AMATH / PMATH 332 Course Notes

Applied Complex Analysis

Haochen Wu

University of Waterloo

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Contents

Chapter 1 Complex Numbers

1.1 Intro, Properties of Complex Numbers

Intro:

- What it's about: not like real analysis; some of intro to calculus on $\mathbb{C}$
- Goal: extend calculus on $\mathbb{R}$ to $\mathbb{C}$ - many results become simpler! (more complete picture here)
- Can be used to solve some $\mathbb{R}$ problems.

The Fundamentals:

- Basic idea: define solutions to $x^2 + 1 = 0$
- Early Mathematicians: $x = \pm\sqrt{-1}$. For $\sqrt{-1}$, should we call it i?
- Note: "$\sqrt{}$" always denotes positive root, e.g. $\sqrt{4} = 2$
- Problem:

$$\sqrt{-1}\sqrt{-1} = -1 \quad \text{by definition of } \sqrt{}$$
$$\sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1 \quad \text{since } \sqrt{ab} = \sqrt{a}\sqrt{b}$$

- Fix: intepret "$\sqrt{}$" differently for complex numbers - it must be multivalued, and define the imaginary unit i by $i^2 = 1$

Definition 1.1. Complex number:

$$z = \underbrace{a}_{\text{"real part" } Re(z)} + i \underbrace{b}_{\text{"imaginary part" } Im(z) \text{ which is real!}} \qquad \text{where } a, b \in \mathbb{R}$$

$\mathbb{C}$ = set of complex numbers. Note that $\mathbb{R} \subset \mathbb{C}$

Definition 1.2. Let $z = a + bi$, and $w = c + di$. Then:

- $z = w$ if and only if $a = c$ and $b = d$
- $z + w = (a + bi) + (c + di) = a + c + (b + d)i$
- $z - w = z + (-w) = (a + bi) + (-c - di) = a - c + (b - d)i$
- $zw = (a + bi)(c + di) = ac + bdi^2 + adi + bci = ac - bd + (ad + bc)i$
- $\dfrac{z}{w} = \dfrac{a + bi}{c + di} \cdot \dfrac{c - di}{c - di} = \dfrac{ac + bd}{c^2 + d^2} + i \cdot \dfrac{bc - ad}{c^2 + d^2}$

Example 1.3.

$$\frac{2+i}{1+2i} = \frac{2+i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{4}{5} - \frac{3}{5}i$$
$$\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = \frac{-i}{-i^2} = -i$$

Theorem 1.4. $z + w = w + z$, $k(z+w) = kz + kw$ apply as usual. $zw = wz$

Note: We can't classify complex numbers as "positive" or "negative", and can't use inequalities, e.g. $z > w$ doesn't make sense.

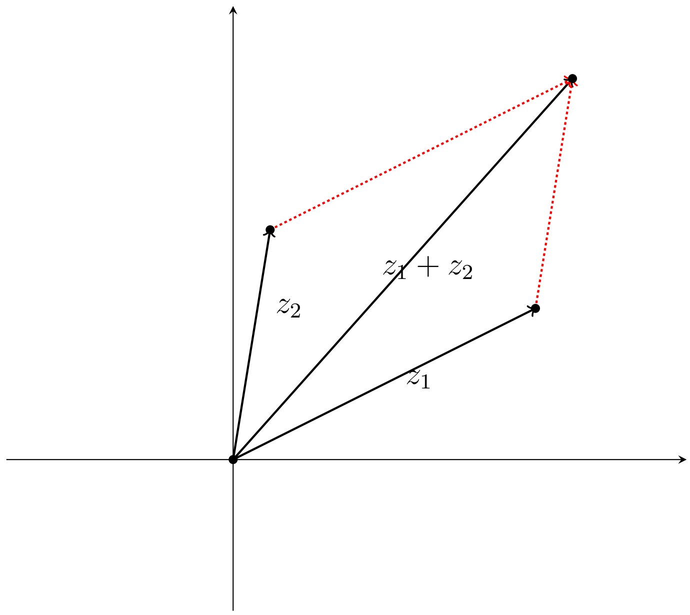
Definition 1.5. Conjugate of $z = a + bi$ is

$$\overline{z} = a - bi$$

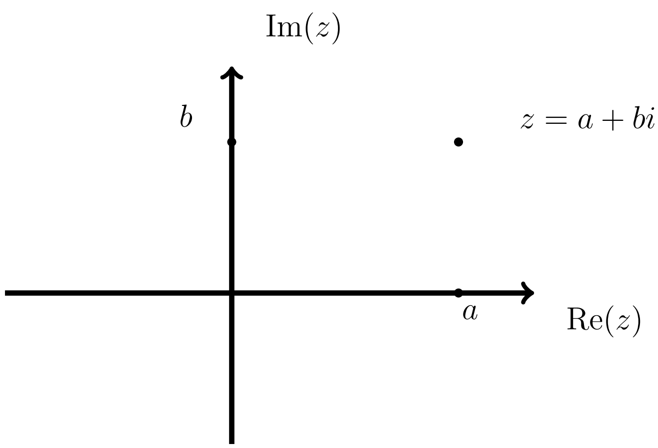
(Sometimes written as z^* as well)

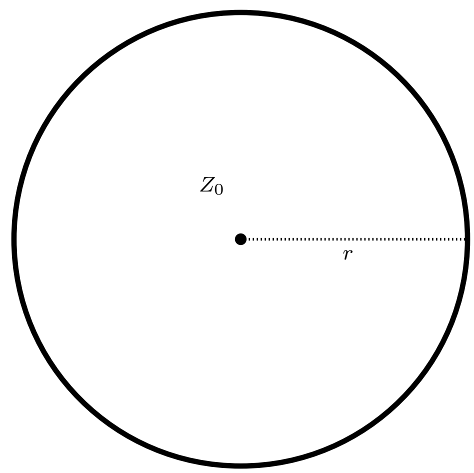
Proposition 1.6. The following rules apply:

1. $\overline{\overline{z}} = z$
2. $\overline{z \pm w} = \overline{z} \pm \overline{w}$
3. $\overline{zw} = \overline{z}\,\overline{w}$ and $\overline{\left(\frac{z}{w}\right)} = \frac{(\overline{z})}{(\overline{w})}$
4. $z + \overline{z} = 2Re(z) \quad \Rightarrow \quad Re(z) = \frac{1}{2}(z + \overline{z})$
5. $z - \overline{z} = 2iIm(z) \quad \Rightarrow \quad Im(z) = \frac{1}{2i}(z - \overline{z})$
6. $z\overline{z} = a^2 + b^2$ which is real!



1.2 The Complex Plane, Polar form





Definition 1.7. The **modulus** of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$

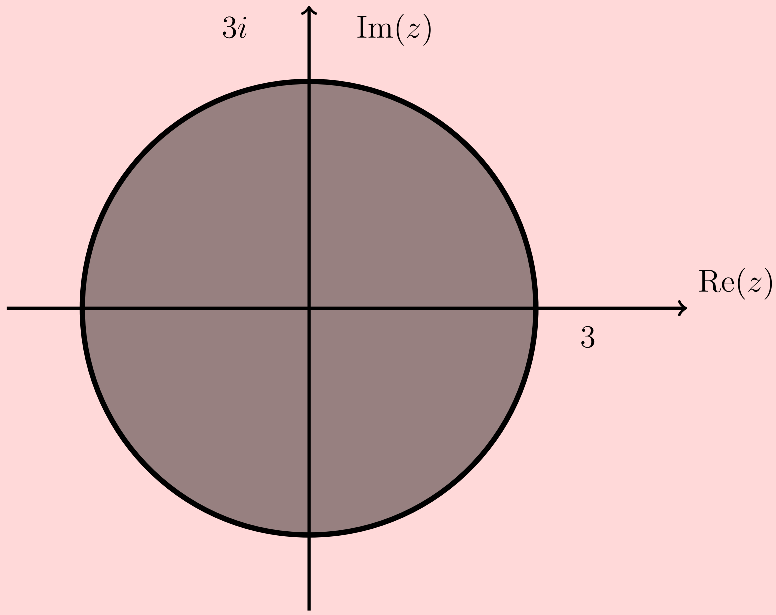
The **distance** between two numbers z and w is $|z - w|$

Notes:

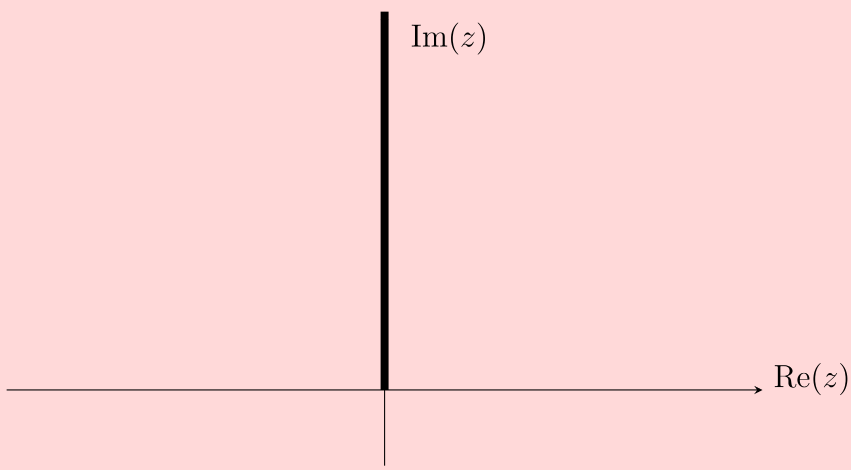
- $|z| \geq 0$ and is real
- $z\overline{z} = a^2 + b^2 = |z|^2$
- $|z - z_0| = r$ describes a circule of radius r centered at z_0

Example 1.8. Sketch the sets:

1. $|z| < 3$



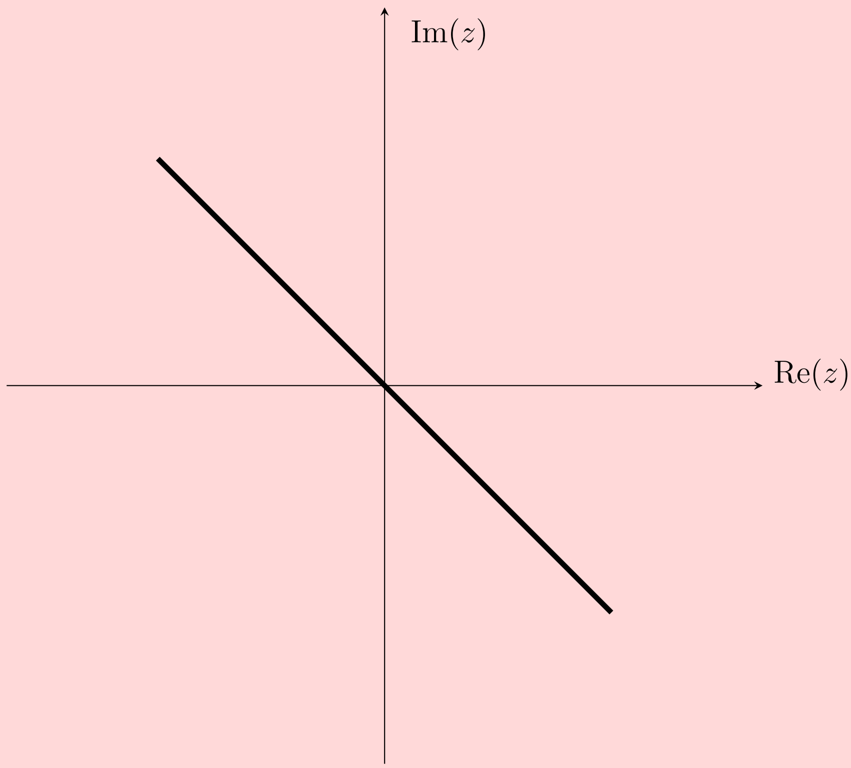
2. $|z| = Im(z)$. Let $z = a + ib$. So, $\sqrt{a^2 + b^2} = b$, which gives $a^2 + b^2 = b^2$, so $a = 0, b \geq 0$



3. $|z - 1| = |z + i|$. So

$$\begin{aligned}
\sqrt{(a-1)^2 + b^2} &= \sqrt{a^2 + (b+1)^2} \\
(a-1)^2 + b^2 &= a^2 + (b+1)^2 \\
a^2 - 2a + 1 + b^2 &= a^2 + b^2 + 2b + 1 \\
b &= -a
\end{aligned}$$

This is the set of points that are equidistant from $z = 1$ and $z = -i$



We will often use $z = x + yi$, so we are in the xy-plane, still not called $\mathbb{R}^2$ though.

Useful inequalities:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

This is known as "Triangle Inequality". This also extends to

$$|z_1 = z_2 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$$

Corollary 1.9.

$$|z_1 + z_2| \geq \Big||z_1| - |z_2|\Big|$$

Proof 1.10.

$$\begin{aligned} |z_1| &= |z_1 + (z_2 - z_2)| \\ &= |(z_1 + z_2) + (-z_2)| \\ &\leq |z_1 + z_2| + |z_2| \end{aligned}$$

$$\begin{aligned} |z_2| &= |z_2 + (z_1 - z_1)| \\ &= |(z_1 + z_2) + (-z_1)| \\ &\leq |z_1 + z_2| + |z_1| \end{aligned}$$

So $|z_1 + z_2| \geq |z_1| - |z_2|$ and $|z_2| - |z_1|$. So

$$|z_1 + z_2| \geq \Big| |z_1| - |z_2| \Big|$$

$\square$

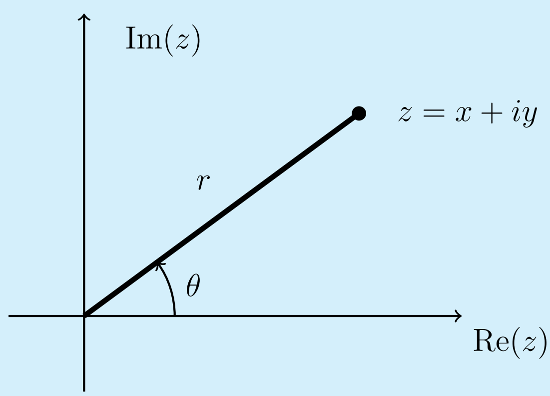
Definition 1.11. Polar Form

$$\boxed{x = r\cos\theta,\ y = r\sin\theta}$$

So,

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= r \underbrace{\text{cis}}_{\text{common abreviation}} \theta \end{aligned}$$

$$\boxed{r = \sqrt{x^2 + y^2},\ \tan\theta = \frac{y}{x}}$$



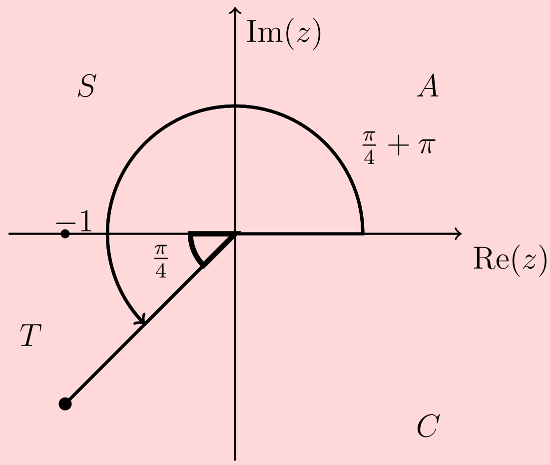
Notes:

- This is not unique. e.g. $z = 2 = 2\,\text{cis}\,0 = 2\,\text{cis}\,2\pi = \cdots$, also $z = 0 = 0\,\text{cis}\,\theta$ for any θ
- $\theta = \tan^{-1}(\frac{y}{x})[\pm 2k\pi]$ if $x > 0$, but must add π if $x < 0$ - Recall principal values

Example 1.12. Say we want to express $z = -1 - i$ in polar form.

We compute $r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$. $\tan\theta = \frac{-1}{-1} = 1$. Note that $\theta \neq \tan^{-1}(1) = \frac{\pi}{4}$, instead, $\theta = \frac{5\pi}{4}$.

So, $z = \sqrt{2}\operatorname{cis}\frac{5\pi}{4}$ or $\sqrt{2}\operatorname{cis}(\frac{5\pi}{4} + 2k\pi)$



Note:

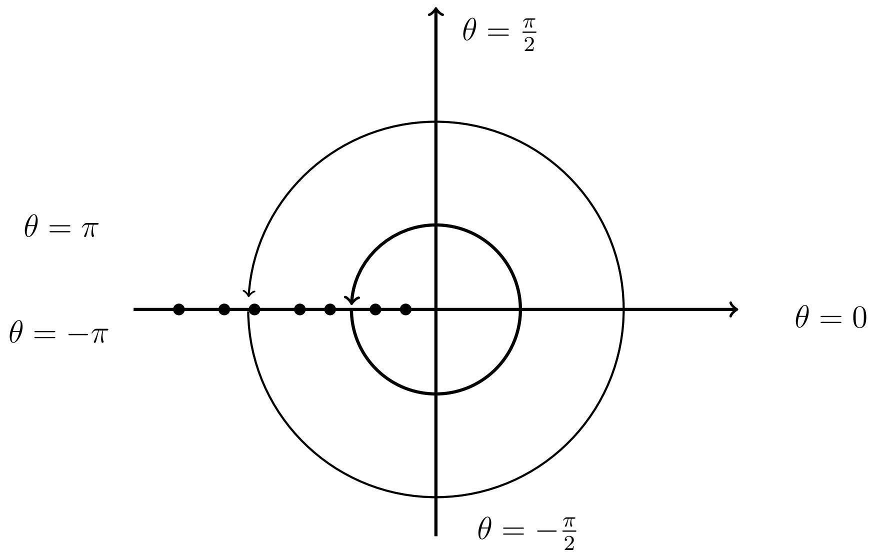
$$z = \underbrace{r}_{=\sqrt{x^2+y^2}, r=|z|, \text{“modulus”}} \operatorname{cis} \underbrace{\theta}_{\text{“argument” of } z}$$

Also, “$\arg z$” = set of all possible values of θ. “$\operatorname{Arg} z$” = principle values of θ, usually in $(\pi, \pi]$

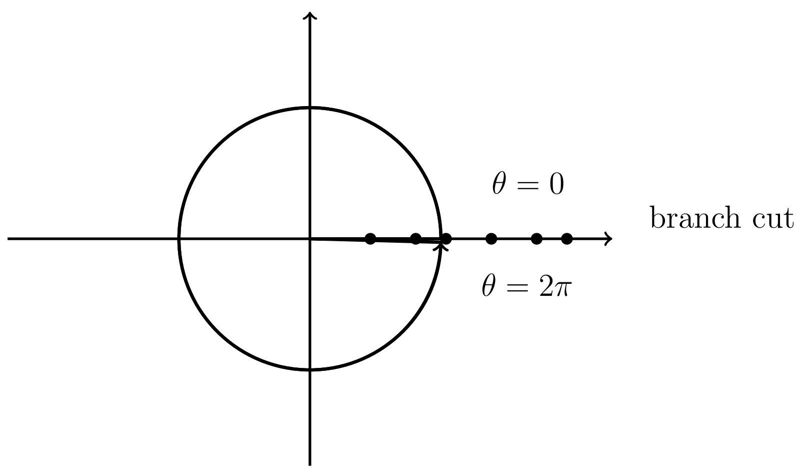
Example 1.13. For $z = -1 + \sqrt{3}i$. $\operatorname{Arg} z = \frac{2\pi}{3}$, $\arg z = \frac{2\pi}{3} + 2k\pi, k \in \mathbb{Z}$

Also, $|z| = 2$, so $-1 + \sqrt{3}i = 2\operatorname{cis}\frac{2\pi}{3}$

We sometimes think of $\arg z$ as a multivalued “function” of z. For a single-valued function, we could use $\operatorname{Arg} z$, but it has discontinuity on negative real axis.



Another way: we can define $\operatorname{Arg}(z)$ to have range $[0, 2\pi)$. In general, $\operatorname{Arg}_{\theta_0} z$ has range $[\theta_0, \theta_0 + 2\pi)$, and usually we use $\operatorname{Arg} z = \operatorname{Arg}_{-\pi} z$



1.3 Complex Exponential, Powers and Roots

Reading textbook Section 1.4, 1.5

Definition 1.14. If $z = x + iy$, then e^z is defined to be the complex number

$$e^z := e^x(\cos y + i \sin y)$$

Proposition 1.15. Euler's equation is formally consistent with the usual Taylor series expansions:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \end{aligned}$$

Proof 1.16. Let's substitute $x = iy$ into the exponential series:

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots \\ &= (1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots) + i(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots) \\ &= \cos y + i \sin y \end{aligned}$$

$\square$

As a result, we may introduce the standard polar representation

$$z = r \operatorname{cis} \theta = r(\cos\theta + i\sin\theta) = re^{i\theta} = |z|e^{i\arg z}$$

Notice that

$$e^{i0} = e^{2\pi i} = e^{-2\pi i} = e^{4\pi i} = e^{-4\pi i} = \cdots = 1$$

$$e^{(\pi/2)i} = i \quad e^{(-\pi/2)i} = -i \quad e^{\pi i} = -1$$

Also notice that

$$\begin{aligned} \cos\theta &= Re(e^{i\theta}) = \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta &= Im(e^{i\theta}) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \end{aligned}$$

Hence,

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$\overline{z} = re^{-i\theta}, \text{ given that } z = re^{i\theta}$$

Example 1.17. Compute the following:

1. $(1+i)/(\sqrt{3}-i)$.

   Notice that $1+i = \sqrt{2}\,\text{cis}(\pi/4) = \sqrt{2}e^{i\pi/4}$, and $\sqrt{3}-i = 2\,\text{cis}(-\pi/6) = 2e^{-i\pi/6}$. So,

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}e^{i\pi/4}}{2e^{-i\pi/6}} = \frac{\sqrt{2}}{2}e^{i5\pi/12}$$

2. $(1+i)^{24}$

   We have

$$(1+i)^{24} = (\sqrt{2}e^{i\pi/4})^{24} = (\sqrt{2})^{24}e^{i24\pi/4} = 2^{12}e^{i6\pi} = 2^{12}$$

Theorem 1.18.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad n = 1, 2, 3, ...$$

Definition 1.19. There are exactly m distinct m-th **roots of unity**, denoted by $1^{1/m}$, and they are given by

$$1^{1/m} = e^{i2k\pi/m} = \cos\frac{2k\pi}{m} + i\sin\frac{2k\pi}{m} \quad (k = 0, 1, 2, ..., m-1)$$

Take $k = 1$ into the above equation, we can get

$$\omega_m := e^{i2\pi/m} = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}$$

So the complete set of roots can be displayed as

$$\{1, \omega_m, \omega_m^2, \cdots, \omega_m^{m-1}\}$$

Note that a number w is said to be a **primitive** m-th root of unity if $w^m = 1$ but $w^k \neq 1$ for $k = 1, 2, ..., m-1$. Clearly, ω_m is a primitive root.

Theorem 1.20.

$$1 + \omega_m + \omega_m^2 + \cdots + \omega_m^{m-1} = 0$$

Proof 1.21. Note that

$$(\omega_m - 1)(1 + \omega_m + \omega_m^2 + \cdots + \omega_m^{m-1}) = (\omega_m - 1) = 0$$

Since $\omega_m \neq 1$, the result follows. $\square$

To obtain the m-th root of an arbitrary (non-zero) complex number $z = re^{i\theta}$, we can obtain the following generalized result.

Definition 1.22. The m-th distinct roots of z are given by

$$z^{1/m} = \sqrt[m]{|z|}e^{i(\theta+2k\pi)/m}$$

Example 1.23. Find all the cube roots of $\sqrt{2} + i\sqrt{2}$

The polar form for $\sqrt{2} + i\sqrt{2}$ is

$$\sqrt{2} + i\sqrt{2} = 2e^{i\pi/4}$$

Putting $|z| = 2, \theta = \pi/4, m = 3$ into the above definition, we obtain

$$(\sqrt{2} + i\sqrt{2})^{1/3} = \sqrt[3]{2}e^{i(\pi/12+2k\pi/3)}, \quad (k = 0, 1, 2)$$

Hence, the three cube roots of $\sqrt{2} + i\sqrt{2}$ are:

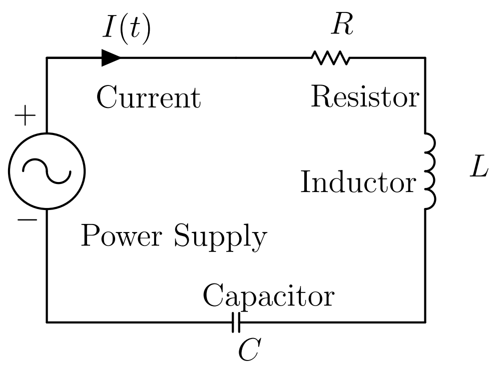
- $\sqrt[3]{2}(\cos \pi/12 + i \sin \pi/12)$
- $\sqrt[3]{2}(\cos 3\pi/4 + i \sin 3\pi/4)$
- $\sqrt[3]{2}(\cos 17\pi/12 + i \sin 17\pi/12)$

1.4 Application to Electrical Circuits

A typical electrical circuits is like the following:

Laws:

1. Resistor: $V = IR$
2. Inductor: $V = L\dfrac{dI}{dt}$
3. Capacitor: $C\dfrac{dV}{dt} = I$



Suppose the current is

$$I(t) = \underbrace{I_0}_{\text{amplitude}} \cos \underbrace{\omega}_{\text{frequency}} t = Re(\underbrace{I_0 e^{i\omega t}}_{\text{call it } \widetilde{I}(t)})$$

Then

1. Law 1 tells us $V = (I_0 \cos \omega t)(R) = Re(\widetilde{I}(t) \cdot R)$. So "complex voltage" is

$$\boxed{\tilde{V} = R\widetilde{I}}$$

2. Law 2 tells us

$$\begin{aligned} V &= L \cdot (-\omega I_0 \sin \omega t) \\ &= -\omega L I_0 \cdot \underbrace{Re(e^{i(\omega t - \frac{\pi}{2})})}_{=\cos(\omega t - \frac{\pi}{2}) = \sin \omega t} \\ &= Re(-\omega L I_0 e^{i\omega t} e^{-i\frac{\pi}{2}}) \\ &= Re(i\omega L I_0 e^{i\omega t}) \end{aligned}$$

   So

$$\boxed{\widetilde{V} = i\omega L\widetilde{I}}$$

3. Law 3 tells us

$$\begin{aligned} V &= \frac{1}{C} \int I(t) \\ &= \frac{I_0}{C\omega} \sin \omega t \\ &= Re(\frac{I_0}{C\omega} e^{i(\omega t - \frac{\pi}{2})}) \\ &= Re(\frac{I_0}{iC\omega} e^{i\omega t}) \end{aligned}$$

So

$$\boxed{\widetilde{V} = \frac{1}{iC\omega}\widetilde{I}}$$

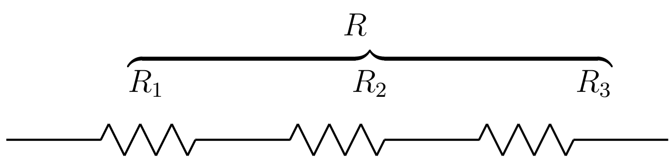
So, with the complex representation, all three circuit elements behave like resistors with a complex "Ohm's Law'

$$\boxed{\widetilde{V} = Z\widetilde{I} \qquad \text{where } Z = \begin{cases} R & \text{for resistors} \\ i\omega L & \text{for inductors} \\ \dfrac{1}{i\omega C} & \text{for inductors} \end{cases}}$$

Moreover, Z is called "impedance"

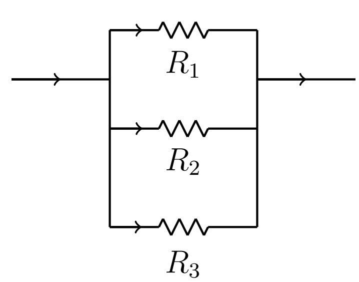
Combining the components:

- In series:



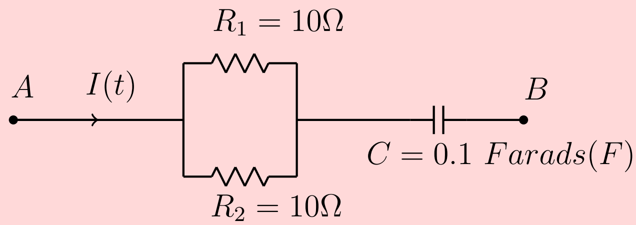
$$\begin{aligned} R &= R_1 + R_2 + R_3 + \cdots \\ L &= L_1 + L_2 + L_3 + \cdots \\ \frac{1}{C} &= \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \\ Z &= Z_1 + Z_2 + Z_3 + \cdots \end{aligned}$$

- In parallel:



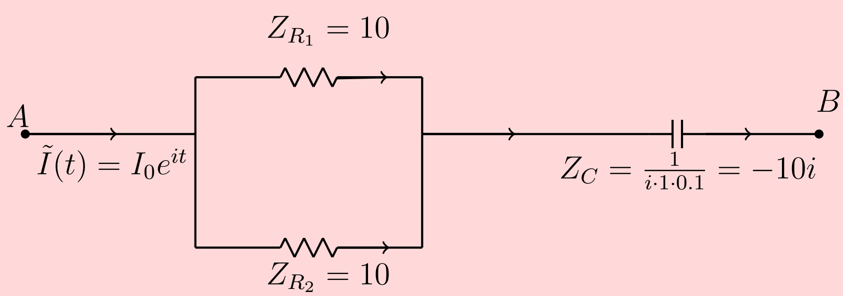
$$
\begin{aligned}
\frac{1}{R} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \\
\frac{1}{L} &= \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots \\
C &= C_1 + C_2 + C_3 + \cdots \\
\frac{1}{Z} &= \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots
\end{aligned}
$$

Example 1.24. Suppose a current $I(t) = I_0 \cos t$, passes through this:



Find $V(t)$, the difference in electrical potential energy between A and B

<u>**Solution**</u>:



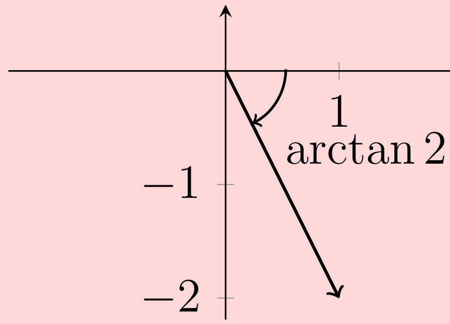
Let's use the complex version of "Ohm's Law". We have $\dfrac{1}{Z_R} = \dfrac{1}{Z_{R_1}} + \dfrac{1}{Z_{R_2}} = \dfrac{1}{10} + \dfrac{1}{10} = \dfrac{1}{5}$, so $Z_R = 5$.

Combine the resistor and capacitor in series: $Z = Z_R + Z_C = 5 - 10i$.

So, the complex voltage is

$$
\begin{aligned}
\widetilde{V} &= Z\widetilde{I} \\
&= (5 - 10i) I_0 e^{it} \\
&= 5I_0(1 - 2i)e^{it} \\
&= 5I_0\sqrt{5} e^{i \arctan -2} e^{it}
\end{aligned}
$$

So, $V(t) = Re(\widetilde{V}(t)) \approx 5\sqrt{5} I_0 \cos(t - 1.107)$



1.5 Sets in the Complex Plane

Definition 1.25. **Neighborhood of z_0** is

$$N_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$$

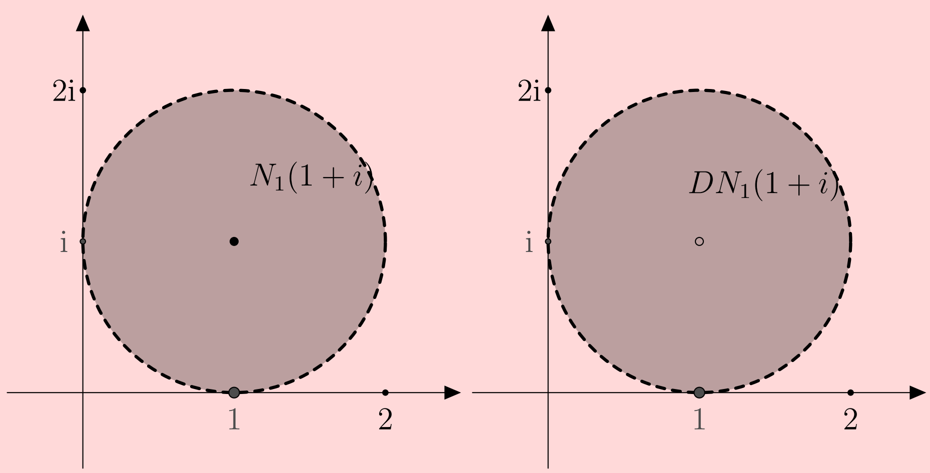
where $\epsilon > 0$ is real

Definition 1.26. **Deleted Neighborhood of z_0** is

$$DN_\epsilon(z_0) = \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$$

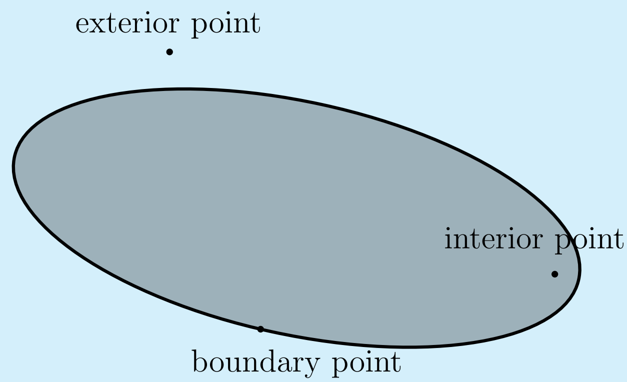
where $\epsilon > 0$ is real

Example 1.27. For $z_0 = 1 + i$, consider $|z - (1 + i)| < 1$. The neighborhood of z_0 and deleted neighborhood of z_0 is as follows:



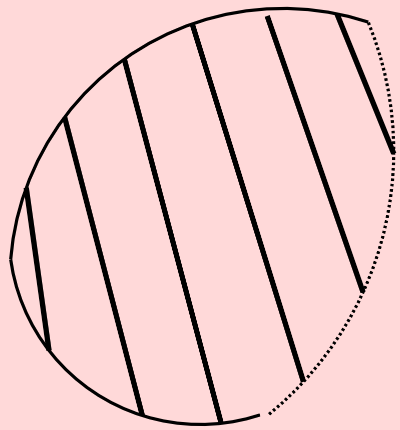
Definition 1.28. Let $S \subseteq \mathbb{C}$:

- z_0 is an **interior point** of S if there exists a neighborhood of z_0 which contains only points in S
- z_0 is an **exterior point** of S if there exists a neighborhood of z_0 which contains no points in S
- z_0 is a **boundary point** of S if every neighborhood of z_0 contains some points in S and some points not.
- **Boundary of S** is the set of all boundary points of S
- S is **open** if it contains none of its boundary points
- S is **closed** if it contains all of its boundary points, equivalently if its complement is open.
- Note that S could be both open and closed, when it does not have any boundary points



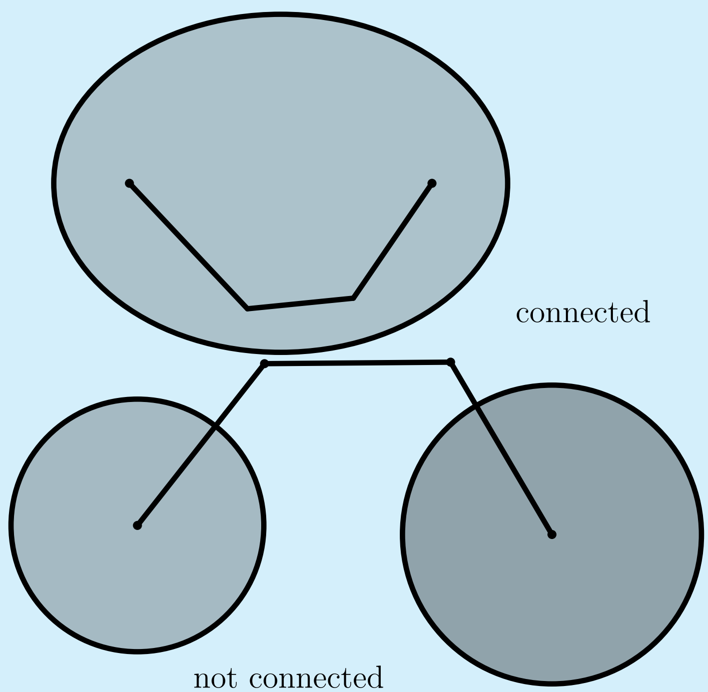
Example 1.29. Note that

- $N_1(1+i)$ is open
- $\mathbb{C}$ is both open and closed
- $|z - z_0| \leq 1$ is closed
- The figure below: it is neither open nor closed.



Definition 1.30. For $S \subseteq \mathbb{C}$:

- **Closure** of S is S plus its boundary.
- An open set S is **connected** if any two points in S can be connected by a polygonal path lying entirely in S
- A **domain** is an open connected set. We should not confuse this with "domain of a function"
- A **region** is a domain plus some, none, or all of its boundary points.
- S is **bounded** if there exists $R \in \mathbb{R}$ such that $|z| < R$ for all $z \in S$

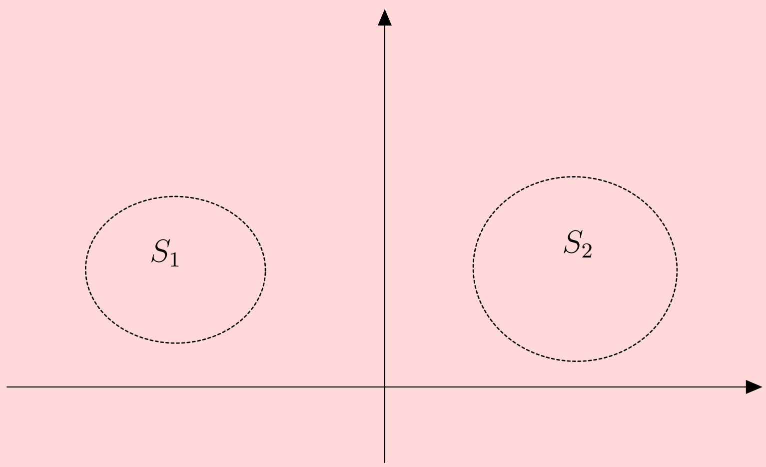


Theorem 1.31. If $u(x, y)$, defined on a domain D, satisfies

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

for all points in D, then $u(x, y) = \text{constant}$ in D.

Example 1.32. Suppose we have S_1 and S_2 like this:



in which we have $u(x, y) = 0$ on S_1 and $u(x, y) = 1$ on S_2.

Then, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0$ on $S_1 \cup S_2$, but $u(x, y)$ is not constant on $S_1 \cup S_2$.

Why does not the theorem hold? Well this is because $S_1 \cup S_2$ is not connected, so it's not a domain.

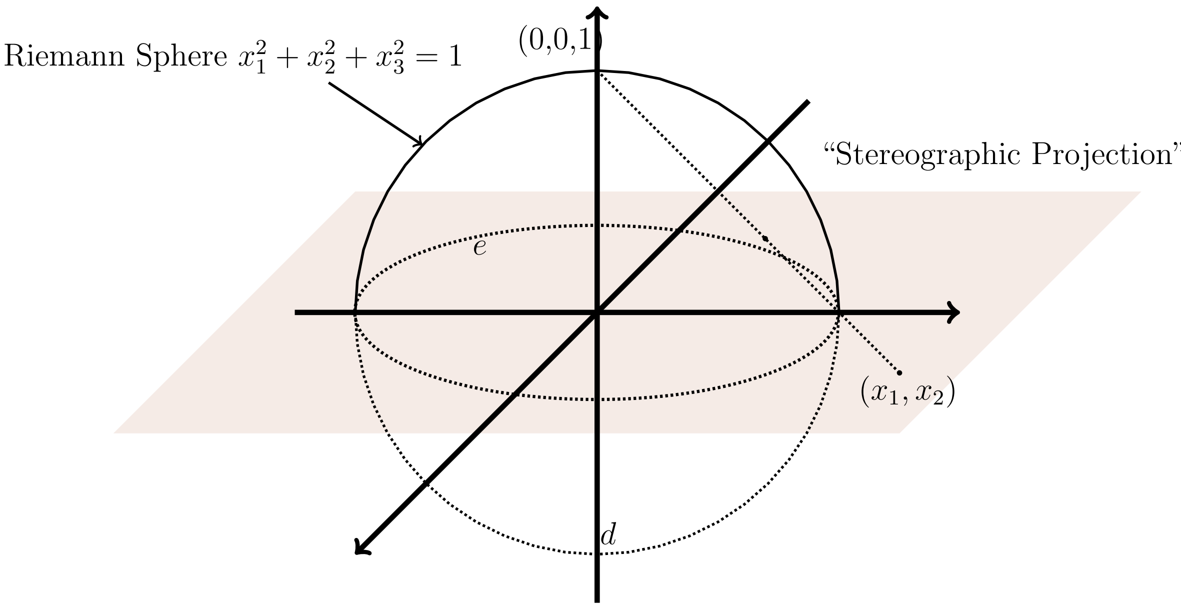
The Extended Complex Plane:

The "neighborhood of ∞" is defined as:

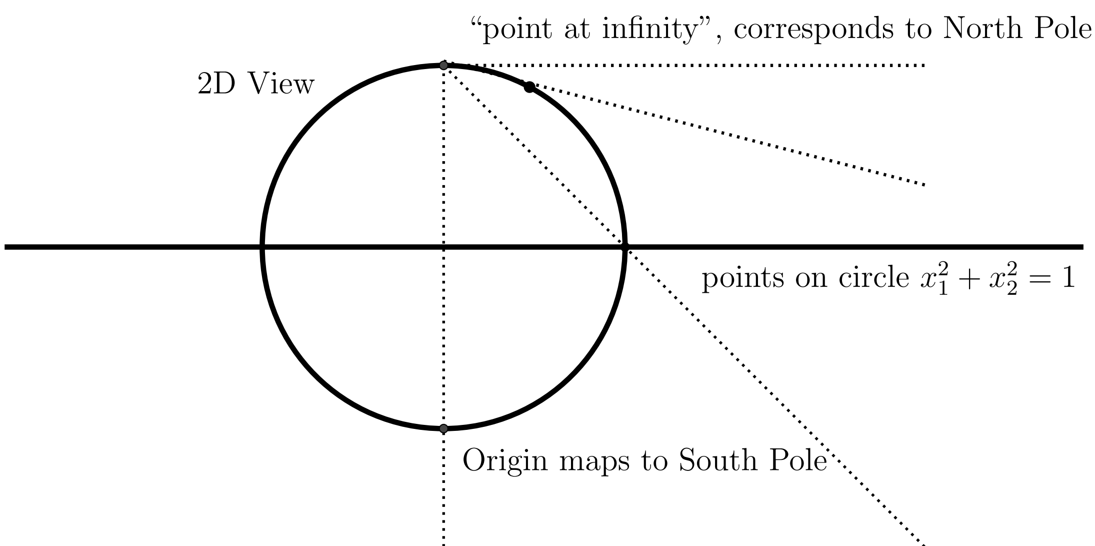
$$N_\epsilon(\infty) = \{z \in \mathbb{C} : |z| > \frac{1}{\epsilon}\}$$

for some real $\epsilon > 0$

The Riemann sphere:



We can define a one-to-one mapping between x_1x_2-plane and the sphere:



See the course text for more detail, in particular:

- Circles and lines all map circles on the sphere
- Lines are just circles which pass through the "point at infinity"

Chapter 2 Analytic Functions

2.1 Functions

For a function on complex numbers:

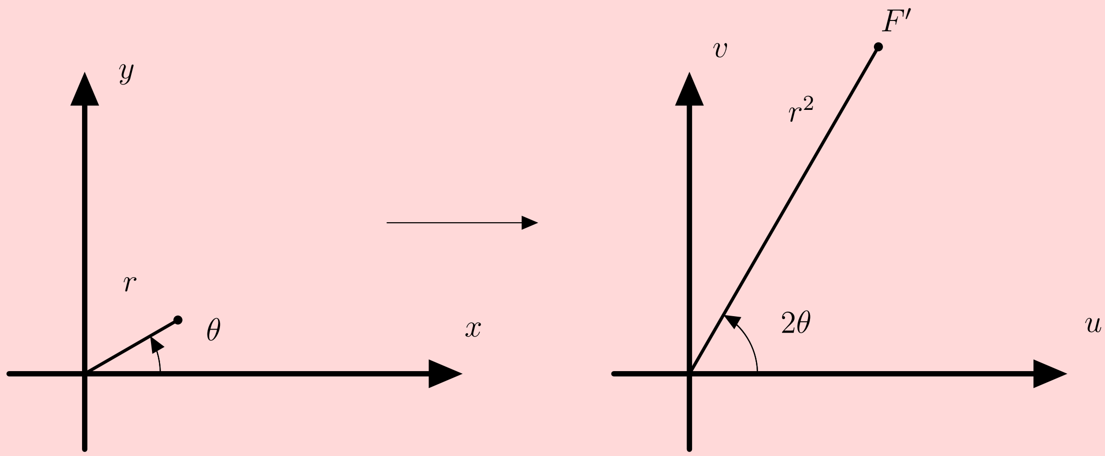
$$\begin{aligned}\omega &= f(z)\\ &= f(x+iy)\\ &= u(x,y)+iv(x,y)\end{aligned}$$

We can think of it as a mapping.

Example 2.1. 1. $f(z)=z^2$. Find the images of

(a) the first quadrant.

$f(z)=(x+iy)^2=\underbrace{(x^2-y^2)}_{u}+i\underbrace{2xy}_{v}$

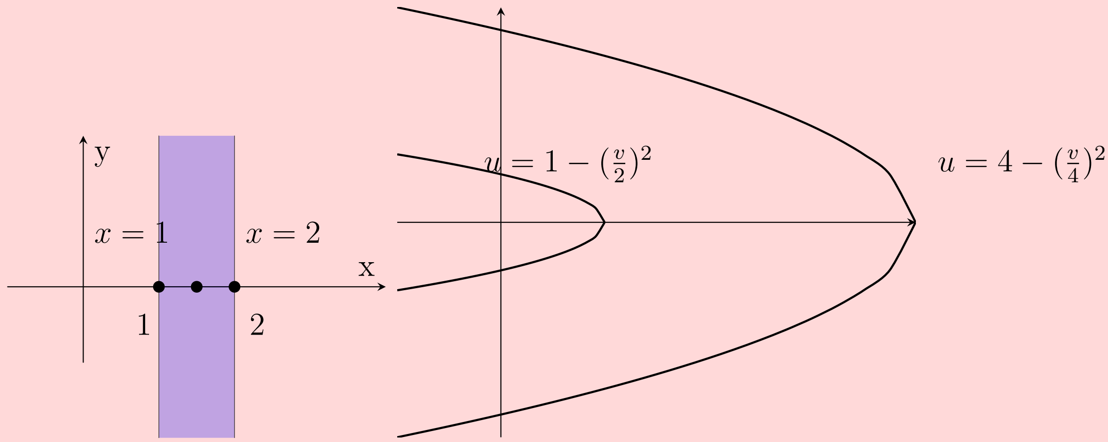


Note that $f(z)=(re^{i\theta})^2=r^2e^{i2\theta}$ (angle is doubled)

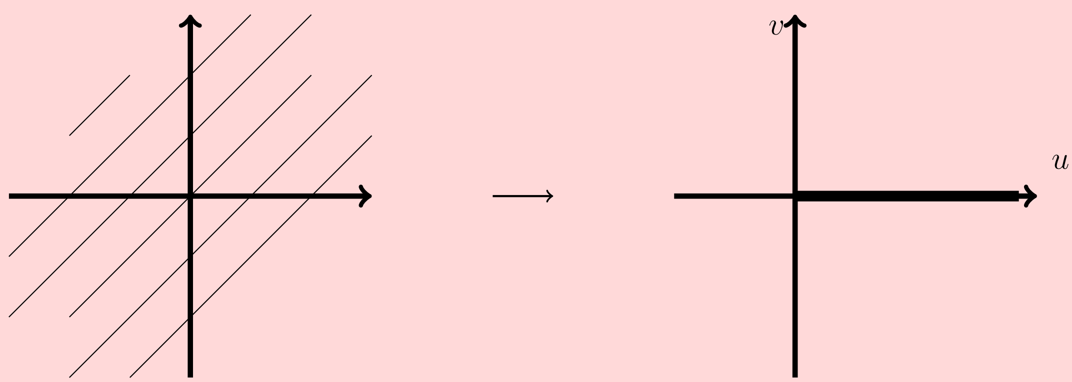
(b) the strip $1\le Re(z)\le 2$

With $1\le x\le 2$, the boundaries become:

- $x=1\Rightarrow \begin{cases}u=1-y^2\\ v=2y\end{cases}\Rightarrow u=1-\left(\frac{v}{2}\right)^2$, which is a parabola
- $x=2\Rightarrow \begin{cases}u=4-y^2\\ v=4y\end{cases}\Rightarrow u=4-\left(\frac{v}{4}\right)^2$, which is a parabola



2. $f(z) = |z|$. This one maps complex plane to non-negative real axis.

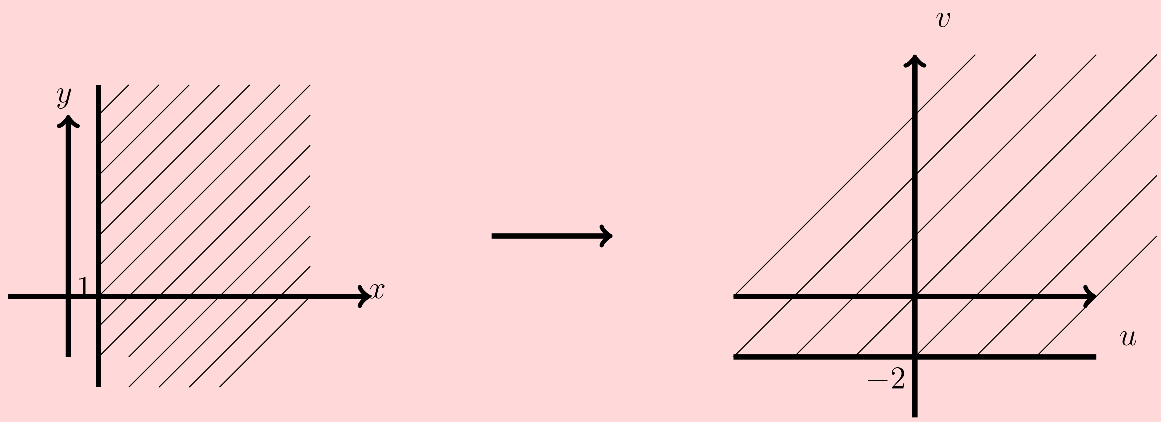


3. $f(z) = z - z_0 = (x + iy) - (x_0 + iy_0) = (x - x_0) + i(y - y_0)$. This is a translation.
4. $f(z) = z_0 z$, so

$$f(z) = r_0 e^{i\theta_0} r e^{i\theta} = \underbrace{r_0}_{\text{magnification}} r e^{i \overbrace{\theta_0}^{\text{rotation}} + \theta} = r_0 r e^{i\theta_0 + \theta}$$

5. $f(z) = \overline{z} = x - iy \to \begin{cases} u = x \\ v = -y \end{cases}$. This is a reflection on y-axis.
6. Find image of half-plane $Re(z) \geq 1$ under the map $\omega = f(z) = iz - 3i$.

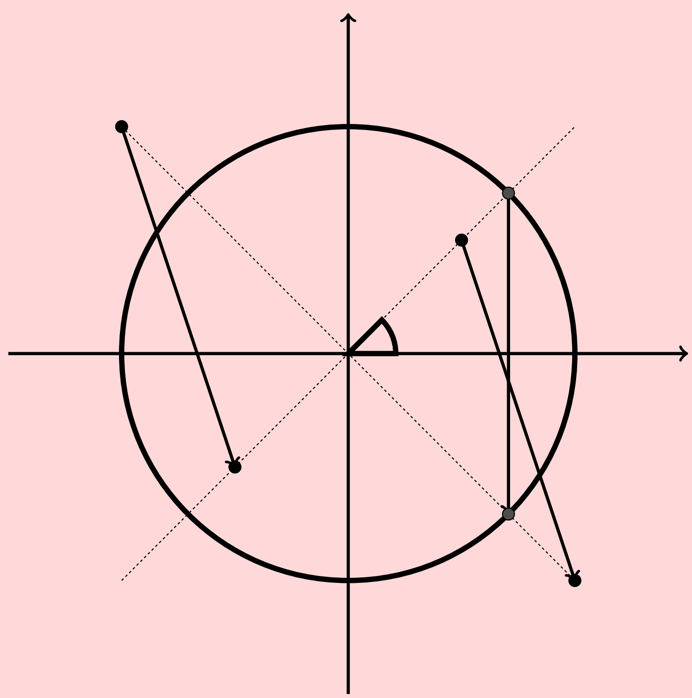
   We can do this step by step. First it's a rotation of $\frac{\pi}{2}$ (comes from the first i), then its a shift down 3 units.



The image is the half-plane $v \geq -2$.

7. Inversion mapping. $f(z) = \dfrac{1}{z} = \dfrac{1}{re^{i\theta}} = \dfrac{1}{r}e^{-i\theta}$. So, it's a scaling by r, and then reflection through the x-axis.

   For this mapping, unit circle maps to the unit circle. Outside points go to inside, and inside points go to outside.



8. Image of circle $(x-1)^2 + y^2 = 1$ under $f(z) = \dfrac{1}{z}$.

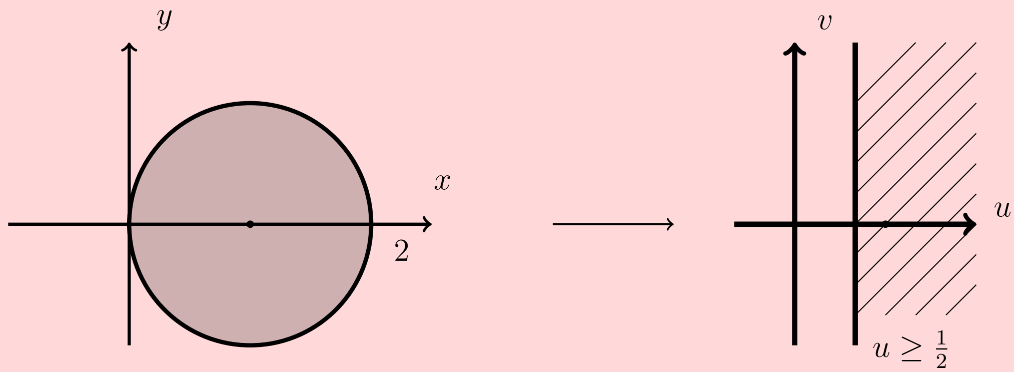
   The trick is to use polar fomulas. Recall $x^2 + y^2 = r^2, x = r\cos\theta, y = r\sin\theta$.

   So, $x^2 - 2x + 1 + y^2 = 1$ yields that $r^2 = 2r\cos\theta$. Since $r \neq 0$, we then have $r = 2\cos\theta$.

To apply the map, replace r with $\dfrac{1}{r}$, and θ with $-\theta$:

$$\frac{1}{r} = 2\cos(-\theta) \Rightarrow r = \frac{1}{2\cos\theta} \Rightarrow r\cos\theta = \frac{1}{2}$$

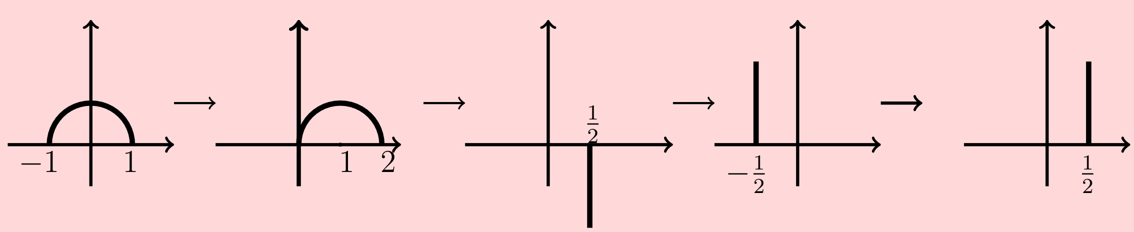
So $u = \dfrac{1}{2}$ since $\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$ in the uv plane.



9. $w = f(z) = \dfrac{z}{z+1}$, find the image of upper-half of unit circle.

First, $f(z) = \dfrac{z+1-1}{z+1} = 1 - \dfrac{1}{z+1}$. This is a sequence of transformations:

$$z \to \underbrace{z+1}_{\text{shift right}} \to \underbrace{\frac{1}{z+1}}_{\text{invert}} \to \underbrace{\frac{-1}{z+1}}_{\text{reflect and rotate }\pi} \to \underbrace{1 - \frac{1}{z+1}}_{\text{shift right}}$$



2.2 Limits and Differentiation

Definition 2.2. Limits:

$$\lim_{z \to z_0} f(z) = w_0$$

means that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z) - w_0| < \epsilon$$

Example 2.3. Prove that $\lim_{z\to 1+i}(2+i)z = 1+3i$.

__**Solution**__: We first do some preliminary work:

$$|(2+i)z - (1+3i)| = |2+i| \cdot |z - \frac{1+3i}{2+i}| = \sqrt{5} \cdot |z - (1+i)|$$

So, let $\epsilon > 0$, with $|z - z_0| < \frac{\epsilon}{\sqrt{5}} (= \delta)$, we have

$$\begin{aligned} |(2+i)z - (1+3i)| &= \sqrt{5} \cdot |z - (1+i)| \\ &< \sqrt{5} \cdot \frac{\epsilon}{\sqrt{5}} \\ &= \epsilon \end{aligned}$$

So, $\lim_{z\to 1+i}(2+i)z = 1+3i$ □

Note that similar definitions apply when dealing with infinity, e.g. $\lim_{z\to z_0} f(z) = \infty$ means that for any $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z)| > \frac{1}{\epsilon}$

Definition 2.4. __**Continuity**__: f is __**continuous**__ at z_0 means that

$$\lim_{z\to z_0} f(z) = f(z_0)$$

The usual limit and continuity theorems hold, e.g.

$$\lim_{z\to z_0} f(z)g(z) = \lim_{z\to z_0} f(z) \cdot \lim_{z\to z_0} g(z)$$

Theorem 2.5. Let $f(z) = u + iv, z_0 = x_0 + iy_0, w_0 = u_0 + iv_0$, then

$$\lim_{z\to z_0} f(z) = w_0 \quad \text{if and only if} \quad \begin{cases} \lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \\ \lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 \end{cases}$$

Definition 2.6. Differentiation:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \left(= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right)$$

Derivative function is

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

For functions with real analogues (e.g. $f(z) = z^2$ analogous to $f(x) = x^2$), the usual rules (power, quotient, etc.) apply, e.g.

$$f(z) = 3z^2 + z^4 \quad \Rightarrow f'(z) = 6z + 4z^3$$

What about functions without real analogues?

Example 2.7. $f(z) = \overline{z}$. Is it differentiable?

Solution:

$$\begin{aligned}
f'(z_0) &= \lim_{z \to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0} \\
&= \lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0} \\
&= \lim_{z \to z_0} \frac{\overline{re^{i\theta}}}{re^{i\theta}} \quad \text{where} \quad z - z_0 = e^{i\theta} \\
&= \lim_{z \to z_0} \frac{e^{-i\theta}}{e^{i\theta}} \\
&= \lim_{z \to z_0} e^{-i2\theta}
\end{aligned}$$

which depends on θ! No unique value, so limit DNE. So, f is not differentiable anywhere.

Theorem 2.8. Cauchy-Riemann Equations: If $f(z) = u(x, y) + iv(x, y)$ and $f'(z_0)$ exists, then

$$\boxed{u_x = v_y \quad \text{and} \quad v_x = -u_y \qquad \text{at } (x_0, y_0)}$$

Note that for notation,

$$u_x = \frac{\partial u}{\partial x}$$
$$u_y = \frac{\partial u}{\partial y}$$
$$v_x = \frac{\partial v}{\partial x}$$
$$v_y = \frac{\partial v}{\partial y}$$

Proof 2.9.

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i \frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y} \right) \end{aligned}$$

Since the limit exists, it must be independent of path, so

- Along $\Delta y = 0$:
$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i(\cdots) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

- Along $\Delta x = 0$:
$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i(\cdots) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary part yields the result. $\square$

2.3 Differentiability Continued

Example 2.10. Is $f(z) = |z|^2$ differentiable? Where?

<u>**Solution**</u>: $f(z) = \sqrt{x^2 + y^2}^2 = \underbrace{x^2 + y^2}_{u} + \underbrace{0}_{v} i$. So, by CRE, we know that

$$\begin{cases} u_x = v_y & \Rightarrow 2x = 0 \\ v_x = -u_y & \Rightarrow 0 = -2y \end{cases}$$

It's clear that this is satisfied only at $x = y = 0$.

So, if $(x, y) \neq (0, 0)$, i.e. $z \neq 0$, then f is not differentiable.

When $z = 0$, $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \dfrac{|\Delta z|^2 - 0}{\Delta z} = 0$. This is because $\left|\dfrac{|\Delta z|^2}{\Delta z} - 0\right| \leq |\Delta z| \to 0$ as $\Delta z \to 0$ (by applying the squeeze theorem).

Hence, CRE are necessary but not sufficient conditions.

Theorem 2.11. Let f be defined in some neighborhood of z_0. If u_x, u_y, v_x, v_y exist in that neighborhood, satisfying CRE at z_0, and are **continuous** at z_0, then f is differentiable at z_0.

Definition 2.12. $f(z)$ is **analytic at z_0** if $f'(z)$ exists at every point in some neighborhood of z_0.

$f(z)$ is **analytic on an open set S** if it is analytic at every point of S.

Example 2.13. $f(z) = z^3 = \cdots = \underbrace{(x^3 - 3xy^2)}_{u(x,y)} + i\underbrace{(3x^2y - y^2)}_{v(x,y)}$.

We have

$$\begin{aligned} u_x &= 3x^2 - 3y^2 \\ u_y &= -6xy \\ v_x &= 6xy \\ v_y &= 3x^2 - 3y^2 \end{aligned}$$

So, CRE satisfied everywhere. All partial derivatives are continuous. By theorem, f is differentiable everywhere, so is analytic everywhere. We refer to "analytic everywhere" as "entire"

Example 2.14. Where is $f(z) = x^2 + iy^2$ analytic?

We have

$$\begin{aligned} u_x &= 2x \\ u_y &= 0 \\ v_x &= 0 \\ v_y &= 2y \end{aligned}$$

We need $x = y$ to satisfy CRE.

- If $x \neq y$, f is not differentiable, so not analytic.
- If $x = y$, f cannot be analytic because we are not on an open set.

So, f is not analytic nowhere.

Theorem 2.15. Sums, products, and compositions of analytic functions are also analytic, except when $\div 0$

Example 2.16. $f(z) = \dfrac{z^3+2}{z^2+1}$ is analytic everywhere except at $z = \pm i$.

$g(z) = f(z^2)$ is analytic everywhere except wherre $z^2 = \pm i$, i.e. except

$$\begin{aligned} z &= e^{i(\frac{n\pi + \pi/2}{2})} \\ &= e^{i(n\pi/2+\pi/4)} \\ &= e^{i(n\pi/4)}, e^{i(n3\pi/4)}, e^{i(n5\pi/4)}, e^{i(n7\pi/4)} \\ &= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \end{aligned}$$

Theorem 2.17. Suppose f is analytic in a domain D. If $f'(z) = 0$ for all $z \in D$, then f is constant in D

Proof 2.18. $f'(z) = u_x + iv_x = v_y - iu_y$. So, $f'(z) = 0 \Rightarrow u_x = v_y = 0 = v_y = u_y$. So, u and v are constant, since D is connected. $\square$

Theorem 2.19. Suppose f is analytic in a domain D. If $|f(z)| = M$ for all $z \in D$, where M is constant, then $f(z)$ is constant in D.

Proof 2.20. $|f(z)|^2 = u^2 + v^2 = M^2$.

We differentiate:

- with respect to x: $2uu_x + 2vv_x = 0 \quad - (1)$

- with respect to y: $2uu_y + 2vv_y = 0 \quad - (2)$

Now $u_x = v_y$, and $v_x = -u_y$, so the (2) gives $-uv_x + vu_x = 0 \quad - (3)$.

Multiply (1) by u_x.

$$\begin{aligned} & uu_x^2 + vu_xv_x = 0 \\ \Rightarrow \; & uux^2 + (uv_x)v_x = 0 \quad \text{by (3)} \\ \Rightarrow \; & u(u_x^2 + v_x^2) = 0 \end{aligned}$$

So, unless $u = 0$ for all $z \in D$, we must have $u_x^2 + v_x^2 = 0$. So, $u_x = v_x = 0$, implying that u, v are constant. Hence, f is constant.

What if $u = 0$ for all $z \in D$? Then, $u_x = u_y = 0$, so $v_x = v_y = 0$ by CRE. f is constant as well. $\square$

2.4 Harmonic Functions

Recap:

$$f'(z) = u_x + iv_x = \frac{u_y + iv_y}{i} = v_y - iu_y$$

$$CRE: \quad u_x = v_y \quad v_x = -u_y$$

Also, "analytic" means differentiable on a open set.

Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Then u and v satisfy CRE.

Also, which will be shown later, $u, v \in C^2$ (continuous under second partial derivatives), and this implies that $u_{xy} = u_{yx}$, and $v_{xy} = v_{yx}$.

From CRE:

$$\underbrace{u_x - v_y}_{\Rightarrow\, u_{xx} = v_{yx}} \quad \text{and} \quad \underbrace{v_x = -u_y}_{\Rightarrow\, u_{yy} = -v_{xy}}$$

Definition 2.21. From the above derivation, we see

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

and

$$v_{xx} + v_{yy} = 0$$

We refer to these as **"Laplace's equation"**

Solution to Laplace's equation are called **"harmonic functions"**

Notes:

- We've shown that if $f(z) = u + iv$ is analytic, then u and v must be harmonic
- Laplace's equation is very useful! We will see that later.
- $u_{xx} + u_{yy} = 0$ is also denoted as $\Delta^2 u = 0$, and we denote Δ as "Laplacian operator".

Example 2.22. Suppose $u(x, y) = e^{-2x} \cos 2y + 2y$. Find $v(x, y)$ such that $f(z) = u + iv$ is analytic.

Solution: u and v must satisfy CRE. So, $v_y = u_x = -2e^{-2x} \cos 2y$. Hence,

$$\begin{aligned} v &= \int -2e^{-2x} \cos 2y dy \\ &= -e^{2x} \sin 2y + C(x) \end{aligned}$$

Note that $C(x)$ is a function of all other variables.

Now we try to make it satisfy other CRE:

$$\begin{aligned} v_x = -u_y &\Rightarrow\ 2e^{-2x} \sin 2y + C'(x) = 2e^{-2x} \sin 2y - 2 \\ &\Rightarrow\ C'(x) = -2 \\ &\Rightarrow\ C(x) = -2x + k \end{aligned}$$

Therefore, $v(x, y) = -e^{-2x} \sin 2y - 2x + k$

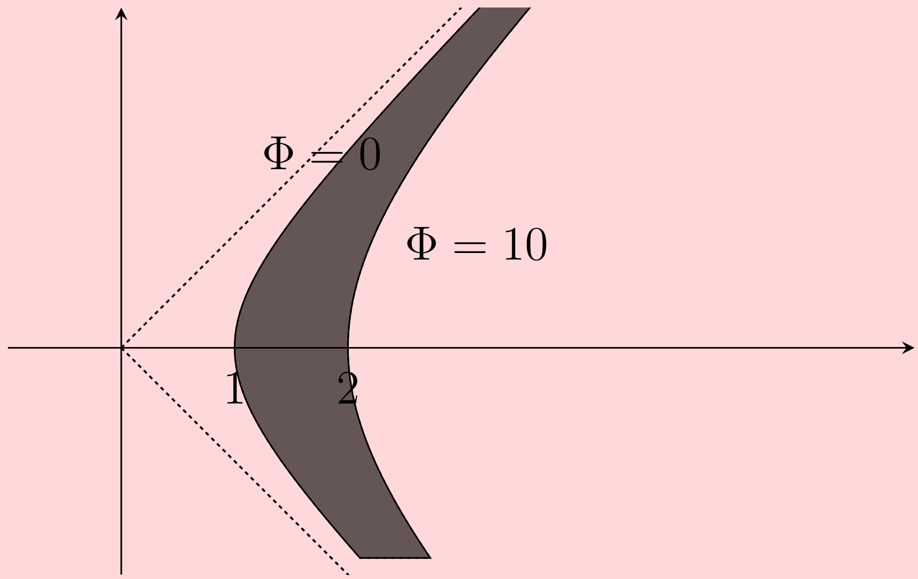
Note that $v(x, y)$ is called the "harmonic conjugate" of u.

Exercise: show that if v is the harmonic conjugate of u, then $-u$ is the harmonic conjugate of v.

Example 2.23. Solve Laplace's equation $\Phi_{xx} + \Phi_{yy} = 0$ on region between hyperbolas $x^2 - y^2 = 1$ and $x^2 - y^2 = 4$, $x > 0$, with "boundary conditions"

$$\begin{cases} \Phi = 0 & \text{on } x^2 - y^2 = 1 \\ \Phi = 10 & \text{on } x^2 - y^2 = 4 \end{cases}$$

i.e. Find $\Phi(x, y)$



Solution: Consider $f(z) = z^2 = (x+yi)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + i\, \underbrace{2xy}_{v(x,y)}$

Since $f(z)$ is already analytic, we have that $u(x,y) = x^2 - y^2$ is harmonic. Boundary curves of region are level curves of a harmonic function.

Is the solution $\Phi(x,y) = x^2 - y^2$? No.

Try $\Phi(x,y) = A \cdot (x^2 - y^2) + B$ (also harmonic by linearity).

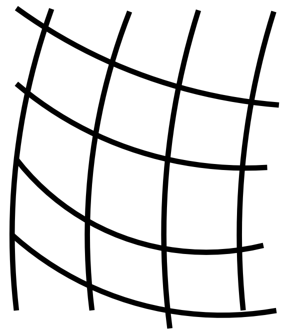
Applying the Boundary Conditions:

$$0 = A \cdot 1 + B \Rightarrow \; B = -A$$
$$10 = A \cdot 4 + B \Rightarrow \; A = \frac{10}{3}, B = -\frac{10}{3}$$

So the solution is $\Phi(x,y) = \frac{10}{3}(x^2 - y^2) - \frac{10}{3}$

Notes:

- It can be used in temperature distribution
- What about more complicated regions?
- Orthogonal trajectories



- list of harmonic functions

Chapter 3 Elementary Functions

3.1 Elementary Functions

Definition 3.1. Polynomials:

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \ , \quad a_i \in \mathbb{C}$$

There are obviously **entire**.

The fundamental theorem of algebra guarantees that we can factor this as

$$p(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_n)$$

Note that z_i are not necessarily distinct.

z_0 is a "zero of multiplicity" k if and only if

$$p(z) = (z - z_0)^k q(z)$$

where $q(z)$ is a polynomial such that $q(z_0) \neq 0$

Definition 3.2. Rational Functions:

$$R(z) = \frac{p(z)}{q(z)} = \frac{a_n(z - z_1)(z - z_2)\cdots(z - z_n)}{b_m(z - w_1)(z - w_2)\cdots(z - w_n)}$$

Suppose all common factors have been cancelled, then

- the roots (or zeroes) of $p(z)$ are called the **roots/zeroes** of $R(z)$
- the roots (or zeroes) of $q(z)$ are called the **poles** of $R(z)$

Example 3.3.

$$R(z) = \frac{3i(z-1)(z-\frac{1}{3}i)^2(z+i)}{(z-i)^3(z-2-i)}$$

Zeroes at 1 and $-i$ (order 1 would be a "simple zero"), and $\frac{1}{3}i$ (order 2).

Poles at i (order 3) and $2+i$ (order 1 would be a "simple pole")

Partial Fractions has simpler rules:

Example 3.4. Decompose $R(z) = \dfrac{1}{(z+4)^2(z^2+1)}$

Solution: Factor and expand

$$\frac{1}{(z+4)^2(z^2+1)} = \frac{A}{z+4} + \frac{B}{(z+4)^2} + \frac{C}{z+i} + \frac{D}{z-i}$$

This gives us

$$1 = A \cdot (z+4)(z+i)(z-i) + B(z+i)(z-i) + C(z+4)^2(z-i) + D(z+4)^2(z+i)$$

We can solve this by:

- set $z = -4$, this gives us $1 = 0 + (-4+i)(-4-i)B + 0 + 0$, so $B = \dfrac{1}{17}$
- set $z = -i$, this gives us $1 = 0+0+(-i+4)^2(-2i)C+0$. Then we conpute $(-2i)(15-8i) = 16-30i$, also $(-16-30i) = \dfrac{(-16-30i)(-16+30i)}{(-16+30i)} = \dfrac{1156}{(-16+30i)} = \dfrac{578}{-8+15i}$.

  Hence, $C = \dfrac{-8+15i}{578}$.
- set $z = -4$, this gives us $1 = 0+0+0+(i+4)^2(2i)D$, so $D = \dfrac{-8-15i}{578}$. The trick to compute things here is that, we can replace i with $-i$ from C since the expression is similar to C.

Now what about A? We can try another z, or just compare the coefficients of z^3. By comparing the coefficients of z^3, we get that

$$0 + A + C + D = A + \frac{-8+15i}{578} + \frac{-8-15i}{578}$$

So $A = \dfrac{16}{578} = \dfrac{8}{289}$

Hence,

$$\frac{1}{(z+4)^2(z^2+1)} = \frac{8/289}{z+4} + \frac{1/17}{(z+4)^2} + \frac{\frac{-8+15i}{578}}{z+i} + \frac{\frac{-8-15i}{578}}{z-i}$$

Actually, often we will only need one of the coefficients, and there's a quick way which will be covered later in the course.

Definition 3.5. **Exponential Function**: We already defined that $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$.

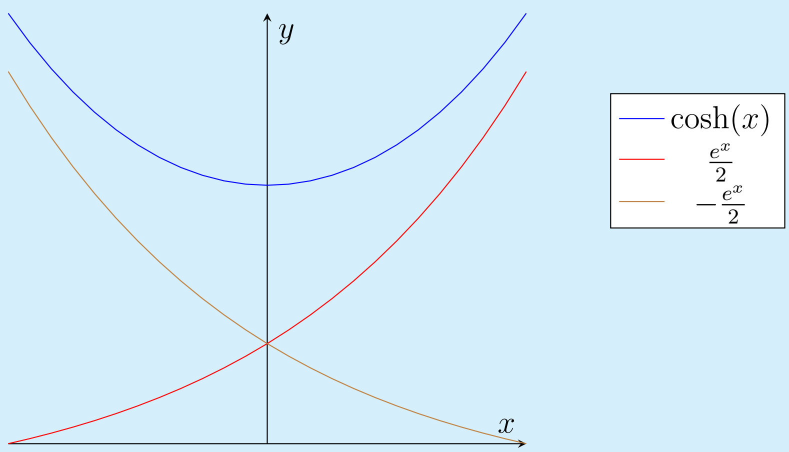
Note that $e^{z_1+z_2} = e^{z_1}e^{z_2}$, $\dfrac{d}{dz}e^z = e^z$. Also, e^z is **periodic** with period $2\pi i$

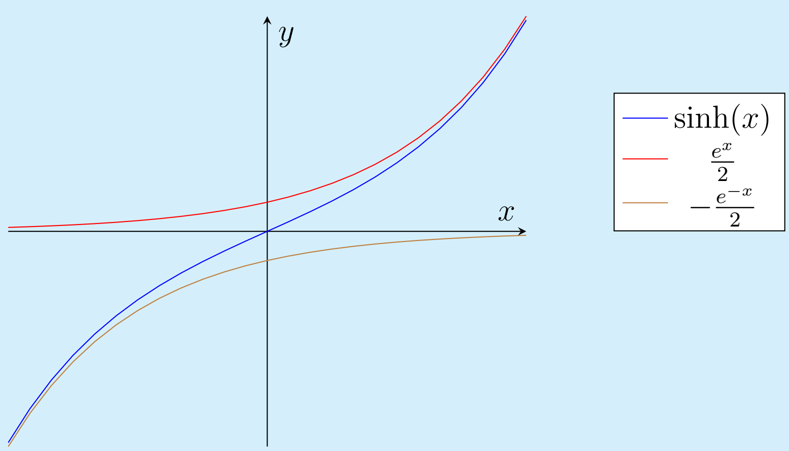
Definition 3.6. **Hyperbolic Functions**: From real calculus, we seen that

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \quad \text{this is the even component of } e^x$$
$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \text{this is the odd component of } e^x$$

It can be shown that

$$\cosh x + \sinh x = e^x$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$\frac{d}{dx}\sinh x = \cosh x$$
$$\frac{d}{dx}\cosh x = \sinh x$$





To extend these to $\mathbb{C}$, we define

$$\boxed{\cosh z = \frac{1}{2}(e^z + e^{-z}) \quad \sinh z = \frac{1}{2}(e^z - e^{-z})}$$

3.2 Trigonometric and Logarithmic Function

Definition 3.7. **Trigonometric Functions**: Recall

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Sum to get $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

We define

$$\boxed{\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \cosh(iz) \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{i}\sinh(iz)}$$

Furthermore:

$$\cos(iz) = \frac{e^{-z} + e^z}{2} = \cosh z$$
$$\sin(iz) = \frac{e^{-z} - e^z}{2i} = i\sinh z$$

$$\text{For real } z, \quad e^z = e^x = \cosh x + \sinh x$$
$$\text{For imaginary } z, \quad e^z = e^{iy} = \cos y + i\sin y$$

The $\cosh x$ and $\cos y$ are the even parts, and $\sinh x$ and $i\sin y$ are the odd parts

Functions	Along Real Axis	Along Imaginary Axis
$e^{iz}, \cos z, \sin z$	periodic	grow exponentially
$e^z, \cosh z, \sinh z$	grow exponentially	periodic

Familiar identities hold true.

Example 3.8.

$$\begin{aligned}\cos^4\theta &= (\frac{e^{i\theta}+e^{-i\theta}}{2})^4 \\ &= \frac{1}{16}(e^{i4\theta}+4e^{2\theta}+6+4e^{-i2\theta}+e^{-i4\theta}) \\ &= \frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}\end{aligned}$$

Example 3.9.

$$\begin{aligned}&\cos^2\theta+\sin^2\theta = 1 \\ \Rightarrow\ &\cos^2(iy)+\sin^2(iy) = 1 \\ \Rightarrow\ &\cosh^2 y+i^2\sinh^2 y = 1 \\ \Rightarrow\ &\cosh^2 y-\sinh^2 y = 1\end{aligned}$$

By using the rules $\begin{cases}\cos(iz)=\cosh z \\ \sin(iz)=i\sinh z\end{cases}$.

Notice the "Obsborne's rule" here: Hyperbolic function satisfy the same identities as trigonometric functions except that <u>we must change the sign of every product of two sines</u>.

Derivatives: e^z is entire, and so is $\cos z, \sin z, \cosh z, \sinh z$. Also,

$$\frac{d}{dz}(\cos z) = \frac{d}{dz}(\frac{e^{iz}+e^{-iz}}{2}) = \frac{ie^{iz}-ie^{-iz}}{2} = \frac{e^{iz}-e^{-iz}}{-2i} = -\sin z$$

Other as expected as well

Note: we can also define $\tan z$, $\sec z$ etc. in the usual ways, and derivatives of them are as expected.

Example 3.10. What is the value of $\sin(\pi+i)$?

<u>**Solution**</u>:

$$\begin{aligned}\sin(\pi+i) &= \sin\pi\cos(i\cdot 1)+\cos\pi\sin(i\cdot 1) \\ &= \sin\pi\cosh 1+\cos\pi i\sinh(1) \\ &= 0+(-1)\cdot i\cdot\sinh(1) \\ &= -\sin i\end{aligned}$$

Example 3.11. Find all solutions of $\sin z = 1000$

Solution: We write $\sin(x+yi) = 1000$, and get that

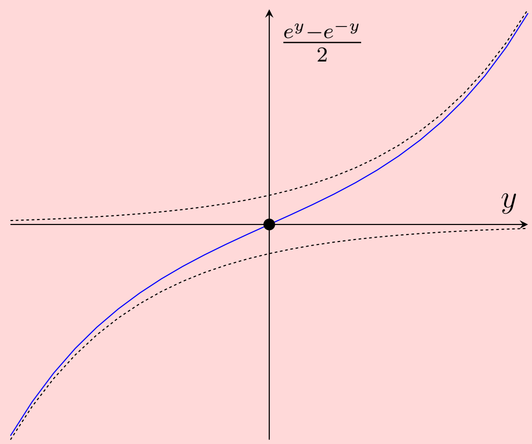
$$\sin x \cosh y + i \cos x \sinh y = 1000$$

So

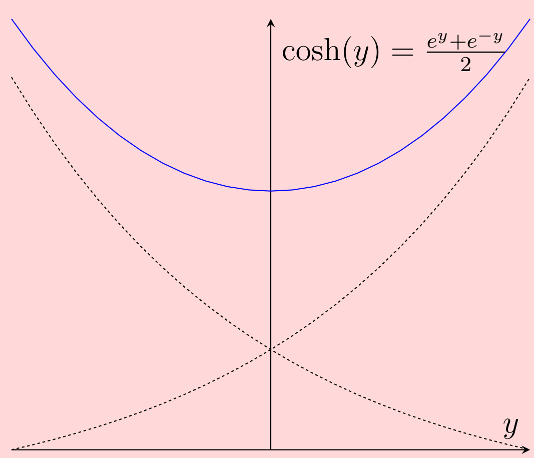
$$\begin{cases} \sin x \cosh y = 1000 & \cdots\cdots(1) \\ \cos x \sinh y = 0 & \cdots\cdots(2) \end{cases}$$

Equation 2 gives that $\cos x = 0$ or $\sinh y = 0$, which yields that $x = (2n+1)\dfrac{\pi}{2}$ or $y = 0$.

The following figure shows that the only x that $\sinh(x) = 0$ is at $x = 0$.



- If $y = 0$, equation 1 gives that $\sin x \cosh(0) = \sin x = 1000$. This is impossible
- If $x = (2n+1)\dfrac{\pi}{2}$, then equation 1 gives $\sin\left((2n+1)\dfrac{\pi}{2}\right)\cosh y = 1000$, so $\cosh y = 1000 \cdot (-1)^n$



But $\cosh y > 0$, so use $n = 2N$ (always even). So $\cosh y = 1000$, and $y = \pm\cosh^{-1}(1000) \approx \pm 7.6$ (There are two solutions, i.e. note the $\pm$ sign, as the figure above shows).

The final answer is that $z = x + iy = (4N+1)\dfrac{\pi}{2} \pm i\cosh^{-1}(1000)$

3.3 Logarithmic Functions

How to define $\log z$? Let $z = e^w$ and solve for w. Note that:

- exponential function is periodic, so log will be a "multi-valued function"
- in $\mathbb{C}$, we use "log" instead of "ln"

Definition 3.12. Now,

$$\begin{aligned} z = e^w &\Rightarrow\ re^{i\theta+2\pi k} = e^{u+iv} \\ &\Rightarrow\ r = e^u,\ \ \theta + 2\pi k = v \\ &\Rightarrow\ u = \ln r,\ \ v = \theta + 2\pi k \end{aligned}$$

So, we define

$$\boxed{\log z = \ln|z| + i \arg z}$$

Example 3.13.

- $\log(1+i) = \ln|1+i| + i\arg(1+i) = \ln\sqrt{2} + i\left(\dfrac{\pi}{4} + 2\pi k\right)$
- $\log(i) = \ln|i| + i\arg(i) = 0 + i\left(\dfrac{\pi}{2} + 2\pi k\right)$

Proposition 3.14. We have the following identity:

$$\begin{aligned} \log(z_1 z_2) &= \ln|z_1 z_2| + i\arg(z_1 z_2) \\ &=^* \ln|z_1| + \ln|z_2| + i(\arg z_1 + \arg z_2) \\ &= \log(z_1) + \log(z_2) \end{aligned}$$

Similarly

$$\log\left(\frac{z_1}{z_2}\right) =^* \log(z_1) - \log(z_2)$$

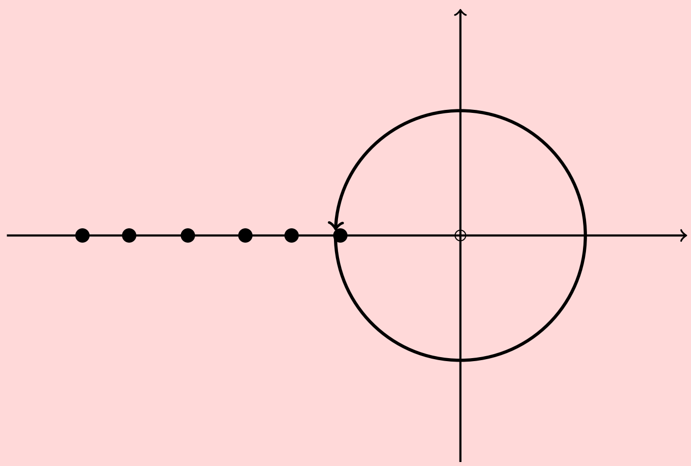
By $=^*$, we actually mean that the set of values of $\log(z_1 z_2)$ is equal to the set of values of $\log(z_1) + \log(z_2)$, due to the multi-valuedness of log.

Definition 3.15. The **principle value of the Logarithm** is

$$\mathrm{Log}(z) = \ln|z| + i \underbrace{\mathrm{Arg}(z)}_{\in(-\pi,\pi] \text{ usually}}$$

Example 3.16.

- $\mathrm{Log}(1+i) = \ln|1+i| + i\,\mathrm{Arg}(1+i) = \ln\sqrt{2} + i\dfrac{\pi}{4}$
- $\mathrm{Log}(i) = \ln|i| + i\,\mathrm{Arg}(i) = 0 + i\pi$
- $\mathrm{Log}\, e^z = z$ if and only if $Im(z) \in (-\pi, \pi]$
- $\mathrm{Log}\, z$ has discontinuity on negative real axis



- $\mathrm{Log}\, z$ is analytic everywhere else, with

$$\frac{d}{dz}\mathrm{Log}\, z = \frac{1}{z}$$

Proof 3.17. Let

$$\begin{aligned} w = \mathrm{Log}\, z &= \ln|z| + i\,\mathrm{Arg}(z) \\ &= \frac{1}{2}\ln(x^2+y^2) + i\left(\arctan(\frac{y}{x}) \pm \pi\right) \\ &= u(x,y) + iv(x,y) \end{aligned}$$

Now,

$$\begin{aligned}\frac{dw}{dz} &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \\ &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \\ &= \frac{x-iy}{x^2+y^2}\cdot\frac{x+iy}{x+iy} \\ &= \frac{1}{z}\end{aligned}$$

□

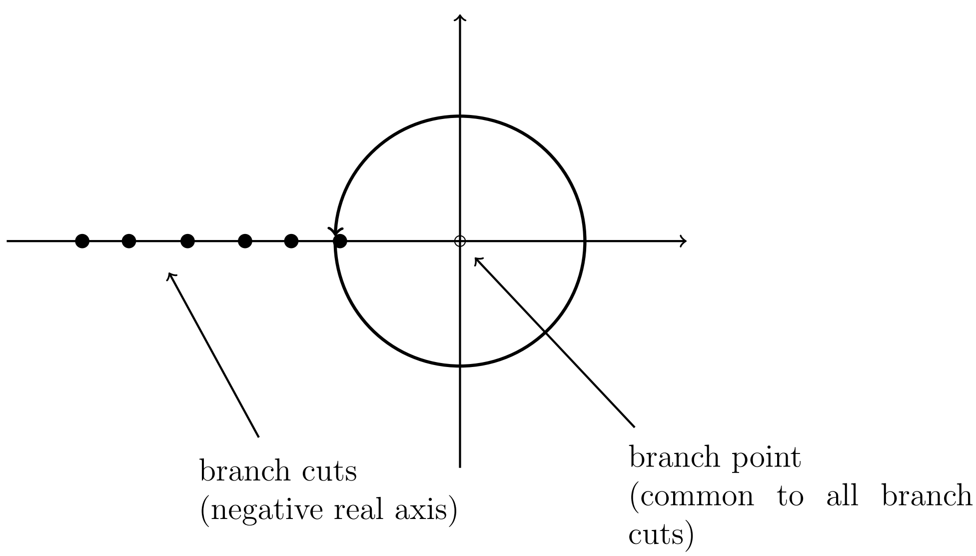
Definition 3.18. **Branch Cuts**: Let $f(z)$ be a multivalued function. $F(z)$ is said to be a **branch** of $f(z)$ on a domain D if $F(z)$ is continuous on D and for each $z \in D$, $F(z)$ is one and only one of the values of $f(z)$.

Example 3.19. $\operatorname{Log} z$ is a branch of $\log z$

We could define different branches of $\log z$ by

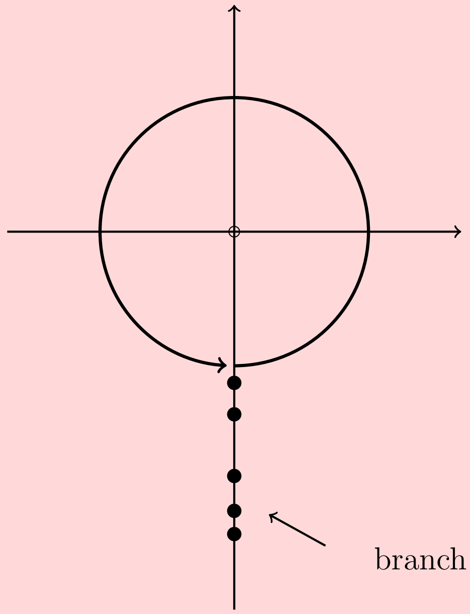
$$\operatorname{Log}_\tau z = \ln|z| + i\operatorname{Arg}_\tau(z)$$

where $\operatorname{Arg}_\tau(z) \in (\tau, \tau + 2\pi]$. Note that $\operatorname{Log} z = \operatorname{Log}_{-\pi}$



Example 3.20.

$$\operatorname{Log}_{-\frac{\pi}{2}} \ln|z| + i\operatorname{Arg}_{-\frac{\pi}{2}}(z)$$



Example 3.21. Find a branch of $f(z) = \log(z+4)$ that is analytic at $z = -5$ and equals $7\pi i$ there.

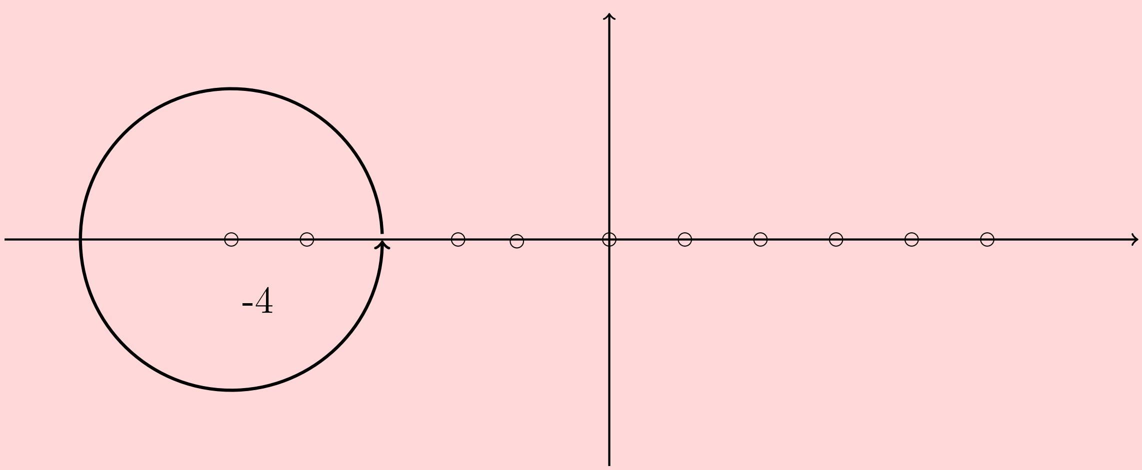
Solution: We want $\mathrm{Log}_\tau(-5+4) = \mathrm{Log}_\tau(-1) = 7\pi i$ for some τ.

So, $\ln|-1| + i\,\mathrm{Arg}_\tau(-1) = 7\pi i$ for some k, i.e.

$$0 + i\underbrace{(\pi + 2k\pi)}_{\in(\tau,\tau+2\pi]} = 7\pi i \quad \text{for some } k$$

Hence, $k = 3$. We can choose $\tau = 6\pi$ so that $7\pi \in (6\pi, 8\pi]$.

The final answer would be $F(z) = \mathrm{Log}_{6\pi}(z+4)$



Example 3.22. Where is $f(z) = \text{Log}(z^2+1)$ analytic?

Solution: We need $z^2+1 \neq 0$ and not equal to negative real number.

So, $z^2+1 = (x+yi)^2+1 = (x^2-y^2+1)+i(2xy)$.

$$z^2+1=0 \text{ when } \begin{cases} x=0 \text{ and } y=\pm 1 \\ \text{or} \\ y=0 \text{ and } x^2+1=0 \text{ This is impossible for } x\in\mathbb{R} \end{cases}$$

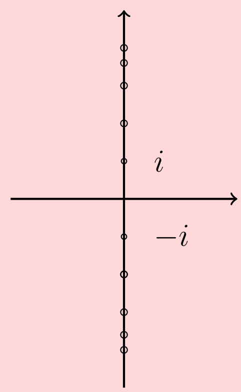
Hence, $z=\pm i$ here.

$$z^2+1<0 \text{ (real) when } \begin{cases} x=0 \text{ and } 1-y^2<0 \quad \Rightarrow y^2>1 \Rightarrow y>1 \text{ or } y<-1 \\ \text{or} \\ y=0 \text{ and } 1+x^2<0 \quad \text{Impossible} \end{cases}$$

Hence, $z=iy$ where $|y|>1$.

For all other points,

$$f'(z) = \frac{2z}{z^2+1}$$



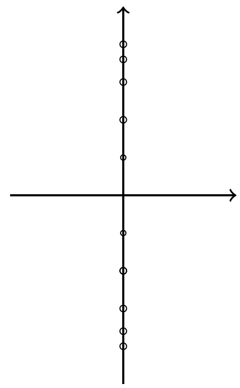
Here is another way to solve the above problem.

$$\text{Log}(z^2+1) = \text{Log}((z+i)(z-i)) = \text{Log}_{\tau_1}(z+i) + \text{Log}_{\tau_2}(z-i)$$

for some τ_1, τ_2

Some possibilities are:

- $\tau_1 = \dfrac{-\pi}{2}$, $\tau_2 = \dfrac{-3\pi}{2}$
- $\tau_1 = \dfrac{3\pi}{2}$, $\tau_2 = \dfrac{-7\pi}{2}$
- $\cdots$

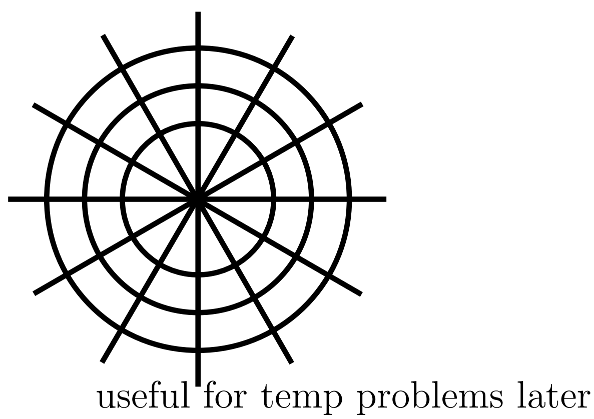


Finally, note that

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$

$\operatorname{Log} z$ is analytic, so $\ln|z|$ and $\operatorname{Arg} z$ are harmonic.

Level curves of $\ln|z| = k$ and $\operatorname{Arg} z = k$ are circles and rays. This would be particularly useful when we deal with temperature problems later.



3.4 Complex Powers and Inverse Trigonometric Functions

Definition 3.23. Complex Powers: We define

$$z^\alpha = e^{\alpha \log z} \quad \text{for } \alpha \in \mathbb{C}, z \neq 0$$

Example 3.24. 1.

$$\begin{aligned}
4^{1/2} &= e^{\frac{1}{2}\log 4} \\
&= e^{\frac{1}{2}(\ln|4|+i\arg(4))} \\
&= e^{\frac{1}{2}\ln 4+i\frac{1}{2}(0+2\pi k)} \\
&= e^{\frac{1}{2}2\ln 2+i\pi k} \\
&= e^{\ln 2}e^{i\pi k} \\
&= 2\cdot(\pm 1) \\
&= \pm 2
\end{aligned}$$

2.

$$\begin{aligned}
(1+i)^3 &= e^{3\log(1+i)} \\
&= e^{3\left(\ln\sqrt{2}+i\arg(1+i)\right)} \\
&= e^{\frac{3}{2}\ln 2}e^{i3\left(\frac{\pi}{4}+2k\pi\right)} \\
&= (e^{\ln 2})^{\frac{3}{2}}\cdot\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right) \\
&= 2^{\frac{3}{2}}\cdot\left(\frac{-1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) \\
&= -2+2i
\end{aligned}$$

3.

$$\begin{aligned}
i^i &= e^{i\ln|i|+i\arg i} \\
&= e^{i\left(0+i(\frac{\pi}{2}+2k\pi)\right)} \\
&= e^{-\left(\frac{\pi}{2}+2\pi k\right)} \\
&= \cdots, e^{\frac{-5\pi}{2}}, e^{\frac{-\pi}{2}}, e^{\frac{3\pi}{2}}, \cdots
\end{aligned}$$

If we want a single value, take the principal branch to be $e^{\alpha\operatorname{Log} z}$, which is analytic everywhere $\operatorname{Log} z$ is, and

$$\frac{d}{dz}z^\alpha = \frac{d}{dz}e^{\alpha\operatorname{Log} z} = e^{\alpha\operatorname{Log} z}\cdot\frac{\alpha}{z} = z^\alpha\cdot\frac{\alpha}{z} = \alpha z^\alpha$$

as expected.

Definition 3.25. **Inverse Trigonometric Functions**: First, we see that $w = \sin^{-1} z$ means $z = \sin w$, etc. Also, we've accepted multivalued functions.

In $\mathbb{R}$, the inverse hyperbolic function can be expressed in terms of logs:

$$\begin{aligned} y = \sinh x &= \frac{1}{2}(e^x - e^{-x}) \\ e^x - 2y - e^{-x} &= 0 \\ (e^x)^2 - 2y(e^x) - 1 &= 0 \quad \text{note that this is a quadratic equation for } e^x \\ e^x = \frac{2y \pm \sqrt{4y^2+4}}{2} &= y \pm \sqrt{y^2+1} \text{ we take the plus sign since } e^x > 0 \end{aligned}$$

So, $x = \ln(y + \sqrt{y^2+1}) = \sinh^{-1} y$.

In $\mathbb{C}$, we define $\sinh^{-1} z = \log(z + \sqrt{z^2+1})$.

Similarly, $\sin^{-1} z = -i\log(iz + (1-z^2)^{\frac{1}{2}})$. Note that for this fefinition, it involves two sets of branches, one with log, and the other one with $(1-z^2)^{\frac{1}{2}}$

Chapter 4 Complex Integration

4.1 Contours

How to integrate in $\mathbb{C}$?

Complex values functions of a real variable are easy to integrate:

$$\int_a^b \Big(u(t) + iv(t)\Big)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Example 4.1. 1.

$$\int_0^1 (t+i)^2 dt \int_0^1 \Big((t^2-1) + i(2t)\Big)dt = \frac{-2}{3} + 2i$$

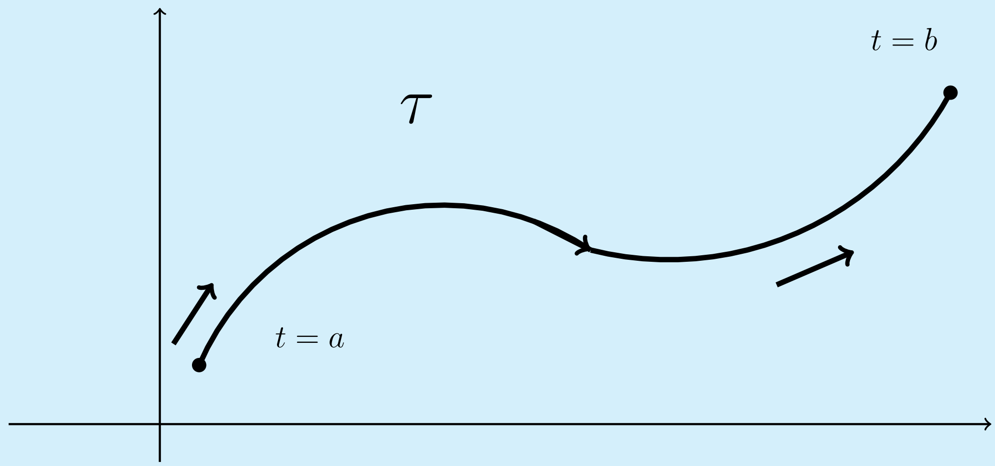
2. We can use a special trick (instead of using integration by parts twice).

$$\begin{aligned}
\int_0^\pi e^{2\pi}\cos x dx &= \int_0^\pi e^2 x(Re(e^i x))dx \\
&= Re\left(\int_0^\pi e^{(2+i)\pi}\right) \\
&= Re\left(\frac{e^{(2+i)}}{2+i}\bigg|_0^\pi\right) \\
&= Re\left(\frac{e^{2x}(\cos x + i\sin x)}{2+i}\cdot\frac{2-i}{2-i}\bigg|_0^\pi\right) \\
&= \left[\frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x\right]\bigg|_0^\pi \\
&= -\frac{2}{5}e^{-2\pi} + \frac{2}{5}
\end{aligned}$$

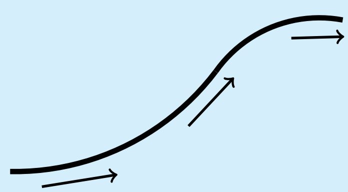
What about integrating a function of a complex variable?

We will replace the intervals with paths.

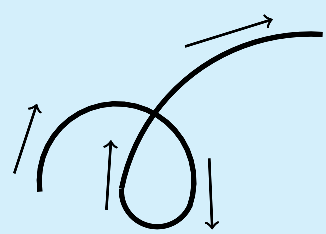
Definition 4.2. Let $z(t) = x(t) + iy(t)$ on $t \in [a,b]$ be continuous. The range is a **curve** $\mathcal{C}$, and is called a **smooth curve** if $z'(t)$ is continuous and non-zero on $[a,b]$



A curve is called **simple** if $z(t_1) \neq z(t_2)$ whenever $t_1 \neq t_2$ for $a < t_i < b$ (basically no self intersection)

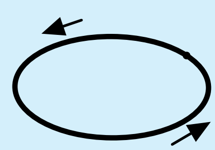


simple



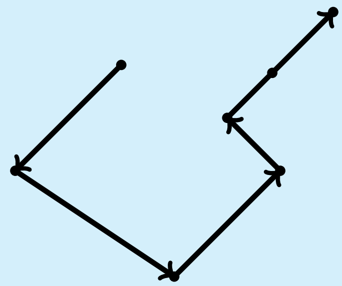
non-simple

If $z(a) = z(b)$, then the curve is called a **closed** curve.

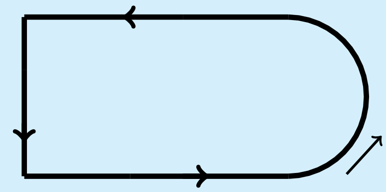


simple closed curve

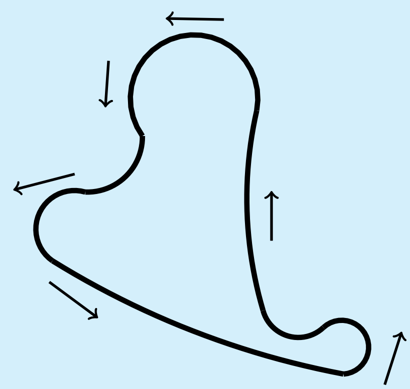
Definition 4.3. **Contour**: a curve that is composed of finitely many smooth curves, joined end-to-end



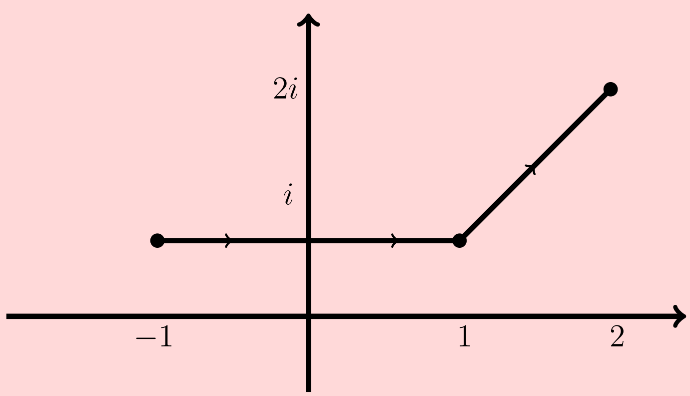
Definition 4.4. **Jordan Curve**: a simple closed contour.



Definition 4.5. **Positively Oriented**: means its interior lies to the **left** as we follow the curve



Example 4.6. Parameterize this:



Solution: Line segment from z_0 to z_1 can be parameterized as: $z(t) = z_0 + (z_1 - z_0)t, t \in [0,1]$.

For the first curve,

$$\begin{aligned} z_1(t) &= (-1+i) + (1+i-(-1+i))t \\ &= -1+i+2t, \quad t \in [0,1] \end{aligned}$$

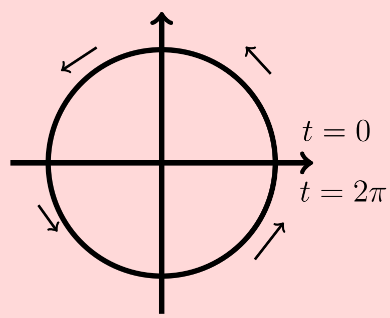
For the second curve,

$$\begin{aligned} z_2(t) &= (1+i) + (2+2i-(1+i))t \\ &= 1+i+(1+i)t, \quad t \in [0,1] \end{aligned}$$

Put everything together we get

$$z(t) = \begin{cases} -1+i+2t & t \in [0,1) \\ 1+i+(1+i)(t-1) & t \in [1,2] \end{cases}$$

Example 4.7. Let $\mathcal{C}$ be a unit circle centered at 0.



Solution: $\mathcal{C} : z(t) = e^{it} \quad t \in [0, 2\pi]$

Example 4.8. Circle, radius r_0, centered at z_0?

Solution: $\mathcal{C} : z(t) = z_0 + r_0 e^{it} \quad t \in [0, 2\pi]$

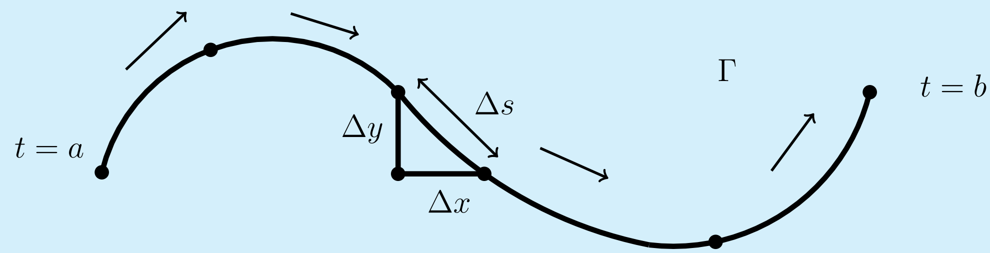
Example 4.9. Parameterize $y = f(x), x \in [a, b]$

Solution: just let $x(t) = t$,

$$z(t) = x(t) + iy(t) = t + if(t), \quad t \in [a, b]$$

For example, $y = x^2$ will be parameterized as $z(t) = t + it^2$

Definition 4.10. **Arclength**: We define the arclength as follows:



Partition the curve

$$\begin{aligned}\Delta s &\approx \sqrt{\Delta x^2 + \Delta y^2} \\ &= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\Delta t\end{aligned}$$

Sum all pieces and let $\Delta t \to 0$ (Performing a Riemann Sum there):

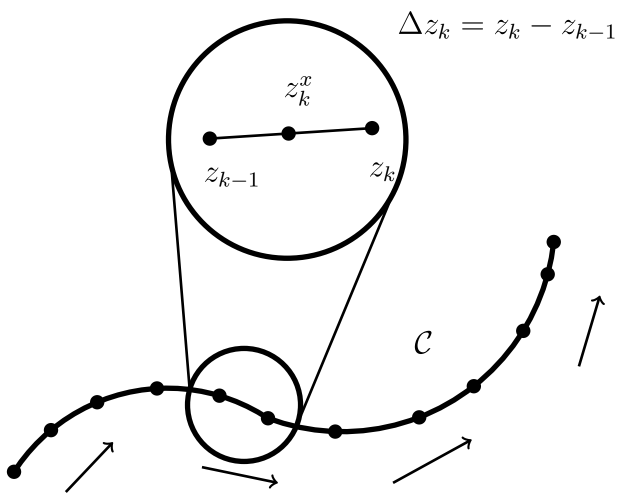
$$\begin{aligned}L = \int_R ds &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_a^b \left|\frac{dz}{dt}\right| dt \quad \text{we use modulus here}\end{aligned}$$

The physical interpretation could be: total_distance $= \int_a^b (\text{speed})\, dt$

Now we are ready to integrate $f(z)$ along a curve.

4.2 Contour Integrals

Partition curve $\mathcal{C}$ as shown.



Sum, and let $\max|\Delta z_k| \to 0$:

$$\int_{\mathcal{C}} f(z)dz = \lim_{\max|\Delta z_k|\to 0} \sum_k f(z_k^*)\Delta z_k$$

See the text for more detail.

If $\mathcal{C}$ is a single point, define $\int_{\mathcal{C}} f(z)dz = 0$.

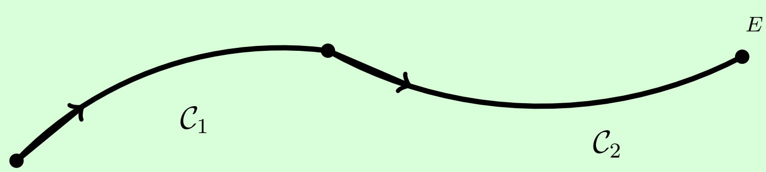
How to calculate?

Definition 4.11. Assume $\mathcal{C}$ has a parameterization. Call it $z(t), t \in [a, b]$. Then:

$$\begin{aligned}\int_{\mathcal{C}} f(z)dz &= \lim_{\max|\Delta z_k|\to 0} \sum_k f(z_k^*)\frac{\overbrace{z(t_k)}^{z_k} - \overbrace{z(t_{k-1})}^{z_{k-1}}}{\Delta t_k}\Delta t_k \\ &= \int_a^b f(z)z'(t)dt\end{aligned}$$

Proposition 4.12. Properties:

- $\int_{\mathcal{C}} \Big(f(z) + g(z)\Big)dz = \int_{\mathcal{C}} f(z)dz + \int_{\mathcal{C}} g(z)dz$
- $\int_{\mathcal{C}} kf(z)dz = k\int_{\mathcal{C}} f(z)dz$
- $\int_{-\mathcal{C}} f(z)dz = -\int_{\mathcal{C}} f(z)dz$. Here $-\mathcal{C}$ means $\mathcal{C}$ traversed in the opposite direction
- $\int_{\mathcal{C}_1+\mathcal{C}_2} f(z)dz = \int_{\mathcal{C}_1} f(z)dz + \int_{\mathcal{C}_2} f(z)dz$. Here it means that we traverse $\mathcal{C}_1$ then traverse $\mathcal{C}_2$.



Is there a triangle inequality? i.e.

$$\left|\int_{\mathcal{C}} f(z)dz\right| \leq_? \int_{\mathcal{C}} |f(z)|\, dz$$

No! LHS is real, but RHS is complex. "$\leq$" does NOT make any sense here.

Proposition 4.13. <u>**The "ML" Inequality**</u>: If $f(z)$ is continuous on a contour $\mathcal{C}$, then

$$\left|\int_{\mathcal{C}} f(z)dz\right| \leq ML$$

where M is an upper bound for $|f(z)|$ on $\mathcal{C}$ and L is the length of $\mathcal{C}$.

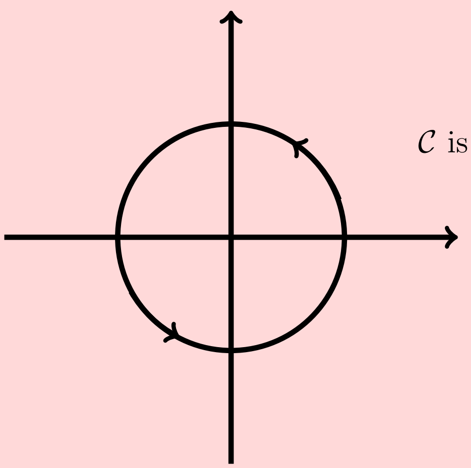
Proof 4.14. Let $z(t)$, $t \in [a,b]$ be a parameterization of $\mathcal{C}$. Then

$$\begin{aligned}
\left|\int_{\mathcal{C}} f(z)dz\right| &= \left|\int_a^b f(z(t))z'(t)dt\right| \\
&\leq \int_a^b \left|f(z(t))z'(t)\right|dt \quad \text{by triangle inequality for integrals w.s.t. real variables} \\
&= M\int_a^b \left|z'(t)\right|dt \\
&= ML
\end{aligned}$$

Second last step: since $|f(z)| \leq M$ on $\mathcal{C}$.

Last step: from last lecture. See Definition 4.10. $\square$

Example 4.15. Find an upper bound on $\left|\int_{\mathcal{C}} e^{\frac{1}{z}}\right|$



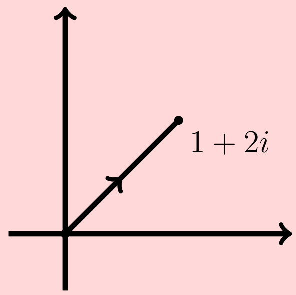
$\mathcal{C}$ is unit circle, traversed <u>once</u> in positive direction

<u>Solution</u>: $M = ?$

$$\begin{aligned}
\left|e^{\frac{1}{z}}\right| &= \left|e^{\frac{1}{x+iy}}\right| \\
&= \left|e^{\frac{x-iy}{x^2+y^2}}\right| \\
&= \left|e^{\frac{x}{x^2+y^2}} \cdot e^{-i\frac{y}{x^2+y^2}}\right| \\
&\leq e^{\frac{x}{1}} \quad \text{since } x^2+y^2=1 \\
&\leq e^1 \quad \text{since } x \leq 1
\end{aligned}$$

Clearly, $L = 2\pi$, so $\left|e^{\frac{1}{z}}\right| \leq e^1 \cdot 2\pi = 2\pi e$ by ML inequality.

Example 4.16. Evaluate $\int_{\mathcal{C}} \cos z dz$ where $\mathcal{C}$ is the line segment from 0 to 1+2i.



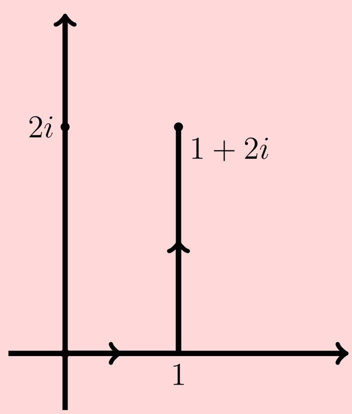
Solution: Parameterize $\mathcal{C}$ by

$$z(t) = 0 + (1 + 2i - 0)t, \quad t \in [0, 1]$$

Then

$$\int_{\mathcal{C}} \cos z dx = \int_0^1 \underbrace{\cos\Big((1+2i)t\Big)}_{f(z(t))} \cdot \underbrace{(1+2i)}_{z'(t)} dt = \sin\Big((1+2i)t\Big)\Big|_0^1 = \sin(1+2i) - 0 = \sin(1+2i)$$

Example 4.17. Evaluate $\int_{\mathcal{C}} \cos z dz$ where $\mathcal{C}$ is:

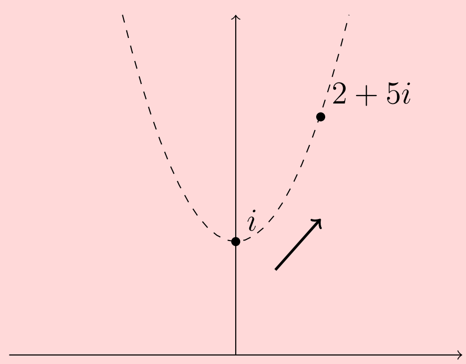


Solution: $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ where $\begin{cases} C_1: & z(t) = t, \quad t \in [0,1) \\ C_2: & z(t) = 1 + (t-1)i, \quad t \in [1,3] \end{cases}$. So

$$\begin{aligned}
\int_{\mathcal{C}} \cos z dx &= \int_{\mathcal{C}_1} \cos z dx + \int_{\mathcal{C}_2} \cos z dx \\
&= \int_0^1 \cos t dt + \int_1^3 \cos(1 + (t-1)i) i dt \\
&= \sin t|_0^1 + \sin(1 + (t-1)i)|_1^3 \\
&= \sin(1) + (\sin(1+2i) - \sin(1)) \\
&= \sin(1+2i)
\end{aligned}$$

As before

Example 4.18. Evaluate $\int_{\mathcal{C}} e^z dz$ where $\mathcal{C}$ is part of $y = x^2 + 1$ from $z = i$ to $z = 2 + 5i$.



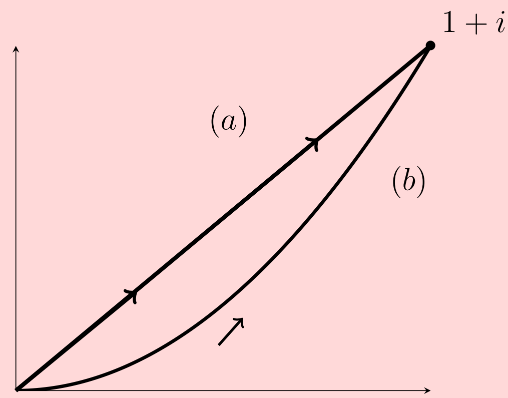
Solution: Let $z(t) = \underbrace{t}_{x} + \underbrace{(t^2+1)}_{y} i, \quad t \in [0,2]$. Then,

$$\begin{aligned}
\int_{\mathcal{C}} e^z dz &= \int_0^2 e^{z(t)} z'(t) dt \\
&= \int_0^2 e^{t^2 + (t^2+1)i} (1 + 2ti) dt \\
&= e^{t^2 + (t^2+1)i} \Big|_0^2 \\
&= e^{2+5i} - e^i \\
&= e^z \Big|_i^{2+5i}
\end{aligned}$$

Does it always work that way? See the following example

Example 4.19. Evaluate $\int_{\mathcal{C}} \overline{z} dz$ where

1. $\mathcal{C}$ is line segment from 0 to $1+i$
2. $\mathcal{C}$ is the smallest arc of circle $x^2 + (y-1)^2 = 1$ from 0 to $1+i$



<u>**Solution**</u>:

1. parameterization: $z(t) = t(1+i), \;\; t \in [0,1]$

$$\begin{aligned}\int_{\mathcal{C}} \overline{z} dz &= \int_0^1 t(1-i) \cdot (1+i) dt \\ &= \int_0^1 2t dt \\ &= 1\end{aligned}$$

2. parameterization: $z(t) = e^{it} + i, \;\; t \in [\frac{-\pi}{2}, 0]$. It's the unit circle, shifted up by 1 unit.

$$\begin{aligned}\int_{\mathcal{C}} \overline{z} dz &= \int_{-\pi/2}^0 (e^{-it} - i)(ie^{it}) dt \\ &= \cdots \\ &= 1 + i(\frac{\pi}{2} - 1) dt \\ &\neq 1\end{aligned}$$

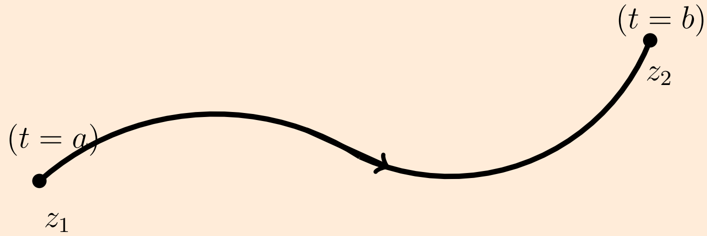
So, the general answer is no. Different paths might yield different results.

4.3 Independence of Path

Theorem 4.20. Complex Extension of Fundamental Theorem of Calculus:

If $f(z)$ is continuous in a domain D and has antiderivative $F(z)$ throughout D, then, for any contour $\mathcal{C}$ lying in D with initial point z_1 and terminal point z_2, we have

$$\int_{\mathcal{C}} f(z)dz = F(z_2) - F(z_1)$$

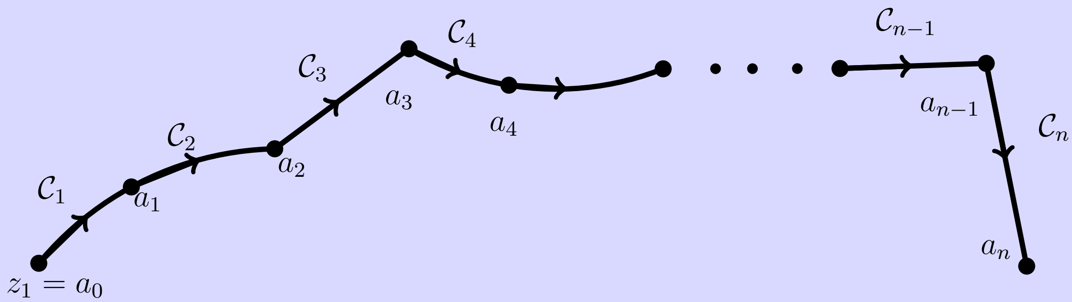


Proof 4.21. First, suppose $\mathcal{C}$ is smooth, i.e. $z'(t) \neq 0$, continuous.

Parameterize by $z(t), t \in [a, b]$. Then,

$$\begin{aligned}
\int_{\mathcal{C}} f(z)dz &= \int_a^b f(z(t))z'(t)dt \\
&= \int_{\mathcal{C}} \frac{d}{dt}\Big(F\big(z(t)\big)\Big)dt \quad \text{by chain rule} \\
&= F(z(t))\Big|_{t=a}^{b} \\
&= F(z(b)) - F(z(a)) \\
&= F(z_2) - F(z_1)
\end{aligned}$$

Next, if $\mathcal{C}$ is not smooth, it has a finite number of smooth pieces, since it's a contour.

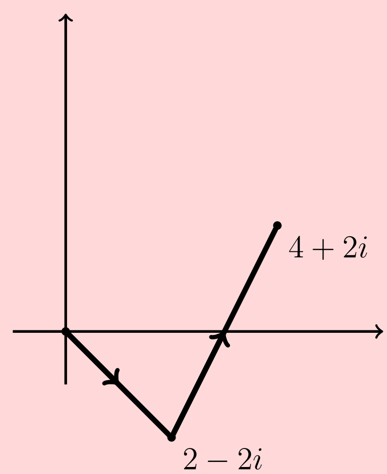


Apply the result above to each piece:

$$\begin{aligned}\int_{\mathcal{C}} f(z)dz &= \int_{\mathcal{C}_1} f(z)dz + \cdots + \int_{\mathcal{C}_n} f(z)dz \\ &= \Big(F(a_1) - F(a_0)\Big) + \Big(F(a_2) - F(a_1)\Big) + \cdots + \Big(F(a_n) - F(a_{n-1})\Big) \\ &= F(a_n) - F(a_0) \\ &= F(z_2) - F(z_1)\end{aligned}$$

□

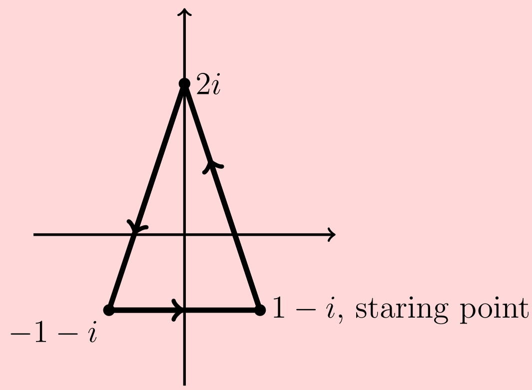
Example 4.22. Evaluate $\int_{\mathcal{C}}(1+z^2)dz$ where $\mathcal{C}$ is:



Solution:

$$\begin{aligned}\int_{\mathcal{C}}(1+z^2)dz &= \left(z + \frac{z^3}{4}\right)\Bigg|_{z=0}^{z=4+2i} \\ &= \cdots \\ &= \frac{28}{3} + \frac{94}{3}i\end{aligned}$$

Example 4.23. Evaluate $\int_{\mathcal{C}} e^z dz$ where $\mathcal{C}$ is:



Solution:

$$\begin{aligned}\int_{\mathcal{C}}(1+z^2)dz &= e^z\Big|_{z=1-i}^{z=1-i}\\ &= e^{1-i}-e^{1-i}\\ &= 0\end{aligned}$$

Theorem 4.24. Let f be a continuous function in a domain D. Then, the following statements are equivalent:

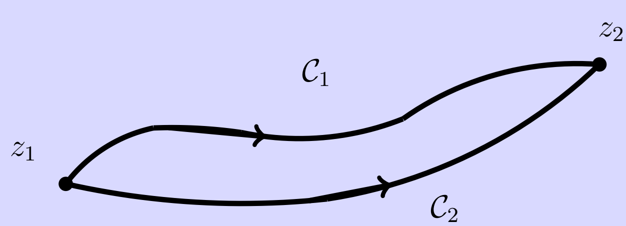
1. f has an antiderivative in D.
2. If $\mathcal{C}$ is any closed contour in D, then $\int_{\mathcal{C}} f(z)dz = 0$.
3. The contour integrals of f are independent of path in D.

Proof 4.25. $1 \Rightarrow 2$: It follows immediately from Theorem 4.20 with $\mathcal{C}$ being a closed contour.

$2 \Rightarrow 3$: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be any two contours in D with same end points. Let $\mathcal{C}$ be the closed contour $\mathcal{C}_1 + (-\mathcal{C}_2)$.

Then, $\int_{\mathcal{C}} f(z)dz = 0$. So $\int_{\mathcal{C}_1} f(z)dz + \int_{-\mathcal{C}_2} f(z)dz = 0$. So $\int_{\mathcal{C}_1} f(z)dz - \int_{\mathcal{C}_2} f(z)dz = 0$, implying that

$$\int_{\mathcal{C}_1} f(z)dz = \int_{\mathcal{C}_2} f(z)dz$$



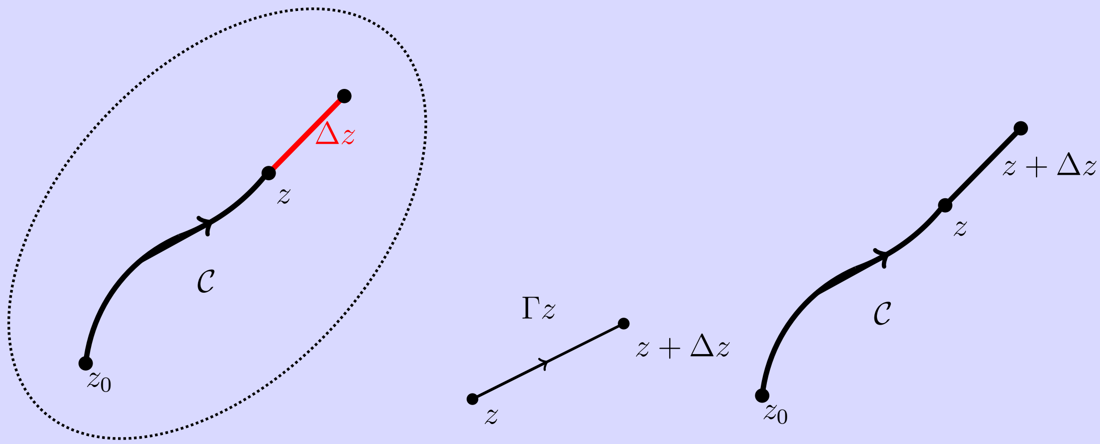
$3 \Rightarrow 1$: Construct the antiderivative. Choose a point $z_0 \in D$, and let $\mathcal{C}$ be the contour as shown. Recall the D is a connected set.

Define $F(z) = \int_{\mathcal{C}} f(w)dw$. By 3, $F(z)$ is single valued; We will show that $F'(z) = f(z)$.

For any point z, choose Δz small enough such that the line segment Γ parameterized by

$$z(t) = z + t\Delta z, \quad t \in [0, 1]$$

is in D (This is possible since D is open)



Then

$$\begin{aligned}
F(z+\Delta z) - F(z) &= \left(\int_{\mathcal{C}} f(w)dw + \int_{\Gamma} f(w)dw\right) - \int_{\mathcal{C}} f(w)dw \\
&= \int_{\Gamma} f(w)dw \\
&= \int_0^1 f(z(t))z'(t)dt \\
&= \int_0^1 f(z+t\Delta z)(\Delta z)dt \\
\Rightarrow \quad \frac{F(z+\Delta z) - F(z)}{\Delta z} &= \int_0^1 f(z+t\Delta z)dt
\end{aligned}$$

Let $\Delta z \to 0$.

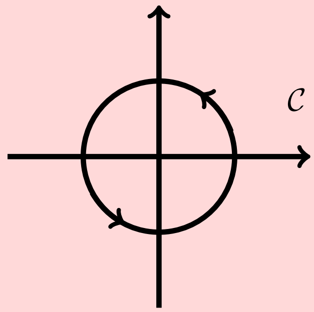
$$F'(z) = \int_0^1 f(z)dt = f(z)\int_0^1 dt = f(z)$$

□

We showed that $\overline{z}$ can be integreated, but the result depends on path. So $\overline{z}$ is integrable, but not anti-differentiable. Also, functions with antiderivatives are easy; for those without, we must parameterize.

4.4 Cauchy's Integral Theorem

Example 4.26. Most Important Example in this Course: Evaluate $\int_{\mathcal{C}} \frac{1}{z} dz$ where $\mathcal{C}$ is the unit circle.

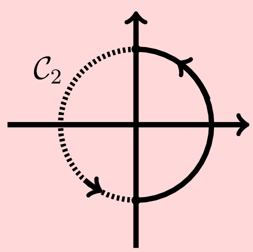


<u>Solution</u>: $\frac{1}{z}$ does not have antiderivative over all of $\mathcal{C}$. Any branch of $\log z$ will have a problem, i.e. $\mathcal{C}$ will cross a branch cut.

Method 1: Parameterize $\mathcal{C}$ by $e^{it}, t \in [0, 2\pi]$. By definition,

$$\int_{\mathcal{C}} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it} dt = 2\pi i$$

Method 2: Split $\mathcal{C}$ in two, and use Theorem 4.20 on each.



$$\begin{aligned}
\int_{\mathcal{C}_1} \frac{1}{z} dz &= \operatorname{Log} z \Big|_{-i}^{i} \quad \text{branch cut at } \theta = -\pi \\
&= \operatorname{Log} i - \operatorname{Log}(-i) \\
&= i\frac{\pi}{2} - i\left(\frac{-\pi}{2}\right) \\
&= \pi i \\
\int_{\mathcal{C}_2} \frac{1}{z} dz &= \operatorname{Log}_0 z \Big|_{i}^{-i} \\
&= \operatorname{Log}_0(-i) - \operatorname{Log}_0(i) \\
&= \frac{3\pi}{2} i - \frac{\pi}{2} i \\
&= \pi i
\end{aligned}$$

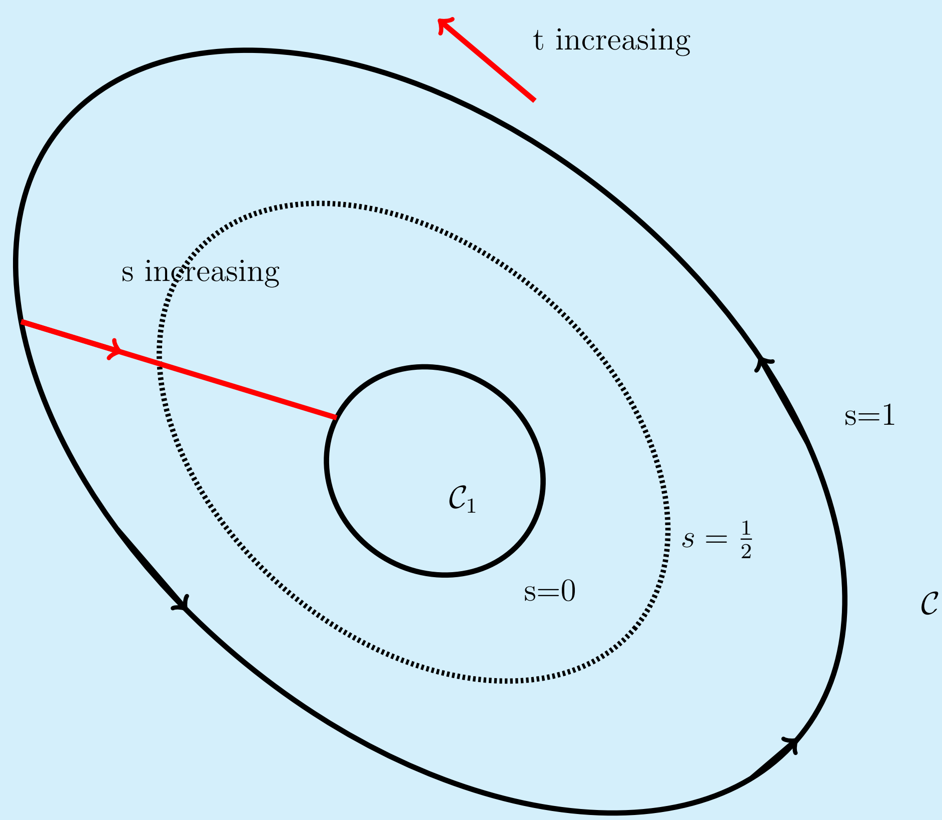
Therefore,

$$\int_{\mathcal{C}} \frac{1}{z} dz = \int_{\mathcal{C}_1} \frac{1}{z} dz + \int_{\mathcal{C}_2} \frac{1}{z} dz = \pi i + \pi i = 2\pi i$$

Go around the contour twice, what's the result? It would be $4\pi i$. Also, going counter-clockwise would yield the result $-2\pi i$

Definition 4.27. A closed contour $\mathcal{C}$ is said to be <u>**continuously deformable**</u> to a contour $\mathcal{C}_1$ in a domain D if there exists a function $z(s,t)$, continuous for $s \in [0,1], t \in [0,1]$, such that

1. $z(s,t)$ is a closed contour in D for each $s \in [0,1]$
2. $z(0,t)$ is a parameterization of $\mathcal{C}$
3. $z(1,t)$ is a parameterization of $\mathcal{C}_1$



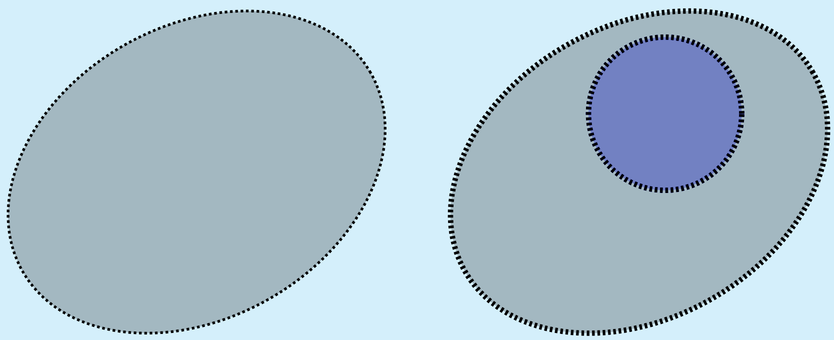
Theorem 4.28. <u>**Deformation Invariance Theorem**</u>: Let f be analytic in a domain D,

containing closed contours $\mathcal{C}_1$ and $\mathcal{C}_2$. If $\mathcal{C}_1$ can be continuously deformed into $\mathcal{C}_2$, then

$$\int_{\mathcal{C}_1} f(z)dz = \int_{\mathcal{C}_2} f(z)dz$$

Proof 4.29. It's too hard - 12 pages long in one text. □

Definition 4.30. A **simply connected domain** is a domain in which every "loop" (closed contour) in D can be continuously deformed to a point (while remaining in D).



simply connected Not simply connected

Theorem 4.31. **Cauchy's Integral Theorem (Cauchy-Goursat Theorem)**:
If f is analytic in a simply connected domain D, and $\mathcal{C}$ is a closed contour in D, then

$$\int_{\mathcal{C}} f(z)dz = 0$$

Proof 4.32. Follows from Theorem 4.28 by shrinking $\mathcal{C}$ continuously to a point. □

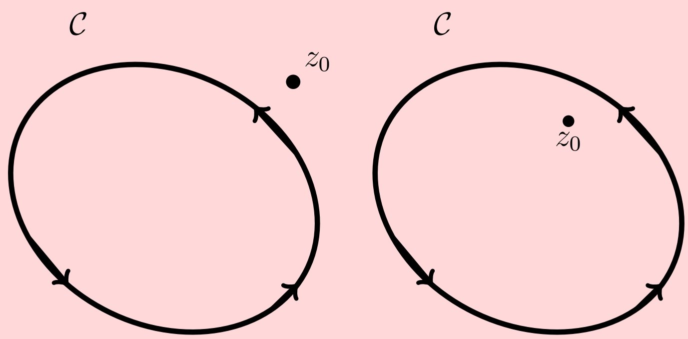
Corollary 4.33. Since $\int_{\mathcal{C}} f(z)dz = 0 \Leftrightarrow f$ has an antiderivative in D, we have that if f is analytic, then f also has an antiderivative, which is analytic. So every analytic function is infinitely antidifferentiable.

Example 4.34. Back to Example 4.26. We know that $\int_{\mathcal{C}} \frac{1}{z}dz = 2\pi i$ for any closed contour enclosing the origin.

Also, $\int_{\mathcal{C}} \frac{1}{z} dz = 0$ for any closed contours not enclosing the origin.

Could shift results:

$$\int_{\mathcal{C}} \frac{1}{z - z_0} dz = \begin{cases} 0 & \text{if } z_0 \text{ is exterior to } \mathcal{C} \\ 2\pi i & \text{if } z_0 \text{ is interior to } \mathcal{C} \end{cases}$$



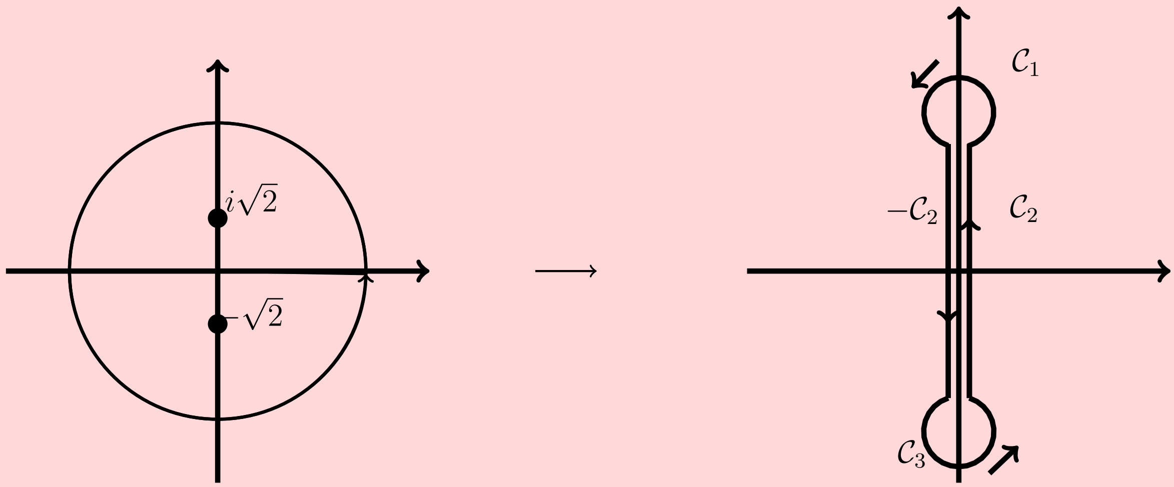
Example 4.35. Evaluate $\int_{\mathcal{C}} \frac{2z}{z^2+2} dz$ where $\mathcal{C}$ is the positively oriented circle of radius 2 centered at origin.

Solution: We can do partial fractions:

$$\frac{2z}{z^2+2} = \frac{1}{z + i\sqrt{2}} + \frac{1}{z - i\sqrt{2}}$$

So we have singularities at $z = \pm i\sqrt{2}$. We can use the **Deformation Invariance Theorem** to deform $\mathcal{C}$ like below. So

$$\begin{aligned} \int_{\mathcal{C}} &= \int_{\mathcal{C}_2} + \int_{\mathcal{C}_1} + \int_{-\mathcal{C}_2} + \int_{\mathcal{C}_3} \\ &= \int_{\mathcal{C}_1} + \int_{\mathcal{C}_3} \end{aligned}$$



And

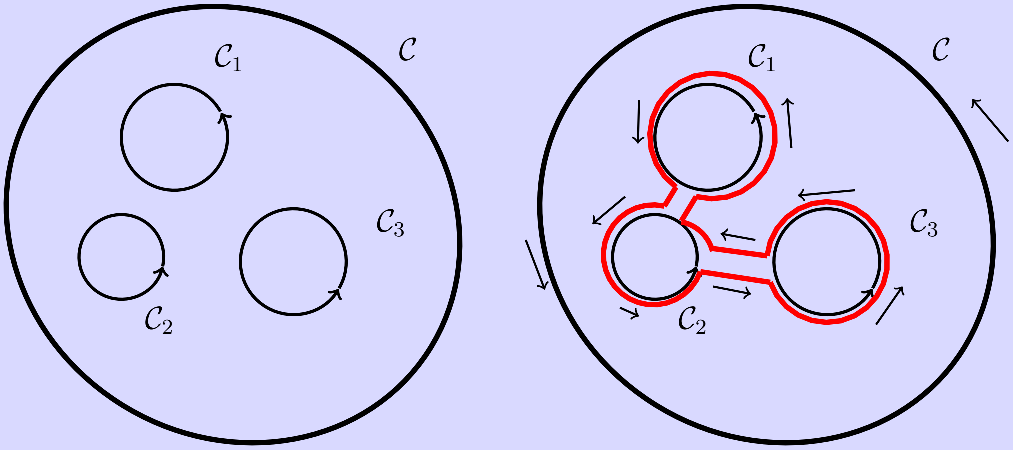
$$\begin{aligned}
\int_{\mathcal{C}} \frac{2z}{z^2+2} dz &= \int_{\mathcal{C}} \left(\frac{1}{z+i\sqrt{2}} + \frac{1}{z-i\sqrt{2}} \right) dz \\
&= \int_{\mathcal{C}_1} \left(\frac{1}{z+i\sqrt{2}} + \frac{1}{z-i\sqrt{2}} \right) dz + \int_{\mathcal{C}_3} \left(\frac{1}{z+i\sqrt{2}} + \frac{1}{z-i\sqrt{2}} \right) dz \\
&= \int_{\mathcal{C}_1} \frac{1}{z+i\sqrt{2}} dz + \int_{\mathcal{C}_1} \frac{1}{z-i\sqrt{2}} dz + \int_{\mathcal{C}_3} \frac{1}{z+i\sqrt{2}} dz + \int_{\mathcal{C}_3} \frac{1}{z-i\sqrt{2}} dz \\
&= 0 + 2\pi i + 2\pi i + 0 \\
&= 4\pi i
\end{aligned}$$

Theorem 4.36. **Extended Cauchy-Goursat Theorem**:

$$\int_{\mathcal{C}} f(z) dz = \sum_{i=1}^{n} \int_{\mathcal{C}_i} f(z) dz$$

Proof 4.37. Ideas (for the case of $n = 3$): Deform $\mathcal{C}$ to Γ as shown:

$$\int_{\mathcal{C}} = \int_{\tilde{\mathcal{C}}} = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3}$$



□

What about $\int_{\mathcal{C}} \frac{1}{(z-z_0)^2} dz$ or other powers of $z - z_0$?

Consider $\int_{\mathcal{C}} (z-z_0)^n dz$ where $n \neq -1$.

- If z_0 is external to $\mathcal{C}$, the integral is zero, by **Cauchy's Integral Theorem**.
- If z_0 is internal to $\mathcal{C}$, deform $\mathcal{C}$ to the unit circle $|z - z_0| = 1$, parameterized by $z = z_0 + e^{it}$, $t \in [0, 2\pi]$. We may use the radius ϵ if the circle is not small enough. The result would be the same.

  Then

$$\begin{aligned}\int_{\mathcal{C}} (z-z_0)^n dz &= \int_{\mathcal{C}} \left(e^{it}\right)^n ie^{it} dt \\ &= \frac{i}{n+1} e^{i(n+1)t}\Big|_0^{2\pi} \\ &= 0\end{aligned}$$

Thus for an interioir point z_0 in $\mathcal{C}$

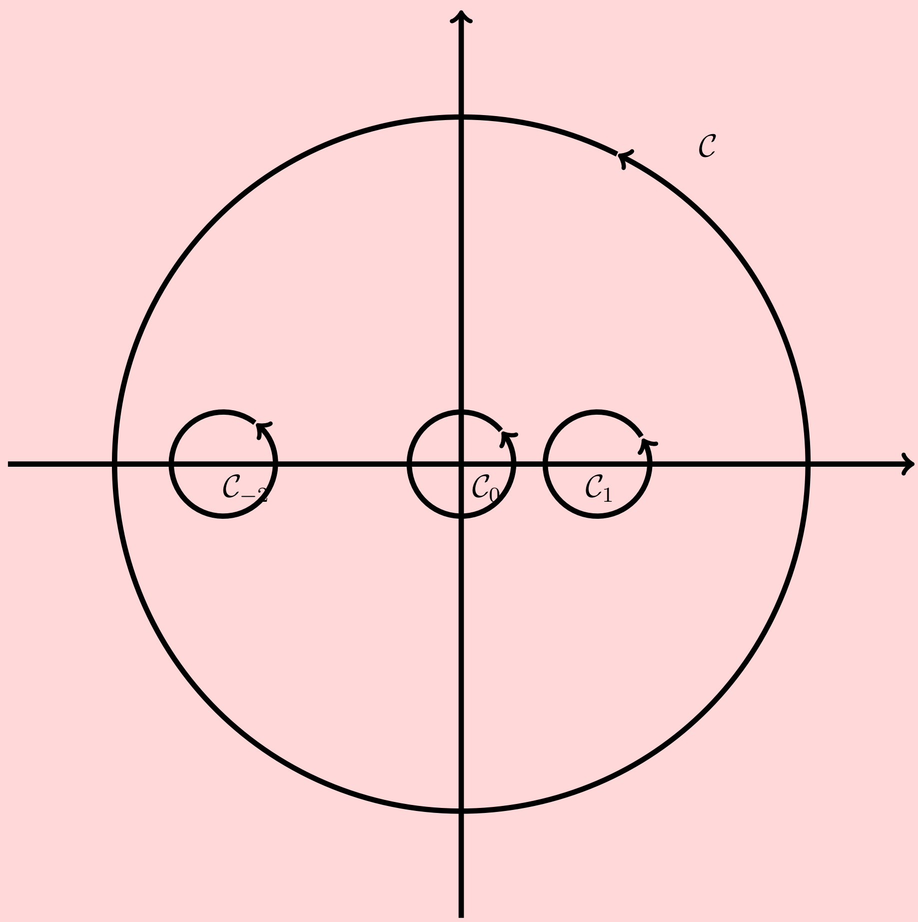
$$\boxed{\int_{\mathcal{C}} (z-z_0)^n dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}}$$

Example 4.38. Let $\mathcal{C}$ be the positively oriented circle of radius 3 centered at the origin. Evaluate $\int_{\mathcal{C}} \frac{3z^3+2}{z^4+z^3-2z^2} dz$

Solution: Note that $z^4 + z^3 - 2z^2 = z^2(z^2 + z - 2) = z^2(z-1)(z+2)$. These give us the location of the singularities.

Note the partial fractions:

$$\frac{3z^3+2}{z^4+z^3-2z^2} = \frac{-1/2}{z} + \frac{-1}{z} + \frac{5/3}{z-1} + \frac{11/6}{z+2}$$



By the **Extended Cauchy-Goursat Theorem**,

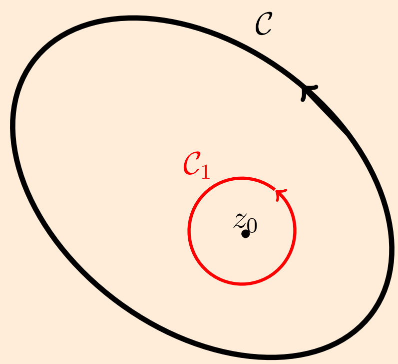
$$\begin{aligned}\int_{\mathcal{C}} f(z)dz &= \int_{\mathcal{C}_{-2}} f(z)dz + \int_{\mathcal{C}_0} f(z)dz + \int_{\mathcal{C}_1} f(z)dz \\ &= \frac{11}{6}\cdot(2\pi i) + \frac{-1}{2}\cdot(2\pi i) + \frac{5}{3}\cdot(2\pi i) \\ &= 6\pi i\end{aligned}$$

4.5 Cauchy's Integral Formula

Theorem 4.39. **Cauchy's Integral Formula (CIF)**: Let $\mathcal{C}$ be a simple, closed, positively-oriented contour. If f is analytic in some simply connected domain D containing

$\mathcal{C}$, and z_0 is any point inside $\mathcal{C}$. Then,

$$\int_{\mathcal{C}} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$



Proof 4.40. deform $\mathcal{C}$ to $\mathcal{C}_r$, a positively oriented circle of radius r centered at z_0: $|z - z_0| = r$. We will let $r \to 0$.

Then,

$$\begin{aligned}
\int_{\mathcal{C}} \frac{f(z)}{z - z_0} dz &= \int_{\mathcal{C}_r} \frac{f(z)}{z - z_0} dz \\
&= \int_{\mathcal{C}_r} \frac{f(z) - f(z_0)}{z - z_0} dz + \int_{\mathcal{C}_r} \frac{f(z_0)}{z - z_0} dz \quad \text{by linearity} \\
&= 0 + 2\pi i f(z_0) \\
&= 2\pi i f(z_0)
\end{aligned}$$

To show $\int_{\mathcal{C}_r} \dfrac{f(z) - f(z_0)}{z - z_0} dz = 0$, we consider the following:

On $\mathcal{C}_r$, we have $|f(z) - f(z_0)| \leq M$ for some M. Then

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq \frac{M}{r} \quad \text{since } |z - z_0| = r \text{ on } C_r$$

By ML inequality,

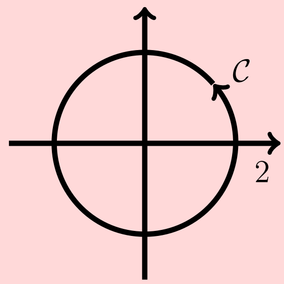
$$\left| \int_{\mathcal{C}_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \frac{M}{r} \cdot \text{length}(\mathcal{C}_r) = \frac{M}{r} 2\pi r = 2\pi M$$

Let $r \to 0$, then $M \to 0$ by continuity of f, and so

$$\int_{\mathcal{C}_r} \frac{f(z) - f(z_0)}{z - z_0} dz = 0$$

$\square$

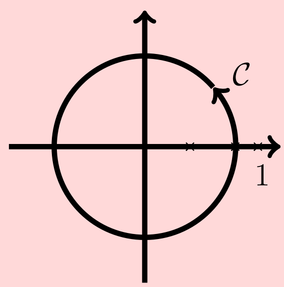
Example 4.41. Evaluate $\int_{\mathcal{C}} \frac{e^z}{z-1}dz$.



Solution: Let $f(z) = e^z$. Since $f(z)$ is entire, and $z_0 = 1$ is inside $\mathcal{C}$, we have, by **CIF**,

$$\int_{\mathcal{C}} \frac{e^z}{z-1}dz = 2\pi i f(1) = 2\pi i e^1 = 2\pi e i$$

Example 4.42. Evaluate $\int_{\mathcal{C}} \frac{e^{i\pi z}}{2z^2-5z+2}dz$.

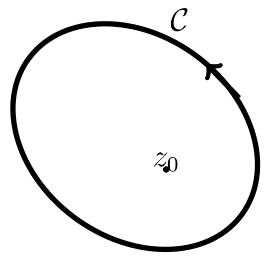


Solution:

$$\begin{aligned}
\int_{\mathcal{C}} \frac{e^{i\pi z}}{2z^2-5z+2}dz &= \int_{\mathcal{C}} \frac{e^{i\pi z}}{2(z-\frac{1}{2})(z-2)}dz \\
&= \int_{\mathcal{C}} \frac{\frac{e^{i\pi z}}{2(z-2)}}{z-\frac{1}{2}}dz \quad \text{regard the numerator as } f(z) \\
&= 2\pi i f(\frac{1}{2}) \quad \text{by CIF} \\
&= 2\pi i \frac{e^{i\pi/2}}{2(\frac{-3}{2})} \\
&= \frac{2\pi}{3}
\end{aligned}$$

From CIF, we know

$$f(z_0) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{z - z_0} dz$$



So the value of f at any point <u>inside</u> $\mathcal{C}$ is determined by the values of f <u>on</u> $\mathcal{C}$

Proposition 4.43. **<u>Mean Value Property</u>**: If $\mathcal{C}$ is a circle of radius R centered at z_0:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f\left(\overbrace{z_0 + Re^{it}}^{z(t)}\right)}{z_0 + Re^{it} - z_0}\overbrace{\left(iRe^{it}\right)}^{z'(t)} \quad \text{by parameterizing circle} \\ &= \frac{\int_0^{2\pi} f\left(z_0 + Re^{it}\right)}{2\pi - 0} \\ &= \text{ averge value of } f \text{ on the circle, recall that } \frac{\int_a^b f(x)dx}{b-a} = \overline{f} \end{aligned}$$

Theorem 4.44. **<u>Derivatives of f</u>**:

$$f(z_0) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(w)}{w - z} dw$$

Differentiate:

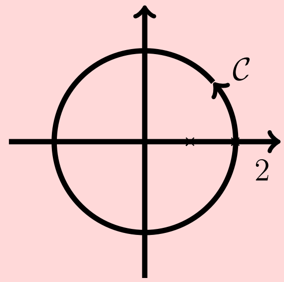
$$\begin{aligned} f'(z) &= \frac{1}{2\pi i}\int_{\mathcal{C}} f(w)\frac{d}{dz}\left(\frac{1}{w-z}\right) dw \quad \text{by Leibniz's rule} \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(w)}{(w-z)^2} dw \end{aligned}$$

which is also differentiable.

Repeating and switch back to z_0 we get **<u>Cauchy's Integral Formula for Derivatives (CIFD)</u>**:

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad \text{where } z_0 \text{ is inside } \mathcal{C}}$$

Example 4.45. Evaluate $\int_{\mathcal{C}} \frac{z^3+2z+1}{(z-1)^3}dz$.



<u>**Solution**</u>: Use **CIFD** with $n = 2$.

$$f''(z_0) = \frac{2!}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^3}dz$$

Let $f(z) = z^3 + 2z + 1$ and $z_0 = 1$. Then we have

$$\begin{aligned}(6z+0+0)\Big|_{z=1} &= \frac{1}{\pi i}\int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^3}dz \\ 6\pi i &= \int_{\mathcal{C}} \frac{z^3+2z+1}{(z-1)^3}dz\end{aligned}$$

4.6 Implication of CIFD

Corollary 4.46. An analytic function is infinitely differentiable. Furthermore, with $f(z) = u(x,y) + iv(x,y)$, u and $v \in C^\infty$ (i.e. have continuous partials of all order)

Proof 4.47. $f = u + iv$, then

$$f' = \begin{cases} u_x + iv_x & \Rightarrow f'' = \begin{cases} u_{xx} + iv_{xx} & \cdots \\ v_{xy} - iu_{xy} & \cdots \end{cases} \\ v_y - iu_y & \Rightarrow f'' = \begin{cases} v_{yx} - iu_{yx} & \cdots \\ -u_{yy} - iv_{yy} & \cdots \end{cases} \end{cases}$$

Existence of f'' implies u_x, u_y, v_x, v_y are all continous. Also, observe that $u_{xx} = -u_{yy}$, $v_{xx} = -v_{yy}$, $v_{xy} = v_{yx}$, $u_{xy} = u_{yx}$ $\square$

Theorem 4.48. Morera's Theorem: (the converse of **Cauchy's Integral Theorem**)

Let f be a continuous function in a simply connected domain D. If $\int_{\mathcal{C}} f(z)dz = 0$ for every closed contour $\mathcal{C}$ in D, then f is analytic in D.

Proof 4.49. We've shown that $\int_{\mathcal{C}} f(z)dz = 0$ for all $\mathcal{C}$ implies that f has antiderivative in D, call it $F(z)$.

Now D is open, and F is differentiable in D ($F' = f$), so therefore F is analytic, therefore $F' = f$ is analytic. $\square$

Lemma 4.50. "Cauchy's Estimate": Let f be analytic on and inside a circle $\mathcal{C}$ of radius R centered at z_0.

If $|f(z)| \leq M$ for all z on $\mathcal{C}$, then $\left|f^{(n)}(z_0)\right| \leq \dfrac{n!M}{R^n}$.

Proof 4.51. From **CIFD**,

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \leq \left|\frac{n!}{2\pi i}\right| \overbrace{\left(\frac{M}{R^{n+1}}\right)}^{\text{“}M\text{''}} \cdot \overbrace{(2\pi R)}^{\text{“}\ell\text{''}}$$

since $|z - z_0| = R$ and the $M\ell$-inequality. $\square$

Theorem 4.52. Liouville's Theorem: If f is entire, and bounded for all $z \in \mathbb{C}$, then f is constant.

Proof 4.53. Have $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Consider $z_0 \in \mathbb{C}$, and let $\mathcal{C}$ be circle of radius R centered at z_0. Cauchy's estimate yields $|f'(z_0)| \leq \dfrac{M}{R}$. True for all R, no matter how large. So $|f'(z_0)| = 0 \Rightarrow f'(z_0) = 0$.

z_0 is arbitrary, so f must be constant. $\square$

Corollary 4.54. Every non-constant, entire function is unbounded.

We can use this to prove the **Fundamental Theorem of Algebra**.

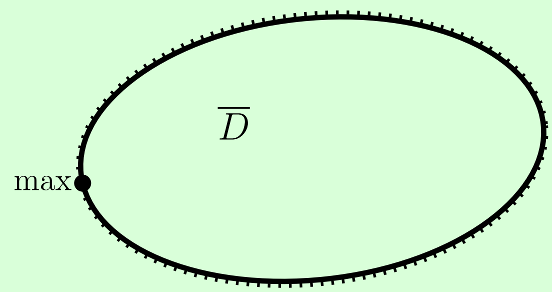
Theorem 4.55. **Fundamental Theorem of Algebra**: Every nonconstant polynomial with complex coefficients has at least one zero.

Proof 4.56. If $P(z)$ has no zeros, then $\dfrac{1}{P(z)}$ is entire. Since it is continuous, we must have $|P(z)| \geq \epsilon$ for some $\epsilon > 0$.

So, $\dfrac{1}{|P(z_0)|} \leq \dfrac{1}{\epsilon}$, implying that $\dfrac{1}{P(z_0)}$ is constant, by **Liouville's Theorem**.

So, $P(z_0)$ is constant. Hence, a non-constant polynomial must have a zero. $\square$

Proposition 4.57. **Maximum Modulus Principle**: If $f(z)$ is analytic on a bounded domain D, and continuous on $\overline{D}$, the closure of D. Then, $|f(z)|$ attains a maximum value on $\overline{D}$ and it occurs on the boundary.



Chapter 5 Series Representation for Analytic Functions

5.1 Sequences and Series

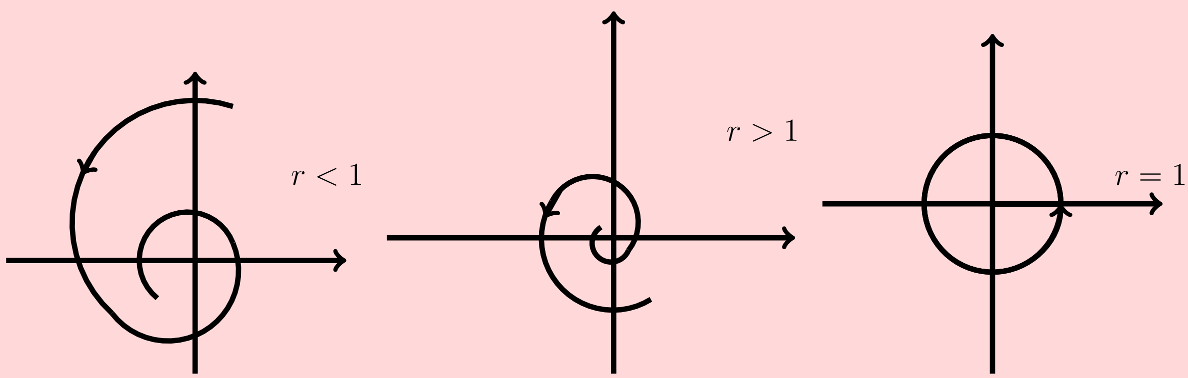
Definition 5.1. A sequence $\{z_n\}_{n=1}^{\infty}$ **converges** to z_0 if for any $\epsilon > 0$, there exists an integer N such that $n > N \Rightarrow |z_n - z_0| < \epsilon$.

Theorem 5.2. Let $z_n = x_n + iy_n$ for $n = 1, 2, ...,$ and $z_0 = x_0 + iy_0$. Then, $z_n \to z_0$ if and only if $x_n \to x_0$ and $y_n \to y_0$ as $n \to \infty$.

Example 5.3. Consider $\{z^n\}_{n=1}^{\infty}$.

Notice that $z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} \to 0$ as $n \to \infty$ if and only if $r < 1$.

In other words, $\{z^n\}_{n=1}^{\infty}$ converges to 0 as $n \to \infty$ if and only if $|z| < 1$.



Definition 5.4. Series:

$$\sum_{n=1}^{\infty} z_n = \lim_{k \to \infty} \sum_{n=1}^{k} z_n$$

is **convergent** if the limit exists (called the **sum** of the series); otherwise is **divergent**.

Note that the LHS does not need to start at $n = 1$. The RHS is just the partial sum.

Proposition 5.5. Divergence Test/nth-term Test:

If $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

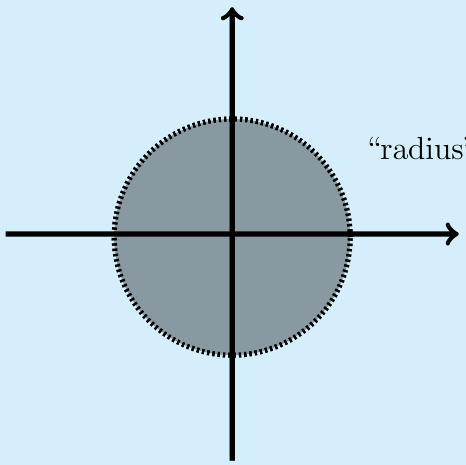
Contrapositive: if $z_n \not\to 0$, then $\sum_{n=1}^{\infty} z_n$ diverges.

Definition 5.6. **Geometric Series**:

$$\sum_{n=0}^{\infty} z_n = \begin{cases} \dfrac{1}{1-z} & \text{if } |z| < 1 \\ \text{divergent} & \text{if } |z| \geq 1 \end{cases}$$

We can see that

- in $\mathbb{R}$, $\sum x^n$ converges for $|x| < 1$.
- in $\mathbb{C}$, $\sum z^n$ converges for $|z| < 1$.



Example 5.7. $\sum_{n=0}^{\infty} \left(\frac{1}{2} + i\right)^n$ and $\sum_{n=0}^{\infty} i^n$ divergent.

$\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n$ is convergent, and equals $\dfrac{1}{1-\frac{i}{2}} = \dfrac{2}{2-i} = \dfrac{4+2i}{3}$

Proposition 5.8. **Comparison Test**: If $\sum_{k=1}^{\infty} M_k$ is a convergent series of real numbers and $|z_k| \leq M_k$ for all sufficiently large k, then $\sum_{k=1}^{\infty} z_k$ converges.

Definition 5.9. $\sum z_n$ is absolutely convergent if $\sum |z_n|$ converges.

Proposition 5.10. **Ratio Test**: If $\lim_{n\to\infty} \left|\dfrac{z_{n+1}}{z_n}\right| = L$, then $\sum_{n=0}^{\infty} z_n$ is absolutely convergent if $L < 1$, and divergent if $L > 1$. No conclusion if $L = 1$.

Example 5.11.

$$\sum_{n=0}^{\infty} \frac{(1-i)^n}{n!} = 1 + \frac{1-i}{1!} + \frac{(1-i)^2}{2!} + \cdots \quad [0! = 1]$$

Ratio is

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(1-i)^{n+1}}{(n+1)n!} \cdot \frac{n!}{(1-i)^n}\right| = \frac{|1-i|}{n+1} = \frac{\sqrt{2}}{n+1} \to 0$$

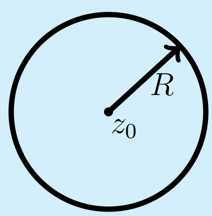
as $n \to \infty$, so the series converges.

Definition 5.12. **Power Series**:

$$\sum_{k=0}^{\infty} c_k (z-z_0)^k$$

This series could

1. Converge only at $z = z_0$
2. Converge for all z; or
3. Converge for all z such that $|z-z_0| < R$, and diverge for all z such that $|z-z_0| > R$.

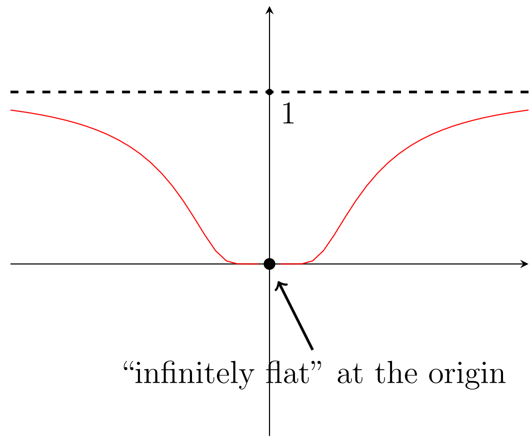


Definition 5.13. The **Taylor Series** of $f(z)$ centered at z_0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

Convergence of Taylor Series: In MATH138, we assumed that $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ converges to $f(x)$ for each x in interval for convenience. This is not always true.

Consider the following famous examle.



$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \Rightarrow \quad f(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}(x-0)^n = \cdots = 0$$

which converges for all x, but $f(x) = g(x)$ at only one point.

Good news is that in $\mathbb{C}$, things make more sense

5.2 Taylor Series and Convergence

Theorem 5.14. If f is analytic in the disk $|z - z_0| < R$, then its Taylor series converges to $f(z)$ for all z in this disc. The convergence is uniform on any closed subdisc $|z-z_0| \leq R_0 < R$.

In $\mathbb{R}$, "analytic" means "has a power series representation".

What's wrong with $g(x)$ above?

- In $\mathbb{R}$, $g^{(n)}(0) = 0$ for all n
- In $\mathbb{C}$, $g^{(n)}(0) = \begin{cases} e^{-1/z^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ is not even **continuous** at $z = 0$.

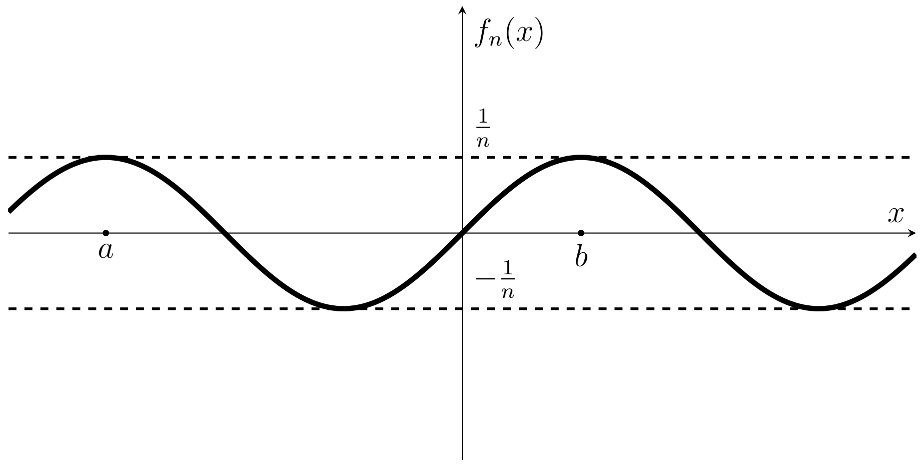
In $\mathbb{R}$, we only saw a cross-section, the "bad behaviour" was missed.

Uniform Convergence: Consider the following sequence of function:

1. $f_n(x) = \dfrac{\sin x}{n}$.

   What happens as $n \to \infty$? $f_n(x) \to 0$ as $n \to \infty$. $f_n \to 0$ "pointwise"

   Over some interval:



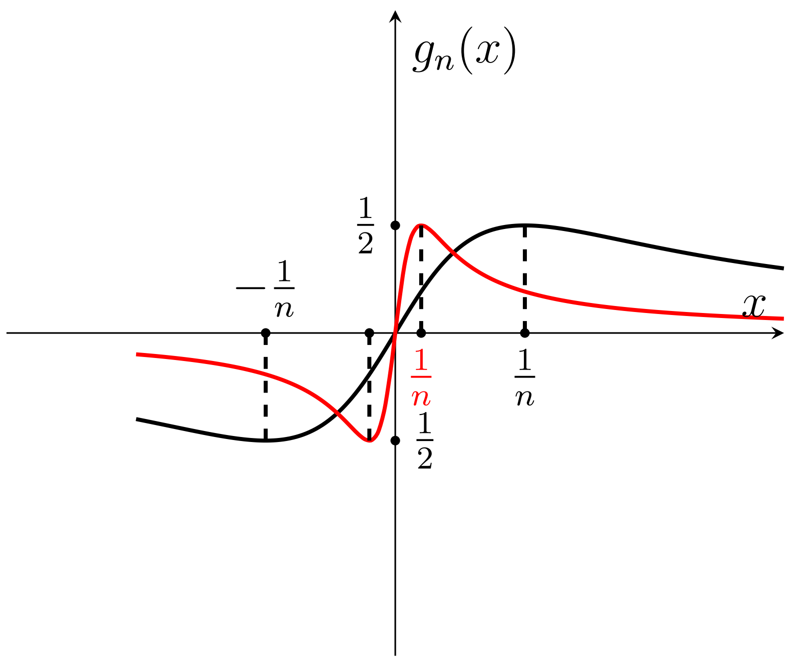
$f_n(x)$ approaches zero function on $[a, b]$.

In this case, we say "$f_n \to 0$ uniformly"

2. $g_n(x) = \dfrac{nx}{1 + n^2x^2}$

   What happens as $n \to \infty$? $g_n(x) \to 0$ as $n \to \infty$. $g_n \to 0$ "pointwise"

   Over some interval:



$g_n(x)$ still has "spikes" even as $n \to \infty$.

In this case, we say "$g_n \to 0$ does not converge uniformly to 0"

For series, we apply the above ideas to the sequence of partial sums: $S_n(x) = \sum_{k=0}^{n} f_k(x)$. AMATH231 covers more on this.

Proposition 5.15. **Manipulation of Taylor Series**: a few known series regarding

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad |z-z_0| < R$$

$$\begin{cases} \dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n & R = 1 \quad \text{radius of convergence} \\ e^z = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} z^n & R = \infty \\ \cos z = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} & R = \infty \\ \sin z = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} & R = \infty \end{cases}$$

Example 5.16. Expand e^z about $z = i$.

Solution: $f(z) = e^z \quad \Rightarrow \quad f^{(n)}(i) = e^i$, so

$$e^z = \sum_{n=0}^{\infty} \frac{e^i}{n!}(z-i)^n$$

Radius? We want

$$\lim_{n\to\infty} \left| \frac{e^i}{(n+1)!}(z-i)^{n+1} \cdot \frac{n!}{e^i} \cdot \frac{1}{(z-i)^n} \right| = \left| \frac{z-i}{n+1} \right| < 1$$

by Ratio test. So the above is true for all z. We have $R = \infty$.

Example 5.17. Maclaurin Series for $\dfrac{1}{8+z^3}$?

Solution:

$$\frac{1}{8+z^3} = \frac{1}{8} \cdot \frac{1}{1 - \left(\frac{-z^3}{8}\right)} = \frac{1}{8} \sum_{n=0}^{\infty} \left(\frac{-z^3}{8} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n z^{3n}}{2^{3n+3}}$$

Radius?

$$\left| \frac{-z^3}{8} \right| < 1 \ \Rightarrow \ |z|^3 < 2^3 \ \Rightarrow \ |z| < 2$$

So $R = 2$