NOW, FACTORIAL DESIGN!

- Suppose you have more than two factors.
- Example: Meredith Corporation, the publisher of Ladies' Home Journal magazine, sends more than a million letters to each potential subscribers hoping to secure as many subscriptions as possible.
- The marketing team looks for the right mix of promotional materials. It wants to experiment with factors such as Brochure message, Enclosed testimonials, Promotional offers (3 factors).
- This is beyond the Two-way ANOVA setting.

TERMINOLOGY

- Run: A particular experiment with each factor at a specified level.

- The notation: When a design is denoted as 2^3 factorial,

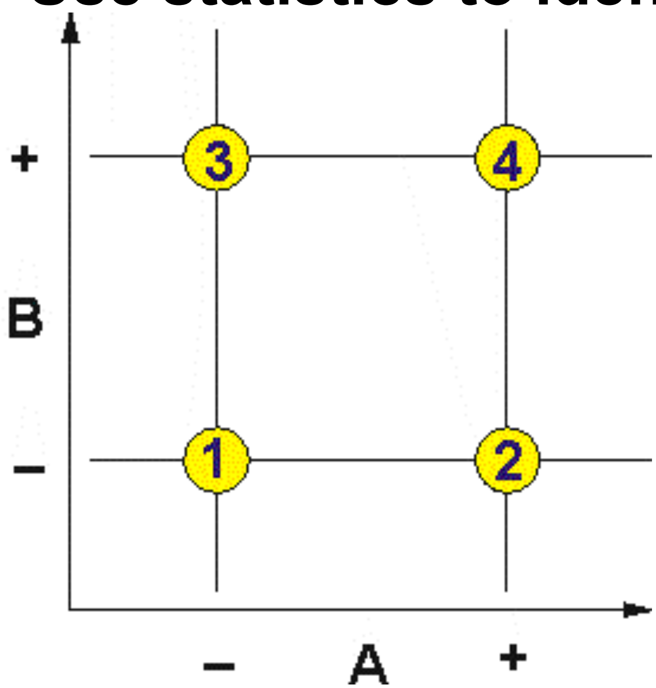
  - the number of factors: 3
  - levels each factor: 2 (conventionally, for the first (or low) level use –, and for the second (or high) level +)

  - total runs: how many experimental conditions there are in the design: $2^3=8$ runs → if there are two replications for each experimental conditions, then 16 runs.

  - a 3^2 factorial design has two factors, each with three levels, and $3^2=9$ experimental conditions.

TWO-LEVEL *FULL* FACTORIAL DESIGN: TWO FACTORS (A, B)

- **Run all high/low combinations of 2 (or more) factors**
- **Use statistics to identify the significant factors**



2^2 Full Factorial

What could be simpler?

$$2^2$$

X_1	X_2
−	−
+	−
−	+
+	+

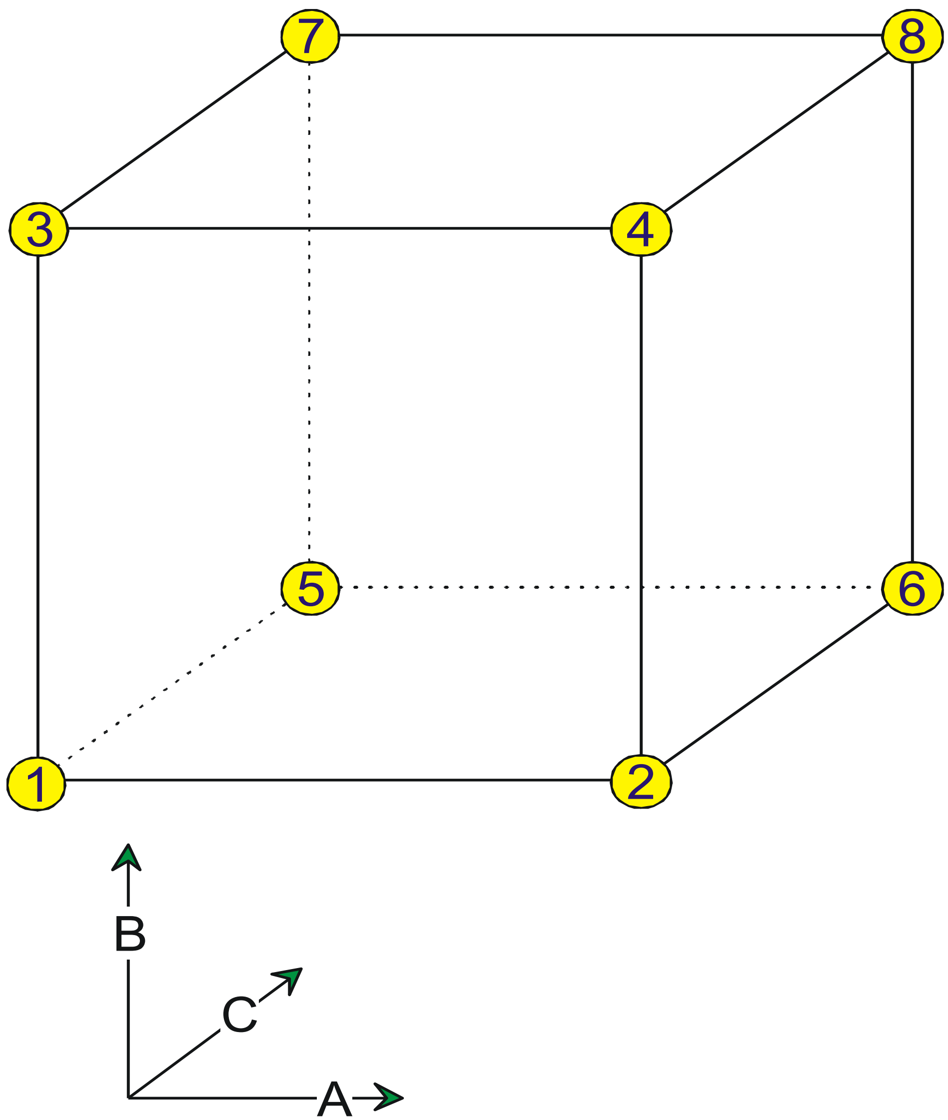
4 combinations that will be run

$$2^3$$

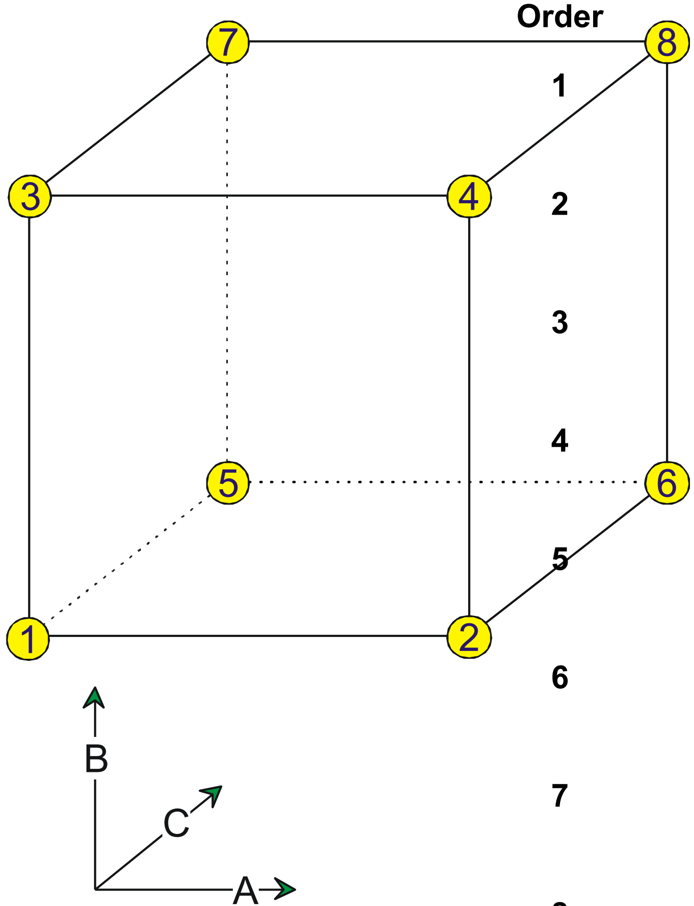
X_1	X_2	X_3
−	−	−
+	−	−
−	+	−
+	+	−
−	−	+
+	−	+
−	+	+
+	+	+

8 combinations that will be run

TWO-LEVEL, THREE FACTORS (A, B, C)-- 2^3 FACTORIAL DESIGN



TWO-LEVEL, THREE FACTORS (A, B, C)-- 2^3 FACTORIAL DESIGN



Standard Order	A	B	C	AB	AC	BC	ABC	Res-ponse
1	–	–	–	+	+	+	–	y_1
2	+	–	–	–	–	+	+	y_2
3	–	+	–	–	+	–	+	y_3
4	+	+	–	+	–	–	–	y_4
5	–	–	+	+	–	–	+	y_5
6	+	–	+	–	+	–	–	y_6
7	–	+	+	–	–	+	–	y_7
8	+	+	+	+	+	+	+	y_8

REAL EXAMPLE: TWO LEVEL FACTORIAL DESIGN

Kitchen scientists* conducted a 2^3 factorial experiment on microwave popcorn. The factors are:

A. Brand of popcorn (Costly, Cheap)

B. Time in microwave (4 min., 6 min

C. Power setting (75%, 100%)

A panel of neighborhood kids rated i) taste from one to ten scale and ii) weighed the un-popped kernels (UPKs).

* For full report, see Mark and Hank Andersons' *Applying DOE to Microwave Popcorn*, PI Quality 7/93, p30. (Uploaded at CANVAS)

TWO LEVEL FACTORIAL DESIGN
AS EASY AS POPPING CORN!

Run Ord	A Brand expense	B Time minutes	C Power percent	R_1 Taste rating*	R_2 UPKs oz.	Std. Ord
1	Costly	4	75	75	3.5	2
2	Cheap	6	75	71	1.6	3
3	Cheap	4	100	81	0.7	5
4	Costly	6	75	80	1.2	4
5	Costly	4	100	77	0.7	6
6	Costly	6	100	32	0.3	8
7	Cheap	6	100	42	0.5	7
8	Cheap	4	75	74	3.1	1

* Average scores multiplied by 10 to make the calculations easier.

TWO LEVEL FACTORIAL DESIGN
AS EASY AS POPPING CORN!

Run Ord	A Brand expense	B Time minutes	C Power percent	R_1 Taste rating	R_2 UPKs oz.	Std. Ord
1	+	–	–	75	3.5	2
2	–	+	–	71	1.6	3
3	–	–	+	81	0.7	5
4	+	+	–	80	1.2	4
5	+	–	+	77	0.7	6
6	+	+	+	32	0.3	8
7	–	+	+	42	0.5	7
8	–	–	–	74	3.1	1

Factors shown in coded values

It is also important to run experiments in random order.

Main Effect of X_i

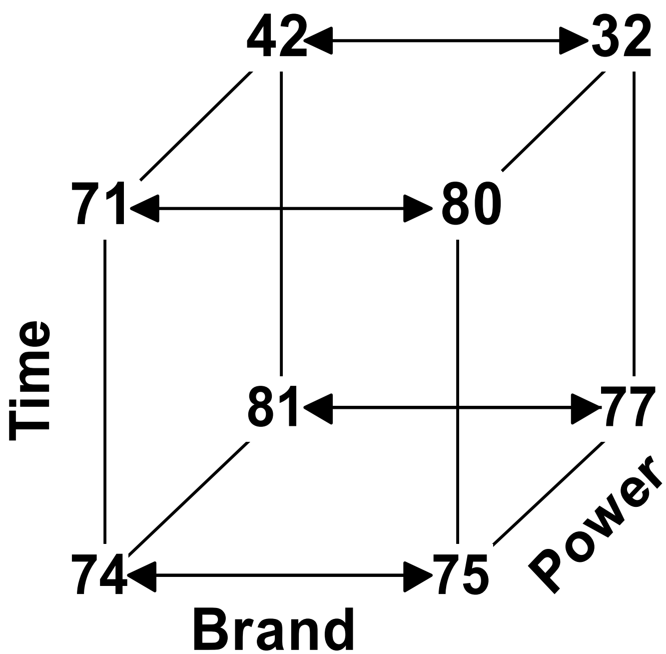
- $(\bar{y} \text{ when } X_i \text{ is high}) - (\bar{y} \text{ when } X_i \text{ is low})$

or in words,

$$\begin{pmatrix}\text{average response} \\ \text{when } X_i \text{ is } +\end{pmatrix} - \begin{pmatrix}\text{average response} \\ \text{when } X_i \text{ is } -\end{pmatrix}$$

R_1 - POPCORN TASTE
AVERAGE A-EFFECT (BRAND)

There are four comparisons of factor A (Brand), where levels of factors B and C (time and power) are the same:



75 – 74 = + 1

80 – 71 = + 9

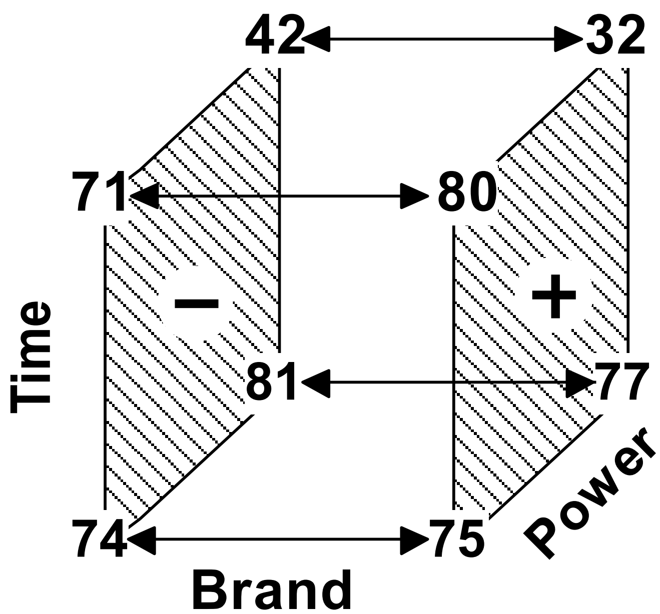
77 – 81 = – 4

32 – 42 = – 10

$$\Delta y_A = \frac{1+9-4-10}{4} = -1$$

R_1 - POPCORN TASTE

AVERAGE A-EFFECT



$$\text{Effect}(\Delta y) = \frac{\sum y_+}{n_+} - \frac{\sum y_-}{n_-}$$

$$\Delta y_A = \frac{75 + 80 + 77 + 32}{4} - \frac{74 + 71 + 81 + 42}{4} = -1$$

R_1 - POPCORN TASTE
ANALYSIS MATRIX IN STANDARD ORDER

Std. Order	A	B	C	AB	AC	BC	ABC	Taste rating
1	–	–	–	+	+	+	–	74
2	+	–	–	–	–	+	+	75
3	–	+	–	–	+	–	+	71
4	+	+	–	+	–	–	–	80
5	–	–	+	+	–	–	+	81
6	+	–	+	–	+	–	–	77
7	–	+	+	–	–	+	–	42
8	+	+	+	+	+	+	+	32

- **A, B and C for main effects (ME's).**
  These columns define the runs.
- **Remainder for factor interactions (FI's)**
  Three 2FI's and One 3FI (we will study later).

POPCORN TASTE
COMPUTE THE EFFECT OF C

Std. Order	A	B	C	AB	AC	BC	ABC	Taste rating
1	–	–	–	+	+	+	–	74
2	+	–	–	–	–	+	+	75
3	–	+	–	–	+	–	+	71
4	+	+	–	+	–	–	–	80
5	–	–	+	+	–	–	+	81
6	+	–	+	–	+	–	–	77
7	–	+	+	–	–	+	–	42
8	+	+	+	+	+	+	+	32
Δy	-1	-20.5		0.5	-6		-3.5	

$$\Delta y = \frac{\sum y_+}{n_+} - \frac{\sum y_-}{n_-}$$

$$\Delta y_C = \frac{\_\_ + \_\_ + \_\_ + \_\_}{4} - \frac{\_\_ + \_\_ + \_\_ + \_\_}{4} = \_\_\_\_$$

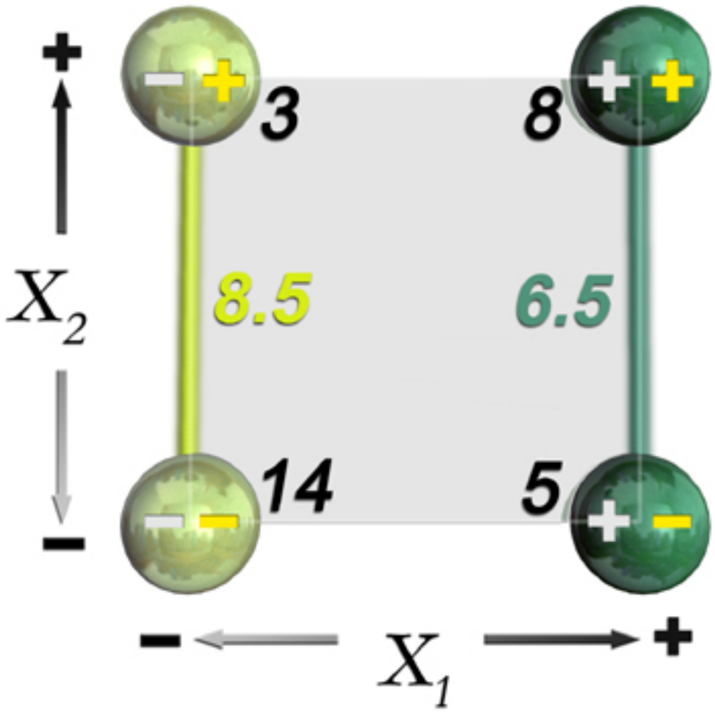
LET'S LOOK AT THE FOLLOWING SIMPLER TWO FACTOR CASE

Examine the idea of interaction effect.

The main question is whether the effect of one factor is independent of the other factor or not.

In other words,
"The effect of one factor may depend on the level of the other factor?"

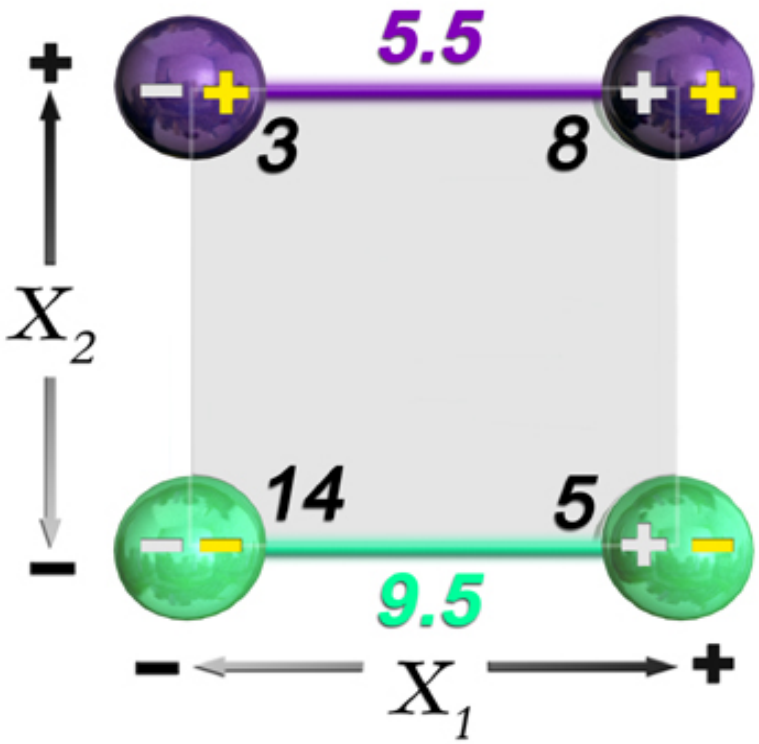
Main Effect of X_1



X_1	X_2	y
−	−	14
+	−	5
−	+	3
+	+	8

$$E_1 = \left(\frac{8+5}{2}\right) - \left(\frac{3+14}{2}\right) = {}^{-}2$$

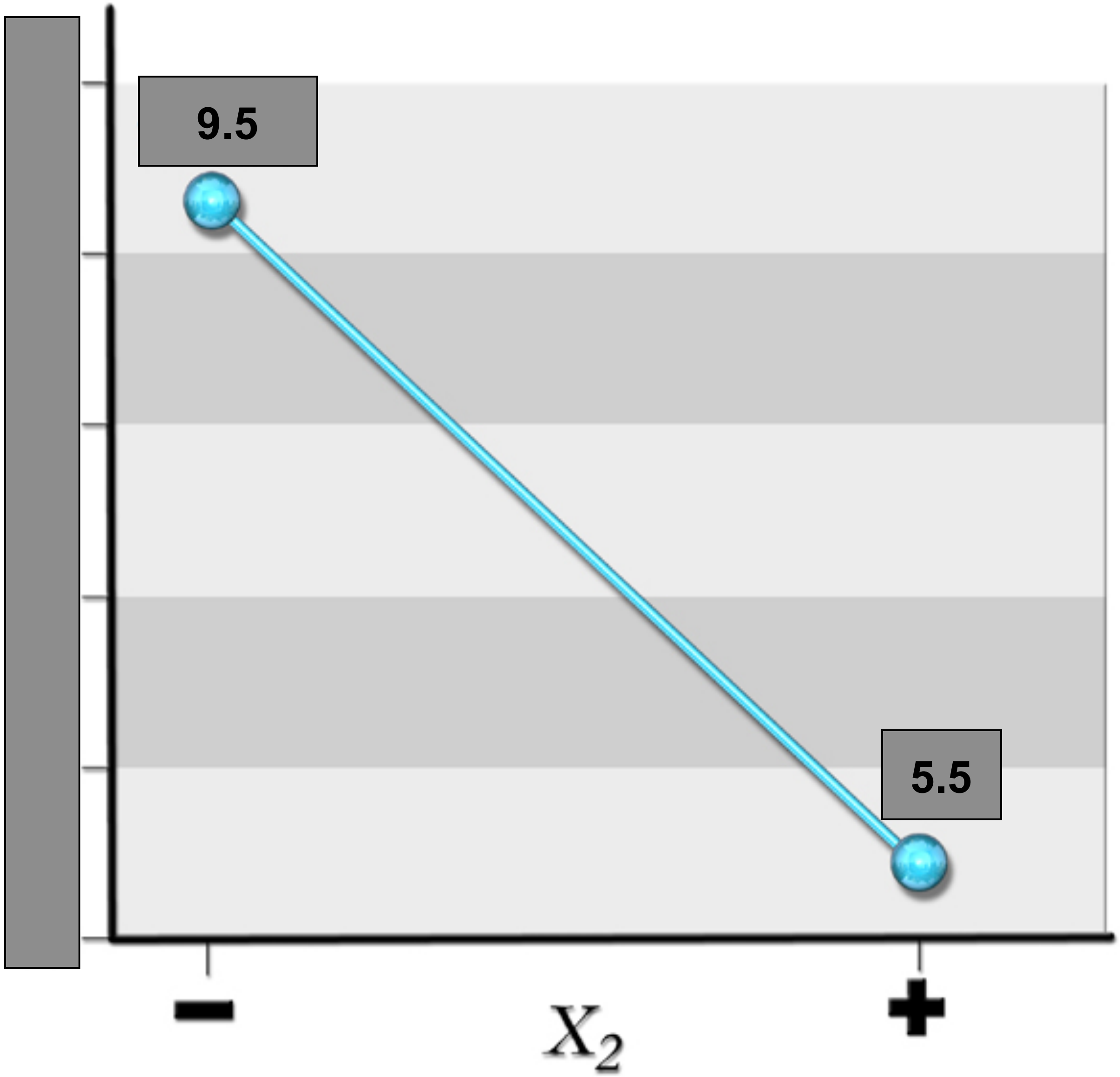
Main Effect of X_2



X_1	X_2	y
−	−	14
+	−	5
−	+	3
+	+	8

$$E_2 = \left(\frac{3+8}{2}\right) - \left(\frac{14+5}{2}\right) = {}^{-}4$$

Main Effect Plot for X_2



RECALL THE DATA

The effect of factor X2 (from its low (-) to high (+)) with X1 at + is:

8-5

The effect of factor X2 (from its low (-) to high (+)) with X1 at - is:

3-14

Interaction effect is:
Difference of differences (that is, difference of slopes)

(3-(-11))/2

INTERACTION PLOT

"The effect of one factor depends on the level of the other factor?"

-- "The magnitude of effect of X2 (from its – to +) is much greater when the other factor X1 is at level –"

-- Moreover, in this case, they exhibit different 'directions'.

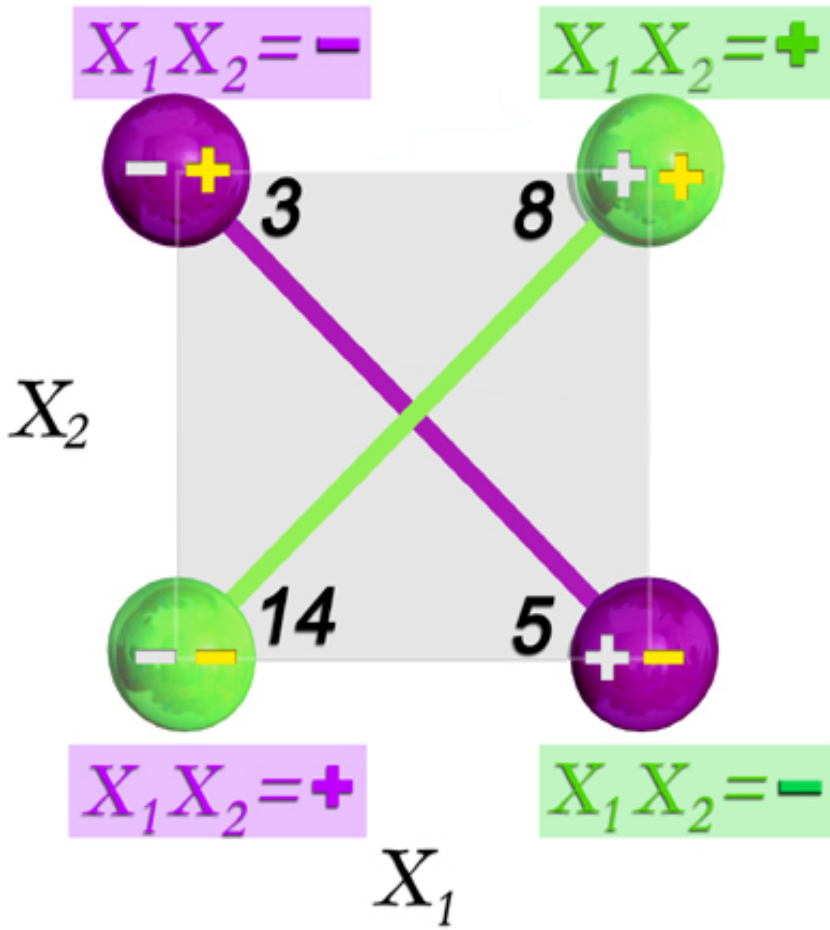
A simple method for calculating effects → next slide

Interaction in a 2^2 Experiment

X_1	X_2		$X_1 X_2$
$-$	$-$	$=$	$+$
$+$	$-$	$=$	$-$
$-$	$+$	$=$	$-$
$+$	$+$	$=$	$+$

Algebra
-1 x -1 = +1
…

$X_1 X_2$ Interaction



$$E_{12}=\left(\frac{8+14}{2}\right)-\left(\frac{3+5}{2}\right)=7$$

Design Matrix

2^3 Experiment

X_1	X_2	X_3	$X_1 X_2$	$X_1 X_3$	$X_2 X_3$	$X_1 X_2 X_3$
−	−	−	+	+	+	−
+	−	−	−	−	+	+
−	+	−	−	+	−	+
+	+	−	+	−	−	−
−	−	+	+	−	−	+
+	−	+	−	+	−	−
−	+	+	−	−	+	−
+	+	+	+	+	+	+

Design Matrix

Full Factorial Design

Calculation of E_1

X_1	X_2	X_3	$X_1 X_2$	$X_1 X_3$	$X_2 X_3$	$X_1 X_2 X_3$	y
−	−	−	+	+	+	−	7
+	−	−	−	−	+	+	9
−	+	−	−	+	−	+	9
+	+	−	+	−	−	−	9
−	−	+	+	−	−	+	8
+	−	+	−	+	−	−	3
−	+	+	−	−	+	−	8
+	+	+	+	+	+	+	3

Average y when X_1 is +:

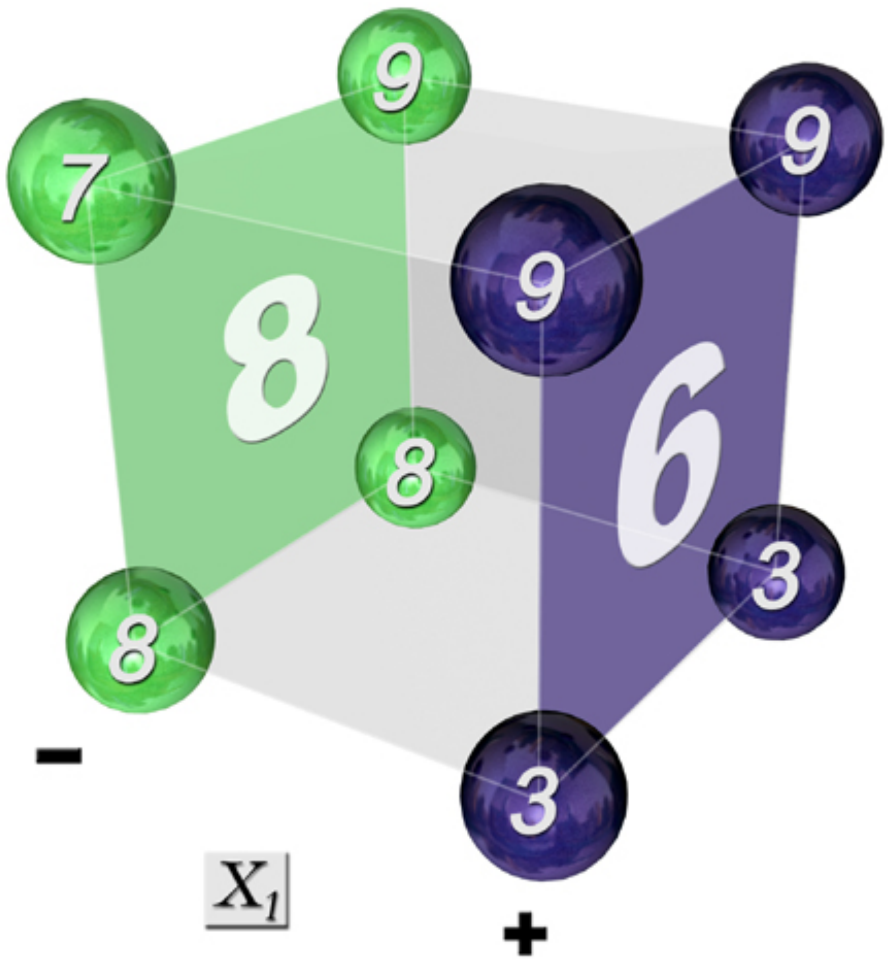
$$\frac{9+9+3+3}{4} = 6$$

Average y when X_1 is −:

$$\frac{7+9+8+8}{4} = 8$$

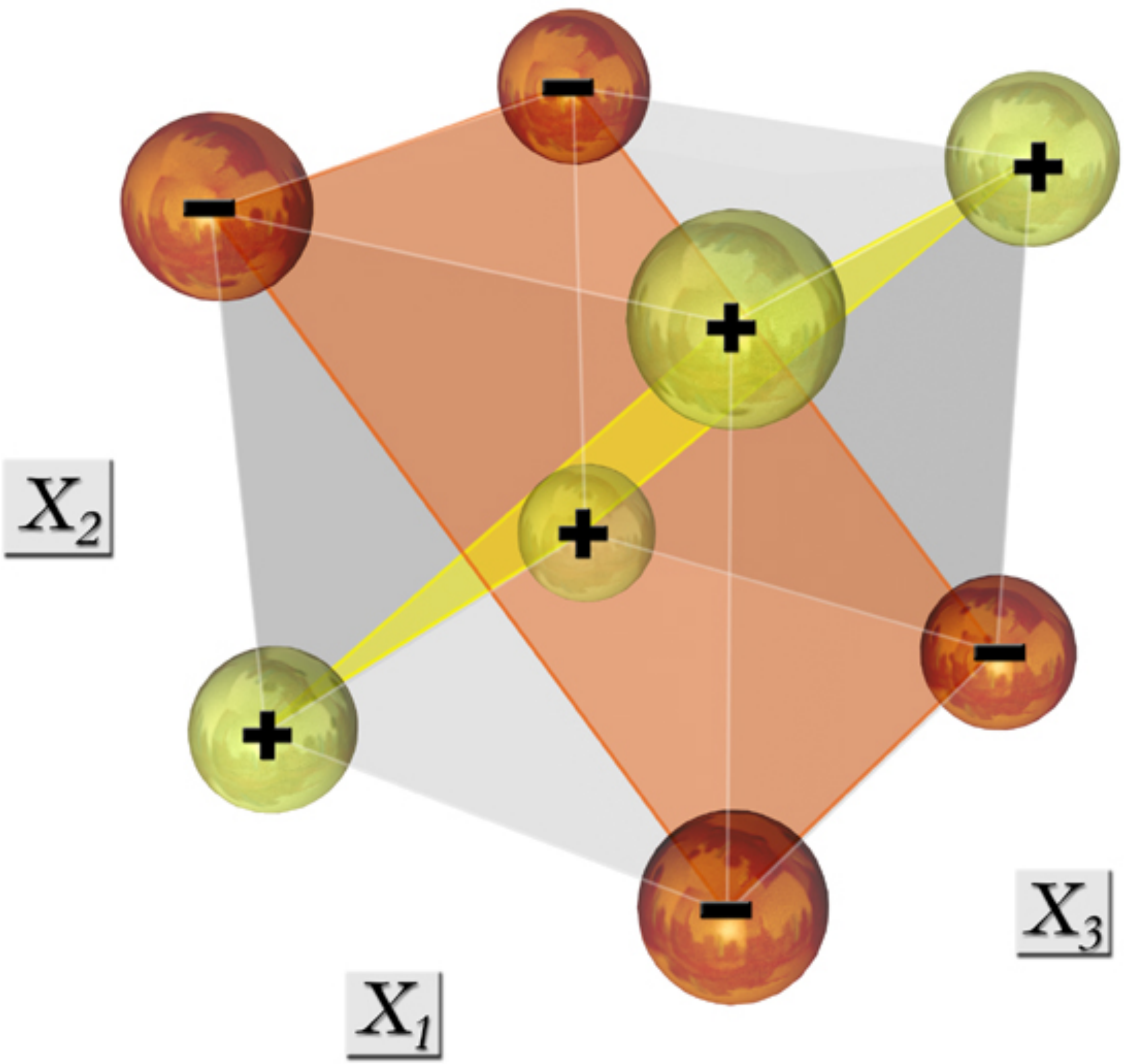
$$E_1 = 6 - 8 = -2$$

E_1

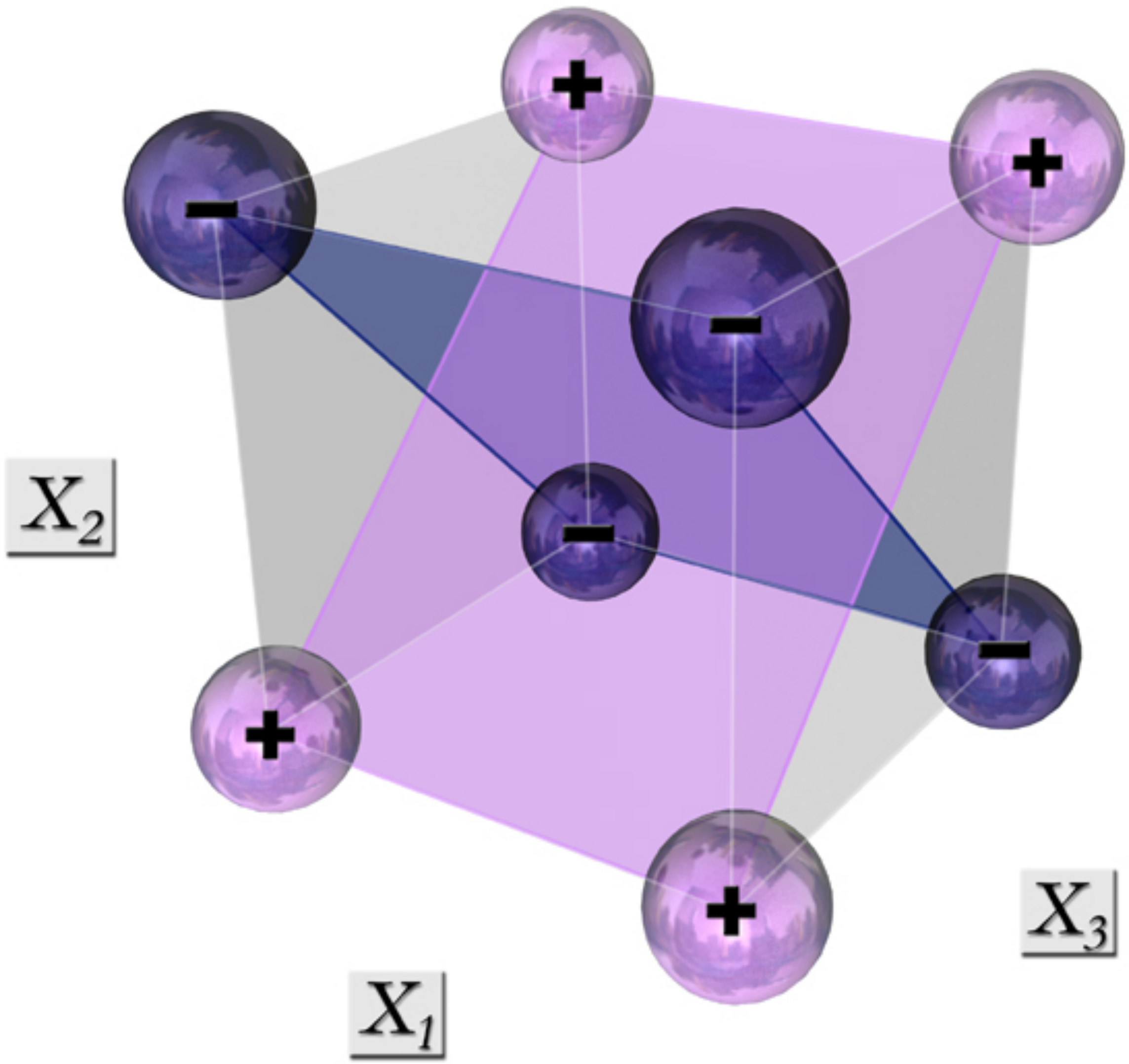


$$E_1 = 6 - 8 = -2$$

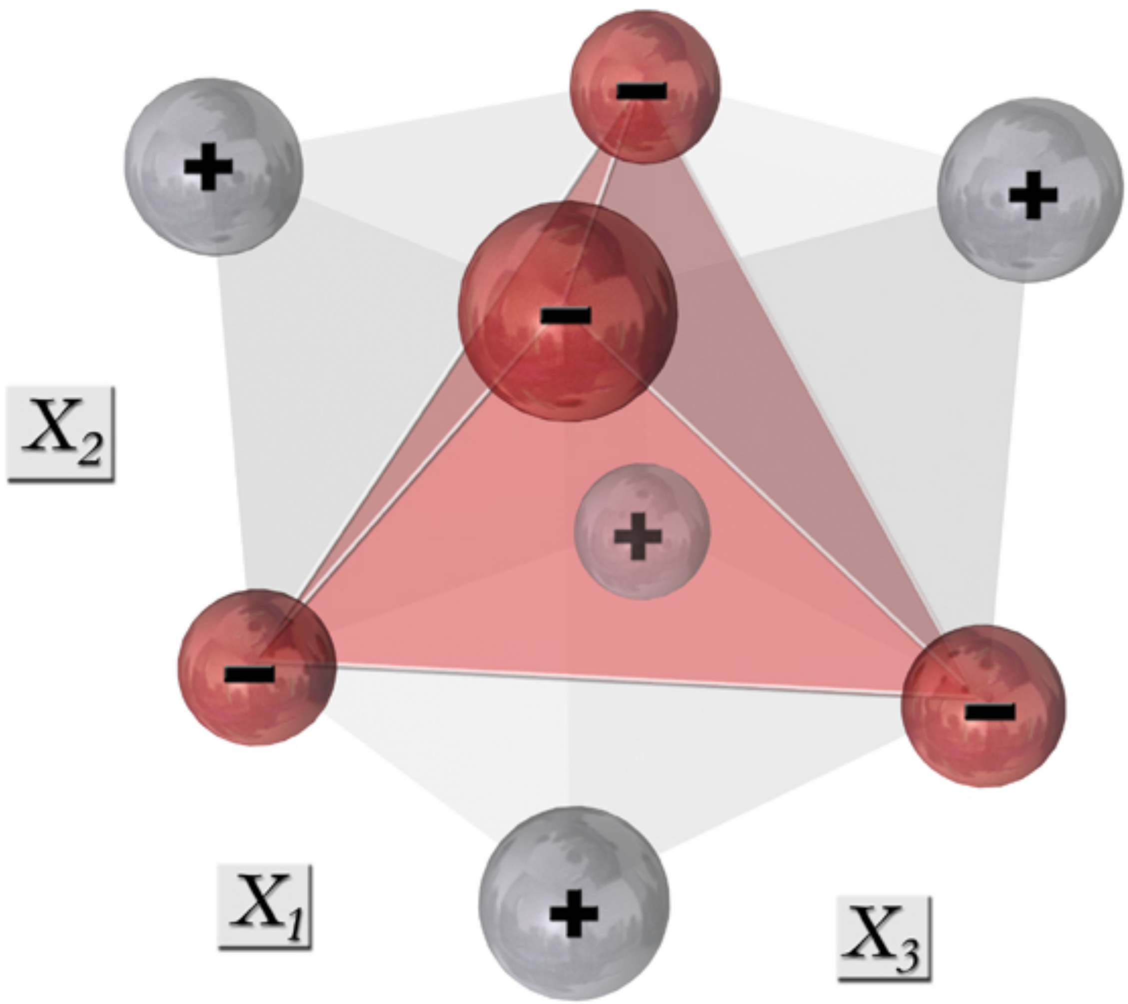
E_{12}



E_{23}



$X_1 X_2 X_3$ is −



BI&A[C.Lee]

2^4 **Full Factorial Design**

x1	x2	x3	x4
-	-	-	-
+	-	-	-
-	+	-	-
+	+	-	-
-	-	+	-
+	-	+	-
-	+	+	-
+	+	+	-
-	-	-	+
+	-	-	+
-	+	-	+
+	+	-	+
-	-	+	+
+	-	+	+
-	+	+	+
+	+	+	+

RECALL: POPCORN TASTE EXAMPLE

COMPUTE THE EFFECT OF BC

Std. Order	A	B	C	AB	AC	BC	ABC	Taste rating
1	–	–	–	+	+	+	–	74
2	+	–	–	–	–	+	+	75
3	–	+	–	–	+	–	+	71
4	+	+	–	+	–	–	–	80
5	–	–	+	+	–	–	+	81
6	+	–	+	–	+	–	–	77
7	–	+	+	–	–	+	–	42
8	+	+	+	+	+	+	+	32
Δy	-1	-20.5		0.5	-6		-3.5	

$$\Delta y = \frac{\sum y_+}{n_+} - \frac{\sum y_-}{n_-}$$

$$\Delta y_{BC} = \frac{\quad + \quad + \quad + \quad}{4} - \frac{\quad + \quad + \quad + \quad}{4} = \underline{\qquad}$$

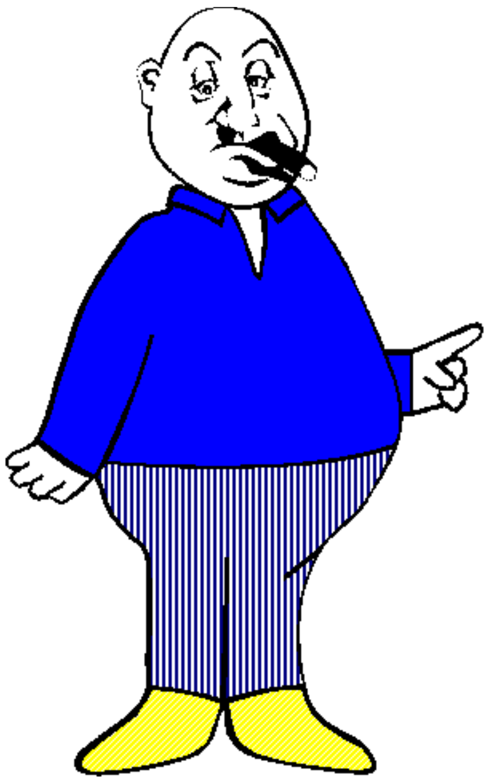
SO… WHAT EFFECTS ARE STATISTICALLY SIGNIFICANT??

Std. Order	A	B	C	AB	AC	BC	ABC	Taste rating
1	–	–	–	+	+	+	–	74
2	+	–	–	–	–	+	+	75
3	–	+	–	–	+	–	+	71
4	+	+	–	+	–	–	–	80
5	–	–	+	+	–	–	+	81
6	+	–	+	–	+	–	–	77
7	–	+	+	–	–	+	–	42
8	+	+	+	+	+	+	+	32
Δy	-1	-20.5		0.5	-6		-3.5	

$$\Delta y = \frac{\sum y_+}{n_+} - \frac{\sum y_-}{n_-}$$

* **Before we proceed, let us compare this design, with the commonly practiced alternative.**

TRADITIONAL APPROACH:
ONE FACTOR AT A TIME (OFAT)



"Lets just vary one thing at a time so we don't get confused."

"I'll investigate that factor next."

"There aren't any interactions."

"We'll worry about the statistics after we've run the experiment."

"It's too early to use statistical methods."

"A statistical experiment would be too large."

"My data are too variable to use statistics."

TRADITIONAL APPROACH:
ONE FACTOR AT A TIME (OFAT)

- **Carry out *successive* experiments in which the levels of each factor are changed *<u>one at a time</u>*.**

1. Start with the current settings (that have been used so far in practice) of the factors and begin <u>by changing the level of the one factor that is considered the most important</u>: For instance, change only Brand in popcorn example.
2. The responses at the low and high settings of this one factor (e.g. Brand) are compared <u>while keeping all other factors fixed</u>.
3. If there is a difference, the level at which the response is best is locked in for the next stage.
4. The factor that is considered <u>second most important is varied next</u>. Again, responses at the low and high levels of this factor are compared, and … (the same as above steps 2-3).
5. This process continues until the last factor is reached.

Question: How many experimental settings? Suppose 10 factors each with 2 levels?

SHORTCOMINGS OF OFAT APPROACH

*** It requires more runs than factorial design, to achieve the same precision in estimating the main effects!**

- For example: Total runs in two level full factorial design vs. Total runs in OFAT with the same precision.

.

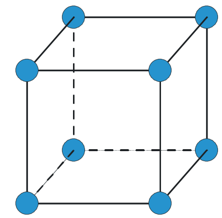
SHORTCOMINGS OF OFAT APPROACH

*** It may miss the optimum!**

- For example: Consider previous case on slides 17, 20.

- It is not possible to estimate an interaction effect!
- Even main effects cannot be generalized because they are effects at specific levels of the other factors! We cannot tell about what would happen at other levels.

2^K FACTORIAL DESIGN
ADVANTAGES



- It is simpler.
- Minimal runs required.
  More factors, more efficient than OFAT.
- <u>Show interactions.</u>
  <u>Key to Success - Extremely important!</u>
- Interpretation is not too difficult.
  Graphs make it easy.
- Form base for more complex designs.

SPARSITY-OF-EFFECTS PRINCIPLE

After computing all effects we then need some way of sorting out which are real.

Two types of effects:

- Vital Few
- Trivial Many

* The **Pareto principle** (also known as the **80–20 rule**, the **law of the vital few,** and the **principle of factor sparsity**) states that, for many events, roughly 80% of the effects come from 20% of the causes (e.g., 20% of the people held 80% of the wealth).

SPARSITY-OF-EFFECTS PRINCIPLE

A 2-level factorial design with k-factors leads to many estimated main effects and interactions.

For example, <u>consider k=5 factors</u>:

There are 5 main effects, 10 two-factor interactions, 10 three-factor Interaction, 5 four-factor interactions, and 1 five-factor interactions.

<u>Fortunately, we can expect the great majority of these effects to be negligible.</u>

Experience shows that
Main effects > 2-factor interaction > 3-factor interaction > ...
<u>4-factor (and higher) interactions are almost certain to be negligible</u>.

DETERMINING WHICH EFFECTS ARE SIGNIFICANT (CASE I)

- Suppose we can obtain replicated runs (that is, multiple runs under the same experimental conditions): This is the most desirable case.

- Step 1: Estimate the variance of the experimental error by weighted sample variances, that is, the pooled sample variance. → Measure of variability in the response of an individual run.
- Note: Degrees of freedom = Sum of each variance estimate.

- Step 2: Estimate the variance of 'effect':

Variance of (effect)

= Variance (mean at level '+' – mean at level '-')

= Variance(mean at level '+') + Variance(…. '-')

DETERMINING WHICH EFFECTS ARE SIGNIFICANT (CASE I)

- Step 2: Estimate the variance of ‘effect’:

Variance of (effect)

= Variance (mean at level ‘+’ – mean at level ‘-’)

= Variance(mean at level ‘+’) + Variance(…. ‘-’)

- Step 3: Use the above to obtain the standard error of an effect.

- Step 4: Construct a Confidence Interval (C.I.) with the center being the estimate of the effect of interest.
  - The effect is significant whenever the C.I. does NOT contain zero.

CRACK POTS EXAMPLE

A company manufactures clay pots:

For one of their newest products, the company has been experiencing an unacceptably high percentage of pots that crack during the manufacturing process.

Company production engineers have identified three key factors they believe will affect cracking, and they decide to run an experiment to learn about the most important factor(s).

The three factors are:

- Rate of cooling (R), whose levels are Slow (-) and Fast (+)
- Temperature (T), whose levels are 2000 F (-) and 2060 F (+)
- Coefficient of expansion (C), whose levels are Low (-) and High (+)

Response variable is:
Percentage of Pots with Cracks

CRACK POTS EXAMPLE

Std. Order	R	T	C	RT	RC	TC	RTC	Percentage Cracked Pots (Avg.)
1	–	–	–	+	+	+	–	8, 3 (6)
2	+	–	–	–	–	+	+	11, 13 (12)
3	–	+	–	–	+	–	+	9, 3 (6)
4	+	+	–	+	–	–	–	17,15 (16)
5	–	–	+	+	–	–	+	19, 13 (16)
6	+	–	+	–	+	–	–	36, 32 (34)
7	–	+	+	–	–	+	–	12, 16 (14)
8	+	+	+	+	+	+	+	33, 35 (34)

The eight test conditions was run twice and the response used in calculating the effects is the average response from two runs.

The 16 runs were performed in random order.

CRACKED POTS EXAMPLE

Std. Order	R	T	C	RT	RC	TC	RTC	Percentage Cracked Pots (Avg.)
1	–	–	–	+	+	+	–	8, 11 (6)
2	+	–	–	–	–	+	+	11, 13 (12)
3	–	+	–	–	+	–	+	9, 3 (6)
4	+	+	–	+	–	–	–	17,15 (16)
5	–	–	+	+	–	–	+	19, 13 (16)
6	+	–	+	–	+	–	–	36, 32 (34)
7	–	+	+	–	–	+	–	12, 16 (14)
8	+	+	+	+	+	+	+	33, 35 (34)

We can then calculate eight separate estimate of the variance of the experimental error:
Estimated sample variances are given by ---

DETERMINING WHICH EFFECTS ARE SIGNIFICANT

Then, the estimated pooled variance (for the experimental error) is given by

Next, what is the variance of *effect (that is, difference of two averages)*?

DETERMINING WHICH EFFECTS ARE SIGNIFICANT

Compute the point estimate of the main effect R, and then carry out a test of significance.

DETERMINING WHICH EFFECTS ARE SIGNIFICANT

Compute the point estimate of interaction effect RT, and then carry out a test of significance.