Measure Theory

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1 Measure Spaces

Definition. Let $X \neq \emptyset$ be a set. $\mathcal{M} \subset \mathcal{P}(X)$ is called a $\sigma-$**algebra** on X if

1. $X \in \mathcal{M}$
2. $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$
3. If $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$

The pair $(X, \mathcal{M})$ is called a **measurable space**. The elements of $\mathcal{M}$ are called **measurable sets**.

Definition. Let $(X, \mathcal{M})$ be a measurable space, (Y, τ) be a topological space. Then $f : X \to Y$ is called **measurable** if $f^{-1}(V) \in \mathcal{M}$ for all $V \in \tau$.

We have the following properties of $\sigma-$algebras.

1.1 Proposition. *1. $\emptyset \in \mathcal{M}$*

2. $A_1, A_2, \ldots, A_n \in \mathcal{M} \Rightarrow A_1 \cup A_2 \cup \cdots \cup A_n \in \mathcal{M}$

3. $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$

4. $A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$

5. f is measurable, $H \subset Y$ is closed, then $f^{-1}(H) \in \mathcal{M}$.

Proof 1. $X \in \mathcal{M} \Rightarrow X^c \in \mathcal{M}$.

2. We can extend this to a countable union by introduction $A_{n+i} = \emptyset$ for $i \in \mathbb{N}$.
3. By DeMorgan's identities, $(\bigcap A_n)^c = \bigcup A_n^c \in \mathcal{M}$.
4. $A \setminus B = A \cap B^c \in \mathcal{M}$.
5. H^c is open implies $f^{-1}(H^c) \in \mathcal{M}$. Then $f^{-1}(H) = (f^{-1}(H^c))^c \in \mathcal{M}$. ■

Extended Real Line

One can define the extended real line as follows: set the space $X = \mathbb{R} \cup \{-\infty, +\infty\}$. Then the topology is given by

$$G \in \tau \Leftrightarrow \begin{cases} \text{for all } x \in G \cap \mathbb{R} & \exists r > 0 \text{ s.t. } (x-r, x+r) \subset G \\ -\infty \in G & \exists b \in \mathbb{R} \text{ s.t. } (-\infty, b) \subset G \\ +\infty \in G & \exists a \in \mathbb{R} \text{ s.t. } (a, \infty) \subset G \end{cases}$$

The same can be done with a single symbol as well. In either case, the extended real line is a compact set. We also extend the general operations so that $a + \infty = \infty$ for any $a \in (0, \infty]$, and $\infty = \sup[0, \infty] = \sup[0, \infty)$, and similarly for $-\infty$.

We define for $(a_i) \subset [0, \infty]$

$$\sum_{i=1}^{\infty} a_i = \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} a_i$$

If $(a_i), (b_i) \subset [0, \infty]$, then

$$\sum_{i=1}^{\infty} (a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$$

Furthermore, if $(a_{ij})_{i=1}^{\infty}{}_{j=1}^{\infty} \subset [0,\infty]$, then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

With this, we can define positive measures.

Definition. Let $(X,\mathcal{M})$ be a measurable space. A function $\mu : \mathcal{M} \to [0,\infty]$ is called a **(positive) measure** if it is countably additive and not constant ∞. In other words,

1. $\mu(\emptyset) = 0$
2. $\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \sum\limits_{n=1}^{\infty} \mu(A_n)$ if $A_i \cap A_j = \emptyset$

The pair $(X,\mathcal{M},\mu)$ is called a **measure space**.

1.2 Proposition. *1. If $A_i \cap A_j = \emptyset$ then $\mu\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$.*
2. $A \subset B$ implies $\mu(A) \leq \mu(B)$ Then $\mu(B \setminus A) = \mu(B) - \mu(A)$.
3. If $A_1, A_2, \ldots \in \mathcal{M}$, then $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$. This is referred to as $\sigma-$subadditivity.
4. $A_1 \subset A_2 \subset A_3 \cdots$ then $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n)$
5. $A_1 \supset A_2 \supset A_3 \cdots$ and $\mu(A_i) < \infty$ then $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_{n=1}^{\infty} A_n)$

Proof 1. Obvious.

2. Follows since $B = A \cup (B \setminus A)$ is a disjoint union.

3. Let $E_1 = A_1$, $E_{n+1} = A_{n+1} \setminus \bigcup_{i=1}^{n} A_i$. Then $E_i \cap E_j = \emptyset$ and if $i \neq j$ and for all $i \in \mathbb{N}$, $E_i \in \mathcal{M}$ and $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} A_i$. Thus

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

4. Define $B_1 := A_1$ and $B_{i+1} = A_{i+1} \setminus A_i$ for $i \geq 1$. Then $B_i \cap B_j = \emptyset$ and $\mu(A_n) = \mu\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} \mu(B_i)$. Similary, $\mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$. Thus the result follows since

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \sum_{n=1}^{\infty} \mu(B_n)$$

5. Without loss of generality $\mu(A_1) < \infty$. Let $C_1 = \emptyset$, $C_n = A_1 \setminus A_n$. Then $C_1 \subset C_2 \subset \cdots$ and $\mu(C_n) + \mu(A_n) = \mu(A_1)$. Let $A = \bigcap_{n=1}^{\infty} A_n$ so $A_1 \setminus A = \bigcup_{n=1}^{\infty} C_n$ and $(\bigcup_{n=1}^{\infty} C_n) \uplus A = A_1$ disjointly. But then $\mu(\bigcup_{n=1}^{\infty} C_n) + \mu(A) = \mu(A_1)$ so that

$$\mu(A_1) - \mu(A) = \mu\left(\bigcup_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \mu(C_n) = \mu(A_1) - \lim \mu(A_n)$$

Since $\mu(A_1)$ is finite, we have $\mu(A) = \lim \mu(A_n)$. ■

Examples of Measures

Definition. A measure space $(X, \mathcal{M}, \mu)$ is called:

1. **finite** if $\mu(X) < \infty$
2. a **probability space** if $\mu(X) = 1$. If $0 < \mu(X) < \infty$, then $\frac{1}{\mu(X)}\mu$ is a probabiliy measure.
3. $\sigma-$**finite** if there is a countable collection $\{X_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$, $\bigcup_{i=1}^{\infty} X_i = X$, and $\mu(X_i) < \infty$.
4. **decomposable** if there is a set $\Pi \subseteq \mathcal{M}$ such that
   a) Π partitions X
   b) If $E \subseteq X$, then $E \in \mathcal{M}$ if and only if $E \cap P \in \mathcal{M}$ for each $P \in \Pi$
   c) $\mu(P) < \infty$ for all $P \in \Pi$
   d) If $E \in \mathcal{M}$ with $\mu(E) < \infty$, then
$$\mu(E) = \sup_{\substack{\mathcal{F} \subseteq \Pi \\ \mathcal{F} \text{ finite}}} \sum_{P \in \mathcal{F}} \mu(E \cap P) := \sum_{P \in \Pi} \mu(E \cap P)$$
5. **semifinite** if for any $E \in \mathcal{M}$ with $\mu(E) > 0$, there is $F \in \mathcal{M}$, $F \subseteq E$ such that $0 < \mu(F) < \infty$ (each set is "finitely approximatable from below")
6. **complete** if whenever $N \subseteq X$ such that $N \subseteq E$, $E \in \mathcal{M}$ and $\mu(E) = 0$, then $N \in \mathcal{M}$.

A common technique that $\sigma-$finiteness allows is to define $E_n = \bigcup_{i=1}^{n} X_i$, so $E_1 \subseteq E_2 \subseteq \cdots$, $X = \bigcup_{i=1}^{\infty} E_i$ and each $\mu(E_i) < \infty$. Alternatively, let $A_1 = X_1$, $A_{n+1} = X_{n+1} \setminus \bigcup_{i=1}^{n} X_i$, so each $A_i \in \mathcal{M}$, $A_i \cap A_j = \emptyset$ if $i \neq j$, each $\mu(A_i) < \infty$, and $X_i = \bigcup_{i=1}^{\infty} A_i$ disjointly.

Note that every $\sigma-$finite measure is both decomposable and semifinite, but decomposability and semifiniteness are not equivalent.

Completeness has some technical usefulness. However, every measure space $(X, \mathcal{M}, \mu)$ extends to a complete measure space, so this property is rather unexciting. Most "natural" constructions of measures give us complete measures.

Here are some common examples of measures.

1. The zero measure. Given a measurable space $(X, \mathcal{M})$, let $\mu(E) = 0$ for $E \in \mathcal{M}$.

2. Counting measure. Let X be any non-empty set. Then $\mathcal{P}(X)$ is a $\sigma-$algebra on X. We let $\gamma : \mathcal{P}(X) \to [0, \infty]$ by
$$\gamma(E) = \begin{cases} |E| & : |E| < \infty \\ \infty & : \text{otherwise} \end{cases}$$
   Then $(X, \mathcal{P}(X), \gamma)$ is a measure space (easy exercise). This space is
   - finite if and only if X is finite
   - $\sigma-$finite if and only if X is countable
   - always decomposable ($\Pi = \{\{x\} : x \in X\}$).
   - always semifinite
   - always complete

   Since $X \neq \emptyset$, if X is finite, let $\nu = \frac{1}{|X|}\gamma$ is the uniform probability.

3. Point mass or Dirac measure. Let $a \in X$ and define $\delta_a : \mathcal{P}(X) \to \{0, 1\} \subset [0, \infty]$ by
$$\delta_a(E) = \begin{cases} 1 & : a \in E \\ 0 & : a \notin E \end{cases}$$
   Again, this is clearly a measure. It is complete, since null sets are those which do not contain a. It is also a probability measure.

4. Let X be a countable set, and let $\mathcal{M}$ be the subsets of X that are countable or have countable complement. Define $\mu : \mathcal{M} \to [0,\infty]$ by $\mu(E) = 0$ if E is countable, and infinity otherwise. The measure is not semifinte, nor decomposable, and naturally not σ−finite. However, it is complete.

5. Let $X = \{x_0\}$, the singleton sets. Then $\mathcal{P}(X) = \{\emptyset, \{x_0\}\}$, and define $\mu(\emptyset) = 0$ and $\mu(\{x_0\}) = \infty$. It is not decomposable nor semifinite.

In the next section, we will investigate how to construct more useful measures.

2 Pre-measures to Outer Measures to Measures

Outer Measures

Definition. Let X be a non-empty set. An **outer measure** on X is a function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ such that

(i) $\mu^*(\emptyset) = 0$

(ii) $A \subseteq B$ implies $\mu^*(A) \le \mu^*(B)$

(iii) $A_1, A_2, \ldots \in \mathcal{P}(X)$, then

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu^*(A_i)$$

Remark. Certainly any measure on $\mathcal{P}(X)$ is an outer measure. While outer measures are easy to construct and have the largest possible domain, they may not be σ−additive.

prop:def-outer

2.1 Proposition. *Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be any family such that $\{\emptyset, X\} \subseteq \mathcal{E}$, and there is a function $\rho : \mathcal{E} \to [0,\infty]$ such that $\rho(\emptyset) = 0$. Then the function μ^* defined for $A \in \mathcal{P}(X)$ by*

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) : E_1, E_2, \ldots \in \mathcal{E}, A \subseteq \bigcup_{i=1}^{\infty} E_i\right\}$$

defines an outer measure on X.

Remark. Unless $(\mathcal{E}, \rho)$ is "nice", we may not be able to recver ρ from μ^*. It certainly holds that for any $E \in \mathcal{E}$, $\mu^*(E) \le \rho(E)$, but we may not get equality.

Proof First, $0 \le \mu^*(\emptyset) \le \rho(\emptyset) = 0$. Second, if $A \subseteq B \subseteq X$, then any countable $\mathcal{E}$−cover of B is evidently an $\mathcal{E}$−cover of A. Finally, suppose $A_1, A_2, \ldots \subseteq X$ and let $\epsilon > 0$. By definition of μ^* to each A_i, get $E_{i1}, E_{i2}, \ldots$ in $\mathcal{E}$ such that $A_i \subseteq \bigcup_{j=1}^{\infty} E_{ij}$ and $\sum_{j=1}^{\infty} \rho(A_i) + \frac{\epsilon}{2^i}$. Then $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} E_{ij}$ so that

$$\begin{aligned}\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) &\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E_{ij} \le \sum_{i=1}^{\infty}\left(\mu^*(A_i) + \frac{\epsilon}{2^i}\right) \\ &= \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the desired inequality holds. ∎

Caratheodory's Theorem

Definition. (Caratheodory) Given an outer measure μ^* on X, we say that a set $A \subseteq X$ is μ^*−**measurable** provided that for any $E \in \mathcal{P}(X)$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$.

Remark. If μ^* is an outer measure, $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \setminus A)$ always happens. In practice, we only need check the converse inequality.

Definition. Given a non-empty set X, an **algebra** on X is a family $\mathcal{A} \subseteq \mathcal{P}(X)$ such that

(i) $X \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow X \setminus A \in \mathcal{A}$

(iii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

By induction, any finite union of sets is in A. As well, $\emptyset \in \mathcal{A}$ and $\mathcal{A}$ is closed under finite intersections.

thm:carat

2.2 Theorem. (Caratheodory) *Given an outer measure $\mu^* : \mathcal{P}(X) \to [0, \infty]$, we have that*

(i) $\mathcal{M} = \left\{A \in \mathcal{P}(X) : \text{for all } E \in \mathcal{E}(X), \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A)\right\}$ *is a σ−algebra.*

(ii) $\mu = \mu^*|_{\mathcal{M}} : \mathcal{M} \to [0, \infty]$ *is a complete measure.*

Claim I *$\mathcal{M}$ is an algebra.*

Proof First note that $X \in \mathcal{M}$: if $E \in \mathcal{P}(X)$, then $\mu^*(E \cap X) + \mu^*(E \setminus X) = \mu^*(E) + \mu^*(\emptyset) \leq \mu^*(E)$.

Now, let $A, B \in \mathcal{M}$ be arbitrary. We have for $E \in \mathcal{P}(X)$ that

$$\mu^*(E \cap (X \setminus A)) + \mu^*(E \setminus (X \setminus A)) = \mu^*(E \setminus A) + \mu^*(E \cap A) \leq \mu^*(E)$$

so that $X \setminus A \in \mathcal{M}$. Furthermore,

$$\begin{aligned}
\mu^*(E) &\geq \mu^*(E \cap A) + \mu^*(E \setminus A) \\
&\geq \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) \setminus B) + \mu^*((E \setminus A) \cap B) + \mu^*((E \setminus A) \setminus B) \\
&= \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \setminus B)) + \mu^*(E \cap (B \setminus A)) + \mu^*(E \setminus (A \cup B)) \\
&\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B))
\end{aligned}$$

by σ−additivity and that $A \cup B = (A \cap B) \cup (A \setminus B) \cup (B \setminus A)$. Thus $A \cup B \in \mathcal{M}$. ∎

Claim II *$\mathcal{M}$ is a σ−algebra.*

Proof It suffices to show closure under countable unions. Let $A_1, A_2, \ldots \in \mathcal{M}$ and $A = \bigcup_{i=1}^{\infty} A_i$. Let $B_1 = A_1$, $B_{n+1} = A_{n+1} \setminus \bigcup_{i=1}^{n} A_i$, so $B_i \cap B_j = \emptyset$. In particular, each $B_i \in \mathcal{M}$ and $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$. We thus have for $E \in \mathcal{P}(X)$ arbitrary

$$\begin{aligned}
\mu^*\left(E \cap \bigcup_{i=1}^{n} B_i\right) \geq \mu^*\left(\left(E \cap \bigcup_{i=1}^{n} B_i\right) \cap B_n\right) + \mu^*\left(\left(E \cap \bigcup_{i=1}^{n} B_i\right) \setminus B_n\right) &= \mu^*(E \cap B_n) + \mu^*\left(E \cap \bigcup_{i=1}^{n-1} B_i\right) \\
&\geq \mu^*(E \cap B_n) + \mu^*(E \cap B_{n-1}) + \mu^*\left(E \cap \bigcup_{i=1}^{n-2} B_i\right) \\
&\vdots \\
&\geq \sum_{i=1}^{n} \mu^*(E \cap B_i)
\end{aligned}$$

by induction. We thus have

$$\begin{aligned}\mu^*(E) &\geq \mu^*\left(E \cap \bigcup_{i=1}^n A_i\right) + \mu^*\left(E \setminus \bigcup_{i=1}^n A_i\right) \geq \mu^*\left(E \cap \bigcup_{i=1}^n B_i\right) + \mu^*(E \setminus A) \\ &\geq \sum_{i=1}^n \mu^*(E \cap B_i) + \mu^*(E \setminus A)\end{aligned}$$

so, taking the limit,

$$\begin{aligned}\mu^*(E) &\geq \sum_{i=1}^\infty \mu^*(E \cap B_i) + \mu^*(E \setminus A) \qquad (2.1) \\ &\geq \mu^*\left(\bigcup_{i=1}^\infty (E \cap B_i)\right) + \mu^*(E \setminus A) \\ &= \mu^*(E \cap A) + \mu^*(E \setminus A)\end{aligned}$$

{eq:car}

so that $A \in \mathcal{M}$. Thus (i) is established. ■

Claim III *μ is a complete measure.*

Proof To show that μ is a measure, it remains to show σ−additivity. Assume $A_1, A_2, \ldots \in \mathcal{M}$ and that $A_i \cap A_j = \emptyset$ for $i \neq j$. Take $E = A$ and $B_i = A_i$ as in *Eq.* (2.1), so that

$$\mu^*(A) \geq \sum_{i=1}^\infty \mu^*(A \cap A_i) + \mu(A \setminus A) = \sum_{i=1}^\infty \mu^*(A_i) \geq \mu^*\left(\bigcup_{i=1}^\infty A_i\right) = \mu^*(A)$$

Finally, let $N \in \mathcal{M}$ with $\mu(N) = 0$, and $E \subseteq N$ arbitrary. It suffices to show that $N \in \mathcal{M}$; indeed, for any $F \in \mathcal{P}(X)$, we have

$$\mu^*(F \cap E) + \mu^*(F \setminus E) \leq \mu^*(N) + \mu^*(F) = \mu(N) + \mu^*(F) = \mu^*(F)$$

and we have (ii). ■

Pre-Measures to Measures

Definition. Let $\mathcal{A}$ be an algebra on X. A **pre-measure** is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ such that

(i) $\mu_0(\emptyset) = 0$

(ii) If $A_1, A_2, \ldots \in \mathcal{A}$ with $A_i \cap A_j = \emptyset$, and $\bigcup_{i=1}^\infty A_i \in \mathcal{A}$, then $\mu_0(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu_0(A_i)$.

A **pre-measure space** is a triple $(X, \mathcal{A}, \mu_0)$.

Since $\mathcal{A}$ is an algebra, μ_0 respects finite unions. As with measures, premeasures are monotone: $A \subseteq B$ in $\mathcal{A}$ implies $\mu_0(A) \leq \mu_0(B)$.

thm:pre-to-m

2.3 Theorem. *Let $(X, \mathcal{A}, \mu_0)$ be a premeasure space and define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by*

$$\mu^*(E) = \inf\left\{\sum_{i=1}^\infty \mu_0(A_i) : A_1, A_2, \ldots \in \mathcal{A}, E \subseteq \bigcup_{i=1}^\infty A_i\right\}$$

Then μ^* is an outer measure, and

(i) $\mu^*|_{\mathcal{A}} = \mu_0$

(ii) The set $\mathcal{M}$ of μ^*–measurable sets contains $\mathcal{A}$. Hence, $\mu = \mu^*|_{\mathcal{M}}$ satisfies $\mu|_{\mathcal{A}} = \mu_0$.

(iii) If $\nu : \mathcal{M} \to [0,\infty]$ is a measure with $\nu_{\mathcal{A}} = \mu_0$, then $\nu(E) \le \mu(E)$ for all $E \in \mathcal{M}$, with $\nu(E) = \mu(E)$ if $\mu(E) < \infty$. In particular, if $(X,\mathcal{M},\mu)$ is σ–finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.

Proof Recall that μ^* is an outer measure by the standard construction in Proposition 2.1.

(i) Let $A \in \mathcal{A}$. Note that $\mu^*(A) \le \mu_0(A)$ immediately by definition of μ^*. Conversely, let $A_1, A_2, \ldots, \in \mathcal{A}$ be such that $A \subseteq \bigcup_{i=1}^\infty A_i$. Let $B_1 = A_1$, $B_{n+1} = A_{n+1} \setminus \bigcup_{i=1}^n A_i$, so $B_i \in \mathcal{A}$, $B_i \cap B_j = \emptyset$ for $i \ne j$. Thus

$$A = A \cap \bigcup_{i=1}^\infty A_i = A \cap \bigcup_{i=1}^\infty B_i = \bigcup_{i=1}^\infty (A \cap B_i)$$

where $(A\cap B_i) \cap (A \cap B_j) = \emptyset$ for $i \ne j$. Hence, by restricted σ–additivity,

$$\mu_0(A) = \mu_0\left(\bigcup_{i=1}^\infty (A \cap B_i)\right) = \sum_{i=1}^\infty \mu_0(A\cap B_i) \le \sum_{i=1}^\infty \mu_0(A_i)$$

Since $\{A_i\}$ was an arbitrary cover, we see that $\mu_0(A) \le \mu^*(A)$ and equality holds.

(ii) Now, let $A \in \mathcal{A}$ and $E \in \mathcal{P}(X)$; we verify Caratheodory's criterion as in Theorem 2.2. Given $\epsilon > 0$, let $A_1, A_2, \ldots \in \mathcal{A}$ be such that $E \subseteq \bigcup_{i=1}^n A_i$ and

$$\sum_{i=1}^\infty \mu_0(A_i) \le \mu^*(E) + \epsilon$$

Then, for each i, $\mu_0(A_i) = \mu_0(A_i \cap A) + \mu_0(A_i \setminus A)$ by finite additivity, and $E \cap A \subseteq \bigcup_{i=1}^\infty (A_i \cap A)$, $E \setminus A \subseteq \bigcup_{i=1}^\infty (A_i \setminus A)$. Thus

$$\begin{aligned}\mu^*(E) + \epsilon &\ge \sum_{i=1}^\infty \mu_0(A_i) = \sum_{i=1}^\infty \mu_0(A_i \cap A) + \sum_{i=1}^\infty \mu_0(A_i \setminus A)\\ &\ge \mu^*(E\cap A) + \mu^*(E \setminus A)\end{aligned}$$

and since ϵ was arbitrary, we see that the desired inequality holds.

(iii) If $E \in \mathcal{M}$ and $A_1, A_2, \ldots \in \mathcal{A}$ are such that $E \subseteq \bigcup_{i=1}^\infty A_i$, then

$$\nu(E) \le \nu\left(\bigcup_{i=1}^\infty A_i\right) \le \sum_{i=1}^\infty \nu(A_i) = \sum_{i=1}^\infty \mu_0(A_i)$$

and it follows from definition of $\mu = \mu^*|_{\mathcal{M}}$ that $\nu(E) \le \mu(E)$.

Now recall that $\mathcal{A}_\sigma = \{\bigcup_{i=1}^\infty A_i : A_1, A_2, \ldots \in \mathcal{A}\}$. In particular, $\nu|_{\mathcal{A}_\sigma} = \mu|_{\mathcal{A}_\sigma}$: to see this, if $A = \bigcup_{i=1}^\infty A_i$, $A_i \in \mathcal{A}$, then

$$\nu(A) = \lim_{n\to\infty} \nu\left(\bigcup_{i=1}^n A_i\right) = \lim_{n\to\infty} \mu_0\left(\bigcup_{i=1}^n A_i\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^n A_i\right) = \mu(A)$$

by continuity from below.

Now, let $E \in \mathcal{M}$ with $\mu(E) < \infty$. Given $\epsilon > 0$, let $A_1, A_2, \ldots \in \mathcal{A}$ with $E \subseteq \bigcup_{i=1}^\infty A_i = A$ and such that

$$\mu(E) + \epsilon = \mu^*(E) + \epsilon > \sum_{i=1}^\infty \mu_0(A_i)$$

Hence, $\mu(E) \le \mu(A) \le \sum_{i=1}^\infty \mu(A_i) = \sum_{i=1}^\infty \mu_0(A_i) < \mu(E)+\epsilon$. Thus $\mu(A \setminus E) = \mu(A) - \mu(E) < \epsilon$. Hence, as $A \in \mathcal{A}_\sigma$, $\mu(A) = \nu(A)$ and we have

$$\begin{aligned}\mu(E) \le \mu(A) = \nu(A) &= \nu(A \cap E) + \nu(A \setminus E) \le \nu(A \cap E) + \mu(A \setminus E) \\ &< \nu(E) + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\mu(E) \le \nu(E)$, so equality must hold.

Finally, if $(X, \mathcal{M}, \mu)$ is σ–finite, we can write $X = \bigcup_{i=1}^\infty X_i$ where $X_i \in \mathcal{M}$, $\mu(X_i) < \infty$, and $X_1 \subseteq X_2 \subseteq \cdots$. If $E \in \mathcal{M}$, then $E = \bigcup_{i=1}^\infty (X_i \cap E)$, so

$$\mu(E) = \lim_{n\to\infty} \mu(X_n \cap E) = \lim_{n\to\infty} \nu(X_n \cap E) = \nu(E)$$

by continuity from below. ■

Remark. The uniqueness also holds if we have that $(X, \mathcal{M}, \mu)$ is semifinite. Indeed, by A1, if $E \in \mathcal{M}$,

$$\begin{aligned}\mu(E) &= \sup\{\mu(F) : F \in \mathcal{M}, F \subseteq E, \mu(F) < \infty\} \\ &= \sup\{\nu(F) : F \in \mathcal{M}, F \subseteq E, \mu(F) < \infty\} \le \nu(E) \le \mu(E)\end{aligned}$$

2.4 Corollary. *Given a measure space $(X, \mathcal{M}, \mu)$, there is a complete measure space $(X, \overline{\mathcal{M}}, \overline{\mu})$ such that $\overline{\mu}|_{\mathcal{M}} = \mu$. Furthermore, if $(X, \mathcal{M}, \mu)$ is a σ–finite, then any $E \in \mathcal{M}$ admits a representation of the form $E = M \cup N$, with $M \in \mathcal{M}$ and $N \subseteq N'$ where $N' \in \mathcal{M}$ has $\mu(N') = 0$.*

Proof We regard $(X, \mathcal{M}, \mu)$ is a pre-measure space. By the previous theorem (Theorem 2.3), we get an outer measure μ^* such that $\mu^*|_{\mathcal{M}} = \mu$ and if

$$\overline{\mathcal{M}} = \{A \in \mathcal{P}(X) : \mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \setminus A) \text{ for all } E \in \mathcal{P}(X)\}$$

then $\mathcal{M} \subseteq \overline{\mathcal{M}}$; we may take $\overline{\mu} = \mu^*|_{\overline{\mathcal{M}}}$.

We now appeal to A1Q4 to see the structure of $E \in \overline{\mathcal{M}}$. We have $X \setminus E \in \overline{\mathcal{M}}$ so that $X \setminus E = A \setminus N$ where $A \in \mathcal{M}_{\sigma\delta}$ and $\mu^*(N) = 0$. For each n, we can find $A_{n1}, A_{n2}, \ldots \in \mathcal{A}$ such that $N \subseteq \bigcup_{i=1}^\infty A_{ni} := A_n$ and $\sum_{i=1}^\infty \mu(A_{ni}) < 1/n = \mu^*(N) + 1/n$. In particular, $N \subseteq A_n$ and $A_n \in \mathcal{M}$. Thus $N \subseteq \bigcap_{n=1}^\infty A_n =: N'$, $N' \in \mathcal{M}$ and $\mu(N') \le \mu(A_n) < 1/n$ for each n so $\mu(N') = 0$. Finally,

$$E = X \setminus (X \setminus E) = X \setminus (A \setminus N) = (X \setminus A) \cup N$$

gives a decomposition of the desired form. ■

The past section has provided an important abstract construction: given a pre-measure space $(X, \mathcal{A}, \mu_0)$, one gets an outer measure μ^*, and by Caratheodory, extract a measure space $(X, \mathcal{M}, \mu)$ such that $\mathcal{M} \supseteq \mathcal{A}$ and $\mu|_{\mathcal{A}} = \mu_0$.

3 Constructing σ–Algebras and Measures

3.1 Lemma. *Let X be a non-empty set.*

(i) *If $\{M_i\}_{i\in I}$ is a family of σ–algebras on X, then $\bigcap_{i\in I} \mathcal{M}_i \subseteq \mathcal{P}(X)$.*

(ii) *Given $\emptyset \neq \mathcal{E} \subseteq \mathcal{P}(X)$, we define*

$$\sigma\langle\mathcal{E}\rangle = \bigcap\Big\{\mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{–algebra on } X, \mathcal{E} \subseteq \mathcal{M}\Big\}$$

*This is the **σ–algebra generated by $\mathcal{E}$**.*

(iii) *If $\emptyset \neq \mathcal{F} \subseteq \sigma\langle\mathcal{E}\rangle$ in $\mathcal{P}(X)$, then $\sigma\langle\mathcal{F}\rangle \subseteq \sigma\langle\mathcal{E}\rangle$.*

Proof (i) It is easy to check the σ–algebra axioms.

(ii) Direct application of (i)

(iii) Since $\sigma\langle\mathcal{E}\rangle$ is a σ–algebra containing $\mathcal{F}$, by (ii), the result follows since $\sigma\langle\mathcal{F}\rangle$ is the smallest σ–algebra containing $\mathcal{F}$. ∎

Definition. We say that the **algebra generated by $\mathcal{E}$** is $\langle\mathcal{E}\rangle = \bigcap\{\mathcal{A} : \mathcal{A}$ is an algebra on $X, \mathcal{E} \subseteq \mathcal{A}\}$.

We now turn our sights towards topology:

Definition. Let (X,τ) be a topological space. The **Borel σ–algebra** is defined $\mathcal{B}(X,\tau) = \mathcal{B}(X) = \sigma\langle\tau\rangle$.

Remark. If $\mathcal{F} = \{F \subseteq X : F \text{ is closed}\}$, then $\mathcal{F} \subseteq \sigma\langle\tau\rangle$ and $\sigma\langle\mathcal{F}\rangle \subseteq \sigma\langle\mathcal{G}\rangle$. The opposite inclusion holds identically, so these sets are equal.

prop:br-gen

3.2 Proposition. *Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ–algebra on $\mathbb{R}$. Consider the following families of subsets of $\mathbb{R}$:*

1. *$\mathcal{O} = \{(a,b) : -\infty \leq a \leq b \leq \infty\}$, $(a,a) = \emptyset$.*
2. *$\mathcal{O}_\infty = \{(a,\infty) : -\infty \leq a \in \mathbb{R}\}$.*
3. *$\mathcal{H} = \{(a,b] : -\infty \leq a \leq b \leq \infty \text{ in } \mathbb{R}\}$, $(a,\infty] = (a,\infty)$, $(a,a] = \emptyset$.*
4. *$\mathcal{C}_\infty\{[a,\infty) : -\infty < a \in \mathbb{R}\}$.*

Then $\mathcal{B}(\mathbb{R}) = \sigma\langle\mathcal{O}\rangle = \sigma\langle\mathcal{O}_\infty\rangle = \sigma\langle\mathcal{H}\rangle = \sigma\langle\mathcal{C}_\infty\rangle$.

Proof This follows since τ has a countable base. ∎

Definition. An **elementary family** of sets on X is any $\mathcal{E} \subseteq \mathcal{P}(X)$ such that

(i) $X \in \mathcal{E}$

(ii) If $E,F \in \mathcal{E}$, $E \cap F = \bigcup_{i=1}^n E_i$ with $E_i \in \mathcal{E}$

(iii) If $E \in \mathcal{E}$, $X \setminus F = \bigcup_{j=1}^m E_j$, $E_1,\ldots,E_j \in \mathcal{E}$.

A simple induction arguement shows that any finite intersection of elements of $\mathcal{E}$ is a finite union of elements in $\mathcal{E}$.

Example. In $\mathbb{R}$, $\mathcal{H} = \{(a,b] : -\infty \leq a \leq b \leq \infty\}$ is an elementary family.

3.3 Lemma. *If $\mathcal{E} \subseteq \mathcal{P}(X)$ is an elementary family, then $\mathcal{E} = \{\bigcup_{i=1}^n E_i, E_i \in \mathcal{E}, n \in \mathbb{N}\}$.*

Proof It suffices to see that the RHS is an algebra. It is clearly closed under finite unions. Let $E_1, \dots, E_n \in \mathcal{E}$, and write each $X \setminus E_i = \bigcup_{j=1}^{m_i} E_{ij}$. We then have

$$\begin{aligned} X \setminus \left(\bigcup_{i=1}^{n} E_i \right) &= \bigcap_{i=1}^{n} (X \setminus E_i) = \bigcap_{i=1}^{n} \bigcup_{j=1}^{m_i} E_{ij} \\ &= \bigcup_{\substack{1 \le j_i \le n \\ 1 \le i \le n}} E_{1j_1} \cap \dots \cap E_{nj_n} \end{aligned}$$

where each finite intersection is a finite union of elements of $\mathcal{E}$. ∎

The following corollary is now clear:

3.4 Corollary. *In $\mathbb{R}$, $\langle \mathcal{H} \rangle = \left\{ \bigcup_{i=1}^{n} (a_i, b_i] : -\infty \le a_i \le b_i \le \infty \right\}$.*

Let $\mathcal{A} = \langle \mathcal{H} \rangle \subseteq \mathcal{P}(\mathbb{R})$. We will build many premeasures on $\mathcal{A}$.

Locally Finite Measures on $\mathbb{R}$

Definition. We define the set of non-decreasing, right-continuous functions

$$\mathrm{ND}_r(\mathbb{R}) = \left\{ F : \mathbb{R} \to \mathbb{R} : x < y \Rightarrow F(x) \le F(y); \lim_{x \to a^+} F(x) = F(x) \right\}$$

lem:pre-ND ***3.5 Lemma.*** *Let $F \in \mathrm{ND}_r(\mathbb{R})$ and $\mathcal{A} = \langle \mathcal{H} \rangle \subset \mathcal{P}(\mathbb{R})$, the algebra generated by half-open half-closed intervals. Then $\mu_{0,F} : \mathcal{A} \to [0, \infty]$ defined by*

$$\mu_{0,F} \left(\dot{\bigcup_{i=1}^{n}} (a_i, b_i] \right) = \sum_{i=1}^{n} (F(b_i) - F(a_i))$$

defines a pre-measure on $\mathcal{A}$. Here, we say that $b - (-\infty) = \infty$ for $-\infty < b \le \infty$.

For simplicity, write $\mu_0 = \mu_{0,F}$. It is evident that μ_0 is well-defined and that $\mu_0(\emptyset) = 0$; it remains to show that μ_0 has restricted σ-additivity.

cl:a1 Claim I *Suppose $(a, b] = \dot{\bigcup}_{j=1}^{\infty} (c_j, d_j]$, $-\infty < a < b < \infty$. Then $\mu_0((a, b]) = \sum_{j=1}^{\infty} \mu_0((c_j, d_j])$.*

Proof First, given $n \in \mathbb{N}$, there is a bijection $\sigma : [n] \to [n]$ such that $c_{\sigma(1)} \le d_{\sigma(1)} \le \dots \le c_{\sigma(n)} \le d_{\sigma(n)}$. Then, as F is non-decreasing, we have

$$\begin{aligned} \sum_{j=1}^{n} \mu_0((c_j, d_j]) = \sum_{j=1}^{n} (F(d_j) - F(c_j)) &= \sum_{j=1}^{n} (F(d_{\sigma(j)}) - F(c_{\sigma(j)})) \\ &= F(d_{\sigma(n)}) \underbrace{- F(c_{\sigma(n)}) + F(d_{\sigma(n-1)})}_{\le 0} + \dots - F(c_{\sigma(1)}) \\ &\le F(d_{\sigma(n)} - F(c_{\sigma(n)}) \le \mu_0((a, b]) \end{aligned}$$

and since $n \in \mathbb{N}$ is arbitrary, $\sum_{j=1}^{\infty} \mu_0((c_j, d_j]) \le \mu_0((a, b])$.

To see the converse inequality, let $\epsilon > 0$ and, since F is right-continuous, we may find

- $\delta_0 > 0$ such that $a + \delta_0 < b$ and $F(a + \delta_0) < F(a) + \epsilon/2$.
- for each j, $\delta_j > 0$ such that $F(d_j + \delta_j) < F(d_j) + \epsilon/2^{j+1}$

Then $\{(c_j, d_j+\delta_j)\}_{j=1}^\infty$ is a cover of $[a+\delta_0, b]$ and hence, by compactness, we have that $[a+\delta_0, b] \subseteq \bigcup_{j=1}^n (c_j, d_j+\delta_j)$ for some n. Let $\sigma : [n] \to [n]$ be as before. Notice that

- $c_{\sigma(1)} < a+\delta_0$ implies $F(c_{\sigma(1)}) \le F(a+\delta_0) < F(a)+\epsilon/2$.
- For $j = 1, \ldots, n-1$, $c_{\sigma(j+1)} < d_{\sigma(j)} + \delta_{\sigma(j)}$ implies $F(c_{\sigma(j+1)}) \le F(d_{\sigma(j)}+\delta_{\sigma(j)}) < F(d_{\sigma(j)}) + \epsilon/2^{\sigma(j)+1}$
- $b < d_{\sigma(n)} + \delta_{\sigma(n)}$ implies $F(b) < F(d_{\sigma(n)}) + \epsilon/2^{\sigma(n)+1}$.

Thus

$$\begin{aligned}
\sum_{j=1}^\infty \mu_0((c_j, d_j]) \ge \sum_{j=1}^n \mu_0((c_j, d_j]) &= \sum_{j=1}^n (F(d_j) - F(c_j)) \\
&= F(d_{\sigma(n)}) + \sum_{j=1}^{n-1} \Big(F(d_{\sigma(j)}) - F(c_{\sigma(j+1)})\Big) - F(c_{\sigma(1)}) \\
&> \left(F(b) - \frac{\epsilon}{2^{\sigma(n)+1}}\right) + \sum_{j=1}^{n-1}\left(-\frac{\epsilon}{2^{\sigma(j)+1}}\right) - \left(F(a) + \frac{\epsilon}{2}\right) \\
&> F(b) - F(a) - \epsilon = \mu_0((a,b]) - \epsilon
\end{aligned}$$

and since $\epsilon > 0$ is arbitrary, our desired inequality holds. ■

c1:a2 CLAIM II *The same applies to* $(-\infty, b]$, $(a, \infty]$.

PROOF Exercise. ■

CLAIM III *Let* $A, A_1, A_2, \ldots \in \mathcal{A}$ *with* $A = \dot\bigcup_{j=1}^\infty A_i$. *Then* $\mu_0(A) = \sum_{j=1}^\infty \mu_0(A_i)$.

PROOF Write $A = \dot\bigcup_{j=1}^\infty (a_i, b_i]$ and for each i, j, $(a_i, b_i] \cap A_j = \dot\bigcup_{k=1}^{m_{ij}} (c_{ijk}, d_{ijk}]$. From Claim I and Claim II, we have that

$$(a_i, b_i] = \dot\bigcup_{j=1}^\infty \dot\bigcup_{k=1}^{m_{ij}} (c_{ijk}, d_{ijk}]$$

so that

$$\mu_0((a_i, b_i]) = \sum_{j=1}^\infty \sum_{k=1}^{m_{ij}} \mu_0((c_{ijk}, d_{ijk}])$$

and

$$\begin{aligned}
\mu_0(A) &= \sum_{i=1}^n \mu_0((a_i, b_i]) \\
&= \sum_{i=1}^n \sum_{j=1}^\infty \sum_{k=1}^{m_{ij}} \mu_0((c_{ijk}, d_{ijk}]) \\
&= \sum_{j=1}^\infty \sum_{i=1}^n \sum_{k=1}^{m_{ij}} \mu_0((c_{ijk}, d_{ijk}]) \\
&= \sum_{j=1}^\infty \mu_0(A_j)
\end{aligned}$$

since each $A_j = \dot\bigcup_{i=1}^n \dot\bigcup_{k=1}^{m_{ij}} (c_{ijk}, d_{ijk}]$. ■

Definition. A measure $\mu : \mathcal{B}(\mathbb{R}) \to [0,\infty]$ is called **locally finite** if $\mu(K) < \infty$ for $K \subset \mathbb{R}$ compact.

Equivalently, $\mu([-a,a]) < \infty$ for any $0 < a \in \mathbb{R}$. Note that any locally finite measure is also σ−finite.

3.6 Theorem. (Locally Finite Measures on $\mathbb{R}$) *The following hold:*

(i) For each F in $\mathrm{ND}_r(\mathbb{R})$, there is a unique locally finite measure $\mu_F : \mathcal{B}(\mathbb{R}) \to [0,\infty]$ such that $\mu_F((a,b]) = F(b) - F(a)$ for any finite a,b.

(ii) Every locally finite measure appears as in (i)

(iii) If $F, G \in \mathrm{ND}_r(\mathbb{R})$, then $\mu_F = \mu_G$ if and only if $F - G$ is constant.

Proof (i) The previous lemma (Lemma 3.5) provides a premeasure $(\mathbb{R}, \langle\mathcal{H}\rangle, \mu_{0,F})$, where $\mu_{0,F}((a,b]) = F(b) - F(a)$ for $-\infty \le a \le b \le \infty$. By the canonical construction in Theorem 2.3, this gives rise to a measure $\mu_F^* : \mathcal{P}(\mathbb{R}) \to [0,\infty]$, and its σ−algebra $\mathcal{F}$ of μ_F^*−measurable sets. From Proposition 3.2, $\mathcal{B}(\mathbb{R}) = \sigma\langle\mathcal{H}\rangle$; since $\mathcal{H} \subseteq \langle H\rangle \subseteq \mathcal{M}_F$, we have that $\mathcal{B}(\mathbb{R}) = \sigma\langle\mathcal{H}\rangle \subseteq \mathcal{M}_F$. Then, we let $\mu_F = \mu_F^*|_{\mathcal{B}(\mathbb{R})} : \mathcal{B}(\mathbb{R}) \to [0,\infty]$. Notice, for $a > 0$ in $\mathbb{R}$, that

$$\mu_F([-a,a]) \le \mu_F((-a-1,a]) = F(a) - F(-a-1) < \infty$$

so μ_F is locally finite, and hence σ−finite. Thus μ_F is the unique extension of $\mu_{0,F}$ to $\mathcal{B}(\mathbb{R})$ (or even to $\mathcal{M}_F$).

(ii) Let $\mu : \mathcal{B}(\mathbb{R}) \to [0,\infty]$ be a locally inite measure. Then for $x \in \mathbb{R}$, we let

$$F(x) = \begin{cases} \mu((0,x]) & : x \ge 0 \\ -\mu((x,0]) & : x < 0 \end{cases}$$

We will see that $F \in \mathrm{ND}_r(\mathbb{R})$. If $x < y$ in $\mathbb{R}$:

- If $x \ge 0$, then $(0,x] \subseteq (0,y]$ so $F(x) = \mu((0,x]) \le \mu((0,y]) = F(y)$
- If $y < 0$, then $(y,0] \subseteq (x,0]$ so $\mu((y,0]) \le \mu((x,0])$, so $F(x) = -\mu((x,0]) \le -\mu((y,0] = F(y)$.
- If $x < 0 \le y$, then $F(x) = -\mu((x,0]) \le 0 \le \mu((0,y]) = F(y)$.

To see right continuity, it suffices to see for $x \in \mathbb{R}$, we have $F(x) = \lim_{n\to\infty} F(x_n)$, where $(x_n) \to x$ monotonically from the right. Thus, given x, $(x_n)_{n=1}^\infty$, we have

$$F(x_n) - F(x) = \mu((x,x_n]) \xrightarrow{n\to\infty} \mu(\emptyset) = 0$$

by continuity from above for measures.

Notice that for $a < b$ in $\mathbb{R}$, $\mu_F((a,b]) = \mu((a,b])$, which by uniqueness in (i) shows that $\mu = \mu_F$.

(iii) $\mu_F = \mu_G$ if and only if for $x \in \mathbb{R}$,

$$\begin{aligned} F(x) - F(0) = \mu_F((0,x]) = \mu_G((0,x]) = G(x) - G(0) \text{ for } x \ge 0 \\ F(0) - F(x) = \mu_F((x,0]) = \mu_G((x,0]) = G(0) - G(x) \text{ for } x < 0 \end{aligned}$$

if and only if $F(x) - G(x) = F(0) - G(0)$ is constant. ■

Remark. Let $F \in \mathrm{ND}_r(\mathbb{R})$, $a < b$ in $\mathbb{R}$,

1. $(a,b) = \bigcup_{n=1}^{\infty}(a, b-1/n]$ so

$$\begin{aligned}\mu_F((a,b)) &= \lim_{n\to\infty} \mu_F((a,b-1/n]) \\ &= \lim_{n\to\infty}[F(b-1/n)-F(a)] \\ &= F(b^-)-F(a)\end{aligned}$$

2. As above,

$$\begin{aligned}\mu_F([a,b]) &= \lim_{n\to\infty} \mu_F((a-1/n,b]) \\ &= F(b)-F(a^-)\end{aligned}$$

In particular, $\mu_F(\{a\}) = \mu_F([a,a]) = F(a)-F(a^-)$, so $\mu_F(\{a\}) = 0$ if and only if F is continuous at a.

Point Mass/Dirac Measure

Fix $a \in \mathbb{R}$. Let $H_a \in \mathrm{ND}_r(\mathbb{R})$ where

$$H_a(x) = 1_{[a,\infty)}(x) = \begin{cases} 1 & : x \in [a,\infty) \\ 0 & : \text{otherwise} \end{cases}$$

Let $\delta_a : \mathcal{B}(\mathbb{R}) \to [0,\infty]$, where

$$\delta_a(A) = \begin{cases} 1 & : a \in A \\ 0 & : a \notin A \end{cases}$$

Notice that if $c < d$ in $\mathbb{R}$, then

$$\delta_a((c,d]) = \begin{cases} 1 & : c < a \leq d \\ 0 & : \text{otherwise} \end{cases} = H_a(d) - H_a(c)$$

so that $\delta_a = \mu_{H_a}$.

Lebesgue Measure

Let $I(x) = x$, $I \in \mathrm{ND}_r(\mathbb{R})$. We let $\lambda = \mu_I$ and $\mathcal{L} = \mathcal{M}_I \supseteq \mathcal{B}(\mathbb{R})$ denote the Lebesgue measure and Lebesgue σ−algebra.

3.7 Theorem. (Characteristic Property of Lebesgue Measure) *The following hold:*

(i) $(\mathbb{R}, \mathcal{L}, \lambda)$ *is translation invariant: for* $x \in \mathbb{R}$, $E \in \mathcal{L}$, *we have* $E + x \in \mathcal{L}$ *and* $\lambda(E+x) = \lambda(E)$.

(ii) If $\mu : \mathcal{B}(\mathbb{R}) \to [0,\infty]$ *is a translation invariant locally finite measure, then* $\mu = c\lambda$ *for some* $c \geq 0$ *in* $\mathbb{R}$.

Proof (i) If $-\infty \leq a \leq b \leq \infty$, then $\lambda((a,b]+x) = \mu_I((a+x,b+x]) = b-a = \lambda((a,b])$. Hence if $A \in \langle H \rangle$, $\mu_I(A+x) = \mu_I(A)$ for $x \in \mathbb{R}$. If $E \in \mathcal{P}(\mathbb{R})$, $E \subseteq \bigcup_{i=1}^{\infty} A_i$, $A_i \in \langle \mathcal{H} \rangle$ if and only $E + x \subseteq \bigcup_{i=1}^{\infty}(A_i + x)$. Thus, by definition of μ_I^*, we see that $\mu_I^*(E+x) = \mu_I^*(E)$. Now, if $A \in \mathcal{L}$, $E \in \mathcal{P}(\mathbb{R})$, then

$$\begin{aligned}\mu_I^*(E \cap (A+x)) + \mu_I^*(E \setminus (A+x)) &= \mu_I^*([(E-x)\cap A]+x) + \mu_I^*([(E-x)\setminus A]+x) \\ &= \mu_I^*((E-x)\cap A) + \mu_I^*((E-x)\setminus A) \\ &\leq \mu_I^*(E-x) = \mu_I^*(E)\end{aligned}$$

so $E + x \in \mathcal{L}$ and $\mu(E+X) = \mu(E)$.

(ii) We let $\mu = \mu_F$ where $F \in \mathrm{ND}_r(\mathbb{R})$. Without loss of generality, $F(0) = 0$, so

$$F(x) = \begin{cases} \mu((0,x]) & : x \geq 0 \\ -\mu((x,0]) & : x < 0 \end{cases}$$

Then for $y \geq 0$, we have

$$F(y) = \mu((0,y]) = \mu((x,x+y]) = F(x+y) - F(x)$$

so $F(x) + F(y) = F(x+y)$. Thus if $x \geq 0$, $F(nx) = nF(x)$ for $n \in \mathbb{N}$. Thus $F(x/n) = F(x)/n$, $0 = F(0) = F(-x) + F(x)$, $x \geq 0$, so $F(-x) = -F(x)$. Thus $F : \mathbb{R} \to \mathbb{R}$ is additive and $F(qx) = qF(x)$ for $x \in \mathbb{R}$, $q \in \mathbb{Q}$. Now, given $x \in \mathbb{R}$, let (q_n) be a rational sequence so $q_n \geq x$, $\lim q_n = x$, and we have

$$F(x) = \lim_{n\to\infty} F(q_n) = \lim_{n\to\infty} q_n \cdot F(1) = F(1) \cdot x$$

Let $c = F(1) = \mu((0,1]) \geq 0$; then by uniqueness, $\mu = \mu_{cI} = c\lambda$. ∎

Cantor's Sets and Functions

Fix $0 < \alpha \leq 1$. Let $I_{01} = [0,1]$ and J_{01} be the open middle of length $\alpha/3$. Notice that $I_{01} \setminus J_{01} = I_{11} \dot\cup I_{12}$, each a closed interval, with $\lambda(I_{1k}) < 1/2$, $k = 1,2$. Having constructed closed intervals $I_{m1}, \ldots, I_{m2^m}$, each of length at most $1/2^m$, we let for each $k = 1, \ldots, 2^m$, J_{mk} denote the open middle of length $\alpha/3^{m+1}$. Then each $I_{mk} \setminus J_{mk} = I_{m+1,2k-1} \dot\cup I_{m+1,2k}$.

Let $C_{\alpha,n} = \bigcup_{k=1}^{2^n} I_{nk}$, so $C_{\alpha,n}$ is compact. Notice that $C_{\alpha,1} \supseteq C_{\alpha,2} \supseteq \cdots$, then $C_\alpha := \bigcap_{n=1}^\infty C_{\alpha,n}$ is empty and compact. If $\alpha = 1$, then $C = C_1$ is called the (middle thirds) **Cantor set**.

Remark. 1. C_α is nowhere dense. Indeed, if $x \in C_\alpha$, $\epsilon > 0$, let n be so $1/2^n < 2\epsilon$ and we see that $(x-\epsilon, x+\epsilon) \subsetneq I_{nk}$ for any $k = 1, \ldots, 2^n$. Thus $(x-\epsilon, x+\epsilon) \cap (\mathbb{R} \setminus C_\alpha) \neq \emptyset$.

2. We can compute

$$\begin{aligned} \lambda(C_\alpha) &= \lambda([0,1]) - \lambda([0,1] \setminus C_\alpha) \\ &= 1 - \lambda\left(\bigcup_{n=1}^\infty \bigcup_{k=1}^{2^n} J_{nk} \right) \\ &= 1 - \sum_{n=1}^\infty \sum_{k=1}^{2^n} \lambda(J_{nk}) \\ &= 1 - \sum_{n=1}^\infty \alpha \frac{\alpha}{3} \left(\frac{2}{3}\right)^n \\ &= 1 - \alpha \end{aligned}$$

In particular, $\lambda(C) = 0$.

Write each $I_{nk} = [a_{nk}, b_{nk}]$. Define $\phi_{\alpha,n} : \mathbb{R} \to \mathbb{R}$ by

$$\phi_{\alpha,n} = \begin{cases} 0 & : x \in (-\infty, 0) \\ \frac{2k-1}{2^{m+1}} & : x \in J_{mk} \\ \frac{1}{2^n(b_{mk} - a_{mk})}(x - a_{mk}) + c_{mk} & : x \in I_{mk} \\ 1 & : x \in (1, \infty) \end{cases}$$

Each $\phi_{\alpha,n}$ is continuous and non-decreasing on $\mathbb{R}$, and $\left\|\phi_{\alpha,n} - \phi_{\alpha,n+1}\right\| = \frac{1}{2^n}$. Thus $(\phi_{\alpha,n})_{n=1}^{\infty}$ is uniformly Cauchy, so $\phi_\alpha := \lim_{n\to\infty} \phi_{\alpha,n}$ exists and is continuous. Furthermore, (1) tells us for $x < y$, $\phi_\alpha(x) \le \phi_\alpha(y)$, so $\phi_\alpha \in \mathrm{ND}_r(\mathbb{R})$ and is, in fact, continuous. We let μ_{ϕ_α} denote the corresponding locally inite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If $\alpha = 1$, $\mu_\phi = \mu_{\phi_1}$ is called the Cantor singular measure.

Note that $\mu_{\phi_\alpha}(C_\alpha) = 1 = \mu_{\phi_\alpha}(\mathbb{R})$, so $\mu_{\phi_\alpha}(\mathbb{R} \setminus C_\alpha) = 0$. We say that μ_{ϕ_α} is **concentrated** on C_α. $\mathcal{M}_{\phi_\alpha} \supseteq \mathcal{P}(\mathbb{R} \setminus C_\alpha)$ as null sets for $\mathcal{M}_{\phi_\alpha}$.

II. Integration Theory

4 Measurable Functions

Let X, Y be sets, $T : X \to Y$. We define the **pullback** of a set $E \in \mathcal{P}(Y)$ by $T^{-1}(E) = \{x \in X : T(x) \subseteq E\}$. If $\mathcal{E} \subseteq \mathcal{P}(Y)$, we write $T^{-1}(\mathcal{E}) = \{T^{-1}(E) : E \in \mathcal{E}\}$. Note that

1. $T^{-1}(Y \setminus E) = X \setminus T^{-1}(E)$
2. $E_1, E_2, \ldots \subseteq Y$, $T^{-1}\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=1}^{\infty} T^{-1}(E_i)$.

4.1 Proposition. *(i) If $\mathcal{N}$ is a σ–algebra on Y, then $T^{-1}(\mathcal{N})$ is a σ–algebra on X (the **pullback σ–algebra**).*
*(ii) If $\mathcal{M}$ is a σ–algebra on X, then $\left\{E \in \mathcal{P}(Y) : T^{-1}(E) \in \mathcal{M}\right\}$ is a σ–algebra on Y (the **pushforward σ–algebra**).*

Proof Exercise. ■

Definition. Let $(X, \mathcal{M})$, $(Y, \mathcal{N})$ be measurable spaces, and $T : X \to Y$. We say that T is **$\mathcal{M}$–$\mathcal{N}$–measurable** provided that $T^{-1}(\mathcal{N}) \subseteq \mathcal{M}$.

4.2 Proposition. *Suppose $(X, \mathcal{M})$, $(Y, \mathcal{N})$, $T : X \to Y$ measurable, and $\mathcal{N} = \sigma\langle\mathcal{E}\rangle$. Then T is $\mathcal{M} - \mathcal{N}$–measurable if and only if $T^{-1}(E) \in \mathcal{M}$ for $E \in \mathcal{E}$.*

Proof The forward direction is obvious. Conversely, as in the previous proposition, $\mathcal{N}' = \{A \in \mathcal{P}(Y) : T^{-1}(A) \in \mathcal{M}\}$ is a σ–algebra. We have that $\mathcal{E} \subseteq \mathcal{N}'$, so $\mathcal{N} = \sigma\langle\mathcal{E}\rangle \subseteq \mathcal{N}'$. ■

cor:br-m

4.3 Corollary. *Let $(X, \mathcal{M})$ be a measurable space, $f : X \to \mathbb{R}$. Then the following are equivalent:*
(i) f is $\mathcal{M} - \mathcal{B}(\mathbb{R})$–measurable
(ii) $f^{-1}(G) \in \mathcal{M}$ for open $G \subseteq \mathbb{R}$.
(iii) $f^{-1}((a, \infty)) \in \mathcal{M}$ for a in $\mathbb{R}$
(iv) $f^{-1}([a, \infty)) \in \mathcal{M}$ for a in $\mathbb{R}$
(v) $f^{-1}((\infty, a)) \in \mathcal{M}$ for a in $\mathbb{R}$
(vi) $f^{-1}((\infty, a]) \in \mathcal{M}$ for a in $\mathbb{R}$

Proof Exercise. ■

Definition. A function $f : X \to \mathbb{R}$ satisfying any of the equivalent conditions above will be called **$\mathcal{M}$–measurable**.

Certainly continuous functions are $\mathcal{B}(\mathbb{R})$–measurable.

For notation, let $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathcal{N}$. We let

$$(\sup_{n \in \mathbb{N}} f_n)(x) = \sup_{n \in \mathbb{N}} f_n(x) \in \overline{\mathbb{R}} \tag{4.1}$$

for $x \in \mathbb{R}$. Let $a \in \mathbb{R}$, $(a, \infty] = \{x \in \overline{\mathbb{R}} : a < x\}$, and let $\mathcal{B}(\overline{\mathbb{R}}) = \sigma\langle\mathcal{G} \cup \{\{-\infty\}, \{\infty\}\}\rangle$. Given a measurable space $(X, \mathcal{M})$, $f : X \to \overline{R}$, we say f is $\mathcal{M}$–measurable if it is $\mathcal{M} - \mathcal{B}(\overline{\mathbb{R}})$–measurable. Notice that if $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, then $\sup_{n \in \mathbb{N}} f_n, \inf_{n \in \mathbb{N}} f_n : X \to \overline{R}$.

4.4 Proposition. *Let $(X,\mathcal{M})$ be a measurable space, $f_n : X \to \overline{\mathbb{R}}$ be measurable for each $n \in \mathbb{N}$. Then the following are measurable:*

(i) $\sup_{n\in\mathbb{N}} f_n$
(ii) $\inf_{n\in\mathbb{N}} f_n$
(iii) $\limsup_{n\to\infty} f_n$
(iv) $\liminf_{n\to\infty} f_n$.

Furthermore, if $\lim_{n\to\infty} f_n$ *exists, it too is measurable.*

Proof To verify measurability, it is fastest to verify any condition as in Corollary 4.3.

(i) Fix $a \in \mathbb{R}$. Then

$$\begin{aligned}\left(\sup_{n\in\mathbb{N}} f_n\right)^{-1}((a,\infty]) &= \left\{x \in X : \sup_{n\in\mathbb{N}} f_n(x) > a\right\}\\ &= \bigcup_{n=1}^{\infty}\left\{x \in X : f_n(x) > a\right\} \in \mathcal{M}\end{aligned}$$

(ii) For $a \in \mathbb{R}$, we have

$$\left(\inf_{n\in\mathbb{N}} f_n^{-1}\right)([-\infty,a)) = \bigcup_{n=1}^{\infty}\left\{x \in X : f_n(x) < a\right\} \in \mathcal{M}$$

(iii) This is immediate since

$$\limsup_{n\to\infty} f_n(x) = \inf_{n\in\mathbb{N}} \underbrace{\sup_{k\geq n} f_k(x)}_{\text{measurable}}$$

(iv) Same as above.

(v) If the limit exists, it is equal to the lim sup and the lim inf. ∎

Product σ-algebras

Definition. If $(X,\mathcal{M})$ and $(Y,\mathcal{N})$ are measurable spaces, we let the **product σ-algebra** of $\mathcal{M}$ and $\mathcal{N}$ be given by

$$\mathcal{M} \otimes \mathcal{N} = \sigma\left\langle \{E \times F : E \in \mathcal{M}, F \in \mathcal{N}\}\right\rangle \subseteq \mathcal{P}(X)$$

4.5 Lemma. *Let π_X, π_Y denote the coordinate projections. Then*

1. $\mathcal{M}\otimes\mathcal{N} = \sigma\langle \pi_X^{-1}(\mathcal{M}) \cup \pi_Y^{-1}(\mathcal{N})\rangle$
2. *If $\mathcal{M} = \sigma\langle\mathcal{E}\rangle, \mathcal{N} = \sigma\langle\mathcal{F}\rangle$, then $\mathcal{M}\otimes\mathcal{N} = \sigma\langle\{E \times F : E \in \mathcal{E}, F \in \mathcal{F}\}\rangle$.*

Proof 1. $E \times F = (E \times F) \cap (X \times F) = \pi_X^{-1}(E) \cap \pi_Y^{-1}(F)$. We see that

$$\{E \times F : E \in \mathcal{M}, F \in \mathcal{N}\} \subseteq \sigma\langle \pi_X^{-1}(\mathcal{M}) \cup \pi_Y^{-1}(\mathcal{N})\rangle$$

and

$$\pi_X^{-1}(\mathcal{M}) \cup \pi_Y^{-1}(\mathcal{N}) \subseteq \sigma\langle\{E \times F : E \in \mathcal{M}, F \in \mathcal{N}\}\rangle$$

2.

$$\begin{aligned}\mathcal{M}\otimes\mathcal{N} &= \sigma\langle\pi_X^{-1}(\mathcal{M})\cup\pi_Y^{-1}(\mathcal{N})\rangle\\ &= \sigma\langle\pi_X^{-1}(\mathcal{E})\cup\pi_Y^{-1}(\mathcal{F})\rangle\end{aligned}$$

since $\sigma\langle\pi_X^{-1}(E)\rangle = \pi_X^{-1}(\mathcal{M})$. ■

Let (X,d) be a metric space, $\mathcal{G}(X)$ denote the open sets in X, and $\mathcal{B}$ the Borel σ−algebra. If ρ is an equivalent metric to d, then these metric generate the same open sets (and thus the same σ−algebra).

4.6 Proposition. *Let $(X,d_X),(Y,d_Y)$ be separable metric spaces, and let ρ be any metric on $X\times Y$ such that $\rho\sim\rho_\infty$ (where $\rho_\infty((x,y),(x',y')) = \max\{d_X(x,x'),d_Y(y,y')\}$. Then $\mathcal{B}(X\times Y,\rho) = \mathcal{B}(X,d_X)\otimes\mathcal{B}(Y,d_Y)$.*

PROOF For $r>0$, $(x,y)\in X\times Y$, we have radius r open balls. Since X,Y are separable, write G as a countable union of products of open balls in X and Y. Thus $\mathcal{G}(X\times Y)\subseteq\mathcal{B}(X)\otimes\mathcal{B}(Y)$, so $\mathcal{B}(X\times Y) = \mathcal{B}(X)\times\mathcal{B}(Y)$. Conversely,

$$\begin{aligned}\mathcal{B}(X)\times\mathcal{B}(Y) &= \sigma\langle\{G\times H : G\subseteq X \text{ open}, H\subseteq Y \text{ open}\}\rangle\\ &\subseteq\sigma\langle\mathcal{G}(X\times Y)\rangle\subseteq\mathcal{B}(X\times Y)\end{aligned}$$ ■

Even without the separability assumption, f always holds. However, the converse inclusion is in doubt. (take $(\mathbb{R},d)$ where d is the discrete metric).

Also note, by induction, $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R})\otimes\cdots\otimes\mathcal{B}(\mathbb{R})$.

4.7 Proposition. *If $(X,\mathcal{M})$, $(Y,\mathcal{N})$ and $(Z,\mathcal{O})$ are measurable spaces, $S:X\to Y$ and $T:Y\to Z$ are measurable, then $T\circ S : X\to Z$ is measurable.*

PROOF If $E\in\mathcal{O}$, then $(T\circ S)^{-1}(E) = S^{-1}(T^{-1}(E))\in\mathcal{M}$. ■

4.8 Proposition. *If $(X,\mathcal{M})$ is a measurable space, and $T:X\to\mathbb{R}^d$, then T is $\mathcal{M}-\mathcal{B}(\mathbb{R})$−measurable if and only if each $\pi_k\circ T : X\to\mathbb{R}$ is $\mathcal{M}$−measurable.*

PROOF If $B\in\mathcal{B}(\mathbb{R})$, then $(\pi_k\circ T)^{-1}(B) = T^{-1}(\pi_k^{-1}(B))$. Let's refer to this by $(*)$.

$(\Rightarrow)$ We have that $\pi_k:\mathbb{R}^d\to\mathbb{R}$ is continuous, so $\pi_k^{-1}(G)$ is open for open G in $\mathbb{R}$, and hence $\pi^{-1}(B)\in\mathcal{B}(\mathbb{R}^d)$ for B above. Hence $T^{-1}(\pi_k^{-1}(B))\in\mathcal{M}$ by $(*)$

$(\Leftarrow)$ We have $(\pi_k\circ T)^{-1}(B)\in\mathcal{M}$ for B above. We have that $\mathcal{B}(\mathbb{R}^d) = \sigma\langle\pi_1^{-1}(\mathcal{B}(\mathbb{R}))\cup\cdots\cup\pi_n^{-1}(\mathcal{B}(\mathbb{R}))\rangle$. Then by $(*)$, we se that T is $\mathcal{M}-\mathcal{B}(\mathbb{R}^d)$−measurable. ■

4.9 Corollary. *$\mathbb{C}\cong\mathbb{R}^2$ and if $(X,\mathcal{M})$ is a measurable space, $T:X\to\mathbb{C}$, then T is $\mathcal{M}-\mathcal{B}(\mathbb{C})$−measurable if and only if $\mathrm{Re}(T)$, $\mathrm{Im}(T) : X\to\mathbb{R}$ is $\mathcal{M}$−measurable.*

Definition. We call an $\mathcal{M}-\mathcal{B}(\mathbb{C})$−measurable function an $\mathcal{M}$−measurable function.

4.10 Corollary. *Arithmetic property of measurable functions. Let $(X,\mathcal{M})$ be a measurable space; $f,g : X\to\mathbb{C}$ each be measurable. Then $f+g, fg : X\to\mathbb{C}$ are each $\mathcal{M}$−measurable.*

PROOF Consider $\alpha:\mathbb{C}^2\to\mathbb{C}$, $m:\mathbb{C}^2\to\mathbb{C}$ given by $\alpha(z,w) = z+2$, $m(z,w) = zw$ are continuous functions and thus $\mathcal{B}(\mathbb{C}^2)-\mathcal{B}(\mathbb{C})$−measurable. We define $F:X\to\mathbb{C}^2$ by $F(x) = (f(x),g(x))$. By a modification of the last proposition, $\mathbb{C}^2$ playing the role of $\mathbb{R}^d$, we see that F is $\mathcal{M}-\mathcal{B}(\mathbb{C})$−measurable. Then $f+g = \alpha\circ F$, $fg = m\circ F$. ■

5 Integration

Definition. If $(X, \mathcal{M})$ is a measurable space, let $\mathcal{S}^+(X, \mathcal{M}) = \{\phi : X \to [0, \infty) : |\phi(x)| < \infty, \phi \text{ is measurable}]\}$.

5.1 Lemma. *(i) If $E \in \mathcal{P}(X)$, then $1_E \in \mathcal{S}^+(X, \mathcal{M})$ if and only if $E \in \mathcal{M}$.*
(ii) If $\phi : X \to [0, \infty)$ then $\phi \in \mathcal{S}^+(X, \mathcal{M})$ if and only if there are $0 \le a_1 < a_2 < \cdots < a_n$, $E_1, \ldots, E_n \in \mathcal{M}$ pairwise disjoint, so that $\phi = \sum_{i=1}^n a_i 1_{E_i}$.

Proof (i) Clearly $1_E(X) \subseteq [0, \infty)$. If $B \in \mathcal{B}(\mathbb{R})$, then

$$1_E^{-1}(B) = \begin{cases} \emptyset & : \{0,1\} \cap B = \emptyset \\ E & : \{0,1\} \cap B = \{1\} \\ X \setminus E & : \{0,1\} \cap B = \{0\} \\ X & : \{0,1\} \subseteq B \end{cases}$$

(ii) $(\Leftarrow)$. Use (i) and arithmetic of measurable functions.
$(\Rightarrow)$ Let $\{a_1, \ldots, a_n\} = \phi(X)$. Then let $E_i = \phi^{-1}(\{a_i\})$. ∎

Definition. If $(X, \mathcal{M}, \mu)$ ie a measure space, define $I_\mu : \mathcal{S}^+(X, \mathcal{M}) \to [0, \infty]$ by $I_\mu(\phi) = \sum_{i=1}^n a_i \mu(E_i)$ where ϕ is in standard form. Here, we say $a \cdot \infty = \infty$ if $a \neq 0$, and $0 \cdot \infty = 0$.

5.2 Proposition. *Let $\phi, \psi \in \mathcal{S}^+(X, \mathcal{M})$. Then*
(i) If $\phi \le \psi$ (pointwise), then $I_\mu(\phi) \le I_\mu(\psi)$.
(ii) If $c \in [0, \infty)$, then $I_\mu(\phi + c\psi) = I_\mu(\phi) + cI_\mu(\psi)$.

Proof Write $\phi = \sum_{i=1}^n a_i 1_{E_i}$, $\psi = \sum_{i=1}^n b_i 1_{F_i}$ in standard forms.
(i)

$$\begin{aligned} I_\mu(\phi) &= \sum_{i=1}^n a_i \mu(E_i) \\ &= \sum_{i=1}^n a_i \sum_{j=1}^m \mu(E_i \cap_i) \\ &= \sum_{j=1}^m \sum_{i=1}^n a_i \mu(E_i \cap F_i) \\ &\le \sum_{j=1}^m \sum_{i=1}^n b_i \mu(E_i \cap F_i) \\ &= \sum_{j=1}^m b_j \mu(F_j) = I_\mu(\psi) \end{aligned}$$

(ii) Notice that $1_E 1_F = 1_{E \cap F}$. We have

$$\begin{aligned} \phi + c\psi &= \sum_{j=1}^m 1_{F_j} \sum_{i=1}^n a_i 1_{E_i} + \sum_{i=1}^n 1_{E_i} \sum_{j=1}^m c b_j 1_{F_j} \\ &= \sum_{i=1}^n \sum_{j=1}^m (a_i + c b_j) 1_{E_i \cap F_j} \end{aligned}$$

Let $\{c_1,\dots,c_p\} = \{a_i + cb_j : i = 1,\dots,n; j = 1,\dots,m\}$ (distinct enumeration) and for $k = 1,\dots,\mu$, and $G_k = \bigcup E_i \cap F_j$ (union over appropriate indices) so $\phi + c\psi = \sum_{k=1}^{p} c_k 1_{G_k}$. Then

$$\begin{aligned} I_\mu(\phi + c\psi) &= \sum_{k=1}^{p} c_k \mu(G_k) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + cb_j)\mu(E_i \cap F_j) \\ &= \sum_{i=1}^{n} a_i \mu(E_i) + c \sum_{j=1}^{m} b_j \mu(F_j) \\ &= I_\mu(\phi) + cI_\mu(\psi) \end{aligned}$$

■

5.3 Theorem. (Monotone Convergence) *Let $f_n : X \to [0,+\infty]$ be measurable, such that*
(i) $0 \le f_1 \le f_2 \le \cdots$
(ii) $f(x) := \lim_{n\to\infty} f_n(x)$ *for all* $x \in X$
Then f is measurable, and $\int_X f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu$.

Proof That f exists and is measurable is clear since $f = \sup_{n\in\mathbb{N}} f_n$. We have $\int_X f_n \,\mathrm{d}\mu \le \int_X f_{n+1} \,\mathrm{d}\mu$ for all n, so $\alpha := \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu$ exists. We also have $f_n \le f$, so $\int_X f_n \,\mathrm{d}\mu \le \int_X f \,\mathrm{d}\mu$ and $\alpha \le \int_X f_n \,\mathrm{d}\mu$. Thus we wish to show $\alpha \ge \int_X f \,\mathrm{d}\mu$. It suffices to prove that $\alpha \ge \int_X s \,\mathrm{d}\mu$ for any simple $s \le f$. Furthermore, if $c \in (0,1)$, it suffices to show that $\alpha \ge \int_X c \cdot s \,\mathrm{d}\mu$.

Define $E_n = \{x \in X : f_n(x) \ge c \cdot s(x)\}$. We have $E_1 \subset E_2 \subset \cdots$ so that $\bigcup_{n=1}^{\infty} E_n = X$. Then

$$\int_X f_n \,\mathrm{d}\mu \ge \int_{E_n} f_n \,\mathrm{d}\mu \ge \int_{E_n} c \cdot s \,\mathrm{d}\mu$$

Let $\phi(E) = \int_E s \,\mathrm{d}\mu$, so $\int_{E_n} s \,\mathrm{d}\mu = \phi(E_n)$. Thus $\lim_{n\to\infty} \phi(E_n) = \phi(X) = \int_X s \,\mathrm{d}\mu$. Thus

$$\alpha \ge c \cdot \lim_{n\to\infty} \phi(E_n) = c \cdot \int_X s \,\mathrm{d}\mu = \int_X c \cdot s \,\mathrm{d}\mu$$

as desired. ■

5.4 Theorem. (Fatou) *Let $(f_n)_{n=1}^{\infty} \subseteq \overline{M}^{+}(X, \mathcal{M})$. Then*

$$\int_X \left(\liminf_{n\to\infty} f_n \right) \mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu$$

Proof For each n, we know that $\int_X \inf_{k\ge n} f_k \,\mathrm{d}\mu \le \int_X f_n \,\mathrm{d}\mu$, so by monotone convergence, we have

$$\int_X \liminf_{n\to\infty} f_n \,\mathrm{d}\mu = \int_X \lim_{n\to\infty} \left(\inf_{k\ge n} f_k \right) \mathrm{d}\mu = \lim_{n\to\infty} \int_X \inf_{k\ge n} f_k \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu$$

■

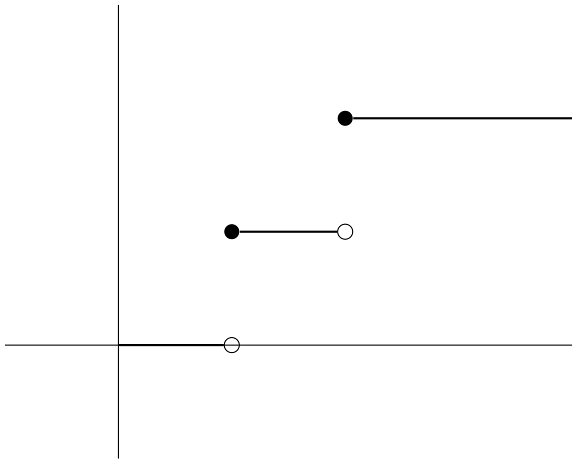
5.5 Theorem. *Let $f : X \to [0,+\infty]$ be nonnegative measurable functions. Then there exists a sequence $s_n : X \to [0,+\infty]$ of simple measurable functions with*

1. *(s_n) is increasing and bounded above by f*
2. *$\lim s_n = f$ pointwise.*

Proof Let $n \in \mathbb{N}$, $t \geq 0$, and define $k_n(t) = [2^n \cdot t]$ (i.e. $k_n(t) \leq 2^n \cdot t < k_n(t) + 1$). Then define

$$\phi_n(t) = \begin{cases} k_n(t) \cdot 2^{-n} & \text{if } t \leq n \\ n & \text{if } t > n \end{cases}$$

I've drawn ϕ_1 below:



Then $t - 2^{-n} \leq \phi_n(t) \leq t$, $\lim \phi_n(t) = t$ uniformly, and $\phi_n \leq \phi_{n+1}$, so the sequence of functions is monotone. Define $s_n = \phi_n \circ f$, so for any $x \in X$, $\lim s_n(x) = \lim \phi_n \circ f(x) = f(x)$. Note that s_n is simple since it has finite range (from ϕ_n), and $s_n \leq s_{n+1}$ because $\phi_n \leq \phi_{n+1}$, and $s_n \leq f$ since $\phi_n(t) \leq t$. Furthermore, ϕ_n is measurable since its level sets are intervals, so $s_n = \phi_n \circ f$ is measurable. ■

5.6 Corollary. 1. *If $f, g \in \overline{M}^+(X, \mathcal{M})$, $c \geq 0$, then $f + cg \in \overline{M}^+(X, \mathcal{M})$ and $\int_X (f + cg)\, \mathrm{d}\mu = \int_X f \,\mathrm{d}\mu + c \int_X g \,\mathrm{d}\mu$.*

2. *If $(f_k)_{k=1}^\infty \subset \overline{M}^+(X, \mathcal{M})$, then $\sum_{k=1}^\infty f_k \in \overline{M}^+(X, \mathcal{M})$ and $\int_X \left(\sum_{k=1}^\infty f_k\right) \mathrm{d}\mu = \sum_{k=1}^\infty \int_X f_k \,\mathrm{d}\mu$.*

3. *If $f \in \overline{M}^+(X, \mathcal{M})$, then $\mu_f : \mathcal{M} \to [0, \infty]$, $\mu_f(E) = \int_X (1_E f)\,\mathrm{d}\mu$ defines a measure.*

Proof 1. Let $(\phi_n)_{n=1}^\infty \subset S_f^+$, so $\phi_1 \leq \phi_2 \leq \cdots$, $\lim_{n\to\infty} \phi_n$ and $(\psi_n)_{n=1}^\infty \subset S_g^+$. Then $(\phi_n + c\psi_n)_{n=1}^\infty \subset S_{f+cg}^+$ with $\phi_1 + c\psi_1 \leq \phi_2 + c\psi_2 \leq \cdots$ and $\lim(\phi_n + c\psi_n) = f + cg$. Thus $f + cg \in \overline{M}^+(X, \mathcal{M})$. Furthermore, MCT provides

$$\begin{aligned} \int_X (f + cg)\,\mathrm{d}\mu &= \lim_{n\to\infty} \int_X (\phi_n + c\psi_n)\,\mathrm{d}\mu \\ &= \lim_{n\to\infty} \left(\int_X \phi_n \,\mathrm{d}\mu + c \int_X \psi_n \,\mathrm{d}\mu \right) \\ &= \lim_{n\to\infty} \int_X \phi_n \,\mathrm{d}\mu + c \lim_{n\to\infty} \int_X \psi_n \,\mathrm{d}\mu \\ &= \int_X \mathrm{d}\mu + c \int_X g \,\mathrm{d}\mu \end{aligned}$$

2. Let $g_n = \sum_{k=1}^n f_k$. Then $g_1 \le g_2 \le \cdots$ with $\sum_{k=1}^\infty f_k = \lim_{n\to\infty} g_n$. We apply (1), and by MCT, we have

$$\begin{aligned}
\int_X \sum_{k=1}^\infty f_k \,\mathrm{d}\mu &= \int_X \lim_{n\to\infty} g_n \,\mathrm{d}\mu \\
&= \lim_{n\to\infty} \int_X g_n \,\mathrm{d}\mu \\
&= \lim_{n\to\infty} \sum_{k=1}^n \int_X f_k \,\mathrm{d}\mu \\
&= \sum_{k=1}^\infty \int_X f_k \,\mathrm{d}\mu
\end{aligned}$$

3. Notice that $1_\emptyset = 0$, so $\mu_f(\emptyset) = 0$. If $E_1, E_2, \ldots \in \mathcal{M}$ are disjoint, then apply (ii) to get $f_k = 1_{E_k}$, noting that $\sum_{k=1}^\infty 1_{E_k} = 1_{\bigcup_{k=1}^\infty E_k}$ to see σ−additivity. ■

Integration of Complex Valued Functions

Let $(X, \mathcal{M}, \mu)$ be a measure space. We let

$$\begin{aligned}
M(X,\mathcal{M}) &= \{f : X \to \mathbb{C} : f \text{ is } \mathcal{M}\text{−measurable}\} \\
M^{\mathbb{R}}(X,\mathcal{M}) &= \{f : X \to \mathbb{R} : f \text{ is } \mathcal{M}\text{−measurable}\} \\
M^{+}(X,\mathcal{M}) &= \{f : X \to [0,\infty) : f \text{ is } \mathcal{M}\text{−measurable}\}
\end{aligned}$$

Remark. 1. If $f \in M^{\mathbb{R}}(X,\mathcal{M})$, then $f^+ := \max\{f, 0\}$, $f^- := \max\{-f, 0\}$ are both in M^+. Thus, we have $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

2. If $f \in M(X,\mathcal{M})$, then $|\cdot| : \mathbb{C} \to [0,\infty)$ is continuous and thus Borel measurable.

Definition. We let $L(X,\mathcal{M},\mu) = L(\mu) := \{f \in M(X,\mathcal{M}) : \int_X |f| \,\mathrm{d}\mu < \infty\}$ denote the μ−**Lebesgue integrable** functions. Notice that $\operatorname{Re} f^+, \operatorname{Re} f^-, \operatorname{Im} f^+, \operatorname{Im} f^- \le |f| \le \operatorname{Re} f^+ + \cdots + \operatorname{Im} f^-$, so we have $f \in L(\mu) \Leftrightarrow \operatorname{Re} f^+, \cdots, \operatorname{Im} f^- \in L(\mu)$. We may therefore define for $f \in L(\mu)$ the **Lebesgue integral** with respect to μ

$$\int_X f \,\mathrm{d}\mu = \int_X \operatorname{Re} f^+ \,\mathrm{d}\mu - \int_X \operatorname{Re} f^- \,\mathrm{d}\mu + i\left(\int_X \operatorname{Im} f^+ \,\mathrm{d}\mu - \int_X \operatorname{Im} f^- \,\mathrm{d}\mu\right)$$

5.7 Proposition. *If $f, g \in L(X,\mathcal{M},\mu)$ and $c \in \mathbb{C}$, then $f + g, cf \in L(\mu)$ with $\int_X (f+g)\,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu + \int_X g \,\mathrm{d}\mu$, $\int_X (cf)\,\mathrm{d}\mu = c\int_X f \,\mathrm{d}\mu$.*

Proof Assume $f, g \in L^{\mathbb{R}}(\mu)$ and $c \in \mathbb{R}$. Then

$$(f+g)^+ - (f+g)^- = f + g = f^+ - f^- + g^+ - g^- \Rightarrow (f+g)^+ + f^- + g^- = f^+ + g^+ + (f+g)^-$$

We then integrate, applying the last corollary, and rearrange. Similarly, $c = cf^+ - cf^-$ if $c \ge 0$, and $|c|f^- - |c|f^+$ if $c < 0$. Then, for example, if $c < 0$, we have $\int_X |c| f^{\pm} \,\mathrm{d}\mu = |c|\int_X f^{\pm} \,\mathrm{d}\mu < \infty$ and $\int_X |c|f^- \,\mathrm{d}\mu - \int_X |c| f^+ \,\mathrm{d}\mu = |c|\int_X f^- \,\mathrm{d}\mu - |c|\int_X f^+ \,\mathrm{d}\mu = c\int_X f \,\mathrm{d}\mu$.

Finally, use $\mathbb{C}$−arithmetic on Re, Im parts. ■

Definition. If $f, g \in M(X, \mathcal{M})$, we say that $f = g$ μ−**almost everywhere** if $\mu(\{x \in X : f(x) \neq g(x)\}) = 0$.

Notice that

$$\{x \in X : f(x) \neq g(x)\} = \begin{cases} (f-g)^{-1}(\mathbb{C}\setminus\{0\}) \\ (f-g)^{-1}((0,\infty)) \cup [f^{-1}(\{\infty\}) \cap g^{-1}([0,\infty))] \cup [f^{-1}([0,\infty)) \cap g^{-1} \cap g^{-1}(\{\infty\}) \end{cases}$$

If $f = g$ μ−a.e., and $g =$ μ−a.e., then $f = h$ μ−a.e. If $(f_n)_{n=1}^{\infty} \subset M(X, \mathcal{M})$, we write $\lim_{n\to\infty} f_n = f$ μ-a.e. if $\mu(\{x \in X : \lim_{n\to\infty} : \lim_{n\to\infty} f_n(x) \neq f(x)\}) = 0$. Notice that

$$\begin{aligned} E &= \{x \in X : \lim_{n\to\infty} f_n(x) \text{ does not exist}\} \\ &= \{x \in X : \liminf_{n\to\infty} \operatorname{Re} f_n \neq \limsup_{n\to\infty} \operatorname{Re} f_n\} \cup \{x \in X : \liminf_{n\to\infty} \operatorname{Im} f_n \neq \limsup_{n\to\infty im f_n}\} \end{aligned}$$

Likewise, $\{x \in X : \lim_{n\to\infty} f(x)$ exists, but is not $f(x)\} \in \mathcal{M}$.

5.8 Lemma. *Let $f \in \overline{M}^+(X, \mathcal{M})$. Then*

1. $\int_X f \,\mathrm{d}\mu < \infty \Rightarrow \mu(f^{-1}(\{\infty\})) = 0$, *i.e.* $f < \infty$ μ*−a.e.*
2. $\int_X f \,\mathrm{d}\mu \Leftrightarrow \mu(f^{-1}((0,\infty]) = 0$, *i.e.* $f = 0$ μ*−a.e.*

Proof 1. For each $N \in \mathbb{N}$, $n1_{f^{-1}(\{\infty\})} \in S_f^+$, so $0 \leq n\mu(f^{-1}(\{\infty\})) \leq \int_X f \,\mathrm{d}\mu < \infty$, so that $\mu(f^{-1}(\{\infty\})) = 0$.

2. $\frac{1}{n}1_{f^{-1}([1/n,\infty])} \in S_f^+$ so

$$0 \leq \frac{1}{n}\mu(f^{-1}([1/n,\infty])) = \int_X \frac{1}{n}1_{f^{-1}([1/n,\infty])} \leq \int_X f \,\mathrm{d}\mu = 0$$

so $\mu(f^{-1}([1/n,\infty])) = 0$. Now,

$$f^{-1}((0,\infty])) = \bigcup_{n=1}^{\infty} f^{-1}([1/n,\infty])$$

so the result holds by σ−subadditivity.
Conversely, let $\phi = \sum_{i=1}^{n} a_i 1_{E_i} \in S_f^+$ in standard form, and $a_i > 0$, then $E_i = f^{-1}(\{a_i\}) \subseteq^{-1} ((0,\infty])$, so $\mu(E_i) = 0$. Thus $\int_X \phi \,\mathrm{d}\mu = 0$ so $\int_X f \,\mathrm{d}\mu = 0$. ■

5.9 Corollary. 1. *If $f \in \overline{M}^+(X, \mathcal{M})$, then $\int_X f \,\mathrm{d}\mu < \infty$ if and only if there is $f_0 \in M^+(X, \mathcal{M})$ so that $f = f_0$ μ−a.e.*

2. *If $f, g \in L(X, \mathcal{M}, \mu)$, then $f = g$ μ−a.e. if and only if $\int_X |f-g| \,\mathrm{d}\mu = 0$.*

Proof Clear from above. ■

5.10 Theorem. (Lebesgue Dominated Convergence) *Let $(f_n) \subseteq L(X, \mathcal{M}, \mu)$, and $f \in M(X, \mathcal{M})$ such that*

- $\lim_{n\to\infty} f_n = f$ *μ−a.e.*
- *There is $g \in L^+(\mu)$ such that $|f_n| \leq g$ μ-a.e. Then $f \in L(\mu)$ and $\lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu$. If, further, $(X, \mathcal{M}, \mu)$ is complete, we may take $f : X \to \mathbb{C}$.*

Proof Let $N = \bigcup_{n=1}^{\infty}(|f_n| - g)^{-1}((0,\infty)) \cup \{x \in X : \lim f_n(x) \neq f(x)\}$, so $\mu(N) = 0$. Replace f_n by $1_N f_n$ and f by $1_N f$, and assume all limits and inequalities are pointwise. Notice if $(X, \mathcal{M}, \mu)$ is complete, then we do not need the assumption that f is measurable to see that $N \in \mathcal{M}$. We thus have that $f \in M(X, \mathcal{M})$ with $|f| = \lim |f_n| \leq |g|$, so $\int_X f \, \mathrm{d}\mu < \infty$.

(I) Assume that each f_n, hence f, is $\mathbb{R}$−valued. Then $(g + f_n)_{n=1}^{\infty}, (g - f_n)_{n=1}^{\infty} \subset M^+(X, \mathcal{M})$. Hence, we may use Fatou's Lemma:

$$\begin{aligned}
\int_X g \, \mathrm{d}\mu \pm \int_X f \, \mathrm{d}\mu = \int_X (g \pm f) \, \mathrm{d}\mu &= \int_X \liminf_{n\to\infty} (g \pm f_n) \, \mathrm{d}\mu \\
&\leq \liminf_{n\to\infty} \int_X (g \pm f_n) \, \mathrm{d}\mu = \liminf_{n\to\infty} \left(\int_X g \, \mathrm{d}\mu \pm \int_X f_n \, \mathrm{d}\mu \right) \\
&= \begin{cases} \int_X g \, \mathrm{d}\mu + \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu & \pm = + \\ \int_X g \, \mathrm{d}\mu - \limsup_{n\to\infty} \int_X f_n \, \mathrm{d}\mu & \pm = - \end{cases}
\end{aligned}$$

Then

- $\pm = +$ provides $\int_X g \, \mathrm{d}\mu + \int_X f \, \mathrm{d}\mu \leq \int_X G \, \mathrm{d}\mu + \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu$. Thus $\int_X f \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu$
- $\pm = -$ implies $\int_X f \, \mathrm{d}\mu \geq \limsup_{n\to\infty} \int_X f_n$.

Thus $\limsup_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \leq \int_X f \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu$, so $\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu$ exists and equals $\int_X f \, \mathrm{d}\mu$.

(II) Here we use (I) to see that $\lim_{n\to\infty} \operatorname{Re} f_n = \operatorname{Re} f$, so $\lim_{n\to\infty} \int_X \operatorname{Re} f_n \, \mathrm{d}\mu = \int_X \operatorname{Re} f \, \mathrm{d}\mu$, and likewis with imaginary parts. Thus

$$\begin{aligned}
\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu &= \lim_{n\to\infty} \int_X \operatorname{Re} f_n \, \mathrm{d}\mu + i \lim_{n\to\infty} \int_X \operatorname{Im} f_n \, \mathrm{d}\mu \\
&= \int_X \operatorname{Re} f \, \mathrm{d}\mu + i \int_X \operatorname{Im} f \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu
\end{aligned}$$

■

Note that MCT and Fatou's lemma also work with assumptions of μ-a.e. convergence. Let $S(X, \mathcal{M}) = \{\phi : X \to \mathbb{C} : \phi \text{is } \mathcal{M}\text{-measurable}, |\phi(X)| < \infty\}$.

5.11 Corollary.

1. *If $(f_n) \subseteq L(\mu)$, $f \in M(X, \mathcal{M})$ with $f = \lim f_n$ μ-a.e. and there is $g \in L^+(\mu)$ with $|f_n| \leq g$ μ–a.e., then $\lim_{n\to\infty} \int_X |f - f_n| \, \mathrm{d}\mu = 0$.*
2. *Given $f \in L(\mu)$, there exists a sequence $(\phi_n) \subseteq S(X, \mathcal{M})$ such that $|\phi_n| \leq |f|$ and $\lim_{n\to\infty} \phi_n = f$. Furthermore, we have that $\int_X f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X \phi_n \, \mathrm{d}\mu$.*
3. *If $f \in L(\mu)$, then $|\int_X f \, \mathrm{d}\mu| \leq \int_X |f| \, \mathrm{d}\mu$.*

Proof

1. We have $\lim_{n\to\infty} |f - f_n| = 0$ μ–a.e., and $|f - f_n| \leq |f| + |f_n| \leq 2g \in L^+(\mu)$. Apply LDCT.

2. An earlier lemma gives us sequences $(\phi_n^{\pm})_{n=1}^{\infty}, (\psi_n^{\pm})_{n=1}^{\infty}$ so that $0 \leq \phi_1^+ \leq \phi_2^+ \leq \cdots$ with $\lim \phi_n^+ = \operatorname{Re} f^+$, $0 \leq \psi_1^- \leq \psi_2^- \leq \cdots$ with $\lim \psi_n^- = \operatorname{Im} f^-$. Let $\phi_n = \phi_n^+ - \phi_n^- + i[\psi_n^+ - \psi_n^-]$. Then

$$\begin{aligned}
|\phi_n| &= [|\phi_n^+ - \phi_n^-|^2 + |\psi_n^+ - \psi_n^-|^2]^{1/2} \\
&\leq [(\phi_n^+ + \phi_n^-)^2 + (\psi_n^+ + \psi_n^-)^2]^{1/2} \qquad \leq \Big[(\operatorname{Re} f^+ + \operatorname{Re} f^-)^2 + (\operatorname{Im} f^+ + \operatorname{Im} f^-)\Big]^{1/2} \\
&= |f|
\end{aligned}$$

and, also, $\lim \phi_n = f$. We have that since $|\phi_n| \leq |f|$, we use LDCT to get a limit of integrals.

3. If $\phi \in S^-(X,M) \cap L(\mu)$, write $\phi = \sum_{i=1}^n c_i 1_{E_i}$. Then

$$|\int_X \phi \, d\mu| = |\sum_{i=1}^n c_i \mu(E_i)| \leq \sum_{i=1}^n |c_i| \mu(E_i) = \int_X |\phi| d\mu$$

Now, if $f \in L(\mu)$, we obtain sequences $(\phi_n)_{n=1}^\infty \subset S(X,\mathcal{M})$. Thus we have

$$|\int_X f \, d\mu| = \lim |\int_X \phi_n \, d\mu| \leq \lim \int_X |\phi_n| d\mu = \int_X |f| d\mu$$

as $|\phi_n| \leq |f|$, $\lim |\phi_n| = |f|$. ■

5.12 Lemma. *Let $(X,\mathcal{A},\mu_0)$ be a premeasure space, and $(X,\mathcal{M},\mu)$ denote the canonical induced measure space. Given $f \in L(\mu)$, $\epsilon > 0$, there is*

$$\phi = \sum_{i=1}^n a_i 1_{B_i}, a_1, \ldots, a_n \in \mathbb{C}, B_1, \ldots, B_n \in \mathcal{A}$$

such that $\int_X |\phi - f| d\mu < \epsilon$.

Proof (I) Let $E \in \mathcal{M}$, with $\mu(E) < \infty$. Then given $\epsilon > 0$, there is $B \in \mathcal{A}$ so that $\mu(B \triangle E) < \epsilon$. To see this, let $A_1, A_2, \ldots \in \mathcal{A}$ so that $E \subseteq \bigcup_{i=1}^\infty A_i$ with $\sum_{i=1}^\infty \mu_0(A_i) < \mu^*(E) + \epsilon = \mu(E) + \epsilon$. Let n be so that $\sum_{i=n+1}^\infty \mu_0(A_i) < \epsilon/2$, and let $B = \bigcup_{i=1}^n A_i \in \mathcal{A}$. Then $B \triangle E \subseteq (\bigcup_{i=1}^\infty A_i \setminus E) \cup (\bigcup_{i=n+1}^\infty A_i)$ and the result follows by σ–subaddivitity.

(II) If $\psi \in S(X,\mathcal{M}) \cap L(\mu)$. Then given $\epsilon > 0$, there is ϕ as above so $\int |\psi - \phi| < \epsilon$. To see this, write $\psi = \sum_{i=1}^n a_i 1_{E_i}$. By (I), we find for each i, B_i in $\mathcal{A}$ such that $\mu(B_i \triangle E_i) < \epsilon/a$, where $a = 1 + \sum_{i=1}^n |a_i|$. Then

$$\int |\phi - \psi| \leq \sum_{i=1}^n |a_i| \int |1_{B_i} - 1_{E_i}| = \sum_{i=1}^n \mu(B_i \triangle E_i) < \epsilon$$

(III) If $f \in L(\mu)$, a corollary to LDCT provides ψ in $S(X,\mathcal{M}) \cap L(\mu)$ such that $\int |f - \psi| < \epsilon.2$. We let ϕ as in (II), so $\int |\psi - \phi| < \epsilon/2$. ■

5.13 Proposition. *Let $(X,\mathcal{M},\mu)$ be a measure space, $f : X \times (a,b) \to \mathbb{C}$ satisfy that*

- *$f(\cdot, s) \in L(\mu)$ for each $s \in (a,b)$*
- *$\frac{\partial}{\partial s} f(x,s) = \lim_{h \to 0} \frac{f(x,s+h) - f(x,s)}{h}$ exists for each (x,s) in $X \times (a,b)$*
- *there is $g \in L^+(\mu)$ so that $\left|\frac{\partial}{\partial s} f(\cdot, s)\right| \leq g$ μ–a.e for each $s \in (a,b)$.*

Then $F(x) = \int_X f(x,s) d\mu(x)$, and F is differentiable on (a,b) with $F'(s) = \int_X \frac{\partial}{\partial s} f(x,s) d\mu(x)$.

Proof We fix $s \in (a,b)$ and an arbitrary sequence $(h_n)_{n=1}^\infty \subset \mathbb{R} \setminus \{0\}$ such that $s + h_n \in (a,b)$ for each n, and $\lim h_n = 0$. Notive that for each $x \in X$, $f(x,\cdot) : (a,b) \to \mathbb{C}$ is continuous on intervals $[s, s+h_n]$, $[s+h_n, s]$ (if $h_n < 0$) for $n \in \mathbb{N}$. Thus, by MVT, we find $c_n, d_n \in (s, s+h_n)$ such that

$$\begin{aligned} |f(x,s+h_n) - f(x,s)| &= \left| \operatorname{Re} \frac{\partial}{\partial s} f(x,c_n) + i \operatorname{im} \frac{\partial}{\partial s} f(x, d+n) \right| |h_n| \\ &\leq 2|g(x)||h_n| \end{aligned}$$

Thus, by LDCT,

$$F'(s) = \lim_{n\to\infty} \frac{F(s+h_n)-F(s)}{h_n} = \lim_{n\to\infty} \int \left(\frac{f(x,s+h_n)-f(x,s)}{h_n} \, \mathrm{d}\mu(x) \right)$$
$$= \int \frac{\partial}{\partial s} f(x,s) \, \mathrm{d}\mu(x) \qquad \blacksquare$$

6 Modes of Convergence

Let $(X, \mathcal{M}, \mu)$ be a measure space, $(f_n), f \in M(X, \mathcal{M})$. We say that $\lim f_n = f$

- **uniformly** if $\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0$
- **pointwise** if $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$ for each $x \in X$
- **pointwise μ-a.e.** if $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$ for each $x \in X \setminus N$, where $\mu(N) = 0$.
- **in $L^1(\mu)$** if $\lim_{n\to\infty} \int_X |f_n - f| \mathrm{d}\mu = 0$.
- **in μ-measure** if for any $\epsilon > 0$ we have $\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) = 0$.

Example. Consider sequences $f_n = \frac{1}{n} 1_{[0,n]}$, $g_n = 1_{[n,n+1]}$, $h_n = n 1_{[0,1/n]}$, $k_n = 1_{[j/2^k,(j+1)/2^k]}$ where $n = 2^k + j$ for $j = 0, \ldots, 2^k - 1$. Then

	uniform	pointwise	pointwise λ–a.e.	in $L^1(\lambda)$	in λ–measure
f_n	✓	✓	✓	×	✓
g_n	×	✓	✓	×	×
h_n	×	×	✓	×	✓
k_n	×	×	×	✓	✓

6.1 Proposition. *If $\lim_{n\to\infty} f_n = f$ in $L^1(\mu)$, then $\lim_{n\to\infty} f_n = f$ in μ-measure.*

Proof Let $\epsilon > 0$, and set $E_n = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$. Then $\int_X |f_n - f| d\mu \geq \int_{E_n} |f_n - f| \mathrm{d}\mu \geq \int_{E_n} \epsilon = \epsilon \mu(E_n)$. Thus $\mu(E_n) \leq \frac{1}{\epsilon} \int_X |f_n - f| \mathrm{d}\mu \to 0$ as n goes to infinity. $\blacksquare$

6.2 Theorem. *Let $(f_n)_{n=1}^\infty$, $f \in M(X, \mathcal{M})$. Then*

*(i) If $\lim_{n\to\infty} f_n = f$ in μ–measure, then $(f_n)_{n=1}^\infty$ is **Cauchy in μ-measure**; i.e., given $\epsilon, \delta > 0$, there is $n_0 \in \mathbb{N}$ (dependent on ϵ, δ) such that whenever $n, m \geq n_0$, $\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}) < \delta$.*

(ii) If $(f_n)_{n=1}^\infty$ is Cauchy in μ–measure, then there is a subsequence $(f_{n_j})_{j=1}^\infty$ such that $\lim_{j\to\infty} f_{n_j} = f_0$ for some $f_0 \in M(X, \mathcal{M})$ μ–a.e. Furthermore, $\lim_{j\to\infty} f_{n_j} = f_0$ in measure.

Proof (i) If $m, n \in \mathbb{N}$, then

$$\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\} \subseteq \{x \in X : |f_n(x) - f(x)| + |f(x) - f_m(x)| \geq \epsilon\}$$
$$\subseteq \{x \in X : |f_n(x) - f(x)| \geq \epsilon/2\} \cup \{x \in X : |f(x) - f_m(x)| \geq \epsilon/2\}$$

and apply convergence in μ–measure to each term.

(ii) Let $n_1 < n_2 < \cdots$ be such that

$$E_j = \{x \in X : |f_n(x) - f_m(x)| \geq 1/2^j \text{ for all } n, m \geq n_j\} \qquad \mu(E_j) < \frac{1}{2^j}$$

by taking $\epsilon, \delta = 1/2^j$.

Let $F_k = \bigcup_{j=k}^{\infty} E_j$, so by σ–subadditivity, $\mu(F_k) \leq 1/2^{k-1}$. If $x \notin F_k$, then for $i > j \geq k$, we have

$$\begin{aligned} \left| f_{n_j}(x) - f_{n_i}(x) \right| &\leq \sum_{p=j}^{i-1} \left| f_{n_p}(x) - f_{n_{p+1}}(x) \right| \\ &< \sum_{p=j}^{i-1} \frac{1}{2^p} \leq \frac{1}{2^{j-1}} \leq \frac{1}{2^{k-1}} \end{aligned}$$

Thus $(f_{n_j})_{j=1}^{\infty}$ is pointwise Cauchy on $X \setminus F_k$. Let $F = \bigcap_{k=1}^{\infty} F_k$, so

$$0 \leq \mu(F) \leq \mu(F_k) \leq \frac{1}{2^{k-1}}$$

and since this holds for any k, $\mu(F) = 0$. Thus for $x \in X \setminus F = \bigcup_{k=1}^{\infty}(X \setminus F_k)$, we have that $(f_{n_j})_{j=1}^{\infty}$ is pointwise Cauchy. Thus there is $\tilde{f} \in M(X \setminus F, \mathcal{M}|_{X \setminus F})$, so $\lim_{j\to\infty} f_{n_j} = \tilde{f}$ on $X \setminus F$. Consider

$$f_0 : X \to \mathbb{C} \qquad f_0(x) = \begin{cases} \tilde{f}(x) & : x \in X \setminus F \\ 0 & : x \in F \end{cases}$$

It is easy to see that $f_0 \in \mathcal{M}(X, \mathcal{M})$. Given $\epsilon > 0$, let k be so $1/2^{k-1} < \epsilon$. Then for $x \in X \setminus F_k$,

$$\left| f_0(x) - f_{n_k}(x) \right| = \lim_{j\to\infty} \left| f_{n_j}(x) - f_{n_k}(x) \right| \leq \frac{1}{2^{k-1}} < \epsilon$$

Thus $\left\{ x \in X : \left| f_0(x) - f_{n_k}(x) \right| \geq \epsilon \right\} \subseteq F_k$, so $\mu(E) \leq \mu(F_k) \leq 1/2^{k-1} < \epsilon$ and the convergence is also in μ–measure. ∎

6.3 Corollary. *If $\lim_{n\to\infty} f_n = f$ in $L^1(\mu)$, then there is a subsequence $(f_{n_j})_{j=1}^{\infty}$ such that $\lim_{j\to\infty} f_{n_j} = f$ μ–a.e.*

Proof By the last proposition, we have $\lim f_n = f$ in μ–measure, and hence by (i), $(f_n)_{n=1}^{\infty}$ is Cauchy in μ–measure. By (ii), there is a subsequence so that $\lim f_{n_j} = f_0$ μ–a.e. As before,

$$E = \{x \in X : |f_0(x) - f(x)| \geq \epsilon\} \subseteq \{x \in X : |f_n(x) - f(x)| \geq \epsilon/2\} \cup \{x \in X : |f(x) - f_m(x)| \geq \epsilon/2\}$$

and since $\lim f_n = f$ in measure and $\lim f_{n_j} = f_0$ in measure, we see that $\mu(E)$ is bounded by arbitrarily small values. ∎

6.4 Corollary. *If $a < b$ in $\mathbb{R}$ $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then $f \in L([a,b], \mathcal{B}([a,b]), \lambda)$ and the Riemann and Lebesgue integral agree.*

Proof Let

$$J_{n,i} = \left[a + \frac{i-1}{n}(b-a), a + \frac{i}{n}(b-a) \right)$$

for $i = 1, \ldots, n$, $I_{n,i} = \overline{J_{n,i}}$, $l_{n,i} = \int_{x \in I_{n,i}} f(x)$, $u_{n,i} = \sup_{x \in I_{n,i}} f(x)$, $\phi_n = \sum_{i=1}^{n} l_{n,i} 1_{J_{n,i}}$, $\psi_n = \sum_{j=1}^{n} u_{n,i} 1_{J_{n,i}}$ and

$$L_n(f) = \int_{[a,b]} \phi_n \, d\lambda, U_n(f) = \int_{[a,b]} \psi_n \, d\lambda$$

Riemann integrability tells us that $\lim_{n\to\infty}(U_n(f)-L_n(f))=0$. Note that $\phi_n \le f \le \psi_n$, and $\int_{[a,b]}|\psi_n-\phi_n|\,\mathrm{d}\lambda = U_n(f)-L_n(f)\to 0$ as $n\to\infty$. Thus $\lim_{n\to\infty}|\psi_n-\phi_n|=0$ in $L^1(\mu)$. Thus, there is a subsequence so $\lim_{j\to\infty}|\psi_{n_j}-\phi_{n_j}|=0$ λ–a.e. Since $\phi_n\le\phi_{n+1}\le f\le\psi_{n+1}\le\psi_n$, we conclude that $f=\lim\phi_{n_j}$ λ–a.e. with integrable majorant $g=|\phi_1|+|\psi_1|$, so $\int_{[a,b]} f\,\mathrm{d}\lambda = \lim_{j\to\infty}L_{n_j}(f)=\int_a^b f$. ∎

More generally, Riemann integrable functions are continuous λ–a.e. If $a<b$ in $\overline{R}$, $f\ge 0$ improperly Riemann integrable, then it is Lebesgue integrable on (a,b).

Definition. If $(f_n)_{n=1}^\infty, f$ are in $M(X,\mathcal{M})$, then $\lim f_n = f$ μ–**almost uniformly** if, given any $\epsilon>0$, there is $E\in M$ with $\mu(E)<\epsilon$ so that $\lim_{n\to\infty}\sup_{x\in X\setminus E}|f_n(x)-f(x)|=0$.

> ***6.5 Theorem. (Egoroff)*** *Suppose $(X,\mathcal{M},\mu)$ is a finite measure space. If $(f_n)_{n=1}^\infty, f$ are in $M(X,\mathcal{M})$ such that $\lim f_n = f$ μ–a.e., then $\lim f_n = f$ μ–almost uniformly.*

Note that finiteness is essential.

Proof Let $N=\{x\in X : \lim f_n(x)$ does not exist, or is not equal to $f(x)\}$, so $\mu(N)=0$. For $k,n\in\mathbb{N}$, let $E_{n,k}=\bigcup_{m=n}^\infty\{x\in X : |f_m(x)-f(x)|\ge 1/k\}$, so $E_{n,k}\in\mathcal{M}$, $E_{n,k}\supseteq E_{n+1,k}$ and $\bigcap_{n=1}^\infty E_{n,k}\subseteq N$. Thus by continuity from above (we assume $\mu(X)<\infty$), we see that $\lim_{n\to\infty}\mu(E_{n,k})=0$.

Given $\epsilon>0$, let n_k so that $\mu(E_{n_k,k})<\epsilon/2^k$. Let $E=\bigcup_{k=1}^\infty E_{n_k,k}$ so $\mu(E)<\epsilon$ and for $x\in X\setminus E=\bigcap_{k=1}^\infty(E\setminus E_{n_k,k})\subseteq E_{n_k,k}$, for any k, we have $|f_n(x)-f(x)|<1/k$ for $n\ge n_k$. Thus $\limsup_{n\to\infty}\sup_{x\in X\setminus E}|f_n(x)-f(x)|\le 1/k$, which gives $\lim_{n\to\infty}\sup_{x\in X\setminus E}|f_n(x)-f(x)|=0$. ∎

III. Product Measures

Let $(X, \mathbb{N}, \mu)$, $(Y, \mathcal{N}, \nu)$ be two measure spaces.

6.6 Proposition. *Let $\mathcal{E} = \{E \times F : E \in \mathcal{M}, F \in \mathcal{N}\} \subseteq \mathcal{P}(X \times Y)$, and let $\mathcal{A} = \langle \mathcal{E} \rangle$. Then*

1. *Each element of $\mathcal{A}$ is of the form $A = \bigcup_{i=1}^{n} E_i \times F_i$ for $E_i \in \mathcal{M}$, $F_i \in \mathcal{N}$, $(E_i \times F_i) \cap (E_j \cap F_j) = \emptyset$ if $i \neq j$.*
2. *We define $(\mu \times \nu)_0 : \mathcal{A} \to [0, \infty]$ by*

$$(\mu \times \nu)_0(A) = \sum_{i=1}^{n} \mu(E_i \cup F_i)$$

if A is as in (i). Then $(\mu \times \nu)_0$ is a pre-measure, hence extends to a measure $\mu \times \nu : M \otimes \mathcal{N} \to [0, \infty]$. If each of μ and ν are σ–finite, $\mu \times \nu$ is σ–finite and this extension is unique.

Proof 1. We see that $\mathcal{E}$ is an elementary family of sets: if $E, E_1 \in \mathcal{M}$, $F, F_1 \in \mathcal{N}$, then

- $(E \times F) \cap (E_1 \times F_1) = (E \cap E_1) \times (F \cap F_1) \in \mathcal{E}$
- $(X \times Y) \setminus (E \times F) = [(X \setminus E) \times F] \cup [E \times (Y \setminus F)] \cup [(X \setminus E) \cup (Y \setminus F)]$.

Thus the result follows from an earlier lemma.

2. We need to establish that the formula for $(\mu \times \nu)_0(A)$ is well-defined. Suppose

$$A = \dot{\bigcup_{i=1}^{n}}(E_i \times F_i) = \dot{\bigcup_{j=1}^{m}}(M_j \times N_j)$$

Then for each $x \in X$ we see that $1_A(x, \cdot) = \sum_{i=1}^{n} 1_{E_i}(x) 1_{F_i} = \sum_{j=1}^{n} 1_{M_j}(x) 1_{F_j}$ and hence

$$\int_Y 1_A(x, y)\, \mathrm{d}\mu(y) = \sum_{i=1}^{n} \nu(F_i) 1_{E_i}(x) = \sum_{j=1}^{m} \mu(N_j) 1_{M_j}(x)$$

and moreover

$$\begin{aligned}\int_X \left[\int_Y 1_A(x, y)\, \mathrm{d}\nu(y)\right] \mathrm{d}\mu(x) &= \sum_{i=1}^{n} \mu(E_i)\nu(F_i) \\ &= \sum_{j=1}^{m} \mu(M_j)\nu(N_j) \qquad (\dagger)\end{aligned}$$

which gives an unambiguous value for $(\mu \times \nu)_0(A)$. Evidently, $\emptyset = \emptyset \times \emptyset$, so $(\mu \times \nu)_0(\emptyset) = 0$. Now suppose $A, (A_n)_{n=1}^{\infty}$ are in $\mathcal{A}$, with $A = \dot{\bigcup}_{n=1}^{\infty} A_n$. But then $1_A = \sum_{n=1}^{\infty} 1_{A_n}$ and for $x \in X$, $1_A(x, \cdot) = \sum_{n=1}^{\infty} 1_{A_n}(x, \cdot)$. Thus, by 2 applications of (a Corollary to) MCT and

(†),

$$\begin{aligned}
(\mu\times\nu)_0(A) &= \int_X\int_Y 1_A(x,y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x)\\
&= \int_X\int_Y\sum_{n=1}^{\infty}1_{A_n}(x,y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x)\\
&= \int_X\left[\sum_{n=1}^{\infty}\int_Y 1_{A_n}(x,y)\,\mathrm{d}\nu(y)\right]\mathrm{d}\mu(x)\\
&= \sum_{n=1}^{\infty}\int_X\int_Y 1_{A_n}(x,y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x)\\
&= \sum_{n=1}^{\infty}(\mu\times\nu)_0(A_n)
\end{aligned}$$

We appeal to the canonical measure construction to get $\mu\times\nu$ on $\mathcal{M}\otimes\mathcal{N}=\sigma\langle\mathcal{E}\rangle=\sigma\langle\mathcal{A}\rangle$. If $(X_n)_{n=1}^{\infty}\subseteq\mathcal{M}$, $(Y_n)_{n=1}^{\infty}\subseteq\mathcal{N}$ show σ–finitenes of μ, (resp. ν), then each $(\mu\times\nu)(X_n\times Y_n)=\mu(X_n)\nu(Y_n)<\infty$ and $X\times Y=\bigcup_{n=1}^{\infty}X_n\times Y_n$, showing σ–finiteness of $\mu\times\nu$. ■

6.7 Theorem. *Let $(X,\mathcal{M},\mu)$, $(Y,\mathcal{N},\nu)$ be σ–finite measure spaces. If $E\in\mathcal{M}\otimes\mathcal{N}$, then*

(i) $x\mapsto\nu(E_x):X\to[0,\infty]$ is $\mathcal{M}$–measurable

(ii) $y\mapsto\mu(E^y):Y\to[0,\infty]$ is $\mathcal{N}$–measurable.

(iii) $\mu\times\nu(E)=\int_Y\mu(E^Y)\,\mathrm{d}\nu(y)=\int_X\nu(E_x)\,\mathrm{d}\mu(x)$.

Proof (I) We assume that $\mu(X)$, $\mu(Y)<\infty$. Set $\mathcal{C}$ be the set of $E\in\mathcal{M}\otimes\mathcal{N}$ for which $(i),(ii),(iii)$ hold. We will establish that $\mathcal{A}=\langle\{M\otimes N:M\in\mathcal{M},N\in\mathcal{N}\}\rangle\subseteq\mathcal{C}$ and that $\mathcal{C}$ is a monotone class. Hence, the Monotone Class lemma show that $M\otimes\mathcal{N}=\sigma\langle\mathcal{A}\rangle=C(\mathcal{A})\subseteq C\subseteq M\mathcal{M}\otimes\mathcal{N}$. If $E\in A$, write $E=\dot{\bigcup}_{i=1}^{n}A_i\times B_i$, $A_i\in\mathcal{M}$, $B_i\in\mathcal{N}$ for $i=1,\dots,n$. Then or $x\in X$, we have

$$E_x=\bigcup_{x\in A_i,i=1}^{n}B_i\implies\nu(E_x)=\sum_{i=1}^{n}\nu(B_i)1_{A_i}(x)$$

Thus it is clear that (i) and part of (iii) hold or E. In the same way, (ii) holds, and the other part of (iii), so $E\in\mathcal{C}$, so $\mathcal{A}\subseteq\mathcal{C}$.

Let's see that $\mathcal{C}$ is a monotone class. Let $E_1\supseteq E_2\supseteq\cdots$ in $\mathcal{C}$. Then, for $x\in X$, $E_{1x}\supseteq E_{2x}\supseteq\cdots$ in $\mathcal{N}$, and $(\bigcap_{n=1}^{\infty}E_n)_x=\bigcap_{n=1}^{\infty}(E_{nx})$. Since $\nu(E_{1x})\le\nu(X)<\infty$, we may appeal to continuity from above to see that

$$\nu\left(\left(\bigcap_{n=1}^{\infty}E_n\right)_x\right)=\nu\left(\bigcap_{n=1}^{\infty}(E_{nx})\right)=\lim_{n\to\infty}\nu(E_{nx})$$

and hence (i) holds for $\bigcap_{n=1}^{\infty}E_n$. Furthermore, by LDCT with integrable majorant $\mu(X)\nu(Y)1_{X\times Y}$

and again by continuity from above,

$$\begin{aligned}(\mu\times\nu)\left(\bigcap_{n=1}^{\infty}E_n\right) &= \lim_{n\to\infty}(\mu\times\nu)(E_n)\\ &= \lim_{n\to\infty}\int_X \nu(E_{nx})\,\mathrm{d}\mu(x)\\ &= \int_X \lim_{n\to\infty}\nu(E_{nx})\,\mathrm{d}\mu(x)\\ &= \int_X \nu\left(\left(\bigcap_{n=1}^{\infty}E_n\right)_x\right)\mathrm{d}\mu(x)\end{aligned}$$

so $\bigcap_{n=1}^{\infty}$ satisfies part of (iii). Likewise, if $E_1 \subseteq E_2 \subseteq \cdots$ in $\mathcal{C}$, we may apply continuity from below, and MCT to see that $\bigcup_{n=1}^{\infty}E_n$ satisfies (i) and part of (iii). Similarly, in each case above, then $y-$sections of intersections of decreasing sequences or unions of increasing sequences are in $\mathcal{C}$.

(II) Now let each of μ,ν be $\sigma-$finite. Hence there are $X_1 \subseteq X_2 \subseteq \cdots$ in $\mathcal{M}$, so $\bigcup_{n=1}^{\infty}X_n = X$, and $Y_1 \subseteq Y_2 \subseteq \cdots$ in $\mathcal{N}$ so $\bigcup_{n=1}^{\infty}Y_n = Y$. If $E \in \mathcal{M}\otimes\mathcal{N}$, then $E\cap(X_1\times Y_1) \subseteq E\cap(X_2\times Y_2) \subseteq \cdots$ and each $E\cap(X_n\times Y_n)$ satisfies (i), (ii), and (iii) in the finite measure space $(\mu\times\nu)|_{X_n\times Y_n}$. Hence, we conclude by continuity from below

$$y \mapsto \mu(E^Y) = \lim_{n\to\infty}\mu(E^Y\cap Y_n)$$

since $(E\cap(X_n\times Y_n))^Y = E^Y\cap Y_n$ is an increasing sequence and this function is $\mathcal{N}-$measurable. Thus, by MCT and again by continuity from below,

$$\begin{aligned}\mu\times\nu(E) &= \lim_{n\to\infty}\mu(E\cap(X_n\times Y_n))\\ &= \lim_{n\to\infty}\int_Y \nu(E^Y\cap Y_n)\,\mathrm{d}\nu(y)\\ &= \int_Y \lim_{n\to\infty}\nu(E^Y\cap Y_n)\,\mathrm{d}\nu(y)\\ &= \int_Y \nu(E^Y)\,\mathrm{d}\nu(y)\end{aligned}$$

Thus, E satisfies (ii) and part of (iii). Likewise, E satisfies (i) and the other part of (iii). ∎

6.8 Theorem. (Tonelli and Fubini) *Let $(X,\mathcal{M},\mu)$ $(Y,\mathcal{N},\nu)$ be $\sigma-$finite measure spaces.*

(Tonelli's Theorem) If $f \in \overline{M}^{+}(X\times Y,\mathcal{M}\otimes\mathcal{N})$, then

$$x \mapsto \int_Y f_x\,\mathrm{d}\nu : X \to [0,\infty] \text{ is } \mathcal{M}-\text{measurable.}$$
$$y \mapsto \int_X f^y\,\mathrm{d}\mu : Y \to [0,\infty] \text{ is } \mathcal{N}-\text{measurable.}$$

and

$$\int_Y\int_X f^y\,\mathrm{d}\mu\,\mathrm{d}\nu(y) = \int_{X\times Y} f\,\mathrm{d}\mu\times\nu = \int_X\int_Y f_x\,\mathrm{d}\nu\,\mathrm{d}\mu(x) \tag{†}$$

(Fubini's Theorem) If $f \in L(\mu \times \nu)$, then

$$\left(x \mapsto \int_Y f_x \, \mathrm{d}\nu\right) \in L(\mu)$$
$$\left(y \mapsto \int_X f^y \, \mathrm{d}\mu\right) \in L(\nu)$$

and (†) *holds.*

Proof For an indicator function, we have

$$\begin{aligned}\int_{X\times Y} 1_E \, \mathrm{d}\mu \times \nu = \mu \times \nu(E) &= \int_X \nu(E_x) \, \mathrm{d}\mu(x) \\ &= \int_X \int_Y 1_{E_x} \, \mathrm{d}\nu \, \mathrm{d}\mu(x) \\ &= \int_X \int_Y (1_E)_x \, \mathrm{d}\nu \, \mathrm{d}\mu(x)\end{aligned}$$

Similarly, this is true for the y–sections and the other itegrated integral. Hence Tonelli holds for $f \in S^+(X \times Y, \mathcal{M} \otimes \mathcal{N})$.

If $f \in \mathcal{M}^+(X \times Y, \mathcal{M} \otimes \mathcal{N})$, we have $(\phi_n)_{n=1}^\infty \subset S^+(X \times Y, \mathcal{M} \otimes \mathcal{N})$ such that $\lim \phi_n = f$. We use MCT.

- $\int_Y f_x \, \mathrm{d}\nu = \int_Y \lim_{n\to\infty} \phi_{nx} \, \mathrm{d}\nu = \lim_{n\to\infty} \int_Y \phi_{nx} \, \mathrm{d}\nu$, so $x \mapsto \int_Y f_x$ is $\mathcal{M}$–measurable, and
-

$$\begin{aligned}\int_{X\times Y} f \, \mathrm{d}\mu \times \nu &= \lim_{n\to\infty} \int_{X\times Y} \phi_n \, \mathrm{d}\mu \times \nu) \\ &= \lim_{n\to\infty} \int_X \int_Y \phi_{nx} \, \mathrm{d}\nu \, \mathrm{d}\mu(x) \\ &= \int_X \lim_{n\to\infty} \int_Y \phi_{nx} \, \mathrm{d}\nu \, \mathrm{d}\mu(x) \\ &= \int_X \int_Y \lim_{n\to\infty} \phi_{nx} \, \mathrm{d}\nu \, \mathrm{d}\mu(x) \\ &= \int_X \int_Y f_x \, \mathrm{d}\nu \, \mathrm{d}\mu(x)\end{aligned}$$

and the same holds for y–sections, and Tonelli's Theorem holds.

For Fubini's Theorem, we proceed as above. Recall that if $f \in L(\mu \times \nu)$, we can find $(\phi_n)_{n=1}^\infty \subset S(X \times Y, \mathcal{M} \otimes \mathcal{N})$ such that each $|\phi_n| \leq f$ and $\lim_{n\to\infty} \phi_n = f$. We use LDCT with integrable majorants to see that

$$\int_{X\times Y} |f| \, \mathrm{d}\mu \times \nu = \int_X \int_Y |f|_x \, \mathrm{d}\nu \, \mathrm{d}\mu(x)$$

so that $x \mapsto \left|\int_Y f_x \, \mathrm{d}\nu\right| \leq \int_Y |f_x| \, \mathrm{d}\nu$, which shows that $x \mapsto \int_Y f_x \, \mathrm{d}\nu$ is in $L(\mu)$. Likewise for the other section. ■

Remark. If $f \in M(X \times Y, \mathcal{M} \otimes \mathcal{Y})$, we may wish to see that $f \in L(\mu \times \nu)$. This is equivalent to saying that $|f| \in L(\mu \times \nu)$, and we may be able to compute this with an integrated integral, using Tonelli's Theorem.

7 Multidimensional Lebesgue Measure

Let $\mathcal{B}(\mathbb{R})$,$\mathcal{L} \subseteq \mathcal{P}(\mathbb{R})$ denote the Borel and Lebesgue $\sigma-$algebras. Recall that the Lebesgue measure is translation invariant.

Remark. If $x, c \in \mathbb{R}$, $c \neq 0$, then the maps $T_x : \mathbb{R} \to \mathbb{R}$ by $y \mapsto x+y$ and $M_c : \mathbb{R} \to \mathbb{R}$ by $y \mapsto cy$ are continuous, hence Borel measurable. Thus if $E \in \mathcal{B}(\mathbb{R})$, $x + E = T_x(E) = T_{-x}^{-1}(E) \in \mathcal{B}(\mathbb{R})$. Similarly, $cE = M_{1/c}^{-1}(E) \in \mathcal{B}(\mathbb{R})$.

7.1 Proposition. *Let $f \in L(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) = L(\lambda)$.*

(i) For $x \in \mathbb{R}$, $f \circ T_x \in L(\lambda)$ with $\int_{\mathbb{R}} f \circ T_x \,\mathrm{d}\lambda = \int_{\mathbb{R}} f \,\mathrm{d}\lambda$.

(ii) For $0 \neq c \in \mathbb{R}$, $f \circ M_c \in L(\lambda)$ with $\int_{\mathbb{R}} f \circ M_c \,\mathrm{d}\lambda = \frac{1}{|c|}\int_{\mathbb{R}} f \,\mathrm{d}\lambda$.

Proof This is a direct application of A2 Q3(b). ∎

Now, recall that $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R})$. Let $\lambda_d = \lambda \times \cdots \times \lambda : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ denote the $d-$dimensional Lebesgue measure. We define $\mathcal{L}_d$ to be the completion of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda_d)$.

Remark. For suitable f, we say

$$\int_{\mathbb{R}^d} f \,\mathrm{d}\lambda_d = \int_{\mathbb{R}^d} f(x_1, \ldots, x_d)\,\mathrm{d}(x_1, \ldots, x_d)$$

Fubini-Tonelli theorem tells us that

$$\int_{\mathbb{R}^d} f \,\mathrm{d}\lambda_d = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \ldots, x_d) dx_{\sigma(1)} \cdots \mathrm{d}x_{\sigma(d)}$$

where $\sigma : [d] \to [d]$ is any bijection.

7.2 Proposition. *Let $f \in L(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda_d) = L(\lambda_d)$.*

(i) For $x \in \mathbb{R}^d$, let $T_x : \mathbb{R}^d \to \mathbb{R}^d$ be given by $T_x(y) = x + y$. Then $f \circ T_x \in L(\lambda)$ with

$$\int_{\mathbb{R}^d} f \circ T)x \,\mathrm{d}\lambda_d = \int_{\mathbb{R}^d} f \,\mathrm{d}\lambda_d$$

(ii) For $A \in (d, \mathbb{R})$, $f \circ A \in L(\lambda)$ with

$$\int_{\mathbb{R}^d} f \circ A \,\mathrm{d}\lambda_d = \frac{1}{|\det A|} \int_{\mathbb{R}^d} f \,\mathrm{d}\lambda_d$$

Proof (i) This follows from the previous proposition as well as Fubini-Tonelli:

$$\begin{aligned}
\int_{\mathbb{R}^d} f \circ T_x \,\mathrm{d}\lambda_d &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1 + y_1, \ldots, x_d + y_d)\,\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_d \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \ldots, x_d + y_d)\,\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_d \\
&\vdots \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \ldots, x_d)\,\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_d \\
&= \int_{\mathbb{R}} f \,\mathrm{d}\lambda_d
\end{aligned}$$

(ii) We can factor $A = A_1 \cdots A_n$ where each A_i is one of the following 3 types:

- (add row to vector) $A_{ij}(x_1,\dots,x_d) = (x_1,\dots,x_i + x_j,\dots,x_d)$.
- (swap) $S_{ij}(x_1,\dots,x_d) = (x_1,\dots,x_j,\dots,x_i,\dots,x_d)$
- (multiply row) $M_{ic}(x_1,\dots,x_d) = (x_1,\dots,cx_i,\dots,x_d)$

Notice that $\det(A_{ij}) = 1 = |\det S_{ij}|$, while $|\det(M_{ic})| = |c|$. If $f \geq 0$, we have for $i < j$

$$\int_{\mathbb{R}^d} f \circ A_{ij} \, \mathrm{d}\lambda_d = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1,\dots,x_i + x_j,\dots,x_d) \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_d = \int_{\mathbb{R}^d} f \, \mathrm{d}\lambda_d$$

by translation invariance. Similarly, $\int_{\mathbb{R}^d} f \circ S_{ij} \, \mathrm{d}\lambda_d = \int_{\mathbb{R}^d} f \, \mathrm{d}\lambda_d$ and $\int_{\mathbb{R}^d} f \circ M_{ic} \, \mathrm{d}\lambda_d = \frac{1}{|c|} \int_{\mathbb{R}^d} f \, \mathrm{d}\lambda_d$. Then

$$\begin{aligned} \int_{\mathbb{R}^d} f \circ A \, \mathrm{d}\lambda_d &= \int_{\mathbb{R}^d} f \circ A_1 \circ \cdots \circ A_n \, \mathrm{d}\lambda_d \\ &= \frac{1}{|\det(A_n)|} \int_{\mathbb{R}^d} f \circ A_1 \circ \cdots \circ A_{n-1} \, \mathrm{d}\lambda_d \\ &= \frac{1}{|\det(A)|} \int_{\mathbb{R}^d} f \, \mathrm{d}\lambda_d \end{aligned}$$

■

IV. Complex Measures

8 Signed Measures

Definition. Let $(X, \mathcal{M}, \mu)$ be a measurable space. A (finite) **signte measure)** on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to \mathbb{R}$ such that

- $\nu(\emptyset) = 0$
- If $E_1, E_2, \ldots \in \mathcal{M}$ are disjoint, then $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i)$.

Remark. 1. It is possible to defined a signed measure into $(-\infty, \infty]$ or $[-\infty, \infty)$. For convenience, we work only with the finite case.
2. As well, note that the series above is always absolutely convergent.
3. If $F \subseteq E$ in $\mathcal{M}$, then $\nu(E \setminus F) = \nu(E) - \nu(F)$.

Example. 1. If $\mu_1, \mu_2 : \mathcal{M} \to [0, \infty)$, then $\nu = \mu_1 - \mu_2$ is a signed measure.
2. If $\mu : \mathcal{M} \to [0, \infty]$ is a measure and $f \in L(\mu)$, we define $f \cdot \mu : \mathcal{M} \to \mathbb{R}$ by $f \cdot \mu(E) = \int_E f \, d\mu = \int_X 1_E f \, d\mu$. This is a signed measure (LDCT).

8.1 Proposition. *(i) If $E_1 \subseteq E_2 \subseteq \cdots$ in $\mathcal{M}$, then $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \nu(E_n)$.*
(ii) If $E_1 \supseteq E_2 \supseteq \cdots$ in $\mathcal{M}$, then $\nu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \nu(E_n)$.

Proof Identical as the proof as for (non-negative) measures. ∎

Definition. Let $(X, \mathcal{M}, \mu)$ be a signed measure space. A set $E \in \mathcal{M}$ is **positive** (or **negative** or **null**) for ν if for any $F \subseteq E$, $F \in \mathcal{M}$, we have $\nu(F) \geq 0$ (or $\nu(F) \leq 0$ or $\nu(F) = 0$).

8.2 Lemma. *(i) If $P \in \mathcal{M}$ is positive and $Q \subseteq P$, then Q is positive.*
(ii) If $P_1, P_2, \ldots \in \mathcal{M}$, then $P = \bigcup_{i=1}^{\infty} P_i$ is positive.

Proof The first statement is clear. For the second, suppose $E \subseteq P$, $E \in \mathcal{M}$, and let $Q_1 = P_1$, $Q_{n+1} = P_{n+1} \setminus \bigcup_{i=1}^{n} P_i$. Each Q_n is positive by (i) and $E = \bigcup_{i=1}^{\infty}(E \cap Q_i)$ as $E \subseteq P$. Thus $\nu(E) = \sum_{i=1}^{\infty} \nu(E \cap Q_i) \geq 0$. ∎

8.3 Theorem. (Hahn Decomposition) *Let $(X, \mathcal{M}, \mu)$ be a signed measure space. Then there exist P, N in $\mathcal{M}$ such that*
(i) P is positive for ν.
(ii) N is negative for ν
(iii) $P \cup N = X$, $P \cap N = \emptyset$.
Furthermore, if P', N' also satisfy the above constraints, then $P \triangle P'$ and $N \triangle N'$ are each null for ν.

Definition. A pair (P, N), as above, is called a **Hahn decomposition** for ν.

Proof Every set named in this proof is assumed to be in $\mathcal{M}$.
I: If $E \in \mathcal{M}$, $\epsilon > 0$, then there is $E_\epsilon \subseteq E$ such that
1. $\nu(E_\epsilon) \geq \nu(E)$
2. for any $B \subseteq E_\epsilon$, $\nu(B) - \epsilon$**.**

If not, then every $A \subseteq E$ satisfying (1), there exists $B \subseteq A$ such that $\nu(B) \leq -\epsilon$. Then, inductively, we find

- $B_1 \subseteq E$ such that $\nu(B_1) \leq -\epsilon$ and $\nu(E \setminus B_1) = \nu(E) - \nu(B_1) > \nu(E)$; hence
- $B_2 \subseteq E \setminus B_1$ such that $\nu(B_2) \leq -\epsilon$ and $\nu(E \setminus (B_1 \cup B_2)) = \nu(E) - \sum_{i=1}^{2} \nu(B_i) > \nu(E)$.
- $B_{n+1} \subseteq E \setminus \bigcup_{i=1}^{n} B_i$, with $\nu(B_{n+1}) \leq -\epsilon$ and $\nu(E \setminus \bigcup_{i=1}^{n+1} B_i) > \nu(E)$.

However, as $B_i \cap B_j = \emptyset$, we would have $\nu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \nu(B_i) = -\infty$, violating finiteness of ν.

II: If $E \in \mathcal{M}$, there is a positive $P \subseteq E$ such that $\nu(P) \geq \nu(E)$. Let $E_0 = E_1$ and we use (I) and induction fo find $E_n \subseteq E_{n-1}$ such that $\nu(E_n) \geq \nu(E_{n-1})$ and if $B \subseteq E_n$, then $\nu(B) > -1/n$. Let $P = \bigcap_{n=1}^{\infty} E_n$. By continuity from above, $\nu(P) = \lim \nu(E_n) \geq \nu(E_0) = \nu(E)$. If $B \subseteq P$, then $B \subseteq E_n$ for each n so $\nu(B) > -1/n$. Thus P is positive for ν.

III: Let $s = \sup\{\nu(E) : E \in \mathcal{M}\}$. Then there is a sequence $E_1, E_2, \ldots$ such that $s = \lim_{n\to\infty} \nu(E_n)$. For each n, find $P_n \subseteq E_n$, which is positive for ν, with $\nu(P_n) \geq \nu(E_n)$. Let $P = \bigcup_{i=1}^{\infty} P_i$. We note that P is positive for ν and we compute

$$s \geq \nu(P) = \lim_{n\to\infty} \nu\left(\bigcup_{i=1}^{\infty} P_i\right) \geq \lim_{n\to\infty} \nu(P_n) \geq \nu(E_n) = s$$

so $\nu(P) = s$. We let $N = X \setminus P$. If there were $E \subseteq N$ with $\nu(E) > 0$, then $\nu(E \cup P) > \nu(E) + \nu(P) > s$, violating definition of s. Thus $\nu(E) \leq 0$, so N is negative.

IV: Essential Uniqueness If P', N' are another Hahn decomposition, then $P \triangle P' \subseteq N' \cup N$. Then $P \triangle P'$ is positive and negative, and thus null. The same result holds for $N' \triangle N$. ■

8.4 Proposition. *Let μ, ν be as above with μ finite. Then $\nu \ll \mu$ if and only if for any $\epsilon > 0$, there is $\delta > 0$ such that for $E \in \mathcal{M}$, $\mu(E) < \delta$ implies $|\nu(E)| < \infty$.*

Proof First, since $|\nu(\bullet)| \leq \operatorname{Re} \nu^+ + \cdots + \operatorname{Im} \nu^-|$, it suffices to show the equivalence for finite measures. Suppose (AC') fails. Then there exists $\epsilon > 0$ such that there is $E_n \in \mathcal{M}$ with $\mu(E_n) < 1/2^n$ while $\nu(E_n) \geq \epsilon$. Let $F_n = \bigcup_{i=n}^{\infty} E_i$ so $F_1 \supseteq F_2 \supseteq \cdots$ with $\mu(F_n) \leq 1/2^{n-1}$ and hence by continuity from above, $\mu(\bigcap_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \mu(F_n)$ while

$$\nu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \nu(F_n) \geq \liminf_{n\to\infty} \nu(E_n) \geq \epsilon$$

so AC fails. Thus AC implies AC'.

If AC' holds, there is $\delta_n > 0$ so for E in $\mathcal{M}$, $\mu(E) < \delta_n$ implies $\nu(E) < 1/n$. Hence if $\mu(E) = 0 < \delta_n$ for all n, then $\nu(E) < 1/n$ for any n, i.e. $\nu(E) = 0$. ■

8.5 Lemma. *Let $\mu, \nu : \mathcal{M} \to [0, \infty)$ be finite measures. Then either $\mu \perp \nu$ or to every $\epsilon > 0$ and $E \in \mathcal{M}$ for which $\mu(E) > 0$ and E is positive $\nu - \epsilon\mu$.*

Proof Let (P_n, N_n) be a Hahn decomposition for $\nu - \frac{1}{n}\mu$ and $P = \bigcup_{n=1}^{\infty} P_n$, $N = X \setminus P = \bigcap_{n=1}^{\infty} N_n$. Then N is negative for each $\nu - \frac{1}{n}$, so $0 \leq \nu(N) \leq \frac{1}{n}\mu(N)$ for each n, so $\nu(N) = 0$. If $\mu(P) = 0$, then $\nu \perp \mu$. Otherwise, $\mu(P) > 0$, so $\mu(P_n) > 0$ for some n, and $E = P_n$ satisfies $\mu(E) > 0$ and $(\nu - \frac{1}{n}\mu)(E) > 0$. ■

8.6 Theorem. (Lebesgue-Radon-Nikodym) *Let $(X, \mathcal{M})$ be a measurable space, $\nu : \mathcal{M} \to \mathbb{C}$ a complex measure and $\mu : \mathcal{M} \to [0, \infty]$ be a σ-finite measure. Then*

(i) There is a unique complex measure $\rho : \mathcal{M} \to \mathbb{C}$ such that $\rho \perp \mu$ and $\nu - \rho \ll \mu$

(ii) There is $f \in L(\mu)$ such that $\nu - \rho = f \cdot \mu$.

Remark. The decomposition $\nu = \rho + (\nu - \rho)$ is called the **Lebesgue decomposition** of ν with respect to μ. The element $f \in L(\mu)$, above, is called the **Radon-Nikodym derivative** of ν with respect to μ. We will often write $f = \frac{d\nu}{d\mu}$.

Proof (I) Assume $\mu, \nu : \mathcal{M} \to [0, \infty)$ are finite measures. Let

$$\mathcal{F} = \{f \in \overline{M}^+(X, \mathcal{M}) : \int_E f \, d\mu \leq \nu(E) \text{ for all } E \text{ in } \mathcal{M}\}$$

Indeed, let $A = \{x \in X : f(x) > g(x)\}$. Then for $E \in \mathcal{M}$,

$$\int_E \max\{f, g\} \, d\mu = \int_{E \cap A} f \, d\mu + \int_{E \setminus A} g \, d\mu \leq \nu(E \cap A) + \nu(E \setminus A) = \nu(E)$$

Thus if $f_1, \ldots, f_n \in \mathcal{F}$, then $\max\{f_1, \ldots, f_n\} \in \mathcal{F}$. Let $s = \sup\{\int_X f \, d\mu : f \in \mathcal{F}\} \leq \nu(X) < \infty$. Hence for each n, there is $f_n \in \mathcal{F}$ such that $s - \frac{1}{n} < \int_X f_n \, d\mu \leq s$. We let $g_n = \max\{f_1, \ldots, f_n\} \in \mathcal{F}$ so $g_n \leq g_{n+1}$, and we let $f = \lim_{n \to \infty} g_n$. Then

$$s \geq \lim_{n \to \infty} \int_X \geq \lim_{n \to \infty} \int_X f_n \, d\mu \geq \lim_{n \to \infty} \left(s - \frac{1}{n}\right) = s$$

so $s = \lim_{n \to \infty} \int_X g_n \, d\mu = \int_X f \, d\mu$ by monotone convergence. In particular, $f \in \overline{L}^+(\mu)$, so we may assume that $f \in L^+(\mu)$ (i.e. $\mathbb{R}$-valued). Again, by MCT,

$$\int_E f \, d\mu = \lim_{n \to \infty} \int_E g_n \, d\mu \leq \lim_{n \to \infty} \nu(E) = \nu(E)$$

so $f \in \mathcal{F}$.

Now, let $\rho = \nu - f \cdot \mu$, which is non-negative as $f \in \mathcal{F}$. If $\rho \not\perp \mu$, then the last lemma provides $\epsilon > 0$ and $E \in \mathcal{M}$ which is positive such that

$$\rho - \epsilon\mu = (\nu - f \cdot \mu) - \epsilon\mu = \nu - (f + \epsilon 1)\mu$$

i.e. for $B \subseteq E$, $B \in \mathcal{M}$, $\int_B (f + \epsilon 1) \, d\mu = (f + \epsilon 1)\mu(B) \leq \nu(B)$. Hence if $A \in \mathcal{M}$, we have

$$\begin{aligned}\int_A (f + \epsilon 1_E) \, d\mu &= \int_{A \setminus E} f \, d\mu + \int_A (f + \epsilon 1_E) \, d\mu \\ &\leq \nu(A \setminus E) + \nu(A \cap E)\end{aligned}$$

so $f + \epsilon 1_E \in \mathcal{F}$. However,

$$\int_X (f + \epsilon 1_E) \, d\mu = \int_X f \, d\mu + \epsilon\mu(E) = s + \epsilon\mu(E) > s$$

But these last two statements contradict definitions of $\mathcal{F}$ and s. Thus $\rho \perp \mu$.

(II) Assume $\nu : \mathcal{M} \to [0, infty)$ and $\mu : \mathcal{M} \to [0, \infty]$ is σ–finite. We get $(X_n)_{n=1}^\infty \subseteq \mathcal{M}$ such that $X = \dot{\bigcup}_{n=1}^\infty X_n$ and each $X_n \in \mathcal{M}$ has $\mu(X_n) < \infty$. Let $\nu_x = \nu_{X_i}$, $\mu_i = \mu_{X_i}$. Apply (I) to pairs (ν_i, μ_i) to obtain measures $\rho_i : \mathcal{M}_{X_i} \to [0, \infty)$ $\rho_i \perp \mu_i$ and $\nu_i - \rho_i = f_i \cdot \mu_i \ll \mu_i$ where $f_i \in L^+(\mu_i)$. Define

- $\rho : \mathcal{M} \to [0, \infty]$ by $\rho(E) = \sum_{i=1}^\infty \rho_i(E \cap X_i)$
- $f : X \to [0, \infty)$ by $f(x) = f_i(x)$ if $x \in X_i$.

It is easily checked that ρ defines a measure and that $f \in M^+(X,\mathcal{M})$. If (E_i,F_i) realize (E_i,F_i) reaizes $\rho_i \perp \mu_i$, then $(\bigcup_{i=1}^{\infty} E_i, \bigcup_{i=1}^{\infty} F_i)$ realizes $\rho \perp \mu$. Furthermore, for $E \in \mathcal{M}$ we have

$$\begin{aligned}\nu(E) &= \sum_{i=1}^{\infty} \nu(E \cap X_i) = \sum_{i=1}^{\infty}\left(\rho_i(E\cap X_i) + \int_{E\cap X_i} f_i \,\mathrm{d}\mu_i\right) \\ &= \rho(E) + \int_E f \,\mathrm{d}\mu\end{aligned}$$

by monotone convergence. In particular, since $\nu(X) < \infty$, we see that ρ is a finite measure and $f \in L^+(\mu)$.

(III) Now suppose $\nu : \mathcal{M} \to \mathbb{C}$, $\mu : \mathcal{M} \to [0,\infty]$ is σ–finite. Apply the Jordan decomposition so that $\nu = (\operatorname{Re}\nu^+ - \operatorname{Re}\nu^-) + i(\operatorname{im}\nu^+ - \operatorname{im}\nu^-)$. Apply (II) to each component to get (ρ_i, f_i) and let $\rho = \rho_1 - \rho_2 + i(\rho_3 - \rho_4)$ and $f = f_1 - f_2 + i(f_3 - f_4)$, which certainly satisfy the properties.

(IV) Uniqueness. Suppose we have $\rho, \rho' : \mathcal{M} \to \mathbb{C}$ satisfying the requiremenets. Since $\rho + (\nu - \rho) = \nu = \rho' + (\nu - \rho')$, we have $\rho - \rho' = (\nu - \rho') - (\nu - \rho)$ simulaneously singular and absolutely continuous with respect to μ, so $\rho - \rho' = 0$. ∎

The Radon-Nikodym Derivative

Definition.

Let us assume above that $\nu \ll \mu$, so (L-)R-N tells us that $\nu = f \cdot \mu$ for some $f \in L(\mu)$.

1. If $f \in L(\mu)$, $f \cdot \mu = 0$ if and only if $1_E f = 0$ μ–a.e. for each $E \in \mathcal{M}$ if and only if $f = 0$ μ–a.e. Hence if $f, g \in L(\mu)$, then $f \cdot \mu = g \cdot \mu$ if and only if $f = g$ μ–a.e.
2. We let $L^1(\mu) = L(\mu)/\sim_\mu$ where $f \sim_\mu g$ if and only if $f = g$ μ–a.e. Pointwise μ–a.e. operations are legal.

If $\nu = f \cdot \mu$ as above, we write $f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ in $L^1(\mu)$, so $\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \cdot \mu$.

Definition. Let $\nu : \mathcal{M} \to \mathbb{C}$ be a complex measure. We let $L(v) = L(\operatorname{Re} v^+) \cap \cdots L(\operatorname{Im} v^-)$ and for $f \in L(v)$, we define the **Lebesgue integral** by

$$\int_X f \,\mathrm{d}\nu = \int_X f \,\mathrm{d}(\operatorname{Re} v^+) - \int_X f \,\mathrm{d}(\operatorname{Re} v^-) + i\left[\int_X f \,\mathrm{d}(\operatorname{Im} v^+) - \int_X f \,\mathrm{d}(\operatorname{Im} v^-)\right]$$

We let $L^1(\nu) = L(\nu)/\sim_\nu$.

8.7 Proposition. *Let ν be a complex measure, μ a finite easure, and λ a σ–finite measure, on a measurable space X. Then*

(i) If $\nu \ll \lambda$, then for $g \in L(\nu)$, $g\frac{\mathrm{d}\nu}{\mathrm{d}\lambda} \in L^1(\lambda)$.

(ii) If $\nu \ll \mu$, $\mu \ll \lambda$, then $\nu \ll \lambda$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$

Proof (roman*) If $E \in \mathcal{M}$, then $\int 1_E \,\mathrm{d}\nu = \nu(E) = \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} \cdot \lambda(E) = \int 1_E \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} \,\mathrm{d}\lambda$. Thus the result holds by LDCT.

(roman*) If $E \in \mathcal{M}$, if $\lambda(E) = 0$, then $\mu(E) = 0$ so $\nu(E) = 0$ so $\nu \ll \lambda$. Then for any $E \in \mathcal{M}$, apply (i) to get

$$\begin{aligned}\int 1_E \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \nu(E) &= \int 1_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu \\ &= \int 1_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \cdot \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\end{aligned}$$

and from above, $\frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ λ–a.e. ■

9 L^p–spaces

Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall that $L^1(\mu) = L(\mu)/\sim_\mu$. Likewise, if $1 < p < \infty$, then we let $L^p(\mu) = \{f \in M(X, \mathcal{M}) : \int_X |f|^p \,\mathrm{d}\mu < \infty\}/\sim_\mu$. Note that the functional $\|\cdot\|_1$ on $L^1(\mu)$ given by $\|f\|_1 = \int_X |f| \mathrm{d}\mu$ is a norm on $L^1(\mu)$.

If $\phi : \mathbb{R} \to \mathbb{R}$ is twice differentiable and for which $\phi'' > 0$, then ϕ is **strictly convex**. If $x < y$ in $\mathbb{R}$, $0 < t < 1$, then $\phi((1-t)x + ty) < (1-t)\phi(x) + t\phi(y)$.

9.1 Proposition. (Young's Inequality) *If $a, b \geq 0$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{1}{p} a^{\frac{1}{q}} b^q$ with equality if $a^p = b^q$.*

Proof By convexity of e^x,

$$ab = e^{\log(ab)} = e^{\frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q)} \leq \frac{1}{p} e^{\log(a^p)} + \frac{1}{q} e^{\log(b^q)} = \frac{1}{p} a^p + \frac{1}{q} b^q$$

and equality holds if and only if $a^p = b^q$. ■

Remark. If $f, g \in L^{\mathbb{R}}(\mu)$, $f \geq g$ μ–a.e. and $f \neq g$ μ–a.e, then $\int_X f \,\mathrm{d}\mu > \int_X g \,\mathrm{d}\mu$. Indeed, $(f - g) \cdot \mu$ is a non-zero (positive) measure.

9.2 Proposition. (Hölder's Inequality) *Let $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\mu)$, $g \in L^q(\mu)$. Then $fg \in L^1(\mu)$ with*

$$\|f\|_1 \leq \|f\|_p \|g\|_q$$

with equality holding onliy if there are $\alpha, \beta \geq 0$ such that $\alpha|f|^p = \beta|g|^q$ μ–a.e.

Proof We may assume that $\|f\|_p \|g\|_q > 0$. By Young's inequality,

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_q^q}$$

Integrate over X and multiply by $\|f\|_p^p \|g\|_q^q$ to see that

$$\begin{aligned} \|fg\|_1 &\leq \frac{1}{p} \cdot \frac{\|f\|_p^p}{\|f\|_p^{p-1}} \|g\|_q + \frac{1}{q} \frac{\|g\|_q^q}{\|g\|_q^{q-1}} \|f\|_p \\ &\leq \left(\frac{1}{p} + \frac{1}{q}\right) \|f\|_p \|g\|_q \end{aligned}$$

with equality holding if and only if $\|g\|_q^q |f|^p = \|f\|_p^p |g|^q$. ■

Remark. We define $\operatorname{sgn} : \mathbb{C} \to \mathbb{C}$ by $\operatorname{sgn}(z) = \frac{z}{|z|}$ if $z \neq 0$, and 0 if $z = 0$.

9.3 Proposition. (Minkowski's Inequality) *If $p > 1$ and $f, g \in L^p(\mu)$, then $f + g \in L^p(\mu)$ with $\|f + g\|_p \leq \|f\|_p + \|g\|_q$ with equality if and only if $\operatorname{sgn} f = \operatorname{sgn} g$ μ–a.e. and there are $\alpha, \beta \geq 0$ so $\alpha|f| = \beta|g|$ μ–a.e.*

Proof We have, by Hölder's inequality used twice,

$$\begin{aligned}|f+g|^p &= |f+g||f+g|^{p-1}\\ &\le (|f|+|g|)|f+g|^{p-1}\\ &\le \|f\|_p \left\||f+g|^{p-1}\right\|_q + \|g\|_q \left\||f+g|^{p-1}\right\|_q\\ &= (\|f\|_p+\|g\|_q)\left\||f+g|^{p-1}\right\|_q \qquad (*)\end{aligned}$$

where equality holds at the first inequality $\operatorname{sgn} f = \operatorname{sgn} g$, and at the second inequality $\alpha|f|^p = \|f\|_p \left\||f+g|^{p-1}\right\|_q$ and $\alpha|g|^p = \|g\|_p \left\||f+g|^{p-1}\right\|_q$ where $\alpha = \left\||f+g|^{p-1}\right\|_q$. Notice that $q(p-1)=p$ so that

$$\left\||f+g|^{p-1}\right\|_q = \left(\int |f+g|^{(p-1)q}\right)^{1/q} = \|f+g\|_p^{p/q}$$

Furthermore, $|f+g|^p \le (|f|+|g|)^p \le 2^p \max\{|f|,|g|\}^p \in L^1(\mu)$. Thus by $(*)$,

$$\|f+g\|_p = \frac{\|f+g\|_p}{\|f+g\|_p^{p/q}} \le \|f\|_p + \|g\|_q$$

and the equality situation is described above. ∎

Remark. This implies that $(L^p(\mu), \|\bullet\|_p)$ is a normed space.

9.4 Lemma. *Let $(L, \|\bullet\|)$ be a normed space. Then $(L,\|\bullet\|)$ is a Banach space if and only $\sum_{k=1}^\infty f_k$ converges in L whenever $\sum_{k=1}^\infty \|f_k\| < \infty$ in $\mathbb{R}$.*

Proof ($\Leftarrow$) Let $(f_n)_{n=1}^\infty$ be Cauchy in $(L,\|\bullet\|)$. Then we can find a subsequence $(f_{n_k})_{k=1}^\infty$ such that $\left\|f_{n_{k+1}} - f_{n_k}\right\| < 1/2^k$ for each k. We then use our assumption to let $f = f_{n_1} + \sum_{k=1}^\infty (f_{n_{k+1}} - f_{n_k}) \in L$. Check that $f = \lim f_{n_k}$, so $f = \lim f_n$. ∎

9.5 Theorem. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \le p < \infty$. Then $(L^p(\mu), \|\bullet\|_p)$ is a Banach space.*

Proof We use the lemma. Let $(f_k)_{k=1}^\infty \subset L^p(\mu)$ such that $s = \sum_{k=1}^\infty \|f_k\|_p < \infty$. We think of each f_k as an element of $M(X,\mathcal{M})$. Let for $n \in \mathbb{N}$ $g_n = \sum_{k=1}^n |f_k$ and $g = \sum_{k=1}^\infty |f_k| \in M^+(X,\mathcal{M})$. Now by Minkowski's inequality,

$$\|g_n\|_p \le \sum_P k = 1^n \|f_k\|_p \le s$$

so

$$\|g_n\|^p \le s^p$$

and hence by monotone convergence

$$\int |g|^p = \lim_{n\to\infty} \int |g_n|^p \le s^p < \infty$$

so $|g|^p \in \overline{L}^+(\mu)$. By replacing values on a null set, we may assume $|g|^q \in L^+(\mu)$. Now, set $f(x) = \sum_{k=1} \infty f_k(x)$ for μ–ae. x in X. Then $|f| \le \sum_{k=1}^\infty |f| \le |g|$ which shows that f is finite and thus μ–a.e. equivalent to an element of $M(X,\mathcal{M})$, which we will also call f. Since $|f|^p \le |g|^p$¡ we see that $f \in L^p(\mu)$. Now for each n,

$$\left| f - \sum_{k=1}^{n} f_k \right|^p \le \left(|f| + \sum_{k=1}^{n} |f_k| \right)^p \le |g|^p \in L(\mu)$$

and $\lim_{n\to\infty} \left| f - \sum_{k=1}^{\infty} f_j \right|^p = 0$ μ–a.e. Thus by LDCT, we have

$$\left\| f - \sum_{k=1}^{n} f_k \right\|_p^p = \int \left| f - \sum_{k=1}^{\infty} f_k \right|^p$$

so $f = \sum_{k=1}^{\infty} f_k \in L^p(\mu)$. ■

Definition. Let $(L, \|\cdot\|)$ be a $\mathbb{C}$–normed Banach space. We let its **dual space** be

$$L^* = \{\Phi : L \to \mathbb{C} \mid \phi \text{ linear and } \left\|\phi\right\|_* = \sup\{|\phi(f)| : f \in L, \|f\| \le 1\} < \infty\}$$

Remark. 1. L^* is itself a $\mathbb{C}$–vector space with norm $\|\|_*$:

$$\begin{aligned} \left\|\phi\right\|_* = 0 &\Leftrightarrow |\Phi(f)| = 0 \text{ for all } f \in L, \|f\| \le 1 \\ &\Leftrightarrow \Phi(f) = \|f\| \Phi\left(\frac{1}{\|f\|} f\right) = 0 \text{ for all } f \in L \setminus \{0\} \\ &\Leftrightarrow \Phi = 0 \end{aligned}$$

Linearity and respecting scalars is obvious.

2. If $\Phi \in L^*$, Φ is Lipschitz, hence continuous. Indeed, if $f \in L \setminus \{0\}$, then $|\Phi(f)| = \|f\| \left|\Phi\left(\frac{1}{\|f\|} f\right)\right| \le \left\|\phi\right\|_* \|f\|$ and hence if $f, g \in L$, $|\Phi(f) - \Phi(g)| = |\Phi(f-g)| \le \|\Phi\|_* \|f - g\|$.

9.6 Theorem. *Let $(X, \mathcal{M}, \mu)$ be a measure space, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) For $g \in L^q(\mu)$ we have $\Phi_g \in L^p(\mu)^*$ given by

$$\Phi_g(f) = \int_X f g \, \mathrm{d}\mu$$

satisfies $\left\|\Phi_g\right\|_* = \|g\|_q$

(ii) If $\Phi \in L^p(\mu)^*$, then $\Phi = \Phi_g$ for some $g \in L^q(\mu)$. Hence, $g \mapsto \Phi_g : L^q(\mu) \to L^p(\mu)^*$ is an isometric surjection.

Proof (i) First notice for $f \in L^p(\mu)$,

$$\int |fg| = \|fg\|_1 \le \|f\|_p \|g\|_q$$

so $fg \in L^1(\mu)$, so $\Phi_g(f) = \int fg$ makes sense. Again, we use Hölder's inequality to see for $f \in L^p(\mu)$ with $\|f\|_p \le 1$, we have

$$\left|\Phi_g(f)\right| = \left|\int fg\right| \le \int |fg| = \|fg\|_1 \le \|f\|_p \|g\|_q \le \|g\|_q$$

so $\|\Phi_g\|_* \leq \|g\|_q$. To see the converse inequality, for $g \neq 0$, let

$$f = \frac{1}{\|g\|_q^{q-1}} |g|^{q-1} \overline{\operatorname{sgn} g}$$

Then $\frac{1}{p} = 1 - \frac{1}{q} = \frac{q-1}{q}$, $q = (q-1)p$ and we have

$$\int |f|^p \leq \frac{1}{\|g\|_q^{(q-1)p}} \int |g|^{(q-1)p} = \frac{1}{\|g\|_q^q} \int |g|^q = 1$$

so $\|f\|_p \leq 1$. Thus

$$\begin{aligned} \|\Phi_g\|_* \geq |\Phi_g(h)| &= \left| \frac{1}{\|g\|_q^{q-1}} \int |g|^{q-1} \overline{\operatorname{sgn} g} g \right| \\ &= \frac{1}{\|g\|_q^{q-1}} \int |g|^q = \frac{\|g\|_q^q}{\|g\|^{q-1}} = \|g\|_q \end{aligned}$$

(ii) Let $\Phi \in L^p(\mu)^*$. (I) Suppose that $\mu(X) < \infty$. Let $\nu : \mathcal{M} \to \mathbb{C}$ be $\nu(E) = \Phi(1_E)$. Then $\nu(\emptyset) = \Phi(1_\emptyset) = 0$. If $E_1, E_2, \ldots \in \mathcal{M}$, $E_i \cap E_j = \emptyset$ for $i \neq j$, then $E = \dot{\bigcup}_{i=1}^\infty E_i$ and we have

$$\begin{aligned} \left\| 1_E - \sum_{i=1}^n 1_{E_i} \right\|_p^p &= \int \left| 1_{\bigcup_{i=n+1}^\infty E_i} \right|^p \mathrm{d}\mu \\ &= \mu\left(\bigcup_{i=n+1}^\infty E_i \right) \\ &= \sum_{i=n+1}^\infty \mu(E_i) \end{aligned}$$

which goes to 0 as $n \to \infty$. Thus $1_E = \lim_{n\to\infty} \sum_{i=1}^n 1_{E_i}$ in $L^p(\mu)$. Thus, as Φ is linear and continuous, we have

$$\nu(E) = \Phi(1_E) = \Phi\left(\lim_{n\to\infty} \sum_{i=1}^n 1_{E_i} \right) = \lim_{n\to\infty} \sum_{i=1}^n \Phi(1_{E_i}) \qquad = \lim_{n\to\infty} \sum_{i=1}^n \nu(E_i) = \sum_{i=1}^\infty \nu(E_i)$$

and thus ν is a $\mathbb{C}$−measure. Furthermore, if $E \in \mathcal{M}$ satisfies $\mu(E) = 0$, then $1_E = 0$ μ−a.e, so $\nu(E) = \Phi(1_E) = \Phi(0) = 0$ and $\nu \ll \mu$. Thus the Radon-Nikodym Theorem provides $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ in $L^1(\mu)$ such that $\nu(E) = \int_E g \,\mathrm{d}\mu$.

We now show that $g \in L^q(\mu)$. First, if $f \in M(X, \mathcal{M})/\sim_\mu$ is essentially bounded, then

$$\int |fg| \,\mathrm{d}\mu \leq \int M|g| \,\mathrm{d}\mu = M\|g\| < \infty$$

so $fg \in L^1(\mu)$. We then note that

$$M(g) \geq \sup\{ \left| \int fg \,\mathrm{d}\mu \right| : f \in M(X, \mathcal{M})/\sim_\mu \text{ is essentially bounded and } \|f\|_p \leq 1 \} \quad (*)$$

For f as in $(*)$, we find $(\psi_n)_{n=1}^{\infty} \subset S(X,\mathcal{M})/\sim_\mu$ such that $f = \lim_{n\to\infty}\psi_n$ μ–a.e. and such that $|\psi_n| \le |f|$. Notice for $\phi \in S(X,\mathcal{M})/\sim_\mu$, $\psi = \sum_{j=1}^{n} c_i 1_{E_i}$ in standard form, that

$$\begin{aligned}\Phi(\psi) &= \sum_{j=1}^{m} c_i \Phi(1_{E_j}) = \sum_{j=1}^{n} c_i \nu(E_j)\\ &= \sum_{j=1}^{m} c_j \int_X 1_{E_j} g \,\mathrm{d}\mu = \int \psi g \,\mathrm{d}\mu\end{aligned}$$

Thus, $|\psi_n - f|^p \le (|\psi_n| + |f|)^p \le 2^p|f|^p \in L^1(\mu)$ so by LDCT,

$$\lim_{n\to\infty} \left\|\psi_n - f\right\|_p^p = \lim_{n\to\infty} |\phi_n - f|^p \,\mathrm{d}\mu = 0$$

and $|\psi_n g| = |\psi_n||g| \le |fg| \in L^1(\mu)$. Thus for such f, using continuity of ϕ, and then LDCT,

$$\Phi(f) = \lim_{n\to\infty} \Phi(\psi_n) = \lim_{n\to\infty} \int \psi_n g \,\mathrm{d}\mu = \int fg \,\mathrm{d}\mu$$

Thus we see that $M(g) \le \|\Phi\|_* < \infty$. Now we let $(\varphi_n)_{n=1}^{\infty} \subset S(X,\mathcal{M})/\sim_\mu$ such that $\lim \varphi_n = g$ and $|\varphi_n| \le |\varphi_{n+1}| \le |g|$. We define

$$f_n = \frac{1}{\|\varphi_n\|_q^{q-1}} |\varphi_n|^{q-1} \overline{\operatorname{sgn} g}$$

which is essntially bounded and with $\int |f_n|^p \le 1$ as above. Furthermore, by MCT,

$$\int |g|^q \,\mathrm{d}\mu = \lim_{n\to\infty} \int |\varphi_n|^q \,\mathrm{d}\mu$$

and we compute

$$\begin{aligned}\|g\|_q = \lim_{n\to\infty} \|\varphi_n\|_q &= \lim_{n\to\infty} \frac{1}{\|\varphi_n\|_q^{q-1}} \int |\varphi_n|^q\\ \lim_{n\to\infty} \int |f_n||\varphi_n| &\le \liminf_{n\to\infty} \int |f_n||g| \,\mathrm{d}\mu\\ &= \liminf \int f_n g \,\mathrm{d}\mu \le \|\Phi\|_\infty < \infty\end{aligned}$$

so $g \in L^q(\mu)$. We see that $\Phi = \Phi_g$ by mimicking the same computation as earlier, but for f not necessarily essentially bounded.

(II) Assume now that μ is a general measure. If $E \in \mathcal{M}$, identify $L^p(\mu_E) \cong 1_E L^p(\mu) \subseteq L^p(\mu)$ and likewise for q. If $F \in \mathcal{M}$, $\mu(F) < \infty$, then (I) provides g_G in $1_F L^p(\mu)$ such that $\phi(1_F f) = \int_F f g_F \,\mathrm{d}\mu = \int_X f g_F \,\mathrm{d}\mu$ as $g_F = 1_F g_F$. Notice that if $F \subseteq F'$, where $F' \in \mathcal{M}$, $\mu(F') < \infty$, then $g_F = g_{F'}$ μ_F–a.e. Hence if $F_1, F_2, \ldots \in \mathcal{M}$, each $\mu(F_i) < \infty$, then on $E = \bigcup_{i=1}^{\infty} F_i$, we may uniquely define g_E so $g_E = g_{F_n}$ μ_{F_n}–a.e. and $1_E g_E = g_E$. Let $E_n = \bigcup_{i=1}^{n} F_i$, and MCT and (I) and (i) provide

$$\int |g_E|^q \,\mathrm{d}\mu = \lim_{n\to\infty} \int |g_{E_n}|^q = \lim_{n\to\infty} \left\|\Phi|_{1_{E_n} L^p(\mu)}\right\|_* \le \|\Phi\|_*$$

so that $g_E \in L^q(\mu)$. In fact, $g_E = 1_E L^q(\mu)$. We then let

$$s = \sup\left\{\int |g_E|^q : E \in \mathcal{M} \text{ is } \sigma\text{−finite for } \mu\right\} \le \|\Phi\|_* < \infty$$

Then let $E_1, E_2, \ldots, \in \mathcal{M}$ each be σ−finite for μ, such that $\lim_{n\to\infty} |g_{E_n}|^q = s$. Then $E = \bigcup_{i=1}^\infty E_i$ is σ−finite, and again using MCT,

$$s \ge \int |g_E|^q \, \mathrm{d}\mu = \lim_{n\to\infty} \int |g_{\bigcup_{i=1}^\infty E_i}|^q \, \mathrm{d}\mu \ge \lim_{n\to\infty} \int |g_{E_n}|^q d\mu = s$$

so that $s = \int |g_E|^q = s$. Now if $E' \in \mathcal{M}$ is σ−finite for μ such that

$$s + \int |g_{E'\setminus E}|^q \, \mathrm{d}\mu = \int |g_E|^q \, \mathrm{d}\mu + \int |g_{E\setminus E}|^q \, \mathrm{d}\mu = \int |g_E|^q \, \mathrm{d}\mu \le s$$

and we conclude that $g_{E'\setminus E} = 0$ μ−a.e.

Finally, if $f \in L^p(\mu)$, we think of f as a function and let

$$E_f = \bigcup_{n=1}^\infty \left\{x \in X : |f(x)|^p < \frac{1}{n}\right\}$$

so E_f is σ−finite. Decompose $E_f \cup E = \bigcup_{i=1}^\infty E_i$, each $E_i \in \mathcal{M}$, $\mu(E_i) < \infty$, $E_1 \subseteq E_2 \subseteq \cdots$ and we have

- $\lim_{n\to\infty} \left\|f - 1_{E_n} f\right\|_p = 0$ (LDCT argument we saw in (I))
- $|f g_{E_n}| \le |f g_E| \in L^1(\mu)$

Thus by continuity of Φ, by LDCT and (I),

$$\Phi(f) = \lim_{n\to\infty} \Phi(1_{E_n} f) = \lim_{n\to\infty} \int 1_{E_n} f g_E \, \mathrm{d}\mu = \int f g_E \, \mathrm{d}\mu$$

Hence $\Phi = \Phi_{g_E}$. ■

10 Radon Measures

Definition. Let (X, d) be a metric space. We say that (X, d) is **locally compact** if for each $x \in X$, there is $\epsilon_x > 0$ such that $\overline{B_{\epsilon_x}}(x)$ is compact.

Example. (i) $\mathbb{R}^d$ with the usual metric is locally compact. Any closed ball $\overline{B_\epsilon(x)}$ is cmpact (Heine-Borel)

(ii) Let X be any non-empty set, d the discrete metric. If $x \in X$, then $B_\epsilon(x) = \overline{B_\epsilon(x)}$ is compact, provided that X is infinite, exactly for $0 < \epsilon \le 1$. Note that we distinguish $\overline{B_\epsilon(x)}$ from $\overline{B}_\epsilon(x) = \{y : d(x, y) < \epsilon\}$.

(iii) If C is a closed subset and U an open subset of a locally compact space, then C, U and $C \cap U$, $C \cup U$ are locally compact.

Definition. Let (X, d) be a locally compact metric space. A measure $\mu : \mathcal{B}(x) \to [0, \infty]$ is called a **Radon measure** if it satisfies

- *(outer regularity)* For $E \in \mathcal{B}(X)$, $\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open}\}$.

- *(locally finite)* For $K \subseteq X$ compact, $\mu(K) < \infty$
- *(inner regular on open sets)* If $U \subseteq X$ is open, then $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ is compact}\}$.

10.1 Proposition. *Let μ be a Radon measure, as above. Then if $E \in \mathcal{B}(X)$ such that $\mu(E) < \infty$, then inner regularity holds for E as well. Thus, if X is σ–finite for μ, then μ is inner regular for each $E \in \mathcal{B}(X)$.*

Proof First assume that $\mu(E) < \infty$. Let $\epsilon > 0$. Let
- $E \subseteq U$, U open, $\mu(E) < \mu(E) + \epsilon$ implies $\mu(U \setminus E) < \epsilon$.
- $F \subseteq U$, F compact, $\mu(U) < \mu(F) + \epsilon$, and
- $U \setminus E \subseteq C$, so V is open and $\mu(V) < \epsilon$.

Let $K = F \setminus V = F \cap (X \setminus V) \subseteq F \setminus (U \setminus E) \subseteq F \cap E \subseteq E$ and is compact with

$$\begin{aligned}\mu(K) &= \mu(F) - \mu(F \cap V)\\ &> \mu(U) - \epsilon - \mu(V) > \mu(E) - 2\epsilon\end{aligned}$$

Now, if E is σ–finite for μ, write $E = \bigcup_{i=1}^{\infty} E_i$, each $E_i \in \mathcal{B}(X)$, $\mu(E_i) < \infty$, $E_1 \subseteq E_2 \subseteq \cdots$. For each n, let $K_n \subseteq E_n$ such that $\mu(K_n) \leq \mu(E_n) < \mu(K_n) + 1/n$. Then by continuity from below, $\mu(E) = \lim \mu(E_n) = \lim \mu(K_n)$ so $\mu(E) = \sup_{n\in\mathbb{N}} \mu(K_n)$. ■

Remark. We say that (X,d) is ***σ*-compact** if $X = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact. If μ is a Radon measure, then σ–compact implies σ–finite.

Riesz Representation Theorem. Let (X,d) be a metric space, $I : C_c(X) \to \mathbb{C}$ a positive linear functional. Then there is a unique Radon measure $\mu : \mathcal{B}(X) \to [0,\infty]$ such that $I(f) = \int_X f\,\mathrm{d}\mu$, $f \in C_c(G)$. We let $U \subseteq C$, $\mu^0(U) = \sup\{I(f) : f \prec U\}$, $E \subseteq X$, $\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu^0(E_i) : U \subseteq \bigcup_{i=1}^{\infty}, U_i \in \tau\right\}$.

(III) We have that $\mathcal{B}(X) \subseteq \mathcal{M}$. In particular, $\mu = \mu^*|_{\mathcal{B}(x)}$ satisfies $\mu(U) = \mu^*(U)$ for U open, and μ is outer regular, by (I), and locally finite, by (II). It suffices to show that $U \in \mathcal{M}$ whenever U is open.

Suppose $V \subseteq X$ is open with $\mu^*(V) < \infty$ (say $\overline{V}$ is compact), and let $\epsilon > 0$. We let
- $f \prec U \cap V$ be so $\mu^*(U) \cap V < I(f) + \epsilon$
- $g \prec V \setminus \operatorname{supp} f$ be such $\mu^*(V \setminus \operatorname{supp} f) < I(g) + \epsilon$

Then $f + g \prec V$ as $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$, and we have

$$\begin{aligned}\mu^*(V \cap U) + \mu^*(V \setminus U) &< I(f) + \epsilon + \mu^*(V \setminus \operatorname{supp} f)\\ &< I(f) + I(g) + 2\epsilon\\ &= I(f+g) + 2\epsilon\\ &\leq \mu^0(V) + 2\epsilon = \mu^*(V) + 2\epsilon\end{aligned}$$

so, since $\epsilon > 0$ is arbitrary, $\mu^*(V \cap U) + \mu^*(V \setminus U) \leq \mu^*(V)$. Now, if $E \subseteq X$, $\mu^*(E) < \infty$, for each $\epsilon > 0$ we find open $V \supseteq E$ such that $\mu^*(V) = \mu^0(E) < \mu^*(E) + \epsilon$. Then

$$\mu^*(E) + \epsilon > \mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$$

and since ϵ is arbitrary, $\mu^0(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$. Notice that this also holds immediately if $\mu^*(E) = \infty$.

(IV) $I(f) = \int_X f\,\mathrm{d}\mu$ for f in $C_c(X)$. First, if $f \in C_c(X)$, we may write $f_1 - f_2 + i(f_3 - f_4)$ where $f_i \geq 0$. Let $M_i = \sup\{f_i(x) : x \in X\}$ and we see that each $_i = (M_i + 1)\frac{1}{M_i+1} f_i$, where $0 \leq \frac{1}{M_i+1} f_i \leq 1$. Hence it suffices to establish this for $0 \leq f \leq 1$. Now let $K_0 = \operatorname{supp} f$, for

$j = 1,\dots,n$, let $K_j = f^{-1}\left(\left[\frac{j}{n},1\right]\right)$ so each $K_0,\dots,K_n$ is compact and $K_0 \supseteq K_1 \supseteq \cdots \supseteq K_n$. Then let $f_j = \min\left\{\max\left\{f - \frac{j-1}{n}1, 0\right\}, \frac{1}{n}\right\}$.

Then $f = \sum_{j=1}^n f_j$ and $1_{K_j} \le nf_j \le 1_{K_{j-1}}$, $j = 1,\dots,n$. Hence, taking integrals, we see $\mu(K_j) \le n\int_X f_j\,\mathrm{d}\mu \le \mu(K_{j-1})$, so that

$$\frac{1}{n}\sum_{j=1}^n \mu(K_j) \le \int_X f\,\mathrm{d}\mu \le \frac{1}{n}\sum_{j=1}^n \mu(K_{j-1}) \tag{$*$}$$

On the other hand, we have $K_j \prec nf_j \prec K_{j-1}^\circ$, so using (II), $\mu(K_j) \le nI(f_j) \le \mu(K_{j-1}^\circ) \le \mu(K_{j-1})$. Thus

$$\frac{1}{n}\sum_{j=1}^n \mu(K_j) \le I(f) \le \frac{1}{n}\mu(K_{j-1}) \tag{$\dagger$}$$

Hence, by ($*$) and ($\dagger$), we obtain

$$|I(f) - \int_X f\,\mathrm{d}\mu| \le \frac{1}{n}(\mu(K_0) - \mu(K_1)) \le \frac{1}{n}\mu(K_0)$$

and this holds for any $n \in \mathbb{N}$, so $I(f) = \int_X f\,\mathrm{d}\mu$.

(V) Inner regularity on open sets. Let $U \subseteq X$ be open. Find $(f_n)_{n=1}^\infty \subseteq C_c(X)$, each $f_n \prec U$ so $\lim_{n\to\infty} I(f_n) = \mu^0(U) = \mu(U)$. Let $K_n = \operatorname{supp} f_n \subseteq U$. Then, by (IV),

$$I(f_n) = \int f_n\,\mathrm{d}\mu \le \int 1_{K_n}\,\mathrm{d}\mu = \mu(K_n) \le \mu(U)$$

and hence, by squeeze, $\lim_{n\to\infty}\mu(K_n) = \mu(U)$, i.e. $\mu(U) \le \sup\{\mu(K) : K \subseteq U, K \text{ compact}\}$ where "$\ge$" is obvious.

(VI) Uniqueness. Let μ' be a Radon measure for which $\int f\,\mathrm{d}\mu' = I(f)$ for $f \in C_c(X)$. Then, if U is open and $K \prec f \prec U$, then

$$\mu'(K) = \int_{1_K}\mathrm{d}\mu' \le \int f\,\mathrm{d}\mu' = I(f) = \int f\,\mathrm{d}\mu \le 1_U\,\mathrm{d}\mu = \mu(U)$$

so

$$\sup\{\mu'(K)LK \subseteq U, K \text{ compact}\} \le \sup\{I(f) : f \prec U\} \le \mu'(U)$$

but, by inner regularity of μ' on open sets and definition of $\mu(U) = \mu^0(U)$, we see $\mu'(U) \le \mu(U) \le \mu'(U)$. Thus $\mu' = \mu$ on open sets. Since each is outer regular, hence $\mu' = \mu$ on $\mathcal{B}(X)$.

10.2 Proposition. *Let (X,d) be a locally compact measure space and $\mu : \mathcal{B}(x) \to [0,\infty]$ a Radon measure. Then for $1 \le p < \infty$, we have that $C_c(X)/\sim_\mu$ is dense in $L^p(\mu)$.*

Proof Note that $C_c(X)/\sim_\mu \subseteq L^p(\mu)$ as μ is lcoally finite. If $E \in \mathcal{B}(X)$, $\mu(E) < \infty$, then by inner and outer regularity we can find for any $\epsilon > 0$ and $\mu(E) < \mu(K) + \epsilon/2$, and $\mu(U) < \mu(E) + \epsilon/2$. Thus $\mu(U \setminus K) = \mu(U \setminus E) + \mu(E \setminus K) < \infty$. Then for any $K \prec f \prec U$, we have

$$\|f - 1_E\|_\mu^p = \int |f - 1_E|^p\,\mathrm{d}\mu \le |1_U - 1_K|^p\,\mathrm{d}\mu = \int 1_{U\setminus K}\,\mathrm{d}\mu < \epsilon$$

Thus simple elements of $L^p(\mu)$ are approximated from $C_c(X)/\sim_\mu$, and hence arbitrary elements. ■

10.3 Theorem. *Let (X,d) be a $\sigma-$compact locally compact metric space. Then every locally finite measure $\nu : \mathcal{B}(x) \to [0,\infty]$ (i.e. $\nu(K) < \infty$, K compact) is a Radon measure. In particular, ν is outer regular and inner regular.*

Proof Since ν is locally finite, each $f \in C_c(X)$ is Borel measurable and $\|\leq 1_{\operatorname{supp} f}$, so $f \in L(\mu)$. Since ν is non-negative, $I(f) = \int_X f \, d\nu$ defines a positive linear function on $C_c(X)$. Hence, the Riesz Representation Theorem provides us with a Radon measure μ such that $\int_X f \, d\nu = \int_X f \, d\mu$. Let's show that $\nu = \mu$.

(I) Let $U \subseteq X$ be open. Since X is $\sigma-$compact, write $X = \bigcup_{n=1}^\infty L_n$, each $L_n \subseteq X$ compact and $L_1 \subseteq L_2 \subseteq \cdots$. For each n, let $F_n = \{x \in U : d(x, X \setminus U) \geq 1/n\}$ and let $K_n = L_n \cap F_n \subseteq U$. Since $F_1 \subseteq F_2 \subseteq \cdots$, $K_1 \subseteq K_2 \subseteq \cdots$. Furthermore, if $x \in U$, there is n_1 so that $d(x, X \setminus U) \geq \frac{1}{n}$, and n_2 such that $x \in L_{n_2}$. Thus for $n \geq \max\{n_1, n_2\}$, we have $x \in K_n \cap L_n$. Thus $U = \bigcup_{n=1}^\infty K_n$. Let's choose $(f_n)_{n=1}^\infty \subset C_c(X)$ inductively:

- $K_1 \prec f_1 \prec U$
- $K_2 \cup \operatorname{supp} f_1 \prec f_2 \prec U$
- $K_{n+1} \cup \operatorname{supp} f_n \prec f_{n+1} \prec U$

Thus $f_1 \leq f_2 \leq \cdots$ and $\lim_{n\to\infty} f_n = 1_U$. Thus by MCT, we have

$$\nu(U) = \int 1_U \, d\nu = \lim_{n\to\infty} f_n \, d\nu = \lim_{n\to\infty} \int f_n \, d\mu = \int 1_U \, d\mu = \mu(U)$$

(II) Now let $E \in \mathcal{B}(X)$, $\mu(E) < \infty$. Given $\epsilon > 0$, find $K \subseteq E \subseteq V$, K compact, V open, so that $\mu(E) < \mu(K) + \epsilon/2$ and $\mu(V) < \mu(E) + \epsilon/2$. Hence by (I),

$$\nu(V) - \nu(K) = \nu(V \setminus K) = \mu(V \setminus K) < \epsilon$$

Thus

$$\nu(E) \leq \mu(V) < \nu(K) + \epsilon \leq \nu(E) + \epsilon$$

Thus $\nu(E) = \inf\{\nu(V) : E \subseteq V, V \text{ open}\} = \inf\{\mu(V) : E \subseteq V, V \text{ open}\} = \mu(E)$.

Finally, by (II) and continuity from below, we have

$$\mu(E) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \nu(E_n) = \nu(E) \qquad \blacksquare$$

10.4 Corollary. *If (X,d) is a $\sigma-$compact locally compact metric space, and $\mu : \mathcal{B}(X) \to \mathbb{C}$, then μ is a linear combination of up to 4 finite Radon measures.*

Proof We consider, for example, the Jordan decomposition, $\mu = \mu_1 - \mu_2 + i[\mu_3 - \mu_4]$. Each μ_k is a finite measure, and hence Radon. $\blacksquare$

10.5 Corollary. *The $d-$dimensional Lebesgue measure $\lambda_d : \mathcal{B}(\mathbb{R}^d) \to [0,\infty]$ is inner and outer regular.*

Proof We note that $\mathbb{R}^d = \bigcup_{n=1}^\infty \overline{B_n(0)}$ is $\sigma-$compact. If $K \subseteq \mathbb{R}^d = \bigcup_{n=1}^\infty (-n,n)^d$ is compact, then $K \subseteq (-n_0, n_0)^d$ for some n_0. Hence $\lambda_d(K) \leq \lambda_d((-n_0,n_0)^d) = (2n_0)^d < \infty$. Thus λ_d is a locally finite measure on a $\sigma-$compact space, hence Radon. $\blacksquare$

Remark. If $\emptyset \neq U \subseteq \mathbb{R}^d$ is open, then $\lambda_d(U) > 0$. Indeed, if $x \in U$, find $\epsilon > 0$ such that $\prod_{j=1}^d (x_j - \epsilon, x_j + \epsilon) = B(x, d_\infty) \subseteq U$, and we have $\lambda_d(U) \geq (2\epsilon)^d > 0$.

TODO: dual of L1 is Linfty (for finite measures)

11 Differentiation in $\mathbb{R}^d$

If $f : (a,b) \to \mathbb{C}$ is continuous and bounded (with $\lim_{t\to\infty} f(t) = f(a)$), then for $x \in (a,b)$,

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_a^t f(s)\,\mathrm{d}s\right] = \lim_{r\to 0^+} \frac{1}{2r}\int_{x-r}^{x+r} f(s)\,\mathrm{d}s$$

We shall generalize this so integrable f and $d > 1$.

If $x \in \mathbb{R}^d$, $r > 0$, we let $B_r(x) = \{y \in \mathbb{R}^d : \|x-y\|_2 < r\}$. In fact, we could replace $\|\cdot\|_2$ with any norm on $\mathbb{R}^d$ and the results will remain true as stated.

11.1 Lemma. (Covering) *Let $\mathcal{C}$ be a collection fo Euclidean balls in $\mathbb{R}^d$, $U = \bigcup_{B\in\mathcal{C}} B$. Then for any $0 < c < \lambda_d(U)$, there exist $B_1,\ldots,B_n$ in $\mathcal{C}$ such that $B_i \cap B_j = \emptyset$ for $I \neq j$ and $3^d \sum_{i=1}^n \lambda_d(B_i)$.*

Proof Since $U \neq \emptyset$, there is c a above. By inner regularity, there is $K \subseteq U$ compact such that $\lambda_d(K) > c$. Since $K \subseteq U = \bigcup_{B\in\mathcal{C}} B$, there is $B_1',\ldots,B_m'$ in $\mathcal{C}$ such that $K \subseteq \bigcup_{j=1}^m B_j'$. Write each $B_j' = B_{r_j'}(x_j')$, we may relabel $r_1' \geq \cdots \geq r_m'$. Then

- $B_1 = B_1'$
- $B_2 = B_{j_2}'$ where $j_2 = \min\{j \in [m] : B_j' \cap B_1 = \emptyset\}$.
- $B_n = B_{j_n}'$ where $j_n = \min\{j \in \{j_{n+1}+1,\ldots,m\} : B_j' \cap \bigcup_{j=1}^{n-1} B_i\}$

where n is determined by where this process stops. If $B_j' \notin \{B_1,\ldots,B_n\}$, then $B_j' \cap B_i = B_{j_i}'$ for some $j_i < j$, ro $r_i := r_{j_i}' \geq r_j'$. If we write $B_i = B_{r_i}(x_i)$, then $B_j' \subseteq B_{3r_i(x_i)}$. Notice that

$$\lambda_d(B_{3r_i}(x_i) = \lambda_d(3I(B_{r_i}(0)) + x_i) = 3^d \lambda_d(B_{r_i(x_i)})$$

Thus

$$c < \lambda_d(K) \leq \lambda_d\left(\bigcup_{j=1}^n B_j'\right) \leq \lambda_d\left(\bigcup_{j=1}^n B_{3r_i}(x_i)\right)$$

$$\leq \sum_{i=1}^n \lambda_d(B_{3r_i}(x_i)) = 3^d \sum_{i=1}^n \lambda_d(B_i) \qquad \blacksquare$$

Definition. If $f \in L(\lambda_d)$, we let $A_r f(x) = \frac{1}{\lambda_d(B_r(x))}\int_{B_r(x)} f(y)\,\mathrm{d}y$ denote the "average value", for $r > 0$, $x \in \mathbb{R}^d$. We let the **Hardy-Littlewood maximal functions**

$$Hf(x) = \sup_{r>0} A_r|f|(x)$$

Remark. (i) $(r,x) \mapsto A_r f(x) : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ is continuous. FIrst, as above, $\lambda_d(B_r(x)) = \lambda_d(rI(B_1(0))) = r^d\lambda_d(B_r(0))$. Second, if $((r_n,x_n))_{n=1}^\infty$ with $\lim_{n\to\infty}(r_n,x_n) = (r,x)$, then $1_{B_{r_n}(x_n)}|f| \leq |f|$ and $|\lim_{n\to\infty} 1_{B_{r_n}}(x_n) f = f|$ pointwise. Hece by LDC,

$$A_{r_n} f(x) = \frac{1}{r_n^d \lambda_d(B_1(0))}\int 1_{B_{r_n}(x_n)} f \overset{n\to\infty}{\longrightarrow} \frac{\int 1_{B_r(x)} f}{r^d \lambda_d(B_1(0))} = A_r f(x)$$

(ii) $Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r\in(0,\infty)\cap\mathbb{Q}} A_r|f|(x)$ so Hf is the supremum of a countable family of continuous functions and hence Borel measurable.

(iii) We may define $A_r f$ and hence Hf for f in

$$L_{loc}(\lambda_d) = \{f \in M(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) : 1_K f \in L(\lambda_d) \text{ for any compact } K \subset \mathbb{R}^d\}$$

11.2 Theorem. (Hardy Littlewood Maximal) *If $f \in L(\lambda_d)$ and $\alpha > 0$, then*

$$\lambda_d\left(Hf^{-1}((\alpha,\infty)]\right) \le \frac{3^d}{\alpha}\int_{\mathbb{R}^d} |f| \,\mathrm{d}\lambda$$

Proof Let $E_\alpha = Hf^{-1}((\alpha,\infty])$. Then for each $x \in E_\alpha$, $Hf(x) > \alpha$ so there is $r_x > 0$ such that $A_{r_x}|f|(x) > \alpha$. Now, $E_\alpha \subseteq \bigcup_{x\in E_\alpha} B_{r_x}(x) = U$, so if $0 < \lambda_d(E_\alpha)$ and $0 < c < \lambda_d(E_\alpha) \le \lambda_d(U)$, the last lemma provides $x_1,\dots,x_n \in E_\alpha$ with $B_i = B_{r_{x_i}}(x_i)$ for $i = 1,\dots,n$ such that $B_i \cap B_j = \emptyset$ and $c < 3^d \sum_{i=1}^n \lambda_d(B_i)$. Then for each i,

$$\frac{1}{\lambda_d(B_i)}\int_{B_i} |f| = A_{r_{x_i}}(x_i) > \alpha \quad \Rightarrow \quad \frac{1}{\alpha}\int_{B_i} |f| > \lambda_d(B_i)$$

and hence

$$c < 3^d \sum_{i=1}^n \lambda_d(B_i) < \frac{3^d}{\alpha}\sum_{i=1}^n \int_{B_i} |f| = \frac{3^d}{\alpha}\int_{\bigcup_{i=1}^n B_i} |f| \le \frac{3^d}{\alpha}\int |f| \qquad \blacksquare$$

11.3 Corollary. *If $f \in \overline{M}^+(X,\mathcal{M})$ and $\mu : \mathcal{M} \to [0,\infty]$ is a measure, and $\alpha > 0$, then*

$$\int_{f^{-1}((\alpha,\infty])} f \,\mathrm{d}\mu \ge \int_{f^{-1}((\alpha,\infty])\alpha 1\{\mu} = \alpha\mu(f^{-1}((\alpha,\infty])$$

so that

$$\frac{1}{\alpha}\int_{f^{-1}}((\alpha,\infty])f \,\mathrm{d}\mu \ge \mu(f^{-1}((\alpha,\infty])$$

11.4 Theorem. (First Differentiation) *If $f \in L_{loc}(\lambda_d)$, then $\lim_{r\to 0^+} A_r f(x) = f(x)$ for λ_d–a.e. in $\mathbb{R}^d$.*

Proof Since $\mathbb{R}^d = \bigcup_{N=1}^\infty B_N(0)$, it suffices to prove this result for $1_{B_N(x)}f$. Hence we may assume $f \in L(\lambda)$. Given $\epsilon > 0$, since λ_d is a Radon measure, there is $h \in C_c(\mathbb{R}^d)$ such that $\int |h - f| < \epsilon$. Notice that

$$\begin{aligned}
|A_r h(x) - h(x)| &= \left|\frac{1}{\lambda_d(B_r(x))}\int_{B_r(x))} (h(y) - h(x)) \,\mathrm{d}\lambda\right| \\
&= \frac{1}{\lambda_d(B_r(x))}\int_{B_r(x)} |h(y) - h(x)| \,\mathrm{d}y \\
&\le \sup_{y\in B_r(x)} |h(y) - h(x)| \to 0
\end{aligned}$$

as $r \to 0^+$. Thus

$$\begin{aligned}
\limsup_{r\to 0^+} |A_r f(x) - f(x)| &\le \limsup_{r\to 0^+} [|A_r f(x) - A_r h(x)| + |A_r h(x)| + |A_r h(x) - h(x)| + |h(x) - f(x)|] \\
&\le \lim_{r\to 0} \sup_{r'\in(0,r)} [A_r|f - h|(x) + |h(x) - f(x)|] \\
&\le H(f - h)(x) + |f(x) - h(x)|
\end{aligned}$$

Given $\delta > 0$, let $E_\delta = \{x \in \mathbb{R}^d : \limsup_{r\to 0^+} |A_r f(x) - f(x)| > \delta\}$. Then

$$E_\delta \subseteq \left\{x \in \mathbb{R}^d : H(f-x)(x) > \frac{\delta}{2}\right\} \cup \left\{x \in \mathbb{R}^d : |f(x) - f(x)| > \frac{\delta}{2}\right\}$$

so by the Hardy-Littlewood maximal theorem and Chebeshev's inequality,

$$\begin{aligned}\lambda_d(E_\delta) &\le \lambda_d(H(f-h)^{-1}((\delta/2,\infty])) + \lambda_d(|h-f|^{-1}((\delta/2,\infty])) \\ &\le \frac{2\cdot 3^d}{\delta}\int |f-h| + \frac{2}{\delta}\int_{|f-h|^{-1}((\lambda/2,\infty])} |f-h| \\ &< \frac{2\cdot 3^d + 2}{\delta}\epsilon\end{aligned}$$

Then, since $\epsilon > 0$ is arbitrary, $\lambda_d(E_\delta) = 0$. Then for $x \in \mathbb{R}^d \setminus \bigcup_{n=1}^\infty E_{1/n}$, we have $\lim_{r\to 0^+} |A_r f(x) - f(x)| = 0$. ■

11.5 Corollary. *For $f \in L_{loc}(\lambda_d)$, we define its **Lebesgue set** to be*

$$L_f = \left\{x \in \mathbb{R}^d : \lim_{r\to 0^+} \frac{1}{\lambda_d(B_r(x))}\int_{B_r(x)} |f(y) - f(x)|\,\mathrm{d}y = 0\right\}$$

Then $\lambda_d^*(\mathbb{R}^D \setminus L_f) = 0$, where λ_d^* is tthe outer measure associated to λ_d.

Proof Let $\overline{\{c_n\}_{n=1}}^\infty = \mathbb{C}$. Let

$$E_n = \left\{x \in \mathbb{R}^d : \limsup_{r\to 0^+} |A_r|f - c_n 1|f(x) - |f(x) - c_n|| > 0\right\}$$

so E_n is a λ_d−null set, and $E = \bigcup_{n=1}^\infty E_n$ is also null. If $x \in \mathbb{R}^d \setminus E$ and $\epsilon > 0$, then $|f(x) - c_n| < \epsilon$ for some n. Thus for any $y \in \mathbb{R}^d$,

$$|f(y) - f(x)| \le |f(y) - c_n| + |c_n - f(x)| < |f(y) - c_n| + \epsilon$$

Thus, as $x \notin E_n$,

$$\begin{aligned}\frac{1}{\lambda_r(B_r(x))}\int_{B_r(x)} |f(y) - f(x)|\,\mathrm{d}y &\le \frac{1}{\lambda_d(B_r(\lambda))}\int_{B_r(x)} (|f(y) - c_n| + \epsilon)\,\mathrm{d}y) \\ &= \frac{1}{\lambda_d(B_r(x))}\int_{B_r(x)} |f(y) - c_n 1(y)|\,\mathrm{d}y + \epsilon \\ &\xrightarrow{r\to 0^+} |f(x) - c_n| + \epsilon < 2\epsilon\end{aligned}$$

Thus as $\epsilon > 0$ is arbitrary, the limit

$$\lim_{r\to 0^+} \frac{1}{\lambda_d(B_r(x))}\int_{B_r(x)} |f(y) - f(x)|\,\mathrm{d}y = 0$$

for $x \in E$. We have $\mathbb{R}^d \setminus E \subseteq L_f$, so $\mathbb{R}^d \setminus L_f \subseteq E$. ■

11.6 Theorem. *Let $\mu : \mathcal{B}(\mathbb{R}^d) \to [0,\infty]$ be a locally finite measure such that $\mu \perp \lambda_d$. Then*

$$\lim_{r\to 0^+} \frac{\mu(B_r(x))}{\lambda_d(B_r(x))} = 0$$

for λ_d−a.e. x.

Proof Let (E,F) be a Borel partition of $\mathbb{R}^d$ such that $\mu(F)=0=\lambda_d(E)$. For $\delta>0$, let

$$F_\delta=\left\{x\in F:\limsup_{r\to 0^+}\frac{\mu(B_r(x))}{\lambda_d(B_r(x))}>\delta\right\}$$

Since μ is a Radon measure, given $\epsilon>0$, there is open $U\supseteq F$ such that $\mu(E)\epsilon$. If $x\in F_\delta\subseteq F\subseteq U$, there is $r_x>0$ be so that

$$B_x:=B_{r_x(x)}\subseteq U \text{ and } \frac{\mu(B_x)}{\lambda_d(B_x)}\geq\delta$$

Then $F_\delta\subseteq\bigcup_{x\in F_\delta}B_x:=V\subseteq U$ and given $0<f<\lambda_d(V)$, we may find $B_{x_1},\dots,B_{x_n},x_1,\dots,x_n\in F_\delta$ such that

$$B_{x_i}\cap B_{x_j}=\emptyset \text{ and } c<3^d\sum_{i=1}^n\lambda_d(B_{x_i})$$

Thus,

$$\begin{aligned}c<e^d\sum_{i=1}^n\lambda_d(B_{x_i})\leq\frac{3^d}{\delta}\sum_{i=1}^n\mu(B_{x_i})=\frac{3^d}{\delta}\sum_{i=1}^n\mu\left(\bigcup_{i=1}^n B_{x_i}\right)&\\ \leq\frac{3^d}{\delta}\mu(V)\leq\frac{3^d}{\delta}\mu(U)<\frac{3^d}{\delta}\epsilon&\end{aligned}$$

But then we have

$$\lambda_d^*(F_\delta)\leq\lambda_d(V)=\lim_{c\to\lambda_d(V)^-}c\leq\frac{3^d}{\delta}\epsilon$$

since $\epsilon>0$ is arbitrary, we see that $\lambda_d^*(F_\delta)=0$. Hence, if $x\in\mathbb{R}^d\setminus\bigcup_{k=1}^\infty F_{1/k}$, then

$$\lim_{r\to 0^+}\frac{\mu(B_r(x))}{\lambda_d(B_r(x))}=0$$

■

Definition. A collection of sets $\{E_r(x):x\in\mathbb{R}^d,r>0\}\subseteq\mathcal{B}(\mathbb{R}^d)$ is called **nicely shrinking** if for each $x\in\mathbb{R}^d$, $r>0$,

- $E_r(x)\subseteq B_r(x)$
- $\lambda_d(E_r(x))>\alpha\lambda_d(B_r(x))$, where α is a fixed constant.

11.7 Corollary. *Let $\nu:\mathcal{B}(\mathbb{R}^d)\to\mathbb{C}$ be a complex measure with Lebesgue-Radon-Nikodym decomposition*

$$\nu=\rho+f\cdot\lambda_d,\rho\perp\lambda_d,f\in L(\lambda_d)$$

Then for any nicely shrinking family $\{E_r(x):x\in\mathbb{R}^d,r>0\}$, we have

$$\lim_{r\to 0^+}\frac{\nu(E_r(x))}{\lambda_d(E_r(x))}=f(x)$$

for λ_d–a.e. x in $\mathbb{R}^d$.

Proof Write $\rho=\operatorname{Re}\rho^+-\operatorname{Re}\rho^-+i[\operatorname{Im}\rho^+-\operatorname{Im}\rho^-]$, $\operatorname{Re}\rho^+,\dots,\operatorname{Im}\rho^-\leq|\rho|\leq\operatorname{Re}\rho^++\cdots+\operatorname{Im}\rho^-$. Thus each $\operatorname{Re}\rho^+,\dots,\operatorname{Im}\rho^-\perp\lambda_d$. By Differentiation Theorem II, we see that

$$\lim_{r\to 0^+}\frac{\mu(E_r(x))}{\lambda_d(E_r(x))}\leq\lim_{r\to 0^+}\frac{\mu(B_r(x))}{\alpha\lambda_d(B_r(x))}=0$$

λ_d–a.e. Hence we conclude the same for ρ. On the other hand,

$$\left|\frac{1}{\lambda_d(E_r(x))}\int_{E_r(x)} f(x)\,\mathrm{d}y - f(x)\right| \le \frac{1}{\lambda_d(E_r(x))}\int_{E_r(x)} |f(y)-f(x)|\,\mathrm{d}y \le \frac{1}{\alpha\lambda_d(B_r(x))}\int_{B_r(x)} |f(y)-f(x)|\,\mathrm{d}y \xrightarrow{r\to 0^+} 0$$

provided that $x \in L_f$. ■

11.8 Proposition. *If $F \in \mathrm{ND}_r(\mathbb{R})$, then $F'(x)$ exists for λ–a.e. x in $\mathbb{R}$.*

Proof If $h \neq 0$, then

$$\frac{F(x+h)-F(x)}{h} = \begin{cases} \frac{\mu_F((x,x+h])}{\lambda_d((x,x+h])} & : h > 0 \\ \frac{\mu_F((x+h,x])}{\lambda_d((x+h,x])} & : h < 0 \end{cases}$$

Since each family $\{(x,x+h] : x \in \mathbb{R}, h > 0\}$ and $\{(x-h,x] : x \in \mathbb{R}, h > 0\}$ is nicely shrinking, we see that

$$\lim_{h\to 0^+}\frac{F(x+h)-F(x)}{h}, \lim_{h\to 0^-}\frac{F(x+h)-F(x)}{h}$$

converge for λ–a.e. x, s right and left limits both exist for such x. However, each is λ–a.e. equal to $\frac{\mathrm{d}\mu_F}{\mathrm{d}\lambda}$, thanks to the last corollary. Hence F' exists λ–a.e. ■

Example. Consider the Cantor ternary function $\phi \in \mathrm{ND}_r(\mathbb{R})$. It is easy to see that $\phi'(x) = 0$ whenever $x \in \mathbb{R}\backslash C$.

Definition. Let $F : \mathbb{R} \to \mathbb{C}$. If $a < b$ in $\mathbb{R}$, we define the **variation** of F on $[a,b]$ by

$$V_F[a,b] = \sup\left\{\sum_{i=1}^{n} |F(a_i) - F(a_{i-1})| : a = a_0 < a_1 < \cdots < a_n = b, n \in \mathbb{N}\right\}$$

Example. Consider $F(x) = x\sin(1/x)$ for $x > 0$, and 0 when $x = 0$. Then $V_F[0,\epsilon] = \infty$ for $\epsilon > 0$.

11.9 Proposition. *(i) If $a < b < c$, then $V_F[a,c] = V_F[a,b] + V_F[b,c]$.*
(ii) If $a' \le a \le b \le b'$, then $V_F[a,b] \le V_F[a',b']$

Definition. Define $V_F(a,b] = \lim_{x\to a V_F[x,b]}$ and $V_F(-\infty,b] = \lim_{x\to-\infty} V_F[x,b]$.

11.10 Proposition. *(i) If F is right continuous at a and $V_F[a,b] < \infty$, then $V_F(a,b] = V_F[a,b]$.*
(ii) If $V_F(-\infty,b] < \infty$, then $\lim_{x\to-\infty}(-\infty,x] = 0$.

Proof (i) Certainly $V_F(a,b] \le V_F[a,b]$. To see the converse inequality, given $\epsilon > 0$, let $\delta > 0$ be such that $a < x < a+\delta$ so $|F(x) - F(a)| < \epsilon$. Now we let $a < a_0 < \cdots < a_n = b$ be so

- $\sum_{i=1}^{n} |F(a_i) - F(a_{i-1})| > V_F[a,b] - \epsilon$
- $a < a+1 < a+\delta$

Then

$$\begin{aligned} V_f[a,b] &< |F(a_1) - F(a_0)| + \sum_{i=2}^{n} |F(a_i) - F(a_{i-1})| + \epsilon \\ &< \epsilon + V_F[a_1,b] + \epsilon \le V_F(a,b] + 2\epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $V_F[a,b] \le V_F(a,b]$.

(ii) For fixed $x < b$, then by (A)

$$\begin{aligned} V_F(-\infty, b] &= \lim_{y\to-\infty} V_F[y,b] \\ &= \lim_{y\to-\infty, y<x} (V_F[y,x] + V_F[x,b]) \\ &= V_F(-\infty, x] + V_F[x,b] \end{aligned}$$

Then take $x \to -\infty$.

Definition. If $V_F(-\infty, x] < \infty$ for each $x \in \mathbb{R}$, we define the **total variation** function of F by $T_F(x) = V_F(-\infty, x] \in [0, \infty)$. If $\sup_{x\in\mathbb{R}} T_F(x) < \infty$, we say that F is of **bounded variation**. Write $F \in \mathrm{BV}(\mathbb{R})$. We further let

$$\mathrm{BV}_r(\mathbb{R}) = \{F \in \mathrm{BV}(\mathbb{R}) : F \text{ is right continuous}\}$$

Remark. (i) It follows (ii) that $T_F(-\infty) = \lim_{x\to-\infty} T_F(x) = 0$.

(ii) If $F \in \mathrm{BV}_r(\mathbb{R})$, then T_F is right continuous. Let $a < x < b$, and we use $(*)$, (A), and part (i) of the last proposition to see that

$$\begin{aligned} T_F(x) - T_F(a) = V_F[a,x] &= V_F[a,b] - V_F[x,b] \\ &= V_F(a,b] - V_F[x,b] \to 0 \end{aligned}$$

so $\lim_{x\to a^+} T_F(x) = T_F(a)$.

11.11 Proposition. *(i)* $F \in \mathrm{BV}(\mathbb{R})$ *if and only if* $\mathrm{Re}\, F$, $\mathrm{Im}\, F \in \mathrm{BV}(\mathbb{R})$

(ii) If $G \in \mathrm{BV}^{\mathbb{R}}(\mathbb{R})$, *then each of* $T_F \pm F$ *is non-decreasing.*

(iii) If $F \in \mathrm{BV}(\mathbb{R})$, *we let*

$$F_1 = \frac{1}{2}(T_{\mathrm{Re}\, F} + \mathrm{Re}\, F), \qquad F_2 = \frac{1}{2}(T_{\mathrm{Re}\, F} - \mathrm{Re}\, F)$$
$$F_3 = \frac{1}{2}(T_{\mathrm{Im}\, F} + \mathrm{Im}\, F), \qquad F_4 = \frac{1}{2}(T_{\mathrm{Im}\, F} - \mathrm{Im}\, F)$$

Then $F = F_1 - F_2 + i[F_3 - F_4]$. *Thus,* F *is bounded and* $F(\pm\infty) = \lim_{x\to\pm\infty} F(x)$ *exists.*

Proof (i) If $x < y$ in $\mathbb{R}$, then by using definitions of V_H, $H = F, \mathrm{Re}\, F, \mathrm{Im}\, F$, we see

$$V_{\mathrm{Re}\, F}[x,y], V_{\mathrm{Im}\, F}[x,y] \le V_F[x,y] \le V_{\mathrm{Re}\, F}[x,y] + V_{\mathrm{Im}\, F}[x,y]$$

Taking $x \to -\infty$, we see that

$$T_{\mathrm{Re}\, F}(x), T_{\mathrm{Im}\, F}(y \le T_F(y) \le T_{\mathrm{Re}\, F}(y) + T_{\mathrm{Im}\, F}(y)$$

and then taking $y \to \infty$ does the job.

(ii) If $x < y \in \mathbb{R}$, then

$$\begin{aligned} (T_G \pm G)(y) - (T_G \pm G)(x) &= T_G(y) - T_G(x) \pm [G(y) - G(x)] \\ &= V_G[x,y] + [G(y) - G(x)] \ge |G(y) - F(x)| \pm [G(y) - F(x)] \ge 0 \end{aligned}$$

Furthermore, $T_G(\pm\infty)$ always exists...

(iii) Obvious ∎

Remark. If F above is right continuous, so too are $\mathrm{Re}\, F$, $\mathrm{Im}\, F$, and ence F_1, F_2, F_3, F_4. If $F : \mathbb{R} \to \mathbb{R}$ is bounded, then $F \in \mathrm{BV}^{\mathbb{R}}(\mathbb{R})$.

11.12 Theorem. (Complex Borel Measures on $\mathbb{R}$) *Let $F \in \mathrm{BV}_r(\mathbb{R})$.*

(i) There is a complex measure $\mu_F : \mathcal{B}(\mathbb{R}) \to \mathbb{C}$ such that

$$\mu_F((a,b]) = F(b) - F(a) \text{ for } a < b \text{ in } \mathbb{R} \tag{†}$$

(ii) If $G \in \mathrm{BV}_r^{\mathbb{R}}(\mathbb{R})$ (real-valued), then |

(iii) Proof (i) Let $F = F_1 - F_2 + i[F_3 - F_4]$. Then each $F_K \in \mathrm{ND}_r(\mathbb{R})$ and corresponds to a measure μ_{F_k} satisfying the analogue of (†). Let $\mu_K = \mu_{F_1} - \mu_{F_2} + i[\mu_{F_3} - \mu_{F_4}]$.

(ii) Let $a < b$ in $\mathbb{R}$. we recall that

- $|\mu_G|((a,b]) = \sup\left\{\sum_{i=1}^{n} |\mu_G(E_i)| : \{E_1,\ldots,E_n\} \text{ is a Borel partition of } (a,b], n \in \mathbb{N}\right\}$
- $\mu_{T_G}((a,b]) = T_G(b) - T_G(a) = V_G[a,b] = \sup\left\{\sum_{i=1}^n |G(a_i) - G(a_{i-1})| : (a,b] = \dot\bigcup_{i=1}^n (a_{i-1}, a_i], n \in \mathbb{N}\right\}$

Hence, it is immediate that $\mu_{T_G}((a,b]) \le |\mu_G((a,b])$.

Now, $\left|\mu_G((a,b])\right| = |G(b) - G(a)| \le V_G[a,b] = T_G(b) - T_G(a) = \mu_{T_G}((a,b])$. We let $\mathcal{H} = \{(c,d] : a \le c < d \le b\}$ and for any $A \in \langle\mathcal{H}\rangle \subseteq \mathcal{P}((a,b])$, we have $A = \dot\bigcup_{i=1}^n (c_i, d_i]$ and hence we have

$$\begin{aligned} |\mu_G(A)| = \left|\sum_{i=1}^n \mu_G((c_i,d_i]))\right| &\le \sum_{i=1}^n |\mu_G((c_i,d_i])| \\ &\le \sum_{i=1}^n \mu_{T_G}((c_i,d_i]) = \mu_{T_G}(A) \end{aligned}$$

We let $\mathcal{C} = \{E \in \mathcal{B}((a,b]) : |\mu_G(E)| \le \mu_{T_G}(E)\}$. Then

- $\langle\mathcal{H}\rangle \subseteq \mathcal{C}$
- If $E_1 \supseteq E_2 \supseteq \cdots$ in $\mathcal{C}$, then by continuity from above,

$$\left|\mu_G\left(\bigcap_{n=1}^\infty E_n\right)\right| = \lim_{n\to\infty} \left|\mu_G(E_n)\right| \le \lim_{n\to\infty} \mu_{T_G}$$

- If $E_1 \subseteq E_2 \subseteq \cdots$ in $\mathcal{C}$, then by continuity from below,

$$\left|\mu_G\left(\bigcup_{n=1}^\infty E_n\right)\right| \le \mu_{T_G}\left(\bigcup_{n=1}^\infty E_n\right)$$

Thus by the Monotone Class Lemma, $C \supseteq \sigma\langle\mathcal{H}\rangle = \mathcal{B}((a,b])$, so $\mathcal{C} = \mathcal{B}((a,b])$. Thus, for any Borel partition $\{E_1,\ldots,E_n\}$ of $(a,b]$, we have

$$\sum_{i=1}^n \left|\mu_G(E_i)\right| \le \sum_{i=1}^n \mu_{T_G}(E_i) = \mu_{T_G}\left(\bigcup_{i=1}^n E_i\right) = \mu_{T_G}((a,b])$$

Thus, $|\mu_G|((a,b]) \le \mu_{T_G}((a,b])$. In conclusion, $|\mu_G|((a,b]) = \mu_{T_G}((a,b])$ and hence, by characterization of (locally) finite Borel measures on $\mathbb{R}$, $|\mu_G| = \mu_{T_G}$.

We have

$$\mu_G^{\pm} = \frac{1}{2}(|\mu_G| \pm \mu_G) = \frac{1}{2}(\mu_{T_G} \pm \mu_G) = \mu_{\frac{1}{2}(T_G \pm G)}$$

(iii) If ν satisfies (††), then we see for $a < b$ in $\mathbb{R}$ that

$$\operatorname{Re}\nu\nu((a,b]) = \operatorname{Re}F(b) - \operatorname{Re}F(a) = \mu_{\operatorname{Re}F}((a,b])$$

By (i), $\operatorname{Re}\nu$, $\mu_{\operatorname{Re}F}$ admit the same Jordan decompsition at least on intervals of the form $(a,b]$. Hence, by uniqueness for measures, $\operatorname{Re}\nu = \mu_{\operatorname{Re}F}$. Likewise, $\operatorname{Im}\nu = \mu_{\operatorname{Im}F}$. ■

Definition. If $F : \mathbb{R} \to \mathbb{C}$ is **absolutely continuous**, write $F \in (\mathbb{R})$, provided: given $\epsilon > 0$, there is $\delta > 0$ such that $a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_n \le b_n$ such that $\sum_{i=1}^n (b_i - a_i) < \delta$, we have $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.

Note that Lipschitz$\Rightarrow$Absolutely continuous$\Rightarrow$uniformly continuous$\Rightarrow$ continuous.

11.13 Proposition. *If $F \in \mathrm{BV}(\mathbb{R}) \cap (\mathbb{R})$, then $T_F \in (\mathbb{R})$.*

Proof Given $\epsilon > 0$, find $\delta > 0$ as in absolute continuity, with $a_i < b_i$. Then as $F \in \mathrm{BV}(\mathbb{R})$, for each $i = 1, \ldots, n$, we find $a_i = t_{i,0} < \cdots < t_{i,m_i} = b_i$ be so

$$\sum_{j=1}^{m_i} |F(t_{i,j}) - F(t_{i,j-1})| > V_F[a_i, b_i] - \epsilon/2^i$$

Then

$$\sum_{i=1}^{n} |T_F(b_i) - T_F(a_i)| = \sum_{i=1}^{n} V_F[a_i, b_i] < \sum_{i=1}^{n} \left(\sum_{j=1}^{m_i} |F(t_{i,j}) - F(t_{i,j-1})| + \frac{\epsilon}{2^i} \right) < 2\epsilon$$

since $\sum_{i=1}^n \sum_{j=1}^{m_i} (t_{i,j} - t_{i,j-1}) = \sum_{i=1}^n (b_i - a_i) < \delta$. ■

11.14 Theorem. (Fundamental Theorem of Calculus) *(i) If $F \in \mathrm{BV}(\mathbb{R}) \cap (\mathbb{R}) \subseteq \mathrm{BV}_r(\mathbb{R})$, then $\mu_F \ll \lambda$.*

(ii) If $f \in L(\lambda)$, then $F(x) = \int_{-\infty}^x f(t)\,\mathrm{d}\lambda(t)$ satisfies $F \in \mathrm{BV}(\mathbb{R}) \cap (\mathbb{R})$.

Proof (i) By Jordan decomposition of F, it suffices to show this for $F \in (\mathbb{R}) \cap \mathrm{ND}(\mathbb{R})$. Let $E \in \mathcal{B}(\mathbb{R})$ be so $\lambda(E) > 0$. Given $\epsilon > 0$, let $\delta > 0$ be as in the definition of absolute continuity. Let $\{(a_i, b_i]\}_{i=1}^\infty$ be so $E \subset \bigcup_{i=1}^\infty (a_i, b_i]$ and $\sum_{i=1}^\infty (b_i - a_i) = \sum_{i=1}^\infty \lambda((a_i, b_i]) < \delta$. Find a sequence $\{(a_i', b_i']\}_{i=1}^\infty$ be such that there are $m_1 < m_2 < \cdots$ such that

$$\bigcup_{i=1}^{n} (a_i, b_i] = \dot{\bigcup_{i=1}^{m_n}} (a_i', b_i'], \qquad (a_i', b_i'] \cap (a_j', b_j'] = \emptyset \text{ if } i \ne j$$

Then for each n, $\sum_{i=1}^{m_n} (b_i' - a_i') \le \sum_{i=1}^n (b_i - a_i) < \delta$ so

$$\begin{aligned} \mu_F(E) \le \mu_F\left(\bigcup_{i=1}^\infty (a_i, b_i] \right) &= \lim_{n\to\infty} \mu_F\left(\bigcup_{i=1}^n (a_i, b_i] \right) \\ &= \lim_{n\to\infty} \mu_F\left(\dot{\bigcup_{i=1}^{m_n}} (a_i', b_i'] \right) \\ &= \lim_{n\to\infty} \sum_{i=1}^n \left[F(b_i') - F(a_i') \right] \le \epsilon \end{aligned}$$

as $\epsilon > 0$, we conclude that $\mu_F(E) = 0$.

(ii) Write $f = \mathrm{Re} f^+ - \mathrm{Re} f^- + i[\mathrm{Im} f^+ - \mathrm{Im} f^-]$ so

$$F(x) = f \cdot \mu((-\infty, x]) = \mathrm{Re} f^+ \cdot \mu((-\infty,]) - \cdots + i \,\mathrm{Im} f^+ \cdot \mu((-\infty, x])$$

is a linear combination of 4 non-decreasing bounded functions. Thus $F \in \mathrm{BV}(\mathbb{R})$.

We recall a proposition proven prior; since $|f| \cdot \lambda \ll \lambda$, the alternate characterization of absolute continuity applies. Hence if $a \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_n \leq b_n$ in $\mathbb{R}$ with

$$\lambda\left(\bigcup_{i=1}^{n}(a_i, b_i]\right) = \sum_{i=1}^{n}(b_i - a_i) < \delta$$

then

$$\begin{aligned}\sum_{i=1}^{n}|F(b_i) - F(a_i)| &= \sum_{i=1}^{n}\left|\int_{(a_i,b_i]} f \,\mathrm{d}\lambda\right| \\ &\leq \sum_{i=1}^{n}\int_{(a_i,b_i]}|f|\,\mathrm{d}\lambda = |f| \cdot \lambda\left(\bigcup_{i=1}^{n}(a_i, b_i])\right) < \epsilon\end{aligned}$$

Hence, $F \in (\mathbb{R})$. ∎

Remark. $F \in \mathrm{BV}(\mathbb{R}) \cap (\mathbb{R})$ if and only if there is $f \in L(\lambda)$ such that $F' = f$ λ–a.e., and $F(x) = \int_{-\infty}^{x} f \,\mathrm{d}\lambda$. Indeed, we saw earlier that $F \in \mathrm{BV}_r(\mathbb{R})$ is λ–a.e. differentiable. Since $F \in \mathrm{BV}(\mathbb{R}) \cap (\mathbb{R})$, $\mu_F \ll \lambda$ implies $\mu_F = f \cdot \lambda$ and hence $F' = f$ λ–a.e. by Differentiation Theorem 1. Converse is just given.