

# Representation Theory of Finite Groups

Alex Rutar\*  
University of Waterloo

Fall 2019<sup>†</sup>

---

\*[arutar@uwaterloo.ca](mailto:arutar@uwaterloo.ca)

<sup>†</sup>Last updated: September 18, 2019



---

# Contents

---

<b>Chapter I</b>	<b>Introduction</b>	
1	Tensor Products . . . . .	4
2	Character Theory . . . . .	4



---

# I. Introduction

---

Let  $G$  be a finite group of order  $n$ , and write  $G = \{g_1, \dots, g_n\}$ . Fix  $g \in G$ ; then  $gg_i = gg_j$  if and only if  $i = j$ . Thus there exists some  $\sigma_g \in S_n$  such that  $gg_i = g_{\sigma_g(i)}$  for all  $i \in \{1, 2, \dots, n\}$ . In particular,  $\phi : G \rightarrow S_n$  by  $\phi(g) = \sigma_g$  is an embedding (injective group homomorphism). This observation is usually referred to as Cayley's Theorem.

Now let  $V$  be an  $n$ -dimensional complex vector space. We then denote  $\text{GL}(V)$  as the group of invertible linear operators  $T : V \rightarrow V$ . Now define  $\psi : S_n \rightarrow \text{GL}(V)$  by  $\psi(\sigma) = T_\sigma$  where if  $\{b_1, \dots, b_n\}$  is a basis for  $V$  and  $T_\sigma(b_i) = b_{\sigma(i)}$ . This is an injective group homomorphism, so  $\psi \circ \phi : G \rightarrow \text{GL}(V)$  is an embedding of  $G$  into  $\text{GL}(V)$ .

**Definition.** Let  $G$  be a finite group, and  $V$  a finite dimensional  $\mathbb{C}$ -vector space. A **representation** of  $G$  is a group homomorphism  $\rho : G \rightarrow \text{GL}(V)$ . We call  $\dim(V)$  the **degree** of the representation.

In particular, if  $V$  is  $n$ -dimensional, then  $\text{GL}(V) \cong \text{GL}_n(\mathbb{C})$ .

*Example.* 1. Consider  $\rho : G \rightarrow \text{GL}(\mathbb{C}) \cong \mathbb{C}^\times$  given by  $\rho(g) = 1$  for all  $g \in G$ . This is called the *trivial representation*.

2. Consider  $\rho : S_n \rightarrow \mathbb{C}^\times$  given by  $\rho(\sigma) = \text{sgn}(\sigma)$ , which is called the *sign representation*.

3. The representation of  $G$  afforded by Cayley's theorem is called the *regular representation* of  $G$ . The next example is a good way to understand the regular rep of  $G$ .

4. Consider  $G$ ,  $X = \{x_1, \dots, x_n\}$ , and  $V = \text{Free}(X)$ . Suppose  $G$  acts on  $X$ . Then  $\rho : G \rightarrow \text{GL}(V)$  given by  $\rho(g)(x_i) = gx_i$ . In particular, if we take  $X = G$ , then this is the regular representation of  $G$ .

5. Consider the 4-gon, with vertices labelled  $a, b, c, d$ . Take  $X = \{a, b, c, d\}$  and the regular representation  $\rho : D_4 \rightarrow \text{GL}(V)$ . This action has a geometric notion.

6. Let  $C_n$  be a cyclic group of order  $n$ ; let us define some  $\rho : C_n \rightarrow \text{GL}(V)$ . Say  $\rho(x) = T$  where  $t \in \text{GL}(V)$ ; then this is a representation if and only if  $T^n = I$ .

**Definition.** We say that two representations  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  are **isomorphic** if there exists an isomorphism  $T : V \rightarrow W$  such that for all  $g \in G$ ,

$$T \circ \rho(g) = \tau(g) \circ T$$

Suppose  $\rho : G \rightarrow \text{GL}(V)$  and  $T : V \rightarrow W$  is an isomorphism. Then we can define  $\tau : G \rightarrow \text{GL}(W)$  by  $\tau(g) = T \circ \rho(g) \circ T^{-1}$ ; this  $\rho \cong \tau$ . In other words, the representation is unique up to isomorphism under change of basis.

*Example.* Consider  $G = \{g_1, \dots, g_n\} = \{h_1, \dots, h_n\}$ , and fix  $g \in G$ . Let  $gg_i = g_{\alpha(i)}$  and  $gh_i = h_{\beta(i)}$  where  $\alpha, \beta \in S_n$ . Fix an  $n$ -dimensional vector space  $V$  with basis  $\{b_1, \dots, b_n\}$ . Then two regular representations are given by

$$\rho_1 : G \rightarrow \text{GL}(V), \rho(g)(b_i) = b_{\alpha(i)}$$

$$\rho_2 : G \rightarrow \text{GL}(V), \rho(g)(b_i) = b_{\beta(i)}$$

Let  $\gamma \in S_n$  be such that  $h_{\gamma(i)} = g_i$ , and define  $T : V \rightarrow V$  by  $T(b_i) = b_{\gamma(i)}$ . Then

$$gg_i = g_{\alpha(i)} = gh_{\gamma(i)} = h_{\beta\gamma(i)} = g_{\gamma^{-1}\beta\gamma(i)}$$

so that  $\alpha = \gamma^{-1}\beta\gamma$ . Thus for each  $b_i$ ,

$$\begin{aligned} T \circ \rho_1(g) \circ T^{-1}(b_i) &= T \circ \rho_1(g)(b_{\gamma^{-1}(i)}) \\ &= T(b_{\alpha\gamma^{-1}(i)})b_{\gamma\alpha\gamma^{-1}(i)} \\ &= b_{\beta(i)} = \rho_2(g)(b_i) \end{aligned}$$

so that  $T \circ \rho_1(g) \circ T^{-1} = \rho_2(g)$ .

Note: conjugate elements have the same cycle type.

## SUBREPRESENTATIONS

What should a subrepresentation of  $\rho : G \rightarrow \text{GL}(V)$  mean?

We would like a subspace  $W \leq V$  such that  $\tau : G \rightarrow \text{GL}(W)$  is a representation given by  $\tau(g)(w) = \rho(g)(w)$  for all  $w \in W$ . Moreover, to make this well-defined, we need  $W$  to be  $\rho(g)$ -invariant for every  $g \in G$  ( $\rho(g)(W) \subseteq W$ ).

Suppose  $T : V \rightarrow V$  is a linear operator, and  $W \leq V$  is a  $T$ -invariant subspace; i.e.  $T(W) \subseteq W$ . In particular, the restriction operator  $T_W : W \rightarrow W$  is well-defined.

**Definition.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation. A subspace  $W \subseteq V$  is said to be  **$G$ -stable** if  $W$  is  $\rho(g)$ -invariant for all  $g \in G$ . A **subrepresentation** of  $\rho$  is a representation  $\rho_W : G \rightarrow \text{GL}(W)$  where for all  $g \in G$  and  $w \in W$ ,  $\rho_W(g)(w) = \rho(g)(w)$  where  $W$  is a  $G$ -stable subspace of  $V$ .

*Example.* Suppose  $\rho : G \rightarrow \text{GL}(V)$  be the regular representation. Take  $W = \text{span}\{\sum_{g \in G} v_g\}$ , which is clearly  $G$ -stable, and  $\rho_W : G \rightarrow \text{GL}(W)$  is isomorphic to the trivial representation.

Similarly, let  $\rho : S_n \rightarrow \text{GL}(V)$  be the regular representation,  $W = \text{span}\{\sum_{\sigma \in S_n} \text{sgn}(\sigma)v_\sigma\}$ ; this is isomorphic to the sign representation.

**0.1 Theorem.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation,  $W \leq V$   $G$ -stable. Then there exists a  $G$ -stable subspace  $W'$  such that  $V = W \oplus W'$ .

**PROOF** Take any inner product  $\langle x, y \rangle$  on  $V$ . Then for any  $x, y \in V$ , define

$$\langle x, y \rangle^* = \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle$$

This is also an inner product. Let  $x, y \in V$  and let  $h \in G$ . Then

$$\begin{aligned} \langle \rho(h)(x), \rho(h)(y) \rangle^* &= \sum_{g \in G} \langle \rho(g)\rho(h)(x), \rho(g)\rho(h)(y) \rangle \\ &= \sum_{g \in G} \langle \rho(gh)(x), \rho(gh)(y) \rangle \\ &= \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle \end{aligned}$$

Thus every  $\rho(h)$  is unitary with respect to  $\langle \cdot, \cdot \rangle^*$ . Let  $W \leq V$  be  $G$ -stable, and take  $W' = W^\perp$  with respect to  $\langle \cdot, \cdot \rangle^*$ . Then  $V = W \oplus W'$ . Let's see that  $W^\perp$  is  $G$ -stable. Let  $x \in W^\perp$ ,  $w \in W$ ,

and  $g \in G$ , so that

$$\begin{aligned} \langle \rho(g)(x), w \rangle^* &= \langle x, \rho(g)^*(w)^* \rangle = \langle x, \rho(g)^{-1}(w) \rangle^* \\ &= \langle x, \underbrace{\rho(g^{-1})(w)}_{\in W} \rangle^* \\ &= 0 \end{aligned}$$

and  $\rho(g)(W^\perp) \subseteq W^\perp$  as required.  $\blacksquare$

**Definition.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation, and  $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$  where each  $W_i$  is  $G$ -stable. For each  $i$ , let  $\rho_i = \rho|_{W_i}$ . For each  $v = \sum w_i \in V$ , we have  $\rho(g)(v) = \sum \rho(g)(w_i) = \sum \rho_i(g)(w_i)$ . In this case, we write

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k$$

and call  $\rho$  a direct sum of the  $\rho_i$ 's.

The previous definition is written as an internal direct sum of  $V$ . Externally, given vector spaces  $W_1, \dots, W_k$  and representations  $\rho_i : G \rightarrow \text{GL}(W_i)$ , we can define

$$(\rho_1 \oplus \cdots \oplus \rho_k) : G \rightarrow \text{GL}(W_1 \oplus \cdots \oplus W_k)$$

by  $(\rho_1 \oplus \cdots \oplus \rho_k)(g)(w_1, \dots, w_k) = (\rho_1(g)(w_1), \dots, \rho_k(g)(w_k))$ . If  $\rho_i : G \rightarrow \text{GL}(W_i)$  is a subrepresentation of  $\rho : G \rightarrow \text{GL}(V)$ , we often say “ $W_i$  is a subrepresentation of  $V$ ”.

**Definition.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation. We say  $\rho$  is **irreducible** if  $V \neq \{0\}$  and the only  $G$ -stable subspaces of  $V$  are  $\{0\}$  and  $V$ .

Clearly,

**0.2 Theorem.** Every representation  $\rho : G \rightarrow \text{GL}(V)$  can be written as a direct sum of irreducible sub-representations.

*Example.* Let  $\rho : S_3 \rightarrow \text{GL}(\mathbb{C}^3)$  be the permutation representation with respect to the standard basis  $\{e_1, e_2, e_3\}$ . Consider  $W_1 = \text{span}\{e_1 + e_2 + e_3\}$  and  $W_2 = \text{span}\{e_1 - e_2, e_2 - e_3\}$ . Is  $W_2$  irreducible?

More generally, if  $V = W_1 \oplus \cdots \oplus W_k$  and  $\dim W_i = 1$  and  $\deg(\rho_i) = 1$ ,

$$\rho(gh)(\sum w_i) = \sum \rho_i(gh)(w_i) = \sum \rho_i(g)\rho_i(h)(w_i) = \sum \rho_i(h)\rho_i(g)(w_i)$$

so that  $\rho(gh) = \rho(hg)$ . In our example, this does not happen, since  $\rho(g) \neq I$  when  $g \neq 1$  and  $S_3$  is not abelian.

*Example.* Let  $\rho : S_3 \rightarrow \text{GL}(V)$  be the regular representation. Let  $W_1 = \text{span}\{\sum_{\sigma \in S_3} v_\sigma\}$  and  $W_2 = \text{span}\{\sum_{\sigma \in S_3} \text{sgn}(\sigma)v_\sigma\}$ , and

$$W_3 = \sum \alpha_\sigma v_\sigma \mid \alpha \begin{matrix} +\alpha_{(123)} + \alpha_{(1,3,2)} \\ = 0 \\ \alpha_{(12)} + \alpha_{(13)} + \alpha_{(23)} \\ = 0 \end{matrix} \in$$

Now let's focus on  $W_3$ . A basis for  $W_3$  is given by

$$\begin{aligned} e_1 &= v_\epsilon - v_{(123)} & e_2 &= v_\epsilon - v_{(123)} \\ e_3 &= v_{(12)} - v_{(13)} & e_4 &= v_{(12)} - v_{(23)} \end{aligned}$$

Recall that  $S_3 = \langle (12), (123) \rangle$ ; suffices to show stability with respect to generators.

$$\begin{aligned} \rho(12) : e_1 &\mapsto e_4, e_2 \mapsto e_3, e_3 \mapsto e_2, e_4 \mapsto e_1 \\ \rho(123) : e_1 &\mapsto e_2 - e_1, e_2 \mapsto -e_1, e_3 \mapsto e_4 - e_3, e_4 \mapsto -e_3 \end{aligned}$$

Let  $U_1 = \text{span}\{e_1 - e_4, e_2 + e_3 - e_1\}$

## 1 TENSOR PRODUCTS

Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  be representations. We define the representation  $\rho \otimes \tau : G \rightarrow \text{GL}(V \otimes W)$

$$(\rho \otimes \tau)(g)(v \otimes w) = \rho(g)(v) \otimes \tau(g)(w)$$

## 2 CHARACTER THEORY

We define the character of  $\rho$  by  $\rho : G \rightarrow \mathbb{C}$  as  $\chi(G) = (\rho(g))$ .

*Remark.* If we choose a basis  $\beta$  for  $V$ , then define  $A(g) = [\rho(g)]_\beta$  and  $\chi(G)$  is given by the sum of the diagonal entries of  $A(g)$ . Furthermore, if  $A, B \in M_n(\mathbb{C})$ , then  $(AB) = (BA)$ .

The remark implies a number of facts:

- (i)  $\rho \cong \tau$ , then  $(\rho(g)) = (\tau(g))$ .
- (ii)  $(T)$  is the sum of eigenvalues of  $T$
- (iii)  $\chi(1) = \dim(V)$ .

**2.1 Proposition.** *For every  $g \in G$  the eigenvalues of  $\rho(g)$  have modulus 1. In particular,  $\chi(g^{-1}) = \overline{\chi(g)}$ .*

**PROOF** Set  $n = |G|$ ; then  $\rho(g)^n = \rho(g^n) = I$  so that  $\lambda^n - 1 = 0$  for any eigenvalue  $\lambda$ , so  $|\lambda| = 1$ . Furthermore,

$$\overline{\chi(g)} = \overline{\sum \lambda_i} = \sum \overline{\lambda_i} = \sum \lambda_i^{-1} = \chi(g^{-1})$$

proving the second component. ■

**2.2 Proposition.** *Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$ . Then  $\chi_{\rho \oplus \tau} = \chi_\rho + \chi_\tau$  and  $\chi_{\rho \otimes \tau} = \chi_\rho \cdot \chi_\tau$ .*

**PROOF** Let  $\beta_1 = \{v_1, \dots, v_n\}$  be a basis for  $V$  and  $\beta_2 = \{w_1, \dots, w_m\}$  a basis for  $W$ . Then a basis for  $V \oplus W$  is given by  $\beta = \{(v_1, 0), \dots, (v_n, 0), (0, w_1), \dots, (0, w_m)\}$ . In particular,

$$[(\rho \oplus \tau)(g)]_\beta = \begin{pmatrix} [\rho(g)]_{\beta_1} & \\ & [\tau(g)]_{\beta_2} \end{pmatrix}$$



and the trace result follows.

A basis for  $V \otimes W$  is given by  $\gamma = \{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$  in lexicographic order. Fix  $g \in G$ , and set  $A = [\rho(g)]_{\beta_1}$ ,  $B = [\rho(g)]_{\beta_2}$ . Fix  $v_i \otimes w_j \in \gamma$ . Then

$$\begin{aligned} (\rho \otimes \tau)(g)(v_i \otimes w_j) &= \rho(g)(v_i) \otimes \tau(g)(w_j) \\ &= (a_{1i}v_1 + \cdots + a_{ni}v_n) \otimes (b_{1j}w_1 + \cdots + b_{mj}w_m) \\ &= \cdots + a_{ii}b_{jj} \cdot (v_i \otimes w_j) + \cdots \\ &= ([\rho \otimes \tau](g))_{\delta} = \sum_{i,j} a_{ii}b_{jj} = (A)() = \chi_{\rho}(g) \cdot \chi_{\tau}(g) \quad \blacksquare \end{aligned}$$

*Example.* Suppose  $\rho : S_n \rightarrow \text{GL}(\mathbb{C}^n)$  is the permutation representation with respect to  $\{e_1, \dots, e_n\}$ . Then  $\chi(\sigma) = |\{e_i : \rho(\sigma)(e_i) = e_i\}| = |\text{Fix}(\sigma)|$ , which is the number of indices  $i$  fixed by  $\sigma$ . Since  $S_n$  acts transitively on  $\{1, \dots, n\}$ , there is exactly 1 orbit, so by Burnside's lemma,

$$n! = |S_n| = \sum_{\sigma \in S_n} \chi(\sigma)$$

*Example.* Let  $\rho : G \rightarrow \text{GL}(V)$  be the regular representation. Note that if  $g \neq 1$ , then for all  $h \in G$ ,  $gh \neq h$ . In particular, this means that  $\chi(g) = 0$  if  $g \neq 1$ , and  $\chi(1) = |G|$  (the dimension of  $V$ ).

*Example.* Let  $\rho : S_3 \rightarrow \text{GL}(V)$  be the regular representation. Recall that  $V = W_1 \oplus W_2 \oplus U_1 \oplus U_2$  where  $W_1$  is the trivial representation,  $W_2$  is the sign representation, and  $U_1, U_2$  are isomorphic. Let  $S_3 = \langle (12), (123) \rangle$ ; then we have

$$\begin{array}{c|cc} x_1 & 1 & 1 \\ x_2 & -1 & 1 \\ x_3 & a & b \\ x_4 & a & b \end{array}$$

In particular,  $\chi(12) = 1 - 1 + 2a = 0$  and  $\chi(123) = 1 + 1 + 2b = 0$ , so  $b = -1$ .

*Example.* Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation. In particular,  $\rho(ghg^{-1}) = \rho(g)\rho(h)\rho(g)$  so that  $\rho(ghg^{-1}) = \rho(h)$  so  $\chi(ghg^{-1}) = \chi(h)$ ; in other words, that characters are constant on conjugacy classes.

**2.3 Lemma. (Schur)** Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  be irreducible representations, and suppose  $T : V \rightarrow W$  is linear such that for all  $g \in G$ ,  $\tau(g) \circ T = T \circ \rho(g)$ . Then either  $T = 0$  or  $T$  is an isomorphism and  $\rho \cong \tau$ . Moreover, if  $V = W$  and  $\rho = \tau$ , then  $T$  is a scalar multiple of the identity.

**PROOF** Assume  $T \neq 0$ .

Let's first see that  $T$  is injective, and let  $v \in \ker(T)$ . Then for any  $g \in G$ ,  $T(\rho(g)(v)) = \tau(g)(T(v)) = 0$ , so  $\rho(g)(v) \in \ker(T)$ . Thus  $\ker(T)$  is  $G$ -stable (with respect to  $\rho$ ). Since  $\rho$  is irreducible and  $T \neq 0$ ,  $\ker(T) = \{0\}$ .

We also have that  $T$  is surjective. Let  $v \in \text{Im}(T)$  and say  $v = T(x)$  with  $x \in V$ . Then for  $g \in G$ ,  $\tau(g)(v) = \tau(g)(T(x)) = T(\rho(g)(x)) \in \text{Im}(T)$  so  $\text{Im}(T)$  is  $G$ -stable, and again by irreducibility of  $\tau$ ,  $\text{Im}(T) = W$ . Thus  $T$  is an isomorphism.

Now let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $T$  and consider  $T' = T - \lambda I$ . Now, note that for  $g \in G$ ,  $\rho(g)T' = T'\rho(g)$ , but  $T'$  has non-trivial kernel, so in fact  $T' = 0$ .  $\blacksquare$

**2.4 Corollary.** Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  be irreducible, and  $T : V \rightarrow W$  linear. Consider

$$T' = \frac{1}{|G|} = \sum_{g \in G} \tau(g)^{-1} T \rho(g)$$

Then

- (i) If  $T' \neq 0$ , then  $\rho \cong \tau$  via  $T'$ .
- (ii) If  $V = W$ ,  $\rho = \tau$ , then  $T' = (T)/\dim(V) \cdot I$ .

PROOF Clearly  $T' : V \rightarrow W$  is linear, and for any  $h \in G$ ,

$$\begin{aligned} \tau(h)T' &= \tau(h) \frac{1}{|G|} \sum_{g \in G} \tau(g)^{-1} T \rho(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \tau(hg^{-1}) T \rho(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \tau(g^{-1}) T(\rho(gh)) \\ &= \frac{1}{|G|} \sum_{g \in G} \tau(g^{-1}) T \rho(g) \rho(h) \\ &= T' \rho(h) \end{aligned}$$

If  $V = W$  and  $\rho = \tau$ , then  $(T') = \frac{1}{|G|}(T) \cdot |G| = (T) = \alpha \dim(V)$ , so  $\alpha = (T)/\dim(V)$ . ■

Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  be irreducible representations, and  $T : V \rightarrow W$  linear. Let  $\beta$  be a basis for  $V$  and  $\gamma$  a basis for  $W$ . Then for  $g \in G$ , let  $[\rho(g)]_\beta = (a_{ij}(g))$ ,  $[\tau(g)]_\gamma = (b_{kl}(g))$ ,  $[T]_\beta^\gamma = (x_{ki})$ , and  $[T']_\beta^\gamma = (x'_{ki})$ .

By matrix multiplication,  $x'_{ki} = \frac{1}{|G|} \sum_g \sum_{j,l} b_{kl}(g^{-1}) x_{lj} a_{ji}(g)$ . If  $\rho \not\cong \tau$ , then  $T' = 0$ , so by viewing the RHS as a polynomial in the  $x_{ij}$ , we have

$$\frac{1}{|G|} \sum_g b_{kl}(g^{-1}) a_{ji}(g) = 0$$