Functional Analysis

Alex Rutar*
University of Waterloo

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*arutar@uwaterloo.ca
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Contents

I. Fundamentals of Functional Analysis

1 Basic Elements of Functional Analysis

Throughout, we denote by $\mathbb{F}$ either the field $\mathbb{R}$ or the field $\mathbb{C}$.

Banach Spaces

Definition. Let X be a vector space over $\mathbb{F}$. A **norm** is a functional $\|\cdot\| : X \to \mathbb{R}$ such that it is

- *(non-negative)* $\|x\| \geq 0$ for any $x \in X$
- *(non-degenerate)* $\|x\| = 0$ if and only if $x = 0$
- *(subadditivity)* $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in X$
- *($|\cdot|$−homogeneity)* $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{F}$, $x \in X$.

We call the pair $(X, \|\cdot\|)$ a **normed vector space**. Furthermore, we say that $(X, \|\cdot\|)$ is a **Banach space** provided that X is complete with respect to the metric $\rho(x, y) = \|x - y\|$.

Example. (i) $(\mathbb{F}, |\cdot|)$ is a Banach space.

(ii) $(\mathbb{F}^b, \|\cdot\|_p)$, $x = (x_j)_{j=1}^n$,

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^n |x_j|^p\right)^{1/p} & 1 \leq p < \infty \\ \max_{j=1,\dots,n} |x_j| & p = \infty \end{cases}$$

(iii) Consider the space

$$L_p^{\mathbb{F}} = \left\{ f : [0,1] \to \mathbb{F} \;\middle|\; f \text{ is Lebesgue measurable}, \left(\int_0^1 |f|^p\right)^{1/p} < \infty \right\} \Big/ {\sim_{\text{a.e.}}}$$

where $1 \leq p < \infty$.

(iv) $L_\infty^{\mathbb{F}}[0,1]$, $\|f\|_\infty = \operatorname{ess\,sup}_{t \in [0,1]} |f(t)|$.

(v) Let (X, d) be a metric space. Then

$$C_b^{\mathbb{F}}(x) = \{ f : X \to \mathbb{F} \mid f \text{ is continuous and bounded} \}, \quad \|f\|_\infty = \sup_{x \in X} |f(x)|$$

is a Banach space.

Here is a more interesting example:

Example. Let (X,d) be a metric space. We define the space of Lipschitz functions

$$\mathrm{Lip}^{\mathbb{F}}(X,d) = \left\{ f : X \to \mathbb{F} \;\middle|\; f \text{ is bounded}, L(f) = \sup_{\substack{x,y\in X \\ x\neq y}} \frac{|f(x)-f(y)|}{d(x,y)} < \infty \right\}$$

We note that for $f : X \to \mathbb{F}$ that

$$f \in \mathrm{Lip}^{\mathbb{F}}(X,d) \Leftrightarrow \text{there is } L \geq 0 \text{ s.t. } |f(x)-f(x)| \leq Ld(x,y) \text{ for all } x,y \in X \tag{1.1}$$

It is easy to verify that $L(f) = \min\{L \geq 0 : (1.1) \text{ holds for } f\}$. It is an easy exercise to see that $\mathrm{Lip}^{\mathbb{F}}$ is a vector space, and that $L : \mathrm{Lip}^{F}(X,d) \to \mathbb{R}$ is a **semi-norm** (non-negative, subadditive, $|\cdot|$–homogeneous). However, we do not have non-degeneracy (for example, constants are taken to 0). We define the Lipschitz norm

$$\|f\|_{\mathrm{Lip}} = \|f\|_\infty + L(f)$$

1.1 Proposition. $(\mathrm{Lip}^{\mathbb{F}}(X,d), \|\cdot\|_{\mathrm{Lip}})$ *is a Banach space.*

Proof Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $(\mathrm{Lip}^{\mathbb{F}}(X,d), \|\cdot\|_{\mathrm{Lip}})$. Since $\|\cdot\|_\infty \leq \|\cdot\|_{\mathrm{Lip}}$ on $\mathrm{Lip}^{F}(X,d)$, we see that $(f_n)_{n=1}^\infty$ is uniformly Cauchy (and bounded), and hence there is $f = \lim_{n\to\infty} f_n$ in $C_b^{\mathbb{F}}(X)$, where the limit is taken with respect to $\|\cdot\|_\infty$, since $(C_b^{\mathbb{F}}(X), \|\cdot\|_\infty)$ is a Banach space. If $x,y \in X$, then

$$\begin{aligned} |f(x)-f(x)| &= \lim_{n\to\infty} |f_n(x)-f_n(y)| \leq \sup_{n\in\mathbb{N}} |f_n(x)-f_n(y)| \\ &\leq \sup_{n\in\mathbb{N}} L(f_n)d(x,y) \leq \sup_{n\in\mathbb{N}} \|f_n\|_{\mathrm{Lip}}\, d(x,y) \end{aligned}$$

Since Cauchy sequences are bounded, we see that $|f(x)-f(y)| \leq Ld(x,y)$, where $L = \sup_{n\in\mathbb{N}} \|f_n\|_{\mathrm{Lip}} < \infty$. Thus by (1.1), $f \in \mathrm{Lip}^{\mathbb{F}}(X,d)$. Exercise: one may verify that $\|f - f_n\|_{\mathrm{Lip}} \to 0$. ■

Another collection of basic examples are given by the sequence spaces. We can define

$$\ell_1^{\mathbb{F}} = \left\{ x = (x_j)_{j=1}^\infty \in \mathbb{F}^{\mathbb{N}} \;\middle|\; \|x\|_1 = \sum_{j=1}^\infty |x_j| < \infty \right\}$$

It is easy to see that $(\ell_1, \|\cdot\|_1)$ is a normed vector space.

For $1 < p < \infty$, and write

$$\ell_p^{\mathbb{F}} = \left\{ x \in \mathbb{F}^{\mathbb{N}} \;\middle|\; \|x\|_p = \left(\sum_{j=1}^\infty |x_j|^p \right)^{1/p} < \infty \right\}$$

Note that $0 \in \ell_p$, $\alpha \in \mathbb{F}$, $\alpha x \in \ell_p$ if $x \in \ell_p$. Let $q = p/(p-1)$ so that $1/p + 1/q = 1$. Then q is called the **conjugate index**. We have

1.2 Proposition. (Young's Inequality) *If $a,b \geq 0$ in $\mathbb{R}$, then $ab \leq a^p/p + b^q/q$, with equality only if $a^p = b^q$.*

and

1.3 Proposition. (Hölder's Inequality) *If $x \in \ell_p$ and $y \in \ell_q$, then $xy = (x_i y_i)_{i=1}^\infty \in \ell_1$, with*

$$\sum_{i=1}^{\infty} |x_i y_i| \le \|x\|_p \|y\|_q$$

with equality exactly when $\operatorname{sgn}(x_i y_i) = \operatorname{sgn}(x_k y_k)$ for all $j,k \in \mathbb{N}$ where $x_i y_i \neq 0 \neq x_k y_k$, and $|x|^p = (|x_j|^p)_{j=1}^\infty$ and $|y|^q$ are linearly dependent in ℓ_1.

and finally

1.4 Proposition. (Minkowski's Inequality) *If $x, y \in \ell_p$, then $\|x+y\|_p \le \|x\|_p + \|y\|_p$ with equality exactly when one of x or y is a non-negative scalar combination of the other.*

Review of Topology

Let X denote a non-empty set, and $\mathcal{P}(X)$ denote the power set of X.

Definition. A **topology** on a set X is a set τ of subsets of X such that
(i) $\emptyset, X \in \tau$
(ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
(iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \le i \le n$, then $\bigcap_{i=1}^n U_i \in \tau$.
The sets $U \in \tau$ are called the **open sets** in X, and sets of the form $X \setminus U$ for some open set U are caled the **closed sets** in X. The pair (X, τ) is called a **topological space**.

The metric topology on a metric space (X, d) is the topology

$$\tau_d = \{\, U \subseteq X \mid \text{ for each } x_0 \in U, \text{ there is } \delta = \delta(x_0) \text{ s.t. } B_\delta(x_0) \subseteq U \,\}$$

Example. (i) Given two metrics d, ρ on X, we say that $d \sim \rho$ if and only if there are $c, C > 0$ such that

$$cd(x,y) \le \rho(x,y) \le Cd(x,y) \text{ for any } x, y \in X$$

Note that $d \sim \rho$ implies that $\tau_d = \tau_\rho$, but the reverse implication is not true. An example of this are the metrics on $X = \mathbb{R}$ given by $d(x,y)$ and $\rho(x,y) = \frac{|x-y|}{1+|x-y|}$. Then $d \nsim \rho$ but $\tau_d = \tau_\rho$.

(ii) "Sorgenfry line" Set $X = \mathbb{R}$, and consider

$$\sigma = \{\, V \subseteq \mathbb{R} \mid \text{ for any } s \in V, \text{ there is } \delta = \delta(s) > 0 \text{ s.t. } [s, s+\delta) \subseteq V \,\}$$

It is an exercise to verify that $\tau_{|\cdot|} \subsetneq \sigma$. We say that σ is **finer** than $\tau_{|\cdot|}$.

(iii) Relative topology: let (X, τ) be a topological space, and $\emptyset \neq A \subseteq X$. Then we can define a topology $\tau|_A = \{U \cap A : U \in \tau\}$.

Definition. Let (X, τ) and (Y, σ) be topological spaces, and $f : X \to Y$. We say that f is $(\tau-\sigma-)$**continuous** at x_0 in X if,

- given $V \in \sigma$ such that $f(x_0) \in V$, then there exists $U \in \tau$ such that $x_0 \in U$ and $f(U) \subseteq V$.

We say that f is $(\tau-\sigma-)$continuous if it is continuous at each x_0 in X.

Space of bounded continuous functions into a normed space

Let $(Y, \|\cdot\|)$ denote a normed space. We let $\tau_{\|\cdot\|}$ denote the topology given by the metric $\rho(x,y) = \left\|x - y\right\|$. Let (X, τ) denote any topological space. Then we write

$$C_b^Y(X) = \left\{ f : X \to Y \,\middle|\, f \text{ is bounded and } \tau - \tau_{\|\cdot\|} - \text{continuous} \right\}$$

With pointwise operations, we see that $C_b^Y(X)$ is a vector space. We also define for $f \in C_b^Y(X)$, $\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}$, making $(C_b^Y(X), \|\cdot\|_\infty)$ a normed vector space.

1.5 Theorem. *If $(Y, \|\cdot\|)$ is a Banach space, then $(C_b^Y(X), \|\cdot\|_\infty)$ is a Banach space.*

Proof Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $(C_b^Y(X), \|\cdot\|_\infty)$. Then for any $x \in X$, we have that $(f_n(x))_{n=1}^\infty$ is Cauchy in $(Y, \|\cdot\|)$ since $\|f_n(x) - f_m(x)\| \leq \|f_n - f_m\|_\infty$, and hence admis a limit $f(x)$. In particular, $x \mapsto f(x)$ defines a function from X to Y. We shall fix $x_0 \in X$ and show that f is continuous at x_0. Given $\epsilon > 0$, we let

- n_1 be so $n, m \geq n_1$ so that $\|f_n - f_m\|_\infty < \epsilon/4$.
- n_2 be so $n \geq n_2$ so that $\|f_n(x_0) - f(x_0)\| < \epsilon/4$.
- $N = \max\{n_1, n_2\}$.
- $U \in \tau$, $x_0 \in U$ such that $f_N(U) \subseteq B_{\epsilon/4}(f(x_0)) \subset Y$.

Then for $x \in U$, we let n_x be so $n_x \geq n_1$ and $n \geq n_x$, so that $\|f_n(x) - f(x)\| < \epsilon/4$. We then have

$$\begin{aligned} \|f(x) - f(x_0)\| &\leq \left\|f(x) - f_{n_x}(x)\right\| + \left\|f_{n_x}(x) - f_N(x)\right\| + \|f_N(x) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| \\ &< \frac{\epsilon}{4} + \left\|f_{n_x} - f_N\right\|_\infty + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon \end{aligned}$$

in other words that $f(U) \subseteq B_\epsilon(f(x_0))$.

Now let us check that $\|f\|_\infty < \infty$. Since $|\|f_n\|_\infty - \|f_m\|_\infty| \leq \|f_n - f_m\|_\infty$, so $(\|f_n\|_\infty)_{n=1}^\infty \subseteq \mathbb{R}$ is Cauchy, hence bounded. If $x \in X$, then

$$\|f(x)\| = \lim_{n\to\infty} \|f_n(x)\| \leq \sup_{n\in\mathbb{N}} \|f_n(x)\| \leq \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$$

so $\|f\|_\infty = \sup_{x\in X} \|f(x)\| < \infty$.

Notice that if ϵ, n_1 are as above, and further x_0, N are as above, we have for $n \geq n_1$

$$\|f_n(x_0) - f(x_0)\| \leq \|f_n(x_0) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| < \frac{\epsilon}{2}$$

so $\|f_n - f\|_\infty = \sup_{x_0\in X} \|f_n(x_0) - f(x_0)\| \leq \epsilon/2 < \epsilon$. This is uniform since n_1 is chosen uniformly in X. ∎

1.6 Corollary. *$(C_b^{\mathbb{F}}(X), \|\cdot\|_\infty)$ is a Banach space.*

Let's first note the following general priniple: let (X, d), (Y, ρ) be metric spaces, where (X, d) is complete. If $\psi : X \to Y$ is a $(d - \rho-)$isometry, then $(\psi(X), \rho|_{\psi(X)})$ is a complete metric space.

Example. (i) Let T be a non-empty set and let

$$\ell_\infty(T) = \left\{ x = (x_t)_{t\in T} \in \mathbb{F}^T \,\middle|\, \|x\|_\infty \right\} < \infty$$

With pointwise operations, $(\ell_\infty, \|\cdot\|_\infty)$ is a normed space. In fact, it is a Banach space. Let us note that

$$f \mapsto (f(t))_{t\in T} : C_b(T, \mathcal{P}(T)) \to \ell_\infty(T)$$

is a surjective linear isometry, and the result follows.

(ii) Let $c = \{ x \in \ell_\infty \mid \lim_{n\to\infty} x_n \text{ exists} \}$. Then $(c, \|\cdot\|_\infty)$ is a Banach space. Consider the topological space given by $\omega = \mathbb{N} \cup \{\infty\}$, with topology

$$\tau_\omega = \mathcal{P}(\mathbb{N}) \cup \bigcup_{n\in\mathbb{N}} \{k \in \mathbb{N} : k \geq n\}$$

The map $f \mapsto (f(n))_{n=1}^\infty : C_b(\omega) \to c$ is a linear surjective isometry.

(iii) $c_0 = \left\{ x \in \mathbb{F}^{\mathbb{N}} \,\middle|\, \lim_{n\to\infty} x_n = 0 \right\} \subseteq c \subseteq \ell_\infty$.

1.7 Lemma. *If $x_0 \in X$ where (X, τ) is a topological space, then*

$$\mathcal{I}(x_0) = \{ f \in C_b(x) \mid f(x_0) = 0 \}$$

is closed, hence complete, subspace of $C_b(X)$.

Proof If $(f_n)_{n=1}^\infty \subseteq \mathcal{I}(x_0)$ and $f = \lim_{n\to\infty} f_n$ with respect to $\|\cdot\|_\infty$ in $C_b(X)$, then $f(x_0) = \lim_{n\to\infty} f_n(x_0) = 0$. Thus $f \in \mathcal{I}(x_0)$, and closed subsets of complete spaces are themselves complete. ■

Now, $f \mapsto (f(n))_{n=1}^\infty : \mathcal{I}(\infty) \to c_0$ is a (linear) surjective isometry.

(iv) Consider the Sorgenfty line $(\mathbb{R}, \sigma)$: verify that

$$c_b(\mathbb{R}, \sigma) = \left\{ f : \mathbb{R} \to \mathbb{F} \,\middle|\, f \text{ is bounded and } \lim_{t\to t_0^+} f(t) = f(t_0) \text{ for } t \in \mathbb{R} \right\}$$

2 Linear operators and linear functionals

Let X, Y be vector spaces. We let $\mathcal{L}(X, Y) = \{ S : X \to Y \mid S \text{ is linear} \}$; this is itself a vector space with pointwise operations. Let $(X, \|\cdot\|)$ be a normed space. We denote

$$\begin{aligned} D(X) &= \{x \in X : \|x\| < 1\} \\ S(X) &= \{x \in X : \|x\| = 1\} \\ B(X) &= \{x \in X : \|x\| \leq 1\} \end{aligned}$$

(Yes, this notation is confusion. No, I didn't choose it.)

2.1 Proposition. *If X, Y are normed spaces and $S \in \mathcal{L}(X, Y)$, then the following are equivalent:*

(i) S is continuous

(ii) S is continuous at some $x_0 \in X$

(iii) $\|S\| = \sup_{x\in D(X)} \|Sx\| < \infty$.

Moreover, in this case, we have

$$\begin{aligned} \|S\| &= \min\{L > 0 : \|Sx\| \leq L\|x\| \textit{ for } x \in X\} \\ &= \sup_{x\in S(X)} \|Sx\| = \sup_{x\in B(X)} \|Sx\| \end{aligned}$$

Proof $(i \Rightarrow ii)$ Obvious

$(ii \Rightarrow iii)$ Note that

$$Sx_0 + D(Y) = \{Sx_0 + y : t \in D(Y)\} = \{y \in Y : \left\|Sx_0 - y'\right\| < 1\}$$

is a neighbourhood of Sx_0. By the definition of metric continuity, there is $\delta > 0$ such that

$$x_0 + \delta D(X) = \{x_0 + \delta x : x \in D(x)\} = \{x' \in X : \left\|x_0 - x'\right\| < \delta\}$$

such that

$$Sx_0 + \delta S(D(X)) = S(x_0 + \delta D(x) \subseteq Sx_0 + D(Y)$$

which implies that $\delta S(D(X)) \subseteq D(Y)$ and $S(D(X)) \subseteq D(Y)/\delta$, in other words that $\|Sx\| \leq 1/\delta$ for $x \in D(X)$.

$(iii \Rightarrow i)$ If $x \in X$ and $\epsilon > 0$, then

$$\|Sx\| = (\|x\| + \epsilon)\left\|S\left(\frac{1}{\|x\| + \epsilon}\|x\|\right)\right\| \leq (\|x\| + \epsilon)\|S\|$$

Then, letting $\epsilon \to 0^+$, we see that

$$\|Sx\| \leq \|x\|\|S\| = \|S\|\|X\|$$

If $x, x' \in X$, then $\|Sx - S'x\| \leq \|S\|\|x - x'\|$ is S is Lipschitz, hence continuous.

To complete the proof, the content of (iii) implies (i) tellus us that the Lipschitz constant $L(S) \leq \|S\|$. Furthermore, if $\|x\| = 1$, the preceding proof gives us that $\|S\|_{S(X)}$.

Conversely,

$$\|S\| = \sup_{x \in D(X) \setminus \{0\}} \|Sx\| = \sup_{x \in D(X) \setminus \{0\}} \|x\| \left\|S\left(\frac{1}{\|x\|}x\right)\right\| \leq \sup_{x \in S(X)} \|Sx\|$$

The remaining equivalence is obvious. ∎

We now let $\mathcal{B}(X, Y) = \{\, S \in \mathcal{L}(X, Y) \mid S \text{ is bounded} \,\}$. We will see that $\|\cdot\|$, above, defines a norm on $B(X, Y)$.

2.2 Theorem. *If X, Y are normed spaces, then $(\mathcal{B}(X, Y), \|\cdot\|)$ is a normed space. Furthermore, if Y is a Banach spaces, then so to is $(\mathcal{B}(X, Y), \|\cdot\|)$.*

Proof Define

$$\Gamma : \mathcal{B}(X, Y) \to C_b^Y(B(X))$$

given by $\Gamma(S) = S|_{B(X)}$. Then, by definition, Γ is linear, with

$$\|\Gamma(S)\|_\infty = \sup_{x \in B(X)} \|Sx\| = \|S\|$$

Thus $\|\cdot\|$ is a norm: if $S, T \in \mathcal{B}(X, Y)$, $\alpha \in \mathbb{F}$,

$$\|S + T\| = \|\Gamma(S + T)\|_\infty = \|\Gamma(S) + \Gamma(T)\|_\infty \leq \|\Gamma(S)\|_\infty + \|\Gamma(T)\|_\infty = \|S\| + \|T\|$$
$$\|\alpha S\| = \|\Gamma(\alpha S)\|_\infty = |\alpha|\|\Gamma(S)\|_\infty = |\alpha|\|S\|.$$

Furthermore, $\Gamma : \mathcal{B}(X,Y) \to C_b^Y(\mathcal{B}(X))$ is an isometry.

Now suppose that Y is a Banach space. We will show that $\Gamma(\mathcal{B}(X,Y))$ is closed in $C_b^Y(B(X))$, and hence $B(X,Y) = \Gamma^{-1}(\Gamma(\mathcal{B}(X,Y)))$ is complete. Let $(S_n)_{n=1}^{\infty} \subset \mathcal{B}(X,Y)$ be $\|\cdot\|$–Cauchy. Then $(\Gamma(S_n))_{n=1}^{\infty}$ is $\|\cdot\|_\infty$–Cauchy in $C_b^Y(B(X))$, and hence there is $f \in C_b^Y(B(X))$ such that $\lim_{n\to\infty} \|\Gamma(S_n) - f\|_\infty = 0$. Then we let $S : X \to Y$ be given by

$$Sx = \begin{cases} \|x\| f\left(\frac{x}{\|x\|}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

If $x, x' \in X$ and $\alpha \in \mathbb{F}$ are all such that $x, x', x + \alpha x' \neq 0$, then

$$\begin{aligned} S(x+\alpha x') &= \left\|x+\alpha x'\right\| f\left(\frac{1}{x+\alpha x'}(x+\alpha x')\right) \\ &= \left\|x+\alpha x'\right\| \lim_{n\to\infty} S_n\left(\frac{1}{x+\alpha x'}(x+\alpha x')\right) \\ &= \lim_{n\to\infty}(S_n x + \alpha S_n x') = \lim_{n\to\infty}\left[\|x\| S_n\left(\frac{1}{\|x\|}x\right) + \alpha \left\|x'\right\| S_n\left(\frac{1}{\|x\|}x'\right)\right] \\ &= \|x\| f\left(\frac{x}{\|x\|}\right) + \alpha \left\|x'\right\| f\left(\frac{x'}{\|x\|}\right) \\ &= Sx + \alpha Sx' \end{aligned}$$

The above computation is easily performed if any of x, x', $x+\alpha x'$ are 0. Hence $S \in \mathcal{L}(X,Y)$. We se that S is continuous (say, at a point on $S(X)$), so $S \in \mathcal{B}(X,Y)$. Finally, as $S|_{B(X)} = f = \lim_{n\to\infty} S_n|_{B(X)}$ (with respect to the uniform norm), we have

$$\|S - S_n\| = \sup_{x \in B(X)} \|(S-S_n)x\| = \|f - \Gamma(S_n)\|_\infty$$

goes to 0 as n goes to infinity. ■

Definition. Given a vector space X, let $X' = \mathcal{L}(X, \mathbb{F})$ denote the **algebraic dual**. If further X is a normed space, we let $X^* = \mathcal{B}(X, \mathbb{F})$ denote the (continuous) dual.

2.3 Corollary. *If X is a normed spaces, then X^* is always a Banach space.*

2.4 Theorem. *Let for $x \in \ell_1$, $f_x : c_0 \to \mathbb{F}$ be given by $f_x(y) = \sum_{j=1}^{\infty} x_j y_j$. Then $f_x \in c_0^*$ with $\|f_x\| = \|x\|_1$. Furthermore, every element of c_0^* arises as above.*

Proof If $x \in \ell_1$ and $y \in c_0 \subseteq \ell_\infty$, then

$$\sum_{j=1}^{\infty} |x_j y_j| \leq \sum_{j=1}^{\infty} |x_j| \left\|y\right\|_\infty = \|x\|_1 \left\|y\right\|_\infty < \infty$$

so $f_x(y) = \sum_{j=1}^{\infty} x_j y_j$ is well-defined. It is obvious that f_x is linear: $f_x(y+\alpha y') = f_x(y) + \alpha f(y')$ for $y, yl \in c_0$ and $\alpha \in \mathbb{F}$. Also, $\|f_x\| \leq \|x\|_1$. We let $y^n = (\overline{\operatorname{sgn} x}, \ldots, \overline{\operatorname{sgn} x_n}, 0, 0, \ldots) \in c_0$, with $\left\|y^n\right\| = 1$. Then

$$\|f_x\| \geq |f_x(y^n)| = \sum_{j=1}^{n} x_i \overline{\operatorname{sgn} x_i} = \sum_{j=1}^{n} |x_i|$$

so that $\|f_x\| \geq \|x\|_1$, and hence equality holds.

Now let $f \in c_0^*$, and write $e_n = (0, \ldots, 0, 1, 0, 0, \ldots) \in c_0$, and let $x_n = f(e_n)$. Then, let $y \in c_0$ and $y^n = (y_1, \ldots, y_n, 0, 0, \ldots)$ and we have

$$\left\|y - y^n\right\|_\infty = \sup_{j \geq n+1} |y_j|$$

which goes to 0 as n goes to infinity. Then since f is continuous, we have

$$f(y) = \lim_{n\to\infty} f(y^n) = \lim_{n\to\infty} \sum_{j=1}^{n} y_j x_j = \sum_{j=1}^{\infty} x_j y_j = f_x(y)$$

We use sequence $(y^n)_{n=1}^\infty$ as in $y^n \in c_0$, to see that

$$\sum_{j=1}^{n} |x_i| = |f(y^n)| \leq \|f\| < \infty$$

so $x \in \ell_1$. Thus $f = f_x$, as desired. ∎

2.5 Corollary. *$\ell_1 \cong c^*$ isometrically isomorphically.*

Proof For $y \in c$, let $L(y) = \lim_{n\to\infty} y_n$. Given $y \in c$, let $y^n = (y_1, \ldots, y_n, L(y), L(y), \ldots) \in c$. Notice that $\left\|y - y^n\right\|_\infty \to 0$ similarly as above.

We let $1 = (1, 1, \ldots)$, and $1_n = (0, \ldots, 0, 1, 1, \ldots)$. If $m < n$, then $1_n - 1_m \in c_0$, so

$$|f(1_n) - f(1_m)| = |f_x(1_n - 1_m)| \leq \sum_{j=m+1}^{n} |x_j|$$

so that $(f(1_n))_{n=1}^\infty$ is Cauchy in $\mathbb{F}$. Let $x_0 = \lim_{n\to\infty} f(1_n)$. Let $\tilde{x} = (x_0, x_1, \ldots) \in \ell_1$. Then letting $x_j = f(e_j)$, we see that

$$f(y) = \lim_{n\to\infty} f(y^n) = \sum_{j=1}^{\infty} x_j y_j + x_0 L(y)$$ ∎

Similarly as above, we may show that $\|f\| = \|\tilde{x}\|_1$.

Remark. We write $c_0^* \cong \ell_1$ isometrically.

2.6 Corollary. *$(\ell_1, \|\cdot\|_1)$ is complete.*

3 Axiom of Choice and the Hahn-Banach Theorem

Definition. Let S be a non-empty set. A **partial ordering** is a binary relation $\leq$ on S which satisfies for $s, t, n \in S$,

(i) *(reflexivity)* $s \leq s$

(ii) *(transitivity)* $s \leq t$, $t \leq u$ implies $s \leq u$

(iii) *(anti-symmetry)* $s \leq t$, $t \leq s$ implies $s = t$

We call the pair $(S,\leq)$ a **partially ordered set**. We say that $(S,\leq)$ is **totally ordered** if, given $s,t\in S$, at least one of $s\leq t$ or $t\leq s$ holds. We say that $(S,\leq)$ is **well-ordered** if given any $\emptyset\neq S_0\subseteq S$, there is some $s_0\in S_0$ such that $s_0\leq s$ for $s\in S_0$. A **chain** in a poset $(S,\leq)$ is any $\emptyset\neq C\subseteq S$ such that $(S,\leq|_C)$ is totally ordered.

Example. (i) $X\neq\emptyset$, $(\mathcal{P}(X),\subseteq)$ is a poset
(ii) $(\mathbb{R},\leq)$ is a totally ordered set
(iii) $(\mathbb{N},\leq)$, $(\omega=\mathbb{N}\cup\{\infty\},\leq)$, are well-ordered sets.

3.1 Theorem. *The following are equivalent:*
(i) (Axiom of Choice 1): For any $x\neq\emptyset$, there is a function $\gamma:\mathcal{P}(X)\setminus\{\emptyset\}\to X$ such that $\gamma(A)\in A$ for each $A\in\mathcal{P})X_{\setminus}\{\emptyset\}$.
(ii) (Axiom of Choice 2): Given any $\{A_\lambda\}_{\lambda\in\Lambda}$ where $A_\lambda\neq\emptyset$ for each λ,

$$\prod_{\lambda\in\Lambda}A_\lambda=\{(a_\lambda)_{\lambda\in\Lambda}:a_\lambda\in A_\lambda\text{ for each }\lambda\}\neq\emptyset$$

(iii) (Zorn's Lemma): In a poset $(S,\leq)$, if each chain $C\subseteq S$ admits an upper bound in S, then $(S,\leq)$ admis a maximal element.
(iv) (Well-ordering principle): Any $S\neq\emptyset$ admits a well-ordering

Proof Exercise. ■

Definition. Let X be a vector space (over k). A subset $S\subseteq X$ is called
- **linearly independent** if for any distinct $x_1,\dots,x_n\in S$, the equation $0=\alpha_1x_1+\cdots+\alpha_nx_n=0$ where $\alpha_i\in k$ implies $\alpha_1=\cdots=\alpha_n=0$.
- **spanning** if each $x\in X$ admits $x_i\in S$ and $\alpha_i\in k$ such that $x=\alpha_1x_1+\cdots+\alpha_nx_n$.
- **Hamel basis** if it is both linearly independent and spanning

3.2 Proposition. *Any vector space X admits a Hamel basis.*

Proof Let $\mathcal{L}=\{L\subseteq X: L\text{ is linearly independent}\}$. Then $(\mathcal{L},\subseteq)$ is a poset. Verify that for any chain $\mathcal{C}\subseteq\mathcal{L}$, that $U=\bigcup_{L\in\mathcal{C}}L\in\mathcal{L}$ and is an upper bound for $\mathcal{C}$. Apply Zorn to find a maximal element M in $(\mathcal{L},\subseteq)$. Verify that M is spanning for X. ■

3.3 Corollary. *If X is an infinite dimensional normed space, then there exists $f\in X'\setminus X^*$.*

Proof Our assumption provides $\{e_n\}_{n=1}^\infty$ which is linearly independent. By normalizing each element, we may and will suppose that each $\|e_n\|=1$. Let

$$\operatorname{span}\{e_n\}_{n=1}^\infty=\left\{\sum_{j=1}^m\alpha_je_{n_j}:m\in\mathbb{N},\alpha_i\in\mathbb{F},n_1<\cdots<n_m\right\}$$

and let B be any linearly independent set containing $\{e_n\}_{n=1}^\infty$. Define $f:X=\operatorname{span}B\to\mathbb{F}$ be given for $x=\sum_{b\in B\setminus\{e_n\}_{n=1}^\infty}\alpha_bb+\sum_{j=1}^n\alpha_je_{n_j}$ by $f(x)=\sum_{j=1}^m\alpha_jn_j$. The point is that $f(e_n)=n$ and $f(e)=0$ for any other $e\in B$. Notice that

$$\|f\|=\sup_{x\in B(X)}|f(x)|\geq\sup_{n\in\mathbb{N}}|f(e_n)|=\sup_{n\in\mathbb{N}}n=\infty$$ ■

Definition. Let X be a $\mathbb{R}-$vector space. A **sublinear functional** is any $\rho : X \to \mathbb{R}$ such that it satisies

- *(non-negative homogenity)* $\rho(tx) = t\rho(x)$ for $t \geq 0$, $x \in X$.
- *(subadditivity)* $\rho(x+y) \leq \rho(x) + \rho(y)$ for $x, y \in X$.

3.4 Theorem. (Hahn-Banach) *Let X be a $\mathbb{R}-$vector space, $\rho : X \to \mathbb{R}$ a sublinear functional, $Y \subseteq X$ a subspace and $f \in Y'$ such that $f \leq \rho|_Y$. Then there exists $F \in X'$ such that $F|_Y = f$ and $F \leq \rho$ on X.*

Proof Given $x \in X \setminus Y$, we wish to find $c \in \mathbb{R}$ such that

$$f(y) + \alpha c \leq \rho(y + \alpha x)$$

for $y \in Y$ and $\alpha \in \mathbb{R}$. In this case, we let $F : \operatorname{span} Y \cup \{x\} \to \mathbb{R}$ be given by $F(y+\alpha x) = f(y) + \alpha c$, and we have that F is linear and satisfies $F \leq \rho$ on $\operatorname{span} Y \cup \{s\}$. To do this, let y_+, y_- in Y and observe that $f(y_+) + f(y_-) = f(y_+ + y_-) \leq \rho(y_+ + y_-) \leq \rho(y_+ + x) + \rho(y_- - x)$ so that $f(y_-) - \rho(y_- - x) \leq \rho(y_+ + x) - f(y_+)$. It thus follows that

$$\sup\{f(y) - \rho(y - x) : y \in Y\} \leq \in \{\rho(y + x) - f(y) : y \in Y\}$$

so we may find $c \in \mathbb{R}$ for which

$$\sup\{f(y) - \rho(y - x) : y \in Y\} \leq c \leq \inf\{\rho(y + x) - f(y) : y \in Y\}$$

If $t > 0$, then for $y \in Y$,

$$c \leq \rho\left(\frac{1}{t}y + x\right) - f\left(\frac{1}{t}y\right) \Rightarrow tc \leq \rho(y + tx) - f(y) \Rightarrow f(y) + tc \leq \rho(y + tx)$$

and if $s > 0$, then for $y \in Y$,

$$f\left(\frac{1}{s}y\right) - \rho\left(\frac{1}{s}y - x\right) \leq c \Rightarrow sc \leq f(y) - \rho(y + sx) \Rightarrow f(y) - sc \leq \rho(y - sx)$$

Clearly, $f(y) + 0 \leq \rho(y + 0x)$. Hence, we have our desired inequality.

To be continued... ■