

PMATH 465

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I. Fundamentals of Manifolds

1 INTRODUCTION TO TOPOLOGY

Definition. A **topology** on a set X is a set τ of subsets of X such that

- (i) $\emptyset \in \tau$ and $X \in \tau$
- (ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
- (iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \leq i \leq n$, then $\bigcap_{i=1}^n U_i \in \tau$.

The sets $U \in \tau$ are called the **open sets** in X , and sets of the form $X \setminus U$ for some open set U are called the **closed sets** in X .

Definition. When X is a topological space and $A \subseteq X$, the **interior** of A (denoted A°) is the union of all open sets contained in A . Similarly, we define the **closure** of A (denoted \overline{A}) as the intersection of all closed sets containing A . Then the **boundary** of A , denoted by ∂A , is the set $\partial A = \overline{A} \setminus A^\circ$.

Example. Let X be any set. The **discrete topology** on X is the topology $\tau = \mathcal{P}(X)$, and the **trivial topology** on X is the topology $\tau = \{\emptyset, X\}$.

Definition. A **basis** for a topology on a set X is a set \mathcal{B} of subsets of X

- (i) $\bigcup_{B \in \mathcal{B}} B = X$
- (ii) for all $a \in X$ and $U, V \in \mathcal{B}$ such that $a \in U \cap V$, then there exists $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$.

When \mathcal{B} is a basis for a topology on X , the topology on X **generated** by \mathcal{B} is the set τ of subsets of X such that for $W \subseteq X$, $W \in \tau$ if and only if for all $a \in W$, there exists $U \in \mathcal{B}$ such that $a \in U \subseteq W$.

Note that τ , as above, is a topology on X since

- (i) $\emptyset \in \tau$ vacuously and $X \in \tau$ obviously.
- (ii) If $A_k \in \tau$ for all $k \in K$ (where K is any set of indices), then given $a \in \bigcup_{k \in K} A_k$, we can choose $\ell \in K$ so that $a \in A_\ell$. Then since $A_\ell \in \tau$, we can choose $U_\ell \in \mathcal{B}$ so that $a \in U_\ell \subseteq A_\ell$. Thus $a \in U_\ell \subseteq A_\ell \subseteq \bigcup_{k \in K} A_k$.
- (iii) By induction, it suffices to prove that if $A, B \in \tau$, then $A \cap B \in \tau$. Suppose $A, B \in \tau$, and let $a \in A \cap B$. Since $A \in \tau$, we can choose $U \in \mathcal{B}$ so that $a \in U \subseteq A$. Since $B \in \tau$, we can choose $V \in \mathcal{B}$ so that $a \in V \subseteq B$. Then we have $a \in U \cap V$. Since \mathcal{B} is a basis, we can choose $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$, so $a \in W \subseteq U \cap V \subseteq A \cap B$.

Note that when τ is the topology on X generated by the basis \mathcal{B} , for $A \subseteq X$, $A \in \tau$ if and only if there exists some $S \subseteq \mathcal{B}$ such that $A = \bigcup_{s \in S} s$. In this sense, the topology τ on X generated by the basis \mathcal{B} is the coarsest topology which contains \mathcal{B} .

Definition. (Subspace Topology) When Y is a topological space and $X \subseteq Y$ is a subset of Y , we define the **subspace topology** on X to be the topology for which a set $U \subseteq X$ is open if and only if $U = X \cap V$ for some open set V .

If \mathcal{C} is a basis for the topology on Y , then $\mathcal{B} = \{X \cap V \mid V \in \mathcal{C}\}$ is a basis for the subspace topology on X .

Definition. (Disjoint Union Topology) If X and Y are topological spaces with $X \cap Y = \emptyset$, then the **disjoint union topology** on $X \cup Y$ is the topology in which a subset $U \subseteq X \cup Y$ is open in $X \cup Y$ if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y .

Definition. (Product Topology) If X and Y are topological spaces, the **product topology** on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} = \{ U \times V \mid U \in \mathcal{C}, V \in \mathcal{D} \}$$

where \mathcal{C} and \mathcal{D} are bases for the topologies on X, Y respectively.

Definition. (Infinite Product Topology) We define the infinite product to be

$$\prod_{k \in K} \left\{ f : K \rightarrow \bigcup_{k \in K} X_k \mid f(k) \in X_k \text{ for all } k \in K \right\}$$

There are two standard topologies on X . The first is the **box topology**,

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \mid U_k \text{ is open in } X_k \right\}$$

and the **product topology**

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \mid \begin{array}{l} U_k \text{ is open in } X_k \\ U_k = X_k \text{ for all but finitely many indices } k \end{array} \right\}$$

Example. (Metric Topology) \mathbb{R}^n has a standard **inner product**, and for $u, v \in \mathbb{R}^n$, $uv = u \cdot v = V^T u = \sum_{i=1}^n u_i v_i$. This gives the standard norm on \mathbb{R}^n for $u \in \mathbb{R}^n$, $\|u\| = \sqrt{uv}$. This gives the standard metric on \mathbb{R}^n : for $a, b \in \mathbb{R}^n$, $d(a, b) = \|b - a\|$.

Given a metric on a set Y , we obtain (by restriction) an induced metric on any subset $X \subseteq Y$. Given a metric space X , we define the **metric topology** on X to be the topology which is generated by the set of open balls

$$B(a, r) = \{ x \in X \mid d(a, x) < r \}$$

where $x \in X$, $r > 0$.

Definition. (Quotient Topology) Let X be a topological space and let \sim be an equivalence relation on X . The set of equivalence classes is denoted X/\sim , and X/\sim is called the **quotient** of X by \sim . The map $q : X \rightarrow X/\sim$ given by $q(a) = [a]$ is called the natural **projection map** or **quotient map**. We define the **quotient topology** on X/\sim by stipulating that for $W \subseteq X/\sim$, W is open in X/\sim if and only if $q^{-1}(W)$ is open in X .

Definition. When X and Y are topological spaces and $f : X \rightarrow Y$, we say that f is **continuous** when it has the property that $f^{-1}(V)$ is open in X for every open set V in Y . We say that $f : X \rightarrow Y$ is a **homeomorphism** when f is bijective and both f and f^{-1} are continuous. Then X, Y are **homeomorphic** if there exists a homeomorphism $f : X \rightarrow Y$.

1.1 Theorem. (Glueing Lemma) Let X and Y be topological spaces, and let $f : X \rightarrow Y$ be a function. Suppose either

- (i) $X = \bigcup_{k \in K} A_k$ where each A_k is open in X , or

(ii) $X = \bigcup_{k=1}^n A_k$ where each A_k is closed in X and each restriction map $f_k : A_k \rightarrow Y$ is continuous, then f is continuous.

PROOF Exercise. ■

Definition. A topological space X is **compact** when it has the property that for every set \mathcal{S} of open subsets of X with $X = \bigcup_{U \in \mathcal{S}} U$, there exists a finite subset $\mathcal{F} \subseteq \mathcal{S}$ such that $X = \bigcup_{F \in \mathcal{F}} F$.

Note that when $X \subseteq Y$ is a subspace, X is compact if and only if X has the property that for every set \mathcal{T} with $X \subseteq \bigcup_{T \in \mathcal{T}} T$, there exists a finite subset $\mathcal{G} \subseteq \mathcal{T}$ such that $X \subseteq \bigcup_{G \in \mathcal{G}} G$.

Definition. A topological space X is **connected** when there do not exist non-empty disjoint open sets $U, V \subseteq X$ such that $X = U \cup V$.

Note that if Y is a metric space and $X \subseteq Y$ is a subspace, then X is connected if and only if there do not exist open sets $U, V \subseteq Y$ such that

$$X \cap U \neq \emptyset, X \cap V \neq \emptyset, U \cap V = \emptyset, \text{ and } X \subseteq U \cup V$$

Definition. A topological space X is called **path connected** when it has the property that for all $a, b \in X$, there exists a continuous map $\alpha : [0, 1] \rightarrow X$ with $\alpha(0) = a$ and $\alpha(1) = b$.

It is easy to see that if X is path connected, then X is connected.

Definition. Let X be a topological space. If we define a relation \sim on X by taking $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a \in A$ and $b \in A$.

It is clear that this is an equivalence relation. Note that when X is a topological space, its connected components are connected, and each connected subspace of X is contained in one of its connected components.

Definition. Let X be a topological space. Define a relation \approx on X by $a \approx b$ if and only if there exists a continuous map $\alpha : [0, 1] \rightarrow X$ with $\alpha(0) = a$ and $\alpha(1) = b$. Such a map α is called a **continuous path**.

One can show that if X is **locally path connected** (which means that X has a basis for its topology which consists of path connected sets), then the path components of X are equal to the connected components of X , and that these components are open.