REPLACE

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[†]Last updated: September 30, 2019

Contents

I. REPLACE

1. For $a, b, k \in \mathbb{N}$,

$$\binom{a+b}{k} = \sum_{j=1}^{k} \binom{a}{j} \cdot \binom{b}{k-j} \tag{0.1}$$

We prove this with a bijection:

$$\mathcal{B}(a+b,k) \leftrightharpoons \bigcup_{j=0}^{k} \mathcal{B}(a,j) \times \mathcal{B}(b,k-j)$$

given by $S \mapsto (S \cap \{1,\ldots,a\}, (S \cap \{a+1,\ldots,a+b\})^{(-a)}$ and $(P,Q) \mapsto P \cup Q^{(a)}$, where $\mathcal{B}(n,i)$ is the set of i−element subsets of $\{1,2,\ldots,n\}$ and for $C \subseteq \mathbb{Z}$ and $q \in \mathbb{Z}$, $C^{(q)} = \{c+q : c \in C\}$. Note that the equation in fact gives the polynomial identity

$$\binom{x+y}{k} = \sum_{j=0}^{k} \binom{x}{j}\binom{y}{k-j}$$

in $\mathbb{Q}[x,y]$. We denote the falling factorial $(x)_i = x(x-1)(x-2)\cdots(x-i+1)$, which has degree i for each $i \in \mathbb{N}$. In particular, $(x)_i = i!\binom{x}{i}$, so multiplying our identity by $k!$, we get

$$(x+y)_k = \sum_{j=0}^{k} \binom{k}{j}(x)_j(y)_{k-j}$$

Compare this with the standard binomial theorem

$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}$$

These are called sequences of binomial type.

2. Here's another identity. For $n \geq 0$ and $s, t \geq 1$,

$$\binom{n+s+t-1}{s+t-1} = \sum_{k=0}^{n} \binom{k+s-1}{s-1}\binom{n-k+t-1}{t-1}$$

Let $\mathcal{M}(m,r)$ denote a multiset of size m with elements of r types, so that $|\mathcal{M}(m,r)| = \binom{m+r-1}{r-1}$. Let's define a bijection

$$\mathcal{M}(n, s+t) \leftrightharpoons \bigcup_{k=1}^{n} \mathcal{M}(k,s) \times \mathcal{M}(n-k,t) \tag{0.2}$$

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$\mu = (m_1,\ldots,m_{s+t}) \mapsto ((m_1,\ldots,m_s),(m_{s+1},\ldots,m_{s+t}))$ and $(\nu,\theta) \mapsto \nu\theta$. Note that if f, g are polynomials of degree d and e respectively, then $\sum_{k=0}^{n} f(k)g(n-k)$ is a polynomial in n of degree $d + e - 1$.

Is there some way to understand (0.2)? It is unclear, with our known techniques, that this corresponds to a polynomial identity since there is a variable n in the exponent. However, we can use generating functions. Define

$$\sum_{n=0}^{\infty} \binom{n+s+t-1}{s+t-1} z^n = \sum_{n=0}^{\infty} |\mathcal{M}(n, s+t)| z^n = \sum_{(m_1,\ldots,m_{s+t})} z^{m_1 + \cdots + m_{s+t}}$$

$$= \left(\sum_{m=0}^{\infty} z^m\right)^{s+t}$$

$$= \frac{1}{(1-z)^{s+t}} = \frac{1}{(1-z)^s}\frac{1}{(1-z)^t}$$

$$= \sum_{k=0}^{\infty} \binom{k+s-1}{s-1} z^k \sum_{\ell=0}^{\infty} \binom{\ell+t-1}{t-1} z^\ell$$

$$= \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} \binom{k+s-1}{s-1}\binom{n-k+t-1}{t-1}\right)$$

Similarly, (0.1) is equivalent to saying $(1+z)^{a+b} = (1+z)^a(1+z)^b$. Note that $(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k = \sum_{k=0}^{\infty} \binom{n}{k} z^k$ for $n \in \mathbb{N}$.

Can we substitute $\frac{1}{(1-q)^t} = (1+z)^n$ where $z = -q$ and $n = -t$?

3. Consider

$$(x_1 + x_2)^n = \sum_{i=0}^{n} \binom{n}{i} x_1^i x_2^{n-i}$$

and

$$(x_1 + x_2)^n = \sum_{f:N_n \to \{1,2\}} \prod_{j=1}^{n} x_{f(j)}$$

More generally, we can consider

$$(x_1 + \cdots + x_k)^n = \sum_{f:N_n \to N_k} \prod_{j \in N_n} x_{f(j)}$$

If we set all $x_1 = \cdots = x_k = 1$, then k^n gives the number of functions from N_n to N_k. If we set $x_i = q^i$ for all $i \in N_k$, then we get

$$\left(\frac{q - q^{k+1}}{1-q}\right)^n = (q + q^2 + \cdots + q^k)^n = \sum_{f:N_n \to N_k} q^{f(1) + \cdots + f(k)}$$

Collect all the terms in $(x_1 + \cdots + x_k)^n$ that produce the same monomial. Given a multiset μ with $m_1 + \cdots + m_k = n$, write $x_1^{m_1} \cdots x_k^{m_k} = \underline{x}^\mu$. Then

$$(x_1 + \cdots + x_k)^n = \frac{n!}{m_1! \cdots m_k!}\underline{x}^\mu = \sum_{\mu \in \mathcal{M}(n,t)} \binom{n}{\mu}\underline{x}^\mu$$

4. How can we interpret

$$P_n(q) = \prod_{i=1}^{n}(1 + q + q^2 + \cdots + q^{i-1})$$

In general, if we set $q = 1$, we see that $P_n(1) = n!$. We might hope that there is some weight function on permutations $w : \mathcal{S}_n \to \mathbb{N}$ such that $P_n(q) = \sum_{\sigma \in \mathcal{S}_n} q^{w(\sigma)}$. Recall the bijection $I_n : \mathcal{S}_n \to \mathcal{Q}_n$ from chapter 1. Let's find some weight function $\nu : \mathcal{Q}_n \to \mathbb{N}$ such that $\sum_{\rho \in \mathcal{Q}_n} x^{\nu(\rho)} = P_n(q)$, then "pull back" the definition of $\nu : \mathcal{Q}_n \to \mathbb{N}$ to get a definition for $\omega : \mathcal{S}_n \to \mathbb{N}$. Note that $\sum_{h \in N_r} q^{h-1} = 1 + q + \cdots + q^{r-1}$. Thus

$$\sum_{\rho = (h_1,\ldots,h_n) \in \mathcal{Q}_n} q^{(h_1-1)+(h_2-1)+\cdots+(h_n-1)} = \prod_{i=1}^{n}(1 + q + \cdots + q^{i-1}) = P_n(q)$$

so we can define $\nu(\rho) = |\rho| - n$ and $\sum_{q \in \mathcal{Q}_n} q^{|\rho|-n} = P_n(q)$. We also have

$$\sum_{\rho \in \mathcal{Q}_n} q^{(h_1-1)+\cdots+(h_n-1)} = (1 + q + \cdots + q^{n-1})(1 + q + \cdots + q^{n-2})\cdots(1 + q)(1)$$

For notation, define $[m]_q = 1 + q + \cdots + q^{m-1} = \frac{1-q^m}{1-q}$. Then $[m]_q! = [m]_q[m-1]_q \cdots [1]_q$.

	1	q	q^2	q^3	q^4	
$q[3]_q$	0	1	1	1		
$[2]_q[3]_q$	1	2	2	1		
$-q[2]_q[3]_q$	0	-1	-2	-2	-1	
$q^2[2]_q[3]_q$	0	0	1	2	2	1
$[6]_q$	1	1	1	1	1	1

so that $[6]_q = (1 - q + q^2)[2]_q[3]_q$. An **inversion** in $\sigma = a_1 \ldots a_n \in \mathcal{S}_n$ is a pair (i,j) of indices $1 \le i < j \le n$ with $a_i > a)j$. Define $\mathrm{Inv}(\sigma)$ as the set of inversions of σ, and $\mathrm{inv}(\sigma) = |\mathrm{Inv}(\sigma)|$. Notice that if $\sigma = a_1 \ldots a_n \mapsto \rho = (h_1,\ldots,h_n)$, then for each $1 \le i \le n$, $h_i - 1$ is the number of inversions of σ with i in the first coordinate. Recall

$$\mathcal{S}_n \leftrightharpoons \mathcal{B}(n,k) \times \mathcal{S}_k \times \mathcal{S}_{n-k}$$
$$\sigma = a_1 \ldots a_n \leftrightarrow (A, \beta, \gamma)$$
$$\mathrm{inv}(\sigma) = w(A) + \mathrm{inv}(\beta) + \mathrm{inv}(\gamma)$$

Assuming such a weight fuction $w(A)$ exists, then

$$[n]!_q = \sum_{\sigma \in \mathcal{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{(A,\beta,\gamma)} q^{w(A)+\mathrm{inv}(\beta)+\mathrm{inv}(\gamma)}$$
$$= [k]!_q \cdot [n-k]!_q \cdot \sum_{A \in \mathcal{B}(n,k)} q^{w(A)}$$

so that

$$\sum_{A \in \mathcal{B}(n,k)} q^{w(A)} = \frac{[n]!_q}{[k]!_q \cdot [n-k]!_q} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

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$$\sum_{S \in \mathcal{B}(n,k)} q^{\mathrm{sum}(S)} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

0.1 Theorem. *Let V be an n–dimensional vector space over a finite field $\mathbb{F}_q$. Then for $0 \le k \le n$, the number of k–dimensional subspaces of V is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

0.2 Lemma. *Let $L : V \to W$ be a linear transformation that is surjective. Then $\dim V = \dim W + \dim(\ker L)$. So if this is over a finite field $\mathbb{R}_q$, every $w \in W$ is the image of exactly $q^{\dim(\ker(L))}$ vectors $v \in V$.*

For every $w \in W$, is the image of exactly q^k vectors in V. The number of ordered bases of V is $q^{\binom{n}{2}}(q-1)^n [n]!_q$.

0.3 Theorem. *Let V be an n–dimensional vector space over a finite field $\mathbb{F}_q$. For $0 \le k \le n$, the number of k–dimensional subspaces of V is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

PROOF Let $\mathrm{OB}(V)$ be the set of ordered bases of V, and let $G(V,k)$ be the set of k–dimensional subspaces of V. Define a function

$$\mathrm{OB}(V) \to \bigcup_{U \in G(V,k)} (\{U\} \times \mathrm{OB}(U) \times \mathrm{OB}(V/U))$$

as follows. Given $(v_1, \dots, v_n)$ an ordered basis of V, let $U = \mathrm{span}_{\mathbb{F}_q}\{v_1, \dots, v_k\}$. Then $(v_1, \dots, v_k) \in \mathrm{OB}(U)$ and $(v_{k+1} + U, \dots, v_n + U) \in \mathrm{OB}(V/U)$. Consider the map $L : V \to V/U$ given by $L(v) = v + U$, so that every $v + U$ in V/U is the image of q^k vectors in V. Thus $(v_{k-1} + U, \dots, v_n + U)$ is the image of $q^{k(n-k)}$ sequences $(z_{k+1}, \dots, z_n)$ of vectors in V. Thus the function $(v_1, \dots, v_n) \mapsto (U, (v_1, \dots, v_k), (v_{k+1} + U, \dots, v_n + U))$ is surjective and hits everything on the RHS $q^{k(n-k)}$ times. But then counting both sides,

$$q^{\binom{n}{2}}(q-1)^n [n]!_q = \sum_{U \in G(V,k)} 1 \cdot q^{\binom{k}{2}}(q-1)^k [k]!_q \cdot q^{\binom{n-k}{2}}(q-1)^{n-k} [n-k]!_q \cdot q^{k(n-k)}$$

$$q^{\binom{n}{2}}[n]!_q = |G(V,k)| \cdot [k]!_q \cdot [n-k]!_q q^{\binom{k}{2} + \binom{n-k}{2} + k(n-k)}$$

$$[n]!_q = |G(V,k)| \cdot [k]!_q \cdot [n-k]!_q$$

giving our desired result. ∎

A **set partition** π of a set V is a collection of subsets $\pi = \{B_1, \dots, B_k\}$ of V such that
- Each B_i is not empty
- $B_i \cap B_j = \emptyset$ if $i \ne j$
- $B_1 \cup \dots \cup B_k = V$

Let $\Pi(n,k)$ be the set of set partitions of N_n with k blocks, and set $S(n,k) = |\Pi(n,k)|$. Certainly $S(0,0) = 1$ for the empty set partition. If $n \ge 1$, then $S(n,0) = 0$, $S(n,n) = 1$, and $S(n,1) = 1$. We can also define a recurrence relation. Let $\Pi'(n,k)$ be those $\pi \in \Pi(n,k)$ in which $\{n\}$ is a block, and $\Pi''(n,k)$ is the set of π in which n is in a block of size at least 2. Note that $\Pi'(n,k) \leftrightarrows \Pi(n-1, k-1)$ by removing or adding the independent element.

Furthemore, the function which removes the element n from a block in $\Pi''(n,k)$ is a surjective function onto $\Pi(n-1,k)$ which hits every element of $\Pi(n-1,k)$ k times. Thus combing these observations, $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$. Thus we can compute

$S(n,k)$	0	1	2	3	4	5	6
0	1	X	X	X	X	X	X
1	0	1	X	X	X	X	X
2	0	1	1	X	X	X	X
3	0	1	3	1	X	X	X
4	0	1	7	6	1	X	X
5	0	1	15	25	10	1	X
6	0	1	31		1		

From homework 2, we have that

$$x^n = \sum_{k=0}^{n} k! S(n,k) \binom{n}{k}$$

Invert this using Binomial Inversion.

0.4 Theorem. (Binomial Inversion) *Let $a_0, a_1, \ldots$ be a sequence.*

Proof For $h \in \mathbb{N}$, let $b_h = \sum_{i=0}^{h} \binom{h}{i} a_i$. Let $A(t) = \sum_{i=0}^{\infty} a_i t^i$ and $B(t) = \sum_{h=0}^{\infty} b_h t^h$. Then

$$\begin{aligned}
B(t) &= \sum_{h=0}^{\infty} t^h \sum_{i=0}^{h} \binom{h}{i} a_i \\
&= \sum_{i=0}^{\infty} a_i t^i \sum_{h=i}^{\infty} \binom{h}{h-i} t^{h-i} \\
&= \sum_{i=0}^{\infty} a_i t^i \sum_{j=0}^{\infty} \binom{i+j}{j} t^j = \sum_{i=0}^{\infty} \frac{a_i t^i}{(1-t)^{i+1}} \\
&= \frac{1}{1-t} \sum_{i=0}^{\infty} a_i \left(\frac{t}{1-t} \right)^i = \frac{1}{1-t} A\left(\frac{t}{1-t} \right)
\end{aligned}$$

Let $z = t/(1-t)$, so that $t = z/(1+z)$. Thus

$$B\left(\frac{z}{1+z} \right) = (1+z)A(z)$$

so that

$$\begin{aligned}
\sum_{i=0}^{\infty} a_i z^i &= \frac{1}{1+z} B\left(\frac{z}{1+z} \right) = \sum_{h=0}^{\infty} b_h \frac{z^h}{(1+z)^{h+1}} \\
&= \sum_{h=0}^{\infty} b_h \sum_{j=0}^{\infty} \binom{j+h}{h} z^h (-z)^j \\
&= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{n-j} b_j
\end{aligned}$$

Thus for all $m \in \mathbb{N}$, $a_m = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} b_j$. ∎

In particular, applying this to Stirling numbers of the second kind, for all $n \in \mathbb{N}$ in $\mathbb{R}[x]$, we have

$$x^n = \sum_{k=0}^{n} S(n,k) \binom{x}{k} k!$$

Let $b_i = i^n$ for $i = 0, 1, 2, \ldots$. If $k > n$ or $k > i$, then $S(n,k)\binom{i}{k} = 0$; thus,

$$i^n = \sum_{k=0}^{n} S(,nk) \binom{i}{k} k! = \sum_{k=0}^{\min(n,i)} S(n,k) \binom{i}{k} k! = \sum_{k=0}^{i} S(n,k) \binom{i}{k} k!$$
$$= \sum_{k=0}^{i} \binom{i}{k} a_k$$

where $a_k = k! S(n,k)$ for all $k \in \mathbb{N}$. But then apply binomial inversion to get

$$a_k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} b_j$$

Suppose $m^n = \sum_{k=0}^{n} S(n,k) \binom{m}{k} k!$. Then $[m]_q^n = \sum_{k=0}^{\infty} S[n,k]_q \begin{bmatrix} m \\ k \end{bmatrix}_q [k]!_q$, where $S[n,k]_q = \sum_{\pi \in \Pi(n,k)} q^{w(\pi)}$. Is there some function $w : \Pi(n,k) \to \mathbb{N}$ that makes this work?

II. Power Series Identities

(i) $\dfrac{1}{(1-z)^h} = \displaystyle\sum_{k=0}^{\infty} \binom{k+h-1}{h-1} z^k$

(ii) Let $a_0, a_1, \ldots$ be a sequence, and $b_h = \sum_{i=0}^{h} \binom{h}{i} a_i$. Then $a_m = \sum_{i=0}^{m} \binom{m}{i}(-1)^{m-i} b_i$.