REPLACE

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Contents

I. Graph Colourings

1 List Colourings

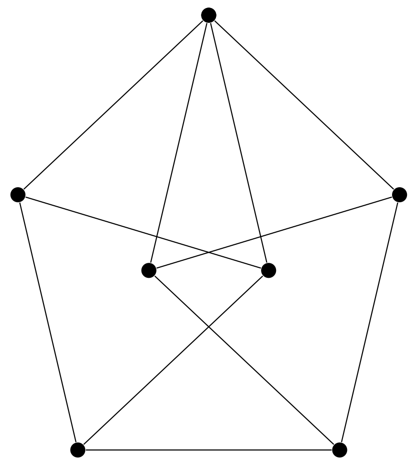
Recall that a colouring of a graph G is an assignment to each $v \in V(G)$ an element $c(v)$ of some set C called "colors" such that if v and v' are neighbours, then $c(v) \neq c(v')$. Then the **chromatic number** $\chi(G)$ is the smallest cardinality $|C|$ such that there exists a colouring of G from C.

There are some basic upper bounds on the chromatic number of a graph:

1. $\chi(G) \leq |V(G)|$, by colouring every vertex distinctly
2. $\chi(G) \leq \Delta(G) + 1$, by randomly colouring the graph based on colours not used on the neighbours

Note that these upper bounds are in fact tight; for example, the complete graph is tight for both, and an odd cycle is tight for (2).

There are some graphs for which the chromatic number is not known: consider the graph given by $V(G) = \mathbb{R}^2$ where vertices are adjacent if they have euclidean distance 1. This graph is not 3–colorable, by taking for example the subgraph



Recently there was a construction showing that the graph is not 4–colourable, and there is an easy upper bound of 7, so that $5 \leq \chi(G) \leq 7$.

We also define the notion of a list colouring:

Definition. A list assignment is an assignment of a set $L(v)$ of colors to each vertex v. Then a graph is k–list-colorable if you can always colour $V(G)$ whenever every vertex has a list of size at least k.

Note that $\chi(G) \leq \chi_\ell(G)$ since asssigning an identical list of size k is a valid list assignmnet and yields a standard coloring. In many cases list colorings can be hard to determine, but in some cases the exact value is known. Consider the complete bipartite graph $K_{k,q}$ where $q \geq k$. We then have the following classification:

1.1 Proposition. *$\chi_\ell(K_{k,q}) \le k$ if and only if $q < k^k$, and $\chi_\ell(K_{k,q}) = k+1$ if and only if $q \ge k^k$.*

Proof Note that $\chi_\ell(K_{k,q}) \le k+1$ always works by taking arbitrary colors on the k–side, and on the q–side, since the lists have size k, there is always a distinct color.

Now $q < k^k$. Try to color the k vertices such that two vertices have the same color. If this works, then for every list of size k on the q–side, there are only $k-1$ disallowed colours, so we may choose a valid color from the corresponding list. Otherwise, every vertex on the k–side has a distinct color; this is forced precisely when all the lists are disjoint. But then since $q < k^k$, there must be some selection of colors from the lists on the k–side such that the set of colors is distinct from every list on the q–side, and we may choose colors from the q–side without issue.

Otherwise if $q \ge k^k$, consider lists given by disjoint sets on the k–side, and then for every possible assignment of colors on the k–side, give a corresponding list for some vertex of the q–side that contains a list with those colors. Since $q \ge k^k$, we will exhaust all possibilities, so there is no valid coloring from those lists. ■

Recall that a planar graph G is one for which there exists an embedding of G into the plane such that each edge is a disjoint curve. Note that it suffices to consider edges which are polygonal curves, which consist of a finite number of straight line segments; in fact we can also do it with straight line segments (requiring that the graph is simple).

1.2 Theorem. (Thomassen) *If G is planar, then $(G) \le 5$.*

In fact, we prove a stronger statement. We call an "almost-triangulation" a planar drawing in which every face except possibly the infinite face is a triangle. We prove this: let w be a given almost-triangulation with lists of available colour $L(v)$ assigned to every vertex v such that

1. $|L(v)| = 5$ for all vertices that are not on the infinite face,
2. two neighbouring vertices of the infinite face, a and b are colored distinctly,
3. and all other vertices of the infinite face have lists of 3 colours.

Then this almost-triangulation has a proper list colouring with respect to the given lists.

This implies the theorem since any planar drawing can be made an almost-triangulation by adding edges, and 5-element lists can be reduced to lists of the size above.

Proof We consider two cases in an induction proof.

1. There is a "long diagonal" connecting two of the vertices of the infinite face (that is not an edge of the infinite face).
2. There is no long diagonal.

The induction is on the number of vertices. When $n = 1, 2$ it is trivial, and when $n = 3$ it is a 3-cycle and it is certainly fine.

Now for the induction step, we have the two cases.

1. Cut the graph along the long diagonal to get G_1, G_2. Without loss of generality, G_1 is exactly as described in the statement, so it can be properly list coloured from the given lists. Then give the endpoints of the copied long diagonal in G_2 so that the endpoint colours are fixed; and by induction, colour it as well. Since the endpoints have the same colouring, we can put the two coloured graphs back together to obtain a proper list colouring of G.

2. Let $u \in V(G)$ be the neighbour of a on the infinite face different from b. Consider the neighbourhood of u, $N(u) = \{a, w, v_1, v_2, \ldots, v_k\}$ where w is on the infinite face different from a. We have $|L(w)| = 3$ and $|L(v_i)| = 5$ for all $i = 1, \ldots, k$ since there is no long diagonal. Choose two different colours γ and Δ in $L(u) \setminus \{\alpha\}$; they certainly exist since $|L(u)| = 3$. Delete γ and δ from all the lists of vertices in $\{v_1, \ldots, v_k\}$, and then by induction we can list colour $G \setminus \{u\}$ from the modified lists. This can be extended to a list colouring of G since u shares no colour in its list with any $\{v_1, \ldots, v_k\}$, and at least one of δ or γ will not be used in w. ∎

n–connected means if you remove any n vertices, the graph remains connected

Take K_4, and have lists with colours $1, 2, 3, 4$ (or any graph which is uniquely 4–colorable). Inscribe a triangle in each face with lists $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$. Always align so that the degree 3 vertex is adjacent to the $1, 2$ and $1, 3$.