**EECS 660 In-class Exam 1 Stable Matching**

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**(1)** Recall the GS algorithm for stable matching. Given men and women as well as their respective rankings as the input, prove that the matching output by the GS algorithm is perfect. That is, every man is matched with exactly one woman, and no free man nor free women upon the termination of the algorithm. (50%)

# Here are some lemmas that will be used in both questions

1. **Proof the algorithm will terminate (Direct proof)**
   1. **Since each man will not propose to the same woman more than once and the number of man and woman are finite, thus the algorithm will terminate.**
2. **Proof the woman will not be freed once she is matched (Direct proof)**
   1. **There is no rule in the algorithm that says woman will be freed, thus the proposition is true.**
3. **Proof a matched woman only trade-up, means she will never get a less preferred partner than her current one (Direct proof)**
   1. **Woman can reject proposal if the man propose to her is less favorite than her current match**

# Proof starts here:

**The proposition will be proofed by contradiction, so first, let an unmatched man m, then there also exists another free woman since one man can only be matched with one woman, the free woman will be called w.**

**Since w is unmatched so she must not be proposed, if so, she will be at least matched with someone. Because the algorithm will terminate eventually (proofed above), it means all men have proposed to all women, that implies that m has proposed to all women, which will contradict the fact that w is never proposed to. Thus, no man or woman will be unmatched upon termination of algorithm.**

**(2)** Given men and women as well as their respective rankings as the input, prove that the matching output by the GS algorithm is stable. (50%)

# proof starts here

**At the very beginning, we will proof the algorithm will terminate and woman will only trade-up. And both are proofed directly in part (1).**

**The proposition will be proofed by contradiction. First, suppose there exits an unstable pair (m, w’) that will cause algorithm unstable. Upon algorithm termination, m is paired with w and m’ is paired with w’. We will get pair (m, w) and (m’, w’) in the matching progress. That implies that m prefers w’ than w and w’ prefers m than m’. From here, it will have two outcomes:**

1. **m has not yet proposed to w’, and man will propose women by favorite descension (most favorite to least favorite), so m must prefer w over w’, this will contradict to that m likes w’ better than w.**
2. **m has proposed to w’, in this case, no matter m got rejected immediately or late, w’ prefers m’ over m since women will not be freed, they only trade-up. This will contradict with the fact that w’ like m better than m’.**

**Since both cases lead to contradiction, thus, we can conclude that there exists no unstable match upon termination of the algorithm.**