

# Representation Theory of Finite Groups

Alex Rutar\*  
University of Waterloo

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\*[arutar@uwaterloo.ca](mailto:arutar@uwaterloo.ca)

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# Contents

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<b>Chapter I</b>	<b>Introduction</b>	
1	Tensor Products . . . . .	4
2	Character Theory . . . . .	4
3	Induced Representations . . . . .	11



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# I. Introduction

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Let  $G$  be a finite group of order  $n$ , and write  $G = \{g_1, \dots, g_n\}$ . Fix  $g \in G$ ; then  $gg_i = gg_j$  if and only if  $i = j$ . Thus there exists some  $\sigma_g \in S_n$  such that  $gg_i = g_{\sigma_g(i)}$  for all  $i \in \{1, 2, \dots, n\}$ . In particular,  $\phi : G \rightarrow S_n$  by  $\phi(g) = \sigma_g$  is an embedding (injective group homomorphism). This observation is usually referred to as Cayley's Theorem.

Now let  $V$  be an  $n$ -dimensional complex vector space. We then denote  $\text{GL}(V)$  as the group of invertible linear operators  $T : V \rightarrow V$ . Now define  $\psi : S_n \rightarrow \text{GL}(V)$  by  $\psi(\sigma) = T_\sigma$  where if  $\{b_1, \dots, b_n\}$  is a basis for  $V$  and  $T_\sigma(b_i) = b_{\sigma(i)}$ . This is an injective group homomorphism, so  $\psi \circ \phi : G \rightarrow \text{GL}(V)$  is an embedding of  $G$  into  $\text{GL}(V)$ .

**Definition.** Let  $G$  be a finite group, and  $V$  a finite dimensional  $\mathbb{C}$ -vector space. A **representation** of  $G$  is a group homomorphism  $\rho : G \rightarrow \text{GL}(V)$ . We call  $\dim(V)$  the **degree** of the representation.

In particular, if  $V$  is  $n$ -dimensional, then  $\text{GL}(V) \cong \text{GL}_n(\mathbb{C})$ .

*Example.* 1. Consider  $\rho : G \rightarrow \text{GL}(\mathbb{C}) \cong \mathbb{C}^\times$  given by  $\rho(g) = 1$  for all  $g \in G$ . This is called the *trivial representation*.

2. Consider  $\rho : S_n \rightarrow \mathbb{C}^\times$  given by  $\rho(\sigma) = \text{sgn}(\sigma)$ , which is called the *sign representation*.

3. The representation of  $G$  afforded by Cayley's theorem is called the *regular representation* of  $G$ . The next example is a good way to understand the regular rep of  $G$ .

4. Consider  $G$ ,  $X = \{x_1, \dots, x_n\}$ , and  $V = \text{Free}(X)$ . Suppose  $G$  acts on  $X$ . Then  $\rho : G \rightarrow \text{GL}(V)$  given by  $\rho(g)(x_i) = gx_i$ . In particular, if we take  $X = G$ , then this is the regular representation of  $G$ .

5. Consider the 4-gon, with vertices labelled  $a, b, c, d$ . Take  $X = \{a, b, c, d\}$  and the regular representation  $\rho : D_4 \rightarrow \text{GL}(V)$ . This action has a geometric notion.

6. Let  $C_n$  be a cyclic group of order  $n$ ; let us define some  $\rho : C_n \rightarrow \text{GL}(V)$ . Say  $\rho(x) = T$  where  $t \in \text{GL}(V)$ ; then this is a representation if and only if  $T^n = I$ .

**Definition.** We say that two representations  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  are **isomorphic** if there exists an isomorphism  $T : V \rightarrow W$  such that for all  $g \in G$ ,

$$T \circ \rho(g) = \tau(g) \circ T$$

Suppose  $\rho : G \rightarrow \text{GL}(V)$  and  $T : V \rightarrow W$  is an isomorphism. Then we can define  $\tau : G \rightarrow \text{GL}(W)$  by  $\tau(g) = T \circ \rho(g) \circ T^{-1}$ ; this  $\rho \cong \tau$ . In other words, the representation is unique up to isomorphism under change of basis.

*Example.* Consider  $G = \{g_1, \dots, g_n\} = \{h_1, \dots, h_n\}$ , and fix  $g \in G$ . Let  $gg_i = g_{\alpha(i)}$  and  $gh_i = h_{\beta(i)}$  where  $\alpha, \beta \in S_n$ . Fix an  $n$ -dimensional vector space  $V$  with basis  $\{b_1, \dots, b_n\}$ . Then two regular representations are given by

$$\rho_1 : G \rightarrow \text{GL}(V), \rho_1(g)(b_i) = b_{\alpha(i)}$$

$$\rho_2 : G \rightarrow \text{GL}(V), \rho_2(g)(b_i) = b_{\beta(i)}$$

Let  $\gamma \in S_n$  be such that  $h_{\gamma(i)} = g_i$ , and define  $T : V \rightarrow V$  by  $T(b_i) = b_{\gamma(i)}$ . Then

$$gg_i = g_{\alpha(i)} = gh_{\gamma(i)} = h_{\beta\gamma(i)} = g_{\gamma^{-1}\beta\gamma(i)}$$

so that  $\alpha = \gamma^{-1}\beta\gamma$ . Thus for each  $b_i$ ,

$$\begin{aligned} T \circ \rho_1(g) \circ T^{-1}(b_i) &= T \circ \rho_1(g)(b_{\gamma^{-1}(i)}) \\ &= T(b_{\alpha\gamma^{-1}(i)})b_{\gamma\alpha\gamma^{-1}(i)} \\ &= b_{\beta(i)} = \rho_2(g)(b_i) \end{aligned}$$

so that  $T \circ \rho_1(g) \circ T^{-1} = \rho_2(g)$ .

Note: conjugate elements have the same cycle type.

## SUBREPRESENTATIONS

What should a subrepresentation of  $\rho : G \rightarrow \text{GL}(V)$  mean?

We would like a subspace  $W \leq V$  such that  $\tau : G \rightarrow \text{GL}(W)$  is a representation given by  $\tau(g)(w) = \rho(g)(w)$  for all  $w \in W$ . Moreover, to make this well-defined, we need  $W$  to be  $\rho(g)$ -invariant for every  $g \in G$  ( $\rho(g)(W) \subseteq W$ ).

Suppose  $T : V \rightarrow V$  is a linear operator, and  $W \leq V$  is a  $T$ -invariant subspace; i.e.  $T(W) \subseteq W$ . In particular, the restriction operator  $T_W : W \rightarrow W$  is well-defined.

**Definition.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation. A subspace  $W \subseteq V$  is said to be  **$G$ -stable** if  $W$  is  $\rho(g)$ -invariant for all  $g \in G$ . A **subrepresentation** of  $\rho$  is a representation  $\rho_W : G \rightarrow \text{GL}(W)$  where for all  $g \in G$  and  $w \in W$ ,  $\rho_W(g)(w) = \rho(g)(w)$  where  $W$  is a  $G$ -stable subspace of  $V$ .

*Example.* Suppose  $\rho : G \rightarrow \text{GL}(V)$  be the regular representation. Take  $W = \text{span}\{\sum_{g \in G} v_g\}$ , which is clearly  $G$ -stable, and  $\rho_W : G \rightarrow \text{GL}(W)$  is isomorphic to the trivial representation.

Similarly, let  $\rho : S_n \rightarrow \text{GL}(V)$  be the regular representation,  $W = \text{span}\{\sum_{\sigma \in S_n} \text{sgn}(\sigma)v_\sigma\}$ ; this is isomorphic to the sign representation.

**0.1 Theorem.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation,  $W \leq V$   $G$ -stable. Then there exists a  $G$ -stable subspace  $W'$  such that  $V = W \oplus W'$ .

**PROOF** Take any inner product  $\langle x, y \rangle$  on  $V$ . Then for any  $x, y \in V$ , define

$$\langle x, y \rangle^* = \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle$$

This is also an inner product. Let  $x, y \in V$  and let  $h \in G$ . Then

$$\begin{aligned} \langle \rho(h)(x), \rho(h)(y) \rangle^* &= \sum_{g \in G} \langle \rho(g)\rho(h)(x), \rho(g)\rho(h)(y) \rangle \\ &= \sum_{g \in G} \langle \rho(gh)(x), \rho(gh)(y) \rangle \\ &= \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle \end{aligned}$$

Thus every  $\rho(h)$  is unitary with respect to  $\langle \cdot, \cdot \rangle^*$ . Let  $W \leq V$  be  $G$ -stable, and take  $W' = W^\perp$  with respect to  $\langle \cdot, \cdot \rangle^*$ . Then  $V = W \oplus W'$ . Let's see that  $W^\perp$  is  $G$ -stable. Let  $x \in W^\perp$ ,  $w \in W$ ,

and  $g \in G$ , so that

$$\begin{aligned} \langle \rho(g)(x), w \rangle^* &= \langle x, \rho(g)^*(w)^* \rangle = \langle x, \rho(g)^{-1}(w) \rangle^* \\ &= \langle x, \underbrace{\rho(g^{-1})(w)}_{\in W} \rangle^* \\ &= 0 \end{aligned}$$

and  $\rho(g)(W^\perp) \subseteq W^\perp$  as required.  $\blacksquare$

**Definition.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation, and  $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$  where each  $W_i$  is  $G$ -stable. For each  $i$ , let  $\rho_i = \rho|_{W_i}$ . For each  $v = \sum w_i \in V$ , we have  $\rho(g)(v) = \sum \rho(g)(w_i) = \sum \rho_i(g)(w_i)$ . In this case, we write

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k$$

and call  $\rho$  a direct sum of the  $\rho_i$ 's.

The previous definition is written as an internal direct sum of  $V$ . Externally, given vector spaces  $W_1, \dots, W_k$  and representations  $\rho_i : G \rightarrow \text{GL}(W_i)$ , we can define

$$(\rho_1 \oplus \cdots \oplus \rho_k) : G \rightarrow \text{GL}(W_1 \oplus \cdots \oplus W_k)$$

by  $(\rho_1 \oplus \cdots \oplus \rho_k)(g)(w_1, \dots, w_k) = (\rho_1(g)(w_1), \dots, \rho_k(g)(w_k))$ . If  $\rho_i : G \rightarrow \text{GL}(W_i)$  is a subrepresentation of  $\rho : G \rightarrow \text{GL}(V)$ , we often say “ $W_i$  is a subrepresentation of  $V$ ”.

**Definition.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation. We say  $\rho$  is **irreducible** if  $V \neq \{0\}$  and the only  $G$ -stable subspaces of  $V$  are  $\{0\}$  and  $V$ .

Clearly,

**0.2 Theorem.** Every representation  $\rho : G \rightarrow \text{GL}(V)$  can be written as a direct sum of irreducible sub-representations.

*Example.* Let  $\rho : S_3 \rightarrow \text{GL}(\mathbb{C}^3)$  be the permutation representation with respect to the standard basis  $\{e_1, e_2, e_3\}$ . Consider  $W_1 = \text{span}\{e_1 + e_2 + e_3\}$  and  $W_2 = \text{span}\{e_1 - e_2, e_2 - e_3\}$ . Is  $W_2$  irreducible?

More generally, if  $V = W_1 \oplus \cdots \oplus W_k$  and  $\dim W_i = 1$  and  $\deg(\rho_i) = 1$ ,

$$\rho(gh)(\sum w_i) = \sum \rho_i(gh)(w_i) = \sum \rho_i(g)\rho_i(h)(w_i) = \sum \rho_i(h)\rho_i(g)(w_i)$$

so that  $\rho(gh) = \rho(hg)$ . In our example, this does not happen, since  $\rho(g) \neq I$  when  $g \neq 1$  and  $S_3$  is not abelian.

*Example.* Let  $\rho : S_3 \rightarrow \text{GL}(V)$  be the regular representation. Let  $W_1 = \text{span}\{\sum_{\sigma \in S_3} v_\sigma\}$  and  $W_2 = \text{span}\{\sum_{\sigma \in S_3} \text{sgn}(\sigma)v_\sigma\}$ , and Now let's focus on  $W_3$ . A basis for  $W_3$  is given by

$$\begin{aligned} e_1 &= v_e - v_{(123)} & e_2 &= v_e - v_{(123)} \\ e_3 &= v_{(12)} - v_{(13)} & e_4 &= v_{(12)} - v_{(23)} \end{aligned}$$

Recall that  $S_3 = \langle (12), (123) \rangle$ ; suffices to show stability with respect to generators.

$$\begin{aligned} \rho(12) : e_1 &\mapsto e_4, e_2 \mapsto e_3, e_3 \mapsto e_2, e_4 \mapsto e_1 \\ \rho(123) : e_1 &\mapsto e_2 - e_1, e_2 \mapsto -e_1, e_3 \mapsto e_4 - e_3, e_4 \mapsto -e_3 \end{aligned}$$

Let  $U_1 = \text{span}\{e_1 - e_4, e_2 + e_3 - e_1\}$

## 1 TENSOR PRODUCTS

Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  be representations. We define the representation  $\rho \otimes \tau : G \rightarrow \text{GL}(V \otimes W)$

$$(\rho \otimes \tau)(g)(v \otimes w) = \rho(g)(v) \otimes \tau(g)(w)$$

## 2 CHARACTER THEORY

We define the character of  $\rho$  by  $\rho : G \rightarrow \mathbb{C}$  as  $\chi(G) = \text{Tr}(\rho(g))$ .

*Remark.* If we choose a basis  $\beta$  for  $V$ , then define  $A(g) = [\rho(g)]_\beta$  and  $\chi(G)$  is given by the sum of the diagonal entries of  $A(g)$ . Furthermore, if  $A, B \in M_n(\mathbb{C})$ , then  $\text{Tr}(AB) = \text{Tr}(BA)$ .

The remark implies a number of facts:

- (i)  $\rho \cong \tau$ , then  $\text{Tr}(\rho(g)) = \text{Tr}(\tau(g))$ .
- (ii)  $\text{Tr}(T)$  is the sum of eigenvalues of  $T$
- (iii)  $\chi(1) = \dim(V)$ .

**2.1 Proposition.** *For every  $g \in G$  the eigenvalues of  $\rho(g)$  have modulus 1. In particular,  $\chi(g^{-1}) = \overline{\chi(g)}$ .*

**PROOF** Set  $n = |G|$ ; then  $\rho(g)^n = \rho(g^n) = I$  so that  $\lambda^n - 1 = 0$  for any eigenvalue  $\lambda$ , so  $|\lambda| = 1$ . Furthermore,

$$\overline{\chi(g)} = \overline{\sum \lambda_i} = \sum \overline{\lambda_i} = \sum \lambda_i^{-1} = \chi(g^{-1})$$

proving the second component. ■

**2.2 Proposition.** *Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$ . Then  $\chi_{\rho \oplus \tau} = \chi_\rho + \chi_\tau$  and  $\chi_{\rho \otimes \tau} = \chi_\rho \cdot \chi_\tau$ .*

**PROOF** Let  $\beta_1 = \{v_1, \dots, v_n\}$  be a basis for  $V$  and  $\beta_2 = \{w_1, \dots, w_m\}$  a basis for  $W$ .

Then a basis for  $V \oplus W$  is given by  $\beta = \{(v_1, 0), \dots, (v_n, 0), (0, w_1), \dots, (0, w_m)\}$ . In particular,

$$[(\rho \oplus \tau)(g)]_\beta = \begin{pmatrix} [\rho(g)]_{\beta_1} & \\ & [\tau(g)]_{\beta_2} \end{pmatrix}$$

and the trace result follows.

A basis for  $V \otimes W$  is given by  $\gamma = \{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$  in lexicographic order. Fix  $g \in G$ , and set  $A = [\rho(g)]_{\beta_1}$ ,  $B = [\tau(g)]_{\beta_2}$ . Fix  $v_i \otimes w_j \in \gamma$ . Then

$$\begin{aligned} (\rho \otimes \tau)(g)(v_i \otimes w_j) &= \rho(g)(v_i) \otimes \tau(g)(w_j) \\ &= (a_{1i}v_1 + \dots + a_{ni}v_n) \otimes (b_{1j}w_1 + \dots + b_{mj}w_m) \\ &= \dots + a_{ii}b_{jj} \cdot (v_i \otimes w_j) + \dots \\ &= \text{Tr}([\rho \otimes \tau](g))_\delta = \sum_{i,j} a_{ii}b_{jj} = \text{Tr}(A)\text{Tr}(B) = \chi_\rho(g) \cdot \chi_\tau(g) \end{aligned} \quad \blacksquare$$



*Example.* Suppose  $\rho : S_n \rightarrow \text{GL}(\mathbb{C}^n)$  is the permutation representation with respect to  $\{e_1, \dots, e_n\}$ . Then  $\chi(\sigma) = |\{e_i : \rho(\sigma)(e_i) = e_i\}| = |\text{Fix}(\sigma)|$ , which is the number of indices  $i$  fixed by  $\sigma$ . Since  $S_n$  acts transitively on  $\{1, \dots, n\}$ , there is exactly 1 orbit, so by Burnside's lemma,

$$n! = |S_n| = \sum_{\sigma \in S_n} \chi(\sigma)$$

*Example.* Let  $\rho : G \rightarrow \text{GL}(V)$  be the regular representation. Note that if  $g \neq 1$ , then for all  $h \in G$ ,  $gh \neq h$ . In particular, this means that  $\chi(g) = 0$  if  $g \neq 1$ , and  $\chi(1) = |G|$  (the dimension of  $V$ ).

*Example.* Let  $\rho : S_3 \rightarrow \text{GL}(V)$  be the regular representation. Recall that  $V = W_1 \oplus W_2 \oplus U_1 \oplus U_2$  where  $W_1$  is the trivial representation,  $W_2$  is the sign representation, and  $U_1, U_2$  are isomorphic. Let  $S_3 = \langle (12), (123) \rangle$ ; then we have

$x_1$	1	1
$x_2$	-1	1
$x_3$	$a$	$b$
$x_4$	$a$	$b$

In particular,  $\chi(12) = 1 - 1 + 2a = 0$  and  $\chi(123) = 1 + 1 + 2b = 0$ , so  $b = -1$ .

*Example.* Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation. In particular,  $\rho(ghg^{-1}) = \rho(g)\rho(h)\rho(g)$  so that  $\text{Tr} \rho(ghg^{-1}) = \text{Tr} \rho(h)$  so  $\chi(ghg^{-1}) = \chi(h)$ ; in other words, that characters are constant on conjugacy classes.

**2.3 Lemma. (Schur)** Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  be irreducible representations, and suppose  $T : V \rightarrow W$  is linear such that for all  $g \in G$ ,  $\tau(g) \circ T = T \circ \rho(g)$ . Then either  $T = 0$  or  $T$  is an isomorphism and  $\rho \cong \tau$ . Moreover, if  $V = W$  and  $\rho = \tau$ , then  $T$  is a scalar multiple of the identity.

**PROOF** Assume  $T \neq 0$ .

Let's first see that  $T$  is injective, and let  $v \in \ker(T)$ . Then for any  $g \in G$ ,  $T(\rho(g)(v)) = \tau(g)(T(v)) = 0$ , so  $\rho(g)(v) \in \ker(T)$ . Thus  $\ker(T)$  is  $G$ -stable (with respect to  $\rho$ ). Since  $\rho$  is irreducible and  $T \neq 0$ ,  $\ker(T) = \{0\}$ .

We also have that  $T$  is surjective. Let  $v \in \text{Im}(T)$  and say  $v = T(X)$  with  $x \in V$ . Then for  $g \in G$ ,  $\tau(g)(v) = \tau(g)(T(x)) = T(\rho(g)(x)) \in \text{Im}(T)$  so  $\text{Im}(T)$  is  $G$ -stable, and again by irreducibility of  $\tau$ ,  $\text{Im}(T) = W$ . Thus  $T$  is an isomorphism.

Now let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $T$  and consider  $T' = T - \lambda I$ . Now, note that for  $g \in G$ ,  $\rho(g)T' = T'\rho(g)$ , but  $T'$  has non-trivial kernel, so in fact  $T' = 0$ . ■

**2.4 Corollary.** Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  be irreducible, and  $T : V \rightarrow W$  linear. Consider

$$T' = \frac{1}{|G|} \sum_{g \in G} \tau(g)^{-1} T \rho(g)$$

Then

- (i) If  $T' \neq 0$ , then  $\rho \cong \tau$  via  $T'$ .
- (ii) If  $V = W$ ,  $\rho = \tau$ , then  $T' = \text{Tr}(T)/\dim(V) \cdot I$ .

PROOF Clearly  $T' : V \rightarrow W$  is linear, and for any  $h \in G$ ,

$$\begin{aligned}\tau(h)T' &= \tau(h) \frac{1}{|H|} \sum_{g \in G} \tau(g^{-1})T\rho(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \tau(hg^{-1})T\rho(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \tau(g^{-1})T(\rho(g)) \\ &= \frac{1}{|G|} \sum_{g \in G} \tau(g^{-1})T\rho(g)\rho(h) \\ &= T'\rho(h)\end{aligned}$$

If  $V = W$  and  $\rho = T$ , then  $\text{Tr}(T') = \frac{1}{|G|} \text{Tr}(T) \cdot |G| = \text{Tr}(T) = \alpha \dim(V)$ , so  $\alpha = \text{Tr}(T)/\dim(V)$ . ■

Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  be irreducible representations, and  $T : V \rightarrow W$  linear. Let  $\beta$  be a basis for  $V$  and  $\gamma$  a basis for  $W$ . Then for  $g \in G$ , let  $[\rho(g)]_\beta = (a_{ij}(g))$ ,  $[\tau(g)]_\gamma = (b_{kl}(g))$ ,  $[T]_\beta^\gamma = (x_{ki})$ , and  $[T']_\beta^\gamma = (x'_{ki})$ .

By matrix multiplication,  $x'_{ki} = \frac{1}{|G|} \sum_g \sum_{j,l} b_{kl}(g^{-1})x_{lj}a_{ji}(g)$ . If  $\rho \not\cong \tau$ , then  $T' = 0$ , so by viewing the RHS as a polynomial in the  $x_{ij}$ , we have

$$\frac{1}{|G|} \sum_g b_{kl}(g^{-1})a_{ji}(g) = 0$$

But now if  $\rho = \tau$ , then  $T' = \lambda I$  where  $\lambda = \text{Tr}(T)/\dim(V)$  so that

$$\frac{1}{|G|} \sum_g \sum_{j,l} a_{kl}(g^{-1})x_{lj}a_{ji}(g) = \lambda \delta_{ki} = \frac{1}{\dim(V)} \sum_{j,l} \delta_{ki} \delta_{jl} x_{lj}$$

Then by equating coefficients of  $x_{lj}$ , we have

$$\frac{1}{|G|} \sum_g a_{kl}(g^{-1})a_{ji}(g) = \frac{1}{\dim(V)} \delta_{ki} \delta_{jl}$$

*Remark.* If  $G$  is a finite group, then consider the vector space of all functions  $\phi : G \rightarrow \mathbb{C}$ . For any  $\phi, \psi$  in this vector space,  $\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_g \phi(g) \overline{\psi(g)}$  defines an inner product. Then if  $\chi_1, \chi_2$  are characters of  $G$ , then

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_g \chi_1(g) \overline{\chi_2(g)}$$

We thus have:

**2.5 Theorem.** If  $\chi$  is a character of an irreducible representation, then  $\langle \chi, \chi \rangle = 1$ , and if  $\chi_1$  and  $\chi_2$  correspond to non-isomorphic representations, then  $\langle \chi_1, \chi_2 \rangle = 0$ .

PROOF Say  $[\rho(g)]_\beta = (a_{ij}(g))$  where  $\rho$  is an irreducible representation with character  $\chi$ . Then

$$\begin{aligned}\langle \chi, \chi \rangle &= \frac{1}{|G|} \sum_g \chi(g) \chi(g^{-1}) = \frac{1}{|G|} \sum_g \chi(g^{-1}) \chi(g) \\ &= \frac{1}{|G|} \sum_g \sum_{i,j} a_{ii}(g^{-1}) a_{jj}(g) = \sum_{i,j} \left( \frac{1}{|G|} \sum_g a_{ii}(g^{-1}) a_{jj}(g) \right) \\ &= \sum_{i,j} \left( \frac{1}{|G|} \sum_g a_{ii}(g^{-1}) a_{ii}(g) \right) \\ &= \sum_i \frac{1}{\dim(V)} = 1\end{aligned}$$

To see the second part,

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_g \chi_1(g) \chi_2(g^{-1}) = \frac{1}{|G|} \sum_g \sum_{i,j} a_{ii}(g) a_{jj}(g^{-1}) = \sum_{i,j} 0 = 0 \quad \blacksquare$$

If  $\chi$  is a character corresponding to an irreducible representation, we say  $\chi$  is irreducible. If  $\rho$  and  $\tau$  are isomorphic representations, we say  $\chi_\rho$  and  $\chi_\tau$  are isomorphic (in fact  $\chi_\rho = \chi_\tau$ ).

**2.6 Corollary.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation with character  $\chi$ . Say  $V = W_1 \oplus \cdots \oplus W_k$  is an irreducible decomposition of  $V$ . If  $\tau : G \rightarrow \text{GL}(W)$  is an irreducible representations with character  $\phi$ , then the number of  $W_i$  isomorphic to  $W$  (i.e.  $\rho_i \cong \tau$ ) is  $\langle \chi, \phi \rangle$ .

PROOF Write  $\chi = n_1 \chi_1 + \cdots + n_l \chi_l$ , where the  $\chi_i$  are pairwise non-isomorphic. Then  $\langle \chi, \chi_i \rangle = n_i$ . ■

Let  $\tau : G \rightarrow \text{GL}(V)$  be irreducible, and let  $\tau$  have character  $\phi$ . Then

$$\langle \chi, \phi \rangle = \sum_{i=1}^k \langle \chi_i, \phi \rangle$$

Now,  $\langle \chi_i, \phi \rangle = 1$  if and only if  $\rho_i \cong \tau$ , so that  $\langle \chi, \phi \rangle$  counts the number of times in which  $\tau$  appears in the irreducible decomposition of  $\rho$ .

**2.7 Corollary.** If two representations of  $G$  have the same character, then they are isomorphic.

PROOF They have the same irreducible decomposition. ■

**2.8 Corollary.** If  $\rho : G \rightarrow \text{GL}(V)$  is a representation and  $\chi$  is a character, then  $\langle \chi, \chi \rangle \in \mathbb{N}$  and  $\langle \chi, \chi \rangle = 1$  if and only if  $\chi$  is irreducible.

PROOF If  $\chi_1, \dots, \chi_k$  are irreducible, write  $\chi = n_1 \chi_1 + \cdots + n_k \chi_k$  so that  $\langle \chi, \chi \rangle = n_1^2 + \cdots + n_k^2 \in \mathbb{N}$ . ■

**2.9 Proposition.** Every irreducible representation of  $G$  occurs as a subgroup for the regular representation of  $G$ , with multiplicity equal to its degree.

PROOF Let  $\chi$  be an irreducible character of  $G$ . Then

$$\langle \chi, \chi_{\text{reg}} \rangle = \frac{1}{|G|} \sum_g \chi(g) \overline{\chi_{\text{reg}}(g)} = \frac{1}{|G|} \chi(1) \overline{\chi_{\text{reg}}(1)} = \frac{1}{|G|} \deg(\chi) \quad \blacksquare$$

**2.10 Corollary.** Let  $\chi_1, \dots, \chi_k$  be the distinct irreducible characters of  $G$ , with  $\deg(\chi_i) = n_i$ . Then  $\sum n_i^2 = |G|$  for  $g \neq 1$ ,  $\sum_{i=1}^k n_i \chi_i(g) = 0$

PROOF Recall that  $\chi_{\text{reg}} = n_1 \chi_1 + \dots + n_k \chi_k$ . Then  $\chi_{\text{reg}}(1) = |G| = n_1^2 + \dots + n_k^2$ , and evaluation at  $g \neq 1$  gives the desired result.  $\blacksquare$

**Definition.** Let  $G$  be a group. A function  $f : G \rightarrow \mathbb{C}$  is called a class function if  $f$  is constant on each conjugacy class, i.e. for all  $a, b \in G$ ,  $f(bab^{-1}) = f(a)$ .

**2.11 Proposition.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation. Then

$$\rho_f = \sum_g f(g) \rho(g)$$

is a linear operator on  $V$ . If  $\rho$  is irreducible of degree  $n$ , then  $\rho_f = \lambda I$ , where  $\lambda = \frac{|G|}{n} \langle f, \bar{\chi} \rangle$  where  $\chi$  is the character of  $\rho$ .

PROOF Note that

$$\begin{aligned} \rho_f \circ \rho(h) &= \sum_g f(g) \rho(g) \rho(h) = \sum_g f(g) \rho(gh) \\ &= \sum_g f(hgh^{-1}) \rho(hg) \\ &= \sum_g f(g) \rho(h) \rho(g) = \rho(h) \circ \rho_f \end{aligned}$$

so by Schur,  $\rho_f = \lambda I$  where  $\lambda = \text{Tr}(\rho_f)/n$ . However,  $\text{Tr}(\rho_f) = \text{Tr}(\sum_g f(g) \rho(g)) = \sum_g f(g) \chi(g) = |G| \langle f, \bar{\chi} \rangle$ .  $\blacksquare$

Recall that

- $\langle \chi, \chi \rangle = 1$  if and only if  $\chi$  is irreducible
- If  $\chi_\rho$  and  $\chi_\tau$  are irreducible then  $\langle \chi_\rho, \chi_\tau \rangle = 0$  if  $\rho \not\cong \tau$ , and 1 otherwise.
- If  $\chi'$  is an irreducible subrepresentation of  $\chi$ , then  $\langle \chi, \chi' \rangle$  is the multiplicity of  $\chi'$  in  $\chi$ .
- $|G| = n_1^2 + \dots + n_k^2$  where  $n_i$  is the multiplicity of  $\chi_i$  as an irreducible subrepresentation of the regular representation.
- Every irreducible character is a character of some subrepresentation of the regular rep?
- ... every irreducible representation is a subrepresentation of the regular rep?

and

$$\rho_f = \sum_g f(g)\rho(g) = \lambda I$$

where  $\lambda = |G|/\dim(V) \cdot \langle f, \bar{\chi} \rangle$ .

**2.12 Proposition.** *Let  $G$  be a group. The irreducible characters of  $G$  form an orthonormal basis for the vector space of class functions on  $G$ .*

PROOF Let  $\beta = \{\chi_1, \dots, \chi_k\}$  be the irreducible characters of  $G$ . We know that  $\beta$  is orthonormal, and hence linearly independent. Let  $W = \text{span}(\beta)$ . To show  $W = V$  where  $V$  is the space of class functions, we prove that  $W^\perp = \{0\}$ . Let  $f \in W^\perp$ , and suppose  $\rho : G \rightarrow \text{GL}(V)$  is irreducible. By A2,  $\bar{\chi}_1, \dots, \bar{\chi}_k$  are all irreducible characters of  $G$ . Thus  $\rho_f = 0$ . By considering irreducible decompositions,  $\rho_f = 0$  for all representations  $\rho : G \rightarrow \text{GL}(V)$ . In particular, when  $\rho$  is the regular representation,

$$0 = \rho_f(v_1) = \sum_g f(g)\rho(g)(v_1) = \sum_g f(g)v_g$$

so by independence of  $\{v_g : g \in G\}$ ,  $f(g) = 0$  for all  $g \in G$ . ■

**2.13 Corollary.** *The number of irreducible characters of  $G$  is equal to the number of conjugacy classes of  $G$ .*

PROOF Let  $C_1, \dots, C_k$  be the conjugacy classes. Then a basis for  $V_{\text{class}} = \{\phi_1, \dots, \phi_k\}$  where each  $\phi_i$  is the indicator for  $C_i$ . Since bases must have the same size, the result follows. ■

**2.14 Proposition.** *Let  $G$  be a group,  $g \in G$ , and  $O_g$  the conjugacy class of  $g$ . Let  $\chi_1, \dots, \chi_k$  be the irreducible characters of  $G$ . Then*

1.  $\sum_{i=1}^k |\chi_i(g)|^2 = |G|/|O_g|$
2. If  $h$  is not conjugate to  $g$ , then  $\sum_{i=1}^k \chi_i(g)\overline{\chi_i(h)} = 0$ .

PROOF Define  $\phi : G \rightarrow \mathbb{C}$  where  $\phi(x)$  is the indicator function for  $O_g$ . Write  $\phi = \sum_{i=1}^k \lambda_i \chi_i$  where

$$\lambda_i = \langle \phi, \chi_i \rangle = \frac{1}{|G|} \sum_x \phi(x)\overline{\chi_i(x)} = \frac{|O_g|\overline{\chi_i(g)}}{|G|}$$

Therefore,

$$\phi(x) = \frac{|O_g|}{|G|} \sum_{i=1}^k \overline{\chi_i(g)}\chi_i(x)$$

Then the result follows by evaluating  $\phi$  at  $g$  and  $h$ . ■

*Example.* Let's compute the character table of  $S_3$ . There are 2 degree 1 representations, and 3 irreducible characters since there are three conjugacy classes (cycle types). In particular,  $|S_3| = 6 = 1^2 + 1^2 + n_3^2$ , so  $n_3 = 2$ .

	$\epsilon$ $(12)$ $(123)$
$(\text{triv})\chi_1$	$1$ $1$ $1$
$(\text{sgn})\chi_2$	$1$ $-1$ $1$
$\chi_3$	$2$ $a$ $b$

Note that the columns must be orthogonal, so by the previous proposition, we have  $a = 0$  and  $b = -1$ .

Let  $\chi_1, \dots, \chi_k$  be the irreducible characters of  $G$ . Then  $\sum_{g \in G} \chi_i(g) = |G|$  and  $\sum_{i=1}^k |\chi_i(g)|^2 = |G|/|O_g|$ .

Let  $G$  be abelian. By A1,  $G$  has  $|G|$  representations of degree 1, and  $[G : [G, G]] = |G|$ . Since  $G$  has  $|G|$  conjugacy classes, these are all of the irreducible representations of  $G$ . Suppose  $G$  is a group whose irreducible representations are all degree one. Since  $n_1^2 + \dots + n_k^2 = |G|$ , then  $k = |G|$ .

**2.15 Proposition.** *Let  $H$  be an abelian subgroup of  $G$ . Then any irreducible representation of  $G$  has degree at most  $[G : H]$ .*

PROOF Let  $\rho : G \rightarrow \text{GL}(V)$  be an irreducible representation of  $G$ . Consider the restriction  $\tilde{\rho} : H \rightarrow \text{GL}(V)$ . Let  $W \leq V$  be an irreducible subrepresentation of  $\tilde{\rho}$ . Since  $H$  is abelian,  $\dim W = 1$ . Suppose  $W = \text{span}\{x\}$ , and let  $W' = \{\rho(g)(x) : g \in G\}$  so that  $W'$  is  $G$ -stable, and in fact  $W' = V$  since  $\rho$  is irreducible.

Take  $g \in G$  and  $h \in H$ , so  $\rho(gh) = \rho(g)\rho(h)(x) = \rho(g)(\alpha x) = \alpha\rho(g)(x)$ . Say  $g_1, \dots, g_m$  are coset representatives of  $H$  in  $G$ . Then  $V = W' = \text{span}\{\rho(g_i)(x) : 1 \leq i \leq m\}$ , then  $\dim(V) \leq m = [G : H]$ . ■

*Example.* Consider  $D_4$ . Then the number of degree 1 representations is  $[D_4 : \langle r^2 \rangle] = 4$ . Since there are 5 conjugacy classes, we know that there are 5 irreducible representations, so that  $n_5^2 = 8$ . Let's make the character table:

$D_4$	1	$r$	$r^2$	$s$	$rs$
$\chi_1$	1	1	1	1	1
$\chi_2$	1	-1	1	1	-1
$\chi_3$	1	1	1	-1	-1
$\chi_4$	1	-1	1	-1	1
$\chi_5$	2	$a$	$b$	$c$	$d$

But then by column orthogonality, we have  $a = 0$ ,  $b = -2$ ,  $c = 0$ ,  $d = 0$ .

*Example.* Consider  $S_4$ . Then  $[S_4 : A_4] = 2$  so there are two degree 1 representations (the trivial and the sign), and the conjugacy classes are given by 1, (12), (12)(34), (123), (1234), so there are 5 irreducible representations. Since  $24^2 = 1^2 + 1^2 + n_3^2 + n_4^2 + n_5^2$ , we have  $22 = n_3^2 + n_4^2 + n_5^2$ , which forces  $n_3 = 2$  and  $n_4 = n_5 = 3$ . Now we have

$D_4$	1	(12)	(12)(34)	(123)	(1234)
$\chi_1$	1	1	1	1	1
$\chi_2$	1	-1	1	1	-1
$\chi_3$	2	1	1	-1	-1
$\chi_4$	3	-1	1	-1	1
$\chi_5$	3	$a$	$b$	$c$	$d$

Note that  $K = \{1, (12)(34), (13)(24), (14)(23)\} \trianglelefteq S_4$  and  $H = \{1, (12), (13), (123), (132), (23)\}$ , so  $S_4 = KH$ . Let  $\rho$  be an irreducible representation of  $H$  of degree 2:

$S_3$	1	(12)	(123)
$\alpha_1$	1	1	1
$\alpha_2$	1	-1	1
$\alpha_3$	2	0	-1

Then  $\rho : S_4 \rightarrow \text{GL}(V)$  by  $\rho(kh) := \rho(h)$  is an irreducible representation of  $S_4$  since  $K \trianglelefteq S_4$ .

### 3 INDUCED REPRESENTATIONS

Given a subgroup  $H \leq G$  and a representation  $\rho : H \rightarrow \text{GL}(V)$ , construct a representation of  $G$ . Let  $H \leq G$  and  $\rho : H \rightarrow \text{GL}(V)$  a representation. Say the cosets of  $H$  in  $G$  are  $g_1H, \dots, g_mH$ . For each  $i$ , let  $g_iV = \{g_iv : v \in V\}$  be an isomorphic copy of  $G$ , and let  $W = \bigoplus_{i=1}^m g_iV$  so that every  $w \in W$  can be uniquely written as  $w = g_1v_1 + \dots + g_mv_m$ , where  $m = [G : H]$ . Fix  $g \in G$ ; then there exists  $\pi \in S_m$  such that for every  $i$ ,  $gg_i = g_{\pi(i)}h_i$ ,  $h_i \in H$ . We then define  $\text{Ind}_H^G(\rho) : G \rightarrow \text{GL}(W)$  by

$$\text{Ind}_H^G(\rho)(g)\left(\sum g_iw_i\right) = \sum g_{\pi(i)}\rho(h_i)v_i$$

*Example.* Let  $\{1\} \leq G$  and suppose  $\rho : \{1\} \rightarrow \text{GL}(\mathbb{C})$  is the trivial representation. Then  $G = \{g_1, \dots, g_n\}$ . Then for  $g \in G$ ,  $gg_i1 \in G$  and

$$\text{Ind}(\rho)(s)\left(\sum_{i=1}^n g_i\alpha_i\right) = \sum ggi\rho(1)(\alpha_i) = \sum ggi\alpha_i$$

so that  $\text{Ind}(\rho)$  is isomorphic to the regular representation.

*Example.* Consider  $\langle r \rangle \leq D_n$ , and let  $\rho : \langle r \rangle \rightarrow \text{GL}(\mathbb{C})$  be given by  $\rho(r)(1) = \zeta_n$ . Let the coset representatives be given by  $\epsilon$  and  $s$ .

- (i) Let  $r \in D_n$  so  $r\epsilon = \epsilon r$  and  $rs = sr^{n-1}$ . Fix  $W = \epsilon\mathbb{C} \oplus s\mathbb{C}$ . Then  $\text{Ind}(\rho) : D_n \rightarrow \text{GL}(W)$  is given by  $\text{Ind}(\rho)(r)(\epsilon\alpha_1 + s\alpha_2) = \epsilon\zeta_n\alpha_1 + 1 + s\zeta_n^{n-1}\alpha_2$ .
- (ii) Let  $s \in D_n$ . Then  $s\epsilon = s\epsilon$  and  $ss = \epsilon\epsilon$ . Then  $\text{Ind}(\rho)(s)(\epsilon\alpha_1 + s\alpha_2) = s\rho(\epsilon)(\alpha_1) + \epsilon\rho(\epsilon)(\alpha_2) = s\alpha_1 + \epsilon\alpha_2$ .

Take the basis  $\beta = \{\epsilon, s\}$  for  $W$ , so we have

$$[\text{Ind}(\rho)(r)]_\beta = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{n-1} \end{pmatrix} \quad [\text{Ind}(\rho)(s)]_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that  $\rho : G \rightarrow \text{GL}(V)$  is faithful if only if the unique zero map on  $v$  is 0.

**Definition.** Let  $M$  be an  $R$ -module. The **annihilator**  $\text{Ann}(M) = \{r \in R : rm = 0\}$ . Then  $M$  is **faithful** if  $\text{Ann}(M) = (0)$ .

**3.1 Proposition.** Let  $M$  be an  $R$ -module. Then  $\text{Ann}(M)$  is a (2-sided) ideal of  $R$ . Moreover,  $M$  is a faithful  $R/\text{Ann}(M)$ -module.

**Definition.** An  $R$ -module  $M$  is **irreducible** if  $M \neq (0)$  and the only submodules of  $M$  are  $(0)$  and  $M$ .

Recall that a division ring is a unital ring such that every non-zero element is invertible.

**3.2 Theorem.** Let  $M$  be an irreducible  $R$ -module. Then  $\text{End}_R(M)$  is a division ring.

**3.3 Theorem.** Let  $M, N$  be  $R$ -modules and let  $\psi : M \rightarrow N$  be a module homomorphism. Then  $M/\ker \psi \cong \psi(M) \leq N$ .

**3.4 Proposition.** *Let  $M$  be an irreducible  $R$ -module, then  $M \cong R/I$ , where  $I$  is a maximal left ideal. Conversely, if  $I$  is a maximal left ideal, then  $R/I$  is irreducible.*

**PROOF** Let  $M$  be an irreducible  $R$ -module and fix  $0 \neq m \in M$ , and define  $\phi : R \rightarrow M$  by  $\phi(r) = rm$ , so  $\phi$  is a homomorphism and  $R/\ker \phi \cong \phi(R) = M$  by irreducibility. But then  $I$  is maximal since  $R/I \cong M$  is simple. ■

**Definition.** Let  $R$  be a ring. Then the **Jacobson radical** of  $R$  is  $J(R) = \bigcap_{\text{irred left } M} \text{Ann}(M)$ .

**Definition.** A left ideal  $I$  of  $R$  is called **left quasiregular** if for all  $a \in I$ ,  $R(1 + a) = R$ .

**3.5 Theorem.** *If  $R$  is a ring, then the following are equivalent:*

- (i)  $a \in J(R)$ .
- (ii)  $Ra$  is left quasiregular
- (iii)  $a \in \bigcap_{I \leq R \text{ maximal}} I$ .