

# REPLACE

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REPLACE<sup>†</sup>

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# I. REPLACE

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1. For  $a, b, k \in \mathbb{N}$ ,

$$\binom{a+b}{k} = \sum_{j=1}^k \binom{a}{j} \cdot \binom{b}{k-j} \quad (0.1)$$

We prove this with a bijection:

$$\mathcal{B}(a+b, k) \rightleftharpoons \bigcup_{j=0}^k \mathcal{B}(a, j) \times \mathcal{B}(b, k-j)$$

given by  $S \mapsto (S \cap \{1, \dots, a\}, (S \cap \{a+1, \dots, a+b\})^{(-a)})$  and  $(P, Q) \mapsto P \cup Q^{(a)}$ , where  $\mathcal{B}(n, i)$  is the set of  $i$ -element subsets of  $\{1, 2, \dots, n\}$  and for  $C \subseteq \mathbb{Z}$  and  $q \in \mathbb{Z}$ ,  $C^{(q)} = \{c+q : c \in C\}$ . Note that the equation in fact gives the polynomial identity

$$\binom{x+y}{k} = \sum_{j=0}^k \binom{x}{j} \binom{y}{k-j}$$

in  $\mathbb{Q}[x, y]$ . We denote the falling factorial  $(x)_i = x(x-1)(x-2)\cdots(x-i+1)$ , which has degree  $i$  for each  $i \in \mathbb{N}$ . In particular,  $(x)_i = i! \binom{x}{i}$ , so multiplying our identity by  $k!$ , we get

$$(x+y)_k = \sum_{j=0}^k \binom{k}{j} (x)_j (y)_{k-j}$$

Compare this with the standard binomial theorem

$$(x+y)^k = \sum_{j=0}^k \binom{k}{j} x^j y^{k-j}$$

These are called sequences of binomial type.

2. Here's another identity. For  $n \geq 0$  and  $s, t \geq 1$ ,

$$\binom{n+s+t-1}{s+t-1} = \sum_{k=0}^n \binom{k+s-1}{s-1} \binom{n-k+t-1}{t-1}$$

Let  $\mathcal{M}(m, r)$  denote a multiset of size  $m$  with elements of  $r$  types, so that  $|\mathcal{M}(m, r)| = \binom{m+r-1}{r-1}$ . Let's define a bijection

$$\mathcal{M}(n, s+t) \rightleftharpoons \bigcup_{k=1}^n \mathcal{M}(k, s) \times \mathcal{M}(n-k, t) \quad (0.2)$$

$\mu = (m_1, \dots, m_{s+t}) \mapsto ((m_1, \dots, m_s), (m_{s+1}, \dots, m_{s+t}))$  and  $(v, \theta) \mapsto v\theta$ . Note that if  $f, g$  are polynomials of degree  $d$  and  $e$  respectively, then  $\sum_{k=0}^n f(k)g(n-k)$  is a polynomial in  $n$  of degree  $d+e-1$ .

Is there some way to understand (0.2)? It is unclear, with our known techniques, that this corresponds to a polynomial identity since there is a variable  $n$  in the exponent. However, we can use generating functions. Define

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{n+s+t-1}{s+t-1} z^n &= \sum_{n=0}^{\infty} |\mathcal{M}(n, s+t)| z^n = \sum_{(m_1, \dots, m_{s+t})} z^{m_1 + \dots + m_{s+t}} \\ &= \left( \sum_{m=0}^{\infty} z^m \right)^{s+t} \\ &= \frac{1}{(1-z)^{s+t}} = \frac{1}{(1-z)^s} \frac{1}{(1-z)^t} \\ &= \sum_{k=0}^{\infty} \binom{k+s-1}{s-1} z^k \sum_{\ell=0}^{\infty} \binom{\ell+t-1}{t-1} z^\ell \\ &= \sum_{n=0}^{\infty} z^n \left( \sum_{k=0}^n \binom{k+s-1}{s-1} \binom{n-k+t-1}{t-1} \right) \end{aligned}$$

Similarly, (0.1) is equivalent to saying  $(1+z)^{a+b} = (1+z)^a (1+z)^b$ . Note that  $(1+z)^n = \sum_{k=0}^n \binom{n}{k} z^k = \sum_{k=0}^{\infty} \binom{n}{k} z^k$  for  $n \in \mathbb{N}$ .

Can we substitute  $\frac{1}{(1-q)^t} = (1+z)^n$  where  $z = -q$  and  $n = -t$ ?

3. Consider

$$(x_1 + x_2)^n = \sum_{i=0}^n \binom{n}{i} x_1^i x_2^{n-i}$$

and

$$(x_1 + x_2)^n = \sum_{f: N_n \rightarrow \{1,2\}} \prod_{j=1}^n x_{f(j)}$$

More generally, we can consider

$$(x_1 + \dots + x_k)^n = \sum_{f: N_n \rightarrow N_k} \prod_{j \in N_n} x_{f(j)}$$

If we set all  $x_1 = \dots = x_k = 1$ , then  $k^n$  gives the number of functions from  $N_n$  to  $N_k$ . If we set  $x_i = q^i$  for all  $i \in N_k$ , then we get

$$\left( \frac{q - q^{k+1}}{1 - q} \right)^n = (q + q^2 + \dots + q^k)^n = \sum_{f: N_n \rightarrow N_k} q^{f(1) + \dots + f(n)}$$

Collect all the terms in  $(x_1 + \dots + x_k)^n$  that produce the same monomial. Given a multiset  $\mu$  with  $m_1 + \dots + m_k = n$ , write  $x_1^{m_1} \dots x_k^{m_k} = \underline{x}^\mu$ . Then

$$(x_1 + \dots + x_k)^n = \frac{n!}{m_1! \dots m_k!} \underline{x}^\mu = \sum_{\mu \in \mathcal{M}(n, k)} \binom{n}{\mu} \underline{x}^\mu$$

## 4. How can we interpret

$$P_n(q) = \prod_{i=1}^n (1 + q + q^2 + \cdots + q^{i-1})$$

In general, if we set  $q = 1$ , we see that  $P_n(1) = n!$ . We might hope that there is some weight function on permutations  $w : \mathcal{S}_n \rightarrow \mathbb{N}$  such that  $P_n(q) = \sum_{\sigma \in \mathcal{S}_n} q^{w(\sigma)}$ . Recall the bijection  $I_n : \mathcal{S}_n \rightarrow \mathcal{Q}_n$  from chapter 1. Let's find some weight function  $v : \mathcal{Q}_n \rightarrow \mathbb{N}$  such that  $\sum_{\rho \in \mathcal{Q}_n} x^{v(\rho)} = P_n(q)$ , then “pull back” the definition of  $v : \mathcal{Q}_n \rightarrow \mathbb{N}$  to get a definition for  $\omega : \mathcal{S}_n \rightarrow \mathbb{N}$ . Note that  $\sum_{h \in \mathbb{N}_r} q^{h-1} = 1 + q + \cdots + q^{r-1}$ . Thus

$$\sum_{\rho=(h_1, \dots, h_n) \in \mathcal{Q}_n} q^{(h_1-1)+(h_2-1)+\cdots+(h_n-1)} = \prod_{i=1}^n (1 + q + \cdots + q^{i-1}) = P_n(q)$$

so we can define  $v(\rho) = |\rho| - n$  and  $\sum_{\rho \in \mathcal{Q}_n} q^{|\rho| - n} = P_n(q)$ . We also have

$$\sum_{\rho \in \mathcal{Q}_n} q^{(h_1-1)+\cdots+(h_n-1)} = (1 + q + \cdots + q^{n-1})(1 + q + \cdots + q^{n-2}) \cdots (1 + q)(1)$$

For notation, define  $[m]_q = 1 + q + \cdots + q^{m-1} = \frac{1-q^m}{1-q}$ . Then  $[m]_q! = [m]_q [m-1]_q \cdots [1]_q$ .

	1	q	q <sup>2</sup>	q <sup>3</sup>	q <sup>4</sup>
$q[3]_q$	0	1	1	1	
$[2]_q[3]_q$	1	2	2	1	
$-q[2]_q[3]_q$	0	-1	-2	-2	-1
$q^2[2]_q[3]_q$	0	0	1	2	2
$[6]_q$	1	1	1	1	1

so that  $[6]_q = (1 - q + q^2)[2]_q[3]_q$ . An **inversion** in  $\sigma = a_1 \dots a_n \in \mathcal{S}_n$  is a pair  $(i, j)$  of indices  $1 \leq i < j \leq n$  with  $a_i > a_j$ . Define  $\text{Inv}(\sigma)$  as the set of inversions of  $\sigma$ , and  $\text{inv}(\sigma) = |\text{Inv}(\sigma)|$ . Notice that if  $\sigma = a_1 \dots a_n \mapsto \rho = (h_1, \dots, h_n)$ , then for each  $1 \leq i \leq n$ ,  $h_i - 1$  is the number of inversions of  $\sigma$  with  $i$  in the first coordinate. Recall

$$\begin{aligned} \mathcal{S}_n &\rightleftharpoons \mathcal{B}(n, k) \times \mathcal{S}_k \times \mathcal{S}_{n-k} \\ \sigma = a_1 \dots a_n &\leftrightarrow (A, \beta, \gamma) \\ \text{inv}(\sigma) &= w(A) + \text{inv}(\beta) + \text{inv}(\gamma) \end{aligned}$$

Assuming such a weight function  $w(A)$  exists, then

$$\begin{aligned} [n]_q! &= \sum_{\sigma \in \mathcal{S}_n} q^{\text{inv}(\sigma)} = \sum_{(A, \beta, \gamma)} q^{w(A) + \text{inv}(\beta) + \text{inv}(\gamma)} \\ &= [k]_q! \cdot [n-k]_q! \cdot \sum_{A \in \mathcal{B}(n, k)} q^{w(A)} \end{aligned}$$

so that

$$\sum_{A \in \mathcal{B}(n, k)} q^{w(A)} = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!} = \left[ \begin{matrix} n \\ k \end{matrix} \right]_q$$

$$\sum_{S \in \mathcal{B}(n,k)} q^{\text{sum}(S)} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

**0.1 Theorem.** Let  $V$  be an  $n$ -dimensional vector space over a finite field  $\mathbb{F}_q$ . Then for  $0 \leq k \leq n$ , the number of  $k$ -dimensional subspaces of  $V$  is  $\begin{bmatrix} n \\ k \end{bmatrix}_q$ .

**0.2 Lemma.** Let  $L : V \rightarrow W$  be a linear transformation that is surjective. Then  $\dim V = \dim W + \dim(\ker L)$ . So if this is over a finite field  $\mathbb{F}_q$ , every  $w \in W$  is the image of exactly  $q^{\dim(\ker L)}$  vectors  $v \in V$ .

For every  $w \in W$ , is the image of exactly  $q^k$  vectors in  $V$ . The number of ordered bases of  $V$  is  $q^{\binom{n}{2}}(q-1)^n[n]_q!$ .

**0.3 Theorem.** Let  $V$  be an  $n$ -dimensional vector space over a finite field  $\mathbb{F}_q$ . For  $0 \leq k \leq n$ , the number of  $k$ -dimensional subspaces of  $V$  is  $\begin{bmatrix} n \\ k \end{bmatrix}_q$ .

**PROOF** Let  $\text{OB}(V)$  be the set of ordered bases of  $V$ , and let  $G(V, k)$  be the set of  $k$ -dimensional subspaces of  $V$ . Define a function

$$\text{OB}(V) \rightarrow \bigcup_{U \in G(V, k)} (\{U\} \times \text{OB}(U) \times \text{OB}(V/U))$$

as follows. Given  $(v_1, \dots, v_n)$  an ordered basis of  $V$ , let  $U = \text{span}_{\mathbb{F}_q}\{v_1, \dots, v_k\}$ . Then  $(v_1, \dots, v_k) \in \text{OB}(U)$  and  $(v_{k+1} + U, \dots, v_n + U) \in \text{OB}(V/U)$ . Consider the map  $L : V \rightarrow V/U$  given by  $L(v) = v + U$ , so that every  $v + U$  in  $V/U$  is the image of  $q^k$  vectors in  $V$ . Thus  $(v_{k+1} + U, \dots, v_n + U)$  is the image of  $q^{k(n-k)}$  sequences  $(z_{k+1}, \dots, z_n)$  of vectors in  $V$ . Thus the function  $(v_1, \dots, v_n) \mapsto (U, (v_1, \dots, v_k), (v_{k+1} + U, \dots, v_n + U))$  is surjective and hits everything on the RHS  $q^{k(n-k)}$  times. But then counting both sides,

$$\begin{aligned} q^{\binom{n}{2}}(q-1)^n[n]_q! &= \sum_{U \in G(V, k)} 1 \cdot q^{\binom{k}{2}}(q-1)^k[k]_q! \cdot q^{\binom{n-k}{2}}(q-1)^{n-k}[n-k]_q! \cdot q^{k(n-k)} \\ q^{\binom{n}{2}}[n]_q! &= |G(V, k)| \cdot [k]_q! \cdot [n-k]_q! q^{\binom{k}{2} + \binom{n-k}{2} + k(n-k)} \\ [n]_q! &= |G(V, k)| \cdot [k]_q! \cdot [n-k]_q! \end{aligned}$$

giving our desired result. ■

A **set partition**  $\pi$  of a set  $V$  is a collection of subsets  $\pi = \{B_1, \dots, B_k\}$  of  $V$  such that

- Each  $B_i$  is not empty
- $B_i \cap B_j = \emptyset$  if  $i \neq j$
- $B_1 \cup \dots \cup B_k = V$

Let  $\Pi(n, k)$  be the set of set partitions of  $N_n$  with  $k$  blocks, and set  $S(n, k) = |\Pi(n, k)|$ . Certainly  $S(0, 0) = 1$  for the empty set partition. If  $n \geq 1$ , then  $S(n, 0) = 0$ ,  $S(n, n) = 1$ , and  $S(n, 1) = 1$ . We can also define a recurrence relation. Let  $\Pi'(n, k)$  be those  $\pi \in \Pi(n, k)$  in which  $\{n\}$  is a block, and  $\Pi''(n, k)$  is the set of  $\pi$  in which  $n$  is in a block of size at least 2. Note that  $\Pi'(n, k) \rightleftharpoons \Pi(n-1, k-1)$  by removing or adding the independent element.



Furthermore, the function which removes the element  $n$  from a block in  $\Pi''(n, k)$  is a surjective function onto  $\Pi(n-1, k)$  which hits every element of  $\Pi(n-1, k)$   $k$  times. Thus combining these observations,  $S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$ . Thus we can compute

$S(n, k)$	0	1	2	3	4	5	6
0	1	X	X	X	X	X	X
1	0	1	X	X	X	X	X
2	0	1	1	X	X	X	X
3	0	1	3	1	X	X	X
4	0	1	7	6	1	X	X
5	0	1	15	25	10	1	X
6	0	1	31		1		

From homework 2, we have that

$$x^n = \sum_{k=0}^n k! S(n, k) \binom{n}{k}$$

Invert this using Binomial Inversion.

**0.4 Theorem. (Binomial Inversion)** Let  $a_0, a_1, \dots$  be a sequence.

PROOF For  $h \in \mathbb{N}$ , let  $b_h = \sum_{i=0}^h \binom{h}{i} a_i$ . Let  $A(t) = \sum_{i=0}^{\infty} a_i t^i$  and  $B(t) = \sum_{h=0}^{\infty} b_h t^h$ . Then

$$\begin{aligned} B(t) &= \sum_{h=0}^{\infty} t^h \sum_{i=0}^h \binom{h}{i} a_i \\ &= \sum_{i=0}^{\infty} a_i t^i \sum_{h=i}^{\infty} \binom{h}{h-i} t^{h-i} \\ &= \sum_{i=0}^{\infty} a_i t^i \sum_{j=0}^{\infty} \binom{i+j}{j} t^j = \sum_{i=0}^{\infty} \frac{a_i t^i}{(1-t)^{i+1}} \\ &= \frac{1}{1-t} \sum_{i=0}^{\infty} a_i \left( \frac{t}{1-t} \right)^i = \frac{1}{1-t} A\left( \frac{t}{1-t} \right) \end{aligned}$$

Let  $z = t/(1-t)$ , so that  $t = z/(1+z)$ . Thus

$$B\left( \frac{z}{1+z} \right) = (1+z)A(z)$$

so that

$$\begin{aligned} \sum_{i=0}^{\infty} a_i z^i &= \frac{1}{1+z} B\left( \frac{z}{1+z} \right) = \sum_{h=0}^{\infty} b_h \frac{z^h}{(1+z)^{h+1}} \\ &= \sum_{h=0}^{\infty} b_h \sum_{j=0}^{\infty} \binom{j+h}{h} z^h (-z)^j \\ &= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{n-j} b_j \end{aligned}$$

Thus for all  $m \in \mathbb{N}$ ,  $a_m = \sum_{j=0}^m \binom{m}{j} (-1)^{m-j} b_j$ . ■

In particular, applying this to Stirling numbers of the second kind, for all  $n \in \mathbb{N}$  in  $\mathbb{R}[x]$ , we have

$$x^n = \sum_{k=0}^n S(n, k) \binom{x}{k} k!$$

Let  $b_i = i^n$  for  $i = 0, 1, 2, \dots$ . If  $k > n$  or  $k > i$ , then  $S(n, k) \binom{i}{k} = 0$ ; thus,

$$\begin{aligned} i^n &= \sum_{k=0}^n S(n, k) \binom{i}{k} k! = \sum_{k=0}^{\min(n, i)} S(n, k) \binom{i}{k} k! = \sum_{k=0}^i S(n, k) \binom{i}{k} k! \\ &= \sum_{k=0}^i \binom{i}{k} a_k \end{aligned}$$

where  $a_k = k! S(n, k)$  for all  $k \in \mathbb{N}$ . But then apply binomial inversion to get

$$a_k = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} b_j$$

Suppose  $m^n = \sum_{k=0}^n S(n, k) \binom{m}{k} k!$ . Then  $[m]_q^n = \sum_{k=0}^{\infty} S[n, k]_q \left[ \begin{matrix} m \\ k \end{matrix} \right]_q [k]_q!$ , where  $S[n, k]_q = \sum_{\pi \in \Pi(n, k)} q^{w(\pi)}$ . Is there some function  $w : \Pi(n, k) \rightarrow \mathbb{N}$  that makes this work?

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## II. Power Series Identities

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- (i)  $\frac{1}{(1-z)^h} = \sum_{k=0}^{\infty} \binom{k+h-1}{h-1} z^k$
- (ii) Let  $a_0, a_1, \dots$  be a sequence, and  $b_h = \sum_{i=0}^h \binom{h}{i} a_i$ . Then  $a_m = \sum_{i=0}^m \binom{m}{i} (-1)^{m-i} b_i$ .
- (iii) General Binomial Series. For  $k \in \mathbb{N}$ , let  $\binom{y}{k} = \frac{y(y-1)\dots(y-k+1)}{k!} \in \mathbb{Q}[y]$ . Then we define

$$(1+x)^y = \sum_{k=0}^{\infty} \binom{y}{k} x^k$$

which is a power series in  $x$ . Each coefficient of  $[x^n]$  is in  $\mathbb{Q}[y]$ . Then by Vandermonde convolution,

$$\begin{aligned} (1+x)^y (1+x)^z &= \sum_{i=0}^{\infty} \binom{y}{i} x^i \sum_{j=0}^{\infty} \binom{z}{j} x^j \\ &= \sum_{n=0}^{\infty} x^n \left( \sum_{i=0}^n \binom{y}{i} \binom{z}{n-i} \right) \\ &= \sum_{n=0}^{\infty} \binom{y+z}{n} x^n = (1+x)^{y+z} \end{aligned}$$

Furthermore, if  $y = -p < 0$  is an integer, then

$$\begin{aligned} (1+x)^{-p} &= \sum_{k=0}^{\infty} \binom{-p}{k} x^k \\ &= \sum_{k=0}^{\infty} \binom{k+p-1}{p-1} x^k \end{aligned}$$

For  $\alpha \in \mathbb{C}$ ,  $f(x) = (1+x)^\alpha$  is analytic for  $|x| < 1$ . In particular, by Taylor's theorem,

$$(1+x)^\alpha = \sum_{k=0}^{\infty} c_k x^k$$

where  $c_k = \frac{1}{k!} \frac{d^k}{dx^k} (1+x)^\alpha \big|_{x=0}$ .