

# AMATH / PMATH 332 Course Notes

## Applied Complex Analysis

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# Chapter 1 Complex Numbers

## 1.1 Intro, Properties of Complex Numbers

Intro:

- What it's about: not like real analysis; some of intro to calculus on  $\mathbb{C}$
- Goal: extend calculus on  $\mathbb{R}$  to  $\mathbb{C}$  - many results become simpler! (more complete picture here)
- Can be used to solve some  $\mathbb{R}$  problems.

The Fundamentals:

- Basic idea: define solutions to  $x^2 + 1 = 0$
- Early Mathematicians:  $x = \pm\sqrt{-1}$ . For  $\sqrt{-1}$ , should we call it  $i$ ?
- Note: “ $\sqrt{\quad}$ ” always denotes positive root, e.g.  $\sqrt{4} = 2$
- Problem:

$$\begin{aligned}\sqrt{-1}\sqrt{-1} &= -1 \quad \text{by definition of } \sqrt{\quad} \\ \sqrt{-1}\sqrt{-1} &= \sqrt{(-1)(-1)} = \sqrt{1} = 1 \quad \text{since } \sqrt{ab} = \sqrt{a}\sqrt{b}\end{aligned}$$

- Fix: interpret “ $\sqrt{\quad}$ ” differently for complex numbers - it must be multivalued, and define the imaginary unit  $i$  by  $i^2 = -1$

**Definition 1.1. Complex number:**

$$z = \underbrace{a}_{\text{“real part” } \operatorname{Re}(z)} + i \underbrace{b}_{\text{“imaginary part” } \operatorname{Im}(z) \text{ which is real!}} \quad \text{where } a, b \in \mathbb{R}$$

$\mathbb{C}$  = set of complex numbers. Note that  $\mathbb{R} \subset \mathbb{C}$

**Definition 1.2.** Let  $z = a + bi$ , and  $w = c + di$ . Then:

- $z = w$  if and only if  $a = c$  and  $b = d$
- $z + w = (a + bi) + (c + di) = a + c + (b + d)i$
- $z - w = z + (-w) = (a + bi) + (-c - di) = a - c + (b - d)i$
- $zw = (a + bi)(c + di) = ac + bdi^2 + adi + bci = ac - bd + (ad + bc)i$
- $\frac{z}{w} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + i \cdot \frac{bc - ad}{c^2 + d^2}$

**Example 1.3.**

$$\frac{2+i}{1+2i} = \frac{2+i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{4}{5} - \frac{3}{5}i$$

$$\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = \frac{-i}{-i^2} = -i$$

**Theorem 1.4.**  $z + w = w + z$ ,  $k(z + w) = kz + kw$  apply as usual.  $zw = wz$

Note: We can't classify complex numbers as "positive" or "negative", and can't use inequalities, e.g.  $z > w$  doesn't make sense.

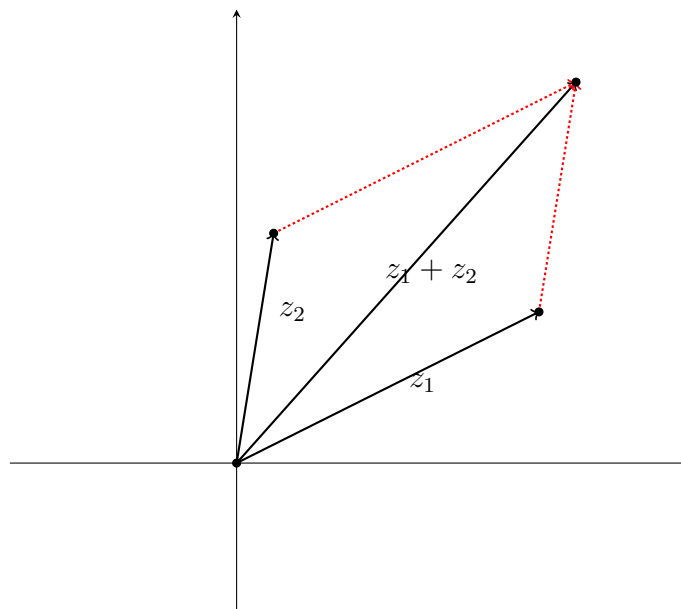
**Definition 1.5.** Conjugate of  $z = a + bi$  is

$$\bar{z} = a - bi$$

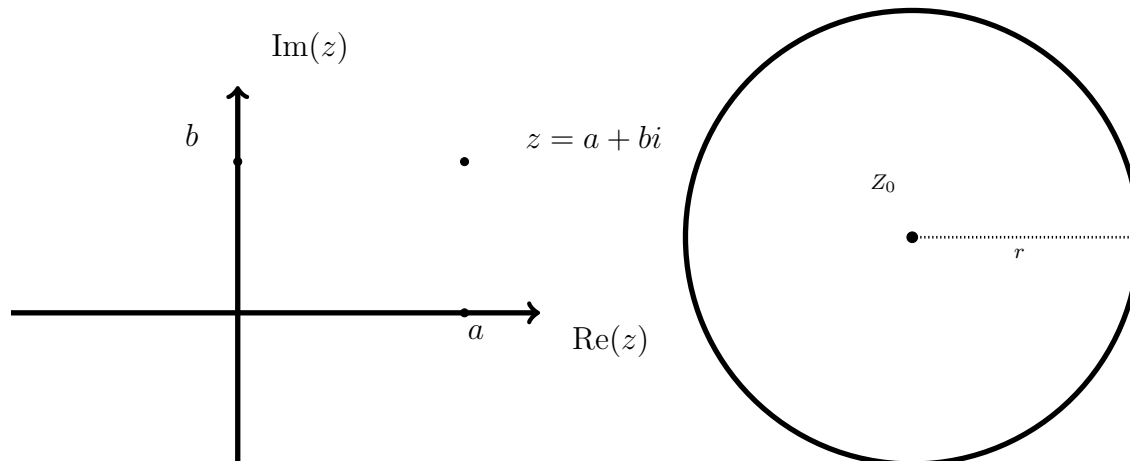
(Sometimes written as  $z^*$  as well)

**Proposition 1.6.** The following rules apply:

1.  $\overline{\bar{z}} = z$
2.  $\overline{z \pm w} = \bar{z} \pm \bar{w}$
3.  $\overline{zw} = \bar{z} \bar{w}$  and  $\overline{\left(\frac{z}{w}\right)} = \frac{(\bar{z})}{(\bar{w})}$
4.  $z + \bar{z} = 2\operatorname{Re}(z) \Rightarrow \operatorname{Re}(z) = \frac{1}{2}(z + \bar{z})$
5.  $z - \bar{z} = 2i\operatorname{Im}(z) \Rightarrow \operatorname{Im}(z) = \frac{1}{2i}(z - \bar{z})$
6.  $z\bar{z} = a^2 + b^2$  which is real!



## 1.2 The Complex Plane, Polar form



**Definition 1.7.** The modulus of  $z = a + bi$  is  $|z| = \sqrt{a^2 + b^2}$

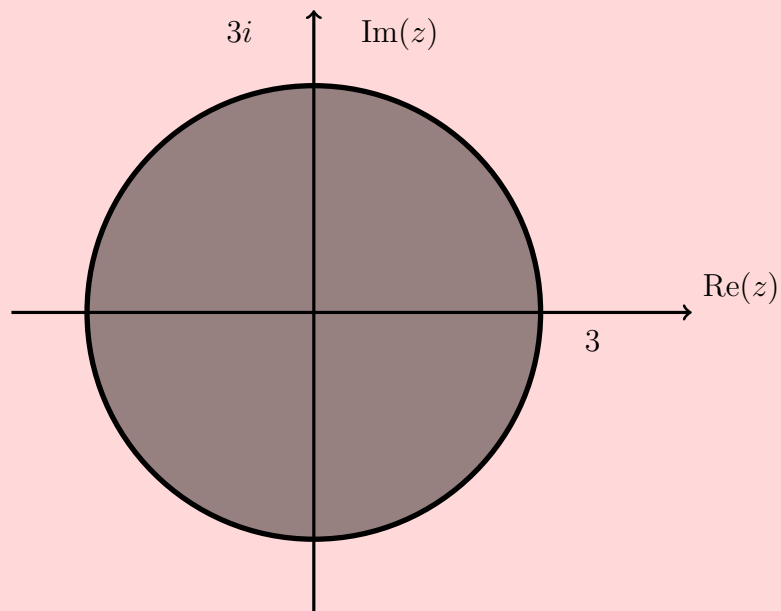
The distance between two numbers  $z$  and  $w$  is  $|z - w|$

Notes:

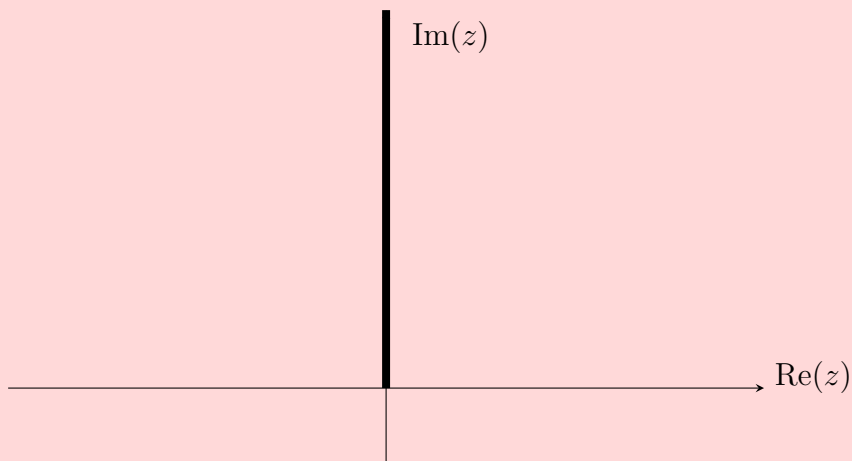
- $|z| \geq 0$  and is real
- $z\bar{z} = a^2 + b^2 = |z|^2$
- $|z - z_0| = r$  describes a circle of radius  $r$  centered at  $z_0$

**Example 1.8.** Sketch the sets:

1.  $|z| < 3$



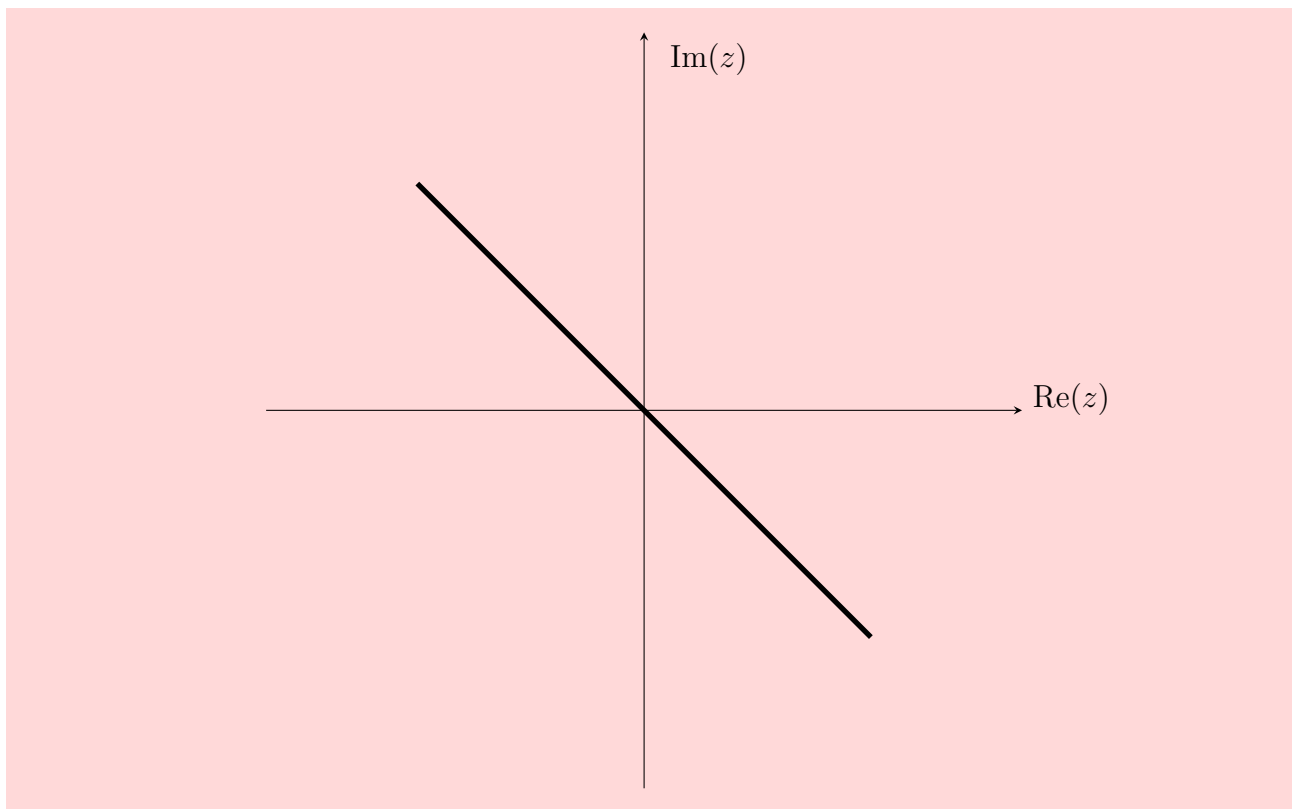
2.  $|z| = \text{Im}(z)$ . Let  $z = a + ib$ . So,  $\sqrt{a^2 + b^2} = b$ , which gives  $a^2 + b^2 = b^2$ , so  $a = 0, b \geq 0$



3.  $|z - 1| = |z + i|$ . So

$$\begin{aligned}\sqrt{(a-1)^2 + b^2} &= \sqrt{a^2 + (b+1)^2} \\ (a-1)^2 + b^2 &= a^2 + (b+1)^2 \\ a^2 - 2a + 1 + b^2 &= a^2 + b^2 + 2b + 1 \\ b &= -a\end{aligned}$$

This is the set of points that are equidistant from  $z = 1$  and  $z = -i$



We will often use  $z = x + yi$ , so we are in the  $xy$ -plane, still not called  $\mathbb{R}^2$  though.

Useful inequalities:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

This is known as “Triangle Inequality”. This also extends to

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$$

**Corollary 1.9.**

$$|z_1 + z_2| \geq \left| |z_1| - |z_2| \right|$$

**Proof 1.10.**

$$\begin{aligned} |z_1| &= |z_1 + (z_2 - z_2)| \\ &= |(z_1 + z_2) + (-z_2)| \\ &\leq |z_1 + z_2| + |z_2| \end{aligned}$$

$$\begin{aligned}
 |z_2| &= |z_2 + (z_1 - z_1)| \\
 &= |(z_1 + z_2) + (-z_1)| \\
 &\leq |z_1 + z_2| + |z_1|
 \end{aligned}$$

So  $|z_1 + z_2| \geq |z_1| - |z_2|$  and  $|z_2| - |z_1|$ . So

$$|z_1 + z_2| \geq \left| |z_1| - |z_2| \right|$$

□

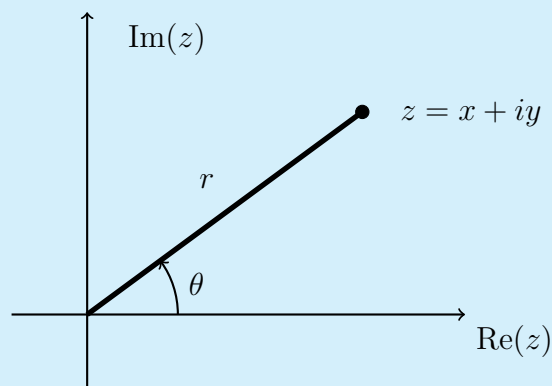
### Definition 1.11. Polar Form

$$x = r \cos \theta, y = r \sin \theta$$

So,

$$\begin{aligned}
 z &= r \cos \theta + ir \sin \theta \\
 &= r(\cos \theta + i \sin \theta) \\
 &= r \underbrace{\text{cis}}_{\text{common abbreviation}} \theta
 \end{aligned}$$

$$r = \sqrt{x^2 + y^2}, \tan \theta = \frac{y}{x}$$



Notes:

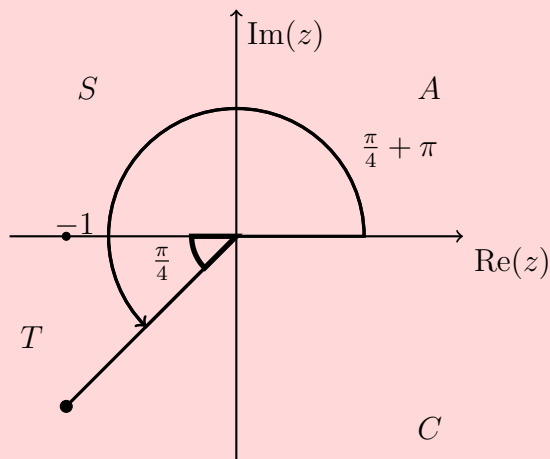
- This is not unique. e.g.  $z = 2 = 2 \text{cis } 0 = 2 \text{cis } 2\pi = \dots$ , also  $z = 0 = 0 \text{cis } \theta$  for any  $\theta$
- $\theta = \tan^{-1}(\frac{y}{x})[\pm 2k\pi]$  if  $x > 0$ , but must add  $\pi$  if  $x < 0$  - Recall principal values



**Example 1.12.** Say we want to express  $z = -1 - i$  in polar form.

We compute  $r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$ .  $\tan \theta = \frac{-1}{-1} = 1$ . Note that  $\theta \neq \tan^{-1}(1) = \frac{\pi}{4}$ , instead,  $\theta = \frac{5\pi}{4}$ .

So,  $z = \sqrt{2} \operatorname{cis} \frac{5\pi}{4}$  or  $\sqrt{2} \operatorname{cis}(\frac{5\pi}{4} + 2k\pi)$



Note:

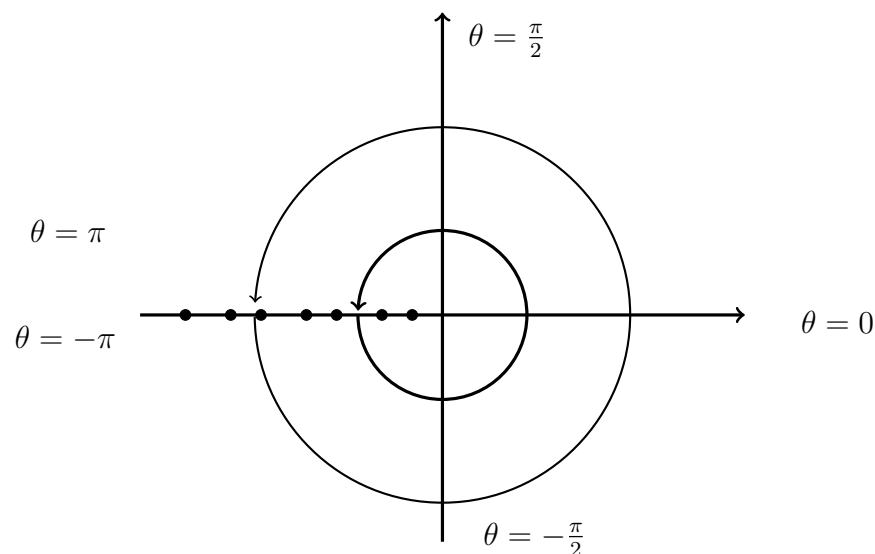
$$z = \underbrace{\sqrt{x^2 + y^2}}_{r=|z|, \text{ "modulus" }} \operatorname{cis} \underbrace{\theta}_{\text{ "argument" of } z}$$

Also, “arg  $z$ ” = set of all possible values of  $\theta$ . “Arg  $z$ ” = principle values of  $\theta$ , usually in  $(-\pi, \pi]$

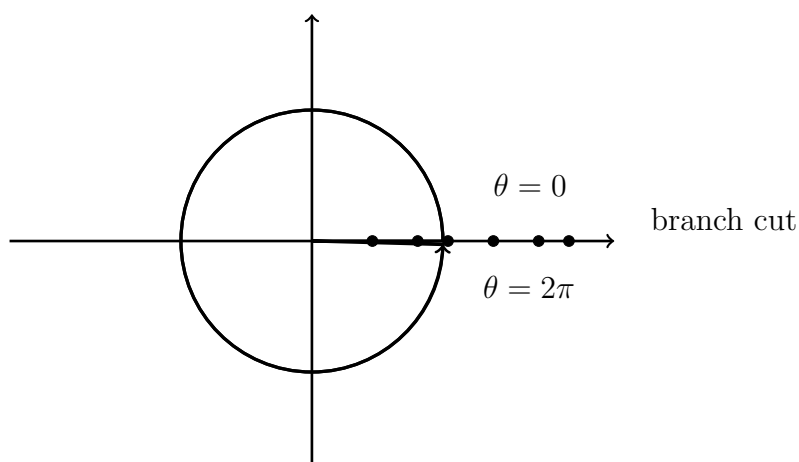
**Example 1.13.** For  $z = -1 + \sqrt{3}i$ .  $\operatorname{Arg} z = \frac{2\pi}{3}$ ,  $\arg z = \frac{2\pi}{3} + 2k\pi, k \in \mathbb{Z}$

Also,  $|z| = 2$ , so  $-1 + \sqrt{3}i = 2 \operatorname{cis} \frac{2\pi}{3}$

We sometimes think of  $\arg z$  as a multivalued “function” of  $z$ . For a single-valued function, we could use  $\operatorname{Arg} z$ , but it has discontinuity on negative real axis.



Another way: we can define  $\text{Arg}(z)$  to have range  $[0, 2\pi)$ . In general,  $\text{Arg}_{\theta_0} z$  has range  $[\theta_0, \theta_0 + 2\pi)$ , and usually we use  $\text{Arg } z = \text{Arg}_{-\pi} z$



### 1.3 Complex Exponential, Powers and Roots

Reading textbook Section 1.4, 1.5

**Definition 1.14.** If  $z = x + iy$ , then  $e^z$  is defined to be the complex number

$$e^z := e^x(\cos y + i \sin y)$$

**Proposition 1.15.** Euler's equation is formally consistent with the usual Taylor series ex-

pansions:

$$\begin{aligned}e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\end{aligned}$$

**Proof 1.16.** Let's substitute  $x = iy$  into the exponential series:

$$\begin{aligned}e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i \sin y\end{aligned}$$

□

As a result, we may introduce the standard polar representation

$$z = r \operatorname{cis} \theta = r(\cos \theta + i \sin \theta) = re^{i\theta} = |z|e^{i \arg z}$$

Notice that

$$\begin{aligned}e^{i0} &= e^{2\pi i} = e^{-2\pi i} = e^{4\pi i} = e^{-4\pi i} = \cdots = 1 \\ e^{(\pi/2)i} &= i \quad e^{(-\pi/2)i} = -i \quad e^{\pi i} = -1\end{aligned}$$

Also notice that

$$\begin{aligned}\cos \theta &= \operatorname{Re}(e^{i\theta}) = \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin \theta &= \operatorname{Im}(e^{i\theta}) = \frac{e^{i\theta} - e^{-i\theta}}{2i}\end{aligned}$$

Hence,

$$\begin{aligned}z_1 z_2 &= (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \\ \bar{z} &= re^{-i\theta}, \text{ given that } z = re^{i\theta}\end{aligned}$$

**Example 1.17.** Compute the following:

1.  $(1 + i)/(\sqrt{3} - i)$ .

Notice that  $1 + i = \sqrt{2} \operatorname{cis}(\pi/4) = \sqrt{2}e^{i\pi/4}$ , and  $\sqrt{3} - i = 2 \operatorname{cis}(-\pi/6) = 2e^{-i\pi/6}$ . So,

$$\frac{1 + i}{\sqrt{3} - i} = \frac{\sqrt{2}e^{i\pi/4}}{2e^{-i\pi/6}} = \frac{\sqrt{2}}{2}e^{i5\pi/12}$$

2.  $(1 + i)^{24}$

We have

$$(1 + i)^{24} = (\sqrt{2}e^{i\pi/4})^{24} = (\sqrt{2})^{24}e^{i24\pi/4} = 2^{12}e^{i6\pi} = 2^{12}$$

**Theorem 1.18.**

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \quad n = 1, 2, 3, \dots$$

**Definition 1.19.** There are exactly  $m$  distinct  $m$ -th roots of unity, denoted by  $1^{1/m}$ , and they are given by

$$1^{1/m} = e^{i2k\pi/m} = \cos \frac{2k\pi}{m} + i \sin \frac{2k\pi}{m} \quad (k = 0, 1, 2, \dots, m-1)$$

Take  $k = 1$  into the above equation, we can get

$$\omega_m := e^{i2\pi/m} = \cos \frac{2\pi}{m} + i \sin \frac{2\pi}{m}$$

So the complete set of roots can be displayed as

$$\{1, \omega_m, \omega_m^2, \dots, \omega_m^{m-1}\}$$

Note that a number  $w$  is said to be a primitive  $m$ -th root of unity if  $w^m = 1$  but  $w^k \neq 1$  for  $k = 1, 2, \dots, m-1$ . Clearly,  $\omega_m$  is a primitive root.

**Theorem 1.20.**

$$1 + \omega_m + \omega_m^2 + \dots + \omega_m^{m-1} = 0$$

**Proof 1.21.** Note that

$$(\omega_m - 1)(1 + \omega_m + \omega_m^2 + \cdots + \omega_m^{m-1}) = (\omega_m - 1) = 0$$

Since  $\omega_m \neq 1$ , the result follows. □

To obtain the  $m$ -th root of an arbitrary (non-zero) complex number  $z = re^{i\theta}$ , we can obtain the following generalized result.

**Definition 1.22.** The  $m$ -th distinct roots of  $z$  are given by

$$z^{1/m} = \sqrt[m]{|z|} e^{i(\theta+2k\pi)/m}$$

**Example 1.23.** Find all the cube roots of  $\sqrt{2} + i\sqrt{2}$

The polar form for  $\sqrt{2} + i\sqrt{2}$  is

$$\sqrt{2} + i\sqrt{2} = 2e^{i\pi/4}$$

Putting  $|z| = 2, \theta = \pi/4, m = 3$  into the above definition, we obtain

$$(\sqrt{2} + i\sqrt{2})^{1/3} = \sqrt[3]{2} e^{i(\pi/12+2k\pi/3)}, \quad (k = 0, 1, 2)$$

Hence, the three cube roots of  $\sqrt{2} + i\sqrt{2}$  are:

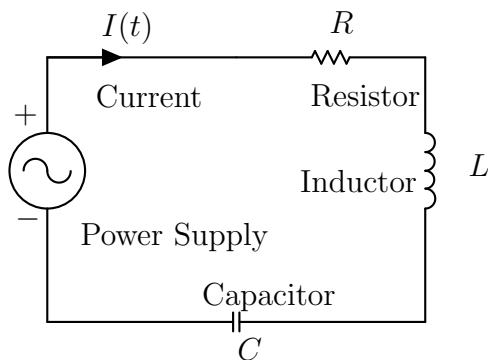
- $\sqrt[3]{2}(\cos \pi/12 + i \sin \pi/12)$
- $\sqrt[3]{2}(\cos 3\pi/4 + i \sin 3\pi/4)$
- $\sqrt[3]{2}(\cos 17\pi/12 + i \sin 17\pi/12)$

## 1.4 Application to Electrical Circuits

A typical electrical circuits is like the following:

Laws:

1. Resistor:  $V = IR$
2. Inductor:  $V = L \frac{dI}{dt}$
3. Capacitor:  $C \frac{dV}{dt} = I$



Suppose the current is

$$I(t) = \underbrace{I_0}_{\text{amplitude}} \cos \underbrace{\omega t}_{\text{frequency}} = \text{Re}(\underbrace{I_0 e^{i\omega t}}_{\text{call it } \tilde{I}(t)})$$

Then

1. Law 1 tells us  $V = (I_0 \cos \omega t)(R) = \text{Re}(\tilde{I}(t) \cdot R)$ . So “complex voltage” is

$$\tilde{V} = R\tilde{I}$$

2. Law 2 tells us

$$\begin{aligned} V &= L \cdot (-\omega I_0 \sin \omega t) \\ &= -\omega L I_0 \cdot \underbrace{\text{Re}(e^{i(\omega t - \frac{\pi}{2})})}_{=\cos(\omega t - \frac{\pi}{2}) = \sin \omega t} \\ &= \text{Re}(-\omega L I_0 e^{i\omega t} e^{-i\frac{\pi}{2}}) \\ &= \text{Re}(i\omega L I_0 e^{i\omega t}) \end{aligned}$$

So

$$\tilde{V} = i\omega L \tilde{I}$$

3. Law 3 tells us

$$\begin{aligned} V &= \frac{1}{C} \int I(t) \\ &= \frac{I_0}{C\omega} \sin \omega t \\ &= \text{Re}\left(\frac{I_0}{C\omega} e^{i(\omega t - \frac{\pi}{2})}\right) \\ &= \text{Re}\left(\frac{I_0}{iC\omega} e^{i\omega t}\right) \end{aligned}$$

So

$$\tilde{V} = \frac{1}{iC\omega} \tilde{I}$$

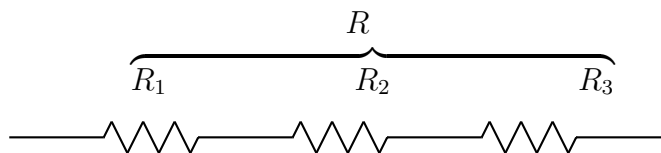
So, with the complex representation, all three circuit elements behave like resistors with a complex “Ohm’s Law”

$$\tilde{V} = Z\tilde{I} \quad \text{where } Z = \begin{cases} R & \text{for resistors} \\ i\omega L & \text{for inductors} \\ \frac{1}{i\omega C} & \text{for capacitors} \end{cases}$$

Moreover,  $Z$  is called “impedance”

Combining the components:

- In series:



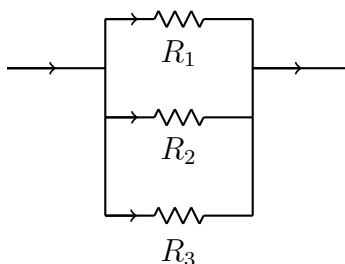
$$R = R_1 + R_2 + R_3 + \cdots$$

$$L = L_1 + L_2 + L_3 + \cdots$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$$

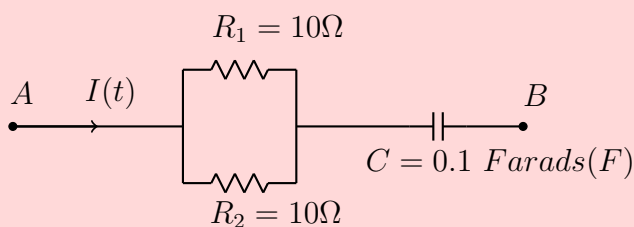
$$Z = Z_1 + Z_2 + Z_3 + \cdots$$

- In parallel:



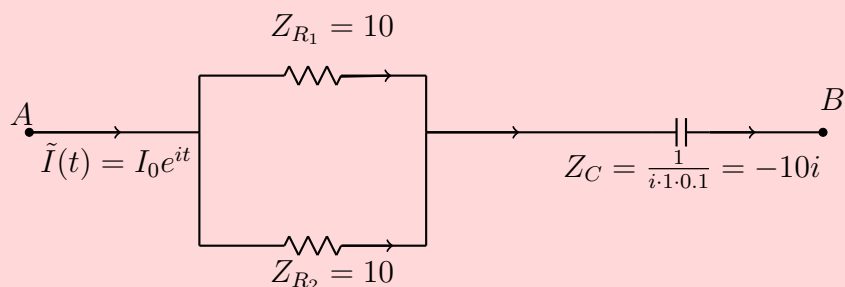
$$\begin{aligned}\frac{1}{R} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \\ \frac{1}{L} &= \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots \\ C &= C_1 + C_2 + C_3 + \cdots \\ \frac{1}{Z} &= \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots\end{aligned}$$

**Example 1.24.** Suppose a current  $I(t) = I_0 \cos t$ , passes through this:



Find  $V(t)$ , the difference in electrical potential energy between A and B

**Solution:**



Let's use the complex version of "Ohm's Law". We have  $\frac{1}{Z_R} = \frac{1}{Z_{R_1}} + \frac{1}{Z_{R_2}} = \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$ , so  $Z_R = 5$ .

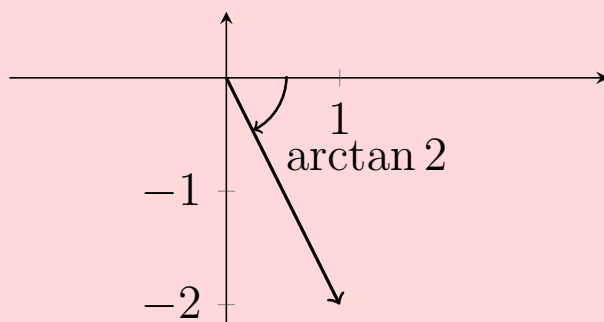
Combine the resistor and capacitor in series:  $Z = Z_R + Z_C = 5 - 10i$ .

So, the complex voltage is

$$\begin{aligned}\tilde{V} &= Z\tilde{I} \\ &= (5 - 10i)I_0 e^{it} \\ &= 5I_0(1 - 2i)e^{it} \\ &= 5I_0\sqrt{5}e^{i \arctan -2}e^{it}\end{aligned}$$

So,  $V(t) = \text{Re}(\tilde{V}(t)) \approx 5\sqrt{5}I_0 \cos(t - 1.107)$





## 1.5 Sets in the Complex Plane

**Definition 1.25.** Neighborhood of  $z_0$  is

$$N_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$$

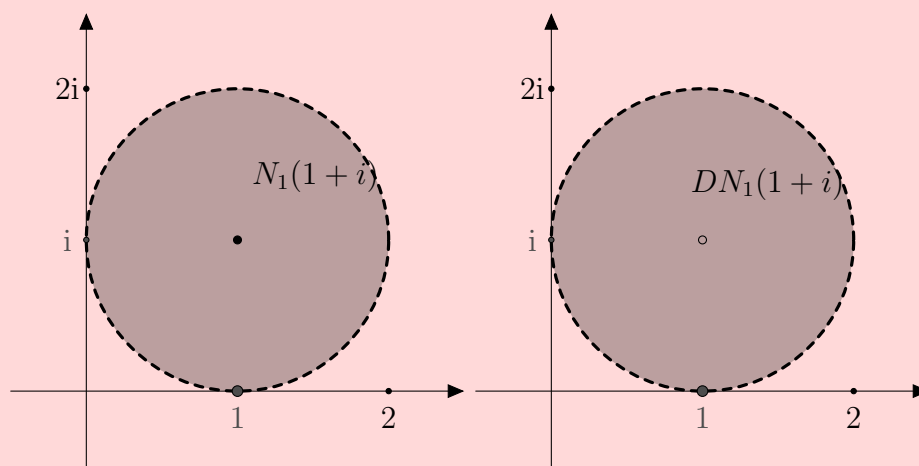
where  $\epsilon > 0$  is real

**Definition 1.26.** Deleted Neighborhood of  $z_0$  is

$$DN_\epsilon(z_0) = \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$$

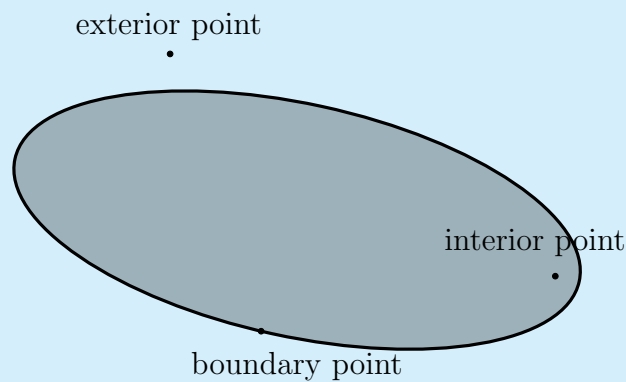
where  $\epsilon > 0$  is real

**Example 1.27.** For  $z_0 = 1 + i$ , consider  $|z - (1 + i)| < 1$ . The neighborhood of  $z_0$  and deleted neighborhood of  $z_0$  is as follows:



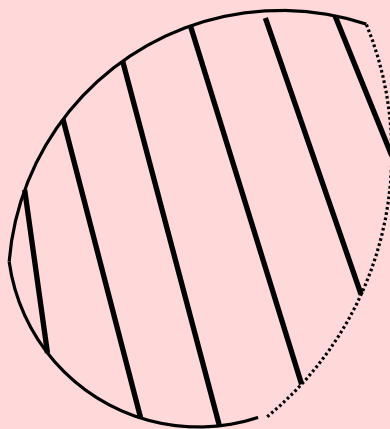
**Definition 1.28.** Let  $S \subseteq \mathbb{C}$ :

- $z_0$  is an **interior point** of  $S$  if there exists a neighborhood of  $z_0$  which contains only points in  $S$
- $z_0$  is an **exterior point** of  $S$  if there exists a neighborhood of  $z_0$  which contains no points in  $S$
- $z_0$  is a **boundary point** of  $S$  if every neighborhood of  $z_0$  contains some points in  $S$  and some points not.
- **Boundary of  $S$**  is the set of all boundary points of  $S$
- $S$  is **open** if it contains none of its boundary points
- $S$  is **closed** if it contains all of its boundary points, equivalently if its complement is open.
- Note that  $S$  could be both open and closed, when it does not have any boundary points



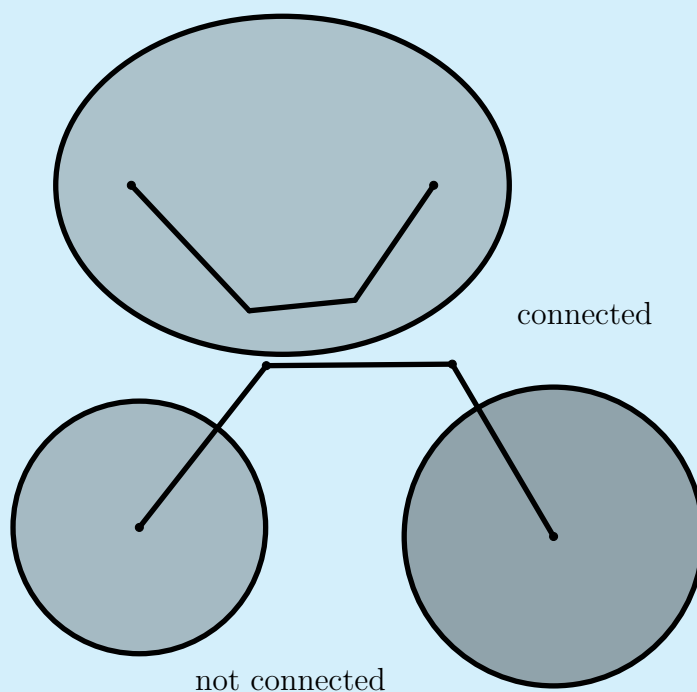
**Example 1.29.** Note that

- $N_1(1 + i)$  is open
- $\mathbb{C}$  is both open and closed
- $|z - z_0| \leq 1$  is closed
- The figure below: it is neither open nor closed.



**Definition 1.30.** For  $S \subseteq \mathbb{C}$ :

- **Closure** of  $S$  is  $S$  plus its boundary.
- An open set  $S$  is **connected** if any two points in  $S$  can be connected by a polygonal path lying entirely in  $S$ .
- A **domain** is an open connected set. We should not confuse this with “domain of a function”
- A **region** is a domain plus some, none, or all of its boundary points.
- $S$  is **bounded** if there exists  $R \in \mathbb{R}$  such that  $|z| < R$  for all  $z \in S$ .

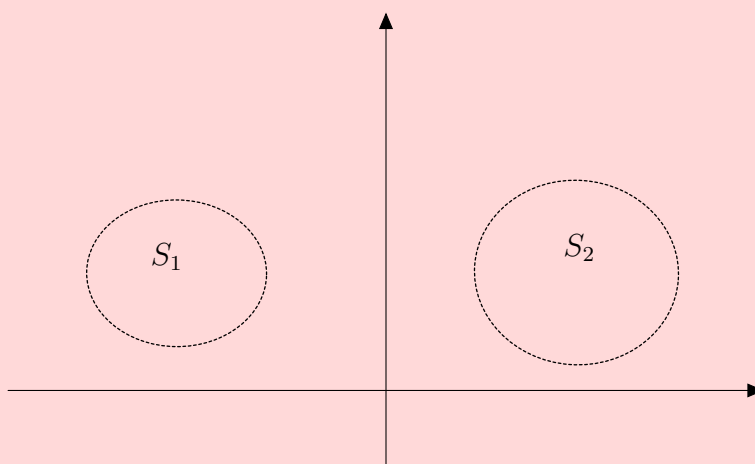


**Theorem 1.31.** If  $u(x, y)$ , defined on a domain  $D$ , satisfies

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

for all points in  $D$ , then  $u(x, y) = \text{constant}$  in  $D$ .

**Example 1.32.** Suppose we have  $S_1$  and  $S_2$  like this:



in which we have  $u(x, y) = 0$  on  $S_1$  and  $u(x, y) = 1$  on  $S_2$ .

Then,  $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$  on  $S_1 \cup S_2$ , but  $u(x, y)$  is not constant on  $S_1 \cup S_2$ .

Why does not the theorem hold? Well this is because  $S_1 \cup S_2$  is not connected, so it's not a domain.

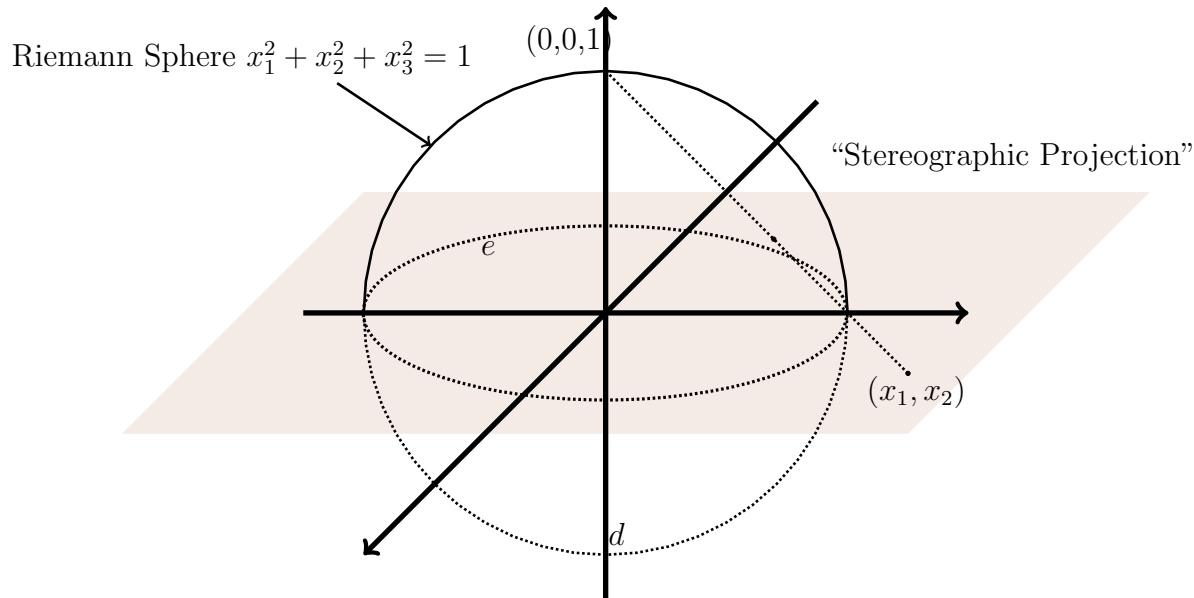
The Extended Complex Plane:

The “neighborhood of  $\infty$ ” is defined as:

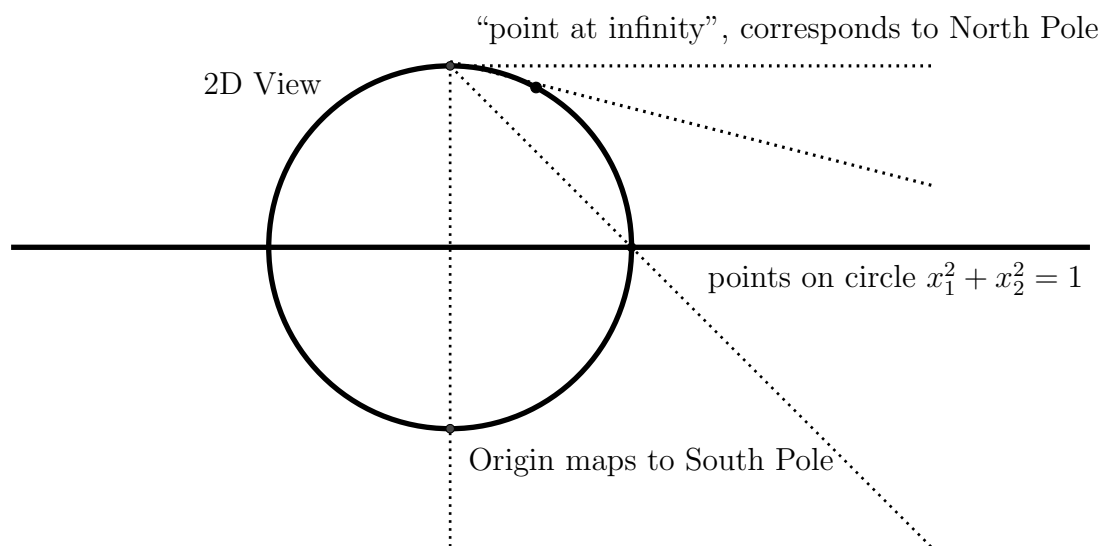
$$N_\epsilon(\infty) = \{z \in \mathbb{C} : |z| > \frac{1}{\epsilon}\}$$

for some real  $\epsilon > 0$

The Riemann sphere:



We can define a one-to-one mapping between  $x_1x_2$ -plane and the sphere:



See the course text for more detail, in particular:

- Circles and lines all map circles on the sphere
- Lines are just circles which pass through the "point at infinity"

## Chapter 2 Analytic Functions

### 2.1 Functions

For a function on complex numbers:

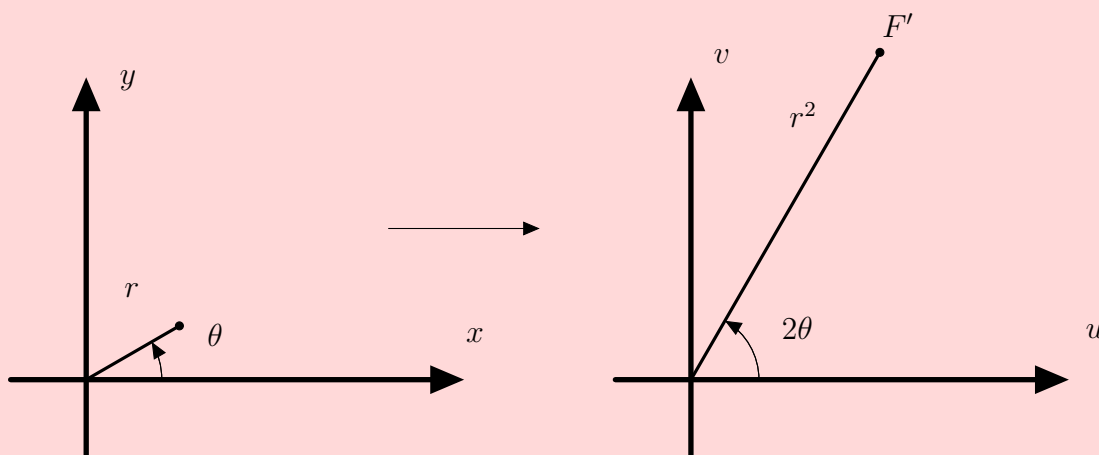
$$\begin{aligned}\omega &= f(z) \\ &= f(x + iy) \\ &= u(x, y) + iv(x, y)\end{aligned}$$

We can think of it as a mapping.

**Example 2.1.** 1.  $f(z) = z^2$ . Find the images of

(a) the first quadrant.

$$f(z) = (x + iy)^2 = \underbrace{(x^2 - y^2)}_u + i \underbrace{2xy}_v$$

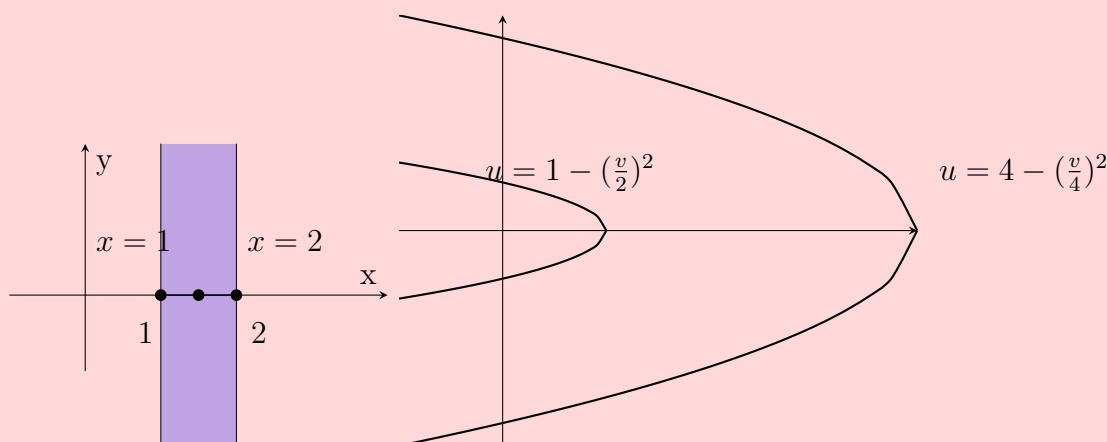


Note that  $f(z) = (re^{i\theta})^2 = r^2e^{i2\theta}$  (angle is doubled)

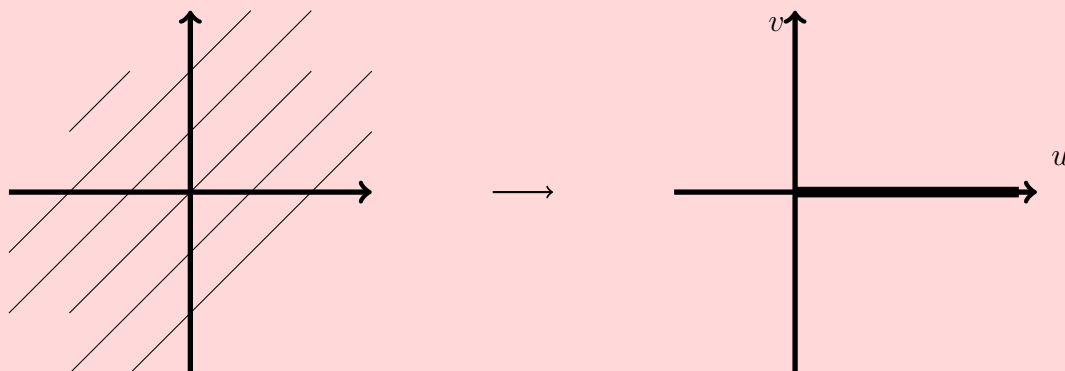
(b) the strip  $1 \leq \operatorname{Re}(z) \leq 2$

With  $1 \leq x \leq 2$ , the boundaries become:

- $x = 1 \Rightarrow \begin{cases} u = 1 - y^2 \\ v = 2y \end{cases} \Rightarrow u = 1 - \left(\frac{v}{2}\right)^2$ , which is a parabola
- $x = 2 \Rightarrow \begin{cases} u = 4 - y^2 \\ v = 4y \end{cases} \Rightarrow u = 4 - \left(\frac{v}{4}\right)^2$ , which is a parabola



2.  $f(z) = |z|$ . This one maps complex plane to non-negative real axis.



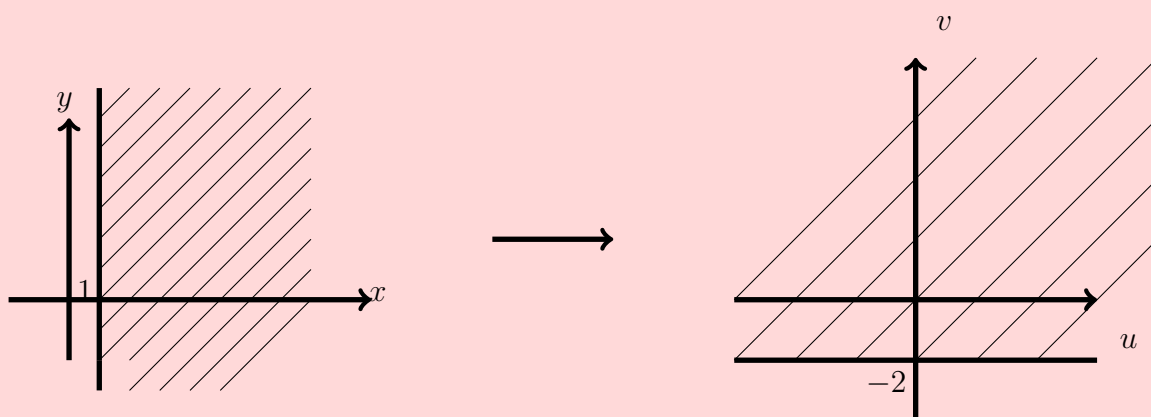
3.  $f(z) = z - z_0 = (x + iy) - (x_0 + iy_0) = (x - x_0) + i(y - y_0)$ . This is a translation.
4.  $f(z) = z_0 z$ , so

$$f(z) = r_0 e^{i\theta_0} r e^{i\theta} = \underbrace{r_0}_{\text{magnification}} r e^{\underbrace{\theta_0}_{\text{rotation}} + \theta} = r_0 r e^{i\theta_0 + \theta}$$

5.  $f(z) = \bar{z} = x - iy \rightarrow \begin{cases} u = x \\ v = -y \end{cases}$ . This is a reflection on  $y$ -axis.

6. Find image of half-plane  $\operatorname{Re}(z) \geq 1$  under the map  $\omega = f(z) = iz - 3i$ .

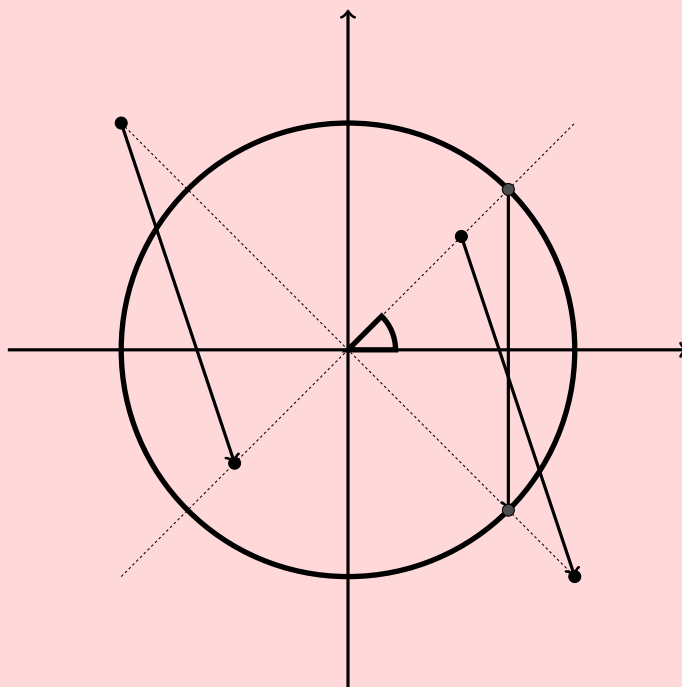
We can do this step by step. First it's a rotation of  $\frac{\pi}{2}$  (comes from the first  $i$ ), then its a shift down 3 units.



The image is the half-plane  $v \geq -2$ .

7. Inversion mapping.  $f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$ . So, it's a scaling by  $r$ , and then reflection through the  $x$ -axis.

For this mapping, unit circle maps to the unit circle. Outside points go to inside, and inside points go to outside.



8. Image of circle  $(x-1)^2 + y^2 = 1$  under  $f(z) = \frac{1}{z}$ .

The trick is to use polar formulas. Recall  $x^2 + y^2 = r^2$ ,  $x = r \cos \theta$ ,  $y = r \sin \theta$ .

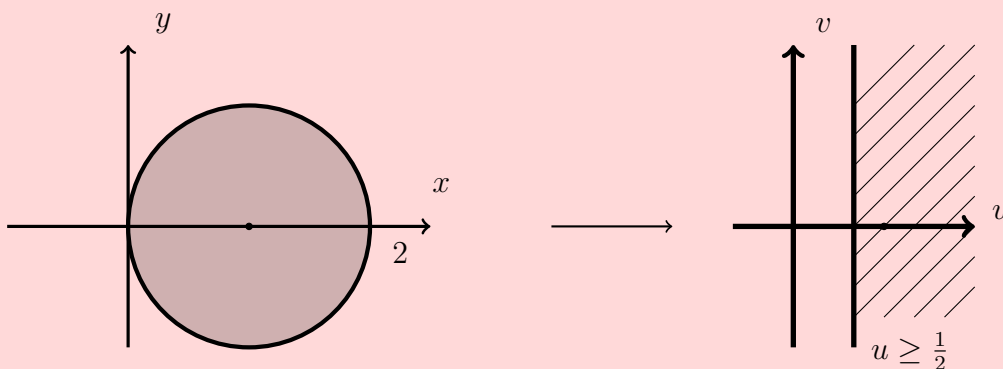
So,  $x^2 - 2x + 1 + y^2 = 1$  yields that  $r^2 = 2r \cos \theta$ . Since  $r \neq 0$ , we then have  $r = 2 \cos \theta$ .



To apply the map, replace  $r$  with  $\frac{1}{r}$ , and  $\theta$  with  $-\theta$ :

$$\frac{1}{r} = 2 \cos(-\theta) \Rightarrow r = \frac{1}{2 \cos \theta} \Rightarrow r \cos \theta = \frac{1}{2}$$

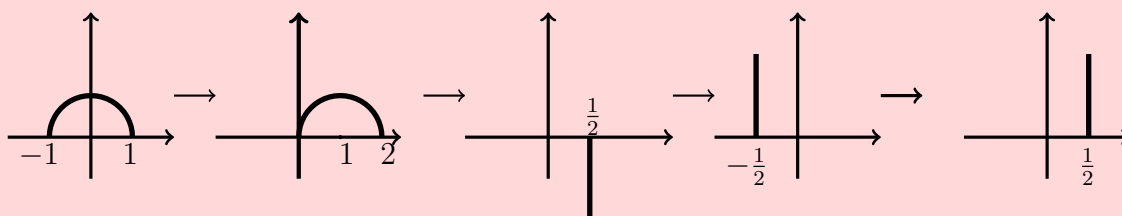
So  $u = \frac{1}{2}$  since  $\begin{cases} u = r \cos \theta \\ v = r \sin \theta \end{cases}$  in the  $uv$  plane.



9.  $w = f(z) = \frac{z}{z+1}$ , find the image of upper-half of unit circle.

First,  $f(z) = \frac{z+1-1}{z+1} = 1 - \frac{1}{z+1}$ . This is a sequence of transformations:

$$z \rightarrow \underbrace{z+1}_{\text{shift right}} \rightarrow \underbrace{\frac{1}{z+1}}_{\text{invert}} \rightarrow \underbrace{\frac{-1}{z+1}}_{\text{reflect and rotate } \pi} \rightarrow \underbrace{1 - \frac{1}{z+1}}_{\text{shift right}}$$



## 2.2 Limits and Differentiation

**Definition 2.2. Limits:**

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

means that for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon$$

**Example 2.3.** Prove that  $\lim_{z \rightarrow 1+i} (2+i)z = 1+3i$ .

**Solution:** We first do some preliminary work:

$$|(2+i)z - (1+3i)| = |2+i| \cdot \left| z - \frac{1+3i}{2+i} \right| = \sqrt{5} \cdot |z - (1+i)|$$

So, let  $\epsilon > 0$ , with  $|z - z_0| < \frac{\epsilon}{\sqrt{5}} (= \delta)$ , we have

$$\begin{aligned} |(2+i)z - (1+3i)| &= \sqrt{5} \cdot |z - (1+i)| \\ &< \sqrt{5} \cdot \frac{\epsilon}{\sqrt{5}} \\ &= \epsilon \end{aligned}$$

So,  $\lim_{z \rightarrow 1+i} (2+i)z = 1+3i$  □

Note that similar definitions apply when dealing with infinity, e.g.  $\lim_{z \rightarrow z_0} f(z) = \infty$  means that for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $0 < |z - z_0| < \delta \Rightarrow |f(z)| > \frac{1}{\epsilon}$

**Definition 2.4. Continuity:**  $f$  is continuous at  $z_0$  means that

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

The usual limit and continuity theorems hold, e.g.

$$\lim_{z \rightarrow z_0} f(z)g(z) = \lim_{z \rightarrow z_0} f(z) \cdot \lim_{z \rightarrow z_0} g(z)$$

**Theorem 2.5.** Let  $f(z) = u + iv$ ,  $z_0 = x_0 + iy_0$ ,  $w_0 = u_0 + iv_0$ , then

$$\lim_{z \rightarrow z_0} f(z) = w_0 \quad \text{if and only if} \quad \begin{cases} \lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y) = u_0 \\ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y) = v_0 \end{cases}$$

**Definition 2.6. Differentiation:**

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \left( = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right)$$

Derivative function is

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

For functions with real analogues (e.g.  $f(z) = z^2$  analogous to  $f(x) = x^2$ ), the usual rules (power, quotient, etc.) apply, e.g.

$$f(z) = 3z^2 + z^4 \Rightarrow f'(z) = 6z + 4z^3$$

What about functions without real analogues?

**Example 2.7.**  $f(z) = \bar{z}$ . Is it differentiable?

**Solution:**

$$\begin{aligned} f'(z_0) &= \lim_{z \rightarrow z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} \\ &= \lim_{z \rightarrow z_0} \frac{\overline{z - z_0}}{z - z_0} \\ &= \lim_{z \rightarrow z_0} \frac{re^{i\theta}}{re^{i\theta}} \quad \text{where } z - z_0 = re^{i\theta} \\ &= \lim_{z \rightarrow z_0} \frac{e^{-i\theta}}{e^{i\theta}} \\ &= \lim_{z \rightarrow z_0} e^{-i2\theta} \end{aligned}$$

which depends on  $\theta$ ! No unique value, so limit DNE. So,  $f$  is not differentiable anywhere.

**Theorem 2.8. Cauchy-Riemann Equations:** If  $f(z) = u(x, y) + iv(x, y)$  and  $f'(z_0)$  exists, then

$$u_x = v_y \quad \text{and} \quad v_x = -u_y \quad \text{at } (x_0, y_0)$$

Note that for notation,

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x} \\ u_y &= \frac{\partial u}{\partial y} \\ v_x &= \frac{\partial v}{\partial x} \\ v_y &= \frac{\partial v}{\partial y} \end{aligned}$$

**Proof 2.9.**

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \left( \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i \frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y} \right) \end{aligned}$$

Since the limit exists, it must be independent of path, so

- Along  $\Delta y = 0$ :

$$f'(z_0) = \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i(\dots) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

- Along  $\Delta x = 0$ :

$$f'(z_0) = \lim_{\Delta y \rightarrow 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i(\dots) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary part yields the result. □

## 2.3 Differentiability Continued

**Example 2.10.** Is  $f(z) = |z|^2$  differentiable? Where?

**Solution:**  $f(z) = \sqrt{x^2 + y^2}^2 = \underbrace{x^2 + y^2}_u + \underbrace{0}_v i$ . So, by CRE, we know that

$$\begin{cases} u_x = v_y & \Rightarrow 2x = 0 \\ v_x = -u_y & \Rightarrow 0 = -2y \end{cases}$$

It's clear that this is satisfied only at  $x = y = 0$ .

So, if  $(x, y) \neq (0, 0)$ , i.e.  $z \neq 0$ , then  $f$  is not differentiable.

When  $z = 0$ ,  $f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{|\Delta z|^2 - 0}{\Delta z} = 0$ . This is because  $\left| \frac{|\Delta z|^2}{\Delta z} - 0 \right| \leq |\Delta z| \rightarrow 0$  as  $\Delta z \rightarrow 0$  (by applying the squeeze theorem).

Hence, CRE are necessary but not sufficient conditions.

**Theorem 2.11.** Let  $f$  be defined in some neighborhood of  $z_0$ . If  $u_x, u_y, v_x, v_y$  exist in that neighborhood, satisfying CRE at  $z_0$ , and are continuous at  $z_0$ , then  $f$  is differentiable at  $z_0$ .

**Definition 2.12.**  $f(z)$  is analytic at  $z_0$  if  $f'(z)$  exists at every point in some neighborhood of  $z_0$ .

$f(z)$  is analytic on an open set  $S$  if it is analytic at every point of  $S$ .

**Example 2.13.**  $f(z) = z^3 = \dots = \underbrace{(x^3 - 3xy^2)}_{u(x,y)} + i \underbrace{(3x^2y - y^3)}_{v(x,y)}$ .

We have

$$u_x = 3x^2 - 3y^2$$

$$u_y = -6xy$$

$$v_x = 6xy$$

$$v_y = 3x^2 - 3y^2$$

So, CRE satisfied everywhere. All partial derivatives are continuous. By theorem,  $f$  is differentiable everywhere, so is analytic everywhere. We refer to “analytic everywhere” as “entire”

**Example 2.14.** Where is  $f(z) = x^2 + iy^2$  analytic?

We have

$$u_x = 2x$$

$$u_y = 0$$

$$v_x = 0$$

$$v_y = 2y$$

We need  $x = y$  to satisfy CRE.

- If  $x \neq y$ ,  $f$  is not differentiable, so not analytic.
- If  $x = y$ ,  $f$  cannot be analytic because we are not on an open set.

So,  $f$  is not analytic nowhere.

**Theorem 2.15.** Sums, products, and compositions of analytic functions are also analytic, except when  $\div 0$

**Example 2.16.**  $f(z) = \frac{z^3 + 2}{z^2 + 1}$  is analytic everywhere except at  $z = \pm i$ .

$g(z) = f(z^2)$  is analytic everywhere except where  $z^2 = \pm i$ , i.e. except

$$\begin{aligned} z &= e^{i\left(\frac{n\pi + \pi/2}{2}\right)} \\ &= e^{i(n\pi/2 + \pi/4)} \\ &= e^{i(n\pi/4)}, e^{i(3\pi/4)}, e^{i(5\pi/4)}, e^{i(7\pi/4)} \\ &= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \end{aligned}$$

**Theorem 2.17.** Suppose  $f$  is analytic in a domain  $D$ . If  $f'(z) = 0$  for all  $z \in D$ , then  $f$  is constant in  $D$

**Proof 2.18.**  $f'(z) = u_x + iv_x = v_y - iu_y$ . So,  $f'(z) = 0 \Rightarrow u_x = v_y = 0 = v_y = u_y$ . So,  $u$  and  $v$  are constant, since  $D$  is connected.  $\square$

**Theorem 2.19.** Suppose  $f$  is analytic in a domain  $D$ . If  $|f(z)| = M$  for all  $z \in D$ , where  $M$  is constant, then  $f(z)$  is constant in  $D$ .

**Proof 2.20.**  $|f(z)|^2 = u^2 + v^2 = M^2$ .

We differentiate:

- with respect to  $x$ :  $2uu_x + 2vv_x = 0 \quad - (1)$

- with respect to  $y$ :  $2uv_y + 2vv_y = 0$  – (2)

Now  $u_x = v_y$ , and  $v_x = -u_y$ , so the (2) gives  $-uv_x + vu_x = 0$  – (3).

Multiply (1) by  $u_x$ .

$$\begin{aligned} u u_x^2 + v u_x v_x &= 0 \\ \Rightarrow u u_x^2 + (u v_x) v_x &= 0 \quad \text{by (3)} \\ \Rightarrow u(u_x^2 + v_x^2) &= 0 \end{aligned}$$

So, unless  $u = 0$  for all  $z \in D$ , we must have  $u_x^2 + v_x^2 = 0$ . So,  $u_x = v_x = 0$ , implying that  $u, v$  are constant. Hence,  $f$  is constant.

What if  $u = 0$  for all  $z \in D$ ? Then,  $u_x = u_y = 0$ , so  $v_x = v_y = 0$  by CRE.  $f$  is constant as well.  $\square$

## 2.4 Harmonic Functions

Recap:

$$f'(z) = u_x + iv_x = \frac{u_y + iv_y}{i} = v_y - iu_y$$

$$CRE: \quad u_x = v_y \quad v_x = -u_y$$

Also, “analytic” means differentiable on a open set.

Suppose  $f(z) = u(x, y) + iv(x, y)$  is analytic in a domain  $D$ . Then  $u$  and  $v$  satisfy CRE.

Also, which will be shown later,  $u, v \in C^2$  (continuous under second partial derivatives), and this implies that  $u_{xy} = u_{yx}$ , and  $v_{xy} = v_{yx}$ .

From CRE:

$$\underbrace{u_x - v_y}_{\Rightarrow u_{xx} = v_{yx}} \quad \text{and} \quad \underbrace{v_x + u_y}_{\Rightarrow u_{yy} = -v_{xy}}$$

**Definition 2.21.** From the above derivation, we see

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

and

$$v_{xx} + v_{yy} = 0$$

We refer to these as “Laplace’s equation”

Solution to Laplace’s equation are called “harmonic functions”

Notes:

- We've shown that if  $f(z) = u + iv$  is analytic, then  $u$  and  $v$  must be harmonic
- Laplace's equation is very useful! We will see that later.
- $u_{xx} + u_{yy} = 0$  is also denoted as  $\Delta^2 u = 0$ , and we denote  $\Delta$  as "Laplacian operator".

**Example 2.22.** Suppose  $u(x, y) = e^{-2x} \cos 2y + 2y$ . Find  $v(x, y)$  such that  $f(z) = u + iv$  is analytic.

**Solution:**  $u$  and  $v$  must satisfy CRE. So,  $v_y = u_x = -2e^{-2x} \cos 2y$ . Hence,

$$\begin{aligned} v &= \int -2e^{-2x} \cos 2y dy \\ &= -e^{2x} \sin 2y + C(x) \end{aligned}$$

Note that  $C(x)$  is a function of all other variables.

Now we try to make it satisfy other CRE:

$$\begin{aligned} v_x = -u_y &\Rightarrow 2e^{-2x} \sin 2y + C'(x) = 2e^{-2x} \sin 2y - 2 \\ &\Rightarrow C'(x) = -2 \\ &\Rightarrow C(x) = -2x + k \end{aligned}$$

Therefore,  $v(x, y) = -e^{-2x} \sin 2y - 2x + k$

Note that  $v(x, y)$  is called the "harmonic conjugate" of  $u$ .

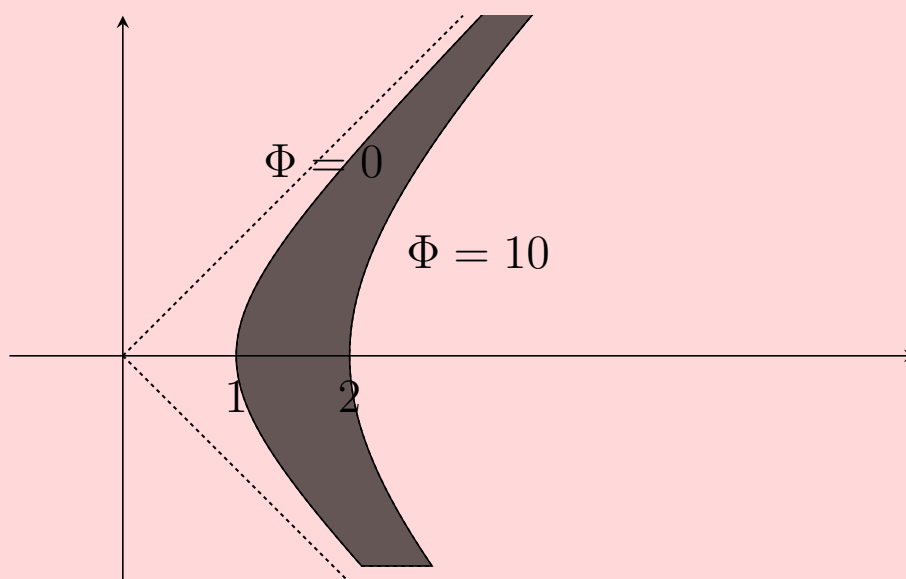
Exercise: show that if  $v$  is the harmonic conjugate of  $u$ , then  $-u$  is the harmonic conjugate of  $v$ .

**Example 2.23.** Solve Laplace's equation  $\Phi_{xx} + \Phi_{yy} = 0$  on region between hyperbolas  $x^2 - y^2 = 1$  and  $x^2 - y^2 = 4$ ,  $x > 0$ , with "boundary conditions"

$$\begin{cases} \Phi = 0 & \text{on } x^2 - y^2 = 1 \\ \Phi = 10 & \text{on } x^2 - y^2 = 4 \end{cases}$$

i.e. Find  $\Phi(x, y)$





**Solution:** Consider  $f(z) = z^2 = (x + yi)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + i \underbrace{2xy}_{v(x,y)}$

Since  $f(z)$  is already analytic, we have that  $u(x, y) = x^2 - y^2$  is harmonic. Boundary curves of region are level curves of a harmonic function.

Is the solution  $\Phi(x, y) = x^2 - y^2$ ? No.

Try  $\Phi(x, y) = A \cdot (x^2 - y^2) + B$  (also harmonic by linearity).

Applying the Boundary Conditions:

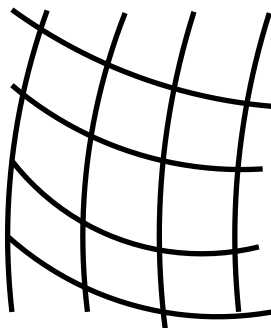
$$0 = A \cdot 1 + B \Rightarrow B = -A$$

$$10 = A \cdot 4 + B \Rightarrow A = \frac{10}{3}, B = -\frac{10}{3}$$

So the solution is  $\Phi(x, y) = \frac{10}{3}(x^2 - y^2) - \frac{10}{3}$

Notes:

- It can be used in temperature distribution
- What about more complicated regions?
- Orthogonal trajectories



- list of harmonic functions

## Chapter 3 Elementary Functions

### 3.1 Elementary Functions

**Definition 3.1. Polynomials:**

$$p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n, \quad a_i \in \mathbb{C}$$

There are obviously entire.

The fundamental theorem of algebra guarantees that we can factor this as

$$p(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n)$$

Note that  $z_i$  are not necessarily distinct.

$z_0$  is a “zero of multiplicity”  $k$  if and only if

$$p(z) = (z - z_0)^k q(z)$$

where  $q(z)$  is a polynomial such that  $q(z_0) \neq 0$

**Definition 3.2. Rational Functions:**

$$R(z) = \frac{p(z)}{q(z)} = \frac{a_n(z - z_1)(z - z_2) \cdots (z - z_n)}{b_m(z - w_1)(z - w_2) \cdots (z - w_m)}$$

Suppose all common factors have been cancelled, then

- the roots (or zeroes) of  $p(z)$  are called the roots/zeroes of  $R(z)$
- the roots (or zeroes) of  $q(z)$  are called the poles of  $R(z)$

**Example 3.3.**

$$R(z) = \frac{3i(z - 1)(z - \frac{1}{3}i)^2(z + i)}{(z - i)^3(z - 2 - i)}$$

Zeroes at 1 and  $-i$  (order 1 would be a “simple zero”), and  $\frac{1}{3}i$  (order 2).

Poles at  $i$  (order 3) and  $2 + i$  (order 1 would be a “simple pole”)

Partial Fractions has simpler rules:

**Example 3.4.** Decompose  $R(z) = \frac{1}{(z+4)^2(z^2+1)}$

**Solution:** Factor and expand

$$\frac{1}{(z+4)^2(z^2+1)} = \frac{A}{z+4} + \frac{B}{(z+4)^2} + \frac{C}{z+i} + \frac{D}{z-i}$$

This gives us

$$1 = A \cdot (z+4)(z+i)(z-i) + B(z+i)(z-i) + C(z+4)^2(z-i) + D(z+4)^2(z+i)$$

We can solve this by:

- set  $z = -4$ , this gives us  $1 = 0 + (-4+i)(-4-i)B + 0 + 0$ , so  $B = \frac{1}{17}$
- set  $z = -i$ , this gives us  $1 = 0 + 0 + (-i+4)^2(-2i)C + 0$ . Then we compute  $(-2i)(15-8i) = 16-30i$ , also  $(-16-30i) = \frac{(-16-30i)(-16+30i)}{(-16+30i)} = \frac{1156}{(-16+30i)} = \frac{578}{-8+15i}$ .

$$\text{Hence, } C = \frac{-8+15i}{578}.$$

- set  $z = -4$ , this gives us  $1 = 0 + 0 + 0 + (i+4)^2(2i)D$ , so  $D = \frac{-8-15i}{578}$ . The trick to compute things here is that, we can replace  $i$  with  $-i$  from  $C$  since the expression is similar to  $C$ .

Now what about  $A$ ? We can try another  $z$ , or just compare the coefficients of  $z^3$ . By comparing the coefficients of  $z^3$ , we get that

$$0 + A + C + D = A + \frac{-8+15i}{578} + \frac{-8-15i}{578}$$

$$\text{So } A = \frac{16}{578} = \frac{8}{289}$$

Hence,

$$\frac{1}{(z+4)^2(z^2+1)} = \frac{8/289}{z+4} + \frac{1/17}{(z+4)^2} + \frac{\frac{-8+15i}{578}}{z+i} + \frac{\frac{-8-15i}{578}}{z-i}$$

Actually, often we will only need one of the coefficients, and there's a quick way which will be covered later in the course.

**Definition 3.5. Exponential Function:** We already defined that  $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$ .

Note that  $e^{z_1+z_2} = e^{z_1}e^{z_2}$ ,  $\frac{d}{dz}e^z = e^z$ . Also,  $e^z$  is periodic with period  $2\pi i$

**Definition 3.6. Hyperbolic Functions:** From real calculus, we seen that

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \quad \text{this is the even component of } e^x$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \text{this is the odd component of } e^x$$

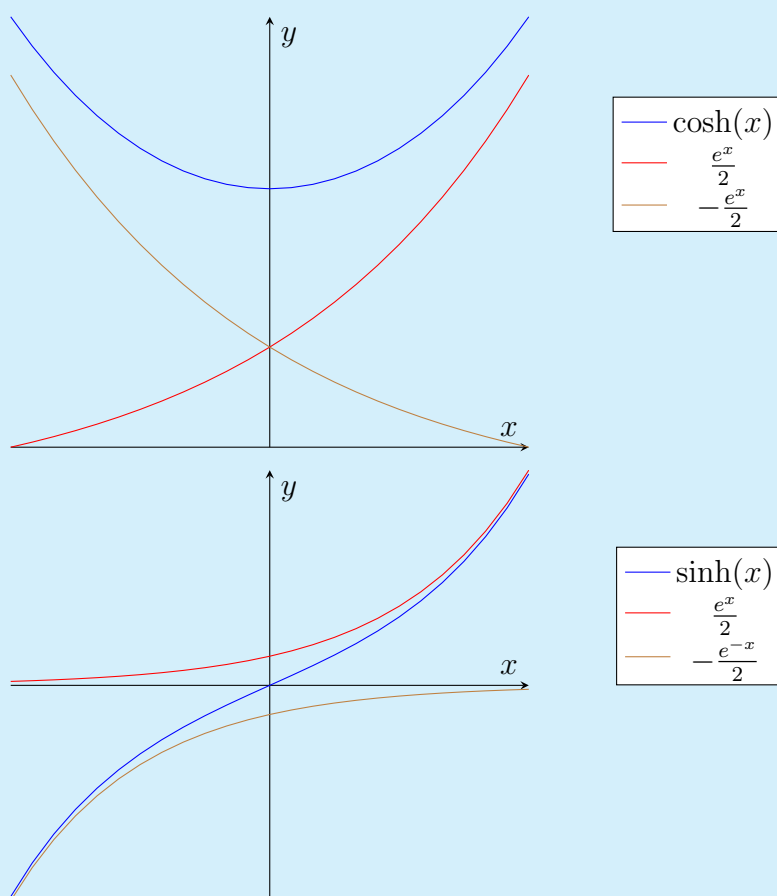
It can be shown that

$$\cosh x + \sinh x = e^x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\frac{d}{dx} \sinh x = \cosh x$$

$$\frac{d}{dx} \cosh x = \sinh x$$



To extend these to  $\mathbb{C}$ , we define

$$\cosh z = \frac{1}{2}(e^z + e^{-z}) \quad \sinh z = \frac{1}{2}(e^z - e^{-z})$$

### 3.2 Trigonometric and Logarithmic Function

**Definition 3.7. Trigonometric Functions:** Recall

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

Sum to get  $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ , and  $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

We define

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \cosh(iz) \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{i} \sinh(iz)$$

Furthermore:

$$\cos(iz) = \frac{e^{-z} + e^z}{2} = \cosh z$$

$$\sin(iz) = \frac{e^{-z} - e^z}{2i} = i \sinh z$$

$$\text{For real } z, \quad e^z = e^x = \cosh x + \sinh x$$

$$\text{For imaginary } z, \quad e^z = e^{iy} = \cos y + i \sin y$$

The  $\cosh x$  and  $\cos y$  are the even parts, and  $\sinh x$  and  $i \sin y$  are the odd parts

Functions	Along Real Axis	Along Imaginary Axis
$e^{iz}, \cos z, \sin z$	periodic	grow exponentially
$e^z, \cosh z, \sinh z$	grow exponentially	periodic

Familiar identities hold true.

**Example 3.8.**

$$\begin{aligned}
\cos^4 \theta &= \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right)^4 \\
&= \frac{1}{16} (e^{i4\theta} + 4e^{2\theta} + 6 + 4e^{-i2\theta} + e^{-i4\theta}) \\
&= \frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}
\end{aligned}$$

**Example 3.9.**

$$\begin{aligned}
\cos^2 \theta + \sin^2 \theta &= 1 \\
\Rightarrow \cos^2(iy) + \sin^2(iy) &= 1 \\
\Rightarrow \cosh^2 y + i^2 \sinh^2 y &= 1 \\
\Rightarrow \cosh^2 y - \sinh^2 y &= 1
\end{aligned}$$

By using the rules  $\begin{cases} \cos(iz) = \cosh z \\ \sin(iz) = i \sinh z \end{cases}$ .

Notice the “Obsborne’s rule” here: Hyperbolic function satisfy the same identities as trigonometric functions except that we must change the sign of every product of two sines.

Derivatives:  $e^z$  is entire, and so is  $\cos z, \sin z, \cosh z, \sinh z$ . Also,

$$\frac{d}{dz}(\cos z) = \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = \frac{ie^{iz} - ie^{-iz}}{2} = \frac{e^{iz} - e^{-iz}}{-2i} = -\sin z$$

Other as expected as well

Note: we can also define  $\tan z, \sec z$  etc. in the usual ways, and derivatives of them are as expected.

**Example 3.10.** What is the value of  $\sin(\pi + i)$ ?

**Solution:**

$$\begin{aligned}
\sin(\pi + i) &= \sin \pi \cos(i \cdot 1) + \cos \pi \sin(i \cdot 1) \\
&= \sin \pi \cosh 1 + \cos \pi i \sinh(1) \\
&= 0 + (-1) \cdot i \cdot \sinh(1) \\
&= -\sin i
\end{aligned}$$

**Example 3.11.** Find all solutions of  $\sin z = 1000$

**Solution:** We write  $\sin(x + yi) = 1000$ , and get that

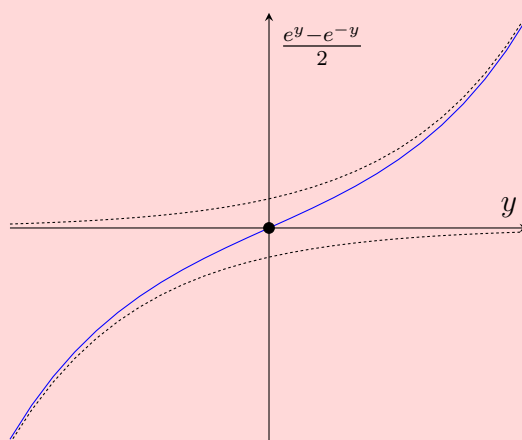
$$\sin x \cosh y + i \cos x \sinh y = 1000$$

So

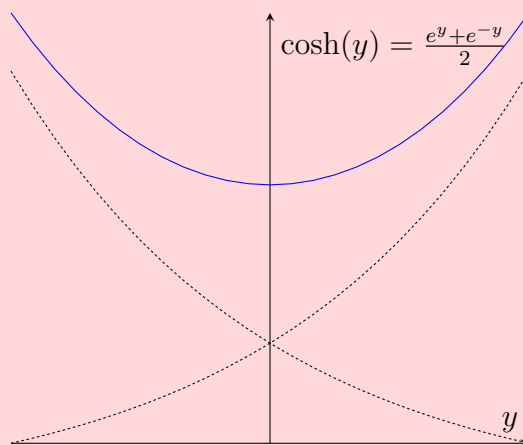
$$\begin{cases} \sin x \cosh y = 1000 & \dots\dots (1) \\ \cos x \sinh y = 0 & \dots\dots (2) \end{cases}$$

Equation 2 gives that  $\cos x = 0$  or  $\sinh y = 0$ , which yields that  $x = (2n + 1)\frac{\pi}{2}$  or  $y = 0$ .

The following figure shows that the only  $x$  that  $\sinh(x) = 0$  is at  $x = 0$ .



- If  $y = 0$ , equation 1 gives that  $\sin x \cosh(0) = \sin x = 1000$ . This is impossible
- If  $x = (2n + 1)\frac{\pi}{2}$ , then equation 1 gives  $\sin\left((2n + 1)\frac{\pi}{2}\right) \cosh y = 1000$ , so  $\cosh y = 1000 \cdot (-1)^n$



But  $\cosh y > 0$ , so use  $n = 2N$  (always even). So  $\cosh y = 1000$ , and  $y = \pm \cosh^{-1}(1000) \approx \pm 7.6$  (There are two solutions, i.e. note the  $\pm$  sign, as the figure



above shows).

The final answer is that  $z = x + iy = (4N + 1)\frac{\pi}{2} \pm i \cosh^{-1}(1000)$

### 3.3 Logarithmic Functions

How to define  $\log z$ ? Let  $z = e^w$  and solve for  $w$ . Note that:

- exponential function is periodic, so  $\log$  will be a “multi-valued function”
- in  $\mathbb{C}$ , we use “ $\log$ ” instead of “ $\ln$ ”

**Definition 3.12.** Now,

$$\begin{aligned} z = e^w &\Rightarrow r e^{i\theta + 2\pi k} = e^{u+iv} \\ &\Rightarrow r = e^u, \quad \theta + 2\pi k = v \\ &\Rightarrow u = \ln r, \quad v = \theta + 2\pi k \end{aligned}$$

So, we define

$$\log z = \ln |z| + i \arg z$$

**Example 3.13.** •  $\log(1 + i) = \ln |1 + i| + i \arg(1 + i) = \ln \sqrt{2} + i \left( \frac{\pi}{4} + 2\pi k \right)$

•  $\log(i) = \ln |i| + i \arg(i) = 0 + i \left( \frac{\pi}{2} + 2\pi k \right)$

**Proposition 3.14.** We have the following identity:

$$\begin{aligned} \log(z_1 z_2) &= \ln |z_1 z_2| + i \arg(z_1 z_2) \\ &=^* \ln |z_1| + \ln |z_2| + i(\arg z_1 + \arg z_2) \\ &= \log(z_1) + \log(z_2) \end{aligned}$$

Similarly

$$\log\left(\frac{z_1}{z_2}\right) =^* \log(z_1) - \log(z_2)$$

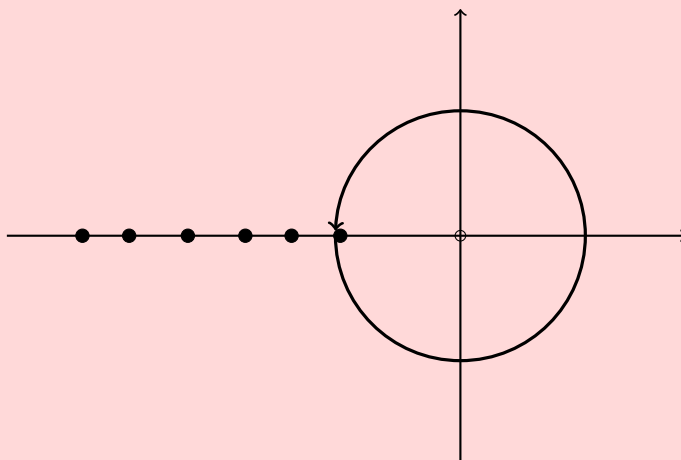
By  $=^*$ , we actually mean that the set of values of  $\log(z_1 z_2)$  is equal to the set of values of  $\log(z_1) + \log(z_2)$ , due to the multi-valuedness of  $\log$ .

**Definition 3.15.** The principle value of the Logarithm is

$$\operatorname{Log}(z) = \ln |z| + i \underbrace{\operatorname{Arg}(z)}_{\in (-\pi, \pi] \text{ usually}}$$

**Example 3.16.** •  $\operatorname{Log}(1+i) = \ln |1+i| + i \operatorname{Arg}(1+i) = \ln \sqrt{2} + i \frac{\pi}{4}$

- $\operatorname{Log}(i) = \ln |i| + i \operatorname{Arg}(i) = 0 + i\pi$
- $\operatorname{Log} e^z = z$  if and only if  $\operatorname{Im}(z) \in (-\pi, \pi]$
- $\operatorname{Log} z$  has discontinuity on negative real axis



- $\operatorname{Log} z$  is analytic everywhere else, with

$$\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}$$

**Proof 3.17.** Let

$$\begin{aligned} w = \operatorname{Log} z &= \ln |z| + i \operatorname{Arg}(z) \\ &= \frac{1}{2} \ln(x^2 + y^2) + i \left( \arctan\left(\frac{y}{x}\right) \pm \pi \right) \\ &= u(x, y) + iv(x, y) \end{aligned}$$

Now,

$$\begin{aligned}
 \frac{dw}{dz} &= \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \\
 &= \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} \\
 &= \frac{x - iy}{x^2 + y^2} \cdot \frac{x + iy}{x + iy} \\
 &= \frac{1}{z}
 \end{aligned}$$

□

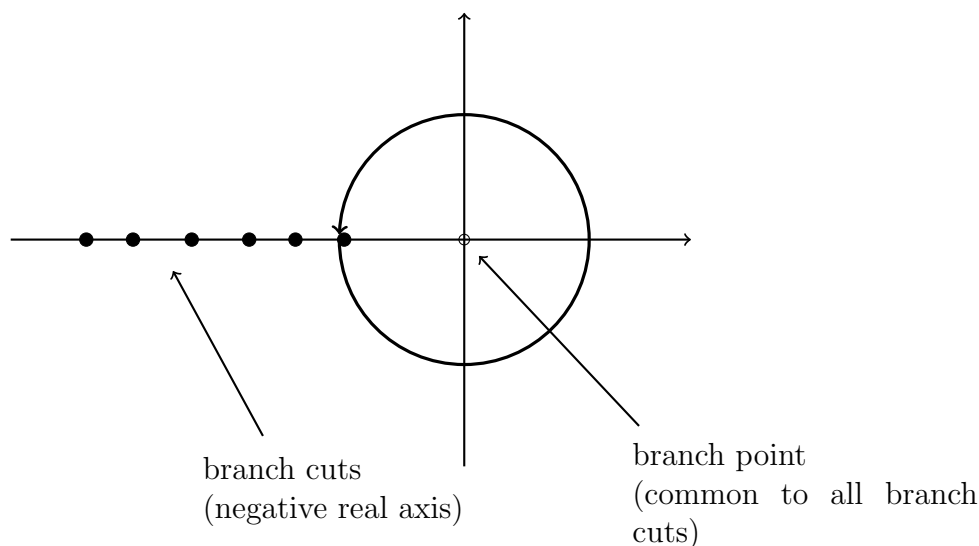
**Definition 3.18. Branch Cuts:** Let  $f(z)$  be a multivalued function.  $F(z)$  is said to be a **branch** of  $f(z)$  on a domain  $D$  if  $F(z)$  is continuous on  $D$  and for each  $z \in D$ ,  $F(z)$  is one and only one of the values of  $f(z)$ .

**Example 3.19.**  $\text{Log } z$  is a branch of  $\log z$

We could define different branches of  $\log z$  by

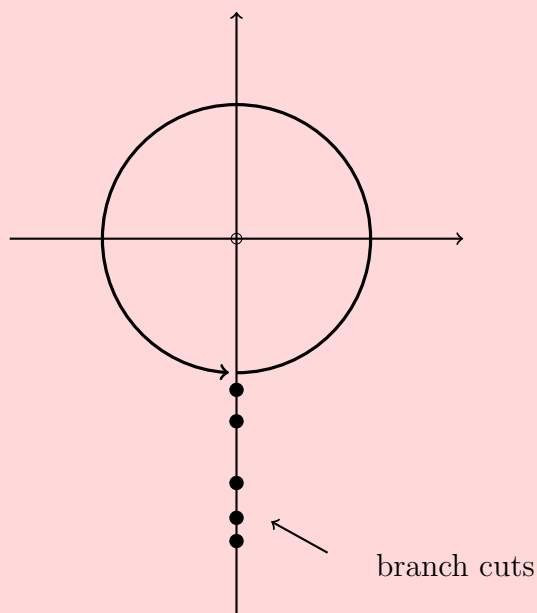
$$\text{Log}_\tau z = \ln |z| + i \text{Arg}_\tau(z)$$

where  $\text{Arg}_\tau(z) \in (\tau, \tau + 2\pi]$ . Note that  $\text{Log } z = \text{Log}_{-\pi}$



**Example 3.20.**

$$\text{Log}_{-\frac{\pi}{2}} \ln |z| + i \text{Arg}_{-\frac{\pi}{2}}(z)$$



**Example 3.21.** Find a branch of  $f(z) = \log(z + 4)$  that is analytic at  $z = -5$  and equals  $7\pi i$  there.

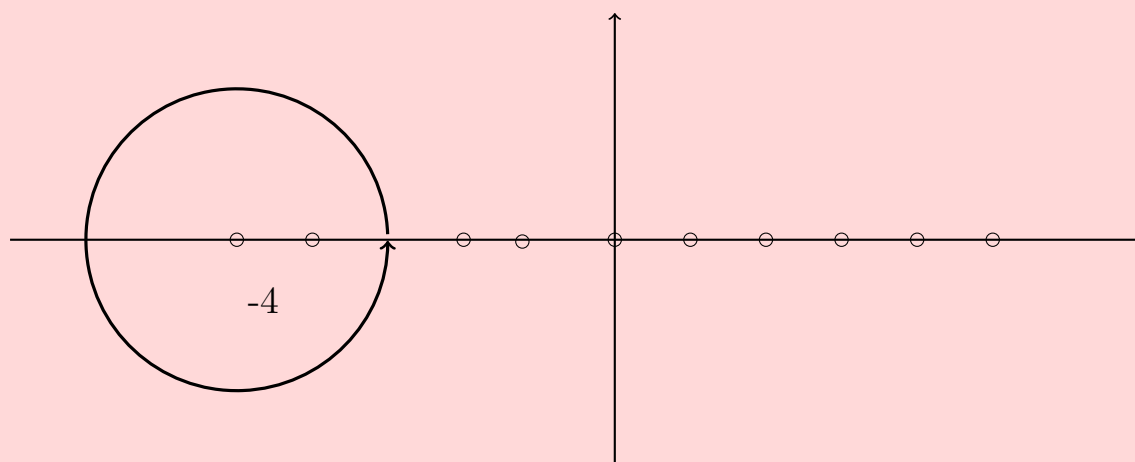
**Solution:** We want  $\text{Log}_\tau(-5 + 4) = \text{Log}_\tau(-1) = 7\pi i$  for some  $\tau$ .

So,  $\ln|-1| + i \text{Arg}_\tau(-1) = 7\pi i$  for some  $k$ , i.e.

$$0 + i \underbrace{(\pi + 2k\pi)}_{\in (\tau, \tau + 2\pi]} = 7\pi i \quad \text{for some } k$$

Hence,  $k = 3$ . We can choose  $\tau = 6\pi$  so that  $7\pi \in (6\pi, 8\pi]$ .

The final answer would be  $F(z) = \text{Log}_{6\pi}(z + 4)$



**Example 3.22.** Where is  $f(z) = \text{Log}(z^2 + 1)$  analytic?

**Solution:** We need  $z^2 + 1 \neq 0$  and not equal to negative real number.

So,  $z^2 + 1 = (x + yi)^2 + 1 = (x^2 - y^2 + 1) + i(2xy)$ .

$$z^2 + 1 = 0 \text{ when } \begin{cases} x = 0 \text{ and } y = \pm 1 \\ \text{or} \\ y = 0 \text{ and } x^2 + 1 = 0 \end{cases} \text{ This is impossible for } x \in \mathbb{R}$$

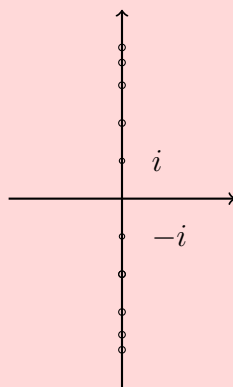
Hence,  $z = \pm i$  here.

$$z^2 + 1 < 0 \text{ (real) when } \begin{cases} x = 0 \text{ and } 1 - y^2 < 0 \Rightarrow y^2 > 1 \Rightarrow y > 1 \text{ or } y < -1 \\ \text{or} \\ y = 0 \text{ and } 1 + x^2 < 0 \end{cases} \text{ Impossible}$$

Hence,  $z = iy$  where  $|y| > 1$ .

For all other points,

$$f'(z) = \frac{2z}{z^2 + 1}$$



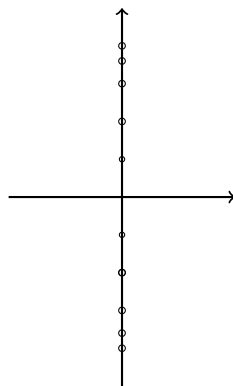
Here is another way to solve the above problem.

$$\text{Log}(z^2 + 1) = \text{Log}((z + i)(z - i)) = \text{Log}_{\tau_1}(z + i) + \text{Log}_{\tau_2}(z - i)$$

for some  $\tau_1, \tau_2$

Some possibilities are:

- $\tau_1 = \frac{-\pi}{2}, \tau_2 = \frac{-3\pi}{2}$
- $\tau_1 = \frac{3\pi}{2}, \tau_2 = \frac{-7\pi}{2}$
- $\dots$

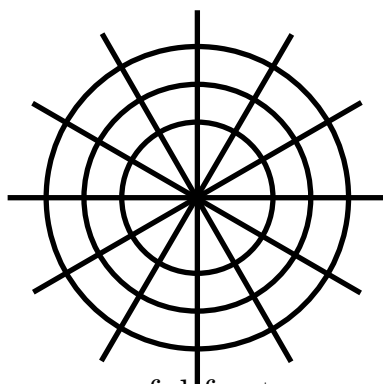


Finally, note that

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z$$

$\operatorname{Log} z$  is analytic, so  $\ln |z|$  and  $\operatorname{Arg} z$  are harmonic.

Level curves of  $\ln |z| = k$  and  $\operatorname{Arg} z = k$  are circles and rays. This would be particularly useful when we deal with temperature problems later.



useful for temp problems later

### 3.4 Complex Powers and Inverse Trigonometric Functions

**Definition 3.23. Complex Powers:** We define

$$z^\alpha = e^{\alpha \log z} \quad \text{for } \alpha \in \mathbb{C}, z \neq 0$$

**Example 3.24.** 1.

$$\begin{aligned}
4^{1/2} &= e^{\frac{1}{2} \log 4} \\
&= e^{\frac{1}{2} (\ln |4| + i \arg(4))} \\
&= e^{\frac{1}{2} \ln 4 + i \frac{1}{2} (0 + 2\pi k)} \\
&= e^{\frac{1}{2} 2 \ln 2 + i\pi k} \\
&= e^{\ln 2} e^{i\pi k} \\
&= 2 \cdot (\pm 1) \\
&= \pm 2
\end{aligned}$$

2.

$$\begin{aligned}
(1+i)^3 &= e^{3 \log(1+i)} \\
&= e^{3 (\ln \sqrt{2} + i \arg(1+i))} \\
&= e^{\frac{3}{2} \ln 2} e^{i3 \left( \frac{\pi}{4} + 2k\pi \right)} \\
&= (e^{\ln 2})^{\frac{3}{2}} \cdot \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \\
&= 2^{\frac{3}{2}} \cdot \left( \frac{-1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) \\
&= -2 + 2i
\end{aligned}$$

3.

$$\begin{aligned}
i^i &= e^{i \ln |i| + i \arg i} \\
&= e^{i \left( 0 + i \left( \frac{\pi}{2} + 2k\pi \right) \right)} \\
&= e^{-\left( \frac{\pi}{2} + 2\pi k \right)} \\
&= \dots, e^{-\frac{5\pi}{2}}, e^{-\frac{\pi}{2}}, e^{\frac{3\pi}{2}}, \dots
\end{aligned}$$

If we want a single value, take the principal branch to be  $e^{\alpha \operatorname{Log} z}$ , which is analytic everywhere  $\operatorname{Log} z$  is, and

$$\frac{d}{dz} z^\alpha = \frac{d}{dz} e^{\alpha \operatorname{Log} z} = e^{\alpha \operatorname{Log} z} \cdot \frac{\alpha}{z} = z^\alpha \cdot \frac{\alpha}{z} = \alpha z^{\alpha-1}$$

as expected.

**Definition 3.25. Inverse Trigonometric Functions:** First, we see that  $w = \sin^{-1} z$  means  $z = \sin w$ , etc. Also, we've accepted multivalued functions.

In  $\mathbb{R}$ , the inverse hyperbolic function can be expressed in terms of logs:

$$\begin{aligned}y &= \sinh x = \frac{1}{2}(e^x - e^{-x}) \\e^x - 2y - e^{-x} &= 0 \\(e^x)^2 - 2y(e^x) - 1 &= 0 \quad \text{note that this is a quadratic equation for } e^x \\e^x &= \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1} \quad \text{we take the plus sign since } e^x > 0\end{aligned}$$

So,  $x = \ln(y + \sqrt{y^2 + 1}) = \sinh^{-1} y$ .

In  $\mathbb{C}$ , we define  $\sinh^{-1} z = \log(z + \sqrt{z^2 + 1})$ .

Similarly,  $\sin^{-1} z = -i \log(iz + (1 - z^2)^{\frac{1}{2}})$ . Note that for this definition, it involves two sets of branches, one with  $\log$ , and the other one with  $(1 - z^2)^{\frac{1}{2}}$



## Chapter 4 Complex Integration

### 4.1 Contours

How to integrate in  $\mathbb{C}$ ?

Complex valued functions of a real variable are easy to integrate:

$$\int_a^b \left( u(t) + iv(t) \right) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt$$

**Example 4.1.** 1.

$$\int_0^1 (t+i)^2 dt = \int_0^1 \left( (t^2 - 1) + i(2t) \right) dt = \frac{-2}{3} + 2i$$

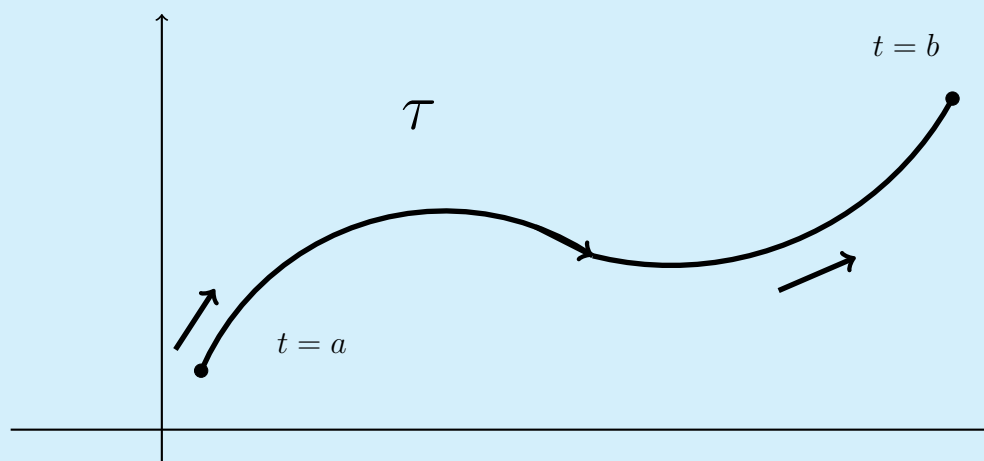
2. We can use a special trick (instead of using integration by parts twice).

$$\begin{aligned} \int_0^\pi e^{2x} \cos x dx &= \int_0^\pi e^{2x} (\operatorname{Re}(e^{ix})) dx \\ &= \operatorname{Re} \left( \int_0^\pi e^{(2+i)x} dx \right) \\ &= \operatorname{Re} \left( \frac{e^{(2+i)x}}{2+i} \Big|_0^\pi \right) \\ &= \operatorname{Re} \left( \frac{e^{2x}(\cos x + i \sin x)}{2+i} \cdot \frac{2-i}{2-i} \Big|_0^\pi \right) \\ &= \left[ \frac{2}{5} e^{2x} \cos x + \frac{1}{5} e^{2x} \sin x \right]_0^\pi \\ &= -\frac{2}{5} e^{-2\pi} + \frac{2}{5} \end{aligned}$$

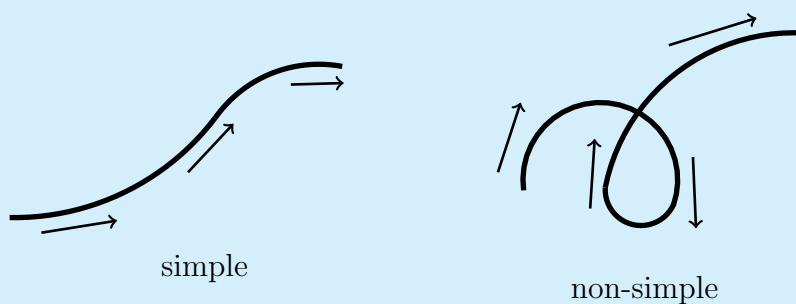
What about integrating a function of a complex variable?

We will replace the intervals with paths.

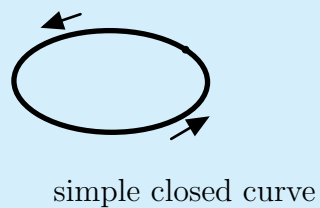
**Definition 4.2.** Let  $z(t) = x(t) + iy(t)$  on  $t \in [a, b]$  be continuous. The range is a curve  $\mathcal{C}$ , and is called a smooth curve if  $z'(t)$  is continuous and non-zero on  $[a, b]$



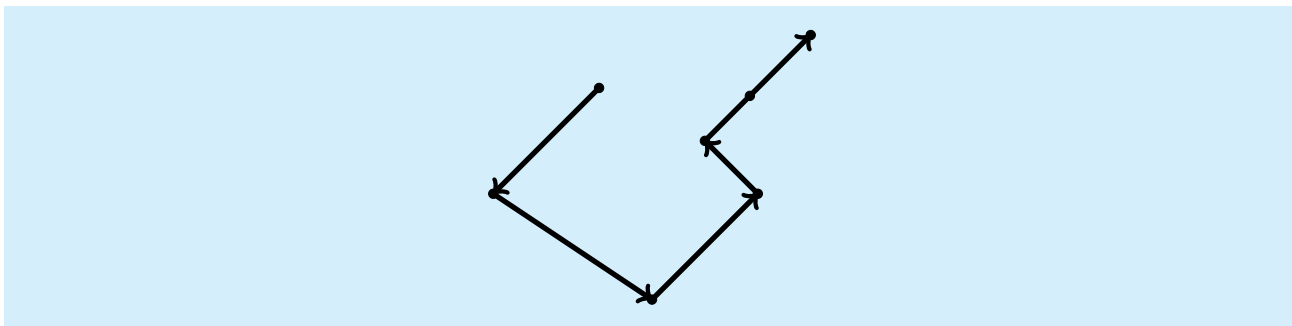
A curve is called **simple** if  $z(t_1) \neq z(t_2)$  whenever  $t_1 \neq t_2$  for  $a < t_i < b$  (basically no self intersection)



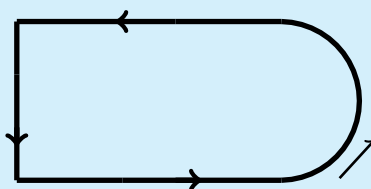
If  $z(a) = z(b)$ , then the curve is called a **closed** curve.



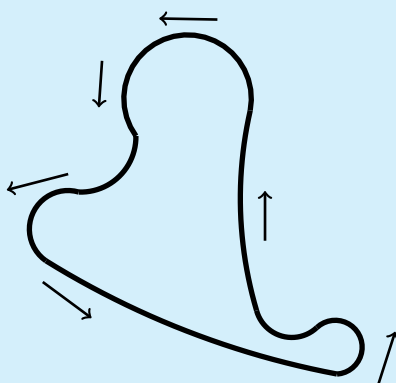
**Definition 4.3. Contour:** a curve that is composed of finitely many smooth curves, joined end-to-end



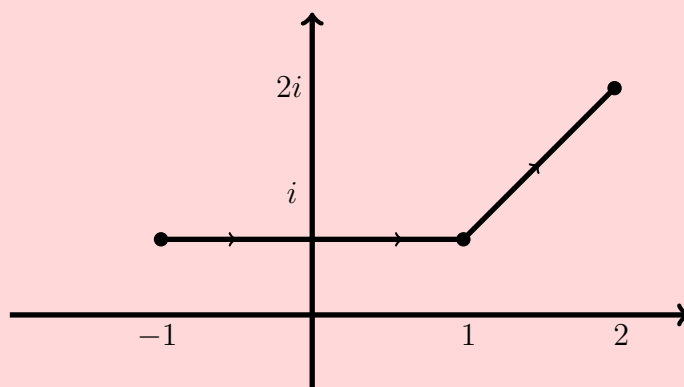
**Definition 4.4. Jordan Curve:** a simple closed contour.



**Definition 4.5. Positively Oriented:** means its interior lies to the left as we follow the curve



**Example 4.6.** Parameterize this:



**Solution:** Line segment from  $z_0$  to  $z_1$  can be parameterized as:  $z(t) = z_0 + (z_1 - z_0)t, t \in [0, 1]$ .

For the first curve,

$$\begin{aligned} z_1(t) &= (-1 + i) + (1 + i - (-1 + i))t \\ &= -1 + i + 2t, \quad t \in [0, 1] \end{aligned}$$

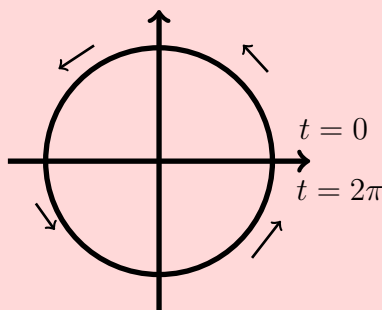
For the second curve,

$$\begin{aligned} z_2(t) &= (1 + i) + (2 + 2i - (1 + i))t \\ &= 1 + i + (1 + i)t, \quad t \in [0, 1] \end{aligned}$$

Put everything together we get

$$z(t) = \begin{cases} -1 + i + 2t & t \in [0, 1) \\ 1 + i + (1 + i)(t - 1) & t \in [1, 2] \end{cases}$$

**Example 4.7.** Let  $\mathcal{C}$  be a unit circle centered at 0.



**Solution:**  $\mathcal{C} : z(t) = e^{it} \quad t \in [0, 2\pi]$

**Example 4.8.** Circle, radius  $r_0$ , centered at  $z_0$ ?

**Solution:**  $\mathcal{C} : z(t) = z_0 + r_0 e^{it} \quad t \in [0, 2\pi]$

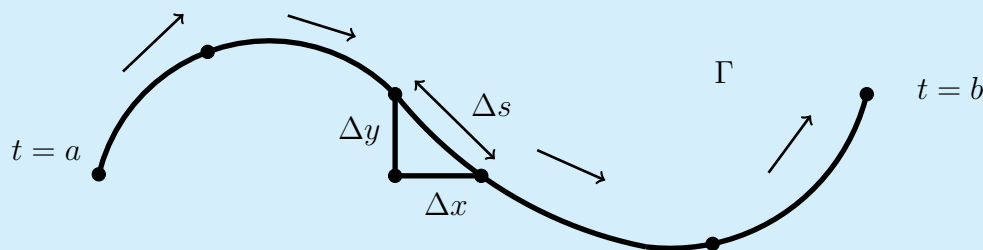
**Example 4.9.** Parameterize  $y = f(x), x \in [a, b]$

**Solution:** just let  $x(t) = t$ ,

$$z(t) = x(t) + iy(t) = t + if(t), \quad t \in [a, b]$$

For example,  $y = x^2$  will be parameterized as  $z(t) = t + it^2$

**Definition 4.10. Arclength:** We define the arclength as follows:



Partition the curve

$$\begin{aligned}\Delta s &\approx \sqrt{\Delta x^2 + \Delta y^2} \\ &= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \Delta t\end{aligned}$$

Sum all pieces and let  $\Delta t \rightarrow 0$  (Performing a Riemann Sum there):

$$\begin{aligned}L &= \int_R ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_a^b \left| \frac{dz}{dt} \right| dt \quad \text{we use modulus here}\end{aligned}$$

The physical interpretation could be:  $\text{total\_distance} = \int_a^b (\text{speed}) dt$

Now we are ready to integrate  $f(z)$  along a curve.

## 4.2 Contour Integrals

Partition curve  $\mathcal{C}$  as shown. Sum, and let  $\max |\Delta z_k| \rightarrow 0$ :

$$\int_{\mathcal{C}} f(z) dz = \lim_{\max |\Delta z_k| \rightarrow 0} \sum_k f(z_k^*) \Delta z_k$$

See the text for more detail.

If  $\mathcal{C}$  is a single point, define  $\int_{\mathcal{C}} f(z) dz = 0$ .

How to calculate?

**Definition 4.11.** Assume  $\mathcal{C}$  has a parameterization. Call it  $z(t), t \in [a, b]$ . Then:

$$\begin{aligned}\int_{\mathcal{C}} f(z) dz &= \lim_{\max |\Delta z_k| \rightarrow 0} \sum_k f(z_k^*) \frac{\overbrace{z(t_k) - z(t_{k-1})}^{z_k}}{\Delta t_k} \Delta t_k \\ &= \int_a^b f(z) z'(t) dt\end{aligned}$$

**Proposition 4.12.** Properties:

- $\int_{\mathcal{C}} (f(z) + g(z)) dz = \int_{\mathcal{C}} f(z) dz + \int_{\mathcal{C}} g(z) dz$
- $\int_{\mathcal{C}} k f(z) dz = k \int_{\mathcal{C}} f(z) dz$
- $\int_{-\mathcal{C}} f(z) dz = - \int_{\mathcal{C}} f(z) dz$ . Here  $-\mathcal{C}$  means  $\mathcal{C}$  traversed in the opposite direction
- $\int_{\mathcal{C}_1 + \mathcal{C}_2} f(z) dz = \int_{\mathcal{C}_1} f(z) dz + \int_{\mathcal{C}_2} f(z) dz$ . Here it means that we traverse  $\mathcal{C}_1$  then traverse  $\mathcal{C}_2$ .

Is there a triangle inequality? i.e.

$$\left| \int_{\mathcal{C}} f(z) dz \right| \leq? \int_{\mathcal{C}} |f(z)| dz$$

No! LHS is real, but RHS is complex. “ $\leq$ ” does NOT make any sense here.

**Proposition 4.13. The “ML” Inequality:** If  $f(z)$  is continuous on a contour  $\mathcal{C}$ , then

$$\left| \int_{\mathcal{C}} f(z) dz \right| \leq ML$$

where  $M$  is an upper bound for  $|f(z)|$  on  $\mathcal{C}$  and  $L$  is the length of  $\mathcal{C}$ .

**Proof 4.14.** Let  $z(t)$ ,  $t \in [a, b]$  be a parameterization of  $\mathcal{C}$ . Then

$$\begin{aligned} \left| \int_{\mathcal{C}} f(z) dz \right| &= \left| \int_a^b f(z(t)) z'(t) dt \right| \\ &\leq \int_a^b \left| f(z(t)) z'(t) \right| dt \quad \text{by triangle inequality for integrals w.s.t. real variables} \\ &= M \int_a^b \left| z'(t) \right| dt \\ &= ML \end{aligned}$$

Second last step: since  $|f(z)| \leq M$  on  $\mathcal{C}$ .

Last step: from last lecture. See Definition 4.10. □

**Example 4.15.** Find an upper bound on  $\left| \int_{\mathcal{C}} e^{\frac{1}{z}} \right|$

**Solution:**  $M = ?$

$$\begin{aligned} \left| e^{\frac{1}{z}} \right| &= \left| e^{\frac{1}{x+iy}} \right| \\ &= \left| e^{\frac{x-iy}{x^2+y^2}} \right| \\ &= \left| e^{\frac{x}{x^2+y^2}} \cdot e^{-i \frac{y}{x^2+y^2}} \right| \\ &\leq e^{\frac{x}{x^2+y^2}} \quad \text{since } x^2 + y^2 = 1 \\ &\leq e^1 \quad \text{since } x \leq 1 \end{aligned}$$

Clearly,  $L = 2\pi$ , so  $\left| e^{\frac{1}{z}} \right| \leq e^1 \cdot 2\pi = 2\pi e$  by ML inequality.

**Example 4.16.** Evaluate  $\int_{\mathcal{C}} \cos z dz$  where  $\mathcal{C}$  is the line segment from 0 to  $1+2i$ .

**Solution:** Parameterize  $\mathcal{C}$  by

$$z(t) = 0 + (1 + 2i - 0)t, \quad t \in [0, 1]$$

Then

$$\int_{\mathcal{C}} \cos z dx = \int_0^1 \underbrace{\cos \left( (1 + 2i)t \right)}_{f(z(t))} \cdot \underbrace{(1 + 2i)}_{z'(t)} dt = \sin \left( (1 + 2i)t \right) \Big|_0^1 = \sin(1+2i) - 0 = \sin(1+2i)$$

**Example 4.17.** Evaluate  $\int_{\mathcal{C}} \cos z dz$  where  $\mathcal{C}$  is:

**Solution:**  $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$  where  $\begin{cases} \mathcal{C}_1 : z(t) = t, & t \in [0, 1) \\ \mathcal{C}_2 : z(t) = 1 + (t - 1)i, & t \in [1, 3] \end{cases}$ . So

$$\begin{aligned} \int_{\mathcal{C}} \cos z dx &= \int_{\mathcal{C}_1} \cos z dx + \int_{\mathcal{C}_2} \cos z dx \\ &= \int_0^1 \cos t dt + \int_1^3 \cos(1 + (t - 1)i) i dt \\ &= \sin t \Big|_0^1 + \sin(1 + (t - 1)i) \Big|_1^3 \\ &= \sin(1) + (\sin(1 + 2i) - \sin(1)) \\ &= \sin(1 + 2i) \end{aligned}$$

As before

**Example 4.18.** Evaluate  $\int_{\mathcal{C}} e^z dz$  where  $\mathcal{C}$  is part of  $y = x^2 + 1$  from  $z = i$  to  $z = 2 + 5i$ .

**Solution:** Let  $z(t) = \underbrace{t}_x + \underbrace{(t^2 + 1)i}_y$ ,  $t \in [0, 2]$ . Then,

$$\begin{aligned} \int_{\mathcal{C}} e^z dz &= \int_0^2 e^{z(t)} z'(t) dt \\ &= \int_0^2 e^{t^2 + (t^2 + 1)i} (1 + 2ti) dt \\ &= e^{t^2 + (t^2 + 1)i} \Big|_0^2 \\ &= e^{2 + 5i} - e^i \\ &= e^z \Big|_i^{2 + 5i} \end{aligned}$$

Does it always work that way? See the following example

**Example 4.19.** Evaluate  $\int_{\mathcal{C}} \bar{z} dz$  where

1.  $\mathcal{C}$  is line segment from 0 to  $1 + i$
2.  $\mathcal{C}$  is the smallest arc of circle  $x^2 + (y - 1)^2 = 1$  from 0 to  $1 + i$

**Solution:**

1. parameterization:  $z(t) = t(1 + i)$ ,  $t \in [0, 1]$



$$\begin{aligned}
 \int_C \bar{z} dz &= \int_0^1 t(1-i) \cdot (1+i) dt \\
 &= \int_0^1 2t dt \\
 &= 1
 \end{aligned}$$

2. parameterization:  $z(t) = e^{it} + i$ ,  $t \in [-\frac{\pi}{2}, 0]$ . It's the unit circle, shifted up by 1 unit.

$$\begin{aligned}
 \int_C \bar{z} dz &= \int_{-\pi/2}^0 (e^{-it} - i)(ie^{it}) dt \\
 &= \dots \\
 &= 1 + i\left(\frac{\pi}{2} - 1\right) \\
 &\neq 1
 \end{aligned}$$

So, the general answer is no. Different paths might yield different results.

### 4.3 Independence of Path

#### **Theorem 4.20. Complex Extension of Fundamental Theorem of Calculus:**

If  $f(z)$  is continuous in a domain  $D$  and has antiderivative  $F(z)$  throughout  $D$ , then, for any contour  $\mathcal{C}$  lying in  $D$  with initial point  $z_1$  and terminal point  $z_2$ , we have

$$\int_{\mathcal{C}} f(z) dz = F(z_2) - F(z_1)$$

**Proof 4.21.** First, suppose  $\mathcal{C}$  is smooth, i.e.  $z'(t) \neq 0$ , continuous.

Parameterize by  $z(t), t \in [a, b]$ . Then,

$$\begin{aligned}
 \int_{\mathcal{C}} f(z) dz &= \int_a^b f(z(t)) z'(t) dt \\
 &= \int_{\mathcal{C}} \frac{d}{dt} \left( F(z(t)) \right) dt \quad \text{by chain rule} \\
 &= F(z(t)) \Big|_{t=a}^b \\
 &= F(z(b)) - F(z(a)) \\
 &= F(z_2) - F(z_1)
 \end{aligned}$$

Next, if  $\mathcal{C}$  is not smooth, it has a finite number of smooth pieces, since it's a contour.

Apply the result above to each piece:

$$\begin{aligned}
 \int_{\mathcal{C}} f(z) dz &= \int_{\mathcal{C}_1} f(z) dz + \cdots + \int_{\mathcal{C}_n} f(z) dz \\
 &= \left( F(a_1) - F(a_0) \right) + \left( F(a_2) - F(a_1) \right) + \cdots + \left( F(a_n) - F(a_{n-1}) \right) \\
 &= F(a_n) - F(a_0) \\
 &= F(z_2) - F(z_1)
 \end{aligned}$$

□

**Example 4.22.** Evaluate  $\int_{\mathcal{C}} (1 + z^2) dz$  where  $\mathcal{C}$  is:

**Solution:**

$$\begin{aligned}
 \int_{\mathcal{C}} (1 + z^2) dz &= \left( z + \frac{z^3}{4} \right) \Big|_{z=0}^{z=4+2i} \\
 &= \dots \\
 &= \frac{28}{3} + \frac{94}{3}i
 \end{aligned}$$

**Example 4.23.** Evaluate  $\int_{\mathcal{C}} e^z dz$  where  $\mathcal{C}$  is:

**Solution:**

$$\begin{aligned}\int_C (1 + z^2) dz &= e^z \Big|_{z=1-i}^{z=1-i} \\ &= e^{1-i} - e^{1-i} \\ &= 0\end{aligned}$$

**Theorem 4.24.** Let  $f$  be a continuous function in a domain  $D$ . Then, the following statements are equivalent:

1.  $f$  has an antiderivative in  $D$ .
2. If  $\mathcal{C}$  is any closed contour in  $D$ , then  $\int_{\mathcal{C}} f(z) dz = 0$ .
3. The contour integrals of  $f$  are independent of path in  $D$ .

**Proof 4.25.**  $1 \Rightarrow 2$ : It follows immediately from Theorem 4.20 with  $\mathcal{C}$  being a closed contour.

$2 \Rightarrow 3$ : Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be any two contours in  $D$  with same end points. Let  $\mathcal{C}$  be the closed contour  $\mathcal{C}_1 + (-\mathcal{C}_2)$ .

Then,  $\int_{\mathcal{C}} f(z) dz = 0$ . So  $\int_{\mathcal{C}_1} f(z) dz + \int_{-\mathcal{C}_2} f(z) dz = 0$ . So  $\int_{\mathcal{C}_1} f(z) dz - \int_{\mathcal{C}_2} f(z) dz = 0$ , implying that

$$\int_{\mathcal{C}_1} f(z) dz = \int_{\mathcal{C}_2} f(z) dz$$

$3 \Rightarrow 1$ : Construct the antiderivative. Choose a point  $z_0 \in D$ , and let  $\mathcal{C}$  be the contour as shown. Recall the  $D$  is a connected set.

Define  $F(z) = \int_{\mathcal{C}} f(w) dw$ . By 3,  $F(z)$  is single valued; We will show that  $F'(z) = f(z)$ .

For any point  $z$ , choose  $\Delta z$  small enough such that the line segment  $\Gamma$  parameterized by

$$z(t) = z + t\Delta z, \quad t \in [0, 1]$$

is in  $D$  (This is possible since  $D$  is open)

Then

$$\begin{aligned}
 F(z + \Delta z) - F(z) &= \left( \int_C f(w)dw + \int_\Gamma f(w)dw \right) - \int_C f(w)dw \\
 &= \int_\Gamma f(w)dw \\
 &= \int_0^1 f(z(t))z'(t)dt \\
 &= \int_0^1 f(z + t\Delta z)(\Delta z)dt \\
 \Rightarrow \frac{F(z + \Delta z) - F(z)}{\Delta z} &= \int_0^1 f(z + t\Delta z)dt
 \end{aligned}$$

Let  $\Delta z \rightarrow 0$ .

$$F'(z) = \int_0^1 f(z)dt = f(z) \int_0^1 dt = f(z)$$

□

We showed that  $\bar{z}$  can be integrated, but the result depends on path. So  $\bar{z}$  is integrable, but not anti-differentiable. Also, functions with antiderivatives are easy; for those without, we must parameterize.

## 4.4 Cauchy's Integral Theorem

**Example 4.26. Most Important Example in this Course:** Evaluate  $\int_C \frac{1}{z} dz$  where  $\mathcal{C}$  is the unit circle.

**Solution:**  $\frac{1}{z}$  does not have antiderivative over all of  $\mathcal{C}$ . Any branch of  $\log z$  will have a problem, i.e.  $\mathcal{C}$  will cross a branch cut.

Method 1: Parameterize  $\mathcal{C}$  by  $e^{it}$ ,  $t \in [0, 2\pi]$ . By definition,

$$\int_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it} dt = 2\pi i$$

Method 2: Split  $\mathcal{C}$  in two, and use Theorem 4.20 on each.

$$\begin{aligned}
 \int_{\mathcal{C}_1} \frac{1}{z} dz &= \text{Log } z \Big|_{-i}^i \quad \text{branch cut at } \theta = -\pi \\
 &= \text{Log } i - \text{Log}(-i) \\
 &= i\frac{\pi}{2} - i\left(\frac{-\pi}{2}\right) \\
 &= \pi i \\
 \int_{\mathcal{C}_2} \frac{1}{z} dz &= \text{Log}_0 z \Big|_i^{-i} \\
 &= \text{Log}_0(-i) - \text{Log}_0(i) \\
 &= \frac{3\pi}{2}i - \frac{\pi}{2}i \\
 &= \pi i
 \end{aligned}$$

Therefore,

$$\int_{\mathcal{C}} \frac{1}{z} dz = \int_{\mathcal{C}_1} \frac{1}{z} dz + \int_{\mathcal{C}_2} \frac{1}{z} dz = \pi i + \pi i = 2\pi i$$

Go around the contour twice, what's the result? It would be  $4\pi i$ . Also, going counter-clockwise would yield the result  $-2\pi i$

**Definition 4.27.** A closed contour  $\mathcal{C}$  is said to be continuously deformable to a contour  $\mathcal{C}_1$  in a domain  $D$  if there exists a function  $z(s, t)$ , continuous for  $s \in [0, 1], t \in [0, 1]$ , such that

1.  $z(s, t)$  is a closed contour in  $D$  for each  $s \in [0, 1]$
2.  $z(0, t)$  is a parameterization of  $\mathcal{C}$
3.  $z(1, t)$  is a parameterization of  $\mathcal{C}_1$

**Theorem 4.28. Deformation Invariance Theorem:** Let  $f$  be analytic in a domain  $D$ , containing closed contours  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . If  $\mathcal{C}_1$  can be continuously deformed into  $\mathcal{C}_2$ , then

$$\int_{\mathcal{C}_1} f(z) dz = \int_{\mathcal{C}_2} f(z) dz$$

**Proof 4.29.** It's too hard - 12 pages long in one text. □

**Definition 4.30.** A simply connected domain is a domain in which every “loop” (closed contour) in  $D$  can be continuously deformed to a point (while remaining in  $D$ ).

**Theorem 4.31.** Cauchy’s Integral Theorem (Cauchy-Goursat Theorem):

If  $f$  is analytic in a simply connected domain  $D$ , and  $\mathcal{C}$  is a closed contour in  $D$ , then

$$\int_{\mathcal{C}} f(z) dz = 0$$

**Proof 4.32.** Follows from Theorem 4.28 by shrinking  $\mathcal{C}$  continuously to a point.  $\square$

**Corollary 4.33.** Since  $\int_{\mathcal{C}} f(z) dz = 0 \Leftrightarrow f$  has an antiderivative in  $D$ , we have that if  $f$  is analytic, then  $f$  also has an antiderivative, which is analytic. So every analytic function is infinitely antiderivable.

**Example 4.34.** Back to Example 4.26. We know that  $\int_{\mathcal{C}} \frac{1}{z} dz = 2\pi i$  for any closed contour enclosing the origin.

Also,  $\int_{\mathcal{C}} \frac{1}{z} dz = 0$  for any closed contours not enclosing the origin.

Could shift results:

$$\int_{\mathcal{C}} \frac{1}{z - z_0} dz = \begin{cases} 0 & \text{if } z_0 \text{ is exterior to } \mathcal{C} \\ 2\pi i & \text{if } z_0 \text{ is interior to } \mathcal{C} \end{cases}$$