AMATH / PMATH 332 Course Notes

Applied Complex Analysis

Haochen Wu

University of Waterloo
Spring 2021

Contents

Chapter 1 Complex Numbers

1.1 Intro, Properties of Complex Numbers

Intro:

- What it's about: not like real analysis; some of intro to calculus on $\mathbb{C}$
- Goal: extend calculus on $\mathbb{R}$ to $\mathbb{C}$ - many results become simpler! (more complete picture here)
- Can be used to solve some $\mathbb{R}$ problems.

The Fundamentals:

- Basic idea: define solutions to $x^2 + 1 = 0$
- Early Mathematicians: $x = \pm\sqrt{-1}$. For $\sqrt{-1}$, should we call it i?
- Note: "$\sqrt{}$" always denotes positive root, e.g. $\sqrt{4} = 2$
- Problem:

$$\sqrt{-1}\sqrt{-1} = -1 \quad \text{by definition of } \sqrt{}$$
$$\sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1 \quad \text{since } \sqrt{ab} = \sqrt{a}\sqrt{b}$$

- Fix: intepret "$\sqrt{}$" differently for complex numbers - it must be multivalued, and define the imaginary unit i by $i^2 = 1$

Definition 1.1. Complex number:

$$z = \underbrace{a}_{\text{"real part" } \Re(z)} + i \underbrace{b}_{\text{"imaginary part" } \Im(z) \text{ which is real!}} \qquad \text{where } a, b \in \mathbb{R}$$

$\mathbb{C}$ = set of complex numbers. Note that $\mathbb{R} \subset \mathbb{C}$

Definition 1.2. Let $z = a + bi$, and $w = c + di$. Then:

- $z = w$ if and only if $a = c$ and $b = d$
- $z + w = (a + bi) + (c + di) = a + c + (b + d)i$
- $z - w = z + (-w) = (a + bi) + (-c - di) = a - c + (b - d)i$
- $zw = (a + bi)(c + di) = ac + bdi^2 + adi + bci = ac - bd + (ad + bc)i$
- $\dfrac{z}{w} = \dfrac{a + bi}{c + di} \cdot \dfrac{c - di}{c - di} = \dfrac{ac + bd}{c^2 + d^2} + i \cdot \dfrac{bc - ad}{c^2 + d^2}$

Example 1.3.

$$\frac{2+i}{1+2i} = \frac{2+i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{4}{5} - \frac{3}{5}i$$
$$\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = \frac{-i}{-i^2} = -i$$

Theorem 1.4. $z+w=w+z$, $k(z+w)=kz+kw$ apply as usual. $zw=wz$

Note: We can't classify complex numbers as "positive" or "negative", and can't use inequalities, e.g. $z>w$ doesn't make sense.

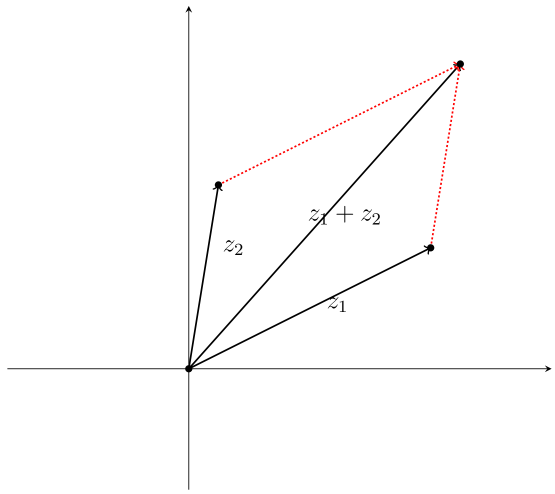
Definition 1.5. <u>**Conjugate**</u> of $z=a+bi$ is

$$\overline{z} = a - bi$$

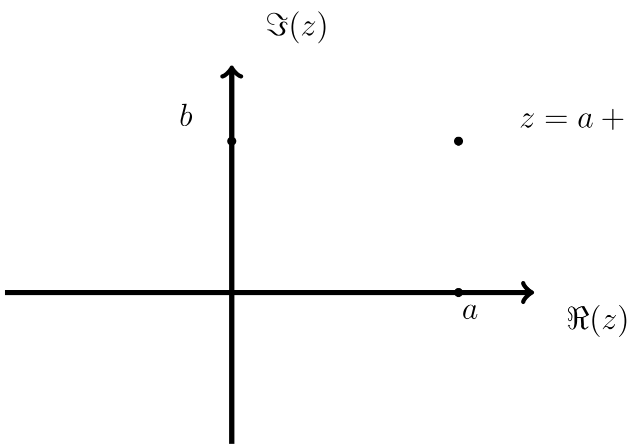
(Sometimes written as z^* as well)

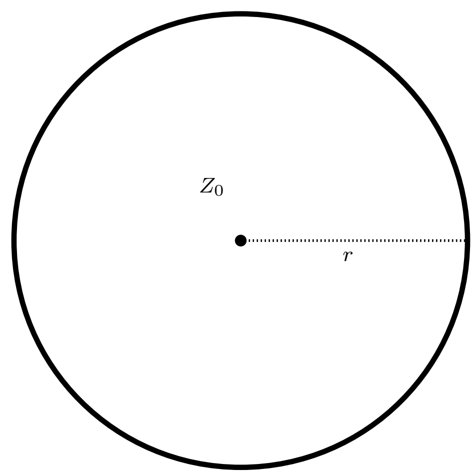
Proposition 1.6. The following rules apply:

1. $\overline{\overline{z}} = z$
2. $\overline{z \pm w} = \overline{z} \pm \overline{w}$
3. $\overline{zw} = \overline{z}\,\overline{w}$ and $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{(\overline{z})}{(\overline{w})}$
4. $z + \overline{z} = 2Re(z) \quad \Rightarrow \quad \Re(z) = \frac{1}{2}(z+\overline{z})$
5. $z - \overline{z} = 2iIm(z) \quad \Rightarrow \quad \Im(z) = \frac{1}{2i}(z-\overline{z})$
6. $z\overline{z} = a^2 + b^2$ which is real!



1.2 The Complex Plane, Polar form





Definition 1.7. The **modulus** of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$

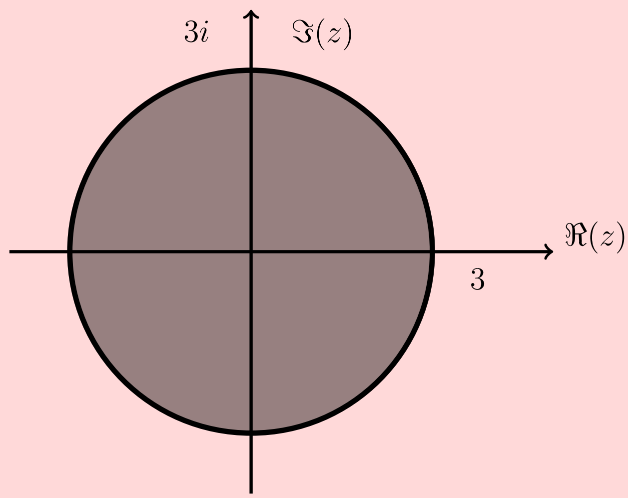
The **distance** between two numbers z and w is $|z - w|$

Notes:

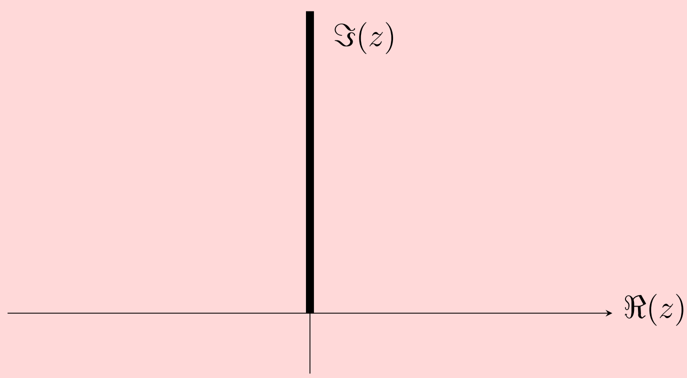
- $|z| \geq 0$ and is real
- $z\overline{z} = a^2 + b^2 = |z|^2$
- $|z - z_0| = r$ describes a circule of radius r centered at z_0

Example 1.8. Sketch the sets:

1. $|z| < 3$



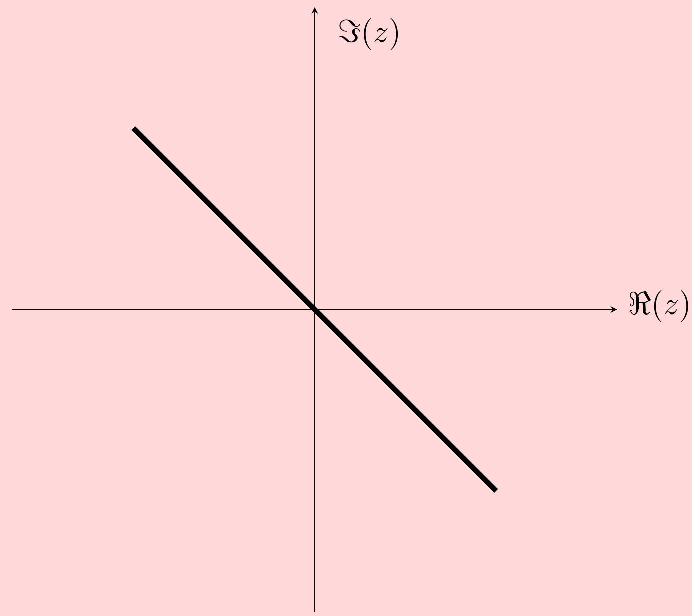
2. $|z| = \Im(z)$. Let $z = a + ib$. So, $\sqrt{a^2+b^2} = b$, which gives $a^2 + b^2 = b^2$, so $a = 0, b \geq 0$



3. $|z-1| = |z+i|$. So

$$\begin{aligned}
\sqrt{(a-1)^2+b^2} &= \sqrt{a^2+(b+1)^2} \\
(a-1)^2+b^2 &= a^2+(b+1)^2 \\
a^2-2a+1+b^2 &= a^2+b^2+2b+1 \\
b &= -a
\end{aligned}$$

This is the set of points that are equidistant from $z = 1$ and $z = -i$



We will often use $z = x + yi$, so we are in the xy-plane, still not called $\mathbb{R}^2$ though.

Useful inequalities:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

This is known as "Triangle Inequality". This also extends to

$$|z_1 = z_2 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$$

Corollary 1.9.

$$|z_1 + z_2| \geq \Big||z_1| - |z_2|\Big|$$

Proof 1.10.

$$\begin{aligned} |z_1| &= |z_1 + (z_2 - z_2)| \\ &= |(z_1 + z_2) + (-z_2)| \\ &\leq |z_1 + z_2| + |z_2| \end{aligned}$$

$$\begin{aligned} |z_2| &= |z_2 + (z_1 - z_1)| \\ &= |(z_1 + z_2) + (-z_1)| \\ &\leq |z_1 + z_2| + |z_1| \end{aligned}$$

So $|z_1 + z_2| \geq |z_1| - |z_2|$ and $|z_2| - |z_1|$. So

$$|z_1 + z_2| \geq \Big||z_1| - |z_2|\Big|$$

$\square$

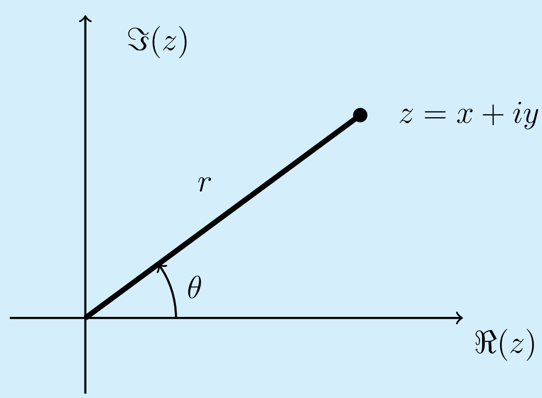
Definition 1.11. **Polar Form**

$$\boxed{x = r\cos\theta,\ y = r\sin\theta}$$

So,

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= r \underbrace{\text{cis}}_{\text{common abreviation}} \theta \end{aligned}$$

$$\boxed{r = \sqrt{x^2 + y^2},\ \tan\theta = \frac{y}{x}}$$



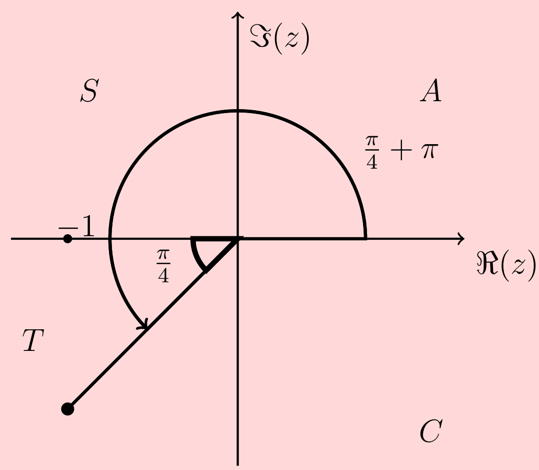
Notes:

- This is not unique. e.g. $z = 2 = 2\,\text{cis}\,0 = 2\,\text{cis}\,2\pi = \cdots$, also $z = 0 = 0\,\text{cis}\,\theta$ for any θ
- $\theta = \tan^{-1}(\frac{y}{x})[\pm 2k\pi]$ if $x > 0$, but must add π if $x < 0$ - Recall principal values

Example 1.12. Say we want to express $z = -1 - i$ in polar form.

We compute $r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$. $\tan\theta = \frac{-1}{-1} = 1$. Note that $\theta \neq \tan^{-1}(1) = \frac{\pi}{4}$, instead, $\theta = \dfrac{5\pi}{4}$.

So, $z = \sqrt{2}\operatorname{cis}\dfrac{5\pi}{4}$ or $\sqrt{2}\operatorname{cis}(\dfrac{5\pi}{4} + 2k\pi)$



Note:

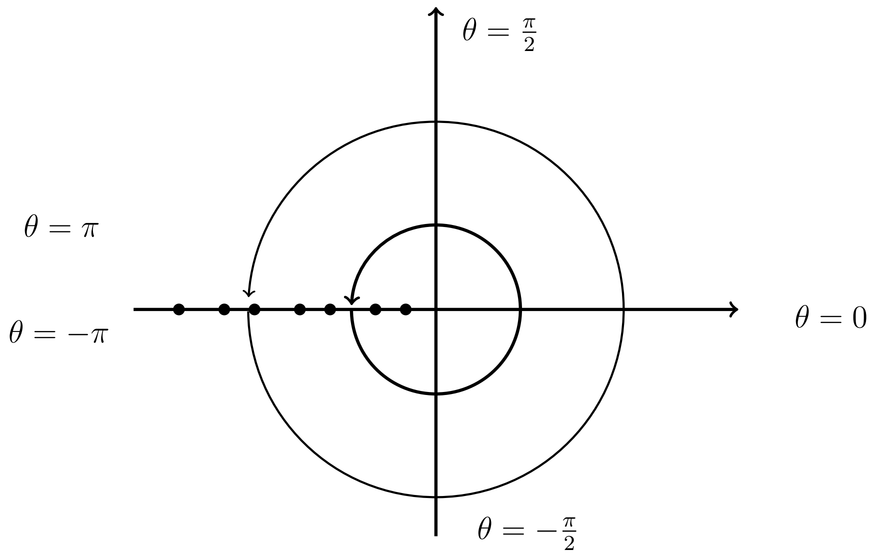
$$z = \underbrace{r}_{=\sqrt{x^2+y^2}, r=|z|, \text{“modulus”}} \operatorname{cis} \underbrace{\theta}_{\text{“argument” of } z}$$

Also, “$\arg z$” = set of all possible values of θ. “$\operatorname{Arg} z$” = principle values of θ, usually in $(\pi, \pi]$

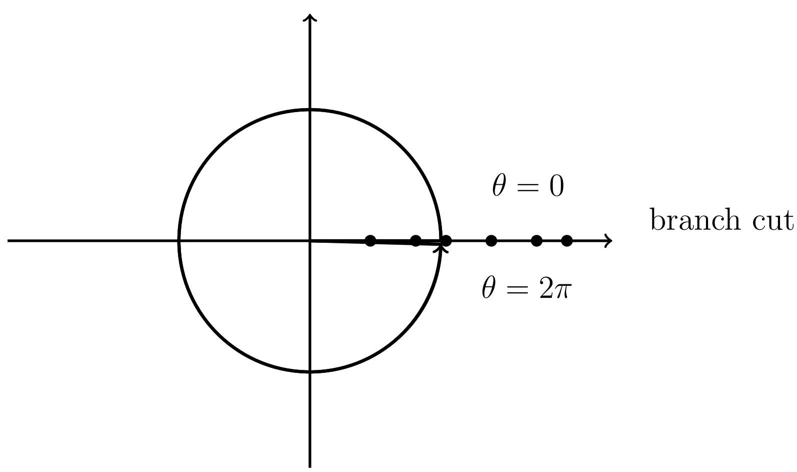
Example 1.13. For $z = -1 + \sqrt{3}i$. $\operatorname{Arg} z = \dfrac{2\pi}{3}$, $\arg z = \dfrac{2\pi}{3} + 2k\pi, k \in \mathbb{Z}$

Also, $|z| = 2$, so $-1 + \sqrt{3}i = 2\operatorname{cis}\dfrac{2\pi}{3}$

We sometimes think of $\arg z$ as a multivalued “function” of z. For a single-valued function, we could use $\operatorname{Arg} z$, but it has discontinuity on negative real axis.



Another way: we can define $\operatorname{Arg}(z)$ to have range $[0, 2\pi)$. In general, $\operatorname{Arg}_{\theta_0} z$ has range $[\theta_0, \theta_0 + 2\pi)$, and usually we use $\operatorname{Arg} z = \operatorname{Arg}_{-\pi} z$



1.3 Complex Exponential, Powers and Roots

Reading textbook Section 1.4, 1.5

Definition 1.14. If $z = x + iy$, then e^z is defined to be the complex number

$$e^z := e^x(\cos y + i \sin y)$$

Proposition 1.15. Euler's equation is formally consistent with the usual Taylor series expansions:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \end{aligned}$$

Proof 1.16. Let's substitute $x = iy$ into the exponential series:

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots \\ &= (1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots) + i(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots) \\ &= \cos y + i \sin y \end{aligned}$$

□

As a result, we may introduce the standard polar representation

$$z = r \operatorname{cis} \theta = r(\cos\theta + i\sin\theta) = re^{i\theta} = |z|e^{i\arg z}$$

Notice that

$$e^{i0} = e^{2\pi i} = e^{-2\pi i} = e^{4\pi i} = e^{-4\pi i} = \cdots = 1$$
$$e^{(\pi/2)i} = i \quad e^{(-\pi/2)i} = -i \quad e^{\pi i} = -1$$

Also notice that

$$\begin{aligned} \cos\theta &= \Re(e^{i\theta}) = \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta &= \Im(e^{i\theta}) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \end{aligned}$$

Hence,

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$
$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$
$$\overline{z} = re^{-i\theta}, \text{ given that } z = re^{i\theta}$$

Example 1.17. Compute the following:

1. $(1+i)/(\sqrt{3}-i)$.

   Notice that $1+i = \sqrt{2}\operatorname{cis}(\pi/4) = \sqrt{2}e^{i\pi/4}$, and $\sqrt{3}-i = 2\operatorname{cis}(-\pi/6) = 2e^{-i\pi/6}$. So,

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}e^{i\pi/4}}{2e^{-i\pi/6}} = \frac{\sqrt{2}}{2}e^{i5\pi/12}$$

2. $(1+i)^{24}$

   We have

$$(1+i)^{24} = (\sqrt{2}e^{i\pi/4})^{24} = (\sqrt{2})^{24}e^{i24\pi/4} = 2^{12}e^{i6\pi} = 2^{12}$$

Theorem 1.18.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad n = 1, 2, 3, ...$$

Definition 1.19. There are exactly m distinct m-th **roots of unity**, denoted by $1^{1/m}$, and they are given by

$$1^{1/m} = e^{i2k\pi/m} = \cos\frac{2k\pi}{m} + i\sin\frac{2k\pi}{m} \quad (k = 0, 1, 2, ..., m-1)$$

Take $k = 1$ into the above equation, we can get

$$\omega_m := e^{i2\pi/m} = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}$$

So the complete set of roots can be displayed as

$$\{1, \omega_m, \omega_m^2, \cdots, \omega_m^{m-1}\}$$

Note that a number w is said to be a **primitive** m-th root of unity if $w^m = 1$ but $w^k \neq 1$ for $k = 1, 2, ..., m-1$. Clearly, ω_m is a primitive root.

Theorem 1.20.

$$1 + \omega_m + \omega_m^2 + \cdots + \omega_m^{m-1} = 0$$

Proof 1.21. Note that

$$(\omega_m - 1)(1 + \omega_m + \omega_m^2 + \cdots + \omega_m^{m-1}) = (\omega_m - 1) = 0$$

Since $\omega_m \neq 1$, the result follows. □

To obtain the m-th root of an arbitrary (non-zero) complex number $z = re^{i\theta}$, we can obtain the following generalized result.

Definition 1.22. The m-th distinct roots of z are given by

$$z^{1/m} = \sqrt[m]{|z|}e^{i(\theta+2k\pi)/m}$$

Example 1.23. Find all the cube roots of $\sqrt{2} + i\sqrt{2}$

The polar form for $\sqrt{2} + i\sqrt{2}$ is

$$\sqrt{2} + i\sqrt{2} = 2e^{i\pi/4}$$

Putting $|z| = 2, \theta = \pi/4, m = 3$ into the above definition, we obtain

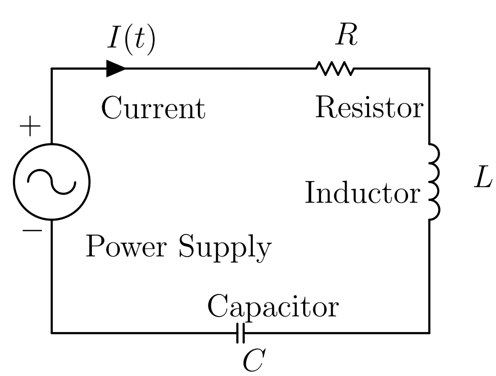
$$(\sqrt{2} + i\sqrt{2})^{1/3} = \sqrt[3]{2}e^{i(\pi/12+2k\pi/3)}, \quad (k = 0, 1, 2)$$

Hence, the three cube roots of $\sqrt{2} + i\sqrt{2}$ are:

- $\sqrt[3]{2}(\cos \pi/12 + i \sin \pi/12)$
- $\sqrt[3]{2}(\cos 3\pi/4 + i \sin 3\pi/4)$
- $\sqrt[3]{2}(\cos 17\pi/12 + i \sin 17\pi/12)$

1.4 Application to Electrical Circuits

A typical electrical circuits is like the following:



Laws:

1. Resistor: $V = IR$
2. Inductor: $V = L\dfrac{dI}{dt}$
3. Capacitor: $C\dfrac{dV}{dt} = I$

Suppose the current is

$$I(t) = \underbrace{I_0}_{\text{amplitude}} \cos \underbrace{\omega}_{\text{frequency}} t = \Re(\underbrace{I_0 e^{i\omega t}}_{\text{call it } \widetilde{I}(t)})$$

Then

1. Law 1 tells us $V = (I_0 \cos \omega t)(R) = \Re(\widetilde{I}(t) \cdot R)$. So "complex voltage" is

$$\boxed{\tilde{V} = R\widetilde{I}}$$

2. Law 2 tells us

$$\begin{aligned} V &= L \cdot (-\omega I_0 \sin \omega t) \\ &= -\omega L I_0 \cdot \underbrace{\Re(e^{i(\omega t - \frac{\pi}{2})})}_{=\cos(\omega t - \frac{\pi}{2}) = \sin \omega t} \\ &= \Re(-\omega L I_0 e^{i\omega t} e^{-i\frac{\pi}{2}}) \\ &= \Re(i\omega L I_0 e^{i\omega t}) \end{aligned}$$

So

$$\boxed{\widetilde{V} = i\omega L\widetilde{I}}$$

3. Law 3 tells us

$$\begin{aligned} V &= \frac{1}{C} \int I(t) \\ &= \frac{I_0}{C\omega} \sin \omega t \\ &= \Re(\frac{I_0}{C\omega} e^{i(\omega t - \frac{\pi}{2})}) \\ &= \Re(\frac{I_0}{iC\omega} e^{i\omega t}) \end{aligned}$$

So

$$\boxed{\widetilde{V} = \frac{1}{iC\omega}\widetilde{I}}$$

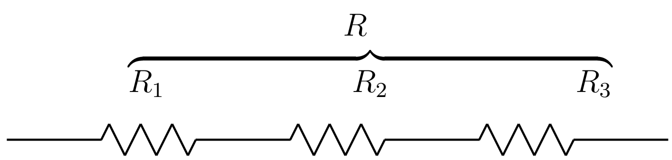
So, with the complex representation, all three circuit elements behave like resistors with a complex "Ohm's Law'

$$\boxed{\widetilde{V} = Z\widetilde{I} \quad \text{where } Z = \begin{cases} R & \text{for resistors} \\ i\omega L & \text{for inductors} \\ \dfrac{1}{i\omega C} & \text{for inductors} \end{cases}}$$

Moreover, Z is called "impedance"

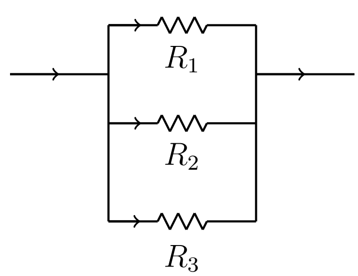
Combining the components:

- In series:



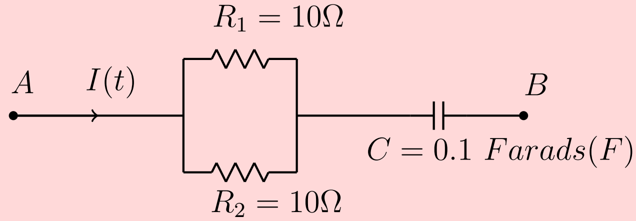
$$\begin{aligned} R &= R_1 + R_2 + R_3 + \cdots \\ L &= L_1 + L_2 + L_3 + \cdots \\ \frac{1}{C} &= \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \\ Z &= Z_1 + Z_2 + Z_3 + \cdots \end{aligned}$$

- In parallel:



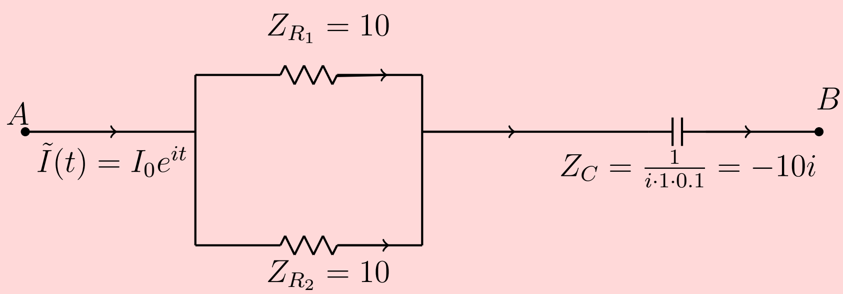
$$\begin{aligned} \frac{1}{R} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \\ \frac{1}{L} &= \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots \\ C &= C_1 + C_2 + C_3 + \cdots \\ \frac{1}{Z} &= \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots \end{aligned}$$

Example 1.24. Suppose a current $I(t) = I_0 \cos t$, passes through this:



Find $V(t)$, the difference in electrical potential energy between A and B

Solution:



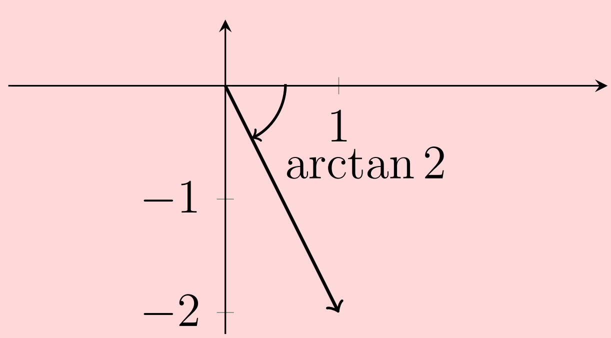
Let's use the complex version of "Ohm's Law". We have $\frac{1}{Z_R} = \frac{1}{Z_{R_1}} + \frac{1}{Z_{R_2}} = \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$, so $Z_R = 5$.

Combine the resistor and capacitor in series: $Z = Z_R + Z_C = 5 - 10i$.

So, the complex voltage is

$$\begin{aligned} \widetilde{V} &= Z\widetilde{I} \\ &= (5 - 10i) I_0 e^{it} \\ &= 5I_0(1 - 2i)e^{it} \\ &= 5I_0\sqrt{5}e^{i \arctan -2} e^{it} \end{aligned}$$

So, $V(t) = \Re(\widetilde{V}(t)) \approx 5\sqrt{5} I_0 \cos(t - 1.107)$



1.5 Sets in the Complex Plane

Definition 1.25. **Neighborhood of z_0** is

$$N_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$$

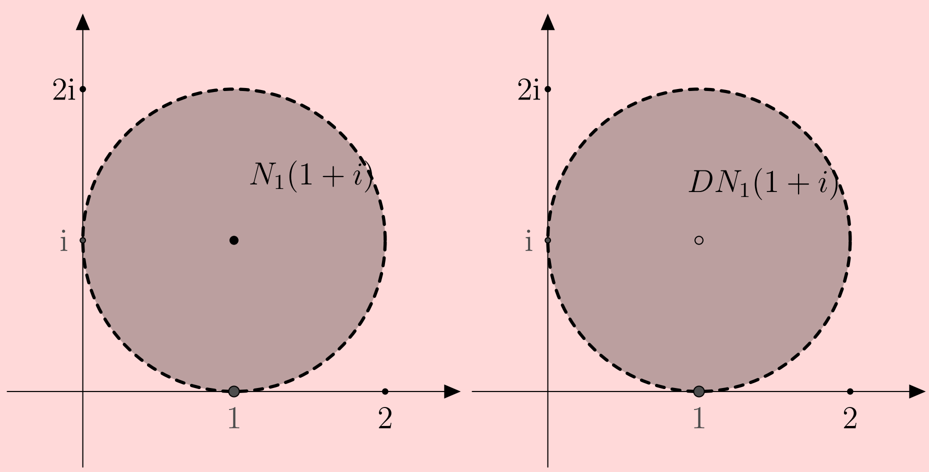
where $\epsilon > 0$ is real

Definition 1.26. **Deleted Neighborhood of z_0** is

$$DN_\epsilon(z_0) = \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$$

where $\epsilon > 0$ is real

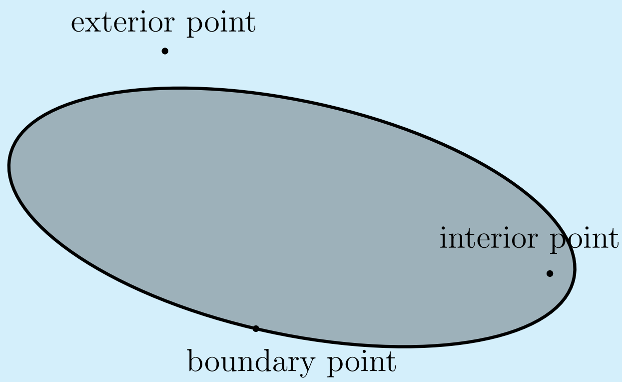
Example 1.27. For $z_0 = 1 + i$, consider $|z - (1 + i)| < 1$. The neighborhood of z_0 and deleted neighborhood of z_0 is as follows:



Definition 1.28. Let $S \subseteq \mathbb{C}$:

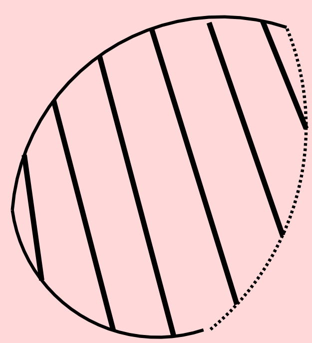
- z_0 is an **interior point** of S if there exists a neighborhood of z_0 which contains only points in S
- z_0 is an **exterior point** of S if there exists a neighborhood of z_0 which contains no points in S
- z_0 is a **boundary point** of S if every neighborhood of z_0 contains some points in S and some points not.
- **Boundary of S** is the set of all boundary points of S
- S is **open** if it contains none of its boundary points

- S is **closed** if it contains all of its boundary points, equivalently if its complement is open.
- Note that S could be both open and closed, when it does not have any boundary points



Example 1.29. Note that

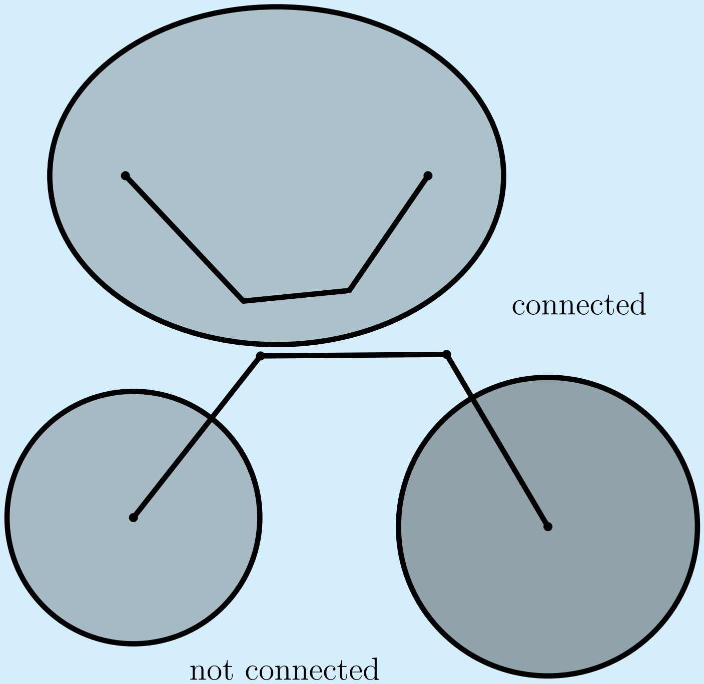
- $N_1(1+i)$ is open
- $\mathbb{C}$ is both open and closed
- $|z - z_0| \leq 1$ is closed
- The figure below: it is neither open nor closed.



Definition 1.30. For $S \subseteq \mathbb{C}$:

- **Closure** of S is S plus its boundary.
- An open set S is **connected** if any two points in S can be connected by a polygonal path lying entirely in S
- A **domain** is an open connected set. We should not confuse this with "domain of a function"
- A **region** is a domain plus some, none, or all of its boundary points.

- S is **bounded** if there exists $R \in \mathbb{R}$ such that $|z| < R$ for all $z \in S$

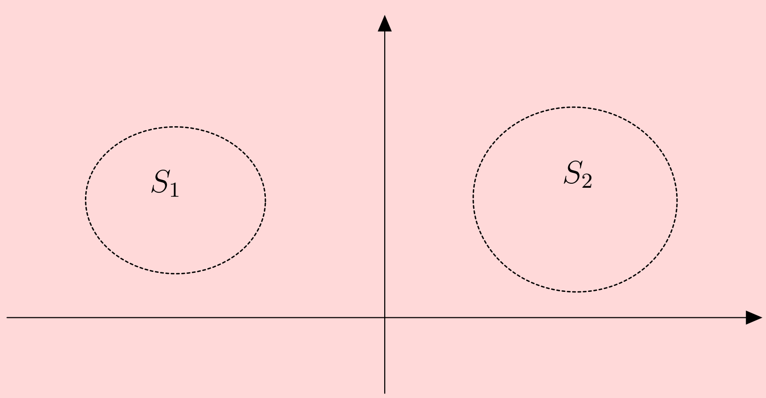


Theorem 1.31. If $u(x, y)$, defined on a domain D, satisfies

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

for all points in D, then $u(x, y) =$ constant in D.

Example 1.32. Suppose we have S_1 and S_2 like this:



in which we have $u(x, y) = 0$ on S_1 and $u(x, y) = 1$ on S_2.

Then, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0$ on $S_1 \cup S_2$, but $u(x, y)$ is not constant on $S_1 \cup S_2$.

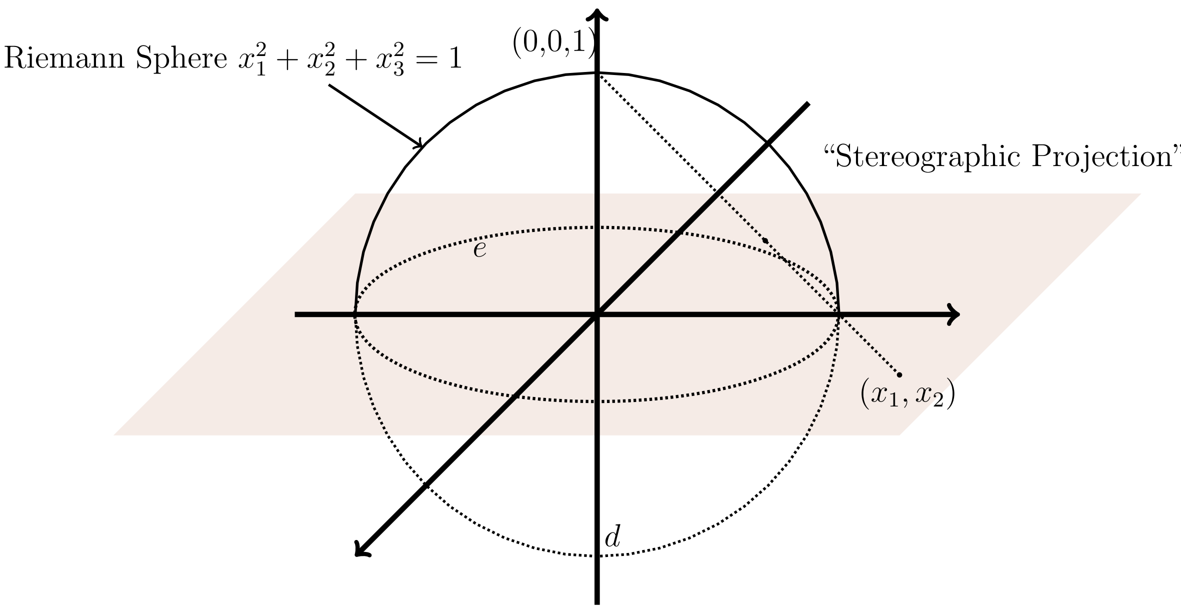
Why does not the theorem hold? Well this is because $S_1 \cup S_2$ is not connected, so it's not a domain.

The Extended Complex Plane:

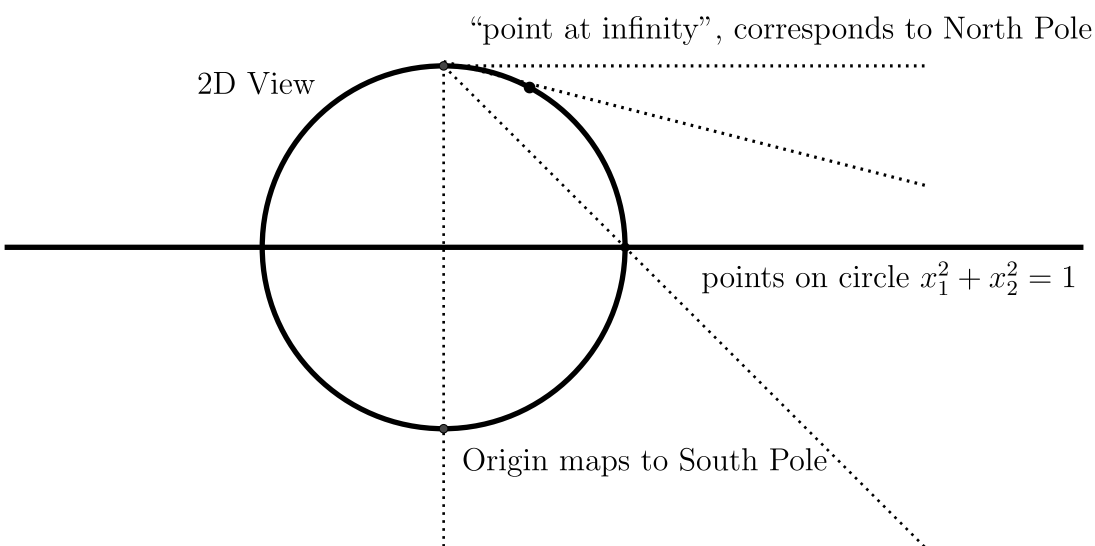
The "neighborhood of ∞" is defined as:

$$N_\epsilon(\infty) = \{z \in \mathbb{C} : |z| > \frac{1}{\epsilon}\}$$

for some real $\epsilon > 0$ The Riemann sphere:



We can define a one-to-one mapping between x_1x_2-plane and the sphere:



See the course text for more detail, in particular:

- Circles and lines all map circles on the sphere
- Lines are just circles which pass through the “point at infinity”

Chapter 2 Analytic Functions

2.1 Functions

For a function on complex numbers:

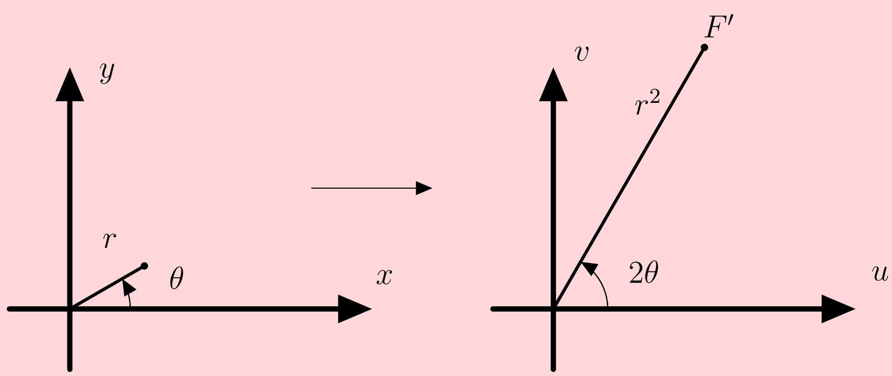
$$\begin{aligned}\omega &= f(z)\\ &= f(x+iy)\\ &= u(x,y)+iv(x,y)\end{aligned}$$

We can think of it as a mapping.

Example 2.1. 1. $f(z) = z^2$. Find the images of

(a) the first quadrant.

$f(z) = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u} + i\underbrace{2xy}_{v}$

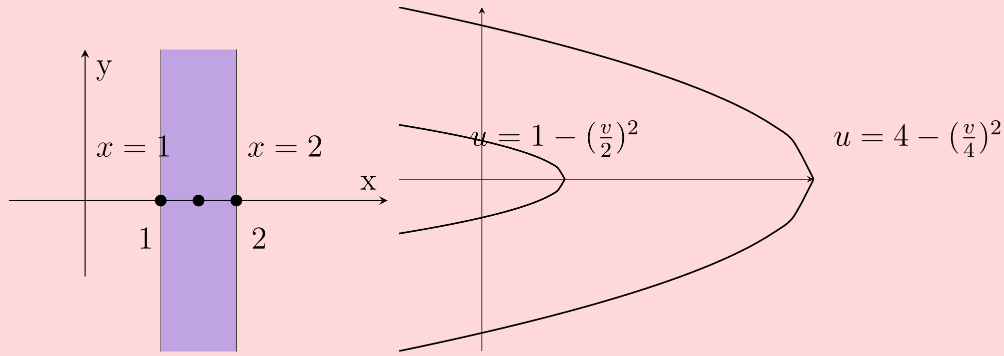


Note that $f(z) = (re^{i\theta})^2 = r^2e^{i2\theta}$ (angle is doubled)

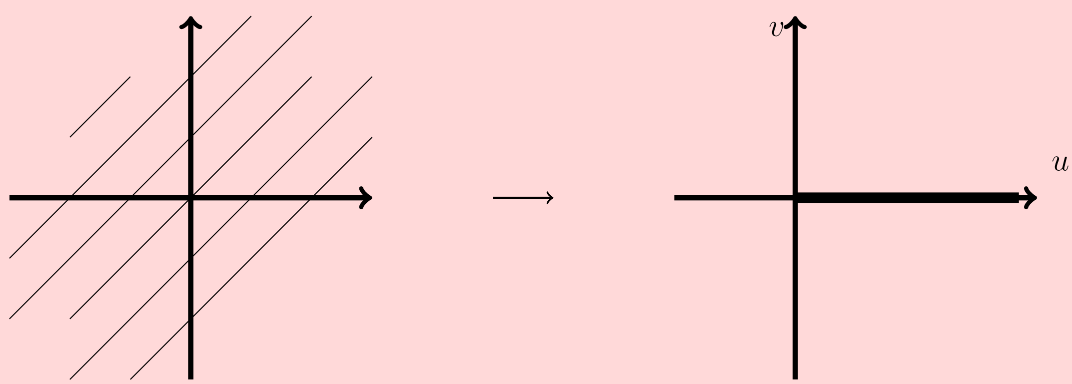
(b) the strip $1 \leq \Re(z) \leq 2$

With $1 \leq x \leq 2$, the boundaries become:

- $x = 1 \Rightarrow \begin{cases} u = 1-y^2 \\ v = 2y \end{cases} \Rightarrow u = 1 - \left(\frac{v}{2}\right)^2$, which is a parabola
- $x = 2 \Rightarrow \begin{cases} u = 4-y^2 \\ v = 4y \end{cases} \Rightarrow u = 4 - \left(\frac{v}{4}\right)^2$, which is a parabola



2. $f(z) = |z|$. This one maps complex plane to non-negative real axis.



3. $f(z) = z - z_0 = (x + iy) - (x_0 + iy_0) = (x - x_0) + i(y - y_0)$. This is a translation.

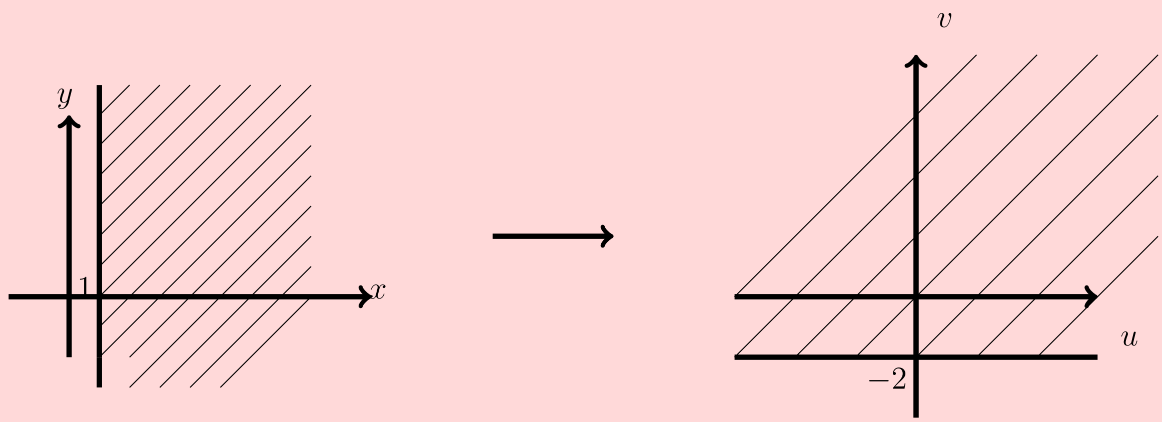
4. $f(z) = z_0 z$, so

$$f(z) = r_0 e^{i\theta_0} r e^{i\theta} = \underbrace{r_0}_{\text{magnification}} r e^{i\underbrace{\theta_0}_{\text{rotation}}+\theta} = r_0 r e^{i\theta_0 + \theta}$$

5. $f(z) = \overline{z} = x - iy \to \begin{cases} u = x \\ v = -y \end{cases}$. This is a reflection on y-axis.

6. Find image of half-plane $\Re(z) \geq 1$ under the map $\omega = f(z) = iz - 3i$.

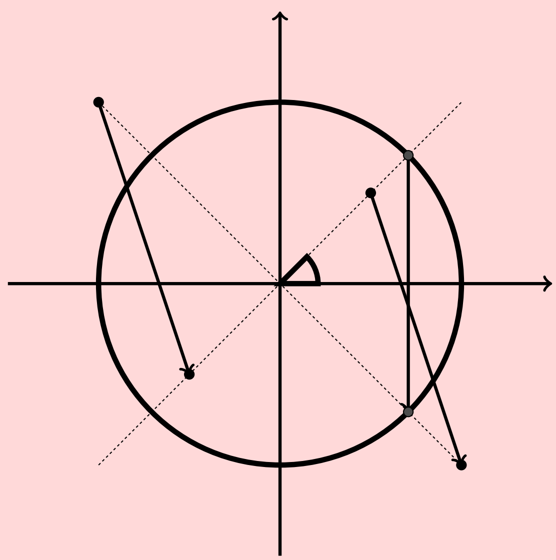
   We can do this step by step. First it's a rotation of $\dfrac{\pi}{2}$ (comes from the first i), then its a shift down 3 units.



The image is the half-plane $v \geq -2$.

7. Inversion mapping. $f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$. So, it's a scaling by r, and then reflection through the x-axis.

   For this mapping, unit circle maps to the unit circle. Outside points go to inside, and inside points go to outside.



8. Image of circle $(x-1)^2 + y^2 = 1$ under $f(z) = \frac{1}{z}$.

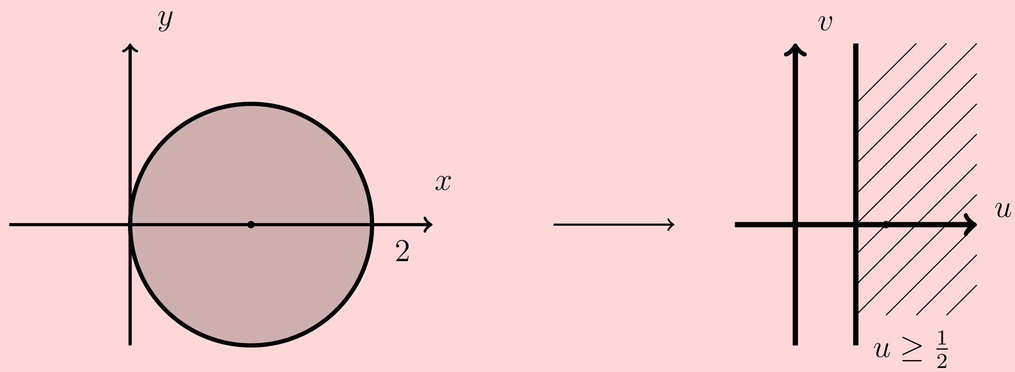
   The trick is to use polar fomulas. Recall $x^2 + y^2 = r^2, x = r\cos\theta, y = r\sin\theta$.

   So, $x^2 - 2x + 1 + y^2 = 1$ yields that $r^2 = 2r\cos\theta$. Since $r \neq 0$, we then have $r = 2\cos\theta$.

   To apply the map, replace r with $\frac{1}{r}$, and θ with $-\theta$:

$$\frac{1}{r} = 2\cos(-\theta) \Rightarrow r = \frac{1}{2\cos\theta} \Rightarrow r\cos\theta = \frac{1}{2}$$

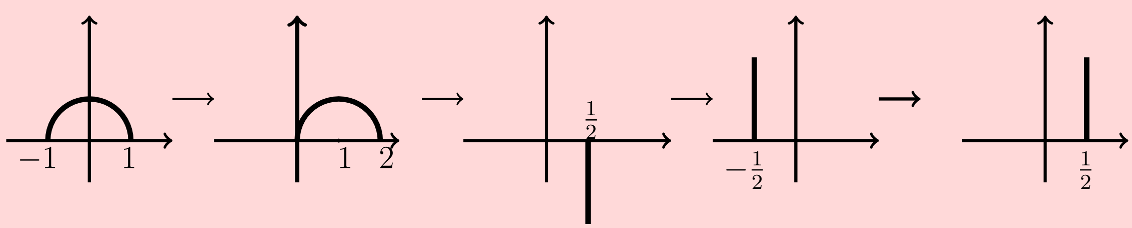
So $u = \frac{1}{2}$ since $\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$ in the uv plane.



9. $w = f(z) = \dfrac{z}{z+1}$, find the image of upper-half of unit circle.

   First, $f(z) = \dfrac{z+1-1}{z+1} = 1 - \dfrac{1}{z+1}$. This is a sequence of transformations:

$$z \to \underbrace{z+1}_{\text{shift right}} \to \underbrace{\frac{1}{z+1}}_{\text{invert}} \to \underbrace{\frac{-1}{z+1}}_{\text{reflect and rotate } \pi} \to \underbrace{1 - \frac{1}{z+1}}_{\text{shift right}}$$



2.2 Limits and Differentiation

Definition 2.2. **Limits**:

$$\lim_{z \to z_0} f(z) = w_0$$

means that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z) - w_0| < \epsilon$$

Example 2.3. Prove that $\lim_{z \to 1+i} (2+i)z = 1 + 3i$.

Solution: We first do some preliminary work:

$$|(2+i)z-(1+3i)| = |2+i|\cdot|z-\frac{1+3i}{2+i}| = \sqrt{5}\cdot|z-(1+i)|$$

So, let $\epsilon > 0$, with $|z-z_0| < \frac{\epsilon}{\sqrt{5}}(=\delta)$, we have

$$\begin{aligned}|(2+i)z-(1+3i)| &= \sqrt{5}\cdot|z-(1+i)| \\ &< \sqrt{5}\cdot\frac{\epsilon}{\sqrt{5}} \\ &= \epsilon\end{aligned}$$

So, $\lim_{z\to 1+i}(2+i)z = 1+3i$ □

Note that similar definitions apply when dealing with infinity, e.g. $\lim_{z\to z_0} f(z) = \infty$ means that for any $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |z-z_0| < \delta \quad \Rightarrow \quad |f(z)| > \frac{1}{\epsilon}$

Definition 2.4. **Continuity**: f is **continuous** at z_0 means that

$$\lim_{z\to z_0} f(z) = f(z_0)$$

The usual limit and continuity theorems hold, e.g.

$$\lim_{z\to z_0} f(z)g(z) = \lim_{z\to z_0} f(z)\cdot\lim_{z\to z_0} g(z)$$

Theorem 2.5. Let $f(z) = u+iv, z_0 = x_0+iy_0, w_0 = u_0+iv_0$, then

$$\lim_{z\to z_0} f(z) = w_0 \quad \text{if and only if} \quad \begin{cases}\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \\ \lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0\end{cases}$$

Definition 2.6. **Differentiation**:

$$f'(z_0) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\left(=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\right)$$

Derivative function is

$$f'(z) = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

For functions with real analogues (e.g. $f(z) = z^2$ analogous to $f(x) = x^2$), the usual rules (power, quotient, etc.) apply, e.g.

$$f(z) = 3z^2 + z^4 \quad \Rightarrow f'(z) = 6z + 4z^3$$

What about functions without real analogues?

Example 2.7. $f(z) = \overline{z}$. Is it differentiable?

Solution:

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0} \\ &= \lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0} \\ &= \lim_{z \to z_0} \frac{\overline{re^{i\theta}}}{re^{i\theta}} \quad \text{where} \quad z - z_0 = e^{i\theta} \\ &= \lim_{z \to z_0} \frac{e^{-i\theta}}{e^{i\theta}} \\ &= \lim_{z \to z_0} e^{-i2\theta} \end{aligned}$$

which depends on θ! No unique value, so limit DNE. So, f is not differentiable anywhere.

Theorem 2.8. **Cauchy-Riemann Equations**: If $f(z) = u(x, y) + iv(x, y)$ and $f'(z_0)$ exists, then

$$\boxed{u_x = v_y \quad \text{and} \quad v_x = -u_y \qquad \text{at } (x_0, y_0)}$$

Note that for notation,

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x} \\ u_y &= \frac{\partial u}{\partial y} \\ v_x &= \frac{\partial v}{\partial x} \\ v_y &= \frac{\partial v}{\partial y} \end{aligned}$$

Proof 2.9.

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i \frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y} \right) \end{aligned}$$

Since the limit exists, it must be independent of path, so

- Along $\Delta y = 0$:

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i(\cdots) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

- Along $\Delta x = 0$:

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i(\cdots) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary part yields the result. $\square$

2.3 Differentiability Continued

Example 2.10. Is $f(z) = |z|^2$ differentiable? Where?

Solution: $f(z) = \sqrt{x^2 + y^2}^2 = \underbrace{x^2 + y^2}_{u} + \underbrace{0}_{v} i$. So, by CRE, we know that

$$\begin{cases} u_x = v_y & \Rightarrow 2x = 0 \\ v_x = -u_y & \Rightarrow 0 = -2y \end{cases}$$

It's clear that this is satisfied only at $x = y = 0$.

So, if $(x, y) \neq (0, 0)$, i.e. $z \neq 0$, then f is not differentiable.

When $z = 0$, $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \dfrac{|\Delta z|^2 - 0}{\Delta z} = 0$. This is because $\left| \dfrac{|\Delta z|^2}{\Delta z} - 0 \right| \leq |\Delta z| \to 0$ as $\Delta z \to 0$ (by applying the squeeze theorem).

Hence, CRE are necessary but not sufficient conditions.

Theorem 2.11. Let f be defined in some neighborhood of z_0. If u_x, u_y, v_x, v_y exist in that neighborhood, satisfying CRE at z_0, and are **continuous** at z_0, then f is differentiable at z_0.

Definition 2.12. $f(z)$ is **analytic at z_0** if $f'(z)$ exists at every point in some neighborhood of z_0.

$f(z)$ is **analytic on an open set S** if it is analytic at every point of S.

Example 2.13. $f(z) = z^3 = \cdots = \underbrace{(x^3 - 3xy^2)}_{u(x,y)} + i\underbrace{(3x^2y - y^2)}_{v(x,y)}$.

We have

$$\begin{aligned} u_x &= 3x^2 - 3y^2 \\ u_y &= -6xy \\ v_x &= 6xy \\ v_y &= 3x^2 - 3y^2 \end{aligned}$$

So, CRE satisfied everywhere. All partial derivatives are continuous. By theorem, f is differentiable everywhere, so is analytic everywhere. We refer to "analytic everywhere" as "entire"

Example 2.14. Where is $f(z) = x^2 + iy^2$ analytic?

We have

$$\begin{aligned} u_x &= 2x \\ u_y &= 0 \\ v_x &= 0 \\ v_y &= 2y \end{aligned}$$

We need $x = y$ to satisfy CRE.

- If $x \neq y$, f is not differentiable, so not analytic.
- If $x = y$, f cannot be analytic because we are not on an open set.

So, f is not analytic nowhere.

Theorem 2.15. Sums, products, and compositions of analytic functions are also analytic, except when $\div 0$

Example 2.16. $f(z) = \dfrac{z^3 + 2}{z^2 + 1}$ is analytic everywhere except at $z = \pm i$.

$g(z) = f(z^2)$ is analytic everywhere except wherre $z^2 = \pm i$, i.e. except

$$\begin{aligned} z &= e^{i(\dfrac{n\pi + \pi/2}{2})} \\ &= e^{i(n\pi/2 + \pi/4)} \\ &= e^{i(n\pi/4)}, e^{i(n3\pi/4)}, e^{i(n5\pi/4)}, e^{i(n7\pi/4)} \\ &= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \end{aligned}$$

Theorem 2.17. Suppose f is analytic in a domain D. If $f'(z) = 0$ for all $z \in D$, then f is constant in D

Proof 2.18. $f'(z) = u_x + iv_x = v_y - iu_y$. So, $f'(z) = 0 \Rightarrow u_x = v_y = 0 = v_y = u_y$. So, u and v are constant, since D is connected. □

Theorem 2.19. Suppose f is analytic in a domain D. If $|f(z)| = M$ for all $z \in D$, where M is constant, then $f(z)$ is constant in D.

Proof 2.20. $|f(z)|^2 = u^2 + v^2 = M^2$.

We differentiate:

- with respect to x: $2uu_x + 2vv_x = 0 \quad - (1)$
- with respect to y: $2uu_y + 2vv_y = 0 \quad - (2)$

Now $u_x = v_y$, and $v_x = -u_y$, so the (2) gives $-uv_x + vu_x = 0 \quad - (3)$.

Multiply (1) by u_x.

$$\begin{aligned} & uu_x^2 + vu_xv_x = 0 \\ \Rightarrow \ & uux^2 + (uv_x)v_x = 0 \quad \text{by (3)} \\ \Rightarrow \ & u(u_x^2 + v_x^2) = 0 \end{aligned}$$

So, unless $u = 0$ for all $z \in D$, we must have $u_x^2 + v_x^2 = 0$. So, $u_x = v_x = 0$, implying that u, v are constant. Hence, f is constant.

What if $u = 0$ for all $z \in D$? Then, $u_x = u_y = 0$, so $v_x = v_y = 0$ by CRE. f is constant as well. □

2.4 Harmonic Functions

Recap:

$$f'(z) = u_x + iv_x = \frac{u_y + iv_y}{i} = v_y - iu_y$$
$$CRE: \quad u_x = v_y \quad v_x = -u_y$$

Also, "analytic" means differentiable on a open set.

Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Then u and v satisfy CRE.

Also, which will be shown later, $u, v \in C^2$ (continuous under second partial derivatives), and this implies that $u_{xy} = u_{yx}$, and $v_{xy} = v_{yx}$.

From CRE:

$$\underbrace{u_x - v_y}_{\Rightarrow u_{xx} = v_{yx}} \qquad \text{and} \qquad \underbrace{v_x = -u_y}_{\Rightarrow u_{yy} = -v_{xy}}$$

Definition 2.21. From the above derivation, we see

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

and

$$v_{xx} + v_{yy} = 0$$

We refer to these as **"Laplace's equation"**

Solution to Laplace's equation are called **"harmonic functions"**

Notes:

- We've shown that if $f(z) = u + iv$ is analytic, then u and v must be harmonic
- Laplace's equation is very useful! We will see that later.
- $u_{xx} + u_{yy} = 0$ is also denoted as $\Delta^2 u = 0$, and we denote Δ as "Laplacian operator".

Example 2.22. Suppose $u(x, y) = e^{-2x} \cos 2y + 2y$. Find $v(x, y)$ such that $f(z) = u + iv$ is analytic.

Solution: u and v must satisfy CRE. So, $v_y = u_x = -2e^{-2x} \cos 2y$. Hence,

$$\begin{aligned} v &= \int -2e^{-2x} \cos 2y dy \\ &= -e^{2x} \sin 2y + C(x) \end{aligned}$$

Note that $C(x)$ is a function of all other variables.

Now we try to make it satisfy other CRE:

$$\begin{aligned} v_x = -u_y \;&\Rightarrow\; 2e^{-2x}\sin 2y + C'(x) = 2e^{-2x}\sin 2y - 2 \\ &\Rightarrow\; C'(x) = -2 \\ &\Rightarrow\; C(x) = -2x + k \end{aligned}$$

Therefore, $v(x,y) = -e^{-2x}\sin 2y - 2x + k$

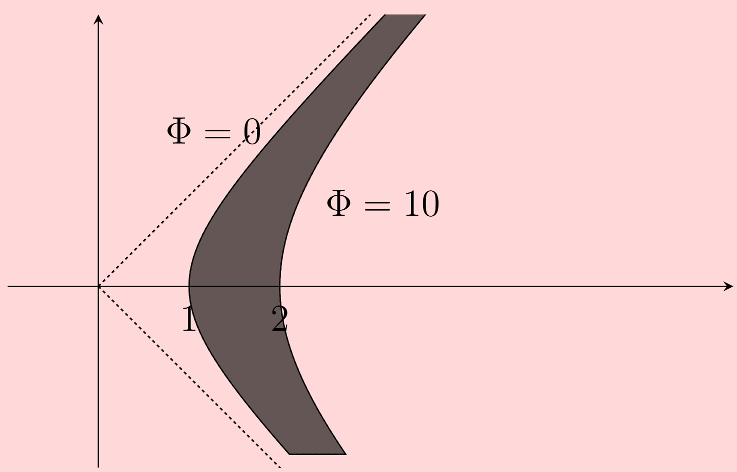
Note that $v(x,y)$ is called the "harmonic conjugate" of u.

Exercise: show that if v is the harmonic conjugate of u, then $-u$ is the harmonic conjugate of v.

Example 2.23. Solve Laplace's equation $\Phi_{xx} + \Phi_{yy} = 0$ on region between hyperbolas $x^2 - y^2 = 1$ and $x^2 - y^2 = 4$, $x > 0$, with "boundary conditions"

$$\begin{cases} \Phi = 0 & \text{on } x^2 - y^2 = 1 \\ \Phi = 10 & \text{on } x^2 - y^2 = 4 \end{cases}$$

i.e. Find $\Phi(x,y)$



<u>**Solution**</u>: Consider $f(z) = z^2 = (x+yi)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$

Since $f(z)$ is already analytic, we have that $u(x,y) = x^2 - y^2$ is harmonic. Boundary curves of region are level curves of a harmonic function.

Is the solution $\Phi(x,y) = x^2 - y^2$? No.

Try $\Phi(x,y) = A \cdot (x^2 - y^2) + B$ (also harmonic by linearity).

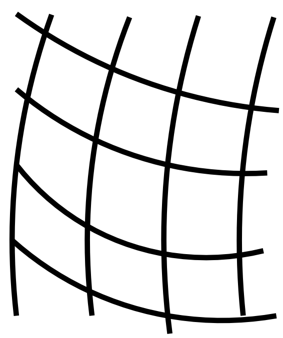
Applying the Boundary Conditions:

$$0 = A \cdot 1 + B \Rightarrow \; B = -A$$
$$10 = A \cdot 4 + B \Rightarrow \; A = \frac{10}{3}, B = -\frac{10}{3}$$

So the solution is $\Phi(x, y) = \frac{10}{3}(x^2 - y^2) - \frac{10}{3}$

Notes:

- It can be used in temperature distribution
- What about more complicated regions?
- Orthogonal trajectories



- list of harmonic functions

Chapter 3 Elementary Functions

3.1 Elementary Functions

Definition 3.1. Polynomials:

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \ , \quad a_i \in \mathbb{C}$$

They are obviously **entire**.

The fundamental theorem of algebra guarantees that we can factor this as

$$p(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n)$$

Note that z_i are not necessarily distinct.

z_0 is a "zero of multiplicity" k if and only if

$$p(z) = (z - z_0)^k q(z)$$

where $q(z)$ is a polynomial such that $q(z_0) \neq 0$

Definition 3.2. Rational Functions:

$$R(z) = \frac{p(z)}{q(z)} = \frac{a_n(z - z_1)(z - z_2) \cdots (z - z_n)}{b_m(z - w_1)(z - w_2) \cdots (z - w_n)}$$

Suppose all common factors have been cancelled, then

- the roots (or zeroes) of $p(z)$ are called the **roots/zeroes** of $R(z)$
- the roots (or zeroes) of $q(z)$ are called the **poles** of $R(z)$

Example 3.3.

$$R(z) = \frac{3i(z-1)(z-\frac{1}{3}i)^2(z+i)}{(z-i)^3(z-2-i)}$$

Zeroes at 1 and $-i$ (order 1 would be a "simple zero"), and $\dfrac{1}{3}i$ (order 2).

Poles at i (order 3) and $2+i$ (order 1 would be a "simple pole")

Partial Fractions has simpler rules:

Example 3.4. Decompose $R(z) = \dfrac{1}{(z+4)^2(z^2+1)}$

Solution: Factor and expand

$$\frac{1}{(z+4)^2(z^2+1)} = \frac{A}{z+4} + \frac{B}{(z+4)^2} + \frac{C}{z+i} + \frac{D}{z-i}$$

This gives us

$$1 = A \cdot (z+4)(z+i)(z-i) + B(z+i)(z-i) + C(z+4)^2(z-i) + D(z+4)^2(z+i)$$

We can solve this by:

- set $z = -4$, this gives us $1 = 0 + (-4+i)(-4-i)B + 0 + 0$, so $B = \dfrac{1}{17}$
- set $z = -i$, this gives us $1 = 0+0+(-i+4)^2(-2i)C+0$. Then we conpute $(-2i)(15-8i) = 16-30i$, also $(-16-30i) = \dfrac{(-16-30i)(-16+30i)}{(-16+30i)} = \dfrac{1156}{(-16+30i)} = \dfrac{578}{-8+15i}$.

  Hence, $C = \dfrac{-8+15i}{578}$.
- set $z = -4$, this gives us $1 = 0+0+0+(i+4)^2(2i)D$, so $D = \dfrac{-8-15i}{578}$. The trick to compute things here is that, we can replace i with $-i$ from C since the expression is similar to C.

Now what about A? We can try another z, or just compare the coefficients of z^3. By comparing the coefficients of z^3, we get that

$$0 + A + C + D = A + \frac{-8+15i}{578} + \frac{-8-15i}{578}$$

So $A = \dfrac{16}{578} = \dfrac{8}{289}$

Hence,

$$\frac{1}{(z+4)^2(z^2+1)} = \frac{8/289}{z+4} + \frac{1/17}{(z+4)^2} + \frac{\frac{-8+15i}{578}}{z+i} + \frac{\frac{-8-15i}{578}}{z-i}$$

Actually, often we will only need one of the coefficients, and there's a quick way which will be covered later in the course.

Definition 3.5. **Exponential Function**: We already defined that $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$.

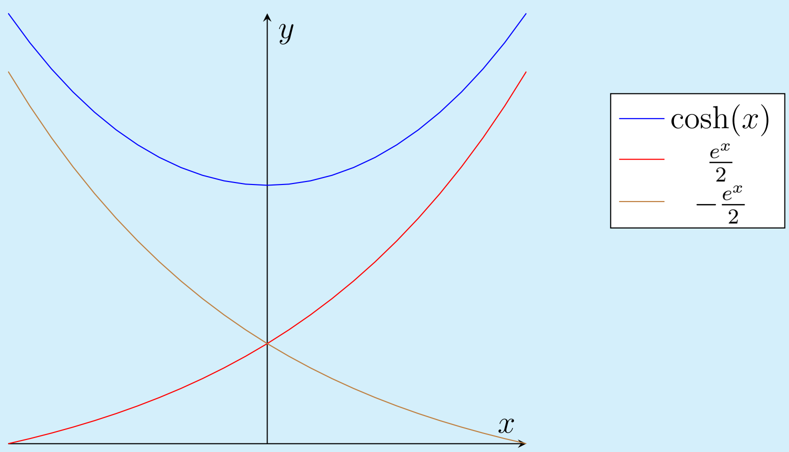
Note that $e^{z_1+z_2} = e^{z_1}e^{z_2}$, $\dfrac{d}{dz}e^z = e^z$. Also, e^z is <u>**periodic**</u> with period $2\pi i$

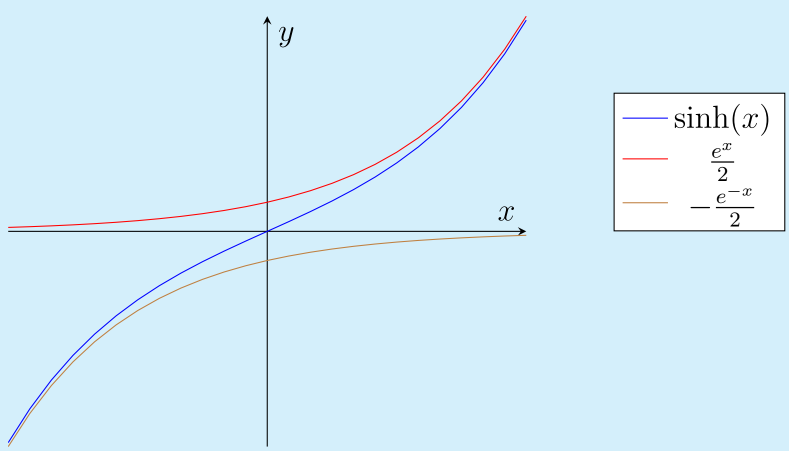
Definition 3.6. <u>**Hyperbolic Functions**</u>: From real calculus, we seen that

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \quad \text{this is the even component of } e^x$$
$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \text{this is the odd component of } e^x$$

It can be shown that

$$\cosh x + \sinh x = e^x$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$\frac{d}{dx}\sinh x = \cosh x$$
$$\frac{d}{dx}\cosh x = \sinh x$$





To extend these to $\mathbb{C}$, we define

$$\boxed{\cosh z = \frac{1}{2}(e^z + e^{-z}) \quad \sinh z = \frac{1}{2}(e^z - e^{-z})}$$

3.2 Trigonometric and Logarithmic Function

Definition 3.7. **Trigonometric Functions**: Recall

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Sum to get $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

We define

$$\boxed{\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \cosh(iz) \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{i}\sinh(iz)}$$

Furthermore:

$$\cos(iz) = \frac{e^{-z} + e^z}{2} = \cosh z$$
$$\sin(iz) = \frac{e^{-z} - e^z}{2i} = i\sinh z$$

$$\text{For real } z, \quad e^z = e^x = \cosh x + \sinh x$$
$$\text{For imaginary } z, \quad e^z = e^{iy} = \cos y + i\sin y$$

The $\cosh x$ and $\cos y$ are the even parts, and $\sinh x$ and $i\sin y$ are the odd parts

Functions	Along Real Axis	Along Imaginary Axis
$e^{iz}, \cos z, \sin z$	periodic	grow exponentially
$e^z, \cosh z, \sinh z$	grow exponentially	periodic

Familiar identities hold true.

Example 3.8.

$$\begin{aligned}\cos^4\theta &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^4 \\ &= \frac{1}{16}(e^{i4\theta}+4e^{2\theta}+6+4e^{-i2\theta}+e^{-i4\theta}) \\ &= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}\end{aligned}$$

Example 3.9.

$$\begin{aligned}&\cos^2\theta+\sin^2\theta = 1 \\ \Rightarrow\ &\cos^2(iy)+\sin^2(iy)=1 \\ \Rightarrow\ &\cosh^2 y + i^2\sinh^2 y = 1 \\ \Rightarrow\ &\cosh^2 y - \sinh^2 y = 1\end{aligned}$$

By using the rules $\begin{cases}\cos(iz)=\cosh z \\ \sin(iz) = i\sinh z\end{cases}$.

Notice the "Obsborne's rule" here: Hyperbolic function satisfy the same identities as trigonometric functions except that <u>we must change the sign of every product of two sines</u>.

Derivatives: e^z is entire, and so is $\cos z, \sin z, \cosh z, \sinh z$. Also,

$$\frac{d}{dz}(\cos z) = \frac{d}{dz}(\frac{e^{iz}+e^{-iz}}{2}) = \frac{ie^{iz}-ie^{-iz}}{2} = \frac{e^{iz}-e^{-iz}}{-2i} = -\sin z$$

Other as expected as well

Note: we can also define $\tan z$, $\sec z$ etc. in the usual ways, and derivatives of them are as expected.

Example 3.10. What is the value of $\sin(\pi+i)$?

<u>**Solution**</u>:

$$\begin{aligned}\sin(\pi+i) &= \sin\pi\cos(i\cdot 1)+\cos\pi\sin(i\cdot 1) \\ &= \sin\pi\cosh 1 + \cos\pi i \sinh(1) \\ &= 0 + (-1)\cdot i\cdot\sinh(1) \\ &= -\sin i\end{aligned}$$

Example 3.11. Find all solutions of $\sin z = 1000$

<u>Solution</u>: We write $\sin(x+yi) = 1000$, and get that

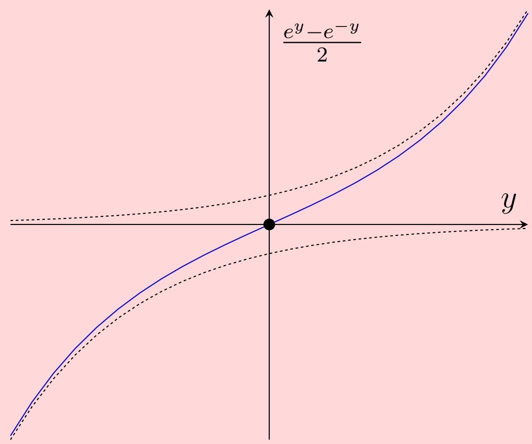
$$\sin x \cosh y + i \cos x \sinh y = 1000$$

So

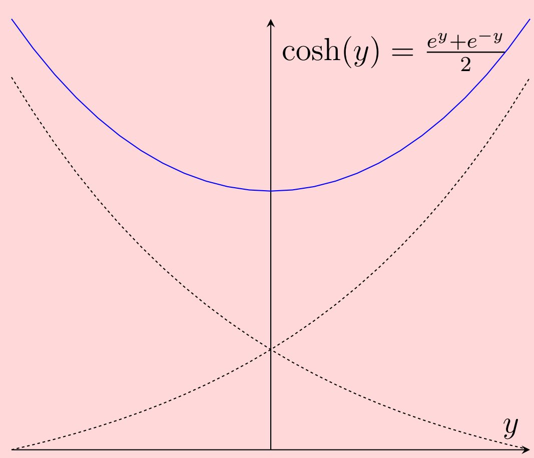
$$\begin{cases} \sin x \cosh y = 1000 & \cdots\cdots(1) \\ \cos x \sinh y = 0 & \cdots\cdots(2) \end{cases}$$

Equation 2 gives that $\cos x = 0$ or $\sinh y = 0$, which yields that $x = (2n+1)\dfrac{\pi}{2}$ or $y = 0$.

The following figure shows that the only x that $\sinh(x) = 0$ is at $x = 0$.



- If $y = 0$, equation 1 gives that $\sin x \cosh(0) = \sin x = 1000$. This is impossible
- If $x = (2n+1)\dfrac{\pi}{2}$, then equation 1 gives $\sin\left((2n+1)\dfrac{\pi}{2}\right)\cosh y = 1000$, so $\cosh y = 1000 \cdot (-1)^n$



But $\cosh y > 0$, so use $n = 2N$ (always even). So $\cosh y = 1000$, and $y = \pm\cosh^{-1}(1000) \approx \pm 7.6$ (There are two solutions, i.e. note the $\pm$ sign, as the figure above shows).

The final answer is that $z = x + iy = (4N+1)\frac{\pi}{2} \pm i\cosh^{-1}(1000)$

3.3 Logarithmic Functions

How to define $\log z$? Let $z = e^w$ and solve for w. Note that:

- exponential function is periodic, so log will be a "multi-valued function"
- in $\mathbb{C}$, we use "log" instead of "ln"

Definition 3.12. Now,

$$\begin{aligned} z = e^w \Rightarrow\ & re^{i\theta + 2\pi k} = e^{u+iv} \\ \Rightarrow\ & r = e^u, \quad \theta + 2\pi k = v \\ \Rightarrow\ & u = \ln r, \quad v = \theta + 2\pi k \end{aligned}$$

So, we define

$$\boxed{\log z = \ln|z| + i \arg z}$$

Example 3.13.

- $\log(1+i) = \ln|1+i| + i\arg(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi k\right)$
- $\log(i) = \ln|i| + i\arg(i) = 0 + i\left(\frac{\pi}{2} + 2\pi k\right)$

Proposition 3.14. We have the following identity:

$$\begin{aligned} \log(z_1 z_2) &= \ln|z_1 z_2| + i\arg(z_1 z_2) \\ &=^* \ln|z_1| + \ln|z_2| + i(\arg z_1 + \arg z_2) \\ &= \log(z_1) + \log(z_2) \end{aligned}$$

Similarly

$$\log\left(\frac{z_1}{z_2}\right) =^* \log(z_1) - \log(z_2)$$

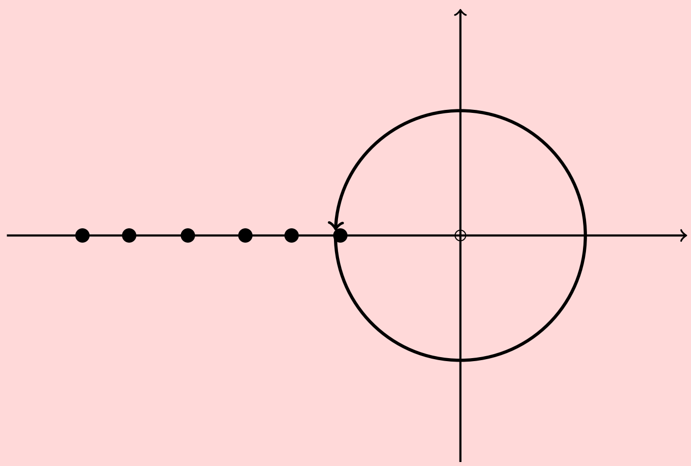
By $=^*$, we actually mean that <u>the set of values</u> of $\log(z_1 z_2)$ is equal to <u>the set of values</u> of $\log(z_1) + \log(z_2)$, due to the multi-valuedness of log.

Definition 3.15. The **principle value of the Logarithm** is

$$\operatorname{Log}(z) = \ln|z| + i \underbrace{\operatorname{Arg}(z)}_{\in(-\pi,\pi] \text{ usually}}$$

Example 3.16.

- $\operatorname{Log}(1+i) = \ln|1+i| + i\operatorname{Arg}(1+i) = \ln\sqrt{2} + i\frac{\pi}{4}$
- $\operatorname{Log}(i) = \ln|i| + i\operatorname{Arg}(i) = 0 + i\pi$
- $\operatorname{Log} e^z = z$ if and only if $\Im(z) \in (-\pi, \pi]$
- $\operatorname{Log} z$ has discontinuity on negative real axis



- $\operatorname{Log} z$ is analytic everywhere else, with

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}$$

Proof 3.17. Let

$$\begin{aligned} w = \operatorname{Log} z &= \ln|z| + i\operatorname{Arg}(z) \\ &= \frac{1}{2}\ln(x^2+y^2) + i\left(\arctan(\frac{y}{x}) \pm \pi\right) \\ &= u(x,y) + iv(x,y) \end{aligned}$$

Now,

$$\begin{aligned}
\frac{dw}{dz} &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \\
&= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \\
&= \frac{x-iy}{x^2+y^2} \cdot \frac{x+iy}{x+iy} \\
&= \frac{1}{z}
\end{aligned}$$

□

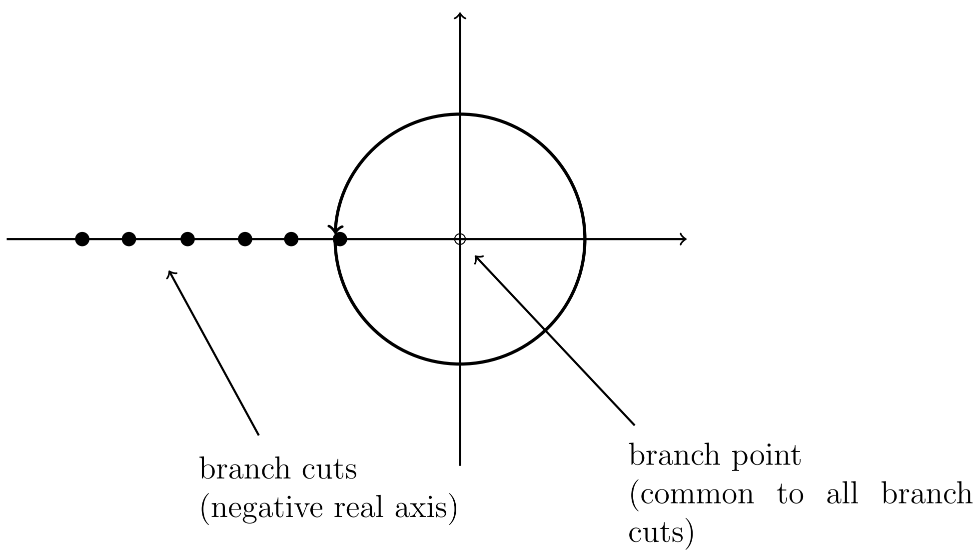
Definition 3.18. **<u>Branch Cuts</u>**: Let $f(z)$ be a multivalued function. $F(z)$ is said to be a **<u>branch</u>** of $f(z)$ on a domain D if $F(z)$ is continuous on D and for each $z \in D$, $F(z)$ is one and only one of the values of $f(z)$.

Example 3.19. $\operatorname{Log} z$ is a branch of $\log z$

We could define different branches of $\log z$ by

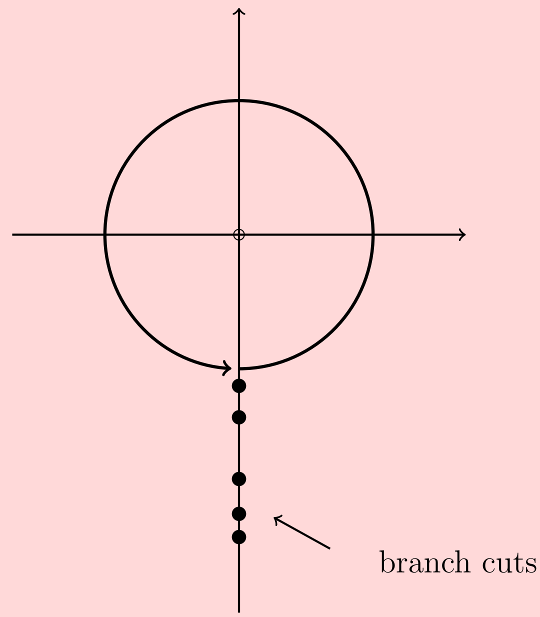
$$\operatorname{Log}_\tau z = \ln|z| + i\operatorname{Arg}_\tau(z)$$

where $\operatorname{Arg}_\tau(z) \in (\tau, \tau+2\pi]$. Note that $\operatorname{Log} z = \operatorname{Log}_{-\pi}$



Example 3.20.

$$\operatorname{Log}_{-\frac{\pi}{2}} \ln|z| + i\operatorname{Arg}_{-\frac{\pi}{2}}(z)$$



Example 3.21. Find a branch of $f(z) = \log(z+4)$ that is analytic at $z = -5$ and equals $7\pi i$ there.

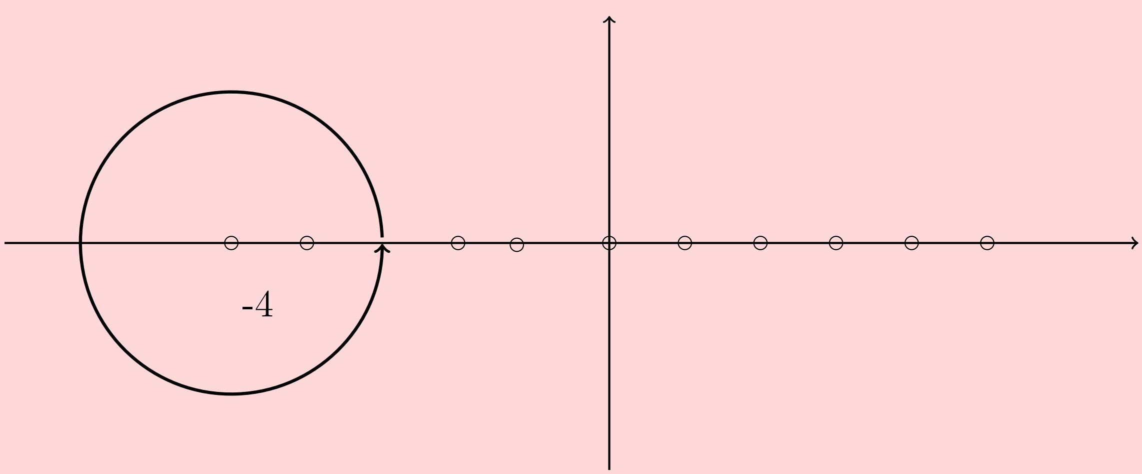
<u>**Solution**</u>: We want $\mathrm{Log}_\tau(-5+4) = \mathrm{Log}_\tau(-1) = 7\pi i$ for some τ.

So, $\ln|-1| + i\,\mathrm{Arg}_\tau(-1) = 7\pi i$ for some k, i.e.

$$0 + i\,\underbrace{(\pi + 2k\pi)}_{\in(\tau,\tau+2\pi]} = 7\pi i \quad \text{for some } k$$

Hence, $k = 3$. We can choose $\tau = 6\pi$ so that $7\pi \in (6\pi, 8\pi]$.

The final answer would be $F(z) = \mathrm{Log}_{6\pi}(z+4)$



Example 3.22. Where is $f(z) = \text{Log}(z^2+1)$ analytic?

Solution: We need $z^2+1 \neq 0$ and not equal to negative real number.

So, $z^2+1 = (x+yi)^2+1 = (x^2-y^2+1)+i(2xy)$.

$$z^2+1=0 \text{ when } \begin{cases} x=0 \text{ and } y=\pm 1 \\ \text{or} \\ y=0 \text{ and } x^2+1=0 \text{ This is impossible for } x \in \mathbb{R} \end{cases}$$

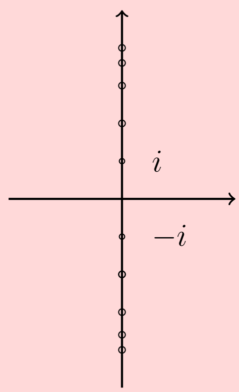
Hence, $z = \pm i$ here.

$$z^2+1<0 \text{ (real) when } \begin{cases} x=0 \text{ and } 1-y^2<0 \quad \Rightarrow y^2>1 \Rightarrow y>1 \text{ or } y<-1 \\ \text{or} \\ y=0 \text{ and } 1+x^2<0 \quad \text{Impossible} \end{cases}$$

Hence, $z = iy$ where $|y|>1$.

For all other points,

$$f'(z) = \frac{2z}{z^2+1}$$



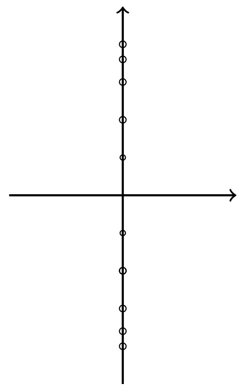
Here is another way to solve the above problem.

$$\text{Log}(z^2+1) = \text{Log}((z+i)(z-i)) = \text{Log}_{\tau_1}(z+i) + \text{Log}_{\tau_2}(z-i)$$

for some τ_1, τ_2

Some possibilities are:

- $\tau_1 = \frac{-\pi}{2}$, $\tau_2 = \frac{-3\pi}{2}$
- $\tau_1 = \frac{3\pi}{2}$, $\tau_2 = \frac{-7\pi}{2}$
- $\cdots$

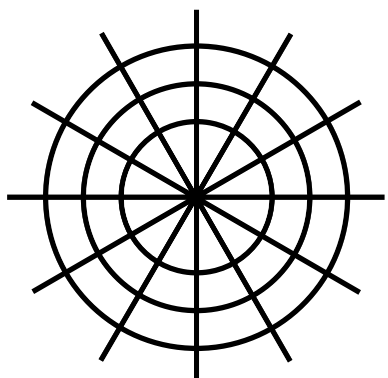


Finally, note that

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$

$\operatorname{Log} z$ is analytic, so $\ln|z|$ and $\operatorname{Arg} z$ are harmonic.

Level curves of $\ln|z| = k$ and $\operatorname{Arg} z = k$ are circles and rays. This would be particularly useful when we deal with temperature problems later.



useful for temp problems later

3.4 Complex Powers and Inverse Trigonometric Functions

Definition 3.23. Complex Powers: We define

$$z^\alpha = e^{\alpha \log z} \quad \text{for } \alpha \in \mathbb{C}, z \neq 0$$

Example 3.24. 1.

$$\begin{aligned}
4^{1/2} &= e^{\frac{1}{2}\log 4} \\
&= e^{\frac{1}{2}(\ln|4|+i\arg(4))} \\
&= e^{\frac{1}{2}\ln 4+i\frac{1}{2}(0+2\pi k)} \\
&= e^{\frac{1}{2}2\ln 2+i\pi k} \\
&= e^{\ln 2}e^{i\pi k} \\
&= 2\cdot(\pm 1) \\
&= \pm 2
\end{aligned}$$

2.

$$\begin{aligned}
(1+i)^3 &= e^{3\log(1+i)} \\
&= e^{3\left(\ln\sqrt{2}+i\arg(1+i)\right)} \\
&= e^{\frac{3}{2}\ln 2}e^{i3\left(\frac{\pi}{4}+2k\pi\right)} \\
&= (e^{\ln 2})^{\frac{3}{2}}\cdot\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right) \\
&= 2^{\frac{3}{2}}\cdot\left(\frac{-1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) \\
&= -2+2i
\end{aligned}$$

3.

$$\begin{aligned}
i^i &= e^{i\ln|i|+i\arg i} \\
&= e^{i\left(0+i(\frac{\pi}{2}+2k\pi)\right)} \\
&= e^{-\left(\frac{\pi}{2}+2\pi k\right)} \\
&= \cdots, e^{\frac{-5\pi}{2}}, e^{\frac{-\pi}{2}}, e^{\frac{3\pi}{2}}, \cdots
\end{aligned}$$

If we want a single value, take the principal branch to be $e^{\alpha\,\mathrm{Log}\,z}$, which is analytic everywhere $\mathrm{Log}\,z$ is, and

$$\frac{d}{dz}z^\alpha = \frac{d}{dz}e^{\alpha\,\mathrm{Log}\,z} = e^{\alpha\,\mathrm{Log}\,z}\cdot\frac{\alpha}{z} = z^\alpha\cdot\frac{\alpha}{z} = \alpha z^\alpha$$

as expected.

Definition 3.25. **Inverse Trigonometric Functions**: First, we see that $w = \sin^{-1} z$ means $z = \sin w$, etc. Also, we've accepted multivalued functions.

In $\mathbb{R}$, the inverse hyperbolic function can be expressed in terms of logs:

$$\begin{aligned} y = \sinh x &= \frac{1}{2}(e^x - e^{-x}) \\ e^x - 2y - e^{-x} &= 0 \\ (e^x)^2 - 2y(e^x) - 1 &= 0 \quad \text{note that this is a quadratic equation for } e^x \\ e^x = \frac{2y \pm \sqrt{4y^2+4}}{2} &= y \pm \sqrt{y^2+1} \text{ we take the plus sign since } e^x > 0 \end{aligned}$$

So, $x = \ln(y + \sqrt{y^2+1}) = \sinh^{-1} y$.

In $\mathbb{C}$, we define $\sinh^{-1} z = \log(z + \sqrt{z^2+1})$.

Similarly, $\sin^{-1} z = -i\log(iz + (1-z^2)^{\frac{1}{2}})$. Note that for this definition, it involves two sets of branches, one with log, and the other one with $(1-z^2)^{\frac{1}{2}}$

Chapter 4 Complex Integration

4.1 Contours

How to integrate in $\mathbb{C}$?

Complex values functions of a real variable are easy to integrate:

$$\int_a^b \Big(u(t) + iv(t)\Big)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Example 4.1. 1.

$$\int_0^1 (t+i)^2 dt \int_0^1 \Big((t^2-1) + i(2t)\Big)dt = \frac{-2}{3} + 2i$$

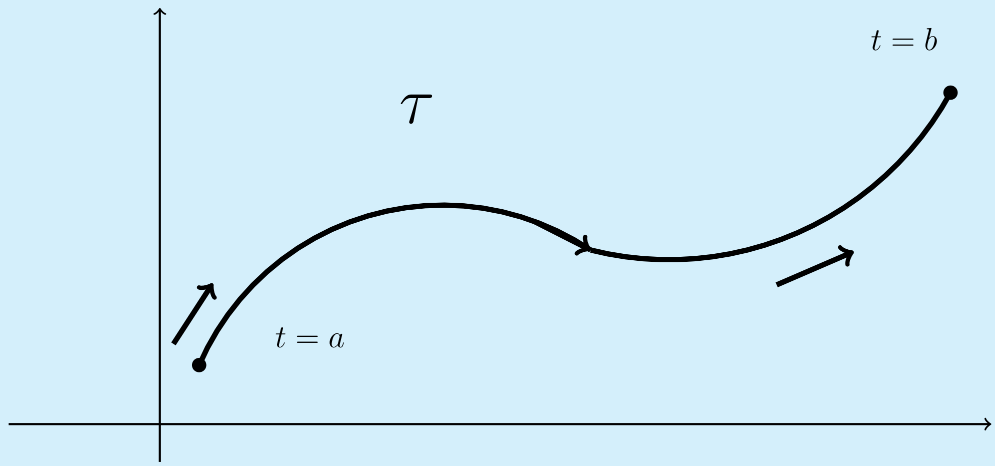
2. We can use a special trick (instead of using integration by parts twice).

$$\begin{aligned}
\int_0^\pi e^{2\pi}\cos x dx &= \int_0^\pi e^2 x(\Re(e^{ix}))dx \\
&= \Re\bigg(\int_0^\pi e^{(2+i)\pi}\bigg) \\
&= \Re\bigg(\frac{e^{(2+i)}}{2+i}\bigg|_0^\pi\bigg) \\
&= \Re\bigg(\frac{e^{2x}(\cos x + i\sin x)}{2+i}\cdot\frac{2-i}{2-i}\bigg|_0^\pi\bigg) \\
&= \bigg[\frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x\bigg]\bigg|_0^\pi \\
&= -\frac{2}{5}e^{-2\pi} + \frac{2}{5}
\end{aligned}$$

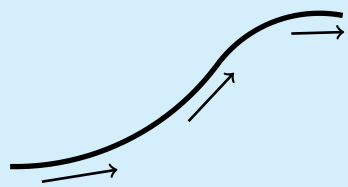
What about integrating a function of a complex variable?

We will replace the intervals with paths.

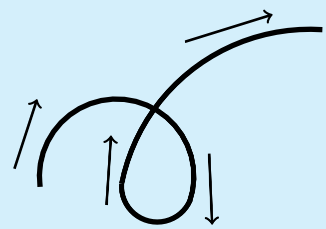
Definition 4.2. Let $z(t) = x(t) + iy(t)$ on $t \in [a, b]$ be continuous. The range is a **curve** $\mathcal{C}$, and is called a **smooth curve** if $z'(t)$ is continuous and non-zero on $[a, b]$



A curve is called **simple** if $z(t_1) \neq z(t_2)$ whenever $t_1 \neq t_2$ for $a < t_i < b$ (basically no self intersection)

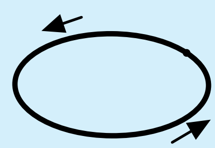


simple



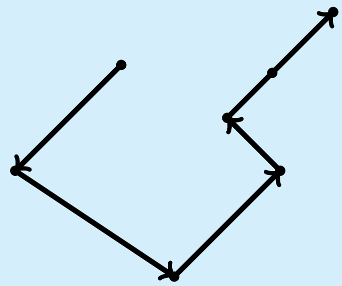
non-simple

If $z(a) = z(b)$, then the curve is called a **closed** curve.

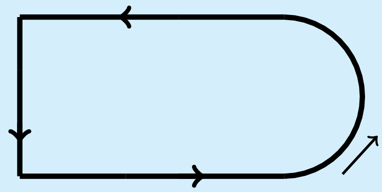


simple closed curve

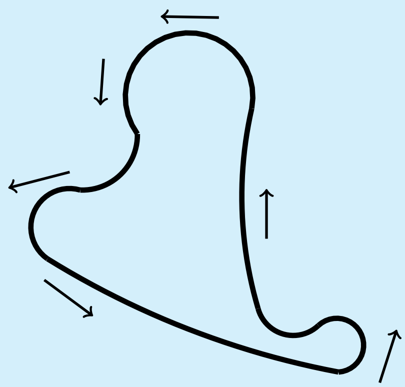
Definition 4.3. **Contour**: a curve that is composed of finitely many smooth curves, joined end-to-end



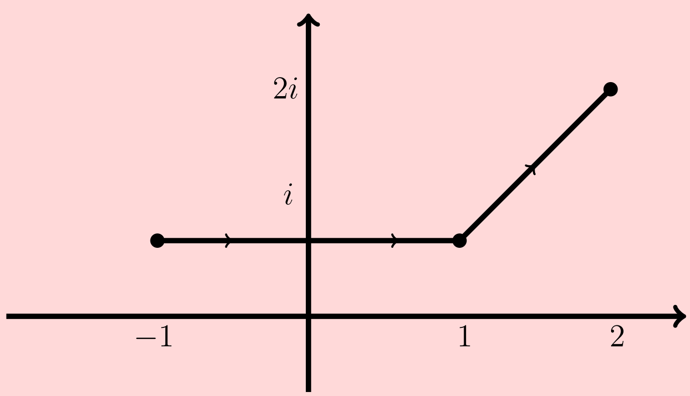
Definition 4.4. **Jordan Curve**: a simple closed contour.



Definition 4.5. **Positively Oriented**: means its interior lies to the **left** as we follow the curve



Example 4.6. Parameterize this:



Solution: Line segment from z_0 to z_1 can be parameterized as: $z(t) = z_0 + (z_1 - z_0)t, t \in [0,1]$.

For the first curve,

$$\begin{aligned} z_1(t) &= (-1+i) + (1+i-(-1+i))t \\ &= -1+i+2t, \quad t \in [0,1] \end{aligned}$$

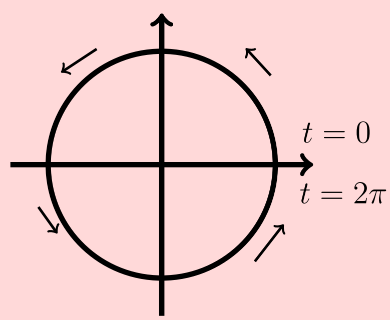
For the second curve,

$$\begin{aligned} z_2(t) &= (1+i) + (2+2i-(1+i))t \\ &= 1+i+(1+i)t, \quad t \in [0,1] \end{aligned}$$

Put everything together we get

$$z(t) = \begin{cases} -1+i+2t & t \in [0,1) \\ 1+i+(1+i)(t-1) & t \in [1,2] \end{cases}$$

Example 4.7. Let $\mathcal{C}$ be a unit circle centered at 0.



Solution: $\mathcal{C} : z(t) = e^{it} \quad t \in [0, 2\pi]$

Example 4.8. Circle, radius r_0, centered at z_0?

Solution: $\mathcal{C} : z(t) = z_0 + r_0 e^{it} \quad t \in [0, 2\pi]$

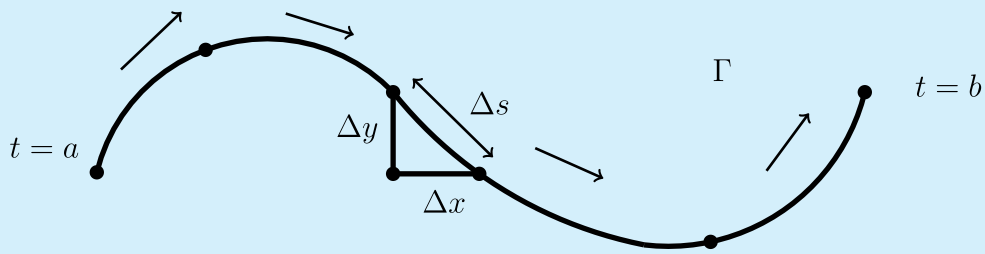
Example 4.9. Parameterize $y = f(x), x \in [a,b]$

Solution: just let $x(t) = t$,

$$z(t) = x(t) + iy(t) = t + if(t), \quad t \in [a,b]$$

For example, $y = x^2$ will be parameterized as $z(t) = t + it^2$

Definition 4.10. **Arclength**: We define the arclength as follows:



Partition the curve

$$\begin{aligned}\Delta s &\approx \sqrt{\Delta x^2 + \Delta y^2} \\ &= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \Delta t\end{aligned}$$

Sum all pieces and let $\Delta t \to 0$ (Performing a Riemann Sum there):

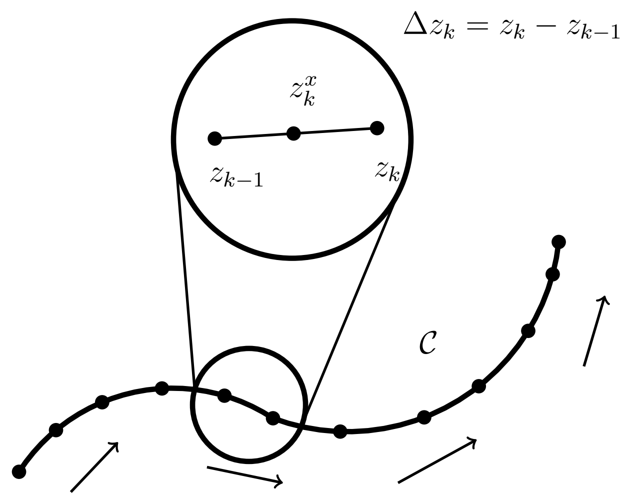
$$\begin{aligned}L = \int_R ds &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_a^b \left|\frac{dz}{dt}\right| dt \quad \text{we use modulus here}\end{aligned}$$

The physical interpretation could be: $\text{total\_distance} = \int_a^b (\text{speed})\, dt$

Now we are ready to integrate $f(z)$ along a curve.

4.2 Contour Integrals

Partition curve $\mathcal{C}$ as shown.



Sum, and let $\max|\Delta z_k| \to 0$:

$$\int_{\mathcal{C}} f(z)dz = \lim_{\max|\Delta z_k|\to 0} \sum_k f(z_k^*)\Delta z_k$$

See the text for more detail.

If $\mathcal{C}$ is a single point, define $\int_{\mathcal{C}} f(z)dz = 0$.

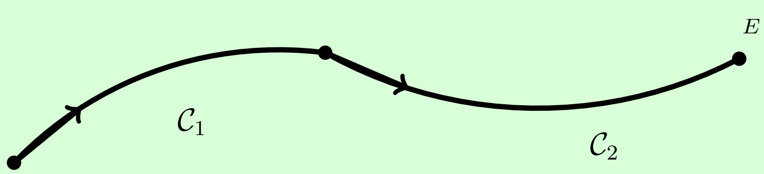
How to calculate?

Definition 4.11. Assume $\mathcal{C}$ has a parameterization. Call it $z(t), t \in [a,b]$. Then:

$$\begin{aligned}\int_{\mathcal{C}} f(z)dz &= \lim_{\max|\Delta z_k|\to 0} \sum_k f(z_k^*)\frac{\overbrace{z(t_k)}^{z_k} - \overbrace{z(t_{k-1})}^{z_{k-1}}}{\Delta t_k}\Delta t_k \\ &= \int_a^b f(z)z'(t)dt\end{aligned}$$

Proposition 4.12. Properties:

- $\int_{\mathcal{C}} \Big(f(z)+g(z)\Big)dz = \int_{\mathcal{C}} f(z)dz + \int_{\mathcal{C}} g(z)dz$
- $\int_{\mathcal{C}} kf(z)dz = k\int_{\mathcal{C}} f(z)dz$
- $\int_{-\mathcal{C}} f(z)dz = -\int_{\mathcal{C}} f(z)dz$. Here $-\mathcal{C}$ means $\mathcal{C}$ traversed in the opposite direction
- $\int_{\mathcal{C}_1+\mathcal{C}_2} f(z)dz = \int_{\mathcal{C}_1} f(z)dz + \int_{\mathcal{C}_2} f(z)dz$. Here it means that we traverse $\mathcal{C}_1$ then traverse $\mathcal{C}_2$.



Is there a triangle inequality? i.e.

$$\left|\int_{\mathcal{C}} f(z)dz\right| \leq_? \int_{\mathcal{C}} |f(z)|\,dz$$

No! LHS is real, but RHS is complex. "$\leq$" does NOT make any sense here.

Proposition 4.13. **The "ML" Inequality**: If $f(z)$ is continuous on a contour $\mathcal{C}$, then

$$\left|\int_{\mathcal{C}} f(z)dz\right| \leq ML$$

where M is an upper bound for $|f(z)|$ on $\mathcal{C}$ and L is the length of $\mathcal{C}$.

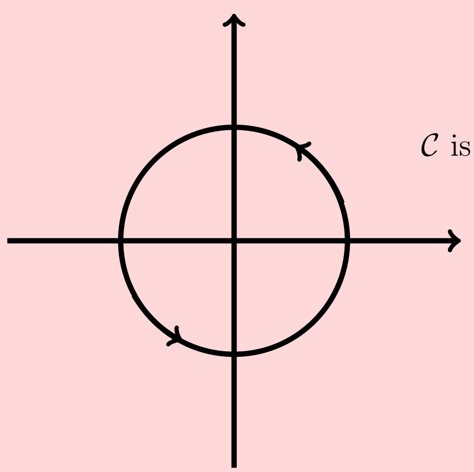
Proof 4.14. Let $z(t)$, $t \in [a,b]$ be a parameterization of $\mathcal{C}$. Then

$$\begin{aligned}
\left|\int_{\mathcal{C}} f(z)dz\right| &= \left|\int_a^b f(z(t))z'(t)dt\right| \\
&\leq \int_a^b \left|f(z(t))z'(t)\right|dt \quad \text{by triangle inequality for integrals w.s.t. real variables} \\
&= M\int_a^b \left|z'(t)\right|dt \\
&= ML
\end{aligned}$$

Second last step: since $|f(z)| \leq M$ on $\mathcal{C}$.

Last step: from last lecture. See Definition 4.10. $\square$

Example 4.15. Find an upper bound on $\left|\int_{\mathcal{C}} e^{\frac{1}{z}}\right|$



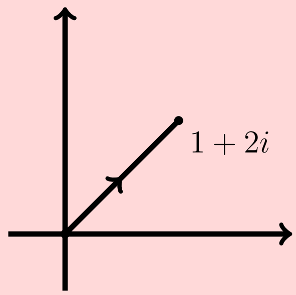
$\mathcal{C}$ is unit circle, traversed <u>once</u> in positive direction

<u>**Solution**</u>: $M = ?$

$$\begin{aligned}
\left|e^{\frac{1}{z}}\right| &= \left|e^{\frac{1}{x+iy}}\right| \\
&= \left|e^{\frac{x-iy}{x^2+y^2}}\right| \\
&= \left|e^{\frac{x}{x^2+y^2}} \cdot e^{-i\frac{y}{x^2+y^2}}\right| \\
&\leq e^{\frac{x}{1}} \quad \text{since } x^2+y^2=1 \\
&\leq e^1 \quad \text{since } x \leq 1
\end{aligned}$$

Clearly, $L = 2\pi$, so $\left|e^{\frac{1}{z}}\right| \leq e^1 \cdot 2\pi = 2\pi e$ by ML inequality.

Example 4.16. Evaluate $\int_{\mathcal{C}} \cos z dz$ where $\mathcal{C}$ is the line segment from 0 to 1+2i.



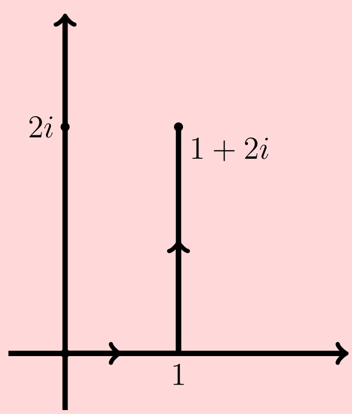
Solution: Parameterize $\mathcal{C}$ by

$$z(t) = 0 + (1 + 2i - 0)t, \quad t \in [0, 1]$$

Then

$$\int_{\mathcal{C}} \cos z dx = \int_0^1 \underbrace{\cos\Big((1+2i)t\Big)}_{f(z(t))} \cdot \underbrace{(1+2i)}_{z'(t)} dt = \sin\Big((1+2i)t\Big)\Big|_0^1 = \sin(1+2i) - 0 = \sin(1+2i)$$

Example 4.17. Evaluate $\int_{\mathcal{C}} \cos z dz$ where $\mathcal{C}$ is:

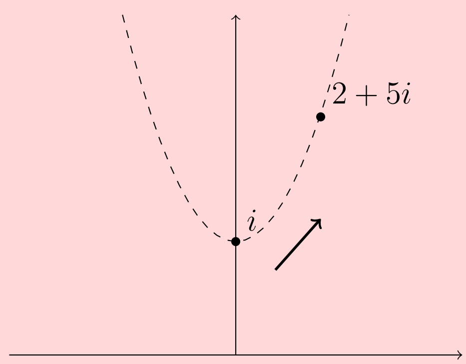


Solution: $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ where $\begin{cases} C_1: & z(t) = t, \quad t \in [0,1) \\ C_2: & z(t) = 1 + (t-1)i, \quad t \in [1,3] \end{cases}$. So

$$\begin{aligned}
\int_{\mathcal{C}} \cos z dx &= \int_{\mathcal{C}_1} \cos z dx + \int_{\mathcal{C}_2} \cos z dx \\
&= \int_0^1 \cos t dt + \int_1^3 \cos(1 + (t-1)i) i dt \\
&= \sin t\big|_0^1 + \sin(1 + (t-1)i)\big|_1^3 \\
&= \sin(1) + (\sin(1+2i) - \sin(1)) \\
&= \sin(1+2i)
\end{aligned}$$

As before

Example 4.18. Evaluate $\int_{\mathcal{C}} e^z dz$ where $\mathcal{C}$ is part of $y = x^2 + 1$ from $z = i$ to $z = 2 + 5i$.



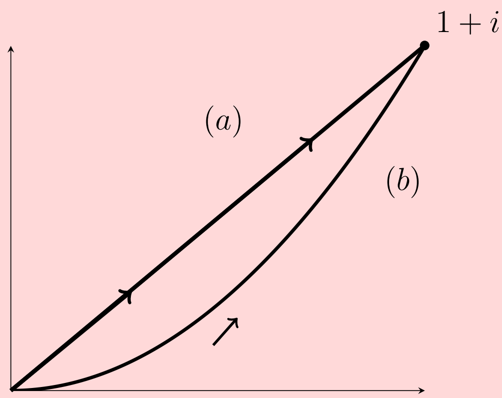
Solution: Let $z(t) = \underbrace{t}_{x} + \underbrace{(t^2+1)}_{y} i, \quad t \in [0,2]$. Then,

$$\begin{aligned}
\int_{\mathcal{C}} e^z dz &= \int_0^2 e^{z(t)} z'(t) dt \\
&= \int_0^2 e^{t^2 + (t^2+1)i} (1 + 2ti) dt \\
&= e^{t^2 + (t^2+1)i}\Big|_0^2 \\
&= e^{2+5i} - e^i \\
&= e^z\Big|_i^{2+5i}
\end{aligned}$$

Does it always work that way? See the following example

Example 4.19. Evaluate $\int_{\mathcal{C}} \overline{z} dz$ where

1. $\mathcal{C}$ is line segment from 0 to $1+i$
2. $\mathcal{C}$ is the smallest arc of circle $x^2+(y-1)^2=1$ from 0 to $1+i$



Solution:

1. parameterization: $z(t) = t(1+i), \;\; t \in [0,1]$

$$\begin{aligned}\int_{\mathcal{C}} \overline{z} dz &= \int_0^1 t(1-i)\cdot(1+i)dt \\ &= \int_0^1 2t dt \\ &= 1\end{aligned}$$

2. parameterization: $z(t) = e^{it}+i, \;\; t \in [\frac{-\pi}{2}, 0]$. It's the unit circle, shifted up by 1 unit.

$$\begin{aligned}\int_{\mathcal{C}} \overline{z} dz &= \int_{-\pi/2}^0 (e^{-it}-i)(ie^{it})dt \\ &= \cdots \\ &= 1 + i(\frac{\pi}{2}-1)dt \\ &\neq 1\end{aligned}$$

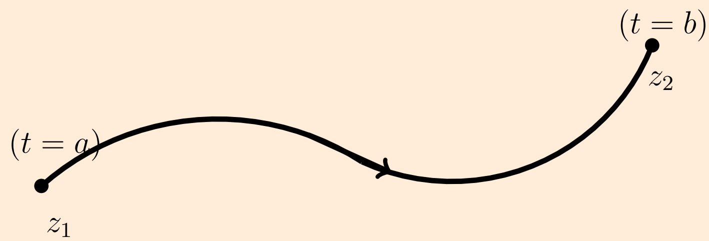
So, the general answer is no. Different paths might yield different results.

4.3 Independence of Path

Theorem 4.20. <u>**Complex Extension of Fundamental Theorem of Calculus**</u>:

If $f(z)$ is continuous in a domain D and has antiderivative $F(z)$ throughout D, then, for any contour $\mathcal{C}$ lying in D with initial point z_1 and terminal point z_2, we have

$$\int_{\mathcal{C}} f(z)dz = F(z_2) - F(z_1)$$

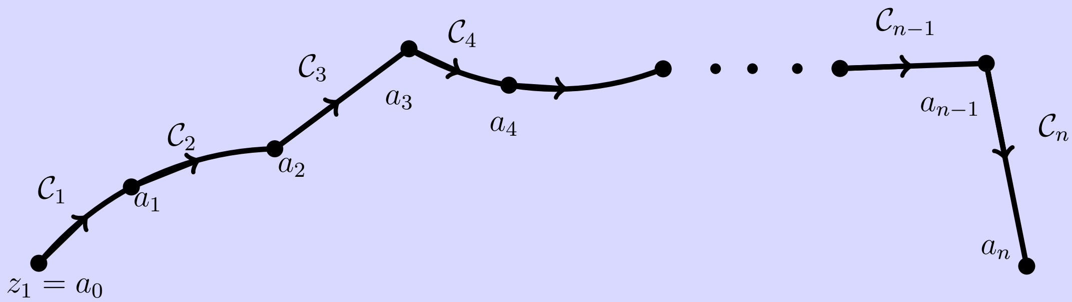


Proof 4.21. First, suppose $\mathcal{C}$ is smooth, i.e. $z'(t) \neq 0$, continuous.

Parameterize by $z(t), t \in [a, b]$. Then,

$$\begin{aligned}\int_{\mathcal{C}} f(z)dz &= \int_a^b f(z(t))z'(t)dt \\ &= \int_{\mathcal{C}} \frac{d}{dt}\Big(F\big(z(t)\big)\Big)dt \quad \text{by chain rule} \\ &= F(z(t))\Big|_{t=a}^{b} \\ &= F(z(b)) - F(z(a)) \\ &= F(z_2) - F(z_1)\end{aligned}$$

Next, if $\mathcal{C}$ is not smooth, it has a finite number of smooth pieces, since it's a contour.

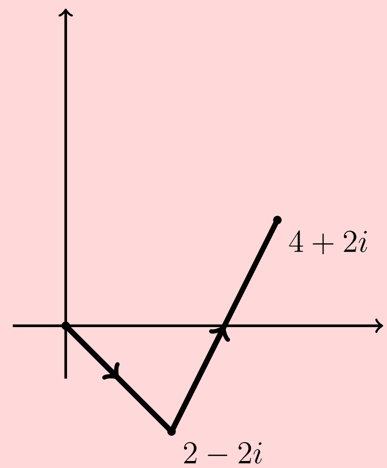


Apply the result above to each piece:

$$\begin{aligned}\int_{\mathcal{C}} f(z)dz &= \int_{\mathcal{C}_1} f(z)dz + \cdots + \int_{\mathcal{C}_n} f(z)dz \\ &= \Big(F(a_1) - F(a_0)\Big) + \Big(F(a_2) - F(a_1)\Big) + \cdots + \Big(F(a_n) - F(a_{n-1})\Big) \\ &= F(a_n) - F(a_0) \\ &= F(z_2) - F(z_1)\end{aligned}$$

□

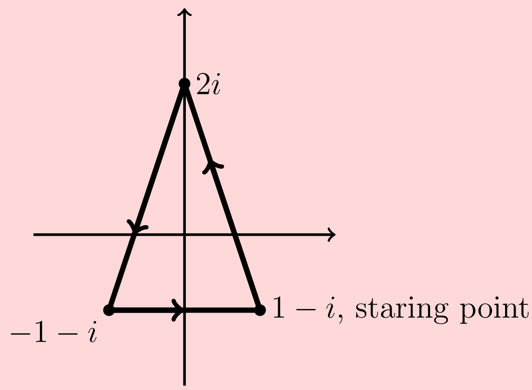
Example 4.22. Evaluate $\int_{\mathcal{C}}(1+z^2)dz$ where $\mathcal{C}$ is:



Solution:

$$\begin{aligned}\int_{\mathcal{C}}(1+z^2)dz &= \left(z + \frac{z^3}{4}\right)\Bigg|_{z=0}^{z=4+2i} \\ &= \cdots \\ &= \frac{28}{3} + \frac{94}{3}i\end{aligned}$$

Example 4.23. Evaluate $\int_{\mathcal{C}} e^z dz$ where $\mathcal{C}$ is:



Solution:

$$\begin{aligned}\int_{\mathcal{C}}(1+z^2)dz &= e^z\Big|_{z=1-i}^{z=1-i}\\ &= e^{1-i}-e^{1-i}\\ &= 0\end{aligned}$$

Theorem 4.24. Let f be a continuous function in a domain D. Then, the following statements are equivalent:

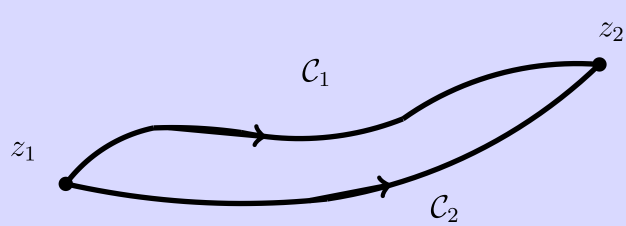
1. f has an antiderivative in D.
2. If $\mathcal{C}$ is any closed contour in D, then $\int_{\mathcal{C}} f(z)dz = 0$.
3. The contour integrals of f are independent of path in D.

Proof 4.25. $1 \Rightarrow 2$: It follows immediately from Theorem 4.20 with $\mathcal{C}$ being a closed contour.

$2 \Rightarrow 3$: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be any two contours in D with same end points. Let $\mathcal{C}$ be the closed contour $\mathcal{C}_1 + (-\mathcal{C}_2)$.

Then, $\int_{\mathcal{C}} f(z)dz = 0$. So $\int_{\mathcal{C}_1} f(z)dz + \int_{-\mathcal{C}_2} f(z)dz = 0$. So $\int_{\mathcal{C}_1} f(z)dz - \int_{\mathcal{C}_2} f(z)dz = 0$, implying that

$$\int_{\mathcal{C}_1} f(z)dz = \int_{\mathcal{C}_2} f(z)dz$$



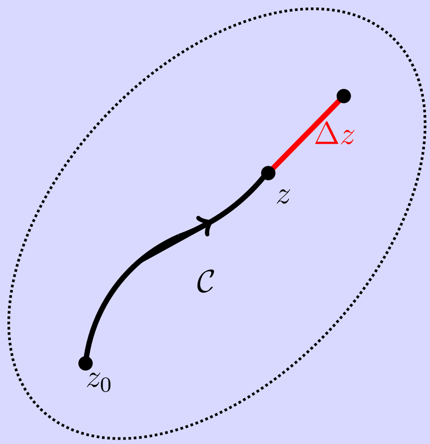
3 $\Rightarrow$ 1: Construct the antiderivative. Choose a point $z_0 \in D$, and let $\mathcal{C}$ be the contour as shown. Recall the D is a connected set.

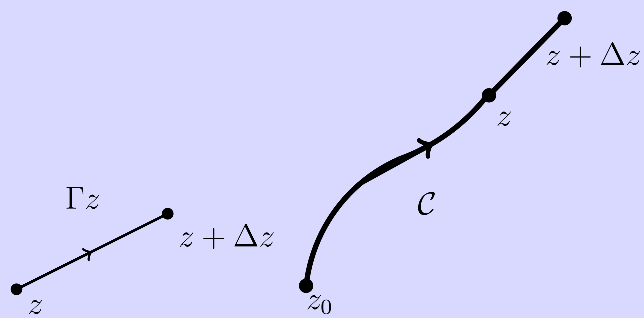
Define $F(z) = \int_{\mathcal{C}} f(w)dw$. By 3, $F(z)$ is single valued; We will show that $F'(z) = f(z)$.

For any point z, choose Δz small enough such that the line segment Γ parameterized by

$$z(t) = z + t\Delta z, \quad t \in [0,1]$$

is in D (This is possible since D is open)





Then

$$\begin{aligned}
F(z+\Delta z) - F(z) &= \left(\int_{\mathcal{C}} f(w)dw + \int_{\Gamma} f(w)dw\right) - \int_{\mathcal{C}} f(w)dw \\
&= \int_{\Gamma} f(w)dw \\
&= \int_0^1 f(z(t))z'(t)dt \\
&= \int_0^1 f(z+t\Delta z)(\Delta z)dt \\
\Rightarrow \quad \frac{F(z+\Delta z) - F(z)}{\Delta z} &= \int_0^1 f(z+t\Delta z)dt
\end{aligned}$$

Let $\Delta z \to 0$.

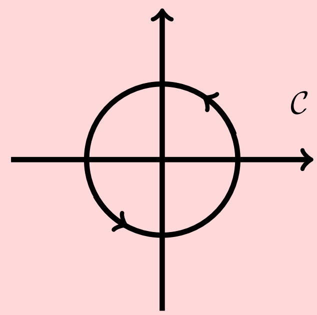
$$F'(z) = \int_0^1 f(z)dt = f(z)\int_0^1 dt = f(z)$$

□

We showed that $\overline{z}$ can be integreated, but the result depends on path. So $\overline{z}$ is integrable, but not anti-differentiable. Also, functions with antiderivatives are easy; for those without, we must parameterize.

4.4 Cauchy's Integral Theorem

Example 4.26. Most Important Example in this Course: Evaluate $\int_{\mathcal{C}} \frac{1}{z} dz$ where $\mathcal{C}$ is the unit circle.

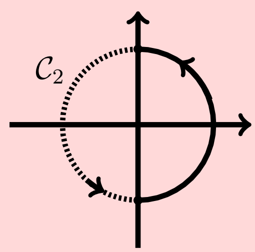


<u>**Solution**</u>: $\frac{1}{z}$ does not have antiderivative over all of $\mathcal{C}$. Any branch of $\log z$ will have a problem, i.e. $\mathcal{C}$ will cross a branch cut.

Method 1: Parameterize $\mathcal{C}$ by $e^{it}, t \in [0, 2\pi]$. By definition,

$$\int_{\mathcal{C}} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it} dt = 2\pi i$$

Method 2: Split $\mathcal{C}$ in two, and use Theorem 4.20 on each.



$$\begin{aligned}
\int_{\mathcal{C}_1} \frac{1}{z} dz &= \operatorname{Log} z\Big|_{-i}^{i} \quad \text{branch cut at } \theta = -\pi \\
&= \operatorname{Log} i - \operatorname{Log}(-i) \\
&= i\frac{\pi}{2} - i\left(\frac{-\pi}{2}\right) \\
&= \pi i \\
\int_{\mathcal{C}_2} \frac{1}{z} dz &= \operatorname{Log}_0 z\Big|_{i}^{-i} \\
&= \operatorname{Log}_0(-i) - \operatorname{Log}_0(i) \\
&= \frac{3\pi}{2} i - \frac{\pi}{2} i \\
&= \pi i
\end{aligned}$$

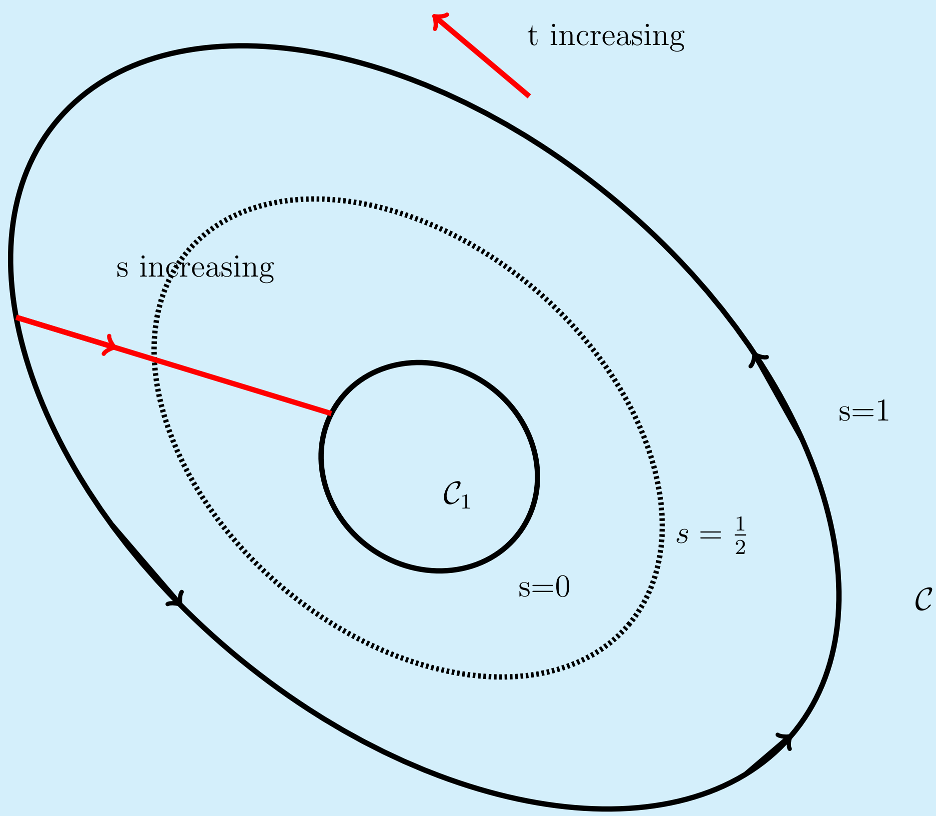
Therefore,

$$\int_{\mathcal{C}} \frac{1}{z}dz = \int_{\mathcal{C}_1} \frac{1}{z}dz + \int_{\mathcal{C}_2} \frac{1}{z}dz = \pi i + \pi i = 2\pi i$$

Go around the contour twice, what's the result? It would be $4\pi i$. Also, going counter-clockwise would yield the result $-2\pi i$

Definition 4.27. A closed contour $\mathcal{C}$ is said to be **continuously deformable** to a contour $\mathcal{C}_1$ in a domain D if there exists a function $z(s,t)$, continuous for $s \in [0,1], t \in [0,1]$, such that

1. $z(s,t)$ is a closed contour in D for each $s \in [0,1]$
2. $z(0,t)$ is a parameterization of $\mathcal{C}$
3. $z(1,t)$ is a parameterization of $\mathcal{C}_1$



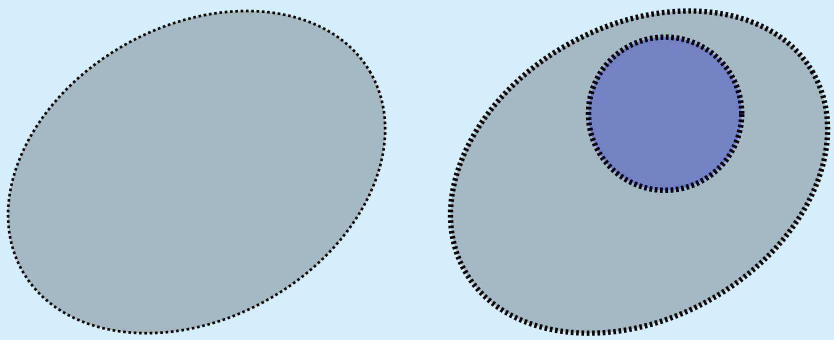
Theorem 4.28. **Deformation Invariance Theorem**: Let f be analytic in a domain D,

containing closed contours $\mathcal{C}_1$ and $\mathcal{C}_2$. If $\mathcal{C}_1$ can be continuously deformed into $\mathcal{C}_2$, then

$$\int_{\mathcal{C}_1} f(z)dz = \int_{\mathcal{C}_2} f(z)dz$$

Proof 4.29. It's too hard - 12 pages long in one text. □

Definition 4.30. A **simply connected domain** is a domain in which every "loop" (closed contour) in D can be continuously deformed to a point (while remaining in D).



simply connected Not simply connected

Theorem 4.31. **Cauchy's Integral Theorem (Cauchy-Goursat Theorem)**:
If f is analytic in a simply connected domain D, and $\mathcal{C}$ is a closed contour in D, then

$$\int_{\mathcal{C}} f(z)dz = 0$$

Proof 4.32. Follows from Theorem 4.28 by shrinking $\mathcal{C}$ continuously to a point. □

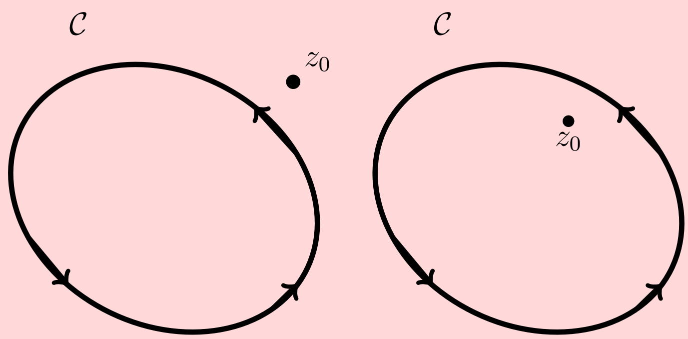
Corollary 4.33. Since $\int_{\mathcal{C}} f(z)dz = 0 \Leftrightarrow f$ has an antiderivative in D, we have that if f is analytic, then f also has an antiderivative, which is analytic. So every analytic function is infinitely antidifferentiable.

Example 4.34. Back to Example 4.26. We know that $\int_{\mathcal{C}} \frac{1}{z} dz = 2\pi i$ for any closed contour enclosing the origin.

Also, $\int_{\mathcal{C}} \frac{1}{z} dz = 0$ for any closed contours <u>not</u> enclosing the origin.

Could shift results:

$$\int_{\mathcal{C}} \frac{1}{z - z_0} dz = \begin{cases} 0 & \text{if } z_0 \text{ is exterior to } \mathcal{C} \\ 2\pi i & \text{if } z_0 \text{ is interior to } \mathcal{C} \end{cases}$$



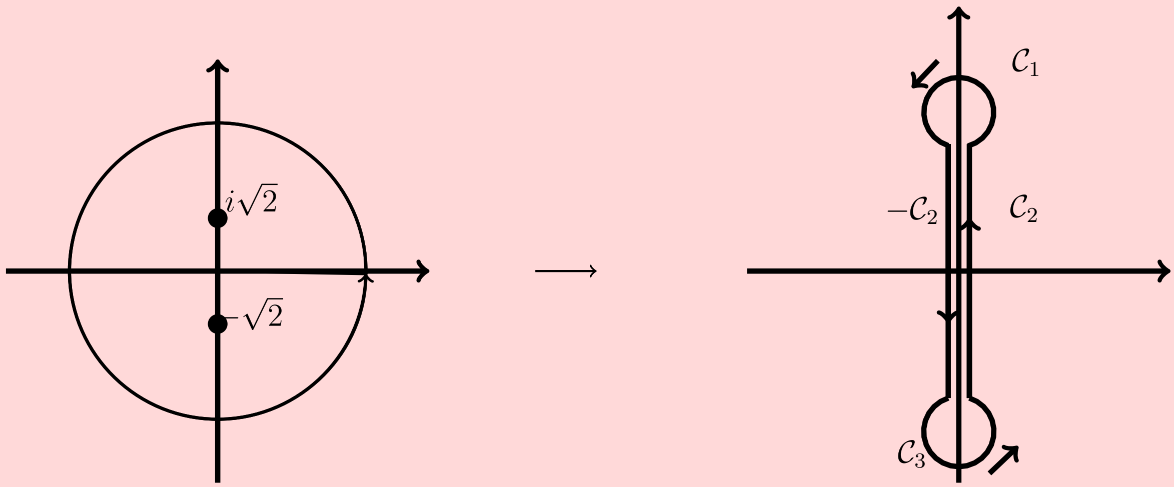
Example 4.35. Evaluate $\int_{\mathcal{C}} \frac{2z}{z^2 + 2} dz$ where $\mathcal{C}$ is the positively oriented circle of radius 2 centered at origin.

<u>**Solution**</u>: We can do partial fractions:

$$\frac{2z}{z^2 + 2} = \frac{1}{z + i\sqrt{2}} + \frac{1}{z - i\sqrt{2}}$$

So we have singularities at $z = \pm i\sqrt{2}$. We can use the **Deformation Invariance Theorem** to deform $\mathcal{C}$ like below. So

$$\begin{aligned} \int_{\mathcal{C}} &= \int_{\mathcal{C}_2} + \int_{\mathcal{C}_1} + \int_{-\mathcal{C}_2} + \int_{\mathcal{C}_3} \\ &= \int_{\mathcal{C}_1} + \int_{\mathcal{C}_3} \end{aligned}$$



And

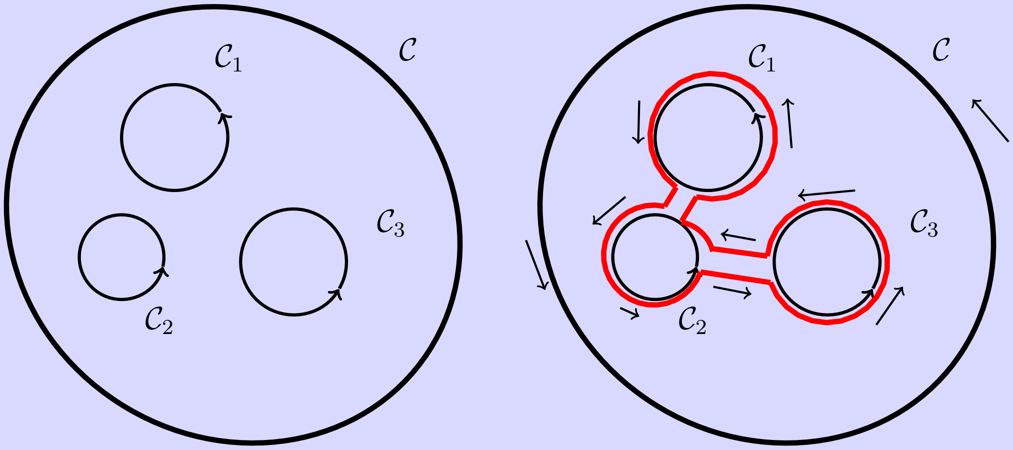
$$\begin{aligned}
\int_{\mathcal{C}} \frac{2z}{z^2+2} dz &= \int_{\mathcal{C}} \left(\frac{1}{z+i\sqrt{2}} + \frac{1}{z-i\sqrt{2}} \right) dz \\
&= \int_{\mathcal{C}_1} \left(\frac{1}{z+i\sqrt{2}} + \frac{1}{z-i\sqrt{2}} \right) dz + \int_{\mathcal{C}_3} \left(\frac{1}{z+i\sqrt{2}} + \frac{1}{z-i\sqrt{2}} \right) dz \\
&= \int_{\mathcal{C}_1} \frac{1}{z+i\sqrt{2}} dz + \int_{\mathcal{C}_1} \frac{1}{z-i\sqrt{2}} dz + \int_{\mathcal{C}_3} \frac{1}{z+i\sqrt{2}} dz + \int_{\mathcal{C}_3} \frac{1}{z-i\sqrt{2}} dz \\
&= 0 + 2\pi i + 2\pi i + 0 \\
&= 4\pi i
\end{aligned}$$

Theorem 4.36. **Extended Cauchy-Goursat Theorem**:

$$\int_{\mathcal{C}} f(z) dz = \sum_{i=1}^{n} \int_{\mathcal{C}_i} f(z) dz$$

Proof 4.37. Ideas (for the case of $n = 3$): Deform $\mathcal{C}$ to Γ as shown:

$$\int_{\mathcal{C}} = \int_{\tilde{\mathcal{C}}} = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3}$$



$\square$

What about $\int_{\mathcal{C}} \frac{1}{(z-z_0)^2} dz$ or other powers of $z - z_0$?

Consider $\int_{\mathcal{C}} (z-z_0)^n dz$ where $n \neq -1$.

- If z_0 is external to $\mathcal{C}$, the integral is zero, by **Cauchy's Integral Theorem** 4.31.
- If z_0 is internal to $\mathcal{C}$, deform $\mathcal{C}$ to the unit circle $|z - z_0| = 1$, parameterized by $z = z_0 + e^{it}$, $t \in [0, 2\pi]$. We may use the radius ϵ if the circle is not small enough. The result would be the same.

  Then

$$\begin{aligned}\int_{\mathcal{C}} (z-z_0)^n dz &= \int_{\mathcal{C}} \left(e^{it}\right)^n ie^{it} dt \\ &= \frac{i}{n+1} e^{i(n+1)t}\Big|_0^{2\pi} \\ &= 0\end{aligned}$$

Thus for an interioir point z_0 in $\mathcal{C}$

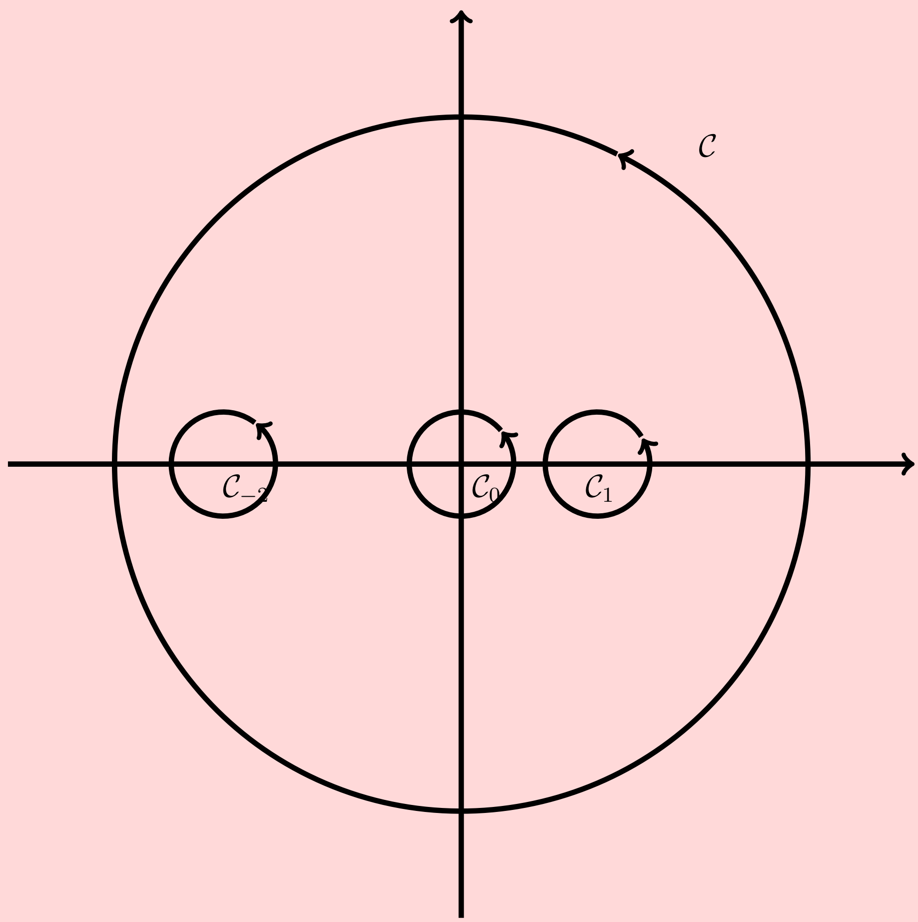
$$\boxed{\int_{\mathcal{C}} (z-z_0)^n dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}}$$

Example 4.38. Let $\mathcal{C}$ be the positively oriented circle of radius 3 centered at the origin. Evaluate $\int_{\mathcal{C}} \frac{3z^3+2}{z^4+z^3-2z^2} dz$

Solution: Note that $z^4 + z^3 - 2z^2 = z^2(z^2 + z - 2) = z^2(z-1)(z+2)$. These give us the location of the singularities.

Note the partial fractions:

$$\frac{3z^3+2}{z^4+z^3-2z^2} = \frac{-1/2}{z} + \frac{-1}{z^2} + \frac{5/3}{z-1} + \frac{11/6}{z+2}$$



By the **Extended Cauchy-Goursat Theorem**,

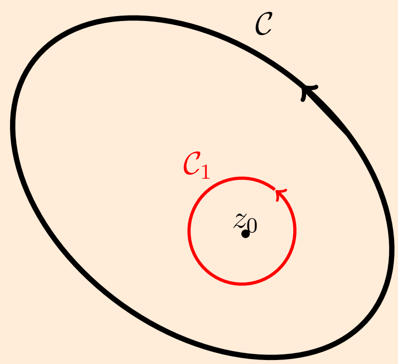
$$\begin{aligned}\int_{\mathcal{C}} f(z)dz &= \int_{\mathcal{C}_{-2}} f(z)dz + \int_{\mathcal{C}_0} f(z)dz + \int_{\mathcal{C}_1} f(z)dz \\ &= \frac{11}{6}\cdot(2\pi i) + \frac{-1}{2}\cdot(2\pi i) + \frac{5}{3}\cdot(2\pi i) \\ &= 6\pi i\end{aligned}$$

4.5 Cauchy's Integral Formula

Theorem 4.39. Cauchy's Integral Formula (CIF): Let $\mathcal{C}$ be a simple, closed, positively-oriented contour. If f is analytic in some simply connected domain D containing

$\mathcal{C}$, and z_0 is any point inside $\mathcal{C}$. Then,

$$\int_{\mathcal{C}} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$



Proof 4.40. deform $\mathcal{C}$ to $\mathcal{C}_r$, a positively oriented circle of radius r centered at z_0: $|z-z_0| = r$. We will let $r \to 0$.

Then,

$$\begin{aligned}
\int_{\mathcal{C}} \frac{f(z)}{z - z_0} dz &= \int_{\mathcal{C}_r} \frac{f(z)}{z - z_0} dz \\
&= \int_{\mathcal{C}_r} \frac{f(z) - f(z_0)}{z - z_0} dz + \int_{\mathcal{C}_r} \frac{f(z_0)}{z - z_0} dz \quad \text{by linearity} \\
&= 0 + 2\pi i f(z_0) \\
&= 2\pi i f(z_0)
\end{aligned}$$

To show $\int_{\mathcal{C}_r} \frac{f(z) - f(z_0)}{z - z_0} dz = 0$, we consider the following:

On $\mathcal{C}_r$, we have $|f(z) - f(z_0)| \leq M$ for some M. Then

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq \frac{M}{r} \quad \text{since } |z - z_0| = r \text{ on } C_r$$

By ML inequality,

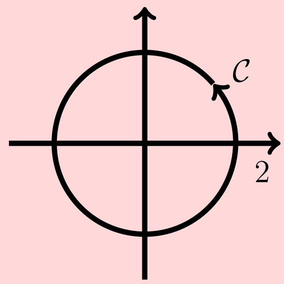
$$\left| \int_{\mathcal{C}_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \frac{M}{r} \cdot \text{length}(\mathcal{C}_r) = \frac{M}{r} 2\pi r = 2\pi M$$

Let $r \to 0$, then $M \to 0$ by continuity of f, and so

$$\int_{\mathcal{C}_r} \frac{f(z) - f(z_0)}{z - z_0} dz = 0$$

$\square$

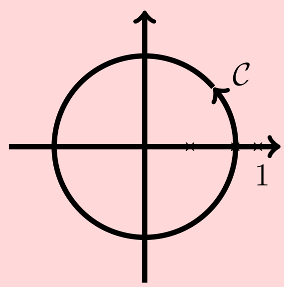
Example 4.41. Evaluate $\int_\mathcal{C} \frac{e^z}{z-1}dz$.



Solution: Let $f(z) = e^z$. Since $f(z)$ is entire, and $z_0 = 1$ is inside $\mathcal{C}$, we have, by **CIF**,

$$\int_\mathcal{C} \frac{e^z}{z-1}dz = 2\pi i f(1) = 2\pi i e^1 = 2\pi e i$$

Example 4.42. Evaluate $\int_\mathcal{C} \frac{e^{i\pi z}}{2z^2 - 5z + 2}dz$.

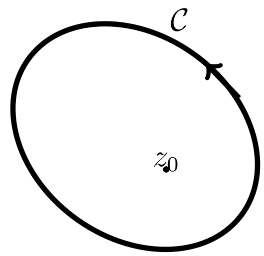


Solution:

$$\begin{aligned}
\int_\mathcal{C} \frac{e^{i\pi z}}{2z^2 - 5z + 2}dz &= \int_\mathcal{C} \frac{e^{i\pi z}}{2(z-\frac{1}{2})(z-2)}dz \\
&= \int_\mathcal{C} \frac{\frac{e^{i\pi z}}{2(z-2)}}{z-\frac{1}{2}}dz \quad \text{regard the numerator as } f(z) \\
&= 2\pi i f(\frac{1}{2}) \quad \text{by CIF} \\
&= 2\pi i \frac{e^{i\pi/2}}{2(\frac{-3}{2})} \\
&= \frac{2\pi}{3}
\end{aligned}$$

From CIF, we know

$$f(z_0) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{z - z_0} dz$$



So the value of f at any point <u>inside</u> $\mathcal{C}$ is determined by the values of f <u>on</u> $\mathcal{C}$

Proposition 4.43. **<u>Mean Value Property</u>**: If $\mathcal{C}$ is a circle of radius R centered at z_0:

$$\begin{aligned}
f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(\overbrace{z_0 + \Re^{it}}^{z(t)}\right)}{z_0 + \Re^{it} - z_0} \overbrace{\left(iRe^{it}\right)}^{z'(t)} \quad \text{by parameterizing circle} \\
&= \frac{\int_0^{2\pi} f\left(z_0 + \Re^{it}\right)}{2\pi - 0} \\
&= \text{ averge value of } f \text{ on the circle, recall that } \frac{\int_a^b f(x)dx}{b - a} = \overline{f}
\end{aligned}$$

Theorem 4.44. **<u>Derivatives of f</u>**:

$$f(z_0) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(w)}{w - z} dw$$

Differentiate:

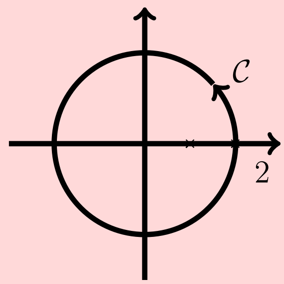
$$\begin{aligned}
f'(z) &= \frac{1}{2\pi i} \int_{\mathcal{C}} f(w) \frac{d}{dz} \left(\frac{1}{w - z} \right) dw \quad \text{by Leibniz's rule} \\
&= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(w)}{(w - z)^2} dw
\end{aligned}$$

which is also differentiable.

Repeating and switch back to z_0 we get **<u>Cauchy's Integral Formula for Derivatives (CIFD)</u>**:

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad \text{where } z_0 \text{ is inside } \mathcal{C}}$$

Example 4.45. Evaluate $\int_{\mathcal{C}} \frac{z^3+2z+1}{(z-1)^3} dz$.



<u>**Solution**</u>: Use **CIFD** with $n=2$.

$$f''(z_0) = \frac{2!}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^3} dz$$

Let $f(z) = z^3 + 2z + 1$ and $z_0 = 1$. Then we have

$$\begin{aligned}(6z+0+0)\Big|_{z=1} &= \frac{1}{\pi i} \int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^3} dz \\ 6\pi i &= \int_{\mathcal{C}} \frac{z^3+2z+1}{(z-1)^3} dz\end{aligned}$$

4.6 Implication of CIFD

Corollary 4.46. An analytic function is infinitely differentiable. Furthermore, with $f(z) = u(x,y) + iv(x,y)$, u and $v \in C^\infty$ (i.e. have continuous partials of all order)

Proof 4.47. $f = u + iv$, then

$$f' = \begin{cases} u_x + iv_x & \Rightarrow f'' = \begin{cases} u_{xx} + iv_{xx} & \cdots \\ v_{xy} - iu_{xy} & \cdots \end{cases} \\ v_y - iu_y & \Rightarrow f'' = \begin{cases} v_{yx} - iu_{yx} & \cdots \\ -u_{yy} - iv_{yy} & \cdots \end{cases} \end{cases}$$

Existence of f'' implies u_x, u_y, v_x, v_y are all continous. Also, observe that $u_{xx} = -u_{yy}$, $v_{xx} = -v_{yy}$, $v_{xy} = v_{yx}$, $u_{xy} = u_{yx}$ $\square$

Theorem 4.48. **Morera's Theorem**: (the converse of **Cauchy's Integral Theorem**)

Let f be a continuous function in a simply connected domain D. If $\int_{\mathcal{C}} f(z)dz = 0$ for every closed contour $\mathcal{C}$ in D, then f is analytic in D.

Proof 4.49. We've shown that $\int_{\mathcal{C}} f(z)dz = 0$ for all $\mathcal{C}$ implies that f has antiderivative in D, call it $F(z)$.

Now D is open, and F is differentiable in D ($F' = f$), so therefore F is analytic, therefore $F' = f$ is analytic. □

Lemma 4.50. **"Cauchy's Estimate"**: Let f be analytic on and inside a circle $\mathcal{C}$ of radius R centered at z_0.

If $|f(z)| \leq M$ for all z on $\mathcal{C}$, then $\left|f^{(n)}(z_0)\right| \leq \dfrac{n!M}{R^n}$.

Proof 4.51. From **CIFD**,

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \leq \left|\frac{n!}{2\pi i}\right| \overbrace{\left(\frac{M}{R^{n+1}}\right)}^{\text{“}M\text{''}} \cdot \overbrace{(2\pi R)}^{\text{“}\ell\text{''}}$$

since $|z - z_0| = R$ and the $M\ell$-inequality. □

Theorem 4.52. **Liouville's Theorem**: If f is entire, and bounded for all $z \in \mathbb{C}$, then f is constant.

Proof 4.53. Have $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Consider $z_0 \in \mathbb{C}$, and let $\mathcal{C}$ be circle of radius R centered at z_0. Cauchy's estimate yields $|f'(z_0)| \leq \dfrac{M}{R}$. True for all R, no matter how large. So $|f'(z_0)| = 0 \Rightarrow f'(z_0) = 0$.

z_0 is arbitrary, so f must be constant. □

Corollary 4.54. Every non-constant, entire function is unbounded.

We can use this to prove the **Fundamental Theorem of Algebra**.

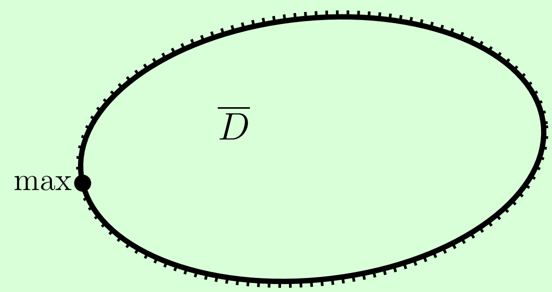
Theorem 4.55. **Fundamental Theorem of Algebra**: Every nonconstant polynomial with complex coefficients has at least one zero.

Proof 4.56. If $P(z)$ has no zeros, then $\dfrac{1}{P(z)}$ is entire. Since it is continuous, we must have $|P(z)| \geq \epsilon$ for some $\epsilon > 0$.

So, $\dfrac{1}{|P(z_0)|} \leq \dfrac{1}{\epsilon}$, implying that $\dfrac{1}{P(z_0)}$ is constant, by **Liouville's Theorem**.

So, $P(z_0)$ is constant. Hence, a non-constant polynomial must have a zero. $\square$

Proposition 4.57. **Maximum Modulus Principle**: If $f(z)$ is analytic on a bounded domain D, and continuous on $\overline{D}$, the closure of D. Then, $|f(z)|$ attains a maximum value on $\overline{D}$ and it occurs on the boundary.



Chapter 5 Series Representation for Analytic Functions

5.1 Sequences and Series

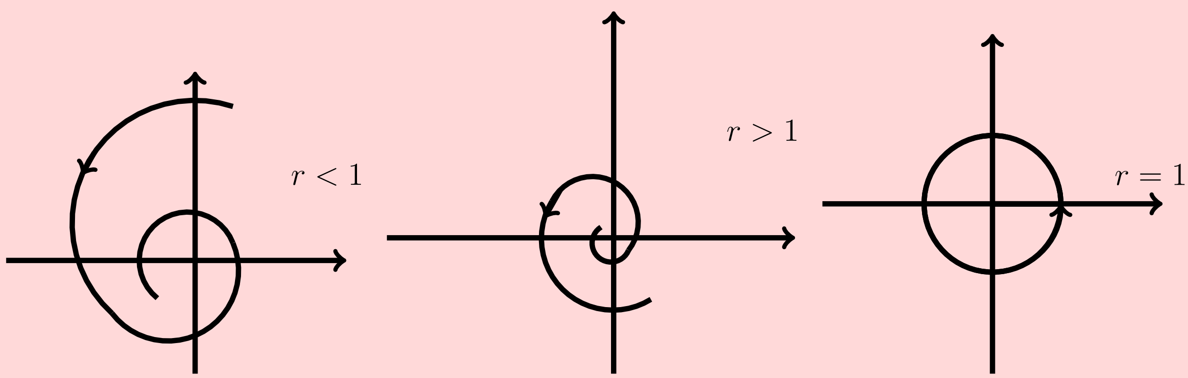
Definition 5.1. A sequence $\{z_n\}_{n=1}^{\infty}$ **converges** to z_0 if for any $\epsilon > 0$, there exists an integer N such that $n > N \Rightarrow |z_n - z_0| < \epsilon$.

Theorem 5.2. Let $z_n = x_n + iy_n$ for $n = 1, 2, ...,$ and $z_0 = x_0 + iy_0$. Then, $z_n \to z_0$ if and only if $x_n \to x_0$ and $y_n \to y_0$ as $n \to \infty$.

Example 5.3. Consider $\{z^n\}_{n=1}^{\infty}$.

Notice that $z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} \to 0$ as $n \to \infty$ if and only if $r < 1$.

In other words, $\{z^n\}_{n=1}^{\infty}$ converges to 0 as $n \to \infty$ if and only if $|z| < 1$.



Definition 5.4. **Series**:

$$\sum_{n=1}^{\infty} z_n = \lim_{k\to\infty} \sum_{n=1}^{k} z_n$$

is **convergent** if the limit exists (called the **sum** of the series); otherwise is **divergent**.

Note that the LHS does not need to start at $n = 1$. The RHS is just the partial sum.

Proposition 5.5. **Divergence Test/nth-term Test**:

If $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

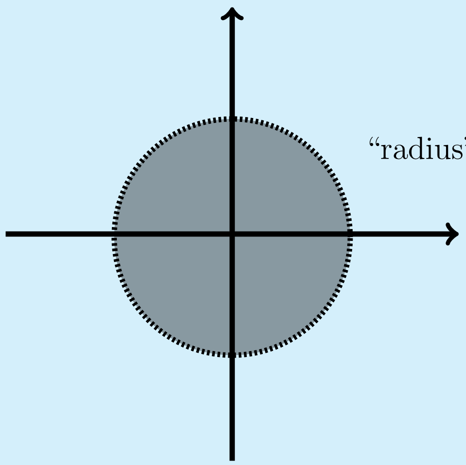
Contrapositive: if $z_n \not\to 0$, then $\sum_{n=1}^{\infty} z_n$ diverges.

Definition 5.6. **Geometric Series**:

$$\sum_{n=0}^{\infty} z_n = \begin{cases} \dfrac{1}{1-z} & \text{if } |z| < 1 \\ \text{divergent} & \text{if } |z| \geq 1 \end{cases}$$

We can see that

- in $\mathbb{R}$, $\sum x^n$ converges for $|x| < 1$.
- in $\mathbb{C}$, $\sum z^n$ converges for $|z| < 1$.



Example 5.7. $\sum_{n=0}^{\infty} \left(\frac{1}{2} + i\right)^n$ and $\sum_{n=0}^{\infty} i^n$ divergent.

$\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n$ is convergent, and equals $\dfrac{1}{1-\frac{i}{2}} = \dfrac{2}{2-i} = \dfrac{4+2i}{3}$

Proposition 5.8. **Comparison Test**: If $\sum_{k=1}^{\infty} M_k$ is a convergent series of real numbers and $|z_k| \leq M_k$ for all sufficiently large k, then $\sum_{k=1}^{\infty} z_k$ converges.

Definition 5.9. $\sum z_n$ is absolutely convergent if $\sum |z_n|$ converges.

Proposition 5.10. **Ratio Test**: If $\lim_{n\to\infty} \left|\dfrac{z_{n+1}}{z_n}\right| = L$, then $\sum_{n=0}^{\infty} z_n$ is absolutely convergent if $L < 1$, and divergent if $L > 1$. No conclusion if $L = 1$.

Example 5.11.

$$\sum_{n=0}^{\infty} \frac{(1-i)^n}{n!} = 1 + \frac{1-i}{1!} + \frac{(1-i)^2}{2!} + \cdots \quad [0! = 1]$$

Ratio is

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(1-i)^{n+1}}{(n+1)n!} \cdot \frac{n!}{(1-i)^n}\right| = \frac{|1-i|}{n+1} = \frac{\sqrt{2}}{n+1} \to 0$$

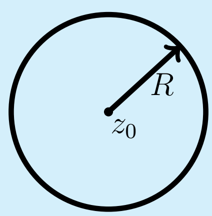
as $n \to \infty$, so the series converges.

Definition 5.12. **Power Series**:

$$\sum_{k=0}^{\infty} c_k (z - z_0)^k$$

This series could

1. Converge only at $z = z_0$
2. Converge for all z; or
3. Converge for all z such that $|z - z_0| < R$, and diverge for all z such that $|z - z_0| > R$.

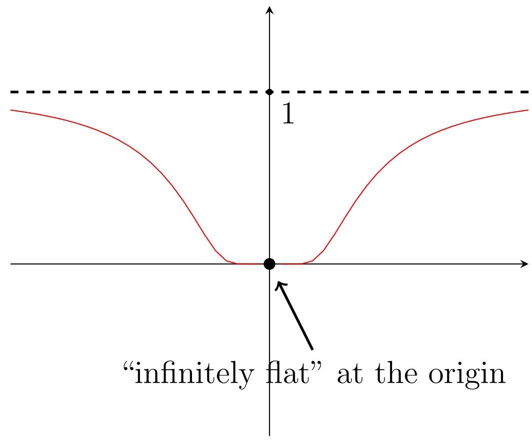


Definition 5.13. The **Taylor Series** of $f(z)$ centered at z_0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Convergence of Taylor Series: In MATH138, we assumed that $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ converges to $f(x)$ for each x in interval for convenience. This is not always true.

Consider the following famous examle.



$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \Rightarrow \quad f(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}(x-0)^n = \cdots = 0$$

which converges for all x, but $f(x) = g(x)$ at only one point.

Good news is that in $\mathbb{C}$, things make more sense

5.2 Taylor Series and Convergence

Theorem 5.14. If f is analytic in the disk $|z - z_0| < R$, then its Taylor series converges to $f(z)$ for all z in this disc. The convergence is uniform on any closed subdisc $|z - z_0| \leq R_0 < R$.

In $\mathbb{R}$, "analytic" means "has a power series representation".

What's wrong with $g(x)$ above?

- In $\mathbb{R}$, $g^{(n)}(0) = 0$ for all n
- In $\mathbb{C}$, $g^{(n)}(0) = \begin{cases} e^{-1/z^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ is not even **continuous** at $z = 0$.

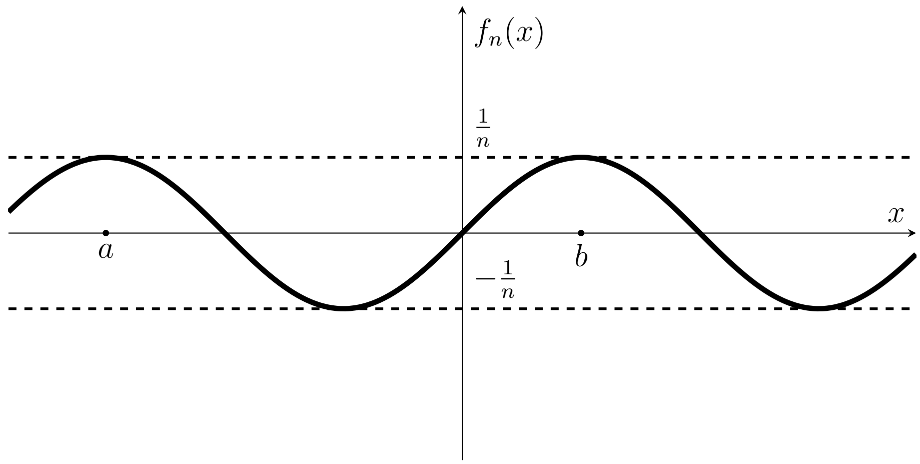
In $\mathbb{R}$, we only saw a cross-section, the "bad behaviour" was missed.

Uniform Convergence: Consider the following sequence of function:

1. $f_n(x) = \dfrac{\sin x}{n}$.

   What happens as $n \to \infty$? $f_n(x) \to 0$ as $n \to \infty$. $f_n \to 0$ "pointwise"

   Over some interval:



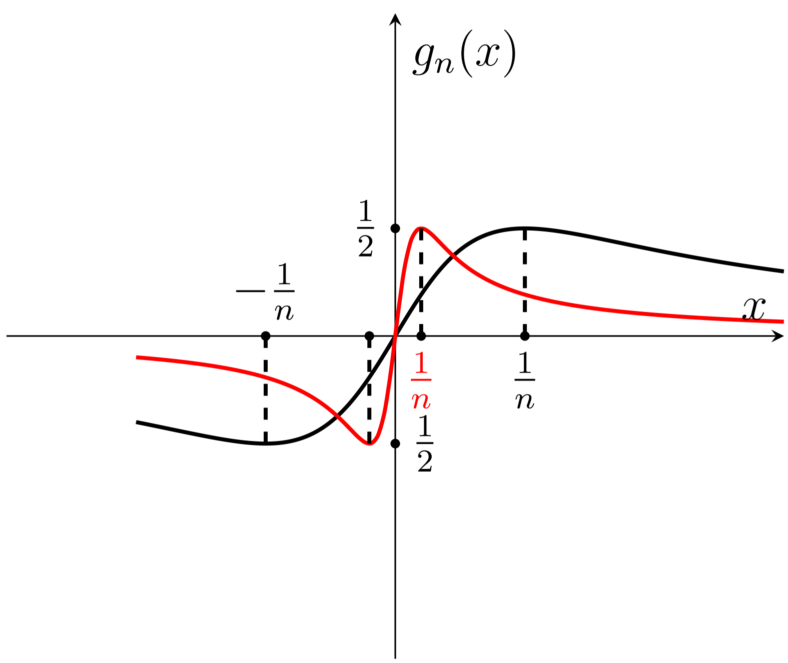
$f_n(x)$ approaches zero function on $[a, b]$.

In this case, we say "$f_n \to 0$ uniformly"

2. $g_n(x) = \dfrac{nx}{1 + n^2x^2}$

   What happens as $n \to \infty$? $g_n(x) \to 0$ as $n \to \infty$. $g_n \to 0$ "pointwise"

   Over some interval:



$g_n(x)$ still has "spikes" even as $n \to \infty$.

In this case, we say "$g_n \to 0$ does <u>not</u> converge uniformly to 0"

For series, we apply the above ideas to the sequence of partial sums: $S_n(x) = \sum_{k=0}^{n} f_k(x)$. AMATH231 covers more on this.

Proposition 5.15. **Manipulation of Taylor Series**: a few known series regarding

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad |z-z_0| < R, \quad \text{(radius of convergence)}$$

$$\begin{cases} \dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n & R = 1 \\ e^z = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} z^n & R = \infty \\ \cos z = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} & R = \infty \\ \sin z = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} & R = \infty \\ \cosh z = \displaystyle\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} & R = \infty \\ \sinh z = \displaystyle\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} & R = \infty \\ (1+z)^\alpha = 1 + \alpha z + \dfrac{\alpha(\alpha-1)}{2!} z^2 + \dfrac{\alpha(\alpha-1)(\alpha-2)}{3!} + \cdots & R = 1, \alpha \in \mathbb{C} \end{cases}$$

we can manipulate the series as follows:

Allowable operations on series with radius R	Radius of Convergence of the result
Multiplication by $c \in \mathbb{C}$	R
Substitution $z \to cz^k, k \in \mathbb{N}$ $\uparrow$ Note: (use $z - z_0$ here if $z_0 \neq 0$)	Get $\lvert cz^k \rvert < R$, so $\lvert z \rvert < \left(\frac{R}{\lvert c \rvert}\right)^{\frac{1}{k}}$
Differentiation	R
Antidifferentiation	R
Addition (series with radii R_1, R_2)	$\min(R_1, R_2)$
Multiplication (series with radii R_1, R_2) $\uparrow$ Note: This is difficult since $\sum a_n \sum b_n \neq \sum a_n b_n$	$\min(R_1, R_2)$

Example 5.16. Expand e^z about $z = i$.

Solution: $f(z) = e^z \quad \Rightarrow \quad f^{(n)}(i) = e^i$, so

$$e^z = \sum_{n=0}^{\infty} \frac{e^i}{n!}(z-i)^n$$

Radius? We want

$$\lim_{n\to\infty}\left|\frac{e^i}{(n+1)!}(z-i)^{n+1}\cdot\frac{n!}{e^i}\cdot\frac{1}{(z-i)^n}\right| = \left|\frac{z-i}{n+1}\right| < 1$$

by Ratio test. So the above is true for all z. We have $R=\infty$.

Example 5.17. Maclaurin Series for $\dfrac{1}{8+z^3}$?

Solution:

$$\frac{1}{8+z^3} = \frac{1}{8}\cdot\frac{1}{1-\left(\frac{-z^3}{8}\right)} = \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{-z^3}{8}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n z^{3n}}{2^{3n+3}}$$

Radius?

$$\left|\frac{-z^3}{8}\right| < 1 \;\Rightarrow\; |z|^3 < 2^3 \;\Rightarrow\; |z| < 2$$

So $R=2$

Example 5.18. Expand $\dfrac{1}{1+z}$ about $z=1$.

Solution: Need powers of $z=1$:

$$\begin{aligned}\frac{1}{1+z} &= \frac{1}{2+(z-1)}\\ &= \frac{1}{2}\cdot\frac{1}{1-(-\frac{z-1}{2})}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z-1}{2}\right)^n \quad \text{by known series}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-1)^n\end{aligned}$$

which converges for $\left|-\dfrac{z-1}{2}\right| < 1 \Rightarrow |z-1| < 2$. So $R=2$.

Example 5.19. Expand $\dfrac{1}{(1-z)^2}$ about $z=0$.

Solution: Key idea: $\dfrac{d}{dz}\left(\dfrac{1}{1-z}\right) = \dfrac{1}{(1-z)^2}$. So

$$\begin{aligned}
\frac{1}{(1-z)^2} &= \frac{d}{dz}\left(\frac{1}{1-z}\right) \\
&= \frac{d}{dz}\sum_{n=0}^{\infty} z^n \quad \text{by known series} \\
&= \sum_{n=1}^{\infty} n z^{n-1} \\
&= \sum_{n=0}^{\infty} (n+1) z^n \quad \text{for } |z| < 1
\end{aligned}$$

Example 5.20. Expand $\dfrac{4}{z^2+2z-3}$ about $z = 0$.

Solution:

$$\begin{aligned}
\frac{4}{z^2+2z-3} &= \frac{1}{z-1} - \frac{1}{z+3} \\
&= \frac{-1}{1-z} - \frac{1}{3}\cdot\frac{1}{1-(\frac{-z}{3})} \\
&= \underbrace{-\sum_{n=0}^{\infty} z^n}_{|z|<1} - \underbrace{\frac{1}{3}\cdot\sum_{n=0}^{\infty}\left(\frac{-z}{3}\right)^n}_{\left|\frac{-z}{3}\right|<1 \Rightarrow |z|<3} \\
&= \sum_{n=0}^{\infty}\left(-1 - \frac{(-1)^n}{3^{n+1}}\right) z^n \quad \text{for } |z| < 1 \quad (R = 1)
\end{aligned}$$

Example 5.21.

$$\begin{aligned}
\frac{\sin z}{z} &= \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}}{z} \\
&= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots
\end{aligned}$$

for $|z| < \infty$, $z \neq 0$.

Definition 5.22. "sinc" **Function**:

$$\operatorname{sinc} z = \begin{cases} \dfrac{\sin z}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

5.3 Laurent Series

What about series with $z^k, k \in \mathbb{Z}$?

Example 5.23.

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{1}{2} z^2 + \cdots \quad \text{for } |z| < \infty$$

Consider

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = 1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \cdots \quad \text{for } |z| > 0$$

Partial sums approximate $e^{\frac{1}{z}}$ well for <u>large</u> $|z|$

Example 5.24.

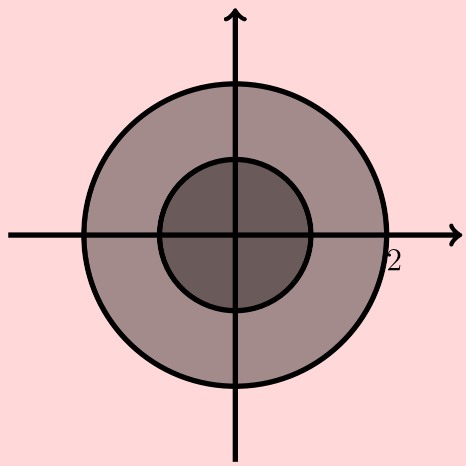
$$f(z) = \frac{3}{2 + z - z^2} = \frac{1}{1+z} + \frac{1}{2-z}$$

$$\frac{1}{1+z} = \begin{cases} \displaystyle\sum_{n=0}^{\infty} (-1)^n z^n & \text{for } |z| < 1 \\ \dfrac{\frac{1}{z}}{1 + \frac{1}{z}} = \dfrac{1}{z} \cdot \displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} & \text{for } |z| > 1 \end{cases}$$

$$\frac{1}{2-z} = \begin{cases} \dfrac{1}{2} \cdot \dfrac{1}{1 - \frac{z}{2}} = \dfrac{1}{2} \displaystyle\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} & \text{for } |z| < 2 \\ \dfrac{-\frac{1}{z}}{1 - \frac{2}{z}} = \dfrac{-1}{z} \displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty} \frac{-2^n}{z^{n+1}} & \text{for } |z| > 2 \end{cases}$$

Put it all together

$$f(z) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \left((-1)^n + \frac{1}{2^{n+1}} \right) z^n & \text{if } |z| < 1 \\ \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} & \text{if } 1 < |z| < 2 \\ \displaystyle\sum_{n=0}^{\infty} ((-1)^n - 2^n) \cdot \frac{1}{z^{n+1}} & \text{if } |z| > 2 \end{cases}$$

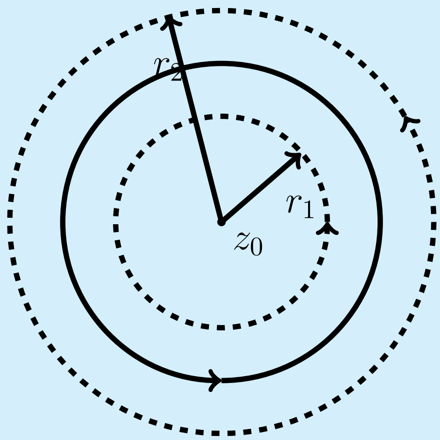


Definition 5.25. Let f be analytic in the annulus $r_1 < |z - z_0| < r_2$. For any z in this domain, f can be expressed as its **Laurent series**:

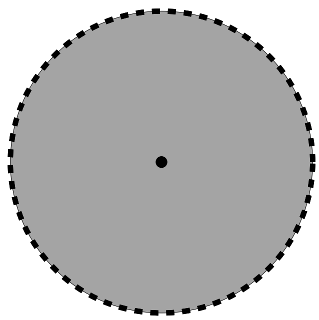
$$f(z) = \sum_{-\infty}^{\infty} c_n (z - z_0)^n$$

where $c_n = \dfrac{1}{2\pi i} \int_{\mathcal{C}} \dfrac{f(z)}{(z - z_0)^{n+1}} dz$ where $\mathcal{C}$ is a positively oriented circle centered at z_0, with radius $r \in (r_1, r_2)$.

The convergence is uniform on any closed subannulus contained in the domain.



Note: if f is analytic throughout the disc $|z - z_0| < r_2$, then $c_n = 0$ for $n \leq -1$ by Cauchy's Integral Theorem 4.31, and we just end up with the Taylor expansion.



5.4 Zeros and Singularities

Definition 5.26. A point where a function $f(z)$ is not analytic but which is the limit of points where f is analytic is called a **singular point** (or **singularity**)

Example 5.27. The poles of a rational function are singularities.

Definition 5.28. A point z_0 is called a **zero of order m** if f is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ but $f^{(m)}(z_0) \neq 0$. If $m = 1$, we have a **simple zero**

Example 5.29. $f(z) = z \sin z^2$. About $z = 0$, we have

$$f(z) = z \cdot (z^2 - \frac{1}{3!}(z^2)^2 + \cdots) = z^3 - \frac{1}{6}z^5 + \cdots$$

So $f(0) = 0, f'(0) = 0, f''(0) = 0$, but $f'''(0) \neq 0$. So f has a zero of order 3 at $z_0 = 0$.

Theorem 5.30. Suppose $f(z)$ is analytic at z_0. Then f has a zero of order m at z_0 if and only if it can be expressed in the form

$$f(z) = (z - z_0)^m g(z)$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$

Proof 5.31.

$$\begin{aligned} f(z) &= \underbrace{0 + 0 + \cdots + 0}_{m \text{ terms}} + \underbrace{c_m}_{\neq 0}(z - z_0)^m + c_{m+1}(z - z_0)^{m+1} + \cdots \\ &= (z - z_0)^m \cdot \underbrace{(c_m + c_{m+1}(z - z_0) + c_{m+2}(z - z_0)^2 + \cdots)}_{\text{analytic function } g(z)} \end{aligned}$$

□

Example 5.32. $f(z) = z^{10}e^z$ has a zero of order 10 at $z = 0$ since e^z is analytic and $e^0 \neq 0$.

Proposition 5.33. Some results follows:

1. If f, g are analytic at z_0 with zeros of order m and n respectively, then fg has zero of order $m + n$ at z_0.
2. If f is analytic at z_0, and $f(z_0) = 0$ then either $f(z) = 0$ in some neighborhood of z_0 or else there is a deleted neighborhood of z_0 in which f has no zeroes.

Proof 5.34.

1. Use the definition
2. Let $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ be Taylor series of f about z_0. We know this converges to $f(z)$ in some neighborhood N_ϵ of z_0.

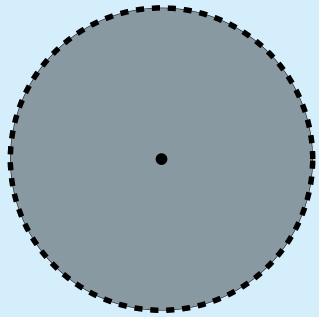
   If $c_n = 0$ for all n, then $f(z) = 0$ in N_ϵ.

   If not, let m be the smallest integer such that $c_m \neq 0$. Then f has zero of order m at z_0, so
   $$f(z) = (z - z_0)^m g(z)$$
   where g is analytic and $g(z_0) \neq 0$. But then g is also continous, so there exist $N_\delta \subseteq N_\epsilon$ in which $g(z) \neq 0$.

Hence $f(z) \neq 0$ in N_δ except at z_0.

$\square$

Definition 5.35. $f(z)$ has an **isolated singularity** at z_0 if f is not analytic at z_0 but is analytic at every other point in some neighborhood of z_0



Definition 5.36. Suppose $f(z)$ has an isolated singularity at z_0 and Laurent series $\sum_{-\infty}^{\infty} c_n(z-z_0)^n$ for $0 < |z-z_0| < R$.

1. If $c_n = 0$ for all $n < 0$, then f has a **removable singularity** at z_0
2. If $c_n = 0$ for all $n < -m$, $(m > 0)$, but $c_{-m} \neq 0$ (think about $\sum_{n=-m}^{\infty} c_n(z-z_0)^n$), then f has a **pole of order** m at z_0. If $m = 1$, we have a **"simple pole"**
3. If $c_n \neq 0$ for infinitely many negative integers n, then f has an **essential singularity** at z_0.

Example 5.37.

- $f(z) = e^{\frac{1}{z}} = 1 + \dfrac{1}{z} + \dfrac{1}{2z^2} + \dfrac{1}{6z^3} + \cdots$ has essential singularity
- $g(z) = \dfrac{\sin z}{z} = 1 - \dfrac{z^2}{6} + \dfrac{z^4}{5!} - \cdots$ has a removable singularity at $z = 0$.
- $h(z) = \dfrac{e^z}{z^3} = \dfrac{1}{z^3} + \dfrac{1}{z^2} + \dfrac{1}{2z} + \dfrac{1}{6} + \dfrac{1}{4!}z + \cdots$ has a pole of order 3 at $z = 0$.

Note: Removable singularities can be "removed" by defining

$$f(z_0) = \lim_{z \to z_0} f(z), \quad \text{e.g.} \quad \operatorname{sinc} z = \begin{cases} \dfrac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

Theorem 5.38. For a pole of order m, $f(z) = \dfrac{c_{-m}}{(z-z_0)^m} + \dfrac{c_{-m+1}}{(z-z_0)^{m-1}} + \cdots$. So

$$\lim_{z \to z_0} (z-z_0)^n f(z) = \begin{cases} \infty & \text{if } n < m \\ c_{-m} & \text{if } n = m \\ 0 & \text{if } n > m \end{cases}$$

which gives useful technique for classifying poles: keep multiplying by $z - z_0$ until the limit exists

Corollary 5.39. If f has a pole of order m at z_0, then the function $h(z) = (z-z_0)^m f(z)$ is analytic and non-zero at z_0. We may write $f(z) = \dfrac{h(z)}{(z-z_0)^m}$

Corollary 5.40. If f is analytic and has a <u>zero</u> of order m at z_0, then $\dfrac{1}{f}$ has a <u>pole</u> of order m at z_0.

Example 5.41. Find and classify isolated singularities:

1. $\dfrac{1}{z^2+2z+1} = \dfrac{1}{(z+1)^2}$: $z = -1$ is a pole of order 2 by definition
2. $f(z) = \dfrac{\sin z}{z^3+z} = \dfrac{\sin z}{z(z^2+1)}$: singularities at $z = 0$ and $z = -1$.
   - $z = 0$: Try

     $$\begin{aligned} \lim_{z \to 0} (z-0)^0 \frac{\sin z}{z^3+z} &= \lim_{z \to 0} \frac{\cos z}{3z^2+1} \quad \text{By L'Hopital's Rule} \\ &= 1 \end{aligned}$$

     So f has a removable singularity at $z = 0$
   - $z = -1$: Try $\lim_{z \to -1} (z-(-1))^0 \dfrac{\sin z}{z^3+z} = \dfrac{1}{2}\sin(1)$. So f has a removable singularity at $z = -1$

Example 5.42. Locate and classify isolated singularities for $f(z) = \dfrac{e^{iz}-1}{z^4(z-2i)^6}$.

<u>**Solution**</u>:

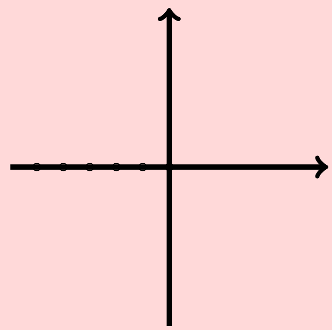
- At $z = 2i$: $\left.\dfrac{e^{iz}-1}{z^4}\right|_{z=2i} \neq 0$, so f has a pole of order 6 at $z = 2i$.
- At $z = 0$,

$$\lim_{z\to 0} z^4 \frac{e^{iz}-1}{z^4(z-2i)^6} = 0$$

$$\lim_{z\to 0} z^3 \frac{e^{iz}-1}{z^4(z-2i)^6} = \lim_{z\to 0} \frac{1 - iz + \cdots - 1}{z(z-2i)^6} \neq 0$$

So $z = 0$ is a pole of order 3.

Example 5.43. Locate and classify isolated singularities for $f(z) = \dfrac{e^z - e}{\operatorname{Log} z}$.



<u>Solution</u>:

- $z = 0$ is not an <u>isolated</u> singularity since it's a branch point.
- $z = 1$ is an isolated singularity.

$$\lim_{z\to 1} \frac{e^z - e}{\operatorname{Log} z} = \lim_{z\to 1} \frac{e^z}{1/z} = e \neq 0$$

So $z = 1$ is a removable singularity.

Example 5.44. Locate and classify isolated singularities for $f(z) = \dfrac{e^z}{\operatorname{Log} z}$.

<u>Solution</u>: Look at $z = 1$ again. $\lim_{z\to 1} \dfrac{e^z}{\operatorname{Log} z} =$ "∞".

$$\lim_{z\to 1} (z-1)\frac{e^z}{\operatorname{Log} z} = \lim_{z\to 1} \frac{e^z + (z-1)e^z}{1/z} = e \neq 0$$

So $z = 1$ is a pole of order 1, i.e. a simple pole.

Solution 2: Let $g(z) = \dfrac{1}{f(z)} = \dfrac{\operatorname{Log} z}{e^z}$. Then, $g(1) = 0$, and

$$g'(1) = \left(\frac{e^z \frac{1}{z} - \operatorname{Log} z \cdot e^z}{(e^z)^2} \right) \Bigg|_{z=1} = \frac{1}{e} \neq 0$$

So g has a simple zero at $z = 1$, implying f has a simple pole at $z = 1$.

Solution 3: Look at the Laurent series.

$$e^z = e + e(z-1) + \frac{1}{2} e(z-1)^2 + \cdots$$
$$\operatorname{Log} z = 0 + 1 \cdot (z-1) + \frac{1}{2}(-1)(z-1)^2 + \cdots$$

So $\dfrac{e^z}{\operatorname{Log} z} \approx \dfrac{e}{z-1}$ for z near 1, implying that $z = 1$ is a simple pole.

Theorem 5.45. Picard's Theorem: The following two statements are equivalent (if and only if relationship):

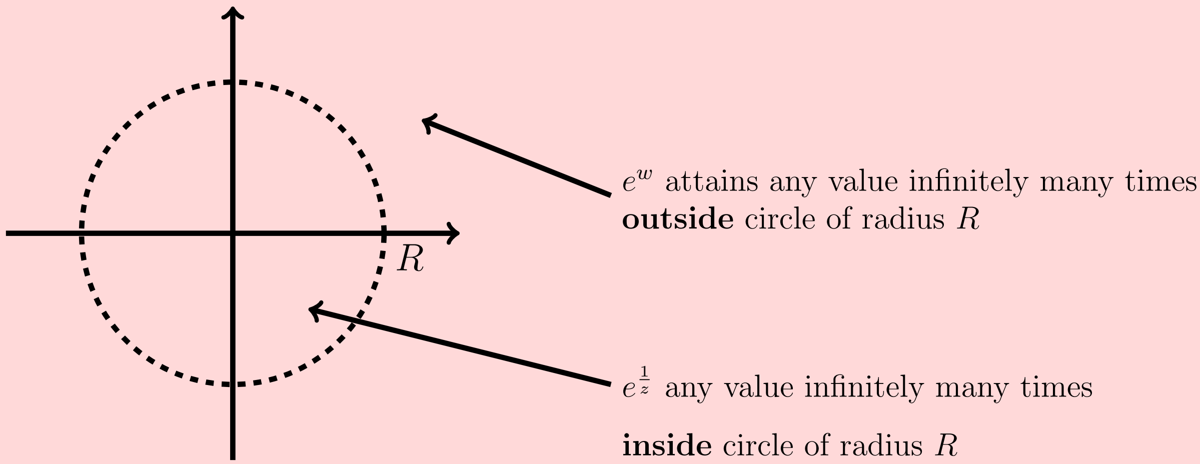
1. z_0 is an essential singularity of f
2. f assumes every value in $\mathbb{C}$, except possibily one exception, infinitely many times in every neighborhood of z_0

Example 5.46. Consider $f(z) = e^{\frac{1}{z}}$.

Since $e^{\frac{1}{z}} = 1 + \left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \cdots$, we have that $z = 0$ is an essential singularity of f.

Observe that $e^{\frac{1}{z}}$ near 0 "mirrors" e^w near $w = \infty$.

e^w can take any value $\neq 0$, this is because, if we let $z = e^w, w = \operatorname{Log} z$ is defined for all z (despite the branch cut). e^w is periodic as well.



We could say that "e^z has an essential singularity at ∞". More generally, $f(z)$ has a pole of order m at ∞ if and only if $f(\frac{1}{z})$ has a pole of order m at 0.

Chapter 6 Residue Theory

6.1 Residues

Why are we studying Laurent series? One reason is that: if $f(z)$ is analytic except at z_0, then

$$\begin{aligned}\int_{\mathcal{C}} f(z)dz &= \int_{\mathcal{C}} \left(\cdots + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \cdots\right) dz \\ &= \cdots + 0 + c_{-1}\cdot 2\pi i + 0 + 0 + \cdots\end{aligned}$$

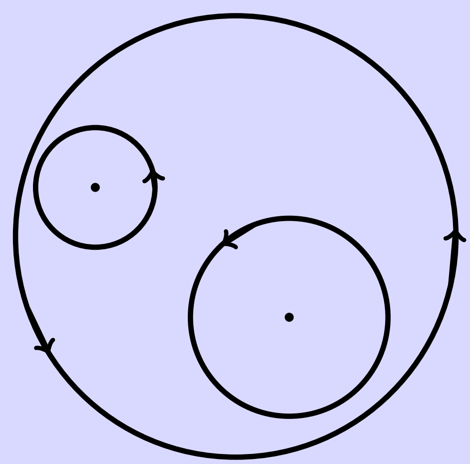
This result depends only on coefficient of $\dfrac{1}{z-z_0}$.

c_{-1} is called the **residue of f at z_0**, denoted $\text{Res}(f, z_0)$ or $\text{Res}(z_0)$

Theorem 6.1. Let D be a simply-connected domain, and $\mathcal{C}$ be a simple, closed, positively-oriented contour in D. If f is analytic inside and on $\mathcal{C}$, except at points $z_1, z_2, ..., z_n$ interior to $\mathcal{C}$, then

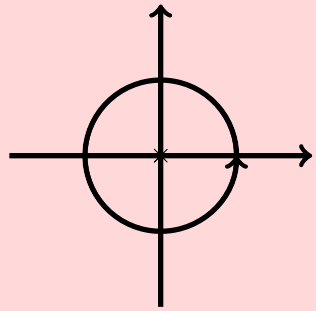
$$\int_{\mathcal{C}} f(z)dz = 2\pi i \cdot \sum_{k=1}^{n} \text{Res}(f, z_k)$$

Proof 6.2. Follows from the above result and the extended Cauchy-Goursat Theorem.



□

Example 6.3. Evaluate $\int_{\mathcal{C}} e^{\frac{2i}{z}} dz$.



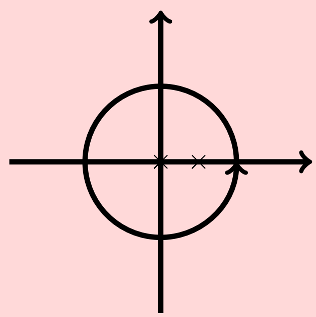
Solution:

$$e^{\frac{2i}{z}} = 1 + \left(\frac{2i}{z}\right) + \frac{1}{2!}\left(\frac{2i}{z}\right)^2 + \cdots$$

which implies $\text{Res}(0) = 2i$

So $\int_{\mathcal{C}} e^{\frac{2i}{z}} dz = 2\pi i \cdot \text{Res}(0) = 2\pi i \cdot 2i = -4\pi$

Example 6.4. Evaluate $\int_{\mathcal{C}} \frac{5z-3}{z(z-1)} dz$.



Solution: Use partial fractions or $\int_{\mathcal{C}} = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2}$, use CIF on each part.

Solution 2: Series expansion of $f(z) = \frac{5z-3}{z(z-1)}$:

- About $z = 0$:

$$\begin{aligned}
&-\frac{5z-3}{z} \cdot \frac{1}{1-z} \\
=&(-5 + \frac{3}{z})(1 + z + z^2 + \cdots), \qquad |z| < 1 \\
=&\frac{3}{z} + (3-5) + (3-5)z + \cdots \\
\Rightarrow& \text{Res}(0) = 3
\end{aligned}$$

- About $z = 1$:

$$
\begin{aligned}
& \frac{1}{z-1} \cdot \left(\frac{5(z-1)+2}{1+(z-1)} \right) \\
=& \left(5 + \frac{2}{z-1}\right) \left(\frac{1}{1-(-(z-1))} \right) \\
=& \left(5 + \frac{2}{z-1}\right) \left(1-(z-1)+(z-1)^2 - \cdots \right) \quad \text{for} |z-1| < 1 \\
=& \frac{2}{z-1} + (5-2) + (5-2)(z-1) + \cdots \\
\Rightarrow & \operatorname{Res}(1) = 2
\end{aligned}
$$

So, by the residue theorem, $\int_{\mathcal{C}} f(z) = 2\pi i(3+2) = 10\pi i$

Actually, we do not need the whole series. Look for shortcuts:

- If f has a simple pole at z_0:

$$
\begin{aligned}
f(z) =& \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \cdots \\
\Rightarrow & (z-z_0)f(z) = c_{-1} + c_0(z-z_0) + c_1(z-z_0)^2 + \cdots \\
\Rightarrow & \operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z-z_0)f(z) = c_{-1}
\end{aligned}
$$

- If f has a pole of order m at z_0:

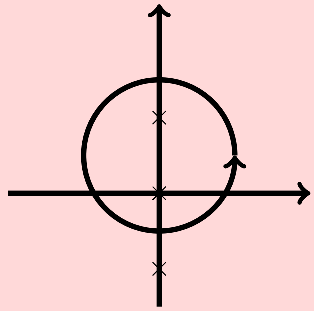
$$
\begin{aligned}
f(z) =& \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \cdots \\
\Rightarrow & (z-z_0)^m f(z) = c_{-m} + c_{-m+1}(z-z_0) + \cdots + c_{-1}(z-z_0)^{m-1} + c_0(z-z_0)^m + \cdots \\
\Rightarrow & \frac{d^{m-1}}{dz^{m-1}} \Big((z-z_0)^m f(z) \Big) = 0 + 0 + \cdots + 0 + (m-1)! c_{-1} + const * (z-z_0) + \cdots
\end{aligned}
$$

  Let $z \to z_0$, and rearrange to get

$$
\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \left(\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z-z_0)^m f(z)) \right)
$$

- If f has a removable singularity at $z = z_0$: $\operatorname{Res}(f, z_0) = 0$
- If f has an essential singularity at $z = z_0$: Need to find the Laurent series. There is no shortcut here.

Example 6.5. Evaluate $\int_{\mathcal{C}} \frac{e^z}{z^4 + z^2} dz$



Solution: $f(z) = \dfrac{e^z}{z^2(z+i)(z-i)}$. It has a simple pole at $z = i$, and pole of order 2 at $z = 0$.

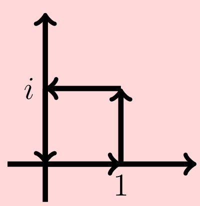
$$\text{Res}(f, i) = \lim_{z \to i}(z-i) \cdot \frac{e^z}{z^2(z+i)(z-i)} = \frac{e^i}{-2i} = \frac{ie^i}{2}$$

$$\text{Res}(f, 0) = \lim_{z \to 0} \frac{1}{(2-1)!} \frac{d}{dz} \left(z^2 \cdot \frac{e^z}{z^2(z+i)(z-i)} \right) = \lim_{z \to 0} \frac{(z^2+1)e^z - e^z \cdot 2z}{(z^2+1)^2} = 1$$

By Residue Theorem,

$$\int_{\mathcal{C}} \frac{e^z}{z^4+z^2} dz = 2\pi i \left(1 + \frac{ie^i}{2} \right)$$

Example 6.6. Evaluate $\int_{\mathcal{C}} \dfrac{dz}{z^4+1}$



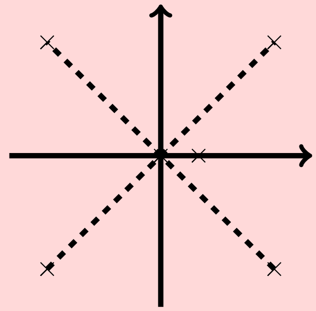
Solution: $f(z) = \dfrac{1}{z^4+1}$. Poles at $z = (-1)^{\frac{1}{4}} = \left(e^{i(\pi + 2k\pi)}\right)^{\frac{1}{4}} = e^{i\left(\frac{\pi}{4} + k\frac{\pi}{2}\right)}$

The only relevant pole is $z_0 = e^{i\pi/4} = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} i$, a simple pole.

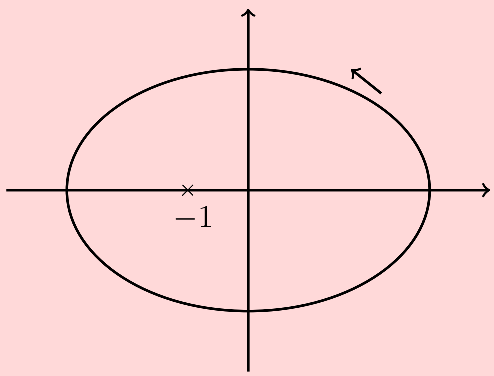
Now, $\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) \dfrac{1}{z^4+1} = \lim_{z \to z_0} \dfrac{1}{4z^3} = \dfrac{1}{4z_0^3} = \dfrac{1}{4} \left(e^{i\pi/4}\right)^3 = \dfrac{1}{4} e^{-i3\pi/4}$

Thus,

$$\int_{\mathcal{C}} \frac{dz}{z^4+1} = 2\pi i \left(\frac{1}{4} e^{-3\pi/4} \right) = \frac{\pi}{2} i \left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right) = \frac{\pi}{2\sqrt{2}} - \frac{\pi}{2\sqrt{2}} i$$



Example 6.7. Evaluate $\int_{\mathcal{C}} \frac{dz}{(z+1)^3(z^2+4)}$



Solution:

$$\begin{aligned}\text{Res}(-1) &= \lim_{z\to-1} \frac{d^2}{dz^2}\left((z+1)^3 \cdot \frac{1}{(z+1)^3(z^2+4)}\right)\\ &= \lim_{z\to-1} \frac{d^2}{dz^2}\left(\frac{1}{z^2+4}\right)\\ &= \frac{-1}{125}\end{aligned}$$

Thus, $\int_{\mathcal{C}} \frac{dz}{(z+1)^3(z^2+4)} = 2\pi i\left(\frac{-1}{125}\right) = \frac{-2\pi i}{125}$

Example 6.8. $f(z) = \frac{\tanh z}{z^2}$. Find Res(0) and Res($\frac{\pi}{2}i$)

Solution:

- At $z = 0$: Pole of what order?

$$f(z) = \frac{\tanh z}{z^2} = \frac{\sinh z}{z^2 \cosh z} = \frac{z + \frac{1}{3!}z^3 + \cdots}{z^2 \cdot (1 + \frac{1}{2!}z^2 + \cdots)} \approx \frac{1}{z^2 \cdot 1} = \frac{1}{z}$$

for $z \approx 0$. So it's a simple pole. By inspection, $\text{Res}(0) = 1$.

- At $z = \frac{\pi}{2}i$: is a simple zero of $\cosh z$ (derivative $\neq 0$ at $\frac{\pi}{2}i$), and $\sinh \neq 0$ and $z^2 \neq 0$ at $\frac{\pi}{2}i$. So, $\frac{\pi}{2}i$ is a simple pole of $f(z)$.

  So

$$\begin{aligned}\text{Res}\left(\frac{\pi}{2}i\right) &= \lim_{z \to \frac{\pi}{2}i} \left(z - \frac{\pi}{2}i\right) \frac{\sinh z}{z^2 \cdot \cosh z} \\ &= \lim_{z \to \frac{\pi}{2}i} \frac{\sinh z + (z - \frac{\pi}{2}i)\cosh z}{2z\cosh z + z^2 \sinh z} \\ &= \frac{\sinh(\frac{\pi}{2}i) + 0}{0 + (\frac{\pi}{2}i)^2 \sinh(\frac{\pi}{2}i)} \\ &= \frac{-4}{\pi^2}\end{aligned}$$

Theorem 6.9. Shortcut for simple poles: With $f(z) = \dfrac{P(z)}{Q(z)}$, in which $P(z)$ is analytic, $\neq 0$ at z_0, and $Q(z)$ has a simple zero at z_0, we know $\text{Res}(f, z_0) = \dfrac{P(z_0)}{Q'(z_0)}$

Proof 6.10.

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)\frac{P(z)}{Q(z)} = \lim_{z \to z_0} (z - z_0)\frac{(1)P(z) + (z - z_0)P(z)}{Q'(z)} = \frac{P(z_0)}{Q'(z_0)}$$

□

We do not need to memorize the theorem. Just use L'Hopital's Rule.

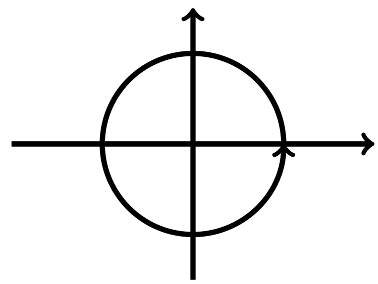
6.2 Application to Real Integrals

Consider

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$$

The key idea is to view it as a parameterized form of a contour integral.

Let $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$, i.e. a unit circle.



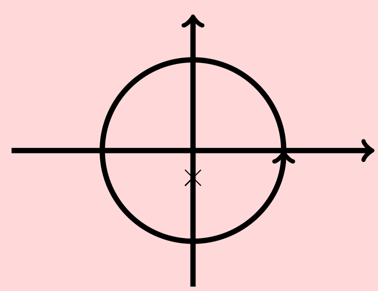
Then $dz = ie^{i\theta}d\theta$, so $d\theta = \dfrac{dz}{iz}$.

Also, $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{z+z^{-1}}{2}$, $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{z-z^{-1}}{2i}$.

Example 6.11. Evaluate $\int_0^{2\pi} \dfrac{1}{8+\sin\theta}d\theta$

Solution: Using the above substitution, we get

$$I = \int_0^{2\pi} \frac{1}{8+\sin\theta}d\theta = \int_{\mathcal{C}} \frac{1}{8+\frac{1}{2i}(z-\frac{1}{z})}\frac{dz}{iz} = \int_{\mathcal{C}} \frac{2}{z^2+16iz-1}dz$$



Poles at $z = \dfrac{-16i \pm \sqrt{-16^2+4}}{2} = -8i \pm \sqrt{63}i$. Only $(-8+\sqrt{63})i$ is inside $\mathcal{C}$, and $\mathrm{Res}(f,(-8+\sqrt{63})i) = \cdots = \dfrac{-i}{\sqrt{63}}$, so by the residue theorem, we have $I = 2\pi i \cdot \left(\dfrac{-i}{\sqrt{63}}\right) = \dfrac{2\pi}{\sqrt{63}}$.

Generalize the process above, we may see that

$$\int_0^{2\pi} \frac{1}{k+\sin\theta} = \frac{2\pi}{\sqrt{k^2-1}}, \quad k > 1$$

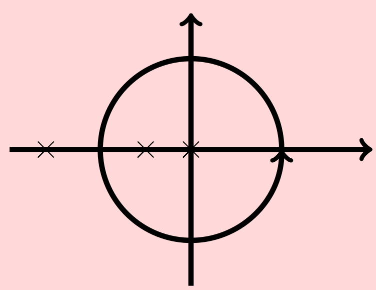
Example 6.12. Evaluate $\int_0^{\pi} \dfrac{\cos\theta}{5+4\cos\theta}d\theta$

Solution: Let $I = \int_0^{\pi} \dfrac{\cos\theta}{5+4\cos\theta} d\theta$. We want to have $\int_0^{2\pi}$. Notice that

$$\begin{aligned}\int_0^{\pi} F(\cos\theta) d\theta &= \frac{1}{2}\int_{-\pi}^{\pi} F(\cos\theta) d\theta \quad \text{by symmetry} \\ &= \frac{1}{2}\int_0^{2\pi} F(\cos\theta) d\theta \quad \text{by periodicity}\end{aligned}$$

So,

$$\begin{aligned} I = \frac{1}{2}\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta} d\theta &= \frac{1}{2}\int_{|z|=1} \frac{\frac{1}{2}(z+z^{-1})}{5+4(z+z^{-1})/2} \frac{dz}{iz} \\ &= \frac{1}{2i}\int_{|z|=1} \frac{z+\frac{1}{z}}{10z+4(z^2+1)} dz \\ &= \frac{-i}{2}\int_{|z|=1} \frac{z^2+1}{4z^3+10z^2+4z} dz \\ &= \frac{-i}{4}\int_{|z|=1} \frac{z^2+1}{z(2z^2+5z+2)} dz \\ &= \frac{-i}{4}\int_{|z|=1} \frac{z^2+1}{z(2z+1)(z+2)} dz \end{aligned}$$



Two simple poles inside $|z| = 1$:

- $z = 0$: $\text{Res}(0) = \lim_{z\to 0}(z-0) \cdot \dfrac{z^2+1}{z(2z+1)(z+2)} = \dfrac{1}{2}$
- $z = -\dfrac{1}{2}$: $\text{Res}\left(-\dfrac{1}{2}\right) = \lim_{z\to -\frac{1}{2}} \left(z+\dfrac{1}{2}\right) \cdot \dfrac{z^2+1}{z(2z+1)(z+2)} = \dfrac{5/4}{-3/2} = \dfrac{-5}{6}$

So by Residue Theorem

$$I = \frac{-i}{4} \cdot 2\pi i \cdot \left(\frac{1}{2} - \frac{5}{6}\right) = \frac{\pi}{2} \cdot \left(-\frac{1}{3}\right) = \frac{-\pi}{6}$$

Example 6.13. Evaluate $\int_0^{2\pi} \dfrac{\cos 2\theta}{5-4\sin\theta} d\theta$

Solution: Let $I = \int_0^{2\pi} \dfrac{\cos 2\theta}{5-4\sin\theta} d\theta$.

Let $z = e^{i\theta}$, so $dz = izd\theta$, and

$$\begin{aligned}
I &= \int_0^{2\pi} \frac{(e^{i2\theta}+e^{-i2\theta})/2}{5-4(e^{i\theta - e^{-i\theta}})/2i} d\theta \\
&= i\int_{|z|=1} \frac{z^2+z^{-2}}{10i-4(z-z^{-1})} \frac{dz}{iz} \\
&= \int_{\mathcal{C}} \frac{z^4+1}{-4z^4+10iz^3+4z^2} dz \\
&= \frac{-1}{2}\int_{\mathcal{C}} \frac{z^4+1}{z^2(2z^2-5iz-2)} dz \\
&= \frac{-1}{2}\int_{\mathcal{C}} \frac{z^4+1}{z^2(2z-i)(z-2i)} dz
\end{aligned}$$

$z = 2i$ is not inside $\mathcal{C}$. So we consider

- $z = 0$:

$$\begin{aligned}
\operatorname{Res}(0) &= \lim_{z\to 0}\left(\frac{d}{dz}\left(z^2 \cdot \frac{z^4+1}{z^2(2z-i)(z-2i)}\right)\right) \\
&= \lim_{z\to 0} \frac{(2z-i)(z-2i)(4z^3)-(z^4+1)(4z-5i)}{(2z-i)^2(z-2i)^2} \\
&= \frac{5i}{4}
\end{aligned}$$

- $z = \dfrac{i}{2}$:

$$\begin{aligned}
\operatorname{Res}\left(\frac{i}{2}\right) &= \lim_{z\to \frac{i}{2}}\left(z-\frac{i}{2}\right)\frac{z^4+1}{z^2(2z-i)(z-2i)} \\
&= \frac{17/16}{\frac{-1}{4}\cdot 2\cdot\left(\frac{-3}{2}i\right)} \\
&= \frac{4}{3i}\cdot\frac{17}{16} \\
&= \frac{17}{12i}
\end{aligned}$$

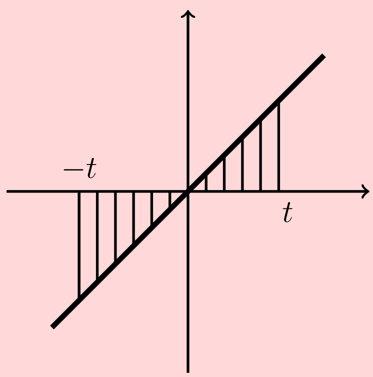
So $I = \dfrac{-1}{2}\cdot 2\pi i\left(\dfrac{5i}{4}+\dfrac{17}{12i}\right) = -\pi\cdot\left(\dfrac{17}{12}-\dfrac{15}{12}\right) = \dfrac{-\pi}{6}$.

6.3 Integral From $-\infty$ to ∞

Recall that $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$ Both term must exist. So, $\int_{-\infty}^{\infty} xdx$ is technically divergent in this case.

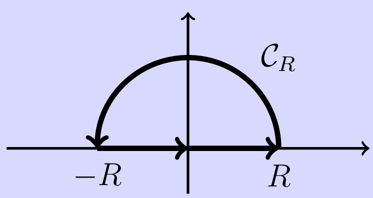
Definition 6.14. The **Cauchy Principle Value** of $\int_{-\infty}^{\infty} f(x)dx$ is $\lim_{t\to\infty} \int_{-t}^{t} f(x)dx$

Example 6.15. The Cauchy Principle Value (cpv) of $\int_{-\infty}^{\infty} xdx$ is 0.



Theorem 6.16. If $\int_{-\infty}^{\infty} f(x)dx$ is convergent, then $\int_{-\infty}^{\infty} f(x)dx = \text{cpv} \int_{-\infty}^{\infty} f(x)dx$

Proof 6.17. Idea: interpret the real axis as part of a large contour:



Then

$$\begin{aligned}\int_{\mathcal{C}} f(z)dz &= \int_{\mathcal{C}_x} f(z)dz + \int_{\mathcal{C}_R} f(z)dz \\ &= \int_{-R}^{R} f(x)dx + \int_{\mathcal{C}_R} f(z)dz\end{aligned}$$

So

$$\text{cpv} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \left[\underbrace{\int_{\mathcal{C}} f(z)dz}_{\text{we know how to compute it}} - \underbrace{\int_{\mathcal{C}_R} f(z)dz}_{\text{approaches 0 under certain conditions}} \right]$$

□

Integrals of form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$$

where $P(x)$, $Q(x)$ are polynomials.

Idea: We want $\dfrac{P(x)}{Q(x)} \to 0$ fast enough as $R \to 0$ to "more than" counter the lengthening of the contour. Need $\deg(P) < \deg(Q)$? -1? -2?

If $\deg Q - \deg P \geq 2$, then $\lim_{|z|\to\infty} \dfrac{zP(z)}{Q(z)} = 0$. So for any $\epsilon > 0$, choose R large enough such that $\left|\dfrac{zP(z)}{Q(z)}\right| < \dfrac{\epsilon}{\pi}$ when $|z| = R$. So $\left|\dfrac{P(z)}{Q(z)}\right| \leq \dfrac{\epsilon}{\pi|z|} = \dfrac{\epsilon}{\pi R}$ on $\mathcal{C}_R$.

Thus, $\left|\int_{\mathcal{C}_R} \dfrac{P(z)}{Q(z)} dz\right| \leq \dfrac{\epsilon}{\pi R} = \epsilon$, and we have shown

$$\lim_{R\to\infty} \int_{\mathcal{C}_R} \frac{P(z)}{Q(z)} dz = 0$$

From the result above and the residue theorem, we now have

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum \text{Res}(f,\ \text{upper half plane}), \qquad \text{if } \deg Q - \deg P \geq 2$$

Example 6.18. Evaluate $\int_{-\infty}^{\infty} \dfrac{1}{(x^2+1)(x^2+4)} dx$

Solution:

$$f(z) = \frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z-i)(z+i)(z-2i)(z+2i)}$$

$$\text{Res}(f, i) = \lim_{z\to i}(z-i)f(z) = \frac{1}{(2i)(-i)(3i)} = \frac{1}{6i}$$

$$\text{Res}(f, 2i) = \frac{1}{(i)(3i)(4i)} = \frac{-1}{12i}$$

So $I = 2\pi i \cdot \left(\dfrac{1}{6i} - \dfrac{1}{12i}\right) = \dfrac{\pi}{6}$

Example 6.19. Evaluate $\int_0^{\infty} \dfrac{1}{x^6+1} dx$

Solution: By symmetry, $\int_0^\infty \frac{1}{x^6+1}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{x^6+1}dx$.

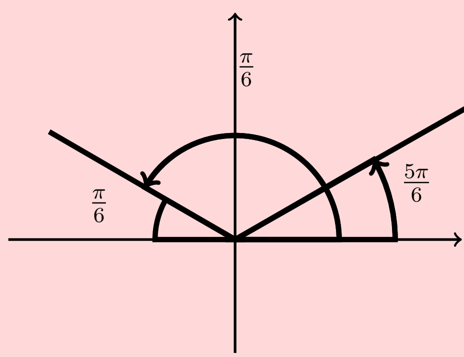
With $f(z) = \frac{1}{z^6+1}$, poles are $(-1)^{1/6} = \left(e^{i(\pi+2k\pi)}\right)^{1/6} = e^{i(\frac{\pi}{6}+k\pi/3)} = e^{i\pi/6}, e^{i\pi/2}, e^{i5\pi/6}$ in upper half plane.

All simple poles, so

$$\begin{aligned}
\text{Res}(e^{i\pi/6}) &= \lim_{z\to e^{i\pi/6}}(z-e^{i\pi 6})\frac{1}{z^6+1} = \lim_{\to e^{i\pi/6}}\frac{1}{6z^5} = \frac{1}{6}e^{-i5\pi/6} \\
\text{Res}(i) &= \cdots = \lim_{z\to i}\frac{1}{6z^5} = \frac{1}{6i} \\
\text{Res}(e^{i5\pi/6}) &= \cdots = \frac{1}{6}e^{-i25\pi/6} = \frac{1}{6}e^{-i\pi/6}
\end{aligned}$$

Thus

$$\begin{aligned}
\int_0^\infty \frac{1}{x^6+1}dx &= \frac{1}{2}2\pi i\cdot\left(\frac{1}{6}e^{-i5\pi/6} + \frac{1}{6i} + \frac{1}{6}e^{-i\pi/6}\right) \\
&= \frac{\pi i}{6}\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6} + \frac{1}{i} + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) \\
&= \frac{\pi i}{6}\left(-2i\sin\frac{\pi}{6} + \frac{1}{i}\right) \\
&= \frac{\pi i}{6}(-i-i) \\
&= \frac{\pi}{3}
\end{aligned}$$



Integrals of the form $\int_{-\infty}^\infty \frac{P(x)}{Q(x)}\cos\alpha x dx$:

We can change the above $\cos\alpha x$ to be $\sin\alpha x$.

Note that $|\sin z|$ and $\cos z$ grow exponentially as $\Im z\to\infty$

Wrtie $\int_{-\infty}^\infty \frac{P(x)}{Q(x)}\cos\alpha x dx = \Re\left(\int_{-\infty}^\infty \frac{P(x)}{Q(x)}e^{i\alpha x}dx\right)$ (or with $\Im()$ for $\sin\alpha x$)

Theorem 6.20. **Jordan's Lemma**: Let $P(z)$, $Q(z)$ be polynomials with $degQ - \deg P \geq 1$, then

$$\lim_{R\to\infty} \int_{\mathcal{C}_R} \frac{P(x)}{Q(x)} e^{i\alpha x} dz = 0$$

Proof 6.21. Ideas: Similar to the previous one, but more complicated. The key ideas are

$$|e^{i\alpha z}| = |e^{i\alpha(x+iy)}| = |e^{i\alpha x - \alpha y}| = e^{-\alpha y}$$

So $e^{i\alpha z} \to 0$ exponentially as $\Im(z) \to \infty$.

Meanwhile, as $\Re(z) \to \infty$, $|e^{i\alpha z}| \to 1$ for $y \approx 0$ so need $\dfrac{P}{Q} \to 0$ as $x \to \infty$.

Now we have

$$\begin{aligned}
\text{cpv} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos \alpha x dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)} \cos \alpha x dx \\
&= \lim_{R\to\infty} \Re \left(\int_{-R}^{R} \frac{P(x)}{Q(x)} e^{i\alpha x} \right) \\
&= \Re \left(\lim_{R\to\infty} \int_{\mathcal{C}} \frac{P(z)}{Q(z)} e^{i\alpha z} dz - \underbrace{\int_{\mathcal{C}_R} \frac{P(z)}{Q(z)} e^{i\alpha z} dz}_{\to 0 \text{ by the lemma}} \right) \\
&= \Re \left(2\pi i \cdot \sum \text{Res} \left(\frac{P(z)}{Q(z)} e^{i\alpha z, \text{ upper-half plane}} \right) \right)
\end{aligned}$$

For $\sin \alpha x$, just change $\Re$ to be $\Im$. $\square$

Example 6.22. Evaluate cpv $\int_{-\infty}^{\infty} \dfrac{x \sin x}{x^2+4} dx$

Note: this might not converge in the usual sense. $\left| \dfrac{x \sin x}{x^2+4} \right| \approx \left| \dfrac{1}{x} \right|$ as $x \to \infty$ (but sort of alternating).

Solution: Let $f(z) = \dfrac{ze^{iz}}{z^2+4}$. Poles at $z = 2i$. $z = -2i$ is not in the upper half plane, so we ignore.

$\text{Res}(2i) = \lim_{z\to 2i}(z-2i)\dfrac{ze^{iz}}{(z+2i)(z-2i)} = \dfrac{2ie^{i(2i)}}{4i} = \dfrac{1}{2}e^{-2}$.

Thus, cpv $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+4} dx = \Im\left(2\pi i \cdot \frac{1}{2} e^{-2}\right) = \frac{\pi}{e^2}$.

Example 6.23. Evaluate $\int_{-\infty}^{\infty} \frac{\cos 3x}{x^4+4} dx$

Note: this would converge in the usual sense, so cpv $\int = \int$ by comparison test.

<u>**Solution**</u>: Let $f(z) = \frac{e^{i3z}}{z^4+4}$. Poles at $\left(4e^{i(\pi+2k\pi)}\right)^{\frac{1}{4}} = \sqrt{2}e^{i\pi/4}, \sqrt{2}e^{i3\pi/4}, \ldots = 1+i, -1+i$ (others not on the upper half plane, so not of interest)

$$
\begin{aligned}
\operatorname{Res}(1+i) &= \lim_{z\to 1+i} (z-(1+i)) \frac{e^{i3z}}{z^4+4} \\
&= \lim_{z\to 1+i} \frac{(1)e^{i3z} - (z-(1+i))(\cdots)}{4z^3} \quad \text{by L'Hospitals Rule} \\
&= \frac{e^{i(1+i)}}{4(1+i)^3} \\
&= \frac{1}{4} e^{-3} e^{3i} \left(\sqrt{2}e^{i\pi/4}\right)^{-3} \\
&= \frac{1}{4e^3} \cdot \frac{e^{3i}}{2\sqrt{2}} \left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right) \\
&= \frac{1}{8\sqrt{2}e^3} \left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)(\cos 3 + i\sin 3) \\
&= \frac{1}{16e^3} \left[(-\cos 3 + \sin 3) + i(-\cos 3 - \sin 3)\right]
\end{aligned}
$$

Similarly, $Res(-1+i) = \frac{1}{16e^3}\left[(\cos 3 - \sin 3) + i(-\cos 3 - \sin 3)\right]$.

Thus,

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos 3x}{x^4+4} dx &= \Re\left(2\pi i \cdot \left(\frac{-i}{8e^3}(\cos 3 + \sin 3)\right)\right) \\
&= \frac{\pi}{4e^3}(\cos 3 + \sin 3)
\end{aligned}
$$

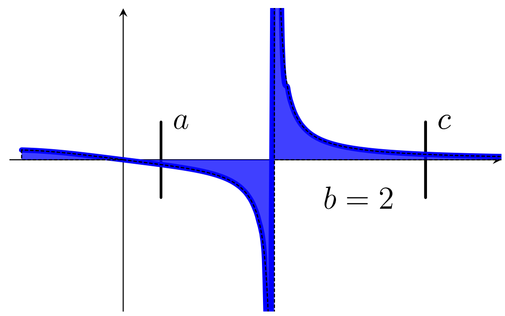
Note that in general,

$$
\int_{-\infty}^{\infty} \frac{\cos kx}{x^4+4} dx = \frac{\pi}{4} \cdot \frac{\cos k + \sin k}{e^k}, \quad k > 0
$$

e.g. $\int_{-\infty}^{\infty} \frac{\cos 100\pi x}{x^4+4} dx = \frac{\pi}{4} e^{-100\pi}$

6.4 Very Improper Integrals

Consider $\int_{-\infty}^{\infty} \frac{x}{x^3-8} dx$. This one does not converge in the usual sense.



Definition 6.24. Let $f(x)$ be continuous on $[a, c]$ except at b. Then

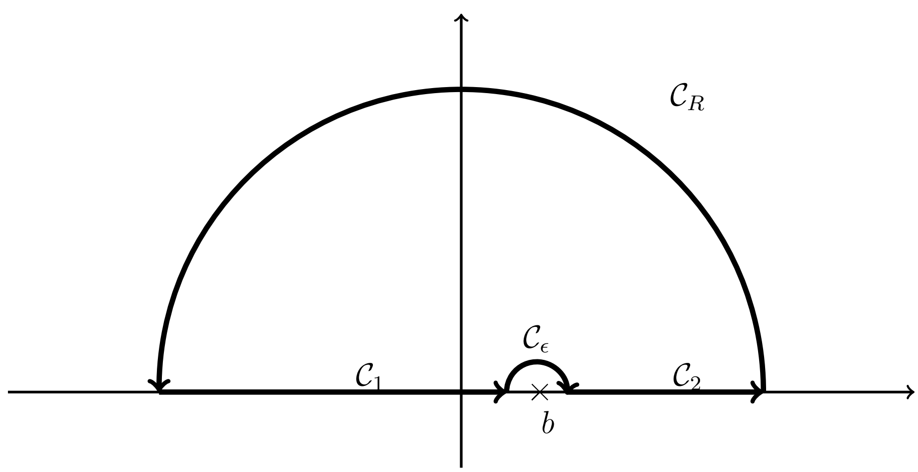
$$\text{cpv} \int_a^c f(x)dx = \lim_{\epsilon \to 0^+} \left(\int_a^{b-\epsilon} f(x)dx + \int_{b+\epsilon}^{c} f(x)dx \right)$$

Also

$$\text{cpv} \int_{-\infty}^{\infty} f(x)dx = \lim_{\epsilon \to 0^+, R \to \infty} \left(\int_{-R}^{b-\epsilon} f(x)dx + \int_{b+\epsilon}^{R} f(x)dx \right)$$

How to use residues when there's a singularity on the real axis?

Go around it!



Consider the following example:

$$\text{cpv} \int_{-\infty}^{\infty} f(x)dx = \lim_{\epsilon \to 0^+, R \to \infty} \left(\int_{\mathcal{C}_1} f(z)dz + \int_{\mathcal{C}_2} f(z)dz \right)$$

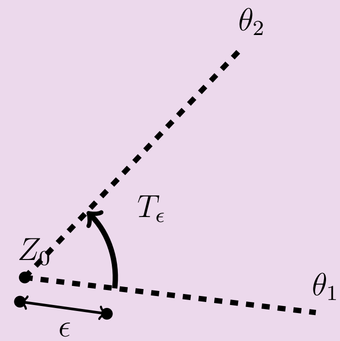
and

$$\int_{\mathcal{C}} f(z)dz = \int_{\mathcal{C}_1} f(z)dz + \int_{\mathcal{C}_\epsilon} f(z)dz + \int_{\mathcal{C}_2} f(z)dz + \int_{\mathcal{C}_R} f(z)dz$$

So

$$\text{cpv} \int_{-\infty}^{\infty} f(x)dx = \lim_{\epsilon\to 0^+, R\to\infty} \left(\underbrace{\int_{\mathcal{C}} f(z)dz}_{\text{equals } \sum \text{Res(upper half plane)}} - \underbrace{\int_{\mathcal{C}_\epsilon} f(z)dz}_{?} - \underbrace{\int_{\mathcal{C}_R} f(z)dz}_{\to 0 \text{ as } R\to 0, \text{ if conditions satisfied}} \right)$$

Lemma 6.25. If f has a simple pole at z_0 and Γ_ϵ is the circular arc $z(t) = z_0 + \epsilon e^{i\theta}$, $\theta_1 \le \theta \le \theta_2$. Then $\lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} f(z)dz = i(\theta_2 - \theta_1)\operatorname{Res}(z_0)$



Proof 6.26. $f(z) = \dfrac{c_{-1}}{z - z_0} + g(z)$, where $g(z)$ is analytic.

Then

$$\int_{\Gamma_\epsilon} f(z) = \int_{\Gamma_\epsilon} \frac{c_{-1}}{z - z_0}dz + \int_{\Gamma_\epsilon} g(z)dz$$

The second term $\to 0$ as $\epsilon \to 0$ by the $M\ell$ inequality: $\left|\int_{\Gamma_\epsilon} g(z)dz\right| \le M \cdot (\theta_2 - \theta_1)\epsilon$ So,

$$\int_{\Gamma_\epsilon} f(z) = \int_{\theta_1}^{\theta_2} \frac{c_{-1}}{\epsilon e^{i\theta}} \cdot \underbrace{\epsilon i e^{i\theta}}_{z'(\theta)} d\theta = ic_{-1}(\theta_2 - \theta_1) = i(\theta_2 - \theta_1)\operatorname{Res}(z_0), \quad \text{as } \epsilon \to 0$$

□

Thus,

$$\int_{\mathcal{C}_\epsilon} f(z)dz = -\pi i \operatorname{Res}(f, b)$$

The negative sign implies negatively oriented, the π implies a semicircle.

So if f satisfies one of the earlier conditions, and $b \in \mathbb{R}$ is a simple pole, then

$$\boxed{\text{cpv} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \cdot \sum \text{Res(upper half plane)} + \pi i \, \text{Res}(b)}$$

Note that we can easily extend this to any finite number of simple poles on the x-axis.

Example 6.27. Evaluate cpv $\int_{-\infty}^{\infty} \frac{x}{x^3-8}dx$

<u>**Solution**</u>: $f(z) = \frac{z}{z^3-8} = \frac{z}{(z-2)(z+1+\sqrt{3}i)(z+1-\sqrt{3}i)}$.

$$\begin{aligned}
\text{Res}(2) &= \lim_{z\to 2}(z-2)\cdot\frac{z}{z^3-8} = \lim_{z\to 2}\frac{2z-2}{3z^2} = \frac{1}{6} \\
\text{Res}(-1+\sqrt{3}i) &= \lim_{z\to -1+\sqrt{3}i}(z+1-\sqrt{3}i)\cdot\frac{z}{(z-2)(z+1+\sqrt{3}i)(z+1-\sqrt{3}i)} \\
&= \lim_{z\to -1+\sqrt{3}i}\frac{1}{2z-2+1+\sqrt{3}i} \\
&= \frac{1}{-3+3\sqrt{3}i} \\
&= \frac{-1-\sqrt{3}i}{12}
\end{aligned}$$

So,

$$\text{cpv}\int_{-\infty}^{\infty}\frac{x}{x^3-8}dx = 2\pi i\cdot\left(\frac{-1-\sqrt{3}i}{12}\right) + \pi i\cdot\frac{1}{6} = \frac{2\sqrt{3}\pi}{6} = \frac{\sqrt{3}\pi}{6}$$

Example 6.28. Evaluate cpv $\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2-16}dx$

<u>**Solution**</u>: $f(z) = \frac{e^{i2z}}{z^2-16}$. We have simple poles at $z = \pm 4$.

$$\begin{aligned}
\text{Res}(4) &= \lim_{z\to 4}(z-4)\cdot\frac{e^{i2z}}{(z+4)(z-4)} = \frac{e^{i8}}{8} \\
\text{Res}(-4) &= \lim_{z\to -4}(z+4)\cdot\frac{e^{i2z}}{(z+4)(z-4)} = \frac{e^{-i8}}{-8}
\end{aligned}$$

So

$$\begin{aligned}
\text{cpv}\int_{-\infty}^{\infty}\frac{e^{i2z}}{z^2-16}dz &= \pi i\cdot\left(\frac{1}{8}e^{i8} - \frac{1}{8}e^{-i8}\right) \\
&= \frac{\pi i}{8}\cdot 2i\sin 8 \\
&= \frac{-\pi}{4}\sin 8
\end{aligned}$$

Thus,

$$\text{cpv}\int_{-\infty}^{\infty}\frac{\cos 2x}{x^2-16}dx = \Re(\frac{-\pi}{4}\sin 8) = \frac{-\pi}{4}\sin 8$$

We can also see that

$$\text{cpv}\int_{-\infty}^{\infty}\frac{\sin 2x}{x^2-16}dx = \Im(\frac{-\pi}{4}\sin 8) = 0$$

as expected by symmetry

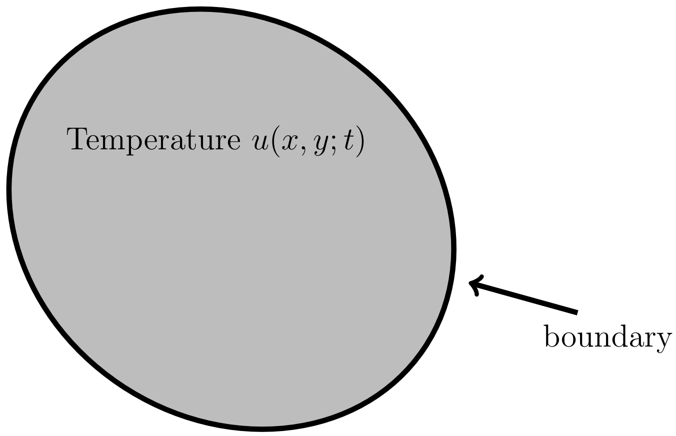
In general, this technique only works for simple poles, e.g.

$$\underbrace{\text{cpv}\int_{-\infty}^{\infty}\frac{1}{x^2}dx}_{\text{clearly }>0,\ =\infty\text{ in fact}} \neq \pi i \cdot \underbrace{\text{Res}(0)}_{=0\text{ by inspection}}$$

Chapter 7 Conformal Mapping

7.1 Application to Dirichlet Problems

Consider a 2D region.



Temperature satisfies the Partial Differential Equation

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

in which K is known as Thermal diffusivity.

As $t \to \infty$, $\dfrac{\partial u}{\partial t} \to 0$, so steady-state temperature satisfies

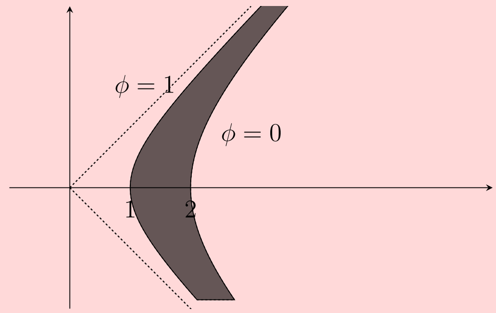
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace's Equation)}$$

also written as $u_{xx} + u_{yy} = 0$, or $\nabla^2 u = 0$.

Definition 7.1. Let D be a domain with a piecewise-smooth boundary. A **Dirichlet Problem** consists of Laplace's Equation, $\nabla^2 f = 0$ within D, together with prescribed values of f on the boundary.

Recall that if $f(z) = u + iv$ is analytic, then u and v are harmonic (i.e. solutions to Laplace's Equation).

Example 7.2. Solve



$$\begin{cases} \nabla^2 \Phi = 0 & \text{on } \rightarrow \\ \Phi = 1 & \text{on } x^2 - y^2 = 1 \\ \Phi = 0 & \text{on } x^2 - y^2 = 4 \end{cases}$$

Solution: Note $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i \cdot 2xy$.

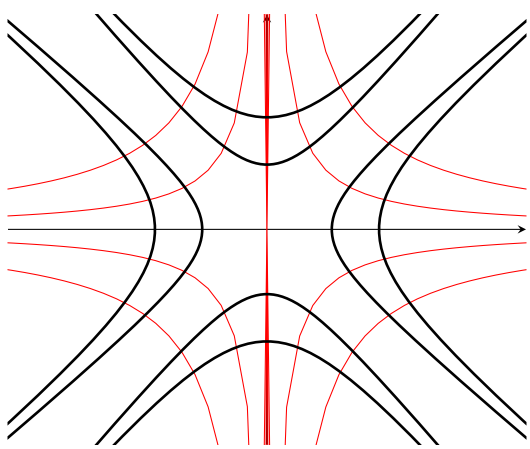
So, set $\Phi(x, y) = x^2 - y^2$? No, second Boundary Condition not satisfies.

Idea: by linearity, $\phi(x, y) = A(x^2 - y^2) + B$ is also harmonic. Applying Boundary Condition:

- $\phi = 1$ on $x^2 - y^2 = 1$ gives $A \cdot 1 + B = 1$, gives $B = -A + 1$
- $\phi = 0$ on $x^2 - y^2 = 4$ gives $A \cdot 4 + B = 0$, gives $3A + 1 = 0$
- So $A = -\frac{1}{3}$, $B = \frac{4}{3}$

So the solution is $\phi(x, y) = -\frac{1}{3}(x^2 - y^2) + \frac{4}{3}$

Note: The curves $x^2 - y^2 = k$ and $xy = k$ intersect orthogonally at every point.

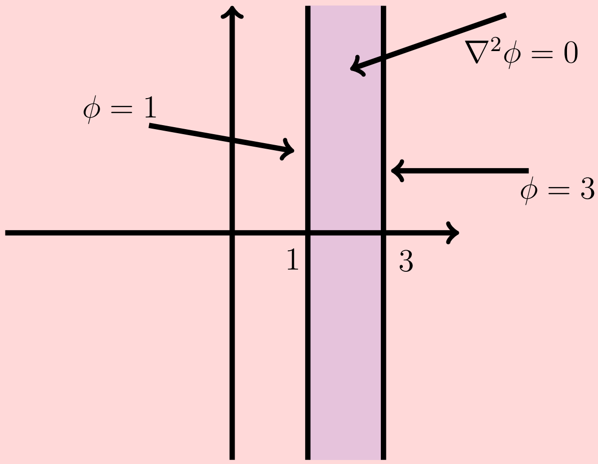


Theorem 7.3. Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D. Then the level curves of u are orthogonal to the level curves for v at every point in D.

Example 7.4. Consider some useful functions:

- $f(z) = z = x + iy$, so $u(x, y) = x$ and $v(x, y) = y$ are harmonic.

This is useful for strips. $\phi(x, y) = Ax + B$.

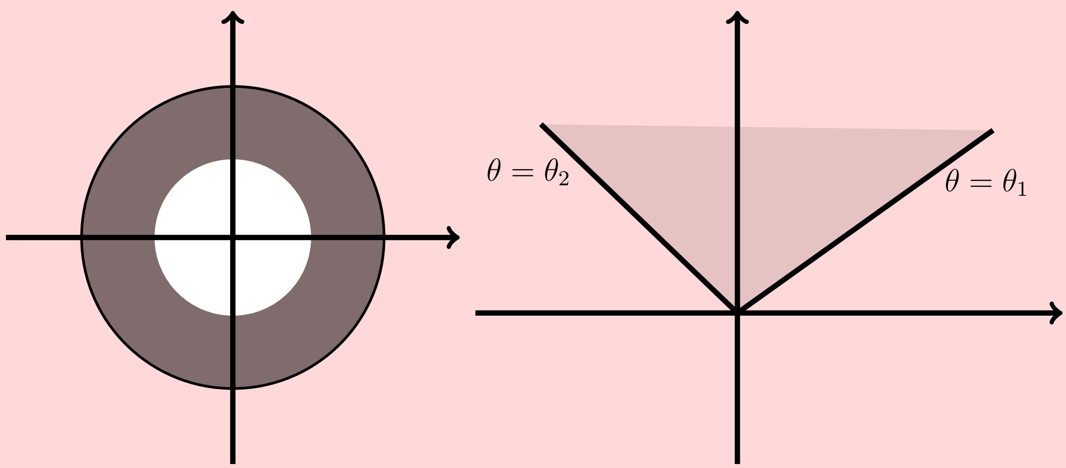


In this example,

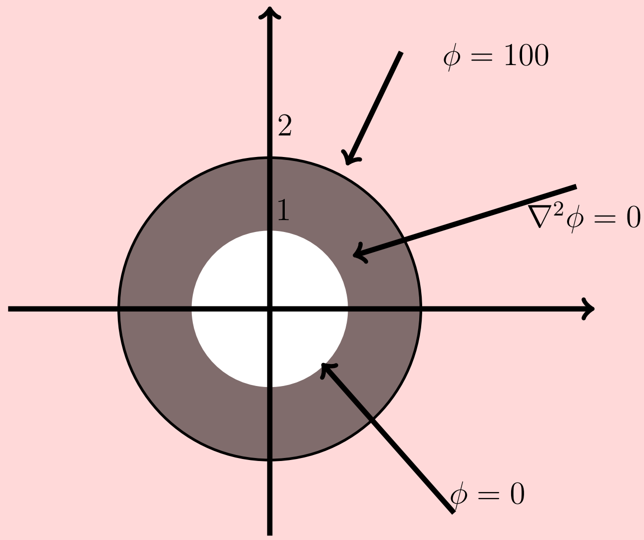
$$\begin{cases} 0 = A + B \\ 10 = 3A + B \end{cases} \Rightarrow A = 5, B = -5$$

So $\phi(x, y) = 5x - 5$

- $f(z) = \log z = \ln|z| + i\arg(z)$ is also analytic, which gives solutions to the problems where boundaries are $\ln|z| = k$ (i.e. $|z|$ is constant) or $\arg(z) = k$.



Example 7.5. Solve



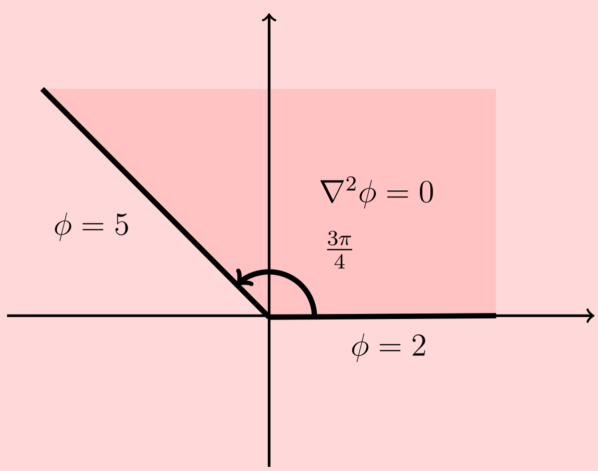
$\phi(x, y) = A \ln |z| + B$

$$\begin{cases} A\ln(1) + B = 0 & \Rightarrow B = 0 \\ A\ln(2) + B = 100 & \Rightarrow A = \dfrac{100}{\ln 2} \end{cases}$$

So

$$\phi(x, y) = \frac{100}{\ln 2} \ln |z| = \frac{100}{\ln 2} \ln(\sqrt{x^2 + y^2}) = \frac{50}{\ln 2} \ln(x^2 + y^2)$$

Example 7.6. Solve



$\phi(x, y) = A \operatorname{Arg} z + B$

$$\begin{cases} A \cdot 0 + B = 2 & \Rightarrow B = 2 \\ A \cdot \dfrac{3\pi}{4} + B = 5 & \Rightarrow A = \dfrac{4}{\pi} \end{cases}$$

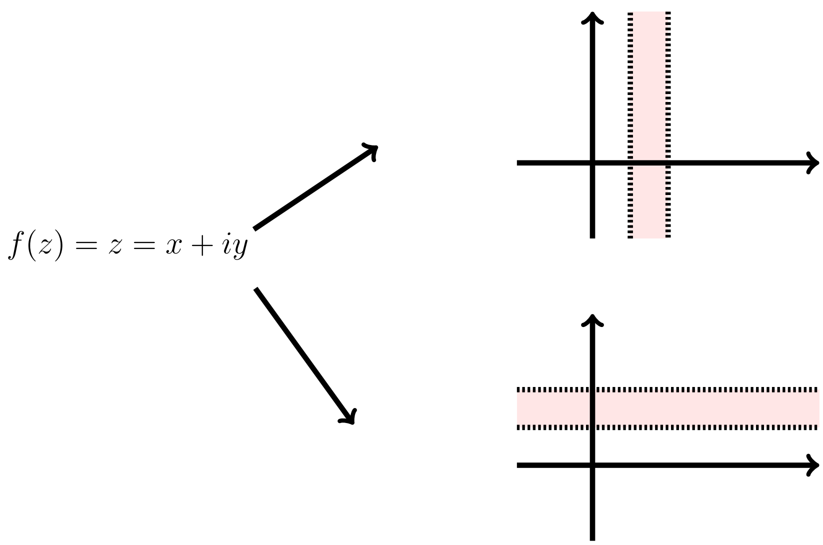
So

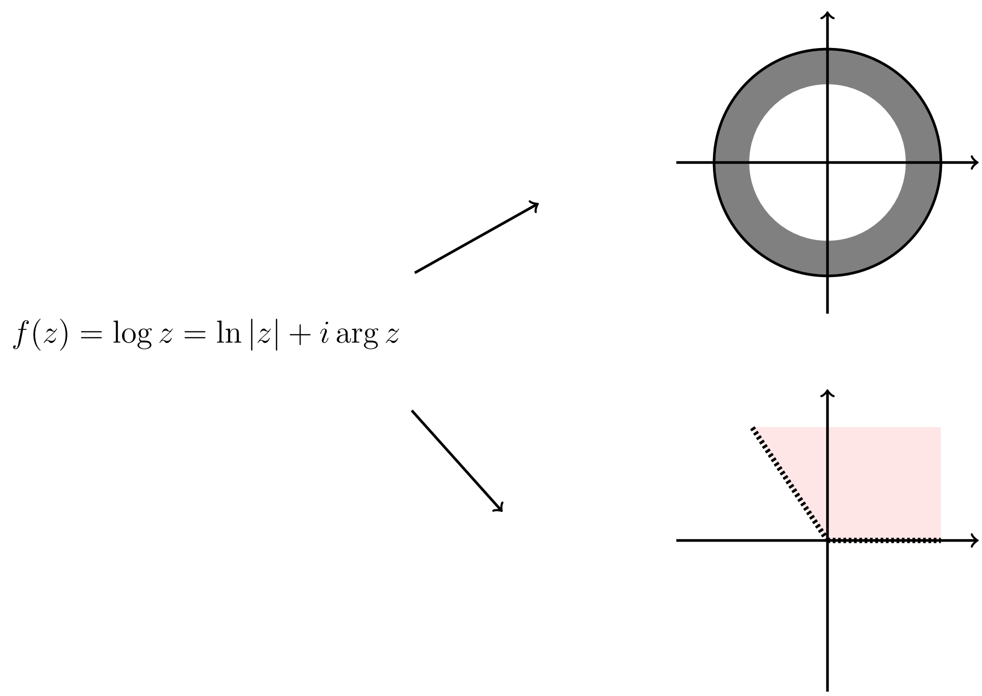
$$\phi(x,y) = \frac{4}{\pi}\operatorname{Arg} z + 2$$

Note that using $\mathbb{C}$ can give nicer notation. With x and y,

$$\operatorname{Arg} z = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{in the first quadrant} \\ \frac{\pi}{2} & \text{on vertical axis} \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{in the second quadrant} \end{cases}$$

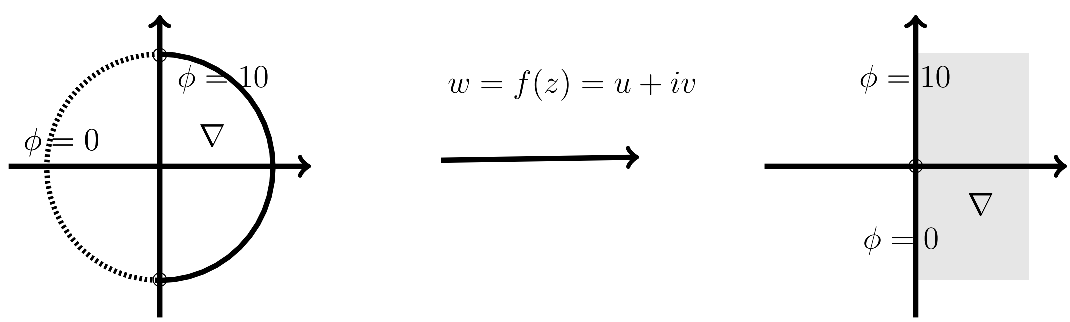
Recall that we already know $f(z) = z$ and $f(z) = \log z$, and we know their real and imaginary part functions. Conveniently, we know





Suppose we want to know more about general boundaries. We may consider try to find some mapping such that the image is one of the four images we get from above.

For example, we may try to find mappings such that:



Does $\phi_{uu} + \phi_{vv} = 0$ implies $\phi_{xx} + \phi_{yy} = 0$? Is the map invertible? Let's have 6 theorems to help answer these questions.

Theorem 7.7. Let $f(z)$ be analytic at z_0. If $f'(t_0) \neq 0$, then there exists neighborhood of z_0 in which $w = f(z)$ is invertible. Also, the derivative of the inverse $z = f^{-1}(w)$ is $\dfrac{dz}{dw} = \dfrac{1}{\underbrace{dw/dz}_{f'(z)}}$.

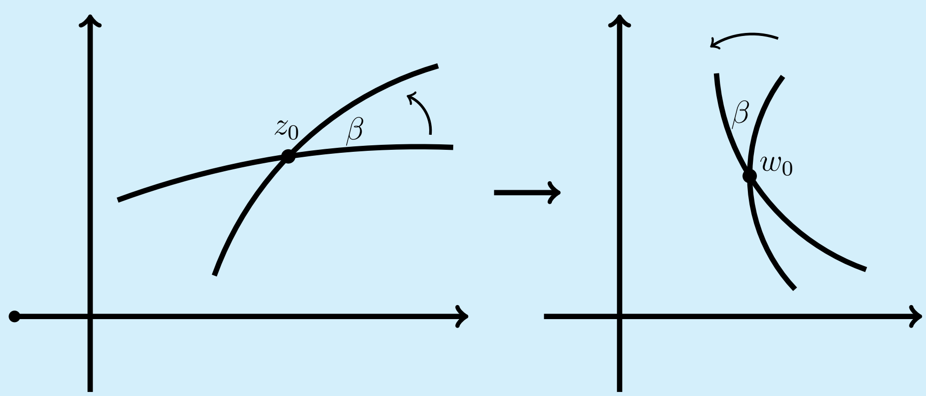
Proof 7.8. Partial proof:

$$\text{Jacobian} = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x v_y - u_y v_x = u_x^2 + u_y^2 = |f'(z_0)|^2 > 0$$

This is not equal to 0 and orientation preserved. □

Theorem 7.9. Suppose that an analytic function $w = f(z) = u(x, y) + iv(x, y)$ maps a domain D_{xy} onto a domain D_{uv}. Suppose also that a scalar field $\Phi(u, v)$ is harmonic in D_{uv}. Then the scalar field $\phi(x, y) = \Phi\big(u(x, y), v(x, y)\big)$ is harmonic in D_{xy}

Definition 7.10. A mapping which preserves the angle of intersection (and the "sense" of that angle) between any two curves intersecting at z_0 is said to be **conformal** at z_0. If it is conformal at every point in a domain D, then it is **conformal** on that domain.



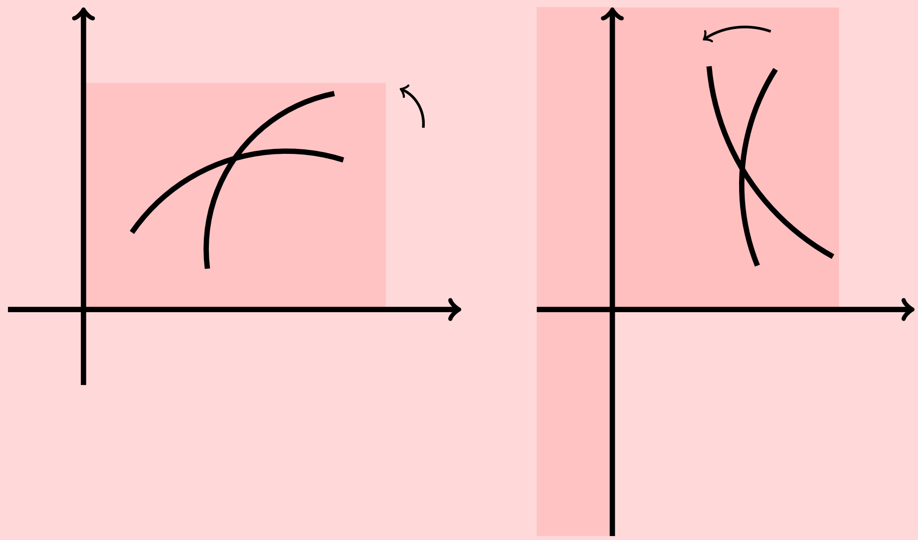
Theorem 7.11. Let $f(z)$ be analytic in a domain D. Then $f(z)$ is conformal at every point in D where $f'(z) \neq 0$

Theorem 7.12. **The Riemann Mapping Theorem**: Let D and D' be any two simply-connected domains, but not the entire plane. Let $z_0 \in D, w_0 \in D'$ and $\alpha \in \mathbb{R}$. There exists a unique conformal mapping $w = f(z)$ of D onto D' such that $f(z_0) = w_0$ and $\operatorname{Arg} f'(z_0) = \alpha$.

We will require $f'(z) \neq 0$ inside D, but $f'(z)$ might be zero on the boundary.

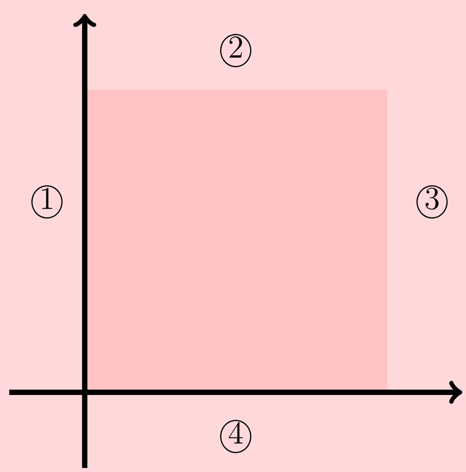
Theorem 7.13. Let f be analytic at z_0. If $f^{(j)}(z_0) = 0$ for $j = 1, 2, ..., k-1$ but $f^{(k)}(z_0) \neq 0$, then the mapping $w = f(z)$ magnifies angles at z_0 by a factor of k.

Example 7.14. $f(z) = z^3$



Angles are preserved everywhere except at the origin (tripled there).

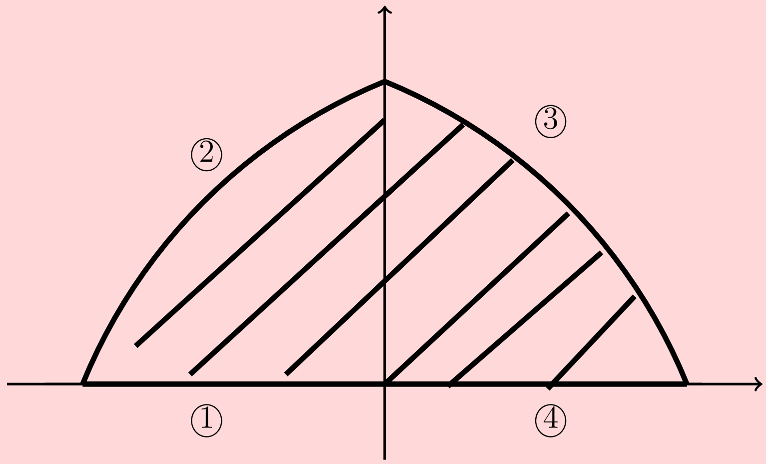
Example 7.15. Find the image of the square shown under $w = f(z) = z^2 = x^2 - y^2 + i \cdot 2xy$



<u>Solution</u>: We look at boundaries:

1. $x = 0, 0 \leq y \leq 1$. Then $\begin{cases} u = -y^2 \\ v = 0 \qquad -1 \leq u \leq 0 \end{cases}$
2. $y = 1, 0 \leq x \leq 1$. Then $\begin{cases} u = x^2 - 1 \\ v = 2x \end{cases}$, so $u = \left(\frac{v}{2}\right)^2 - 1, -1 \leq u \leq 0.$
3. $x = 1, 0 \leq y \leq 1$. Then $\begin{cases} u = 1 - y^2 \\ v = 2y \end{cases}$, so $u = 1 - \left(\frac{v}{2}\right)^2, 0 \leq v \leq 2$

4. $y = 0, 0 \le x \le 1$. Then $\begin{cases} u = x^2 \\ v = 0 \quad\quad 0 \le u \le 1 \end{cases}$



This looks inside maps to inside. Take a point to check: $f\left(e^{i\pi/4}\right) = e^{i\pi/2} = i$.

Note: Angle at $z = 0$ is doubled ($f''(0) \neq 0$), rest are preserved.

Theorem 7.16. If f is analytic, then it will map closed curves to closed curves.

This holds true given the assumption that we can include the point at infinity. One example would be: circles can map to lines.

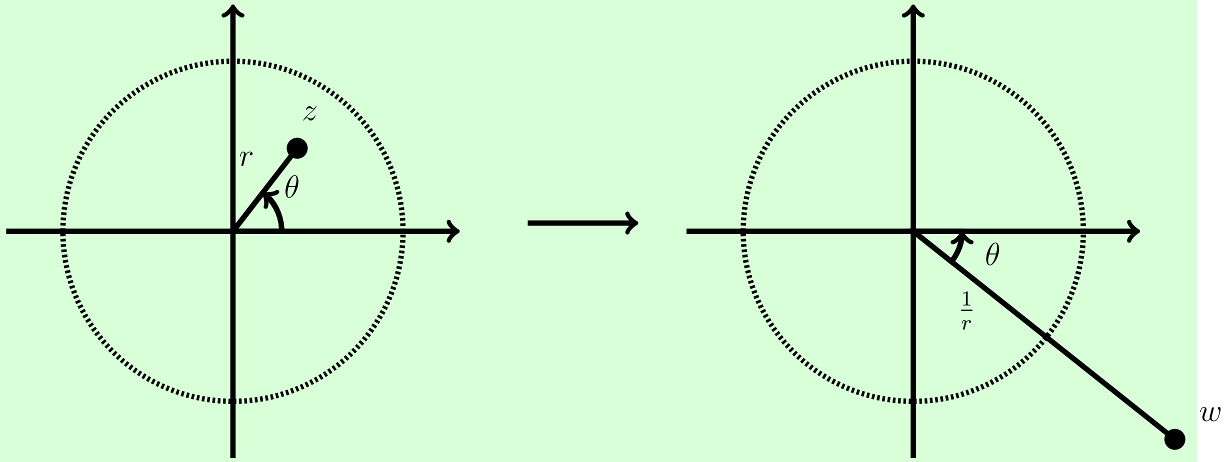
This thereom also implies that f will map domains to domains.

Proposition 7.17. Let's consider some common mappings:

- **Linear Transformations**: $w = f(z) = az + b$, $a, b \in \mathbb{C}, a \neq 0$. a is for scaling and rotation, and b is for translation.

  This maps circles to circles, and lines to lines.

- **Inversion Mapping**: $w = \frac{1}{z}$. Recall that $w = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$.



This maps circles and lines to circles or lines.

- **Mobius Transformation**: aka "bilinear", "linear fractional". $w = f(z) = \dfrac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$.

For the lasp mapping, we have the following properties:

1. $$f(z) = \frac{a(cz+d)+bc-ad}{c(cz+d)} = \frac{a}{c} + \frac{bc-ad}{c} \cdot \frac{1}{cz+d}$$
   in which we assume $bc - ad \neq 0$, otherwise f is constant. This is a composition of linear, inversion, then linear maps.

2. $$f'(z) = \frac{a(cz+d)-c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2 \neq 0}$$
   since $bc - ad \neq 0$. So f is analytic and conformal except at $z = \dfrac{-d}{c}$.

3. We can find
   $$f^{-1} : w = \frac{az+b}{cz+d} \Rightarrow \cdots \Rightarrow z = \frac{b-dw}{cw-a} = f^{-1}(w)$$
   which is also Mobius.

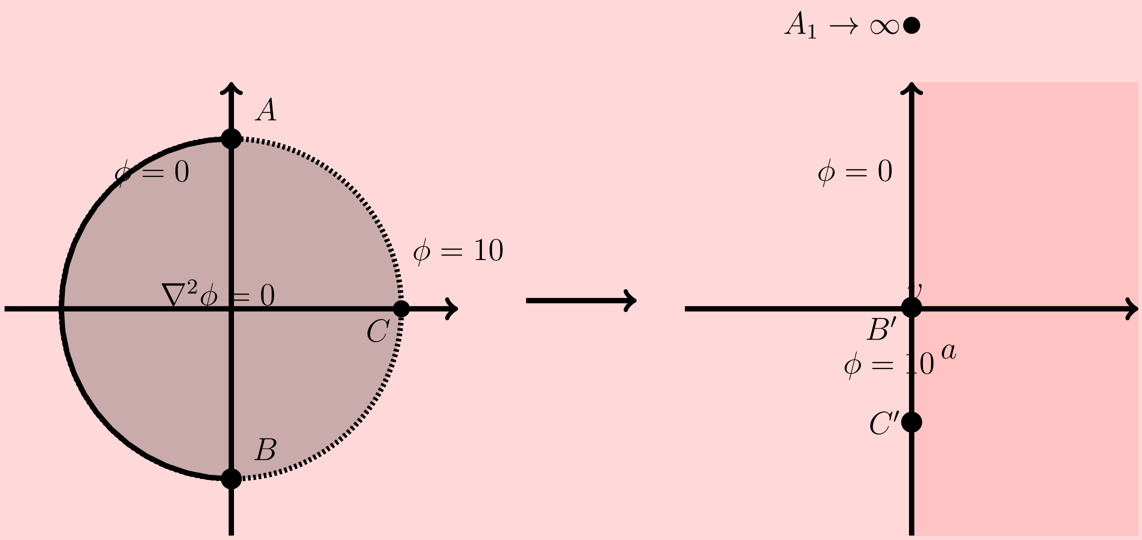
   If we define
   $$\begin{cases} f\left(\dfrac{-d}{c}\right) = \infty \\ f(\infty) = \dfrac{a}{c} \\ f^{-1}(\infty) = \dfrac{-d}{c} \\ f^{-1}\left(\dfrac{a}{c}\right) = \infty \end{cases}$$
   then f is invertible on entire extended complex plane.

4. Circles and lines map to circles or lines.
5. There exists Mobius transformation which maps three distinct points z_1, z_2, z_3 to 3 distinct points w_1, w_2, w_3.

   This is because there are really only 3 constants

$$f(z) = \frac{z + \frac{b}{a}}{\frac{c}{a}z + \frac{d}{a}} = \frac{z + A}{Bz + C}$$

Example 7.18. Solve the Dirichlet Problem:



__**Solution**__: Map circle to a line using Mobius transform.

Send discontinuities (points A and B) to convenient locations on Imaginary axis. Say $f(i) = \infty$ and $f(-i) = 0$.

Put the third point somewhere else on the line, say $f(1) = -i$.

Note: orientation is preserved (on left as $A' \to B' \to C'$)

Find $\Phi(u, v)$: region is a wedge, so $\Phi(u, v) = c_1 \operatorname{Arg}(w) + c_2$.

Apply BC:

$$\begin{cases} c_1 \cdot \left(-\dfrac{\pi}{2}\right) + c_2 = 10 \\ c_1 \cdot \left(\dfrac{\pi}{2}\right) + c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = \dfrac{-10}{\pi} \\ c_2 = 5 \end{cases}$$

So $\Phi(u, v) = 5 - \dfrac{10}{\pi} \operatorname{Arg} w$. How to get $\phi(x, y)$? We need the map.

Let

$$f(z) = \frac{z+a}{bz+c} \quad \Rightarrow \quad \begin{cases} f(i) = \infty & \Rightarrow bi + c = 0 \Rightarrow c = -bi \\ f(-i) = 0 & \Rightarrow \dfrac{-i+a}{bi+c} = 0 \Rightarrow a = i \\ f(1) = -i & \Rightarrow \dfrac{i+a}{b+c} = -i \Rightarrow \cdots \Rightarrow b = -1, c = i \end{cases}$$

So $w = \dfrac{z+i}{-z+i}$, and so $\phi(x,y) = 5 - \dfrac{10}{\pi} \operatorname{Arg}\left(\dfrac{z+i}{-z+i}\right)$

Note that

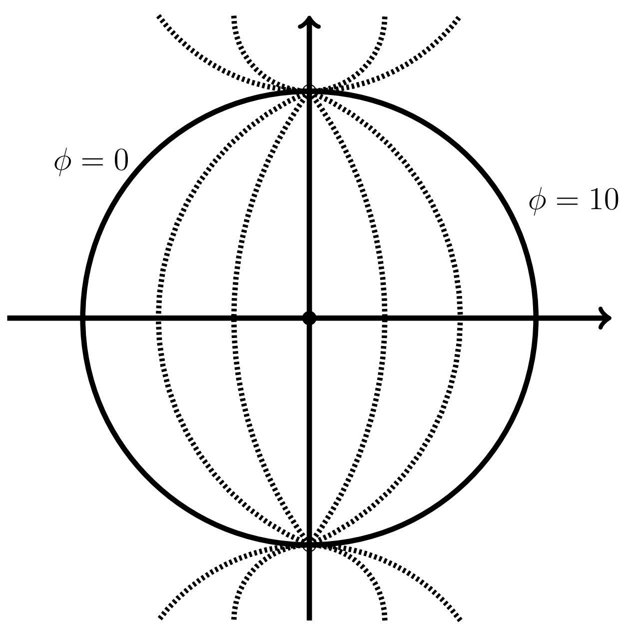
$$w = \frac{(x+iy)+i}{-(x+iy)+i} = \cdots = \frac{-x^2-y^2+1-2xi}{x^2+(y-1)^2} = u + iv$$

Then

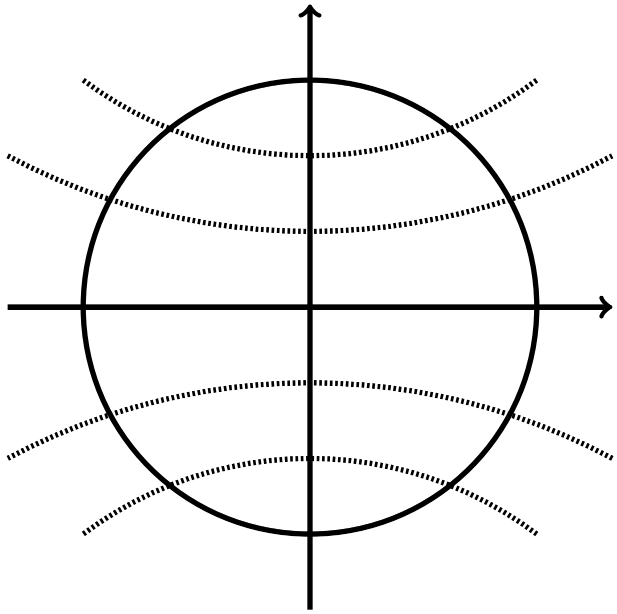
$$\begin{aligned} \operatorname{Arg} w &= \arctan\left(\frac{v}{u}\right) \\ &= \arctan\left(\frac{-2x}{-x^2-y^2+1}\right) \\ \phi(x,y) &= 5 - \frac{10}{\pi} \arctan\left(\frac{2x}{x^2+y^2-1}\right) \end{aligned}$$

There are many ways to solve the above problem, but it will always have a unique solution when expressed in terms of x and y.

We could look at isotherms: $\Phi = \text{const}$, so $\arctan\left(\dfrac{2x}{x^2+y^2-1}\right) = \text{const}$, so $x^2 + (\text{const})x + y^2 = 1$. This is circles through $(0, \pm 1)$.



Harmonic conjugate function $\Psi(x, y)$ will have $\Psi(x, y) = k$ curves orthogonal, called "flow lines".



Fluids: $\vec{v} = \Delta\phi$

Proposition 7.19. **<u>The Cross-Ratio Formula</u>**: To map z_1, z_2, and z_3 to w_1, w_2, and w_3 respectively, set

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Shorthand notation: $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$.

Note: if we use ∞ as one of the points, e.g. $z_3 = \infty$, then RHS simplifies to

$$\frac{z - z_1}{z_2 - z_1} \cdot \underbrace{\frac{z_2 - z_3}{z - z_3}}_{\to 1} = \frac{z - z_1}{z_2 - z_1}$$

Example 7.20. Find a mapping which takes circle $|z + 1| = 1$ to the real axis.

<u>Solution</u>: Choose points on circle: $z_1 = -2, z_2 = -1 - i, z_3 = 0$, and map these to (respectively) $w_1 = -1, w_2 = 0, w_3 = \infty$.

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \quad \Rightarrow \quad \frac{w + 1}{1} = \frac{(z + 2)(-1 - i)}{z(1 - i)}$$

So

$$w = \frac{(-1 + i)z + (-2 - 2i)}{(1 - i)z}$$

Note that interior points maps to upper half plane.

Example 7.21. This is a "pipe within a pipe"

Find $\phi(x, y)$

<u>**Solution**</u>: Key idea: $z = 0$ in on both circles, so send it to ∞ so that we get lines.

Which other points? One option: notice orthogonal intersection with y-axis, so if we map y-axis to v-axis

So we get horizontal lines.

Choose points:

$$\begin{cases} z_1 = 0 & \Rightarrow w_1 = \infty \\ z_2 = 2i & \Rightarrow w_2 = 0 \\ z_3 = 4i & \Rightarrow w_3 = i \end{cases}$$

The mapping is

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \quad \Rightarrow \quad \frac{-i}{w - i} = \frac{z(2i - 4i)}{(z - 4i)(2i)}$$

So

$$w = i + i \cdot \frac{z - 4i}{z} = \frac{iz + iz + 4}{z} = \frac{2iz + 4}{z}$$

The image is

Is it mapping inside to inside? Let's check a point. $f(3i) = \dfrac{2i(3i) + 4}{3i} = \dfrac{2i}{3}$. So yes.

The solution in w-plane is (using the function $g(w) = w$ which is analytic everywhere)

$$\Phi(u, v) = A \cdot v + B$$

Applying Boundary Conditions:

$$\begin{cases} A \cdot 1 + B = 15 & \Rightarrow A = -50 \\ A \cdot 0 + B = 65 & \Rightarrow B = 65 \end{cases} \quad \Rightarrow \quad \Phi(u, v) = -50v + 65 = -50\Im(w) + 65$$

So, $\phi(x, y) = -50\Im\left(\dfrac{2iz + 4}{z}\right) + 65$. To write in terms of x and y, ntoe that

$$\frac{2iz + 4}{z} = \frac{2i(x + iy) + 4}{x + iy} = \cdots = \frac{\text{stuff } + i(2x^2 + 2y^2 - 4y)}{x^2 + y^2}$$

So

$$\begin{aligned}
\phi(x,y) &= -50\Im\left(\frac{2iz+4}{z}\right)+65\\
&= -50\cdot\left(\frac{2(x^2+y^2-2y)}{x^2+y^2}\right)+65\\
&= \cdots\\
&= \frac{200y}{x^2+y^2}-65
\end{aligned}$$