Boundary Reduction of Helmholtz Scattering Problem for exterior domains

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Let D be an open Lipschitz bounded domain, $D^c := \mathbb{R}^d \backslash \overline{D}$ and $\Gamma := \partial D$, $d = 2, 3$ with exterior normal field $\mathbf{n}$ pointing by convention toward the exterior domain. We use the superscript $i = 1$ (resp. $i = 2$) when referring to D^c (resp. D). For a given scalar field U, we introduce the Dirichlet (resp. Neumann) traces on Γ as:

$$\gamma_0 \mathrm{U}^i := \mathrm{U}|_\Gamma \text{ and } \gamma_1 \mathrm{U} := \langle \mathbf{n}, \nabla \mathrm{U}|_\Gamma \rangle. \tag{1}$$

For a given κ, we define the Helmholtz equation operator $L_\kappa : \mathrm{U} \mapsto \Delta \mathrm{U} + \kappa^2 \mathrm{U}$. For an index $\beta \in \{0, 1\}$ us consider the following exterior boundary value problems $(\mathbf{EP}_\beta)$, which consist in finding the scattered field induced in response to a given incident field:

Exterior Problems

Given $\kappa > 0$ and $\mathrm{U}^{inc} \in H^1_{\text{loc}}(D^c)$ with $L_\kappa \mathrm{U}^{inc} = 0$ in D^c, find $\mathrm{U} := (\mathrm{U}^{scat} + \mathrm{U}^{inc}) \in H^1_{\text{loc}}(D^c)$ such that

$$\begin{cases} \Delta \mathrm{U} + \kappa^2 \mathrm{U} = 0 & \text{in } D^c, \\ \gamma_\beta \mathrm{U} = 0 & \text{on } \Gamma, \\ \left| \frac{\partial}{\partial r} \mathrm{U}^{scat} - i\kappa \mathrm{U}^{scat} \right| = o(r^{\frac{1-d}{2}}) & \text{for } r \to \infty. \end{cases} \tag{2}$$

Let us introduce the boundary integral operators $\mathcal{V}_\kappa, \mathcal{K}_\kappa, \mathcal{W}_\kappa, \mathcal{K}'_\kappa$, $\mathcal{I}$ defined on Γ. Then, we have the following propositions:

Proposition 1. *The* sound-soft *problem* $(\mathbf{EP_0})$ *is equivalent to each of the following BIE given on* Γ*:*

$$\begin{cases} \mathcal{V}_\kappa \gamma_1 U & = \gamma_0 U^{inc} \ \textit{and} \\ \mathcal{V}_\kappa \gamma_1 U & = -(\mathcal{K}_\kappa - \frac{1}{2} \mathcal{I}) \gamma_0 U^{inc}. \end{cases} \tag{3}$$

Proposition 2. *The* sound-hard *problem* $(\mathbf{EP_1})$ *is equivalent to each of the following BIE given on* Γ*:*

$$\begin{cases} \mathcal{W}_\kappa \gamma_0 U & = \gamma_1 U^{inc} \ \textit{and} \\ \mathcal{W}_\kappa \gamma_0 U & = (\mathcal{K}'_\kappa + \frac{1}{2} \mathcal{I}) \gamma_1 U^{inc}. \end{cases} \tag{4}$$