REPLACE

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Contents

I. REPLACE

1. For $a, b, k \in \mathbb{N}$,

$$\binom{a+b}{k} = \sum_{j=1}^{k} \binom{a}{j} \cdot \binom{b}{k-j} \tag{0.1}$$

We prove this with a bijection:

$$\mathcal{B}(a+b,k) \leftrightarrows \bigcup_{j=0}^{k} \mathcal{B}(a,j) \times \mathcal{B}(b,k-j)$$

given by $S \mapsto (S \cap \{1,\ldots,a\}, (S \cap \{a+1,\ldots,a+b\})^{(-a)}$ and $(P,Q) \mapsto P \cup Q^{(a)}$, where $\mathcal{B}(n,i)$ is the set of i–element subsets of $\{1,2,\ldots,n\}$ and for $C \subseteq \mathbb{Z}$ and $q \in \mathbb{Z}$, $C^{(q)} = \{c+q : c \in C\}$. Note that the equation in fact gives the polynomial identity

$$\binom{x+y}{k} = \sum_{j=0}^{k} \binom{x}{j} \binom{y}{k-j}$$

in $\mathbb{Q}[x,y]$. We denote the falling factorial $(x)_i = x(x-1)(x-2)\cdots(x-i+1)$, which has degree i for each $i \in \mathbb{N}$. In particular, $(x)_i = i!\binom{x}{i}$, so multiplying our identity by $k!$, we get

$$(x+y)_k = \sum_{j=0}^{k} \binom{k}{j} (x)_j (y)_{k-j}$$

Compare this with the standard binomial theorem

$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}$$

These are called sequences of binomial type.

2. Here's another identity. For $n \geq 0$ and $s, t \geq 1$,

$$\binom{n+s+t-1}{s+t-1} = \sum_{k=0}^{n} \binom{k+s-1}{s-1} \binom{n-k+t-1}{t-1}$$

Let $\mathcal{M}(m,r)$ denote a multiset of size m with elements of r types, so that $|\mathcal{M}(m,r)| = \binom{m+r-1}{r-1}$. Let's define a bijection

$$\mathcal{M}(n,s+t) \rightleftharpoons \bigcup_{k=1}^{n} \mathcal{M}(k,s) \times \mathcal{M}(n-k,t) \tag{0.2}$$

$\mu = (m_1, \ldots, m_{s+t}) \mapsto ((m_1, \ldots, m_s), (m_{s+1}, \ldots, m_{s+t}))$ and $(\nu, \theta) \mapsto \nu\theta$. Note that if f, g are polynomials of degree d and e respectively, then $\sum_{k=0}^{n} f(k)g(n-k)$ is a polynomial in n of degree $d + e - 1$.

Is there some way to understand (0.2)? It is unclear, with our known techniques, that this corresponds to a polynomial identity since there is a variable n in the exponent. However, we can use generating functions. Define

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{n+s+t-1}{s+t-1} z^n = \sum_{n=0}^{\infty} |\mathcal{M}(n, s+t)| z^n &= \sum_{(m_1, \ldots, m_{s+t})} z^{m_1 + \cdots + m_{s+t}} \\
&= \left(\sum_{m=0}^{\infty} z^m \right)^{s+t} \\
&= \frac{1}{(1-z)^{s+t}} = \frac{1}{(1-z)^s} \frac{1}{(1-z)^t} \\
&= \sum_{k=0}^{\infty} \binom{k+s-1}{s-1} z^k \sum_{\ell=0}^{\infty} \binom{\ell+t-1}{t-1} z^\ell \\
&= \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} \binom{k+s-1}{s-1} \binom{n-k+t-1}{t-1} \right)
\end{aligned}$$

Similarly, (0.1) is equivalent to saying $(1+z)^{a+b} = (1+z)^a (1+z)^b$. Note that $(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k = \sum_{k=0}^{\infty} \binom{n}{k} z^k$ for $n \in \mathbb{N}$.

Can we substitute $\frac{1}{(1-q)^t} = (1+z)^n$ where $z = -q$ and $n = -t$?

3. Consider

$$(x_1 + x_2)^n = \sum_{i=0}^{n} \binom{n}{i} x_1^i x_2^{n-i}$$

and

$$(x_1 + x_2)^n = \sum_{f : N_n \to \{1,2\}} \prod_{j=1}^{n} x_{f(j)}$$

More generally, we can consider

$$(x_1 + \cdots + x_k)^n = \sum_{f : N_n \to N_k} \prod_{j \in N_n} x_{f(j)}$$

If we set all $x_1 = \cdots = x_k = 1$, then k^n gives the number of functions from N_n to N_k. If we set $x_i = q^i$ for all $i \in N_k$, then we get

$$\left(\frac{q - q^{k+1}}{1-q} \right)^n = (q + q^2 + \cdots + q^k)^n = \sum_{f : N_n \to N_k} q^{f(1) + \cdots + f(k)}$$

Collect all the terms in $(x_1 + \cdots + x_k)^n$ that produce the same monomial. Given a multiset μ with $m_1 + \cdots + m_k = n$, write $x_1^{m_1} \cdots x_k^{m_k} = \underline{x}^\mu$. Then

$$(x_1 + \cdots + x_k)^n = \frac{n!}{m_1! \cdots m_k!} \underline{x}^\mu = \sum_{\mu \in \mathcal{M}(n,t)} \binom{n}{\mu} \underline{x}^\mu$$

4. How can we interpret

$$P_n(q) = \prod_{i=1}^{n}(1 + q + q^2 + \cdots + q^{i-1})$$

In general, if we set $q = 1$, we see that $P_n(1) = n!$. We might hope that there is some weight function on permutations $w : \mathcal{S}_n \to \mathbb{N}$ such that $P_n(q) = \sum_{\sigma\in\mathcal{S}_n} q^{w(\sigma)}$. Recall the bijection $I_n : \mathcal{S}_n \to \mathcal{Q}_n$ from chapter 1. Let's find some weight function $\nu : \mathcal{Q}_n \to \mathbb{N}$ such that $\sum_{\rho\in\mathcal{Q}_n} x^{\nu(\rho)} = P_n(q)$, then "pull back" the definition of $\nu : \mathcal{Q}_n \to \mathbb{N}$ to get a definition for $\omega : \mathcal{S}_n \to \mathbb{N}$. Note that $\sum_{h\in N_r} q^{h-1} = 1 + q + \cdots + q^{r-1}$. Thus

$$\sum_{\rho=(h_1,\dots,h_n)\in\mathcal{Q}_n} q^{(h_1-1)+(h_2-1)+\cdots+(h_n-1)} = \prod_{i=1}^{n}(1 + q + \cdots + q^{i-1}) = P_n(q)$$

so we can define $\nu(\rho) = |\rho| - n$ and $\sum_{q\in\mathcal{Q}_n} q^{|\rho|-n} = P_n(q)$. We also have

$$\sum_{\rho\in\mathcal{Q}_n} q^{(h_1-1)+\cdots+(h_n-1)} = (1 + q + \cdots + q^{n-1})(1 + q + \cdots + q^{n-2})\cdots(1 + q)(1)$$

For notation, define $[m]_q = 1 + q + \cdots + q^{m-1} = \frac{1-q^m}{1-q}$. Then $[m]_q! = [m]_q[m-1]_q\cdots[1]_q$.

	1	q	q^2	q^3	q^4	
$q[3]_q$	0	1	1	1		
$[2]_q[3]_q$	1	2	2	1		
$-q[2]_q[3]_q$	0	−1	−2	−2	−1	
$q^2[2]_q[3]_q$	0	0	1	2	2	1
$[6]_q$	1	1	1	1	1	1

so that $[6]_q = (1 - q + q^2)[2]_q[3]_q$. An **inversion** in $\sigma = a_1 \dots a_n \in \mathcal{S}_n$ is a pair (i, j) of indices $1 \le i < j \le n$ with $a_i > a)j$. Define $\mathrm{Inv}(\sigma)$ as the set of inversions of σ, and $\mathrm{inv}(\sigma) = |\mathrm{Inv}(\sigma)|$. Notice that if $\sigma = a_1 \dots a_n \mapsto \rho = (h_1, \dots, h_n)$, then for each $1 \le i \le n$, $h_i - 1$ is the number of inversions of σ with i in the first coordinate. Recall

$$\begin{aligned} \mathcal{S}_n &\leftrightarrows \mathcal{B}(n,k) \times \mathcal{S}_k \times \mathcal{S}_{n-k} \\ \sigma = a_1 \dots a_n &\leftrightarrow (A, \beta, \gamma) \\ \mathrm{inv}(\sigma) &= w(A) + \mathrm{inv}(\beta) + \mathrm{inv}(\gamma) \end{aligned}$$

Assuming such a weight fuction $w(A)$ exists, then

$$\begin{aligned} [n]!_q &= \sum_{\sigma\in\mathcal{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{(A,\beta,\gamma)} q^{w(A)+\mathrm{inv}(\beta)+\mathrm{inv}(\gamma)} \\ &= [k]!_q \cdot [n-k]!_q \cdot \sum_{A\in\mathcal{B}(n,k)} q^{w(A)} \end{aligned}$$

so that

$$\sum_{A\in\mathcal{B}(n,k)} q^{w(A)} = \frac{[n]!_q}{[k]!_q \cdot [n-k]!_q} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

$$\sum_{S\in\mathcal{B}(n,k)} q^{\mathrm{sum}(S)} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

0.1 Theorem. *Let V be an $n-$dimensional vector space over a finite field $\mathbb{F}_q$. Then for $0 \le k \le n$, the number of $k-$dimensional subspaces of V is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

0.2 Lemma. *Let $L : V \to W$ be a linear transformation that is surjective. Then* $\dim V = \dim W + \dim(\ker L)$. *So if this is over a finite field $\mathbb{R}_q$, every $w \in W$ is the image of exactly $q^{\dim(\ker(L)}$ vectors $v \in V$.*

For every $w \in W$, is the image of exactly q^k vectors in V. The number of ordered bases of V is $q^{\binom{n}{2}}(q-1)^n[n]!_q$.

0.3 Theorem. *Let V be an $n-$dimensional vector space over a finite field $\mathbb{F}_q$. For $0 \le k \le n$, the number of $k-$dimensional subspaces of V is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

Proof Let $\mathrm{OB}(V)$ be the set of ordered bases of V, and let $G(V,k)$ be the set of $k-$dimensional subspaces of V. Define a function

$$\mathrm{OB}(V) \to \bigcup_{U\in G(V,k)} (\{U\}\times \mathrm{OB}(U)\times \mathrm{OB}(V/U))$$

as follows. Given $(v_1,\ldots,v_n)$ an ordered basis of V, let $U = \mathrm{span}_{\mathbb{F}_q}\{v_1,\ldots,v_k\}$. Then $(v_1,\ldots,v_k) \in \mathrm{OB}(U)$ and $(v_{k+1}+U,\ldots,v_n+U) \in \mathrm{OB}(V/U)$. Consider the map $L : V \to V/U$ given by $L(v) = v+U$, so that every $v+U$ in V/U is the image of q^k vectors in V. Thus $(v_{k-1}+U,\ldots,v_n+U)$ is the image of $q^{k(n-k)}$ sequences $(z_{k+1},\ldots,z_n)$ of vectors in V. Thus the function $(v_1,\ldots,v_n) \mapsto (U,(v_1,\ldots,v_k),(v_{k+1}+U,\ldots,v_n+U))$ is surjective and hits everything on the RHS $q^{k(n-k)}$ times. But then counting both sides,

$$\begin{aligned}
q^{\binom{n}{2}}(q-1)^n[n]!_q &= \sum_{U\in G(V,k)} 1\cdot q^{\binom{k}{2}}(q-1)^k[k]!_q\cdot q^{\binom{n-k}{2}}(q-1)^{n-k}[n-k]!_q\cdot q^{k(n-k)} \\
q^{\binom{n}{2}}[n]!_q &= |G(V,k)|\cdot[k]!_q\cdot[n-k]!_q q^{\binom{k}{2}+\binom{n-k}{2}+k(n-k)} \\
[n]!_q &= |G(V,k)|\cdot[k]!_q\cdot[n-k]!_q \qquad \blacksquare
\end{aligned}$$

giving our desired result. ■