

When X, Y are any toplogical spaces and $\pi : X \to Y$ is surjective, we can define an equivalence relation X by $a \sim b$ if and only if $\pi(a) = \pi(b)$. We then have a natural bijection from Y to $X/\sim$ in which $y \in Y$ corresponds to the fibre $\pi^{-1}(y) \in X/\sim$.

If Y has the topology such that for $W \subseteq Y$, W is open in Y if and only if $q^{-1}(W)$ is open in X. In this case, we also use the terminology "quotient map" for π.

Remark. Let X be a topological space and let $\sim$ be an equivalence relation on X. Let Y be any set. If $f : X \to Y$ is constant on the equivalence classes, then f induces a well-defined map $\overline{f} : X/\sim \to Y$ given by define $\overline{f}([a]) = f(a)$.

Example. Define an equivalence class on $[0,1] \subseteq \mathbb{R}$ by $s \sim t$ if and only if $s = t$ or $\{s,t\} = \{0,1\}$. Then $[0,1]\big/\!\sim\; \cong SS^1$. Define $f : [0,1] \to \S^1$ by $f(t) = e^{i2\pi t}$. Note that $f(0) = f(1)$, so f induces a continuous map $\overline{f} : [0,1]\big/\!\sim\; \to SS^1$. The inverse map can be constructed as follows. We define $g : SS^1 \to [0,1]\big/\!\sim$ by

$$g(x,y) = \begin{cases} \left[\frac{1}{2\pi}\cos^{-1} x\right] & : y \geq 0 \\ \left[1 - \frac{1}{2\pi}\cos^{-1} x\right] & : y \leq 0 \end{cases}$$

Then g is continuous by the Glueing lemma.

In particular, the same proof shows that $\mathbb{R}/\mathbb{Z}$ is homeomorphic to SS^1.

Example. The projective space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ can be defined in several ways. $\mathbb{P}^n$ is the set of all 1−dimensional vector subspaces of $\mathbb{R}^{n+1}$, or $\mathbb{P}^n = \mathbb{R}^{n+1}\setminus\{0\}\big/\mathbb{R}^\times$, or $\mathbb{P}^n = SS^n/\pm 1$ where $SS^n = \{u \in \mathbb{R}^{n+1} : |u| = 1\}$.

Let us show that $\mathbb{R}^{n+1}\setminus\{0\}\big/\mathbb{R}^\times$ is homeomorphic to $SS^n/\pm 1$. Define $f : \mathbb{R}^{n+1}\setminus\{0\} \to SS^n$ by $f(x) = x/|x|$, and $g = \pi \circ f$. Then g is given by $g(x) = \{\pm x/|x|\}$. Note that for $t \in \mathbb{R}^\times$,

$$g(tx) = \left[\frac{t}{|t|} \cdot \frac{x}{|x|}\right] = \left[\frac{x}{|x|}\right]$$

since $t/|t| = \pm 1$. Thus g induces a continuous map $\overline{g}$ on the quotient. We construct the inverse map in a similar way.