Department of Mathematics at University of California, San Diego

# Computational Finance

Homework #3b

Instructor: Professor Ruth Williams [Lecture **6**]

Due by 9 pm (PST), Wednesday, February 16, 2022.

**All of the derivatives considered in this homework are of European type.**

**1.** [**Lookback Options**]

The payoffs of Lookback options depend on the maximum or the minimum underlying asset price reached during the life of the option. There are several types of such options. The two basic types are the floating strike Lookback options and the fixed strike Lookback options.

The payoff of a floating strike Lookback call option is the difference between the minimum underlying asset price $S_{\min} = \min_{0\le t\le T} S_t$ achieved during the life of the option and the final asset price $S_T$. Thus the payoff is

$$\text{Float Call } = (S_T - \min_{0\le t\le T} S_t).$$

Similarly, the payoff of a floating strike Lookback put option is the difference between the maximum underlying asset price $S_{\max} = \max_{0\le t\le T} S_t$ achieved during the life of the option and the final asset price $S_T$. Hence, the payoff is

$$\text{Float Put } = (\max_{0\le t\le T} S_t - S_T).$$

The payoff of a fixed strike Lookback option has similar payoff to that of a standard option, with strike price $K$, except that the final underlying asset price $S_T$ is replaced by the maximum and minimum asset price reached during the life of the option for a call and put, respectively. Thus, the payoffs of the fixed strike Lookback options are the following

$$\text{Fix Put } = \max(K - \min_{0\le t\le T} S_t, 0), \qquad \text{Fix Call } = \max(\max_{0\le t\le T} S_t - K, 0).$$

In this homework we first apply the dynamic Monte Carlo simulation for approximate pricing of lookback options in the context of the Black Scholes model. When the sampling frequency increases, the results of the Monte Carlo simulation approach the theoretical value.

(a) Using a dynamic Monte Carlo approximation for the stock price in the Black-Scholes model, create a Jupyter notebook to approximate the initial price of a floating strike Lookback call. The initial arbitrage free price is given by

$$e^{-rT}\mathbb{E}^*\left[S_T - \min_{0\le t\le T} S_t\right],$$

where the option maturity is $T$ years, and the stock, under the risk neutral probability, follows the Black-Scholes model, i.e.

$$S_t = S_0 \exp\left\{(r-\sigma^2/2)t + \sigma W(t)\right\}, \qquad t\in[0,T],$$

where $r$ is the risk-free interest rate and $\{W(t), t\in[0,T]\}$ is a standard Brownian motion.

(b) Use the code from part (a) to price a floating strike Lookback call with $S_0 = 40$, interest rate $r = 0.1$, volatility $\sigma = 0.2$ and $T = 10$.

**2.** [**Discrete time Asian Options**]
Asian options are options in which the payoff involves an average (over time) of the underlying asset. Asian options are of particular importance for commodity products which have low trading volumes (e.g. crude oil), since price manipulation is inhibited. Since there are no known closed form analytical expressions for pricing Asian options in general, numerical methods are usually applied.

(a) Create a Jupyter notebook for pricing an Asian put option whose payoff at $T$ is given by

$$\max\left\{K - \frac{1}{T}\int_0^T S_t\,dt, 0\right\},$$

and the stock price follows the Black-Scholes model (same as in Exercise 1). The initial arbitrage free price for the option is given by

$$e^{-rT}\mathbb{E}^*\left[\max\left(K - \frac{1}{T}\int_0^T S_t\,dt, 0\right)\right].$$

(b) For the same parameters as in part (b) of Exercise 1, find the arbitrage free initial price for the Asian put option described in 2(a) with strike price $K = 29$.

**3.** [**Constant elasticity of variance model.**] The constant elasticity of variance (CEV) model was introduced by John Cox. It attempts to capture stochastic volatility and leverage effects. The model is used by practitioners in the financial industry, especially for modelling equities and commodities.

The CEV model specifies the price for a single risky asset $\{S_t, t\in[0,T]\}$ and has an accompanying riskless asset price process $\{B_t = e^{rt}, t\in[0,T]\}$, the same as in the Black-Scholes model. Under a risk neutral probability, the risky asset evolves according to the following stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t^{\gamma} dW_t,$$

where $\sigma > 0$, $\gamma > 0$. The case $\gamma = 1$ corresponds to the Black-Scholes model. When $\gamma \neq 1$, we can think of $\sigma S_t^{\gamma-1}$ as a state dependent volatility (replace $\sigma$ by $\sigma S_t^{\gamma-1}$ in the Black-Scholes equation). When $\gamma < 1$, we see the so-called leverage effect, commonly observed in equity markets, where the volatility of a stock increases as its price falls. Conversely, in commodity markets, we often observe $\gamma > 1$, the so-called inverse leverage effect, whereby the volatility of the price of a commodity tends to increase as its price increases. When $\gamma < 1$, $S_t$ will eventually reach zero and stay there forever after, corresponding to bankruptcy. When $\gamma \geq 1$, $S_t$ will never reach zero starting from $S_0 > 0$.

(a) For $\gamma > 1$, create a Jupyter notebook to use an Euler approximation to approximate the sample paths of $S_t$ under the CEV model. Draw some sample plots for $T = 5$, $\gamma = 1.4$, $r = 0.1$, $S_0 = 20$, $\sigma = 0.1$.

(b) Add to the code developed in 3(a) to approximate the value of

$$E^*[(\max_{0 \leq t \leq T} S_t - K)^+]$$

for the CEV model. Illustrate for the parameters in (a) and for $K = 18$.