CO342: Introduction to Graph Theory **Fall 2020**

Lecture 1: September 8

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 1.1. A graph G consists of a vertex set $V(G)$ (usually finite, assume unless otherwise specified) and a set of edges $E(G)$ where each edge is a subset of $V(G)$ of size 2. Usually write ab instead of $\{a, b\}$.

No loop and no multiple edges between two vertices.

Definition 1.2. Recall definitions for

- adjacent
- incident
- neighbour
- neighbourhood
- degree
- complete graph
- bipartite graph
- k-regular
- subgraph
- path
- cycle
- connected graph
- component
- tree
- planar graph
- subdivision
- face of a planar graph

Definition 1.3. Let G be a connected graph, and let $x \in V(G)$

We say x is a **cut vertex** of G if the graph $G - x$ obtained from G by deleting the vertex x is disconnected.

Definition 1.4. For a subset $W \subseteq V(G)$, we write $G - W$ for the graph obtained from G by deleting every vertex of W from G.

We say $W \subseteq V(G)$ is a **vertex cut** of the connected graph G if $G - W$ is disconnected.

Note that any complete graph has no vertex cut.

Exercise: If G is a connected graph but not complete, then G has a vertex cut.

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Definition 2.5. Let G be a connected graph, and let $k \geq 1$ be an integer. We say G is **k-connected** if

1. $|V(G)| \geq k + 1$
2. G has no vertex cut of size $\leq k - 1$

The **connectivity** of a graph G is the largest k such that G is k-connected.

Definition 2.6. The minimum degree of $\delta(G)$ is $\min\{d(v) : v \in V(G)\}$, where $d(v)$ is the degree of v.

Lemma 2.7. If G is k-connected, then $\delta(G) \geq k$.

Proof 2.8. Let $x \in V(G)$ be a vertex of degree $\delta(G)$. By definition, $|V(G)| \geq k + 1$. There are two cases:

- If $|V(G)| \geq \delta(G) + 2$, then the neighbourhood of x $N(x)$ is a vertex cut of G. So $|N(x)| = \delta(G) \geq k$
- If $|V(G)| \neq \delta(G) + 1$, then $k + 1 \leq |V(G)| \leq \delta(G) + 1$

So in both cases, $k \leq \delta(G)$ $\square$

Note that the converse of the above lemma is not true, i.e. $\delta(G)$ does NOT imply that G is k connected.

Lemma 2.9. Let G be a graph with n vertices, and let $1 \leq k \leq n - 1$. If $\delta(G) \geq \frac{n+k-2}{2}$, then G is k-connected.

Proof 2.10. Say $|V(G)| \geq k+1$ by assumption. We suppose on the contrary that G has a vertex cut W with $|W| \leq k-1$. Let H be the smallest component of $G - W$. Then $|H| \leq \frac{n-|W|}{2}$.

For $v \in V(H)$, we see that $d(v) \leq |W| + (|H|-1)$. So $\delta(G) \leq d(v) \leq |W| + (|H|-1) \leq |W| + \frac{n-|W|}{2} - 1 \leq \frac{n}{2} + \frac{|W|}{2} - 1 \leq \frac{n}{2} + \frac{k-1}{2} - 1 = \frac{n+k-3}{2} < \frac{n+k-2}{2}$.

We reach a contradiction, which shows that the assumption G does not have a vertex cut of size $< k-1$. So, by definition, G is k-connected. $\square$

Note that the converse of the above lemma is not true, i.e. G is k connected does not imply that $\delta(G) \geq \frac{n+k-2}{2}$ where $n = |V(G)|$

Note that if G is k-connected, and $1 \leq \ell \leq k$m, then G is also ℓ connected.

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Lecture 3: September 8

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Lemma 3.11. If G is a 2-connected graph, and $xy, xz \in E(G)$, then there exists a cycle in G containing edges xy and xz

Proof 3.12. G is 2-connected, so $G - x$ is a connected graph. Then y and z are joined by a path p in $G - x$. So $P \cup \{x\}$ is the vertex set of a cycle in G that contains xy and xz. □

In particular, every edge e in a 2-connected graph lies in a cycle. So, G is 2-connected implies that $\delta(G) \geq 2$. So there exists edge f different from e sharing a vertex in e, so the cycle in G containing both edges e and f.

Lemma 3.13. Let G be a graph, suppose that

- edges e_1 and e_2 both lie in a cycle C_1
- edges e_2 and e_3 both lie in a cycle C_2

Then there exists a cycle C_3 in G that contains both edges e_1 and e_3

Proof 3.14. Let x and y be the first two vertices of C_1 reached by waling around C_2 starting at e_3 and going in either direction. Then $x \neq y$ since e_2 is in both cycles C_1 and C_2. Then the (x, y) segment of C_2 containing e_3 together with the (y, x) segment of C_1 containing e_1, form a cycle in G containing e_1 and e_3. □

Theorem 3.15. Let G be a graph with $|V(G)| \geq 3$. Then the following are equivalent

(a) G is 2-connected

(b) G has no isolated vertices, and any two edges of G lies in a common cycle

(c) any two vertices of G lie in a common cycle.

It means that $(a) \Leftrightarrow (b) \Leftrightarrow (c)$

Proof 3.16. $(a) \Rightarrow (b)$: Assume G is 2-connected. Then G has no isolated vertices since $\delta(G) \geq 2$. Let e and f are distinct edges of G. Since G is connected, then there exists a path from an end point of e to an end point of f. Therefore, there exists a sequence of edges $e_0, e_1, ..., e_k$ in G such that $e_0 = e, e_k = f$, and e_i shares a vertex with e_{i+1} for each i.

Since G is 2-connected, by Lemma 3.11, e and e_1 lie in a common cycle. Similarly, e_i and e_{i+1} lie in a common cycle for each i. By Lemma 3.13 (apply it $k-1$ times), we can conclude that e and f lie in a common cycle.

$(b) \Rightarrow (c)$: Let x and y be vertices of G. Since G has no isolated vertices, then there exists edges e_1 incident to x and e_2 incident to y. Then G contains a cycle C that contains both e_1 and e_2. So C contains both x and y as required.

$(c) \Rightarrow (a)$: First, we have $|V(G)| \leq 3$ by assumption. Suppose on the contrary that x is a cut vertex of G. Let y and z be vertices in distinct components of $G - x$. THen there is no path from y to z in $G - x$. But y and z lie in a common cycle. Removing x cannot disonnect y and z. This contradiction shows that x cannot exist. So G is 2-connected. $\square$

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Lecture 4: September 15

Lecturer: Penny Haxell *Noted By: Haochen Wu*

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Definition 4.17. A **block** is a connected graph with no cut vertex.

Every block is

1. a single vertex (aka the trivial block)
2. two vertices joined by an edge
3. 2-connected

Definition 4.18. a block of a graph G is a subgraph H that is maximal with respect to being a block.

The above definition means that it is a block, and H is not contained in any larger subgraph J that is also a block

Definition 4.19. Let G be a connected graph. The **Block-cut vertex forest** F for G is the bipartite graph with vertex classes B ad C, where B is the set of blocks of G, and C is the set of cut vertices of G. The edge set of F is

$$\{bc : \text{block } b \text{ contains cut vertex } c\}$$

Theorem 4.20. Let G be a connected graph. Then, the block-cut vertex forest F of G is a tree. (F is connected and has no cycles)

Proof 4.21. The proof of F is connected is left as an exercise.

To prove F contains no cycles, suppose on the contrary that $c_1b_1c_2b_2...c_tb_t$ is a cycle in F. Note that $t \geq 2$. Since each block b_i is connected, there is a path joining c_i to c_{i+1} inside b_i for each i. Their union gives a cycle C in G containing all of these cut vertices $c_1, c_2, ..., c_t$. Note that C has ≥ 3 vertices.

We claim that $C \cup b_i$ is 2-connected:

- It has ≥ 3 vertices since it contains C
- C has no cut vertex since it is a cycle
- b_1 has no cut vertex since it is a block
- $C \cap b_1$ contains 2 vertices. It contains c_1 and c_2.

So, for any vertex x,

- $C - x$ is connected
- $b_1 - x$ is connected
- $(C - x) \cap (b_1 - x)$ is nonempty

This implies that x is not a cut vertex of $C \cup b_1$. This contradicts the fact that b_1 is a block of G since $C \cup b_1$ is 2 connected, and it contains b_1, and $C \cup b_1$ is larger than b_1.

Hence, F does not contain a cycle. $\square$

Recall that a leaf of a tree is a vertex of degree 1. Every cut vertex of G has at least 2 blocks as neighbours in F. So every leaf of F is a block of G. They are called the **end-blocks** of G.

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Lecture 5: September 15

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We see that if a graph is connected but not 2-connected, then its blocks form a "tree-like" structure. This will help us to prove the general statements about connected graphs G by allowing us to assume G is 2-connected. That is, if we can prove statement X for a 2-connected graph G, then we can conclude that X is true for all connected graphs, by using induction on the number of blocks of G.

Recall from MATH239, let $\tilde{G}$ be a planar drawing of a planar graph G and let f be a face of $\tilde{G}$. Then G has a planar drawing in which f is the outer face.

Theorem 5.22. Theorem K1: Let G be a connected graph, and suppose that every block of G is planar, then G is planar

Proof 5.23. By induction on the number t of blocks of G.

Base Case: $t = 1$, Then G itself is the only block, hence by assumption G is planar.

Inductive Hypothesis: Assume $t \geq 2$, and that if it is a connected graph with $\leq k-1$ blocks, all of which are planar, then H is planar.

Inductive Step: Let b_1 be an end block of G (so it has degree 1 in the block-cut vertex forest of G). So b_1 contains exactly one cut vertex c_1 of G. Then

- b_1 is planar by assumption
- the graph $H = G - (b_1 - c_1)$ has $t-1$ blocks, all of which are blocks of G, hence planar by assumption. Therefore, it is planar by inductive hypothesis.

Take a planar drawing of b_1 with c_1 on the outer face. Also, take a planar drawing of H with c_1 on the outer face. We can glue these two drawings together at c_1. So this is a planar drawing of G, which tells us G is planar as required. □

Definition 5.24. The graph $G \setminus e$ has vertex set $V(G)$ and edge set $E(G) \setminus \{s\}$.

Definition 5.25. The graph G/e has vertex set $V(G) \setminus \{x, y\} \cup \{z\}$ where $e = xy$ and z is a new vertex, and edge set

$$\{uv \in E(G) : \{u, v\} \cap \{x, y\} = \emptyset\} \cup \{uz : ux \in E(G) \setminus \{e\} \text{ or } uy \in E(G) \setminus \{e\}\}$$

Definition 5.26. The graph H is said to be a **minor** of G if it can be obtained from G by a sequence of edge deletions and edge contractions. (plus a detection of isolated vertices)

Definition 5.27. **Subdivision**: subdividing the edges means take edges of a graph H and replace it by a path, i.e. add some vertices that have degree 2 on the edges, and we can obtain H_1, a subdivision of H.

Note the the vertices on the edges (newly introduced ones) are called **path vertices**. Unchanged vertices are called **branch vertices**.

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Lecture 6: September 15

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Lemma 6.28. If H_1 is a subdivision of H, and H_2 is a subdivision of H_1, then H_2 is a subdivision of H.

For any subdivision H of graph G, we can obtain from G to H via edge deletions and contractions, i.e. G contains a subvidision of H implies that G has H as a minor. However, the converse is not true.

Definition 6.29. A graph is a said to be **cubic** if every vertex has degree exactly 3 (i.e. the same as 3 regular).

Lemma 6.30. Let H be a cubic graph. If G has H as a minor, then G contains a subdivision of H.

Proof 6.31. By induction of $|E(G)|$.

Base Case: $|E(G)| = |E(H)|$, then $G = H$ (plus possibly isolated vertices), note that H is a subdivision of itself.

Inductive Hypothesis: Assume $|E(G)| > |E(H)|$ and every graph G' that has H as a minor, where $|E(G')| < |E(G)|$ contains a subdivision of H

Inductive Conclusion: Since G has H as a minor, there is a sequence of edge deletions and contractions starting with G and ending with H. Let $e = xy$ be the first edge in this sequence.

Case 1: The sequence starts by deleting e. Then the rest of the sequence shows that $G \setminus e$ has H as a minor. By inductive hypothesis, since $|E(G)| > |E(G \setminus e)|$, we know that $G \setminus e$ contains a subdivision of H. Hence so does G.

Case 2: The sequence starts by contracting $e = xy$. Then the rest of the sequence shows that G/e has H as a minor. By inductive hypothesis, since $|E(G)| > |E(G/e)|$, we know that G/e contains a subdivision of H. If the image z if e under contraction is not a vertex of H_1, then H_1 is also a subgraph of G, then H_1 is a subdivision of G.

So we may assume z is a vertex of it. Let H_2 be the subgraph of G such that $H_2/e = H_1$. Since H is cubic, z has degree ≤ 3 in H_1. Look at the ≤ 3 edges of H_1 incident to z. One of x and y is incident to ≤ 1 of them (in H_2). It follows that H_2 is a subdivision of H_1, or contains H_1. Hence, H_2, or a subgraph of H_2, is a subdivision of H contained in G as required.

Note: we use degree 3 here is essentially using the fact that if $3 = a + b$, a, b are non-negative integers, then one of a, b is ≤ 1. $\square$

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Lecture 7: September 22

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 7.32. Let G_1 and G_2 be graphs with at least 3 vertices. Suppose that

- $V(G_1) \cap V(G_2) = \{u, v\}$
- $uv \in E(G_1) \cap E(G_2)$

The **2-sum** $G_1 \oplus G_2$ with respect to the edge uv is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \setminus \{uv\}$

Note that any 2-sum of two cycles is a cycle. $C_k \oplus C_\ell = C_{k+\ell-2}$

Lemma 7.33. If $G = G_1 \oplus G_2$ and G_2 is 2-connected. Then G contains a subdivision of G_1.

Proof 7.34. Since G_2 is 2-connected, by a previous lemma, we know that every edge of G_2 lies in a cycle. Let G be a cycle in G_2 containing uv. So $P = C \setminus uv$ is a path joining u and v in G_2, which together with $G_1 \setminus \{uv\}$ forms a subdivision of G_1. □

Lemma 7.35. If $G = G_1 \oplus G_2$ and G_1 and G_2 are both 2-connected. Then G is 2-connected.

Proof 7.36. We will show that G has no isolated vertices, and any two edges lie in a common cycle. Then we may use a previous lemma to show that G is 2-connected.

The proof is as follows: First, G has no isolated vertices. Since G_1 and G_2 are both 2-connected, we know, by a previous lemma, $\delta(G_1)$ and $\delta(G_2)$ are both ≥ 2. Since only one edge is removed in forming G, there is definitely no isolated vertices.

Then, let $e, f \in E(G)$. If $e, f \in E(G_1)$, let C_1 be a cycle in G_1 containing e, f (such cycle exists since G_1 is 2-connected). If $uv \notin E(C_1)$, then C_1 is the required cycle in G. If $uv \in E(C_1)$, then $C_1 \oplus C$ is the required cycle in G, where C is a cycle in G_2 containing uv.

The case is similar if $e, f \in E(G_2)$.

The other possibility is $e \in E(G_1), f \in E(G_2)$. Let C_1 be a cycle in G_1 containing e and uv since G_1 is 2-connected. Similarly, let C_2 be a cycle in G_2 containing f and uv since G_2 is 2-connected. Then, $C_1 \oplus C_2$ is the required cycle in G containing e and f. Thus we have shown that G is 2-connected. $\square$

Lemma 7.37. Let G be a 2-connected graph. Suppose $\{x, y\}$ is a vertex cut of G. Let C_1 be a component of $G - \{x, y\}$. Then, the graph H with vertex set $V(C_1) \cup \{x, y\}$ and edge set $\{wz \in E(G) : w, z \in V(H)\} \cup \{xy\}$ is 2-connected.

Proof 7.38. Note that $|V(H)| \geq 3$. Since C_1 is nonempty. Since C_1 is a component, it is connected, and there exists edges joining C_1 to $\{x, y\}$ since G is connected. Thus, H is connected.

Suppose for a contradiction that H has a cut vertex w.

Suppose $w \notin \{x, y\}$, note x, y are in the same component of $H - w$ since $xy \in E(H)$. Let K be a component of $H - w$ not containing $\{x, y\}$, then K is a component of $G - w$. This is not possible since G is 2-connected. Contradiction.

If $w = x$, then let L be a component of $H - x$ not containing y. Then L is a component of $G - x$, contradicting G being 2-connected. Case is similar if $w = y$.

Hence, H has no cut vertices, and hence is 2-connected as required. $\square$

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Lecture 8: September 22

Lecturer: Penny Haxell *Noted By: Haochen Wu*

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Lemma 8.39. Let G_1, G_2 be k-connected graphs such that $|V(G_1) \cap V(G_2)| \geq k$ for some $k \geq 2$. Then $G_1 \cup G_2$ is k-connected.

Proof 8.40. Clearly, $G_1 \cup G_2$ is connected and $|V(G_1 \cup G_2)| \geq |V(G_1)| \geq k+1$. Let $W \subset V(G_1 \cup G_2)$ be a set of size $\leq k-1$. Then

- $G_1 - W$ is connected because G_1 is k-connected.
- $G_2 - W$ is connected because G_2 is k-connected.
- $V(G_1 - W) \cap V(G_2 - W) \neq \emptyset$ since $|V(G_1) \cap V(G_2)| \geq k$.

Hence, $G_1 \cup G_2 - W$ is connected. So $G_1 \cup G_2$ has no vertex cut of size $\leq k-1$. Hence, $G_1 \cup G_2$ is k-connected. $\square$

Theorem 8.41. A structure theorem for 2-connected graphs that have vertex cuts of size 2: **Decomposition Theorem**

Suppose the 2-connected graph G has a vertex cut $\{x, y\}$. Then there exists graphs G_1 and G_2 with the following properties.

- $V(G_1) \cap V(G_2) = \{x, y\}$
- $xy \in E(G_1) \cap E(G_2)$
- G_1 and G_2 are both 2-connected.
- If $xy \notin E(G)$, then $G = G_1 \oplus G_2$
- If $xy \in E(G)$, then $G \setminus e = G_1 \oplus G_2$

Proof 8.42. Let $C_1, ..., C_r$ be the components of $G - \{x, y\}$. For each i, let H_i be the graph with vertex set $V(C_i) \cup \{x, y\}$ and edge set $\{wz \in E(G) : w, z \in V(H_i)\} \cup \{xy\}$. By a previous lemma, we know that H_i is 2-connected, for each i.

Let $G_1 = H_1$, and $G_2 = H_2 \cup \cdots \cup H_r$. Then G_2 is 2-connected since $|V(H_i) \cap V(H_j)| \geq 2$ for each $i, j \in \{2, ..., r\}$ by a previous lemma.

By now, G_1 and G_2 satisfy the first 3 properties. Then, if $xy \notin E(G)$, then $G = G_1 \oplus G_2$ by definition. If $xy \in E(G)$, then $G - e = G_1 \oplus G_2$ by definition. $\square$

Corollary 8.43. Let G be a 2-connected graph with at least 4 vertices. For any edge $e \in E(G)$, either $G \setminus e$ is 2-connected or G/e is 2-connected.

Proof 8.44. Consider $e = xy$. Suppose G/e is not 2-connected. Let z be the image of e in G/e. Note that G/e is connected (For any $u, v \in V(G/e)$, we need to find a path joining them together. Then, if the original path in G does not use e, then the same path works. if the original path in G uses e, then the path still works but just shortened by one edge).

Since G/e is not 2-connected, and has ≥ 3 vertices. It has a cut vertex w. If $w \neq z$, then w is cut vertex of G, contradiction. Hence, z is the only cut vertex of G/e.

Hence, $\{x, y\}$ is a vertex cut of G. Let G_1, G_2 be as in the decomposition theorem. Then G_1 and G_2 are 2-connected. and $G \setminus e = G_1 \oplus G_2$. By a previous lemma, $G \setminus e$ is 2-connected. $\square$

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Lecture 9: September 22

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

The Decomposition Theorem allows us to reduce the proof of Kuratowski's Theorem to the case of 3-connected graphs:

Theorem 9.45. **Theorem K2**: Suppose that every 3-connected graph that does not contain a subdivision of K_5 or $K_{3,3}$ is planar, then every 2-connected graph that does not contain a subdivision of K_5 or $K_{3,3}$ is planar.

Proof 9.46. Let G be a 2-connected graph without subdivision of K_5 or $K_{3,3}$. We will use induction of $|V(G)|$

Base Case: $|V(G)| \leq 5$. It is trivially true. Since the only non-planar graph with ≤ 5 vertices is K_5.

Inductive Hypothesis: Assume $|V(G)| \geq 6$, and every 2-connected graph H with $|V(H)| < |V(G)|$ with no subdivision of K_5 or $K_{3,3}$ is planar.

Inductive Conclusion: If G is 3-connected, then G is planar by assumption.

If G is not 3-connected, then it has a vertex cut $\{x, y\}$. By our decomposition theorem, there exists 2-connected graphs G_1 and G_2, each containing the edge xy with $G = G_1 \oplus G_2$ or $G \setminus xy = G_1 \oplus G_2$.

Note that G_1 and G_2 each have fewer vertices than G. Since G_2 is 2-connected, then by a previous lemma, $G = G_1 \oplus G_2$ or $G \setminus e = G_1 \oplus G_2$ contains a subdivision of G_1. So either way, G contains a subdivision of G_1. Therefore G_1 does not contain a subdivision of K_5 or $K_{3,3}$ (Recall a subdivision of a subdivision of K_5 or $K_{3,3}$ is itself a subdivision of K_5 or $K_{3,3}$). That's not possible.

Therefore, by inductive hypothesis, G_1 is planar. Similarly, G_2 is planar.

Take a planar drawing of $\tilde{G_1}$ of G_1 with xy on its outer face. And, take a planar drawing of $\tilde{G_2}$ of G_2 with xy on its outer face. Then gluing $\tilde{G_1}$ and $\tilde{G_2}$ along xy gives a planar drawing of G, or of $G \cup \{xy\}$, which contains G. $\square$

Example 9.47. The 2-sum of H with a cycle C with respect to the edge uv is the graph obtained from H by subdividing uv. If C has length k, then we add $k-2$ new path vertuices to uv.

How to "build" a 2-connected graph

Theorem 9.48. Let G be a 2-connected graph. Then, at least one of the following holds:

1. G is a cycle
2. there exists $e \in E(G)$ such that $G \setminus e$ is 2-connected
3. G is the 2-sum of a 2-connected graph and a cycle. (This can also be written as, is obtained by subdividing an edge of a 2-connected graph)

Proof 9.49. Call a graph **good** if it satisfies at least one of the above properties. Note that any good graph is 2-connected. If the first property is satisfied, then it is 2-connected automatically. If the second property is satisfied, then we add an edge to the graph won't ruin the 2-connectivity. If the third property, then by an earlier lemma, it is 2-connected.

Fix a 2-connected graph G. We know G contains a cycle with any edge we want, by an earlier lemma.

Let H be a good subgraph of G with the largest possible number of edges. If $H = G$, then G is good, and then we are done. Assume $|E(H)| < |E(G)|$. If $|V(H)| = |V(G)|$ then there exists $e \in E(G) \setminus E(H)$, so $G \setminus e$ contains H. Thus $G \setminus 2$ is 2-connected, so G satisfies the second property. Hence G is good, contradicting that H has maximal edges.

So we may assume that $|V(G)| > |V(H)|$. Then there exists a vertex $x \in V(G) \setminus V(H)$ and an edge xu with $u \in V(H)$ since G is connected. Since G is 2-connected, u is not a cut vertex of G. So there exists a path P in $G - u$ from x to some vertex $v \in V(H)$, such that $V(P) \cap V(H) = \{v\}$.

Let $H' = \begin{cases} H & \text{if } uv \in E(H) \\ H \cup \{uv\} & \text{if } uv \notin E(H) \end{cases}$. Then, H' is 2-connected since H is 2-connected.

Let C be the cycle $P \cup \{u\} \cup \{xu, uv\}$. Then $J = H' \oplus C$ is a subgraph of G. Since H' is 2-connected, J satisfies the third property. But $|E(J)| = |E(H')| + |E(C)| - 2 \geq |E(H)| + 1 > |E(H)|$, and J is good, contradicting that H has maximal edges.

Hence, G is good, as required. $\square$

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Lecture 10: September 29

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

From Theorem 9.48, we know that any 2-connected graph could be "built" by starting from a cycle, and adding edges joining 2 vertices already present, or subdividing edges.

Lemma 10.50. Let G be a 2-connected planar graph. In every planar drawing of G, every face is bounded by a cycle, and every edge is incident to exactly two distinct faces.

Proof 10.51. By induction on $|E(G)|$.

Base Case: Since G is 2-connected, $\delta(G) \geq 2$. So $|E(G)| = \frac{1}{2}\sum_{x \in V(G)} d(x) \geq \frac{1}{2} \cdot 2|V(G)| = |V(G)|$. So the base case is $|E(G)| = |V(G)|$, so G is a cycle. There is only one way to draw a cycle, and we have exactly two faces. The statement is trivially true.

Inductive Hypothesis: Assume that $|E(G)| > |V(G)|$, and every planar drawing of a 2-connected planar graph H with $|E(H)| < |E(G)|$ is such that every face is bounded by a cycle, and every edge is incident to exactly two distinct faces.

Inductive Conclusion: Fix a planar drawing of $\tilde{G}$ of the 2-connected planar graph G.

Case 1: There exists $e = xy \in E(G)$ such that $H = G \setminus e$ is 2-connected. Then erasing e from $\tilde{G}$ gives a planar drawing $\tilde{H}$ of H, in which x and y lie in the boundary of a common face F. By IH (apply to H), every face of $\tilde{H}$ is bounded by a cycle, and every edge is incident to exactly 2 distinct faces. So F is bounded by a cycle. Thus in $\tilde{G}$, e cuts F into 2 distinct faces, both of which are bounded by cycles. Hence $\tilde{G}$ satisfies the conclusion of the theorem.

Case 2: G is obtained by subdividing one edge of a 2-connected graph H. (i.e. $G = H \oplus C$ where C is a cycle). Then $|E(H)| < |E(G)|$ since $|E(C)| \geq 3$.

We can obtain a planar drawing $\tilde{H}$ of H by "suppressing" the interior vertices in the path P in $\tilde{G}$ that replaced f in $\tilde{H}$ (delete vertices, reverse engineering of subdividing). By IH (apply to H), every face of $\tilde{H}$ is bounded by a cycle, and every edge is incident to exactly two distinct faces. Then the same is true for G, since the edges of P inherit the desired property from f. $\square$

Lemma 10.52. Let G be a k-connected graph. Suppose G has a vertex cut W of size k. Then for each component C of $G - W$ and each $w \in W$, there is an edge from w to C.

Proof 10.53. Suppose on the contrary that there is no edge from $w_0 \in W$ to C. Then $W \setminus \{w_0\}$ is a vertex cut of G separating C from the rest of G. But $|W \setminus \{w_0\}| = k - 1$, contradicting the fact that G is k-connected. $\square$

CO342: Introduction to Graph Theory **Fall 2020**

Lecture 11: September 29

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 11.54. Let G be a 3-connected graph and let $e \in E(G)$. We say that e is **contractible** if G/e is 3-connected.

Theorem 11.55. Every 3-connected graph with ≥ 5 vertices has a contractible edge. (Exclude the graph of K_4)

Proof 11.56. Suppose on the contrary that no such edge exists. Since G/e has ≥ 4 vertices for any e, it must have a vertex cut of size at most 2.

Let $e = xy \in E(G)$. Let z be the image of e under contraction. Then

- any vertex cut of G/e of size ≤ 2 contains z, otherwise it would be a vertex cut of G of size at most 2.
- any vertex cut of G/e of size ≤ 2 is of size 2, otherwise $\{x, y\}$ would be a vertex cut of G of size 2, contradicting that G is 3-connected.

Hence, for every edge $e = xy$ of G, there exists $w \in V(G) \setminus \{x, y\}$ such that $\{w, z\}$ is a vertex cut of G/e. Then, $\{x, y.w\}$ is a vertex cut of G.

Choose

- the edge xy
- the vertex w
- the component C of $G - \{x, y, w\}$ such that C has the smallest possible number of vertices.

Let v be a neighbour of w in C. Note that v exists because $\{x, y, w\}$ is a 3-vertex cut in 3-connected graph G (by lemma 10.52). Since $wv \in E(G)$, there exists a vertex $u \in V(G) \setminus \{w, v\}$ such that $\{u, v, w\}$ is a vertex cut of G. We will use $wv \in E(G)$ and $u \in V(G)$ to show that our choice of (xy, w, C) was wrong.

We can see that $\{w, v, u\}$ is a vertex cut of G. We claim that some component D of $G - \{w, v, u\}$ is disjoint from $\{x, y\}$. To see this, we know that

- $G - \{w, v, u\}$ has ≥ 2 components
- xy is an edge

Let q be a neighbour of v in D. Again, we know q exists by lemma 10.52, since $\{w, v, u\}$ is a 3-vertex cut in G.

Then $q \in C$ because $v \in C$ since all neighbours of C are in $C \cup \{x, y, w\}$. But we know $q \notin \{x, y, w\}$ since q is in D, which is disjoint from $\{x, y\}$, and w is not in $G - \{w, v, u\}$. So $q \in C$.

Thus, $D \cap C \neq \emptyset$, so $D \subseteq C$ since $D \cap \{x, y, w\} = \emptyset$.

But $v \in C \setminus D$, hence $|D| < |C|$. Then, (uv, u, D) contradicts our choice of (xy, w, C) (minimality on number of vertices in C). Hence, there exists some contractible edge in G. $\square$

CO342: Introduction to Graph Theory **Fall 2020**

Lecture 12: September 29

Lecturer: Penny Haxell *Noted By: Haochen Wu*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They may be distributed outside this course only with the permission of the instructors.*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Lemma 12.57. Subdivisions of K_5: Let e be an edge of a graph G, and suppose G/e contains a subdivision of K_5. Then G contains a subdivision of K_5 or $K_{3,3}$.

Proof 12.58. Let z be the image of e under contraction. Let K be a subdivision of K_5 in G/e. If $e \notin V(K)$, then K is a subdivision of K_5 in G as well. So we may assume $z \in V(K)$.

If z is a path vertex (degree 2) in K, in all cases (consider how $e = xy$ is contracted and connected to the remaining part of K, there are only 4 possibilities), G contains a subdivision of K, which is a subdivision of K_5.

If z is a branch vertex (degree 4) in K, there would be 8 cases similar to the analysis above. Let's say $e = xy$, and the four connecting vertices are u, v, w, q (from left to right). The tricky case would be x connects to u and v, and y connects to w and q. In this case, we would obtain a $K_{3,3}$. We actually have a hexagon, each vertex is x, a, b, c, d, y, and $e = xy$ is the edge to be contracted. We would obtain a subdivision of $K_{3,3}$ (consider the bipartite drawing of $K_{3,3}$), where (b, a, y) is on one side of $K_{3,3}$, and (x, c, d) is on the other side of $K_{3,3}$.

So in all cases, G gets a subdivision of either K_5 or $K_{3,3}$. $\square$

Theorem 12.59. Kuratowski's Theorem for 3-connected graph (K3): Let G be a 3-connected graph that does not contain a subdivision of K_5 or $K_{3,3}$. Then G is planar.

Proof 12.60. By induction on $|V(G)|$.

Base Case: $|V(G)| \leq 5$ since the only non-planar graph with ≤ 5 vertices is K_5. Hence, the statement holds in this case.

Inductive Hypothesis: Assume $|V(G)| \geq 6$, and any 3-connected graph with fewer than $|V(G)|$ vertices, that does not not contain a subdivision of K_5 or $K_{3,3}$, is planar.

Inductive Conclusion: Since $|V(G)| \geq 6$, by theorem 11.55, G has a contractible edge e. Then G/e is 3-connected and has $|V(G)| - 1$ vertices.

We claim that G/e does not contain a subdivision of K_5 or $K_{3,3}$:

- Suppose G/e contains a subdivision of K_5, then G contains a subdivision of K_5 or $K_{3,3}$ by lemma 12.57. Contradiction.
- Suppose G/e contains a subdivision of $K_{3,3}$. Then $K_{3,3}$ is a minor of G/e. and hence a minor of G. Since $K_{3,3}$ is cubic, by lemma 6.30, G contains a subdivision of $K_{3,3}$. Contradiction.

Therefore G/e satisfies the inductive hypothesis. Hence $G/e = G'$ is planar. Fix a planar drawing $\tilde{G}'$ of G'. Let z be the image of e under contraction. Then erasing z and its incident edges from $\tilde{G}'$ gives a planar drawing of $G' - z$.

Note $G' - z$ is 2-connected since G' is 3-connected. Removing 1 vertex from a 3-connected graph won't generate any cut vertices. By lemma 10.50, every face of the drawing $\tilde{G}' - z$ is bounded by a cycle.

Let C denote the cycle bounding the face of $\tilde{G}' - z$ that contained z in G'. Then all neighbours of x and y in G are neighbours of z in G', so they lie on the cycle C.

Label the neighbours $x_1, x_2, ..., x_t$ of x in order around C.

Denote by P_i, the path in C from x_i to x_{i+1} (P_{t+1} is P_1). There are four cases:

- Case I: All neighbours of y are in the same P_i, Then we can complete the drawing to a drawing of G. So G is planar.
- Case II: y has a neighbour in the interior of some P_i, and a neighbour not in P_i. Then G has a subdivision of $K_{3,3}$. The branch vertices are $\{a, b, x\} \cup \{y, x_i, x_{i+1}\}$ (a, b are neighbours of y, where a lies on P_i, b lies on P_{i+1}). Contradiction.
- Case III: y has x_i and x_j as neighbours, where x_i and x_j are not endpoints of the same P_ℓ. Then G has a subdivision of $K_{3,3}$. The branch vertices are $\{x_i, x, x_j\} \cup \{y, x_{j-1}, x_{-1}\}$. Contradiction.
- Caie IV: y has three of the x_i's as n neighbours. Then G has a subdivision of K_5. The pentagon would have vertices $\{y, x, x_i, x_j, x_k\}$ where x_i, x_j, x_k are neighbours of y. Contradiction.

Since only case I can occur, we proved that G is planar. $\square$

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Lecture 13: October 6

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Theorem 13.61. **Kuratowski's Theorem**: A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$

Proof 13.62. $\Rightarrow$: Since K_5 or $K_{3,3}$ are not planar (from MATH239), if G has a subdivision of K_5 or $K_{3,3}$, then it is not planar.

$\Leftarrow$: Suppose G does not contain a subdivision of K_5 or $K_{3,3}$.

- If G is 3-connected, then G is planar by **Theorem K3**.
- If G is 2-connected, then G is planar by **Theorem K2**.
- If G is connected but not 2-connected, then none of its blocks contain a subdivision of K_5 or $K_{3,3}$, otherwise G would. Each block is either an edge joining two vertices, or 2-connected. Hence, it is planar by **Theorem K2**, so every block of G is planar. By Theorem K1, then G is planar.
- If G is not connected, all components are planar.

□

Definition 13.63. **Internally Disjoint Path**: A set of paths, $\{P_1, ..., P_t\}$, with common end points x and y is said to be **internally-disjoint** if $V(P_i) \cap V(P_j) = \{x, y\}$ for all $i \neq j$.

Definition 13.64. **Vertex Cuts separating vertices x and y**: Let x and y be vertices in a graph G. A vertex cut W of G is said to **separate** x and y if x and y are in different component of $G - W$. (Note that $x, y \notin W$)

Corollary 13.65. If x, y are joined by a set of k internally disjoint paths in G, then there is no vertex cut of size $\leq k-1$ in G that separates x and y.

Theorem 13.66. Menger's Theorem: Let a and b be distinct non-adjacent vertices in a graph G. Let s be the minimum size of a vertex cut of G that separates a and b. Then G contains a set of s internally disjoint paths joining a and b. We will write it as "(a,b)-paths"

Proof 13.67. If $s \leq 1$ then the statement is true. So we may assume $s \geq 2$. We use induction on $|E(G)|$.

Base Case: $|E(G)| = 0$. Statement is true trivially.

Inductive Hypothesis: Assume $|E(G)| \geq 1$ and for every graph with $|E(H)| < |E(G)|$ and all a, b non-adjacent in H, there exists a set of t internally disjoint (a,b)-paths in H where t is the minimum size of a vertex cut separating a and b in H.

Inductive Conclusion:

- Case I: All edges of G are incident to a or b. Then $s = |N(a) \cap N(b)|$

  There is a set of s internally disjoint (a,b)-paths, namely $\{aub : u \in N(a) \cap N(b)\}$.

- Case II: There exists $e \in E(G)$, say $e = xy$, such that $\{x, y\} \cap \{a, b\} = \emptyset$. Let $H = G \setminus e$. Let S be a vertex cut of minimum size separating a and b in H.

  If $|S| \geq s$, then by inductive hypothesis, there exists a set of s internally disjoint (a,b)-paths in H, and hence the same paths are also in G.

  So, we may assume $|S| < s$. Note $S \cup \{x\}$ separates a and b in G. Hence, by definition of s, $|S \cup \{x\}| \geq s$. Therefore, $|S| = s-1$. Also, $S \cup \{x\}$ and $S \cup \{y\}$ are vertex cuts of size s separating a and b in G.

  Since $|S| < s$, then there is an (a,b)-path in $G - S$. Hence, it must use the edge xy since there is no (a,b)-path in H. Without loss of generality, x is closer to a on this path than y is.

  By applying inductive hypothesis to $H = G \setminus e$, we found $S \cup \{x\}$ and $S \cup \{y\}$ are both vertex cuts of size s separating a and b in G.

  Let G_b be the graph obtained from G by contracting all edges in the component C_b of $G - (S \cup \{x\})$ that contains b. Then $|E(G_b)| < |E(G)|$. Since y and b and the path joining them are in C_b.

  Call the new vertex z_b.

  So, we can apply inductive hypothesis to G_b, with vertices a, z_b.

  If T is a vertex cut separating a and z_b in G_b, then T is a vertex cut separating a and z_b in G. Thus, the minimum size of a vertex cut separating a and b in G_b is $\geq s$, by definition of s. To be more precise, we can say $= s$ since $S \cup \{x\}$ is such a cut.

  By inductive hypothesis applying to G_b, there exists a set $\{P_1, ..., P_s\}$ of s internally disjoint (a, z_b)-paths in G_b. Note that for each i, the neighbour of z_b on P_i is in $S \cup \{x\}$ since $N(Z_b) \subseteq S \cup \{x\}$.

  Similarly, we define G_a and find internally disjoint (z_a, b)-paths $\{Q_1, ..., Q_s\}$ in G_a.

  Without loss of generality, P_1 contains x, Q_1 contains y, and for $2 \leq i \leq s$, P_i and Q_i both contain the same vertex $u_i \in S$.

Note that $V(P_i) \backslash (S \cup \{x, z_b\})$ is contained in the component of a in $H - S$. Similarly, $V(Q_i) \backslash (S \cup \{y, z_a\})$ is contained in the component of b in $H - S$.

Therefore, $V(P_i) \cap V(Q_i) = \begin{cases} ui & \text{if } i = j \geq 2 \\ \emptyset & \text{otherwise} \end{cases}$. Hence, the set $\{P_1 \cup Q_1 \cup \{xy\}, P_2 \cup Q_2, ..., P_s \cup Q_s\}$ is a set of s internally disjoint (a, b)-paths in G as required.

□

Theorem 13.68. **Menger-Whitney Theorem**: Let G be a graph with ≥ 2 vertices, then G is k-connected if and only if for every pair of vertices a and b, there exists a set of k internally disjoint (a, b) paths.

Proof 13.69. For $k = 1$, the statement is true by definition. So, we may assume $k \geq 2$.

$\Rightarrow$: Assume G is k-connected. Then $|V(G)| \geq k + 1$. Let a and b be distinct vertices in G.

- If $ab \notin E(G)$, then the minimum size s of a vertex cut separating a and b is $\geq k$ since G is k connected. Hence, by **Menger's Theorem**, there exists a set of $s \geq k$ internally disjoint (a, b)-paths in G as required.
- If $ab \in E(G)$, then we claim that $G' = G \setminus ab$ is $(k - 1)$-connected.

  The proof of the claim is as follows: Suppose not. Then G' has a vertex cut of size $\leq k - 2$. Then a and b are in different components of $G' - Y$, otherwise Y is a vertex cut of G. Since $|V(G')| \geq k + 1$, a and b can't be a component by their own.

  Without loss of generality, some component of $G' - Y$ not containing b has a vertex $c \neq a$. Then, $Y \cup \{a\}$ is a vertex cut of G of size $\leq k - 1$, separating b and c, contradicting G being k-connected. Hence, G' is $(k - 1)$-connected.

  Hence, we can apply Menger's Theorem to the vertices a and b in $G \setminus ab$ to find that there exists a set S of $k - 1$ internally disjoint (a, b)-paths in $G \setminus ab$. So, $S \cup \{ab\}$ is the required set of k internally disjoint (a, b)-paths in G.

$\Leftarrow$: Assume G has the property that every pair of a, b of vertices is joined by a set of k internally disjoint (a, b)-paths. Then G has no vertex cut of size $\leq k - 1$, as we observed before.

Moreover, $|V(G)| \geq k + 1$ since all but at most one of k internally disjoint (a, b)-paths for fixed a and b each contain a distinct vertex (from each other and from a and b).

Hence, G is k-connected. □

CO342: Introduction to Graph Theory **Fall 2020**

Lecture 14: October 6

Lecturer: Penny Haxell *Noted By: Haochen Wu*

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See **Proof 13.67**.

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Lecture 15: October 6

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Extending a k-connected graph.

Lemma 15.70. **The Extension Lemma**: Let G be a k-connected graph, and let $Y \subset V(G)$ be a set of size k. Then, the graph H with vertext set $V(H) = V(G) \cup \{x\}$ and edge set $E(H) = E(G) \cup \{xy : y \in Y\}$ is k-connected. Here x is a new vertex.

Proof 15.71. We know $|V(H)| > |V(G)| \geq k + 1$.

Suppose W is a vertex cut of H, and assume for a contradiction that $|W| \leq k - 1$. Then $x \notin W$, otherwise $W - \{x\}$ is a vertex cut of G of size $\leq k - 2$, contradiction.

Let z be a vertex in a component of $H - W$ that does not contain x. There exists $y \in Y \setminus W$, since $|Y| > |X|$. So y is in the component of x in $H - W$. Then W separates y and z in G, contradicting that G is k-connected. Hence, W cannot exist. So H is k-connected. $\square$

Definition 15.72. **Fans**: Let G be a graph, let $x \in V(G)$ and let $Y \subseteq V(G) \setminus \{x\}$. An (x, Y)-**fan** in G is a set $S = \{P_1, ..., P_k\}$ of paths in G from x to Y such that $k = |Y|$ and $V(P_i) \cap V(P_j) = \{x\}$ for all $i \neq j$.

Lemma 15.73. **The Fan Lemma**: Let G be a k-connected graph, let $x \in V(G)$, and let $Y \subseteq V(G) \setminus \{x\}$ be such that $|Y| = k$. Then G contains an (x, Y)-fan.

Proof 15.74. Let H be the graph formed by adding a new vertex z to G and new edges $\{zy : y \in Y\}$. Then by the extension lemma (Lemma 15.70), H is k-connected. By the Menger-Whitney Theorem, applying it to H, there exists a set $\{P_1, ..., P_k\}$ of internally disjoint (x, z)-paths in H. Since $N(z) = Y$.

The neighbour of z on P_i is $y_i \in Y$, and $\{y_1, ..., y_k\} = Y$. Hence, $\{Q_1, ..., Q_k\}$ is an (x, Y)-fan where $Q_i = P_i - z$ for each i. $\square$

Lemma 15.75. **The Cycle Lemma**: Let G be a k-connected graph, where $k \geq 2$. Let $Y \subset V(G)$ be such that $|Y| = k$. Then, there exists a cycle in G that contains every vertex of Y.

Proof 15.76. Let C be a cycle in G that contains the maximum possible number of vertices of Y. Let $C \cap Y = \{y_1, ..., y_m\}$.

If $m = k$, then we are done. So, we may assume $m < k$. Let $y \in Y \setminus \{y_1, ..., y_m\}$. Note that $m \geq 2$ by our characterization theorem for 2-connected graphs.

- Case I: $V(C) = \{y_1, ..., y_m\}$. We apply the Fan lemma for m-connected graphs to G with vertex y and set $V(C)$, to get a $(y, V(C))$-fan $\{P_1, ..., P_m\}$ where P_i joins to (without loss of generality) $y_i \in Y$ (Recall G is k-connected implies that G is m-connected since $m < k$).

  Note that since $m \geq 2$, $C \setminus \{y_1 y_2\} \cup P_1 \cup P_2$ contains $\{y_1, ..., y_m\}$ plus y contradicting the maximality of C. So this case cannot happen.

- Case II: $W = \{x, y_1, ..., y_m\} \subseteq V(C)$ where $x \notin \{y_1, ..., y_m\}$. By the Fan Lemma, for $(m+1)$-connected graphs (Note that $m + 1 \leq k$), there exists a (y, W)-fan $\{P_0, P_1, ..., P_m\}$ in G.

  For $0 \leq i \leq m$. Let u_i be the vertex of C that is closest to y on P_i. Then, some segment (y_i, y_{i+1}) on C must contain two of the vertices, say u_j and u_ℓ, since there are m y_i's and $m + 1$ u_i's.

  Then, the cycle formed by deleting the (u_j, u_ℓ)-segment of C and adding the (u_j, y)-segment of P_j and the (u_ℓ, y)-segment of P_ℓ is a cycle containing $\{y_1, ..., y_m, y\}$, also contradicting the maximality of C.

  Hence, $m = k$ as required.

□

Lemma 15.77. Every 3-connected graph G contains a subdivision of K_4.

Proof 15.78. Fix $x \in V(G)$. Then $G - x$ is 2-connected since G is 3-connected. Let C be a cycle in $G - x$ by the cycle lemma.

Let $y_1, y_2, y_3 \in V(C)$ be distinct vertices. Then, by the Fan Lemma, there exists an $(x, \{y_1, y_2, y_3\})$-fan $\{P_1, P_2, P_3\}$ in G. Let u_i be the vertex of P_i closest to x on P_i for each i, and let Q_i be the (x, u_i)-segment of P_i. Then $Q_1 \cup Q_2 \cup Q_3 \cup C$ is a subdivision of K_4 in G. □

CO342: Introduction to Graph Theory **Fall 2020**

Lecture 16: October 20

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 16.79. A **matching** in a graph G is a set M of disjoint edges (i.e. no two edges in M are incident to a common vertex).

Definition 16.80. A matching M **saturates** $v \in V(G)$ if v is incident to an edge of M.

Definition 16.81. A **maximum matching** in G is a matching of maximum size.

Definition 16.82. A **perfect matching** in G is a matching that saturates every vertex of G.

Note that not every graph has a perfect matching. If G does have a perfect matching, then $|V(G)|$ is even. But it is not sufficient.

The Greedy Algorithm for finding a Matching:
Input: a graph G
Output: A matching in G
    Set $M = \emptyset$, $H = G$
    while true:
        if H has no edges **then**
            STOP and **Output** M
        else
            Choose an edge $xy \in H$, add xy to M
            $H = H - \{x, y\}$

The features of the above algorithm:

- Good ones are: it is very simple and efficient to implement
- Bad ones are: it does not always find a maximum matching.

Definition 16.83. The maximum size of a matching in G is denoted as $\nu(G)$

Proposition 16.84. The greedy algorithm always finds a matching of size at least $\frac{1}{2}\nu(G)$ in G.

Proof 16.85. Let M be a matching in G found by the greedy algorithm. Let M^* be a maximum matching in G. So $|M^*| = \mu(G)$

- Every edge of G has a vertex that is saturated by M, since the algorithm does not terminate until H has no edges.
- M saturates $2|M|$ vertices. Hence $|M^*| \leq 2|M|$ since no two edges of M^* can be incident to the same vertex.

Thus, $\mu(G) = |M^*| \leq 2|M|$ as required □

For a matching in Bipartite Graph G, whose vertices are partitioned to (X, Y), if G has a matching M saturating X, then $|S|$ vertices of S are matched by M to $|S|$ of the vertices of Y, which have to be neighbour of S.

Theorem 16.86. If G has a matching M saturating X, then for every subset $S \subseteq X$, $|N(S)| \geq |S|$. This is called Hall's Condition.

CO342: Introduction to Graph Theory **Fall 2020**

Lecture 17: October 20

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Theorem 17.87. **Hall's Theorem**: Let G be a bipartite graph with vertex classes X and Y, Then G has a matching saturating X if and only if

$$|N(S)| \geq |S|$$

for every $S \subseteq X$

Proof 17.88. We have observed that if G has a matching saturating X, then the hall's condition holds.

$\Leftarrow$: Assume the Hall's condition holds, we use induction on $|X|$.

Base Case: $|X| = 1$. Then G has a matching of size 1. The base case holds.

Inductive Hypothesis: Assume $|X| \geq 2$, and any bipartite graph H with vertex classes X' and Y' satisfying the Hall's condition for X', where $|X'| < |X|$, has a matching saturating X'.

Inductive Conclusion: There are several cases:

- Case I: Every $S \subseteq X$ with $\emptyset \neq S \neq X$ satisfies $|N(S)| > |S|$. Let $xy \in E(G)$. Let $H = G - \{x, y\}$.

  Since $N_H(S) = N_G(S) \setminus \{y\}$ and $|N_G(S)| > |S|$.

  Then, for every $S \subseteq X \setminus \{x\}$ we have $|N_H(S)| \geq |S|$.

  Therefore, by the inductive hypothese applying to H, H has a matching M' saturating $X \setminus \{x\}$. Then $M' \cup \{xy\}$ is a matching in G saturating X as required.

  So we may assume that Case I does not hold.

- Case II: there exists some $S_1 \subseteq X$ with $\emptyset \neq S_1 \neq X$ such that $|N(S_1)| = |S_1|$.

  Let G_1 be the subgraph of G induced by $S_1 \cup N(S_1)$. Then G_1 satisfies the Hall's condition. If $S \subseteq S_1$, then $N(S) \subseteq N(S_1)$, all of them are in G_1.

  Let G_2 be the subgraph of G induced by $(X \setminus S_1) \cup (Y \setminus N(S_1))$.

  Consider $S \subset X \setminus S_1$. Then $N_{G_2}(S) = N_G(S \cup S_1) \setminus N(S_1)$.

  So

$$\begin{aligned}
|N_{G_2}(S)| &= |N_G(S \cup S_1)| - |N(S_1)| \\
&\geq |S \cup S_1| - |N(S_1)| \quad \text{By Hall's condition in } G \text{ applied to } S \cup S_1 \\
&= |S \cup S_1| - |S_1| \\
&= |S|
\end{aligned}$$

Hence G_2 satisfies the Hall's Condition as well. Then, since $S_1 \neq X$, we can apply the inductive hypothesis to G_1 and since $S_1 \neq \emptyset$ we can apply the inductive hypothesis to G_2, to get disjoint matchings $M_1 \in G_1$ and $M_2 \in G_2$. Then, $M_1 \cup M_2$ in G saturates x.

□

Theorem 17.89. Let G be a regular bipartite graph of positive degree. Then G has a perfect matching.

Proof 17.90. Let $k \geq 1$ be the degree of each vertex of G, and let X and Y be the vertex classes of G.

For $S \subseteq X$, we write $E(S, N(S))$ for the set of edges of G that join S to $N(S)$.

Then, since G is k-regular, $|E(S, N(S))| = k$.

Also, since G is k-regular by looking at the other side, $|E(S, N(S))| \leq k|N(S)|$.

So $k|N(S)| \geq |E(S, N(S))| = k|S|$, which implies $|N(S)| \geq |S|$. So, G has Hall's condition. By Hall's theorem, G has a matching M saturating X.

Note that this matching is also perfect. Since $|E(G)| = k|X| = k|Y|$, so $|X| = |Y|$. So M is a perfect matching. □

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Lecture 18: October 20

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Theorem 18.91. **The defect version of Hall's Theorem**: Let $d \geq 0$, and let G be a bipartite graph with vertex classes X and Y. Then G has a matching of size $\geq |X| - d$ if and only if

$$|N(S)| \geq |S| - d$$

for every $S \subseteq X$

Proof 18.92. $\Rightarrow$: If G has a matching M of size $|X| - d$, then M matches $\geq |S| - d$ vertices of S into $N(S)$ for each $S \subseteq X$.

$\Leftarrow$: Assume the defected Hall's condition holds. Form the graph H by adding a set D of d new vertices and joining them to all vertices of X. Then, for $S \subseteq X$, $N_H(S) = N_G(S) \cup D$. So $|N_H(S)| \geq \underbrace{|S| - d}_{\in G} + d$, $N_H(S) \geq |S|$. Hence by Hall's Theorem, H has a matching M saturating X. Since $\leq d$ edges of M are not edges of G, M has at least $|X| - d$ edges left in G.

So, G has a matching of size $\geq |X| - d$ $\square$

Definition 18.93. A **vertex cover (of the edges)** of a graph G is a set W of vertices of G such that $G - W$ has no edges, i.e. every edge of G is incident to a vertex in W.

Also, we denote the minimum size of a vertex cover of a graph G as $\tau(G)$.

Proposition 18.94. For every graph G, we have $\tau(G) \geq \nu(G)$

Proof 18.95. For any matching M, and any vertex cover W, W must contain a distinct vertex from each edge of M, so $|W| \geq |M|$. $\square$

Theorem 18.96. **<u>Konig's Theorem</u>**: If G is a bipartite graph, then $\tau(G) = \nu(G)$

Proof 18.97. Let X and Y be the vertex classes of G. Define H by $V(H) = V(G) \cup \{x, y\}$, and $E(H) = E(G) \cup \{xz : z \in X\} \cup \{yz : z \in Y\}$. The minimum size of a vertex cut W separating x and y in H is $\tau(G)$. This is because W is actually a vertex cover of G.

Therefore, by Menger's Theorem applying to H, there exists a set of $\geq \tau(G)$ internally disjoint (x, y)-paths in H. Taking the second edge from each of these forms a matching in G.

Hence, G has a matching of size $\geq \tau(G)$. Therefore $\nu(G) \geq \tau(G)$. We already observed that $\nu(G) \leq \tau(G)$ for every G. So $\nu(G) = \tau(G)$. $\square$

Exercise: Prove that for every graph G, $\tau(G) \leq 2\nu(G)$.

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Lecture 19: October 27

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Theorem 19.98. **Konig's Theorem**: If G is a bipartite graph, then $\tau(G) = \nu(G)$

We give another proof here.

Proof 19.99. We know already $\tau(G) \geq \nu(G)$ for every graph G. Let T be a vertex cover of G with $|T| = \tau(G)$.

Let X and Y be the vertex classes of G. Let G_1 be the subgraph of G induced by $(T \cap X) \cup (Y \setminus T)$. For each $S \subseteq T \cap X$, $|N_{G_1}(S)| \geq |S|$, otherwise, $T \setminus S \cup N_{G_1}(S)$ would be a vertex cover of G that is smaller than T.

Therefore, by **Hall's Theorem**, G_1 has a matching M of size $|T \cap X|$. Similarly, the graph G_2 induced by $(T \cap Y) \cup (X \setminus T)$ has a matching of size $|T \cap Y|$.

G_1 and G_2 are disjoint, so $M_1 \cap M_2 = \emptyset$ as well. So $M_1 \cup M_2$ is a matching in G of size $|T \cap X| + |T \cap Y| = |T|\tau(G)$. Hence, $\nu(G) \geq \tau(G)$ as required.

So, $\nu(G) \geq \tau(G)$ □

Definition 19.100. A set $W \subseteq V(G)$ is called **independent** if no two vertices of W are joined by an edge of G. The maximum size of an independent set in G is denoted as $\alpha(G)$.

Definition 19.101. A set $S \subseteq E(G)$ is an **edge cover** (of the vertices) if every vertex of G is incident to an edge of S. We denote the minimum size of an edge cover of an edge cover by $\rho(G)$.

Lemma 19.102. $\alpha(G) + \tau(G) = |V(G)|$ for every graph G.

Proof 19.103. Note that T is a vertex cover of G if and only if $V(G) \setminus T$ is independent. Hence, T is a minimum vertex cover if and only if $V(G) \setminus T$ is a maximum independent set. $\square$

Lemma 19.104. **<u>Gallai's Lemma</u>**: If G has no isolated vertices, then $\nu(G) + \rho(G) = |V(G)|$.

Proof 19.105. Let $|V(G)| = n$. Let M be a matching in G with $|M| = \nu(G)$. Let $V(M)$ denote the set of vertices saturated by M. Since M is maximum, $V(G) \setminus V(M)$ is independent.

Construct an edge cover as follows:

- start with M
- for each $x \in V(G) \setminus V(M)$, take an edge incident to x

We get $|M| + (n - 2|M|) = n - |M| = n - \nu(G)$ edges. Hence, $\rho(G) \leq n - \nu(G)$, so $\rho(G) + \nu(G) \leq n$.

Then, let F be an edge cover of G with $|F| = \rho(G)$. Let H be the graph with $V(H) = V(G)$ and $E(H) = F$. Then, each $e \in F$ is incident to a vertex of degree in H since F is minimum (Otherwise, $F - \{e\}$ is a smaller edge cover).

So H has no cycles, i.e. it is a forest. We know $|V(H)| = n$, $|E(H)| = |F| = \rho(G)$. So, the number of components of the forest H is $n - \rho(G) = c$ (recall from MATH239).

H has no isolated vertices, since F is an edge cover of G. So, we may take (at least) one edge from each component of H to get a matching in G of size $n - \rho(G)$. Hence, $\nu(G) \geq n - \rho(G)$. So $\nu(G) + \rho(G) \geq n$

So, $\nu(G) + \rho(G) = n$ as desired. $\square$

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Lecture 20: October 27

Lecturer: Penny Haxell *Noted By: Haochen Wu*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They may be distributed outside this course only with the permission of the instructors.*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 20.106. Let **M-alternating** path in G is a path in G in which every second edge is in M.

Definition 20.107. A vertex of G is said to be **M-exposed** if it is not saturated by M.

Definition 20.108. An **M-augmenting path** is an M-alternating path of length at least 1 whose endpoints are both M-exposed.

Note that every M-augmenting path P has odd length, and $|M \cap E(P)| < |E(P) \setminus M|$. We denote by $M\Delta E(P)$ the matching $[M \setminus (M \cap E(P))] \cup [E(P) \setminus M]$. This is the matching obtained by (from M) switching on P.

If P is an M-augmenting path, then $|M\Delta E(P)| > |M|$, so M is not a maximum matching.

Suppose M_1 and M_2 are two matchings in the same graph G. Let H be the graph with $V(H) = V(G)$, and $E(H) = M_1 \cup M_2$. Then

- the maximum degree of H is ≤ 2.
- Every component of H is a path or a cycle
- Each cycle component is even, and its edges alternate between M_1 and M_2. It has the same number of M_1-edges as M_2-edges.
- Each path component is either
  - a single edge in $M_1 \cap M_2$
  - alternating between edges in M_1 and M_2, and the number of M_1-edges differs from the number of M_2-edges by at most 1.

Theorem 20.109. **Berge's Theorem**: A matching M in a graph is a maximum matching if and only if G does not contain an M-augmenting path

Proof 20.110. $\Rightarrow$: We've already observed that if M is maximum, then there is no M-augmenting path in G.

$\Leftarrow$: Assume there is no M-augmenting path in G. Let M^* be a maximum matching in G, and suppose for a contradiction that $|M^*| > |M|$. Define H by $V(H) = V(G)$ and $E(H) = M \cup M^*$. Then, as observed previously, every component of H contains the same number of M-edges as M^*-edges unless it is a path component P of odd length.

Since $|M^*| > |M|$, such a path component P must exist, that starts and ends with an M^*-edge. Then, the endpoints of P are M-exposed, so P is an M-augmenting path. Contradiction. So $|M^*| = |M|$, which shows M is maximum. $\square$

Theorem 20.111. **<u>Erdo's Posa's Theorem</u>**: For any graph G, $\nu(G) \geq \min\{\delta(G), \lfloor|\frac{V(G)}{2}|\rfloor\}$.

Proof 20.112. Let M be a maximum matching and let $V(M)$ denote the set of vertices saturated by M. If $|M|$ is $\lfloor|\frac{V(G)}{2}|\rfloor$, then we're done. So, assume $|M| < \lfloor|\frac{V(G)}{2}|\rfloor$. Then, $|V(G) \setminus V(M)| \geq 2$.

Let v and w be M-exposed. For any $xy \in M$, if $vx \in E(G)$, then $wy \notin E(G)$, otherwise $wyxv$ is an M-augmenting path, contradicting Berge's Theorem.

Similarly, $vy \in E(G) \Rightarrow wx \notin E(G)$. So the number of edges joining $\{x, y\}$ and $\{w, v\} \leq 2$. Hence, the total number L of edges from $\{w, v\}$ to $V(M)$ is $\leq 2|M|$. But $N(\{w, v\}) \subseteq V(M)$ since M is maximum. So, $2\delta(G) \leq d(v) + d(w) \leq L \leq 2|M|$. So $\delta(G) \leq |M|$. $\square$

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Lecture 21: October 27

Lecturer: Penny Haxell *Noted By: Haochen Wu*

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This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 21.113. Let G be a bipartite graph with vertex classes X and Y. A set of **preference lists** consists of a linear order $L(z)$ of $N(z)$ for each $z \in X \cup Y$.

Definition 21.114. In a bipartite graph G with preference lists L, a **matching** M is said to be **stable** with respect to L if, for every edge $xy \in E(G) \setminus M$, either

- $y' > y$ in $L(x)$ where $xy' \in M$, i.e. x prefers its partner y' in the matching M to y.
- or $x' > x$ in $L(y)$ where $x'y \in M$, i.e. y prefers its partner x' in the matching M to x

Stable matching exists for all bipartite graphs, but they are not always maximum matchings.

Gale and Shapley's Greedy Algorithm(A bipartite graph G, vertex classes X, Y, and preference list L)
    Set $K(x) := L(x)$ for all $x \in X$. Set $M := \emptyset$.
    while (true)
        if for each $x \in X$, either $K(x) = \emptyset$ or M saturates x **then**
            STOP, and **output** M.
        choose M-exposed $x \in X$ with $K(x) \neq \emptyset$
        let y be the max in $K(x)$:
        if y prefers x to x' where $x'y \in M$, **then**
            set $M := M \setminus \{x'y\} \cup \{xy\}$
        else if y is M-exposed, **then**
            set $M := M \cup \{xy\}$
            set $K(x) := K(x) \setminus \{y\}$

Note that $\sum_{x \in X} |K(x)|$ decreases in each iteration, so the algorithm terminates.

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Lecture 22: November 3

Lecturer: Penny Haxell *Noted By: Haochen Wu*

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We prove the correctness of **Gale and Shapley's Greedy Algorithm**.

Proof 22.115. We ensured that M is a matching at every iteration. Also, observe that at each iteration, the situation improves, or stays the same for every $y \in Y$, and deteriorates or stays the same for every $x \in X$.

To show the final matching M^* is stable: Consider an edge $x_0y_0 \notin M^*$.

- Case I: $y_0 = y$ for some iteration with $x = x_0$. "x_0 proposes to y_0".
  - If x_0y_0 is put into M in this iteration, then it is removed from M in a later iteration, and replaced by some x_1y_0 where y_0 prefers x_1 to x_0. Thus, by the observation at the very beginning, y_0 prefers its partner in M^* to x_0
  - If x_0y_0 is not put into M in this iteration, then y_0 is already matched by M to some x_1 it prefers to x_0. Thus, by the same observation, y_0 prefers its partner in M^* to x_0
- Case II: It never happens that $y_0 = y$ with $x = x_0$. "x_0 does not propose to y_0".

  Since initially $y_0 \in K(x_0)$, when the algorithm terminates, x_0 is matched to some y_1 it prefers to y_0, by the stopping rule.

Thus, M^* is a stable matching. □

Note that stable matchings are not unique.

Theorem 22.116. Let G be a bipartite graph with preference lists. Then all stable matchings of G saturate the same set of vertices.

This theorem implies that all stable matchings in G have the same size.

Proof 22.117. Let M_1 and M_2 be stable matchings in G. Let H be the graph $V(H) = V(G)$ and $E(H) = M_1 \cup M_2$. Suppose $P = x_1y_1x_2...$ is a path component of H, where (without loss of generality) $x_1y_1 \in M_1$ and $x_2y_2 \in M_2$. Then, x_1 is M_2-exposed.

So, y_1 prefers x_2 to x_1 since $x_1y_1 \notin M_2$. x_2 prefers y_2 to y_1 since $x_2y_1 \notin M_1$. y_2 prefers x_3 to x_2 since $x_2y_2 \notin M_2$, and so on.

If x_4 is M_1-exposed, then x_4y_3 contradicts M_1 being stable. So, x_4 is matched to y_4 where x_4 prefers y_4 to y_3. If y_4 is M_2-exposed, then x_4y_4 contradicts M_2 being stable.

But then P cannot end. So, no such P can exist in H. Hence M_1 and M_2 saturates the same set of vertices in G. $\square$

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Lecture 23: November 3

Lecturer: Penny Haxell *Noted By: Haochen Wu*

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This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 23.118. Let G be a bipartite graph with preference lists. A stable matching M_0 is said to be **X-optimal** if, for every stable matching M, and every $x \in X$, if $xy \in M$, then there exists $y' \in Y$ such that $xy' \in M_0$ and $y' \geq y$ in $L(x)$

i.e. this means that every $x \in X$ is matched by M_0 the best possible partner it could get in any stable matching

The stable matching M_0 is **X-pessimal** if, for every stable matching M, and every $x \in X$, if $xy \in M_0$, then there exists $y' \in Y$ such that $xy' \in M$ and $y' \geq y$ in $L(x)$

i.e. this means that every $x \in X$ is matched by M_0 the worst possible partner it could get in any stable matching

We prove that **Gale and Shapley's Greedy Algorithm** is X-optimal

Proof 23.119. Let M^* denote the stable matching found by Gale and Shapley's Greedy Algorithm. For any edge $xy \notin M^*$, if x prefers y to its partner in M^*, then, since M^* is stable, y prefers its partner in M^* to x.

Suppose, for a contradiction, that there exists a stable matching M' and $x_0y_0 \in M'$ where x_0 strictly prefers y_0 to its partner in M^*. Then, by the observation made, $x_1y_0 \in M^*$ for some x_1 that y_0 prefers to x_0.

So x_1y_0 was put into the matching M in the Gale and Shapley's Greedy Algorithm in some iteration.

This shows that the following set is non-empty: The set of iterations I of the implementation of Gale and Shapley's Greedy Algorithm that produced M^*, in which an edge x_1y_0 is put into M, where $x_0y_0 \in M', x_0y_0 \notin M^*$, and x_0 strictly prefers y_0 to its partner in M^*.

Let I be the earliest iteration in this set. I is the earliest iteration in which

- we add an edge x_1y_0 to M where
- $x_0y_0 \in M'$ and x_0 strictly prefers y_0 to its partner in M^*.

Since $x_1y_0 \notin M'$, $x_1y_1 \in M'$ for some y_1 that x_1 prefers to y_0.

In itration I, x_1 proposes to y_0 (and is accepted). Since x_1 prefers y_1 to y_0, so x_1 has already been rejected by y_1 in an earlier iteration I_1. In I_1, y_1 received and accepted a proposal from some x_2 that y_1 prefers to x_1. Thus, in I_1

- We add edge x_2y_1 to M, where
- $x_1y_1 \in M'$ and x_1 strictly prefers y_1 to its partner in M^*.

This contradicts our choice of I.

Hence, M^* is X-optimal, as required. $\square$

Note: Gale and Shapley's Greedy Algorithm finds the same matching, independent of the order in which the vertices $x \in X$ are considered.

What about stable matching in non-bipartite graphs? Stable matchings may not exist. Consider directed odd cycle in the sense of prefernce lists.

CO342: Introduction to Graph Theory **Fall 2020**

Lecture 24: November 3

Lecturer: Penny Haxell *Noted By: Haochen Wu*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They may be distributed outside this course only with the permission of the instructors.*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 24.120. An **odd component** of a graph G is a component with an odd number of vertices.

We denote the number of odd components of G by $odd(G)$.

Suppose G has a perfect matching M. Let $T \subseteq V(G)$. Then, for any odd component C of $G-T$. There must be an edge of M joining C to T. Hence $odd(G-T) \leq |T|$.

Theorem 24.121. Tutte's Theorem: A graph G has a perfect matching if and only if $odd(G-T) \leq |T|$ for every $T \subseteq V(G)$

Notes:

- The condition is called **Tutte's condition**.
- We know that if G has a perfect matching, then $|V(G)|$ is even. By applying Tutte's condition with $T = \emptyset$, we find $odd(G) = odd(G - \emptyset) \leq |\emptyset| = 0$, so $odd(G) = 0$, i.e. every component of G is even. So $|V(G)|$ is even.

Lemma 24.122. Let G be a graph satisfying $odd(G-T) \leq |T|$ for all $T \subseteq V(G)$. Suppose G is a subgraph of H where $V(H) = V(G)$, then H satisfies $odd(H-T) \leq |T|$ for all $T \subseteq V(H)$.

Proof 24.123. Let $T \subseteq V(H) = V(G)$. Then $G-T$ has $\leq |T|$ odd components.

Adding edges inside components of $G-T$, inside T, or joining T to components of $G-T$ does not change the number of odd components.

Adding edges between components can only decrease the number of odd components, or leave it unchanged.

So, $odd(H-T) \leq |T|$ for each $T \subseteq V(H)$ □

Definition 24.124. A graph G is **type-0** if $V(G)$ has a partition $X \cup Y$ such that $X = \{x \in V(G) : xz \in E(G) \text{ for all } z \in V(G) \setminus \{x\}\}$ and every component of $G - X$ is a complete graph.

Lemma 24.125. If G is type-0, and $odd(G - T) \leq |T|$ for all $T \subseteq V(G)$, then G has a perfect matching.

Proof 24.126. Setting $T = X$ shows $odd(G - X) \leq |X|$. Beging constructing a matching by taking disjoint edges, one joining a vertex of each odd component of $G - X$ to X. Match the remainder of each odd component and every even component inside the component itself.

Last, complete to a perfect matching inside X (recall that $|V(G)|$ is even, by Tutte's condition). $\square$

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Lecture 25: November 10

Lecturer: Penny Haxell *Noted By: Haochen Wu*

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This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Proof of **Tutte's Theorem**:

Proof 25.127. $\Rightarrow$: We've already seen that if G has a perfect matching, then for every $T \subseteq V(G)$ we have $odd(G - T) \leq |T|$.

$\Leftarrow$: Let G be a graph such that $odd(G - T) \leq |T|$ for all $T \subseteq V(G)$. Suppose on the contrary that G does not have a perfect matching.

Let H be a graph such that

- $V(H) = V(G)$
- $E(G) \subseteq E(H)$
- H has no perfect matching
- $H \cup \{ab\}$ has a perfect matching for every a, b with $ab \notin E(H)$

We can construct it from G by adding edges one by one if necessary until the last condition is satisfied.

Then, H satisfies $odd(H - T) \leq |T|$ for every $T \subseteq V(G)$ by **lemma 24.122**.

Aim: to get a contradiction by showing that it is type-0, this would contradict **lemma 24.125**.

Let $X = \{x \in V(H) : xz \in E(H) \text{ for all } z \in V(H) \setminus \{x\}\}$. Note that $X = \emptyset$ is possible. If $X = V(H)$ we are done, so we may assume $X \neq V(H)$.

We want to show that all components of $H - X$ are complete graphs.

Let C be a component of $H - X$. If C has only 1 or 2 vertices, then it is complete.

We claim that there exist vertices $a, b, c \in C$, and $d \in V(H) \setminus \{a, b, c\}$ such that $ab, bc \in E(H)$, and $ac, bd \notin E(H)$. The proof is as follows: since C is not complete, there exists distinct vertices $a, x \in V(C)$ where $ax \notin E(H)$. Take a shortest path P in C from a to x and let b, c be the second and the third vertices on P. Then, $ab, bc \in E(H)$ and $ac \notin E(H)$, since P is shortest.

There exists $d \notin \{a, b, c\}$ where $bd \notin E(H)$, since $b \notin X$. This proves the claim.

By the special (maximality) property of H, the graph $H \cup \{ac\}$ has a perfect matching M_1 and $H \cup \{bd\}$ has a perfect matching M_2. The subgraph of H with vertex set $V(H)$ and edge set $M_1 \cup M_2$ has the property that every component is

- a single edge in $M_1 \cap M_2$ or

- an alternating cycle

since there are no M_1-exposed vertices in $H \cup \{ac\}$ or M_2 exposed vertices in $H \cup \{bd\}$.

The edge ac lies in an alternating cycle component, say U (since $ac \notin M_2$). If $bd \notin E(U)$, then $M_1 \Delta E(U)$ is a perfect matching of H. Hence, $bd \in E(U)$.

Since a and c are symmetric, we may assume, without loss of generality, there is a path P in U having d and a as endpoints, that does not contain b or c. But then $U' = P \cup \{db\} \cup \{ab\}$ is an M_2-alternating cycle. So, $M_2 \Delta E(U')$ is a perfect matching of H.

This contradiction shows H is type-0, completing the proof. □

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Lecture 26: November 10

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Hall's Theorem implies Tutte's Theorem when G is bipartite.

Proof 26.128. Suppose G is bipartite and $odd(G-T) \leq |T|$ for every $T \subseteq V(G)$.

Let X and Y denote the vertex classes of G. We will show that G has a perfect matching using **Hall's Theorem**.

- Take $T = X$. Note $G - X$ consists of $|Y|$ isolated vertices in Y. Since each isolated vertex is an odd component, $|Y| = odd(G-X) \leq |X|$ by Tutte's condition. Similarly, $|X| \geq |Y|$. Thus, $|X| = |Y|$.
- Let $S \subseteq X$, then each $x \in S$ is an isolated vertx in $G - N(S)$. Hence, $|S| \leq odd(G - N(S)) \leq |N(S)|$ by Tutte's Condition

So G satisfies the Hall's Condition. Thus, by **Hall's Theorem**, G has a perfect matching. □

Parity of $|T|$ and $odd(G-T)$

Lemma 26.129. Let G be a graph with an even number of vertices. Then for every $T \subseteq V(G)$, we have $odd(G-T) \equiv |T| \pmod 2$

Proof 26.130. Since $|V(G)| \equiv 0 \pmod 2$, we have $|T| + |V(G-T)| = |V(G)| \equiv 0 \pmod 2$. Then, $|T| \equiv -|V(G-T)| \equiv |V(G-T)| \pmod 2$.

But,

$$\begin{aligned} |V(G-T)| &= \sum_{C \text{ as odd component of } G-T} |V(C)| + \sum_{C \text{ as even component of } G-T} |V(C)| \\ &\equiv \sum_{C \text{ as odd component of } G-T} 1 + 0 \pmod 2 \\ &\equiv odd(G-T) \pmod 2 \end{aligned}$$

Thus, $|T| \equiv |V(G-T)| \equiv odd(G-T) \pmod 2$. □

Definition 26.131. Recall from MATH239 that an edge $e \in E(G)$ of a **connected** graph G is called a **bridge** if $G \setminus e$ is disconnected

Theorem 26.132. Petersen's Theorem: Let G be a connected cubic graph with at most 2 bridges. Then G has a perfect matching.

Proof 26.133. First observe that $|V(G)|$ is even since G is cubic, or by handshake lemma.

Suppose on the contrary that G has no perfect matching. By **Tutte's Theorem**, there exists $T \subseteq V(G)$ with $odd(G-T) > |T|$. By our previous lemma, in fact $odd(G-T) \geq |T|+2$.

We claim that for each odd component C of $G-T$, the number of edges m_c from C to T is odd.

The proof is as follows: $\sum_{v \in V(C)} d(v) = 2|E(C)| - m_c$. So $3|V(C)| = \sum_{v \in V(C)} d(v) \equiv m_c \pmod 2$. Hence, m_c is odd since $|V(C)|$ is odd. This proves the claim.

If $m_c = 1$, then the edge joining C to T is a bridge.

Hence, for $\geq odd(G-T)-2$ odd components, we have $m_c \geq 3$. Thus, the total number of edges from $V(G) \setminus T$ to T is

$$\begin{aligned}
\geq \underbrace{3(odd(G-T)-2)}_{m_c \geq 2} + \underbrace{1 \cdot (2)}_{m_c \geq 1} &= 3odd(G-T) - 4 \\
&\geq 3(|T|+2) - 4 \text{ since } odd(G-T) \geq |T|+2 \\
&= 3|T| + 2
\end{aligned}$$

But each vertex has degree 3, so this is impossible. Hence, T cannot exist. So, G satisfies Tutte's condition. Hence, G has a perfect matching. $\square$

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Lecture 27: November 10

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Theorem 27.134. Defect Version of Tutte's Theorem: Let G be a graph, and let $d = \max_{T\subseteq V(G)}\{odd(G-T)-|T|\}$. Then, G has a matching that saturates at least $V(G)-d$ vertices.

Proof 27.135. Note that $d \geq 0$ since $d \geq odd(G-\emptyset)-|\emptyset| \geq 0$

If $d=0$, then we are done by Tutte's Theorem. Let's assume $d \geq 1$.

For every $T \subseteq V(G)$ we have $odd(G-T) \leq |T|+d$. Construct a new graph H by adding a set A of d new vertices and all edges $\{az : z \in V(G) \cup A \setminus \{a\}\}$. We should see that $|V(H)|$ is even since there is some T such that $odd(G-T)-|T|=d$. Then we can see that $|V(H)| = |V(G-T)|+|T|+d \equiv odd(G-T)-|T|+d \equiv 2d \equiv 0 \pmod 2$.

- If $\emptyset \neq S \subseteq V(H)$ and $A \neq S$, then $H-S$ has at most 1 component. So, $odd_H(H-S) \leq 1 \leq |S|$.
- If $A \subseteq S$, $odd_H(H-S) = odd(G-(S\setminus A)) \leq |S-A|+d = |S|-d+d = |S|$ by condition on T.
- If $S=\emptyset$, then $odd_H(H-\emptyset)=0$ since H is connected and $|V(H)|$ is even.

So, by **Tutte's Theorem**, H has a perfect matching M. At most d edges of M failed to be edges of G. The rest of M is a matching in G, which saturates $\geq |V(G)|-d$. □

Theorem 27.136. Tutte-Berge Formula: Let G be a graph and let $d = \max_{T\subseteq V(G)}\{odd(G-T)-|T|\}$. THen, $\nu(G) = \frac{|V(G)|-d}{2}$.

Recall from previous proof, $|V(G)|-d \equiv |V(G)|+d \equiv 0 \pmod 2$. So the number in the formula is an integer.

Proof 27.137. By the **defect version of Tutte's Theorem** we know that G has a matching M that saturates at least $|V(G)|-d$ vertices of G. So, $\nu(G) \geq |M| \geq \frac{|V(G)|-d}{2}$

Now, consider a maximum matching M^* in G. Let $T \subseteq V(G)$ be a subset for which $d = odd(G-T)-|T|$.

At most $|T|$ odd components of $G - T$ can have an M^* edge joining it to T. Each of the remaining d odd components have an M^*-exposed vertex. So, M^* saturates at most $|V(G)| - d$ vertices.

Thus, $\nu(G) = |M^*| \leq \frac{|V(G)|-d}{2}$ $\square$

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Lecture 28: November 17

Lecturer: Penny Haxell — *Noted By: Haochen Wu*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They may be distributed outside this course only with the permission of the instructors.*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 28.138. Let G be a graph and let M be a matching in G. An odd cycle C in G of length $2k+1$ is said to be **shrinkable** with respect to M if

- $|M \cap E(C)| = k$
- C contains an M-exposed vertex.

Lemma 28.139. Let M be a matching in G and let C be an odd cycle in G that is shrinkable with respect to M. Let G; be the grpah obtained from G by contracting all edges in $E(C)$ (This is the operation of "shrinking" C in G).

Then, $M' = M \setminus E(C)$ is maximum in G' if and only if M is maximum in G.

Let c be the image of the contraction of C. Note that c is M'-exposed in G'.

Proof 28.140. $\Rightarrow$: Suppose M' is maximum in G'. Suppose on the contrary that M is not maximum in G. Then by **Berge's Theorem**, there is an M-augmenting path P in G. Then $P \cap |V(C)| = \emptyset$, otherwise P would be an M'-augmenting path in G'.

At least one endpoint x of P is not in $V(C)$, since C is shrinkable. Let $z \in V(C)$ be the vertex of C closest to x on P.

Then, the (x, z)-segment of P becomes an M'-augmenting path from x to c (the image of C under contraction) in G'. Recall that c is M'-exposed. This contradicts to **Berge's Theorem** in G'.

Hence, M is maximum in G.

$\Leftarrow$: Suppose M is maximum in G. Suppose on the contrary that some matching N' in G' satisfies $|N'| > |M'|$. Then, N' corresponds to a matching N in G that saturates at most one vertex of C.

Then, we can add k more edges in C to N, where C has length $2k+1$ to get a matching in G that is larger than M. This contradicts the maximality of M. ☐

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Lecture 29: November 17

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Why is the condition that the shrinkable cycle contains M-exposed vertex necessary? If M' is maximum in G' and M is not maximum in G, then the problem is that G has no M-exposed vertex.

Note that the proof of the Cycle Lemma does NOT say that if N' is a maximum matching in G', then adding k edges of C to N gives a maximum matching in G. It says only that this is a **larger** matching in G.

Definition 29.141. Let M be a matching in a graph G. An **M-alternating tree** in G is a subgraph T of G with the following properties

- T is a tree
- T contains exactly one M-exposed vertex, called the root s
- Each edge of T at an odd distance in T from s is in M (the distance from the closest vertex of the edge to s)
- each vertex of T at an odd distance in T from s has degree 2 in T. These are called the **inner** vertices of T. The remaining vertices are the **outer** vertices. Note that the root s is outer

Definition 29.142. Let M be a matching in G. An **M-alternating forest** in G is a subgraph F of G such that every component of F is an M-alternating tree.

The set of **inner** vertices of F is

$$I(F) = \bigcup_{T \text{ as a component of } F} I(T)$$

and the set of **outer** vertices of F

$$O(F) = \bigcup_{T \text{ as a component of } F} O(T)$$

where $I(T)$ and $O(T)$ denote the sets of inner and outer vertices of T respectively.

Definition 29.143. We say that F is **maximal M-alternating forest** if there is no larger M-alternating forest that contains F.

In general, there are no edges in G joining two outer vertices of F where F is a maximal M-alternating forest.

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Lecture 30: November 17

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Lemma 30.144. Let M be a matching in a graph G. Let F be a maximal M-alternating forest in G. Suppose there is no edge of G joining two outer vertices of F. Then, M is a maximum matching in G.

Proof 30.145. Note that every M-exposed vertex of G is in F. Otherwise, we can add it to F as a component of size 1, and this contradicts to the maximality of F.

Also, if $xy \in M$ and x is in a component T of F, then y is also in T.

We claim that if $z \in O(F)$, then $N_G(z) \subseteq I(F)$. Here is the proof. Suppose $yz \in E(G)$. Then, $y \notin O(F)$ by the assumption of the lemma.

If $y \notin V(F)$, then y is saturated by M by the first observation. Let's say $xy \in M$. Then, by the second observation, $x \notin V(F)$. But then, $F \cup \{yz, yx\}$ is an M-alternating forest that contains F, contradicting the maximality of F. Hence, $y \in I(F)$.

For each component T of F, we have $|O(T)| = |I(T)| + 1$ since each outer vertex of T except the roots is matched by M to an inner vertex of T.

So, $|O(F)| = |I(F)| + k$, where k is the number of component of F, and this is the numer of M-exposed vertices in G.

Then, in $G - I(F)$, every vertex in $O(F)$ is an isolated vertex, i.e. a component of size 1, by the claim we proved above. Hence, $odd(G - I(F)) \geq |O(F)| = |I(F)| + k$.

But, $\nu(G) = \frac{|V(G)|-d}{2}$ where $d = \max_{T \subseteq V(G)}\{odd(G-T) - |T|\}$ by **Tutte-Berge's Formula**. Hence, $d \geq odd(G - I(F)) - |I(F)| \geq k$. So, $\nu(G) \leq \frac{|V(G)|-k}{2}$.

But $|M| = \frac{|V(G)|-k}{2}$, so $|M| \geq \nu(G)$, and hence M is maximum. $\square$

Lemma 30.146. Let M be a matching in G, and let T_1 and T_2 be distinct components of an M-alternating forest F in G. If there exists $xy \in E(G)$ where $x \in O(T_1)$ and $y \in O(T_2)$, then G contains an M-augmenting path.

Proof 30.147. Let s_1 and s_2 denote the roots of T_1 and T_2 respectively. Then s_1 and s_2 are M-exposed. Then, the unique path joining s_1 and s_2 in $T_1 \cup T_2 \cup \{xy\}$ is an M-augmenting path. $\square$

Lemma 30.148. Let M be a matching in a graph G. Let T be a component of an M-alternating forest F in G, and suppose there exists $xy \in E(G)$ where $x, y \in O(T)$. Then, there exists a matching $\bar{M}$ with $|\bar{M}| = |M|$ and an odd cycle C in G that is shrinkable with respect to $\bar{M}$.

Proof 30.149. Let C be the unique cycle in $T \cup \{xy\}$, and ley P be the unique path from C to the root s in T. Set $\bar{M} = M \Delta E(P)$.

Then, C is shrinkable with respect to $\bar{M}$. $\square$

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Lecture 31: November 24

Lecturer: Penny Haxell *Noted By: Haochen Wu*

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This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

> Edmond's Algorithm (G): Finds a maximum matching
>
> start with a matching M in G (M could be empty, or more ideally, be a greedy matching)
> **construct** an M-alternating forest F
> **if** there exists $xy \in E(G)$ joining two outer verticeds in the same component of F
>   **shrink** the odd cycle C
>   **replace** G by the resulting graph and M by $\bar{M} \setminus E(C)$
>   **go back** to the construction of M-alternating forest step
> **if** there exists $xy \in E(G)$ joining two outer verticeds in different component of F
>   There exists a M-augmenting path P
>   **replace** M by $M \Delta E(P)$
>   **expand** shrunk cycles back to the original input graph with the new larger matching
>   We need to relabel the shrunk cycle with the new matching.
>   **go back** to the construction of M-alternating forest step
>
> Continue **constructing** F until no $xy \in E(G)$ joins two outer vertices.
> **Then** we STOP, **output** current initial matching in original graph.

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Lecture 32: November 24

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 32.150. Let G be a grpah and let $f : V(G) \to \{0, 1, 2, ...\}$ be a function. An **f-factor** of G is a spanning subgraph H of G such that $d_H(v) = f(v)$ for every $v \in V(G)$. By spanning, we mean that $V(H) = V(G)$

For example, a perfect matching in G is 1-factor, where 1 denotes the function whose value is 1 for every $v \in V(G)$

Note that if $f(v) > d_G(v)$ for some $v \in V(G)$, then G does not have an f-factor.

Definition 32.151. Let G be a graph and let $f : V(G) \to \{0, 1, 2, ...\}$, where $f(v) \leq d_G(v)$ for each $v \in V(G)$. The graph $H(G, f)$ has vertex set

$$\bigcup_{v \in V(G)} (A(v) \cup B(v))$$

where

- $A(v) \cap B(w) = \emptyset$ for all $v, w \in V(G)$
- $|A(v)| = d_G(v)$
- $|B(v)| = d_G(v) - f(v)$

The edge set of $H(G, f)$ is obtained by

- adding one edge e_{vw} from $A(v)$ to $A(w)$ for each edge $vw \in E(G)$, such that $\{e_{vw} : vw \in E(G)\}$ is a matching in $H(G, f)$
- adding add edges xy with $x \in A(v)$ and $y \in B(v)$ for each $v \in V(G)$.

So, the problem of if G has a f factor is equivalent to if $H(G, f)$ has a perfect matching

Theorem 32.152. Let G be a graph and $f : V(G) \to \{0, 1, 2, ...\}$ be such that $f(v) \leq d_G(v)$ for each $v \in V(G)$. Then, G has a f-factor if and only if $H(G, f)$ has a perfect matching.

Proof 32.153. $\Rightarrow$: Assume G has a f-factor J. Then, $\{e_{vw} : vw \in E(J)\}$ is a matching $M_j \in H(G, f)$ that saturates exactly $f(v)$ vertices of $A(v)$ for each v. The remaining $d_G(v) - f(v)$ vertices in $A(v)$ can be matched to $B(v)$ to form a perfect matching of $H(G, f)$.

$\Leftarrow$: Assume $H(G, f)$ has a perfect matching M. Since $N(B(v)) = A(v)$, M matches $B(v)$ to exactly $d_G(v) - f(v)$ vertices of $A(v)$ for each v. So, exactly $f(v)$ edges of M are incident to the rest of $A(v)$. So, there are edges of the form e_{vw}, correponding to an f-factor in G. $\square$

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Lecture 33: November 24

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 33.154. A graph G is said to be **even** if every vertex of G has even degree

Definition 33.155. An **Euler tour** of a graph G is a sequence $v_0e_1v_1e_2v_2....e_kv_k$ of vertices and edges of G such that

- $e_i = v_{i-1}v_i$
- $v_k = v_0$
- $k = |E(G)|$
- $e_i \neq e_j$ for all $i \neq j$

Remark:

- If G has an Euler tour, then G is connected (except possibly it contains isolated vertices), and G is even.
- If G is a connected even graph, then it has an Euler tour.

Definition 33.156. A **k-factor** in a grpah G is an f-factor for the function f defined by $f(v) = k$ for each $v \in V(G)$.

Theorem 33.157. Petersen's 2-factor Theorem: Let $d \geq 2$ be even. Then every d-regular graph has a 2-factor.

Proof 33.158. We may assume G is connected, otherwise consider components separately. Then, G has an Euler tour Q.

Define a bipartite graph $B(G)$ with vertex classes V_0 and V_1 each of which is a copy of $V(G)$. Let v_0w_1 be an edge of $B(G)$ if and only if $vw \in E(G)$ and Q traverses vw in the direction $v \to w$.

$B(G)$ is a $\frac{d}{2}$-regular bipartite graph. So, $B(G)$ has a perfect matching. M gives a 2-factor in G since each vertex $v \in V(G)$ is incident to 2 edges of M. They are all distinct since we have an Euler tour that goes through each edge exactly once. $\square$

CO342: Introduction to Graph Theory **Fall 2020**

Lecture 34: December 1

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Recall that. the principle properties of a vector space W over a field $\mathbb{F}$:

- W contains a zero vectors
- any $v, w \in W$ can be added to get a vector $v + w \in W$
- any $v \in W$ can be multiplied by any $c \in \mathbb{F}$ to get $cv \in W$.

If $U \subseteq W$ and U is also a vector space over $\mathbb{F}$, we say U is a subspace of W.

Definition 34.159. Let G be a graph and let S be a spanning subgraph of G. The **characteristic vector** w_S of S in the $\{0,1\}$-vector with $|E(G)|$ coordinates, indexed by $E(G)$, whose e_i-th coordinate is $\begin{cases} 1 & \text{if } e_i \in E(S) \\ 0 & \text{if } e_i \notin E(S) \end{cases}$

Definition 34.160. Let G be a graph. The **flow space** of G is the set of all characteristic vectors of even spanning subgraphs of G.

Theorem 34.161. The flow space $W(G)$ of G is a vector space over $\mathbb{Z}_2$.

Proof 34.162. We first check principle properties for vector space over $\mathbb{Z}_2$:

- The zero vector $(0, ..., 0) \in W$. This is w_S where $E(S) = \emptyset$
- If $w_S \in W$ and w_T, then $w_S + w_T \in W$, where addition is done in $\mathbb{Z}_2$. $w_S + w_T$ is the characteristic vector of the spanning subgraph $S \oplus T$ of G, with edge set $E(S) \cup E(T) \setminus (E(S) \cap E(T))$. Note $S \oplus T$ is even.

  This is because, for any $v \in V(G)$, $d_{S \oplus T}(v) = d_S(v) + d_T(v) - 2d_{S \cap T}(v)$ is even.
- Scalar multiplication: $0v = (0, ..., 0)$, $1 \cdot v = v$ for all $v \in W$.

$\square$

Theorem 34.163. Let G be a graph and let H be an even spanning subgraph of G. Then there exists cycles $C_1, ..., C_k$ in G such that $E(H) = \bigcup_{i=1}^{k} E(C_i)$ and $E(C_i) \cap E(C_j) = \emptyset$ for all $i \neq j$

Proof 34.164. We do induction on $|E(H)|$

Base Case: If $|E(H)| = 0$, then we take $k = 0$

Inductive Hypothesis. Assume $|E(H)| > 0$, and every even spanning subgraph J of G with $|E(J)| < |E(H)|$ is the union of edge-disjoint cycles in G.

Inductive Step: For some component H_1 of H, $d_H(v) \geq 2$ for all $v \in V(H_1)$ since each degree in H is positive and even. Recall from MATH239 that H_1 contains a cycle C_1.

Then, $J = H \setminus E(C_1)$ is an even spanning subgraph of $G \setminus E(C_1)$ since $d_J(v) = d_H(v)$ if $v \notin V(C_1)$, and $d_J(v) = d_H(v) - 2$ otherwise.

So, by Inductive Hypothesis, $E(J) = \bigcup_{i=2}^{k} E(C_i)$ for some edge-disjoint cycles $C_2, ..., C_k$ in $G \setminus E(C_1)$.

Hence, $E(H) = \bigcup_{i=1}^{k} E(C_i)$ as required. $\square$

This tells us that the flow space $W(G)$ is spanned by the characteristic vectors of cycles, i.e.

$$w_H = w_{C_1} + \cdots + w_{C_k}$$

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Lecture 35: December 1

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 35.165. Let G be a connected graph and let T be a spanning tree of G. Let $e \in E(G) \setminus E(T)$. Then, the unique cycle C_e in $T \cup \{e\}$ is called the **fundamental cycle** for e with respect to T.

Theorem 35.166. Let G be a connected graph and let T be a spanning tree of G. Then, $B_T = \{w_C : C \in F(T)\}$ is a basis for the flow space $W(G)$ of G, where $F(T)$ denotes the set of all fundamental cycles in G with respect to T.

Proof 35.167. We first prove that B_T is linearly independent in $W(G)$.

Let $F(T) = \{C_e : e \in E(G) \setminus E(T)\}$, so C_e is the cycle in $T \cup \{e\}$.

Suppose $w_\lambda = \sum_{e \in E(G)\setminus E(T)} \lambda_e w_{C_e} = 0$, where $\lambda_e \in \mathbb{Z}_2$ for each e.

Suppose $\lambda_{e_0} = 1$, then e_0 is an edge of the graph $\oplus_{e \in E(G)\setminus E(T)} \lambda_e C_e$ since $e_0 \in E(C_{e_0})$, but $e_0 \notin E(C_e)$ for $e \neq e_0$. So the e_0-coordinate of w_λ is 1. Contradiction.

Hence, $\lambda_e = 0$ for each e. So, B_T is linearly independent.

We then prove that B_T spans $W(G)$. Consider an arbitrary element $w_H \in W(G)$, where H is an even spanning subgraph of G.

We claim that $w_H = \sum_{e \in E(G)\setminus E(T)} w_{C_e}$. Note that RHS is the characteristic vector of the graph $\oplus_{e \in E(H)\setminus E(T)} C_e = J_H$.

The proof is as follows: Consider $J_H \oplus H$. Each edge $e \in E(G) \setminus E(T)$ is in exactly one fundamental cycle C_e. Since $e \in E(G)$ as well, the edge e is not in $E(J_H \oplus H)$ since $E \in E(J_H) \cap E(H)$.

Thus, $J_H \oplus H$ is a subgraph of T. If $J_H \oplus H$ has any edges, it's a forest, with ≥ 1 edge, so it is a vertex of degree 1. This is not possible since it is even.

Hence, $J_H \oplus H$ has no edges, i.e. $J_H = H$.

So, B_T is a basis of $W(G)$. $\square$

Corollary 35.168. Let G be a connected graph with $|V(G)| = n$ and $|E(G)| = m$. Then, $dim(W(G)) = m - n + 1$ and $|W(G)| = 2^{m-n+1}$

Proof 35.169. Any spanning tree T has $n - 1$ edges, so $|B_T| = m - n + 1$. $\square$

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Lecture 36: December 1

Lecturer: Penny Haxell *Noted By: Haochen Wu*

This lecture's notes tend to be supplementary (add-on notes) of the course notes provided.

Definition 36.170. A **binary code** of length m is a subspace U of $\mathbb{Z}_2^m$

Definition 36.171. The **minimum distance** of a binary code U is the smallest t such that some $u \in U$ with $u \neq 0$ has exactly t coordinates that are 1.

Definition 36.172. The **hamming distance** between u and v in a binary code U is the number of coordinates that are 1 in $u + v$.

Lemma 36.173. Let U be a binary code with minimum distance t. Then any two distinct elements $u, v \in U$ are at Hamming distance at least t.

Proof 36.174. Let $u, v \in U$, $u \neq v$. Then $u + v \in U$ since U is a vector space. So $u + v$ has at least t 1-coordinates by definition of t. □

A code with high minimum distance t provides good "error correction": if no more than $\lfloor \frac{t-1}{2} \rfloor$ error bits occur during the transimission, then the original element of the code can be recovered.

Definition 36.175. The **girth** of a graph G is the length of a shortest cycle in G. We say the girth is infinite if G has no cycles.

Lemma 36.176. Let G be a connected graph of finite girth g. Then, the flow space $W(G)$ is a binary code of length $m = |E(G)|$, dimension $|E(G)| - |V(G)| + 1$, and minimum distance g.

Proof 36.177. We know that B_T is a basis for $W(G)$, where T is a spanning tree of G. So, $dim(W(G))$ is $|E(G)| - |V(G)| + 1$. Length is m be definition.

Let $w_H \in W(G)$ be nonzero. Then, H contains a cycle C since H is even, and $|E(H)| > 0$. Therefore $|E(H)| \geq |E(C)| \geq g$ by definition of girth. So, $W(G)$ has minimum distance $\geq g$ ($= g$ since there exists a cycle C_0 of length g such that $w_{C_0} \in W(G)$). $\square$

If we have a d-regular graph G, with girth g odd, then, the number of vertices $\geq 1+d+d(d-1)+d(d-1)^{\frac{g-1}{2}} > d^{\frac{g+1}{2}}$.