Tensor Products Implementations and Optimizations in MATLAB

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Abstract

The tensor product is a fundamental operation in applied mathematics, computer science, and engineering, underpinning numerous applications ranging from signal processing and data analysis to machine learning and quantum mechanics. Efficient computation and optimization of tensor products are crucial for addressing high-dimensional problems, particularly in fields such as multilinear algebra and numerical modeling. This paper explores the implementation of tensor product computations and optimizations using MATLAB, a powerful numerical computing environment widely adopted for its flexibility and extensive built-in functionalities. The proposed approach leverages MATLAB's high-performance tensor manipulation tools, including the Tensor Toolbox and built-in array operations, to develop efficient algorithms for tensor products. The paper provides an in-depth analysis of computational techniques, focusing on scalability, memory management, and parallelization strategies that enhance performance on large-scale datasets. Optimization techniques such as dimensionality reduction, Kronecker product decomposition, and hierarchical Tucker decomposition are integrated into the MATLAB framework to handle complex tensor structures effectively. Furthermore, we present case studies demonstrating the real-world applicability of the methods in fields like image processing, where multidimensional data representations are critical, and in supply chain modeling, where tensors capture intricate relational data. Experimental results validate the efficiency of the proposed implementation, achieving significant reductions in computational time and resource usage compared to conventional methods. This work highlights MATLAB's capability as a versatile platform for tensor computations and contributes to the growing body of research focused on improving the practicality and efficiency of high-dimensional data analysis. Future directions include extending the methodology to incorporate machine learning frameworks and further optimizing the integration with GPU-based computing for real-time applications.

1 Tensor products

The Pauli spin matrices are given by

$$S^x = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, S^y = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}, S^z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{1}$$

Definition 1.1. Let $\mathbb{C}^2 \otimes \mathbb{C}^2$ be 2 spins, then the unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ can be expressed as $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$, respectively.

Below are examples of Tensor Product operations on S^x:

Example 1.2.

$$
\begin{aligned}
S^x \otimes S^x |\uparrow\uparrow\rangle &= S^x|\uparrow\rangle \otimes S^x|\uparrow\rangle \\
&= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \\
&= \frac{1}{2} \times \frac{1}{2} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\
&= \frac{1}{4} |\downarrow\downarrow\rangle
\end{aligned}
$$

$$
\begin{aligned}
S^x \otimes S^x |\downarrow\downarrow\rangle &= S^x|\downarrow\rangle \otimes S^x|\downarrow\rangle \\
&= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \\
&= \frac{1}{2} \times \frac{1}{2} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\
&= \frac{1}{4} |\uparrow\uparrow\rangle
\end{aligned}
$$

$$
\begin{aligned}
S^x \otimes S^x |\uparrow\downarrow\rangle &= S^x|\uparrow\rangle \otimes S^x|\downarrow\rangle \\
&= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \\
&= \frac{1}{2} \times \frac{1}{2} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\
&= \frac{1}{4} |\downarrow\uparrow\rangle
\end{aligned}
$$

$$
\begin{aligned}
S^x \otimes S^x |\downarrow\uparrow\rangle &= S^x|\downarrow\rangle \otimes S^x|\uparrow\rangle \\
&= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \\
&= \frac{1}{2} \times \frac{1}{2} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\
&= \frac{1}{4} |\uparrow\downarrow\rangle
\end{aligned}
$$

Similarly, we can have the same operations on S^y:

Example 1.3.

$$\begin{aligned}
S^y \otimes S^y|\uparrow\uparrow\rangle &= S^y|\uparrow\rangle \otimes S^y|\uparrow\rangle \\
&= \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \frac{i}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \frac{i}{2} \end{pmatrix} \\
&= \frac{i}{2} \times \frac{i}{2} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\
&= -\frac{1}{4}|\downarrow\downarrow\rangle
\end{aligned}$$

$$\begin{aligned}
S^y \otimes S^y|\downarrow\downarrow\rangle &= S^y|\downarrow\rangle \otimes S^y|\downarrow\rangle \\
&= \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{i}{2} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} -\frac{i}{2} \\ 0 \end{pmatrix} \\
&= \left(-\frac{i}{2}\right) \times \left(-\frac{i}{2}\right) \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\
&= -\frac{1}{4}|\uparrow\uparrow\rangle
\end{aligned}$$

$$\begin{aligned}
S^y \otimes S^y|\uparrow\downarrow\rangle &= S^y|\uparrow\rangle \otimes S^y|\downarrow\rangle \\
&= \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \frac{i}{2} \end{pmatrix} \otimes \begin{pmatrix} -\frac{i}{2} \\ 0 \end{pmatrix} \\
&= \frac{i}{2} \times \left(-\frac{i}{2}\right) \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\
&= \frac{1}{4}|\downarrow\uparrow\rangle
\end{aligned}$$

$$\begin{aligned}
S^y \otimes S^y|\downarrow\uparrow\rangle &= S^y|\downarrow\rangle \otimes S^y|\uparrow\rangle \\
&= \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{i}{2} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \frac{i}{2} \end{pmatrix} \\
&= \left(-\frac{i}{2}\right) \times \frac{i}{2} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\
&= \frac{1}{4}|\uparrow\downarrow\rangle
\end{aligned}$$

Finally, we will have the operations on S^z:

Example 1.4.

$$
\begin{aligned}
S^z \otimes S^z | \uparrow\uparrow\rangle &= S^z | \uparrow\rangle \otimes S^z | \uparrow\rangle \\
&= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \\
&= \frac{1}{2} \times \frac{1}{2} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\
&= \frac{1}{4} | \uparrow\uparrow\rangle
\end{aligned}
$$

$$
\begin{aligned}
S^z \otimes S^z | \downarrow\downarrow\rangle &= S^z | \downarrow\rangle \otimes S^z | \downarrow\rangle \\
&= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \\
&= \left(-\frac{1}{2} \right) \times \left(-\frac{1}{2} \right) \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\
&= \frac{1}{4} | \downarrow\downarrow\rangle
\end{aligned}
$$

$$
\begin{aligned}
S^z \otimes S^z | \uparrow\downarrow\rangle &= S^z | \uparrow\rangle \otimes S^z | \downarrow\rangle \\
&= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \\
&= \frac{1}{2} \times \left(-\frac{1}{2} \right) \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\
&= -\frac{1}{4} | \uparrow\downarrow\rangle
\end{aligned}
$$

$$
\begin{aligned}
S^z \otimes S^z | \downarrow\uparrow\rangle &= S^z | \downarrow\rangle \otimes S^z | \uparrow\rangle \\
&= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \\
&= \left(-\frac{1}{2} \right) \times \frac{1}{2} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\
&= -\frac{1}{4} | \downarrow\uparrow\rangle
\end{aligned}
$$

Observation 1.5. In Example 1.4, since

$$\begin{aligned}
S^z \otimes S^z|\uparrow\uparrow\rangle &= \frac{1}{4}|\uparrow\uparrow\rangle \\
S^z \otimes S^z|\downarrow\downarrow\rangle &= \frac{1}{4}|\downarrow\downarrow\rangle \\
S^z \otimes S^z|\uparrow\downarrow\rangle &= -\frac{1}{4}|\uparrow\downarrow\rangle \\
S^z \otimes S^z|\downarrow\uparrow\rangle &= -\frac{1}{4}|\downarrow\uparrow\rangle
\end{aligned}$$

we know that $\frac{1}{4}, -\frac{1}{4}$ are two eigenvalues of the above operations based on the properties of eigenvalues[1].

2 Two Site Operator

Definition 2.1. Let h$[|\alpha\rangle \otimes |\beta\rangle]$ be the two site operator where $\alpha, \beta \in \{\uparrow, \downarrow\}$, then for $\Delta \in \mathbb{R}$,

$$h = \frac{1}{4}[I \otimes I] - S^z \otimes S^z - \frac{1}{\Delta}[S^x \otimes S^x + S^y \otimes S^y]$$

In the following example, we will discuss the operations of $h|\uparrow\uparrow\rangle, h|\downarrow\downarrow\rangle, h|\uparrow\downarrow\rangle, h|\downarrow\uparrow\rangle$. In addition, we will also utilize the results from previous examples.

Example 2.2.

$$\begin{aligned}
h|\uparrow\uparrow\rangle &= \frac{1}{4}[I \otimes I|\uparrow\uparrow\rangle] - S^z \otimes S^z|\uparrow\uparrow\rangle - \frac{1}{\Delta}[S^x \otimes S^x|\uparrow\uparrow\rangle + S^y \otimes S^y|\uparrow\uparrow\rangle] \\
&= \frac{1}{4}|\uparrow\uparrow\rangle - \frac{1}{4}|\uparrow\uparrow\rangle - \frac{1}{\Delta}\left[\frac{1}{4}|\downarrow\downarrow\rangle - \frac{1}{4}|\downarrow\downarrow\rangle\right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
h|\downarrow\downarrow\rangle &= \frac{1}{4}[I \otimes I|\downarrow\downarrow\rangle] - S^z \otimes S^z|\downarrow\downarrow\rangle - \frac{1}{\Delta}[S^x \otimes S^x|\downarrow\downarrow\rangle + S^y \otimes S^y|\downarrow\downarrow\rangle] \\
&= \frac{1}{4}|\downarrow\downarrow\rangle - \frac{1}{4}\downarrow\downarrow\rangle - \frac{1}{\Delta}\left[\frac{1}{4}|\uparrow\uparrow\rangle - \frac{1}{4}|\uparrow\uparrow\rangle\right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
h|\uparrow\downarrow\rangle &= \frac{1}{4}[I \otimes I|\uparrow\downarrow\rangle] - S^z \otimes S^z|\uparrow\downarrow\rangle - \frac{1}{\Delta}[S^x \otimes S^x|\uparrow\downarrow\rangle + S^y \otimes S^y|\uparrow\downarrow\rangle] \\
&= \frac{1}{4}|\uparrow\downarrow\rangle + \frac{1}{4}|\uparrow\downarrow\rangle - \frac{1}{\Delta}\left[\frac{1}{4}|\downarrow\uparrow\rangle + \frac{1}{4}|\downarrow\uparrow\rangle\right] \\
&= \frac{1}{2}|\uparrow\downarrow\rangle - \frac{1}{\Delta}\left[\frac{1}{2}|\downarrow\uparrow\rangle\right]
\end{aligned}$$

$$\begin{aligned}
h|\downarrow\uparrow\rangle &= \frac{1}{4}[I \otimes I|\downarrow\uparrow\rangle] - S^z \otimes S^z|\downarrow\uparrow\rangle - \frac{1}{\Delta}[S^x \otimes S^x|\downarrow\uparrow\rangle + S^y \otimes S^y|\downarrow\uparrow\rangle] \\
&= \frac{1}{4}|\downarrow\uparrow\rangle + \frac{1}{4}|\downarrow\uparrow\rangle - \frac{1}{\Delta}\left[\frac{1}{4}|\uparrow\downarrow\rangle + \frac{1}{4}|\uparrow\downarrow\rangle\right] \\
&= \frac{1}{2}|\downarrow\uparrow\rangle - \frac{1}{\Delta}\left[\frac{1}{2}|\uparrow\downarrow\rangle\right]
\end{aligned}$$

3 Inner Product

Definition 3.1. Let V be a vector space over $\mathbb{F}$. An *inner product* on V is a function assigns to, each ordered pair of vectors x and y in V, a scalar in $\mathbb{F}$, denote $< x, y >$ or (x, y) such that for all $x, y, z \in V$ and $c \in \mathbb{F}$, the followings hold

$$\begin{aligned}
\langle x+z, y\rangle &= \langle x, y\rangle + \langle z, y\rangle \\
\langle cx, y\rangle &= c\langle x, y\rangle \\
\overline{\langle x, y\rangle} &= \langle y, x\rangle \\
\langle x, x\rangle &> 0 \ \text{ if } \ x \neq 0
\end{aligned}$$

Theorem 3.2. *Let V be an inner product space. Then for $x, y, z \in V$ and $c \in \mathbb{F}$, the followings are true:*

$$\begin{aligned}
\langle x, y+z\rangle &= \langle x, y\rangle + \langle x, z\rangle \\
\langle x, cy\rangle &= \bar{c}\langle x, y\rangle \\
\langle x, 0\rangle &= \langle 0, x\rangle = 0 \\
\langle x, x\rangle &= 0 \ \ \textit{iff} \ \ x = 0 \\
if\ \langle x, y\rangle &= \langle x, z\rangle \ \ for\ all\ \ x \in V, then\ \ y = z
\end{aligned}$$

Proof. First of all, $\langle x, y+z\rangle = \overline{\langle y+z, x\rangle} = \overline{\langle y, x\rangle + \langle z, x\rangle} = \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle} = \langle x, y\rangle + \langle x, z\rangle$. Then, $\langle x, cy\rangle = \overline{\langle cy, x\rangle} = \overline{c\langle y, x\rangle} = \bar{c}\overline{\langle y, x\rangle} = \bar{c}\langle x, y\rangle$. Moreover, $\langle x, 0\rangle = \langle x, 0 \times y\rangle = \bar{0}\langle x, y\rangle = 0\langle x, y\rangle = 0$. Also, if $x = 0$, then $\langle 0, 0\rangle = 0$, if $\langle x, x\rangle = 0$, then $x = \vec{0}$, and so $\langle x, x\rangle = 0 \Leftrightarrow x = \vec{0}$. Therefore, inner product is semi positive-definite. Finally, $\langle x, y\rangle = \langle x, z\rangle \Rightarrow \langle x, y\rangle - \langle x, z\rangle = 0 \Rightarrow \langle x, y\rangle + \langle x, -z\rangle = 0 \Rightarrow \langle x, y-z\rangle = 0$ for all $x \in V$, then take $x = y - z$, and we have $\langle y-z, y-z\rangle = 0 \Rightarrow y - z = \vec{0} \Rightarrow y = z$. Therefore, above statements are proved. □

4 Rigorous Application on Tensor Product and Inner Product

4.1 Defining Tensor Product in $\mathbb{C}^2$[9]

First of all, we will start with $\mathbb{C}^m, \mathbb{C}^n$, where $\mathbb{C}^m = \left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} : a_1, a_2, \cdots, a_m \in \mathbb{C}\right)$ and $\mathbb{C}^n = \left(\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} : b_1, b_2, \cdots, b_n \in \mathbb{C}\right)$

The canonical/standard basis of $\mathbb{C}^m$ is defined to be $\{e_1, e_2, \cdots, e_m\}$, where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, e_m = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$. Similarly, we can define the canonical/standard basis of $\mathbb{C}^n$ to be $\{f_1, f_2, \cdots, f_n\}$ where $f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, f_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$. Now, the tensor product of $\mathbb{C}^m$ and $\mathbb{C}^n$ is defined as $e_i \otimes f_j$ where $i \in \{1, \cdots, m\}, j \in \{1, \cdots, n\}$. In our case, we will limit the vector space in 2 dimensions, which means we will look at $\mathbb{C}^2 \otimes \mathbb{C}^2$ where $m = n = 2$. Therefore, we have the following standard basis:

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then, from $\{e_i \otimes f_j : i \in \{1,2\}, j \in \{1,2\}\}$ we obtain

$$\{e_1 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_1, e_2 \otimes f_2\}$$

by plugging e_1, e_2, f_1, f_2, we get

$$\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

and clearly,

$$\begin{aligned} \mathbb{C}^2 \otimes \mathbb{C}^2 &= \operatorname{span}\{e_1 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_1, e_2 \otimes f_2\} \\ &= \{a(e_1 \otimes f_1) + b(e_1 \otimes f_2) + c(e_2 \otimes f_1) + d(e_2 \otimes f_2)\}, a, b, c, d \in \mathbb{C} \end{aligned}$$

Remark 4.1. According to previous sections, we have used:

$$\begin{aligned} e_1 \otimes f_1 &= |\uparrow\uparrow\rangle \\ e_1 \otimes f_2 &= |\uparrow\downarrow\rangle \\ e_2 \otimes f_1 &= |\downarrow\uparrow\rangle \\ e_2 \otimes f_2 &= |\downarrow\downarrow\rangle \end{aligned}$$

which can be plugged into the equations for $\mathbb{C}^2 \otimes \mathbb{C}^2$ to simplify the calculations.

4.2 Defining Inner Product in $\mathbb{C}^2$[4]

First of all, we start by introducing the definitions of orthogonal vectors and orthonormal vectors.

Definition 4.2. Let V be an inner product space and vectors $\vec{x}$ and $\vec{y}$ are in V, then $\vec{x}, \vec{y}$ are *orthogonal (perpendicular)* if $\langle \vec{x}, \vec{y} \rangle = 0$.

Remark 4.3. A subset S of V is orthogonal if any two distinct vectors in S are orthogonal. Moreover, a vector $\vec{x}$ in *in* V is a unit vector if $||\vec{x}|| = 1$.

Definition 4.4. A subset S of V is *orthonormal* if S is orthogonal and consists entirely of unit vectors.

Next, we will define the inner product on complex space. Given two vectors $\vec{a}, \vec{b} \in \mathbb{C}^2$ such that

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

and we define the inner product $\langle \vec{a}, \vec{b} \rangle$ to be

$$\langle \vec{a}, \vec{b} \rangle = a_1 \bar{b}_1 + a_2 \bar{b}_2$$

Example 4.5. Let $\vec{a} = \begin{pmatrix} 1 \\ i \end{pmatrix}, \vec{b} = \begin{pmatrix} i \\ 1 \end{pmatrix}$, then

$$\langle \vec{a}, \vec{b} \rangle = \left\langle \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix} \right\rangle = 1 \times \bar{i} + i \times \bar{1} = -i + i = 0$$

Remark 4.6. Based on the result of Example 4.5, we know $\vec{a}$ and $\vec{b}$ are orthogonal.

What about the inner product in $\mathbb{C}^2 \otimes \mathbb{C}^2$?

Definition 4.7. In general, we define the inner product in $\mathbb{C}^2 \otimes \mathbb{C}^2$ for the basis elements such that

$$\langle e_{i1} \otimes f_{j1}, e_{i2} \otimes f_{j2} \rangle = \langle e_{i1}, e_{i2} \rangle * \langle f_{j1}, f_{j2} \rangle$$

Below are several examples that illustrate how such operation works.

Example 4.8.

$$\begin{aligned}
\langle e_1 \otimes f_2, e_2 \otimes f_1 \rangle &= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle * \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \\
&= \langle e_1, e_2 \rangle * \langle f_2, f_1 \rangle \\
&= 0
\end{aligned}$$

Example 4.9.

$$\begin{aligned}
\langle e_1 \otimes f_1, e_1 \otimes f_2 \rangle &= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle * \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \\
&= \langle e_1, e_1 \rangle * \langle f_1, f_2 \rangle \\
&= 0
\end{aligned}$$

According to Example 4.8 and Example 4.2, it is easy to prove that given e_i, e_k, f_j, f_l, for some integers i, k, j, l. If $\langle e_i \otimes f_j, e_k \otimes f_l \rangle \neq 0$, then $e_i = e_k, f_j = f_l$. Alternatively, the equation is true only when $i = k, j = l$. Therefore, this indicates that $\{e_1 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_1, e_2 \otimes f_2\}$ is an orthonormal basis in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

5 The Adjoint of a Linear Operator[3]

In this section, we will begin by defining the adjoint of a linear operator as well as self-adjoint operator, which could be helpful for later calculations.

Definition 5.1. Let V be an inner product space, and $y \in V$. The function $g : V \to \mathbb{F}$ defined by $g(x) = \langle x, y \rangle$ is linear.

Proof. Let $x_1, x_2 \in V$, $c \in \mathbb{F}$, then $gc(x_1, x_2) = \langle cx_1 + x_2, y \rangle = c\langle x_1, y \rangle + \langle x_2, y \rangle = cg(x_1) + g(x_2)$. So g is linear. □

Theorem 5.2. *[3] Let V be a finite-dimensional inner-product space over $\mathbb{F}$, and let $g : V \to \mathbb{F}$ be a linear transformation. Then there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$.*

Proof. Let $\beta = \{v_1, v_2, \cdots, v_n\}$ be an orthonormal basis for V. Take $y = \sum_{i=1}^{n} \overline{g(v_i)} v_i$. Define $h : V \to \mathbb{F}$ by $h(x) = \langle x, y \rangle$, which is clearly linear. Moreover, for $1 \leq j \leq n$ we have

$$h(v_j) = \langle v_j, y \rangle = \left\langle v_j, \sum_{i=1}^{n} \overline{g(v_i)} v_i \right\rangle = \sum_{i=1}^{n} g(v_i) \langle v_j, v_i \rangle = g(v_i)\delta_{ji} = g(v_j)$$

Since g and h both agree on β, we have that $g = h$. To show that y is unique, suppose that $g(x) = \langle x, y' \rangle$ for all x. Then, $\langle x, y \rangle = \langle x, y' \rangle$ for all x, so $y = y'$. □

Here is an example that helps illustrate Theorem 5.2.

Example 5.3. Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(a_1, a_2) = 2a_1 + a_2$. Let $\beta = \{e_1, e_2\}$ and let $y = g(e_1)e_1 + g(e_2)e_2 = 2e_1 + e_2 = (2, 1)$. Therefore, based on Theorem 5.2, we have $g(a_1, a_2) = \langle (a_1, a_2), (2, 1)\rangle = 2a_1 + a_2$

Theorem 5.4. *[3] Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Then there exists a unique function $T^* : V \to V$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$. Moreover, we have T^* is linear.*

Proof. Let $y \in V$. Define $g : V \to \mathbb{F}$ by $g(x) = \langle T(x), y\rangle$ for all $x \in V$. we first show that g is linear. Let $x_1, x_2 \in V$ and $c \in \mathbb{F}$. Then we can obtain

$$g(cx_1 + x_2) = \langle T(cx_1 + x_2), y\rangle = \langle cT(x_1) + T(x_2), y\rangle = c\langle T(x_1), y\rangle + \langle T(x_2), y\rangle = cg(x_1) + g(x_2)$$

therefore, g is linear. Now apply Theorem 5.2 to obtain a unique vector $y' \in V$ such that $g(x) = \langle x, y'\rangle$, which is $\langle T(x), y\rangle = \langle x, y'\rangle$ for all $x \in V$. Define $T^* : V \to V$ by $T^*(y) = y'$, then we have $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$. To show that T^* is linear, let $y_1, y_2 \in V$ and $c \in \mathbb{F}$. Then for any $x \in V$ we have

$$\begin{aligned}\langle x, T^*(cy_1 + y_2)\rangle &= \langle T(x), cy_1 + y_2\rangle \\ &= \bar{c}\langle T(x), y_1\rangle + \langle T(x), y_2\rangle \\ &= \bar{c}\langle x, T^*(y_1)\rangle + \langle x, T^*(y_2)\rangle \\ &= \langle x, cT^*(y_1) + T^*(y_2)\rangle\end{aligned}$$

Since x is arbitrary, $T^*(cy_1+y_2) = cT^*(y_1)+T^*(y_2)$. Finally, we need to show that T^* is unique. Suppose that $U : V \to V$ is linear and that it satisfies $\langle T(x), y\rangle = \langle x, U(y)\rangle$ for all $x, y \in V$. Then, $\langle x, T^*(y)\rangle = \langle x, U(y)\rangle$ for all $x, y \in V$, so $T^* = U$. □

Here, the linear operator T^* described in the Theorem 5.4 is called the *adjoint* of the operator T. Therefore, T^* is the unique operator on V satisfying $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$.

Remark 5.5. Based on the above definition, we also have

$$\langle x, T(y)\rangle = \overline{\langle T(y), x\rangle} = \overline{\langle y, T^*(x)\rangle} = \langle T^*(x), y\rangle$$

Therefore, $\langle x, T(y)\rangle = \langle T^*(x), y\rangle$ for all $x, y \in V$.
For an infinite-dimensional inner product space, the adjoint of a linear operator T may be defined to be the function T^* such that $\langle x, T(y)\rangle = \langle T^*(x), y\rangle$ for all $x, y \in V$ if it exists. Although the uniqueness and linearity of T^* follow as before, the existence of the adjoint is not guaranteed.

Theorem 5.6. *Let V be a finite-dimensional inner product space, and let β be an orthonormal basis for V. If T is a linear operator on V, then we have*

$$[T^*]_\beta = [T]^*_\beta$$

Proof. Let $A = [T]_\beta$, $B = [T^*]_\beta$, and $\beta = \{v_1, v_2, \cdots, v_n\}$. Then we have

$$B_{ij} = \langle T*(v_j), v_i\rangle = \overline{\langle v_i, T*(v_j)\rangle} = \overline{\langle T(v_i), v_j\rangle} = \overline{A_{ji}} = (A^*)_{ij}$$

and so $B = A^*$ □

Corollary 5.7. Let A be an $n \times n$ matrix. Then $L_{A^*} = (L_A)^*$.

Proof. If β is the standard ordered basis for $\mathbb{F}^n$,then we have $[L_A]_\beta = A$. Therefore, $[(L_A)^*]_\beta = [L_A]^*_\beta = A^* = [L_{A^*}]_\beta$, and so $(L_A)^* = L_A$. □

Example 5.8. Let T be the linear operator on $\mathbb{C}^2$ defined by $T(a_1, a_2) = (2ia_1 + 3a_2, a_1 - a_2)$. If β is the standard ordered basis for $\mathbb{C}^2$, then

$$[T]_\beta = \begin{pmatrix} 2i & 3 \\ 1 & -1 \end{pmatrix}$$

and so

$$[T^*]_\beta = [T]^*_\beta = \begin{pmatrix} -2i & 1 \\ 3 & -1 \end{pmatrix}$$

therefore

$$T^*(a_1, a_2) = (-2ia_1 + a_2, 3a_1 - a_2)$$

The following theorem suggests several basic operations for adjoint linear operators.

Theorem 5.9. *Let V be an inner product space, and let T and U be linear operators on V. The the following are true:*

$$\begin{aligned}(T+U)^* &= T^* + U^* \\ (cT)^* &= \overline{c}T^* \text{ for any } c \in \mathbb{F} \\ (TU)^* &= U^*T^* \\ T^{**} &= T \\ I^* &= I\end{aligned}$$

Here, we will prove the first and fourth operations and the rest will be similar to these two proofs.

Proof. First, let $x, y \in V$, since

$$\begin{aligned}\langle x, (T+U)^*(y)\rangle &= \langle (T+U)(x), y\rangle \\ &= \langle T(x) + U(x), y\rangle \\ &= \langle T(x), y\rangle + \langle U(x), y\rangle \\ &= \langle x, T^*(y)\rangle + \langle x, U^*(y)\rangle \\ &= \langle x, T^*(y) + U^*(y)\rangle \\ &= \langle x, (T^* + U^*)(y)\rangle\end{aligned}$$

$T^* + U^*$ has the property unique to $(T+U)^*$. Therefore, $T^* + U^* = (T+U)^*$ Now, for the fourth item, since

$$\langle x, T(y)\rangle = \langle T^*(x), y\rangle = \langle x, T^{**}(y)\rangle$$

Hence proved. ☐

Results obtained from Theorem 5.9 can also be applied to matrices.

Corollary 5.10. Let A and B be $n \times n$ matrices. Then

$$\begin{aligned}(A+B)^* &= A^* + B^* \\ (cA)^* &= \overline{c}A^* \text{ for all } c \in \mathbb{F} \\ (AB)^* &= B^*A^* \\ A^{**} &= A \\ I^* &= I\end{aligned}$$

Here is a proof for the third item, and the rest of the items can be proved similarly.

Proof. Since $L_{(AB)^*} = (L_{AB})^* = (L_A L_B)^* = (L_B)^*(L_A)^* = L_{B^*}L_{A^*} = L_{B^*A^*}$, so we have $(AB)^* = B^*A^*$. ☐

Theorem 5.11. *(The Spectral Theorem)[7] Suppose that T is a linear operator on a finite-dimensional inner product space V over $\mathbb{F}$ with the distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$. Assume that T is normal if $\mathbb{F} = \mathbb{C}$ and that T is self-adjoint if $\mathbb{F} = \mathbb{R}$. For each $i (1 \leq i \leq k)$, let W_i be the eigenspace of T corresponding to the eigenvalue λ_i, and let T_i be the orthogonal projection of V on W_i. Then the following statements are true.*

1. *$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$*
2. *If W_i' denotes the direct sum of subspace W_j for $j \neq i$, then $W_i^\perp = W_i'$*
3. *$T_iT_j = \delta_{ij}T_i$ for $1 \leq i,\ j \leq k$*
4. *$I = T_1 + T_2 + \cdots + T_k$*
5. *$T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$*

Before proving this theorem, please recall our original goal of finding an orthonormal basis of eigenvectors of a linear operator T on a finite-dimensional inner product space V. Note that if such an orthonormal basis β exists, then $[T]_\beta$ is a diagonal matrix, and hence $[T^*]_\beta = [T]_\beta^*$ is also a diagonal matrix. Because diagonal matrices commute, we conclude that T and T^* commute. Therefore, if V possesses an orthonormal basis of eigenvectors of T, then $TT^* = T^*T$. Now, consider the following theorems.

Theorem 5.12. *[5] Let T be a linear operator on a finite-dimensional complex inner product space V. Then T is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.*

Proof. Suppose that T is normal. By the fundamental theorem of algebra, the characteristic polynomial of T splits. So we may apply Schur's theorem to obtain an orthonormal basis $\beta = \{v_1, v_2, \cdots, v_n\}$ for V such that $[T]_\beta = A$ is upper triangular. We also know that v_1 is an eigenvector of T because A is upper triangular. Assume that $v_1, v_2, \cdots, v_{k-1}$ are eigenvectors of T. We claim that v_k is also an eigenvector of T. It then follows by mathematical induction on k that all of the v_i's are eigenvectors of T. Consider any $j < k$, and let λ_j denote the eigenvalue of T corresponding to v_j. By previous theorem, $T^*(v_j) = \overline{\lambda}_j v_j$. Since A is upper triangular,

$$T(v_k) = A_{1k}v_1 + A_{2k}v_2 + \cdots + A_{jk}v_j + \cdots + A_{kk}v_k$$

Moreover, we know that

$$A_{jk} = \langle T(v_k), v_j\rangle = \langle v_k, T^*(v_j)\rangle = \langle v_k, \overline{\lambda}_j v_j\rangle = \lambda_j\langle v_k, v_j\rangle = 0$$

It follows that $T(v_k) = A_{kk}v_k$, and hence v_k is an eigenvector of T. So by induction, all the vectors in β are eigenvectors of T. ☐

Theorem 5.13. *Let T be a linear operator on a finite-dimensional real inner product space V. Then T is self-adjoint if and only if there exists an orthonormal basis β for V consisting of eigenvectors of T.*

Proof. Suppose that T is self-adjoint. Then we may apply Schur's theorem to obtain an orthonormal basis β for V such that the matrix $A = [T]_\beta$ is upper triangular. But

$$A^* = [T]_\beta^* = [T^*]_beta = [T]_\beta = A$$

Therefore, A and A^* are both upper triangular, and so A is a diagonal matrix. Thus β must consist of eigenvectors of T. The converse can be proved through similar techniques. ☐

Theorem 5.14. *[8] A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if V is the direct sum of the eigenspaces of T.*

Proof. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the distinct eigenvalues of T. First suppose that T is diagonalizable, and for each i choose an ordered basis γ_i for the eigenspace E_{λ_i}. Then $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is a basis for V, and hence the V is a direct sum of the E_{λ_i}'s. Conversely, assume that V is a direct sum of the eigenspaces of T. For each i, choose an ordered basis γ_i of E_{λ_i}. Then the union $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is a basis for V. Since this basis consists of eigenvectors of T, we conclude that T is diagonalizable. ☐

Now, we can use Theorem 5.12, Theorem 5.13, and Theorem 5.14 to prove the spectral theorem.

Proof. (a) By Theorem 5.12 and Theorem 5.13, T is diagonalizable, so

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

by Theorem 5.14.
(b) If $x \in W_i$ and $y \in W_j$ for some $i \neq j$, then $\langle x, y\rangle = 0$. It follows from this result that $W_i' \subseteq W_i^\perp$. From (a), we have

$$\dim(W_i') = \sum_{j\neq i}\dim(W_j) = \dim(V) - \dim(W_i)$$

On the other hand, we have $\dim(W_i^\perp) = \dim(V) - \dim(W_i)$. So $W_i' = W_i^\perp$. Hence proved.
(c) Suppose that T is a linear operator on a finite-dimensional inner product space V over $\mathbb{F}$ with the distinct

eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$. Assume that T is normal if $\mathbb{F} = \mathbb{C}$ and that T is self-adjoint if $\mathbb{F} = \mathbb{R}$. For each i where $1 \leq i \leq k$, let W_i be the eigenspace of T corresponding to the eigenvalue λ_i and let T_i be the orthogonal projection of V on W_i. Then $T_iT_j = \delta_{ij}T_i$ for $1 \leq i, j \leq k$. It is known that element in the space can be written as

$$v = x_1 + x_2 + \cdots + x_k$$

such that $x_i \in W_i$. If $i \neq j$ then we can write

$$T_iT_j(v) = 0 = \delta_{ij}T_i(v)$$

By the definition of T_i's and theorem which states that if W_i' denotes the direct sum of the subspaces W_j for $j \neq i$, then $W_i^{\perp} = W_i'$. Similarly, if $i = j$, we have

$$T_iT_j(v) = x_i = \delta_{ij}T_i(v)$$

So they are the same.
(d) Since T_i is the orthogonal projection of V on W_i, it follows from (b) that $N(T_i) = R(T_i)^{\perp} = W_i^{\perp} = W_i'$. Hence for $x \in V$, we have $x = x_1 + x_2 + \cdots + x_k$, where $T_i(x) = x_i \in W_i$.
(e) For $x \in V$, write $x = x_1 + x_2 + \cdots + x_k$, where $x_i \in W_i$. Then

$$\begin{aligned} T(x) &= T(x_1) + T(x_2) + \cdots + T(x_k) \\ &= \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k \\ &= \lambda_1 T_1(x) + \lambda_2 T_2(x) + \cdots + \lambda_k T_k(x) \\ &= (\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k)(x) \end{aligned}$$

□

Additional problem: Show that the tensor product of two selfadjoint operators is selfadjoint

Proof. Let S_1 and S_2 be two self-adjoint operators on H_1 and H_2 respectively. Then we have $S_1^* = S_1$ and $S_2^* = S_2$. Without the loss of generosity, assume that S_1 and S_2 are bounded so that H_1 is the domain of S_1 and H_2 is the domain of S_2. Then it follows that

$$\begin{aligned} \langle (S_1 \otimes S_2)(\phi_1 \otimes \phi_2), \varphi_1 \otimes \varphi_2 \rangle &= \langle (S_1, \phi_1) \otimes (S_2, \phi_2), \varphi_1 \otimes \varphi_2 \rangle \\ &= \langle S_1\phi_1, \varphi_1 \rangle_1 \langle S_2\phi_2, \varphi_2 \rangle_2 \\ &= \langle \phi_1, S_1^*\varphi_1 \rangle_1 \langle \phi_2, S_2^*\varphi_2 \rangle_2 \\ &= \langle \phi_1 \otimes \phi_2, (S_1^*, \varphi_1) \otimes (S_2^*, \varphi_2) \rangle \\ &= \langle \phi_1 \otimes \phi_2, (S_1^* \otimes S_2^*)(\varphi_1 \otimes \varphi_2) \rangle \end{aligned}$$

Therefore, $(S_1 \otimes S_2)^* = S_1^* \otimes S_2^* = S_1 \otimes S_2$. □

6 MatLab Implementation

In this section, we will be utilizing MatLab programming language in order to compute the eigenvalues, eigenvectors, as well as diagonalize the matrices obtained from the calculations of kronecker products, inner products, and spin operators. Finally, we will try to discover the relationships among each elements (i.e eigenvalues, eigenvectors, matrix diagonalization, kronecker products, inner products, and spin operators) based on the results from MatLab.[11][2]

6.1 Code and Graphs Part 1

Now, we first create a *for loop* and the the condition for g is set to be range from 0 to 1 with step size 0.01. Such condition for this *for loop* acts like a test-run for our algorithms, meaning that we will use smaller step size run several experiments and based on the results that we have found, we will increase the step size in order to generalize our findings. Then, we create three matrices, which are h, B, H, and then we

```
s_x=[0,1/2;1/2,0];
s_y=[0,-i/2;i/2,0];
s_z=[1/2,0;0,-1/2];
D=0.5;
%this displays first two columns of matrix v when g varies
for g=0:0.01:1
    %decreasing the step size of g does not affect the pattern of distribution, only the density
    h=-1/D*((1+g)*kron(s_x,s_x)+(1-g)*kron(s_y,s_y))+eye(4)/4-kron(s_z,s_z);
    B=-1/D*((1+g)*kron(s_x,kron(eye(4),s_x))+(1-g)*kron(s_y,kron(eye(4),s_y)))+eye(16)/4-kron(s_z,kron(eye(4),s_z));
    H=kron(eye(2),kron(h,eye(2)))+kron(eye(4),h)+kron(h,eye(4));
    u=eig(H+B);
    u1=u(1:4);
    [v, lambda]=eig(H+B);
    disp(vpa(v(1:16,1:2)))
    %this shows more detailed digits for particular entries
    hold all
    scatter(g,u,'X') %the returned graph shows both the upper bound and lower bound of the eigenvalues
    line(g,u)
end
```

Figure 1: S^x, S^y,and S^z are preset in this code, which correspond the values that are calculated in Section 1. D, which is the value of Δ, is set to 0.5 to test the accuracy of the computation. *Note*: This code represents the computation of spin 5[12]

define another variable u which equals to the eigenvalues of the matrix $H + B$, and then we calculate the eigenvectors corresponding to each eigenvalue of the matrix. Also note that since we just start our algorithm and we are still exploring every possibility of the algorithm, we are only interested in the first 4 eigenvalues and each of their eigenvectors.

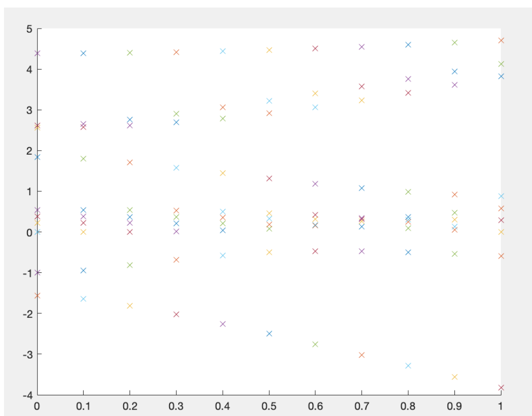

Figure 2: This is the graph that corresponds to the code shown above except the step size of g is 0.1. Its x axis ranges from 0 to 1, which corresponds to the values of g. The y axis ranges from -4 to 5, and it corresponds to the values of u, the eigenvalues of the matrix $H + B$[10]

As we can see from Figure 2, it displays the overall variations in terms of values of g and u, and we can also see the intersections between two individual points. However, since the step size defined in this case is only 0.1, the points are too sparse to determine whether the results are valid. Therefore, in Figure 3, we will redefine the step size of g to be 0.01 and also improve the visualization of each individual point.

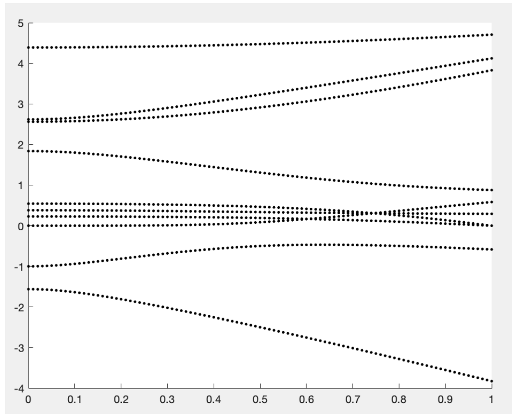

Figure 3: Algorithms that are used to generate this graph is the same as the one shown at the beginning of Section 6.1. The differences are that we change the step size of g so that the graph will show a clearer pattern of the variation. Moreover, the visualization of each point is also optimized so that it will be more pleasant for readers to read the graph

As Figure 3 shows, the overall pattern[6] of each value is clearer to see by only changing the step size of g. Moreover, we can see the starting point of each eigenvalue of the matrix $H+B$, and as the step size changes, the values of h, B, H have also been changed. We can tell that there are few intersections between the step size value 0.7 and 0.8. It is important to notice that these intersections may indicate certain features of spins as well as the nature of our algorithms, and we will continue to investigate and explore this finding in later sections. In the following parts, we will show the code for higher spins as well as each of their corresponding graphs.

Remark 6.1. In Figure 4, we have changed the step size to be 0.001 while the range stays the same. As the figure shows, the distance between each individual dot has become so close that the over change of each eigenvalue has become a “solid” line. The purpose of Figure 4 is to demonstrate how the small changes in step size can largely affect the resulting graph.

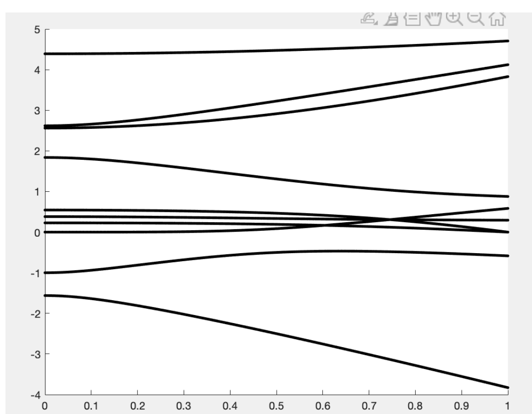

Figure 4: The step size of g has been changed to 0.001 in order to show a clearer pattern of the changes of each eigenvalue. However, the range of g is not changed and other conditions are also kept the same

```
s_x=[0,1/2;1/2,0];
s_y=[0,-i/2;i/2,0];
s_z=[1/2,0;0,-1/2];
for q=0:0.1:1
    g=q;
    %1/D=q;
    h=-q*((1+g)*kron(s_x,s_x)+(1-g)*kron(s_y,s_y))+(eye(4)/4-kron(s_z,s_z));
    B=-q*((1+g)*kron(s_x,kron(eye(8),s_x))+(1-g)*kron(s_y,kron(eye(8),s_y)))+(eye(32)/4-kron(s_z,kron(eye(8),s_z)));
    H=kron(eye(2),kron(h,eye(4)))+kron(eye(8),h)+kron(h,eye(8))+kron(eye(4),kron(h,eye(2)));
    u=eig(H+B);
    u1=u(1:4);
    [v, lambda]=eig(H+B);
    disp(vpa(v(1:32,1:2)))
    hold all
    scatter(g,u,'.k')
    line(g,u)
end
```

Figure 5: Same algorithms as spin 5. The difference is that value of g is set to be as the same as q, which ranges from 0 to 1 with step size 0.1, and $\frac{1}{D}$ is also changed to be equal to q in order to increase the performance of the algorithm

6.2 Code and Graphs Part 2

In this section, we will discuss the algorithms of spin 6 as well as the corresponding graphs.

```
s_x=[0,1/2;1/2,0];
s_y=[0,-i/2;i/2,0];
s_z=[1/2,0;0,-1/2];
D=0.5;
for g=0:0.1:1
    h=-1/D*((1+g)*kron(s_x,s_x)+(1-g)*kron(s_y,s_y))+(eye(4)/4-kron(s_z,s_z));
    B=-1/D*((1+g)*kron(s_x,kron(eye(16),s_x))+(1-g)*kron(s_y,kron(eye(16),s_y)))+(eye(64)/4-kron(s_z,kron(eye(16),s_z)));
    H=kron(eye(2),kron(h,eye(8)))+kron(eye(16),h)+kron(h,eye(16))+kron(eye(16),h)+kron(h,eye(16))+kron(eye(8),kron(h,eye(2)));
    u=eig(H+B);
    u1=u(1:4);
    [v, lambda]=eig(H+B);
    disp(vpa(v(1:64,1:2)))
    hold all
    scatter(g,u,'.k')
    line(g,u)
end
```

Figure 6: This is the original algorithms for spin 6, and as the code shows, all of the parameters stay the same while the variables h, B, H have changed since spin 6 requires the computation at a higher level (i.e. the required Kronecktor products are more than that of spin 5 and the dimension of each matrix has also increased

For spin 6, we still define S_x, S_y, and S_z the same way and the value of D stays the same as well. However, since the matrices h, B, H have become larger than that of spin 5, the time it takes to produce the eigenvalues and graphs is also longer even though the range and step size of g remain unchanged. Based on the actual performance of this algorithm, the time it takes to complete the calculation is similar to that of spin 5.

7 Conclusion

This research represents a foundational step toward the efficient computation and optimization of tensor products using MATLAB. While the methods and techniques presented in this paper provide a robust framework, they also open up a multitude of avenues for future exploration. This initial work lays the groundwork for continued investigation and refinement, ensuring that the challenges associated with high-dimensional tensor computations are progressively addressed. One promising direction is the integration of machine learning algorithms into MATLAB's tensor computation framework. Tensor-based machine learning methods, such as tensor regression, decomposition, and factorization, are gaining traction in data-driven fields. By implementing these algorithms in MATLAB and optimizing them for high-performance environments, researchers can explore novel applications in areas like recommendation systems, biomedical imaging, and natural language processing. Another area for future work involves the incorporation of GPU and distributed computing capabilities. While this study explored basic parallelization strategies, fully leveraging MATLAB's Parallel Computing Toolbox and GPU acceleration has the potential to further improve computational efficiency, particularly for large-scale tensor datasets. Additionally, exploring distributed tensor computations across cloud platforms would enable scalability for big data applications, such as climate modeling and financial analytics. The exploration of advanced decomposition techniques, including tensor networks, hierarchical Tucker decompositions, and tensor train decompositions, is another promising direction. These methods are essential for reducing the computational complexity of high-dimensional tensors and could benefit from MATLAB's ability to handle multidimensional arrays and integrate with external libraries. Expanding the use of such decompositions will also facilitate applications in quantum computing, control systems, and simulation-based optimization. Furthermore, ongoing development of user-friendly MATLAB toolboxes and visualization tools tailored for tensor computations could enhance accessibility for researchers and practitioners from diverse domains. Intuitive interfaces, real-time visualization of tensor manipulations, and automated workflow generation would greatly accelerate research and application development. The research presented in this paper marks the beginning of an ambitious journey. Future studies will build upon the results and insights gained here, aiming to address the limitations encountered and to explore new methodologies. Collaboration with interdisciplinary teams and the adaptation of MATLAB's tensor tools to emerging fields will be crucial in ensuring that this line of research evolves dynamically. By continuing to enhance computational techniques, integrating cutting-edge optimization methods, and expanding the range of applications, this research will contribute significantly to the ongoing advancement of tensor analysis and its practical implementations in scientific and engineering domains.

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