# Example_Week6

February 14, 2022

### 0.0.1 Simulate stock prices under BS model

1. Simulate $\{Z_i\}_{i=1}^n$ that are i.i.d $\mathcal{N}(0,1)$.
2. $S_i(T) = S_0 \exp(\sigma\sqrt{T}Z_i - \frac{1}{2}\sigma^2 T + rT), i = 1, \ldots, n$. Then we have a sample of size $n$.
3. Compute the payoff $X$ based on the stock price for each $i = 1, \ldots, n$: $X_i = H(S_i(T))$ where $H$ is a function.
4. Compute the unique arbitrage free initial price: $V_0 = e^{-rT}E^*[X]$.

Note: another way to write $S_i(T) = S_0 \exp((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z_i), i = 1, \ldots, n$.

[1]:
```
import numpy as np
```

[2]:
```
# Set parameters
T = 7  # time to maturity
S0 = 20  # initial stock price
K = 20  # strike price
r = 0.1  # interest rate
sig = 1  # standard deviation (volatility)
M = 10000  # number of Monte Carlo sample
```

[3]:
```
# Simulate the stock prices for European derivative (with final payoff)
np.random.seed(123)
Z = np.random.normal(0, 1, M)
ST = S0 * np.exp(sig * np.sqrt(T) * Z - 0.5 * sig ** 2 * T + r * T)
```

[4]:
```
Z.shape
```

[4]:
```
(10000,)
```

[5]:
```
ST.shape
```

[5]:
```
(10000,)
```

[6]:
```
# European Call Option
X = np.maximum(ST - K, 0)
ECO = np.exp(-r * T) * np.mean(X)
```

[7]:
```
print('The unique arbitrage free initial price for this European call option is:
 ↪', round(ECO, 2))
```

The unique arbitrage free initial price for this European call option is: 17.31

**What if the option pricing is path dependent?**

$$S_t = S_{t-\Delta t}\exp((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}z_t)$$

where Z_t is a sample from $\mathcal{N}(0,1)$.

Recursively, one can compute and get

$$\begin{aligned}
S_{\Delta t} &= S_0\exp((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}z_1) && (1)\\
S_{2\Delta t} &= S_0\exp((r - \frac{1}{2}\sigma^2)2\Delta t + \sigma\sqrt{\Delta t}z_1 + \sigma\sqrt{\Delta t}z_2) && (2)\\
&= S_0\exp((r - \frac{1}{2}\sigma^2)2\Delta t + \sigma\sqrt{\Delta t}(z_1 + z_2))... && (3)\\
S_{k\Delta t} &= S_0\exp((r - \frac{1}{2}\sigma^2)k\Delta t + \sigma\sqrt{\Delta t}\sum_{i=1}^{k} z_k) && (4)
\end{aligned}$$

[12]:
```
# Simulate the stock prices for Asian call option (depending on the price path)
dt = 0.1  # the increment in discretization
N = int(T / dt)  # the number of discrete times
Zt = np.random.normal(0, 1, size=(M, N))  # simulate all Zi's (Brownian Motion␣
↪terms)
St = S0 * np.exp((r - sig ** 2 / 2) * np.arange(dt, T+dt, dt) + sig * np.
↪sqrt(dt) * np.cumsum(Zt, axis=1))
```

[13]:
```
print('St shape:', St.shape)
```

```
St shape: (10000, 70)
```

[16]:
```
St[0,:]
```

[16]:
```
array([32.86463877, 36.06587221, 48.19699436, 23.69478839, 15.15812405,
        9.72692876,  7.9625108 ,  7.207133  ,  7.31171033,  6.82549455,
        4.24056315,  4.91874443,  7.4312107 ,  9.64734507, 15.81770908,
       12.51928609, 12.96868407, 13.89162223, 13.66111984,  8.52411252,
       11.31594824, 11.33865305, 10.49972985, 10.91038888, 12.47046165,
       10.76632689,  7.31460034,  4.83259778,  4.6280983 ,  6.77520074,
        9.42866497,  8.53444996,  5.03865529,  4.83189404,  4.8335845 ,
        2.74604709,  3.18680749,  2.85385836,  4.81883222,  5.94167054,
        5.35307431,  4.40530547,  2.88742722,  2.09821812,  1.57122675,
        1.33452674,  0.78954519,  0.47372575,  0.59536307,  0.68498016,
        0.53673413,  0.37819426,  0.24950075,  0.27107007,  0.17496992,
        0.13616812,  0.21183195,  0.0845522 ,  0.10286046,  0.14632622,
        0.14664024,  0.20122238,  0.16525073,  0.11299399,  0.12117562,
        0.12891047,  0.10337493,  0.09096926,  0.12126403,  0.13112629])
```

[17]:
```
# Monte Carlo
AsianCall = np.maximum(np.mean(St,axis=1) - K, 0);  # simulated payoff at␣
↪maturity
AsC0 = np.mean(AsianCall) / np.exp(T * r);  #estimated value at time zero
```

[18]:
```
AsC0
```

[18]: 9.175569917887975

## 0.1 Simulate Stock Prices under BS model

[19]:
```
import numpy as np
import seaborn as sns
import matplotlib.pyplot as plt
%matplotlib inline
```

**option pricing is path dependent (dynamic Monte Carlo)**

$$S_t = S_{t-\Delta t}\exp((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}z_t)$$

where Z_t is a sample from $\mathcal{N}(0,1)$.

Parameters from the problem - T = ? - S0 = ? - K = ? - r = ? - sig = ?

Other parameters - dt = ? the increment in discretization - M = ? number of Monte Carlo sample path - N = ? number of steps

### 0.1.1 Example: Up-and-out call option

$$X = \text{ final payoff at } T = \max\left(S_T - K, 0\right)\mathbf{1}_{\{S_t \leq b \text{ for all } t\in[0,T]\}}.$$

Approximate $X = F(S_t : 0 \leq t \leq T)$ by

$$\tilde{X} = \max\left(\tilde{S}_{t_N} - K, 0\right)\mathbf{1}_{\{\tilde{S}_{t_0}\leq b,\ldots,\tilde{S}_{t_N}\leq b\}} = \tilde{F}(\tilde{S}_{t_0},\ldots,\tilde{S}_{t_N}).$$

$$E^*X = E^*[\tilde{X}] = \frac{1}{n}\sum_{i=1}^{n}\tilde{X}^{(i)}$$

[20]:
```
# Set parameters
T = 5 # time to maturity
S0 = 20  # initial stock price
K = 23  # strike price
r = 0.1  # interest rate
sig = 0.2  # standard deviation (volatility)
M = 10000  # number of Monte Carlo sample paths
```

[21]:
```
# Simulate the stock prices
# Dynamic Monte Carlo
dt = 0.1  # the increment in discretization
N = int(T / dt)  # the number of discrete time steps
np.random.seed(7)
Zt = np.random.normal(0, 1, size=(M, N))  # simulate all Zi's (Brownian Motion␣
↪terms)

# One way: Recursive step hidden in np.cumsum and np.arange
St = S0 * np.exp((r - sig ** 2 / 2) * np.arange(dt, T+dt, dt) + sig * np.
↪sqrt(dt) * np.cumsum(Zt, axis=1))
```

$$S_t = S_{t-\Delta t} \exp((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} z_t)$$

where Z_t is a sample from $\mathcal{N}(0,1)$.

[22]:
```
# The other way (following the recursive formula, directly)
St = np.zeros((M, N+1))
St[:,0] = S0
for i in range(1, N+1, 1):
    St[:, i] = St[:, i-1] * np.exp((r - 0.5 * sig ** 2) * dt + sig * np.
↪sqrt(dt) * Zt[:, i-1])

print('stock price matrix dimension:', St.shape)
```

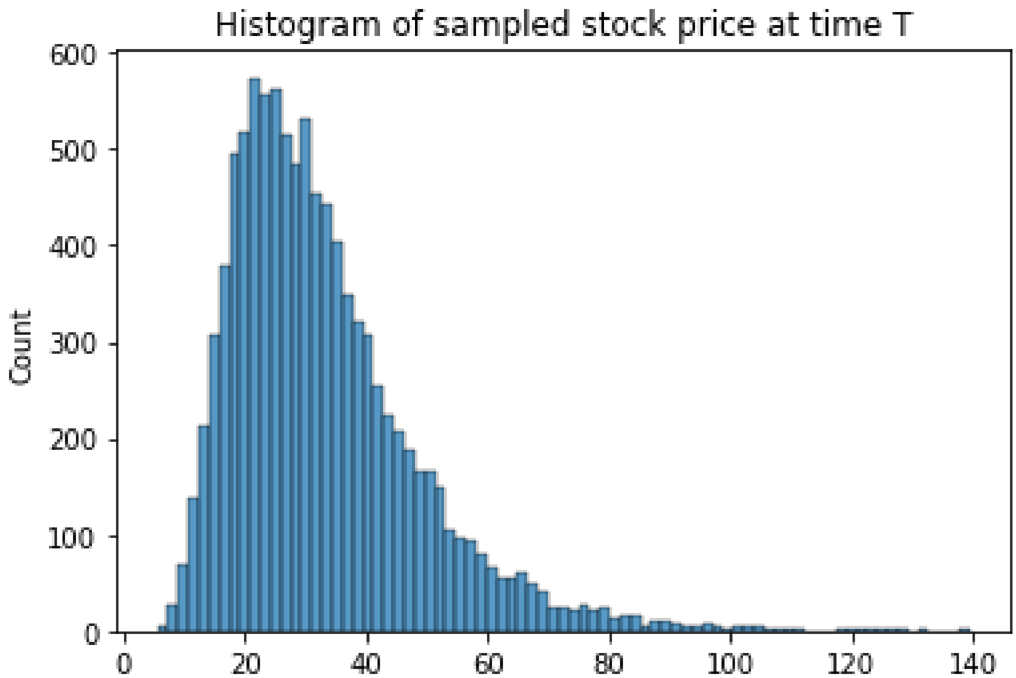
```
stock price matrix dimension: (10000, 51)
```

[23]:
```
Zt.shape
```

[23]:
```
(10000, 50)
```

[24]:
```
sns.histplot(St[:, -1]).set(title='Histogram of sampled stock price at time T')
```

[24]:
```
[Text(0.5, 1.0, 'Histogram of sampled stock price at time T')]
```




[25]:
```
b = 80
```

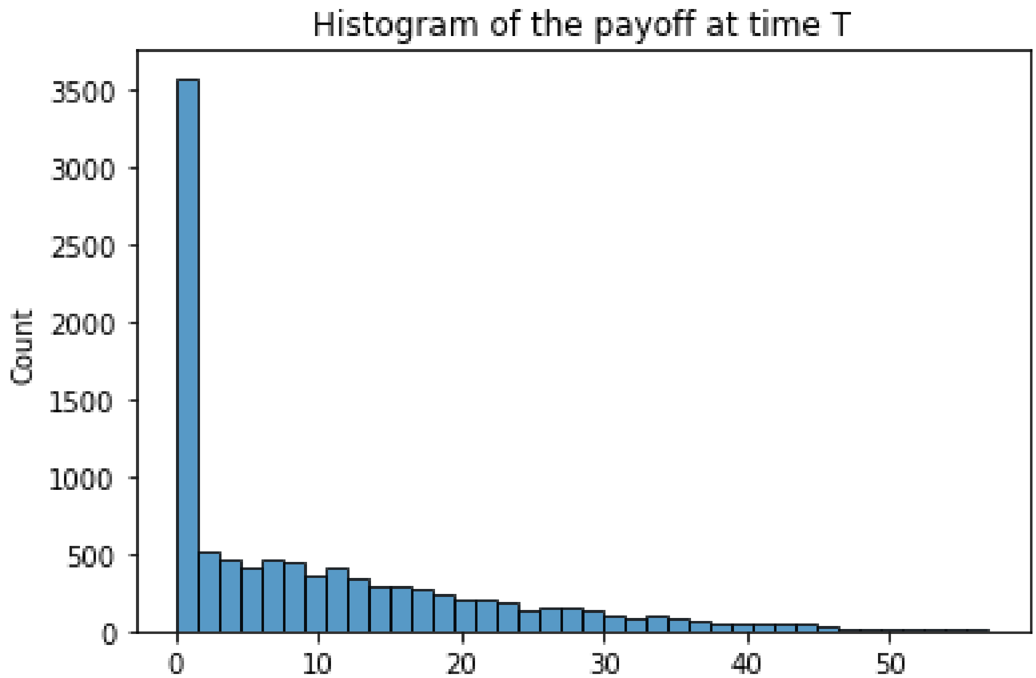
[26]:
```
X = (np.max(St, axis=1) <= b) * np.maximum(St[:,-1] - K, 0)
est_X = np.mean(X)
```

[27]:
```
len(X)
```

[27]: 10000

[28]:
```
sns.histplot(X).set(title='Histogram of the payoff at time T')
```

[28]: [Text(0.5, 1.0, 'Histogram of the payoff at time T')]




```
[29]: est_X
```

```
[29]: 10.078868042106162
```

### 0.1.2 Example: Asian Call Option

$$X = \max\left(\frac{1}{T}\int_0^T S_t dt - K, 0\right)$$

- Let $A_T = \frac{1}{T}\int_0^T S_t dt$ and approximate $A_T$ by $\tilde{A}_T = \frac{1}{N}\sum_{j=1}^N \tilde{S}_{t_j-1}$. - Approximate $X$ by $\tilde{X} = \max\left(\frac{1}{N}\sum_{j=1}^N \tilde{S}_{t_j-1} - K, 0\right)$. - $E^*[X] \approx E^*[\tilde{X}]$.

```
[30]: # Monte Carlo
AsianCall = np.maximum(np.mean(St,axis=1) - K, 0);  # simulated payoff at␣
↪maturity
AsC0 = np.mean(AsianCall) / np.exp(T * r);  # estimated arbitrage free initial␣
↪price
```

```
[31]: AsC0
```

```
[31]: 2.6545247527884634
```

## 0.2 Simulations of solutions of stochastic differential equations

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dWt, \quad t \in [0, T]$$

For Black-Scholes model, $X_t = S_t, \mu(t, X_t) = rX_t, \sigma(t, X_t) = \sigma X_t$.

**Example**

$$dX_t = \alpha(\beta - X_t)dt + \sigma X_t^\gamma dW_t,$$

where $\alpha, \beta, \gamma$ are non-negative constants.

**Euler Scheme**

Fix $N > 0$, let $\Delta t = \frac{T}{N}$ and $t_j = j\Delta t, j = 0, 1, \ldots, N$.

Goal: Approximate solutions of $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dWt, \quad t \in [0, T]$.

Idea: $\Delta X_t \approx \mu(t, X_t)\Delta t + \sigma(t, X_t)\Delta W_t$

Steps: 1. $\tilde{X}_0 = X_0$ 2. $\tilde{X}_{t_j} = \tilde{X}_{t_{j-1}} + \mu(t_{j-1}, \tilde{X}_{t_{j-1}})\Delta t + \sigma(t_{j-1}, \tilde{X}_{t_{j-1}})\sqrt{\Delta t}Z_j$ where $\{Z_j, j = 1, \ldots, N\}$ are i.i.d. $\mathcal{N}(0, 1)$. 3. For $t \in [0, T]$, let $\tilde{X}_t = \tilde{X}_{t_{j-1}}$ for $t_{j-1} \le t < t_j$.

[32]:
```
alpha = 1
beta = 1
gamma = 0.5
X0 = 3
T = 6
N = 10000
M = 10 #10 sample paths
sigma = 1
```

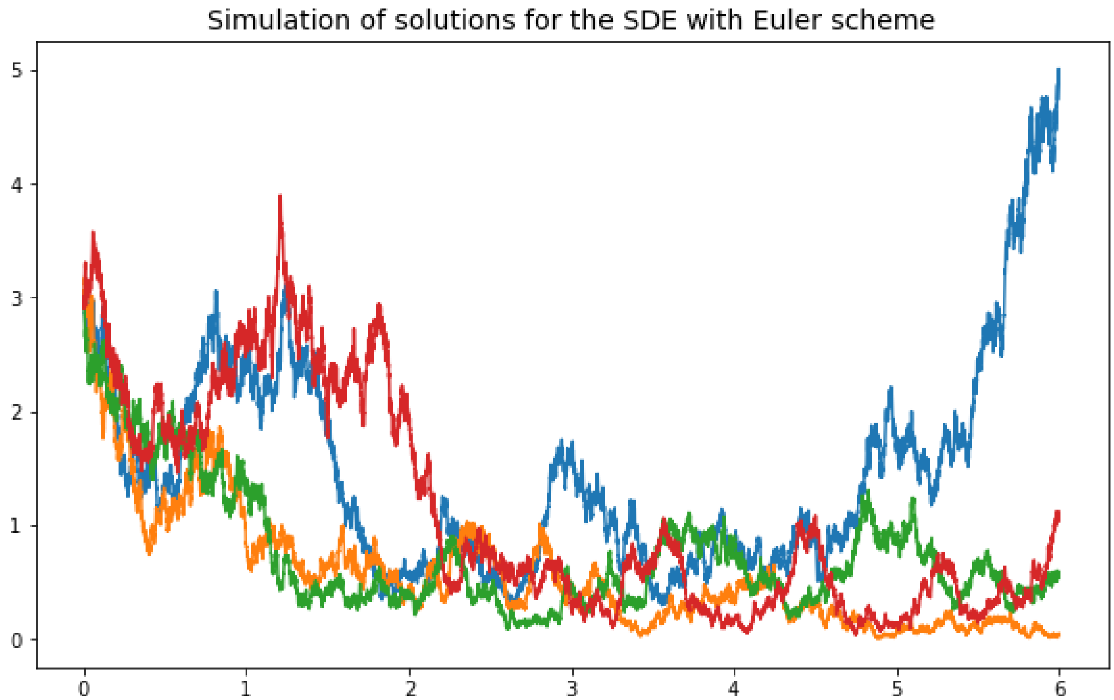
[33]:
```
step = T / N #dt
np.random.seed(111)
Z = np.random.normal(0, 1, size=(M, N))
paths = np.zeros((M, N+1))
paths[:, 0] = X0
for i in range(1, N+1, 1):
    paths[:, i] = paths[:, i-1] + alpha * (beta - paths[:, i-1]) * step + sigma␣
 ↪* paths[:, i-1] ** gamma * np.sqrt(step) * Z[:, i-1]
```

[34]:
```
paths.shape
```

[34]: (10, 10001)

[35]:
```
plt.figure(figsize=(10,6))
t = np.linspace(0, T, N+1)
plt.plot(t, paths[0, :])
plt.plot(t, paths[1, :])
plt.plot(t, paths[2, :])
plt.plot(t, paths[3, :])
plt.title('Simulation of solutions for the SDE with Euler scheme', fontsize=14)
```

[35]: Text(0.5, 1.0, 'Simulation of solutions for the SDE with Euler scheme')

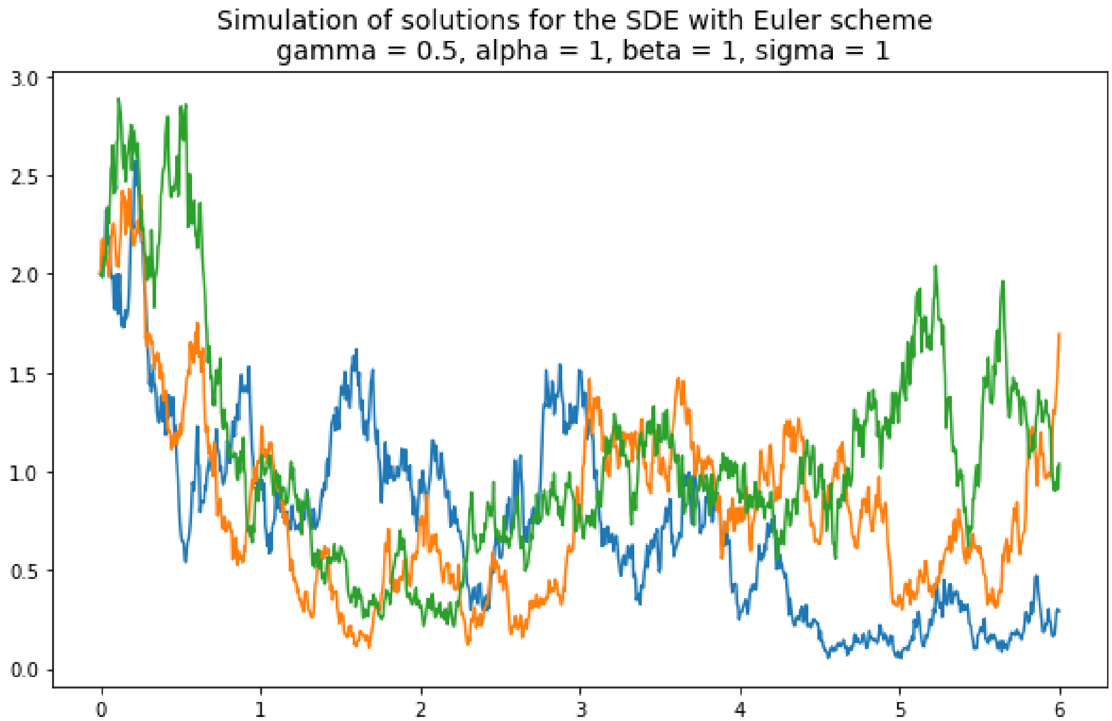



[36]:
```
def generate_paths(alpha, beta, gamma, X0, T, N, M, sigma, seed):
    step = T / N
    np.random.seed(seed)
    Z = np.random.normal(0, 1, size=(M, N))
    paths = np.zeros((M, N+1))
    paths[:, 0] = X0
    for i in range(1, N+1, 1):
        paths[:, i] = paths[:, i-1] + alpha * (beta - paths[:, i-1]) * step +␣
↪sigma * paths[:, i-1] ** gamma * np.sqrt(step) * Z[:, i-1]
    return paths
```

[37]:
```
paths = generate_paths(1, 1, 0.5, 2, 5, 10**3, 10, 1, 7)
```

[38]:
```
plt.figure(figsize=(10,6))
t = np.linspace(0, T, 10**3+1)
plt.plot(t, paths[1, :])
plt.plot(t, paths[3, :])
plt.plot(t, paths[5, :])
plt.title("Simulation of solutions for the SDE with Euler scheme \n gamma = 0.
↪5, alpha = 1, beta = 1, sigma = 1",
          fontsize=14)
```

[38]:
```
Text(0.5, 1.0, 'Simulation of solutions for the SDE with Euler scheme \n gamma =
0.5, alpha = 1, beta = 1, sigma = 1')
```




**Example: Multi-dimensional/multi-factor models**

$$dr_t = \alpha(\mu - r_t)dt + \sqrt{\nu_t}dW_t^{(1)}$$
$$d\nu_t = \beta(\bar{\mu} - \nu_t)dt + \sigma\sqrt{\nu_t}dW_t^{(2)}$$

**Euler scheme steps**: 1. $\Delta t = \frac{T}{N}, \tilde{r}_{t_0} = r_0, \tilde{\nu}_{t_0} = \nu_0$. 2. for $j = 1, \ldots, N$,

$$\tilde{r}_{t_j} = \tilde{r}_{t_{j-1}} + \alpha(\mu - \tilde{r}_{t_{j-1}})\Delta t + \sqrt{\tilde{\nu}_{t_{j-1}}}\sqrt{\Delta t}Z_j^{(1)} \tag{5}$$
$$\tilde{\nu}_{t_j} = \tilde{\nu}_{t_{j-1}} + \beta(\bar{\mu} - \tilde{\nu}_{t_{j-1}})\Delta t + \sigma\sqrt{\tilde{\nu}_{t_{j-1}}}\sqrt{\Delta t}Z_j^{(2)}, \tag{6}$$

where $\{(Z_j^{(1)}, Z_j^{(2)}\}_{j=1}^N$ are from i.i.d bivariate (2-dim) normal distribution with covariance matrix $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, $\rho \in (-1, 1)$. To achieve this, we can let $\{Y_j^{(1)}\}_{j=1}^N, \{Y_j^{(2)}\}_{j=1}^N$ be two independent sequences of i.i.d $\mathcal{N}(0,1)$ random variables. Then define $Z_j^{(1)} = Y_j^{(1)}$ and $Z_j^{(2)} = \rho Y_j^{(1)} + \sqrt{1-\rho^2}Y_j^{(2)}$.

[39]:
```
alpha = 0.2; beta = 0.5; mu = 1; bmu = 2;
r0 = 10; v0 = 15; T = 10; N = 10000
M = 10 #10 sample path pairs
sigma = 0.1
rho = 0.3
dt = T / N
```

[40]:
```
np.random.seed(111)
Y = np.random.normal(0, 1, size=(M, 2 * N))
Z1 = Y[:, 0:N]
Z2 = rho * Y[:, 0:N] + np.sqrt(1 - rho ** 2) * Y[:, N:]
```

[41]:
```
r = np.zeros((M, N+1))
v = np.zeros((M, N+1))
r[:, 0] = r0
v[:, 0] = v0
for i in range(1, N+1, 1):
    r[:, i] = r[:, i-1] + alpha * (mu - r[:, i-1]) * dt + np.sqrt(v[:, i-1]) *␣
↪np.sqrt(dt) * Z1[:, i-1]
    v[:, i] = v[:, i-1] + beta * (bmu - v[:, i-1]) * dt + sigma * np.sqrt(v[:,␣
↪i-1]) * np.sqrt(dt) * Z2[:, i-1]
```

[42]:
```
r.shape
```

[42]:
```
(10, 10001)
```

[43]:
```
v.shape
```

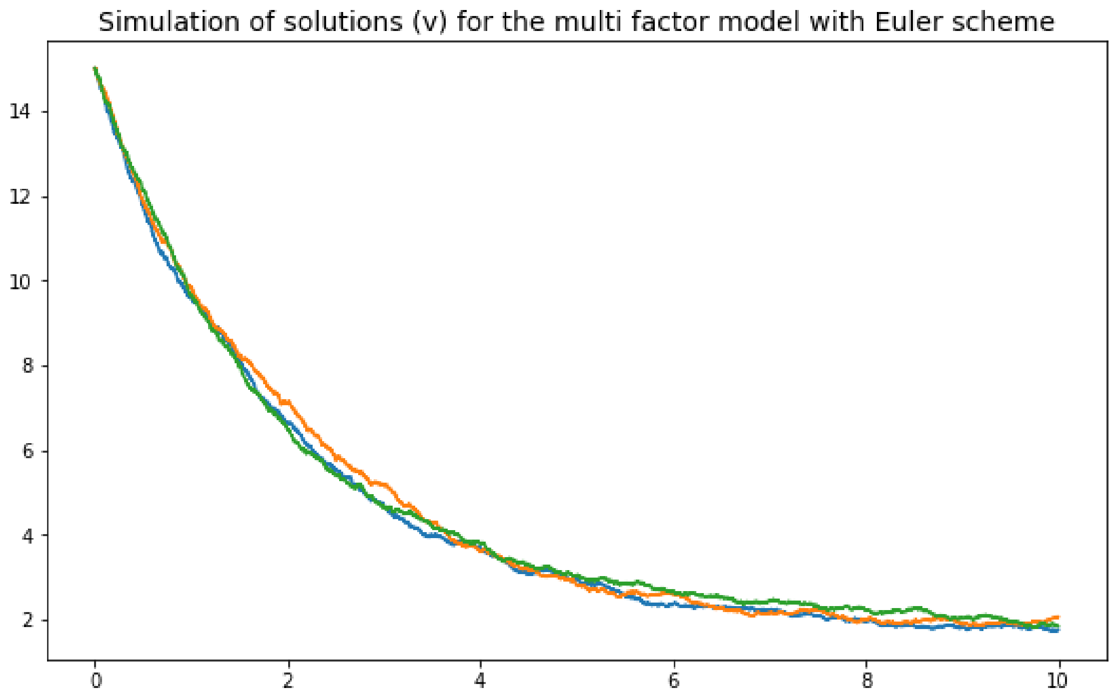
[43]:
```
(10, 10001)
```

[44]:
```
plt.figure(figsize=(10,6))
t = np.linspace(0, T, N+1)
plt.plot(t, r[0, :])
plt.plot(t, r[1, :])
plt.plot(t, r[2, :])
plt.title('Simulation of solutions (r) for the multi factor model with Euler␣
↪scheme', fontsize=14)
```

[44]:
```
Text(0.5, 1.0, 'Simulation of solutions (r) for the multi factor model with
Euler scheme')
```


[45]:
```
plt.figure(figsize=(10,6))
t = np.linspace(0, T, N+1)
plt.plot(t, v[0, :])
plt.plot(t, v[1, :])
plt.plot(t, v[2, :])
plt.title('Simulation of solutions (v) for the multi factor model with Euler␣
 ↪scheme', fontsize=14)
```

[45]:
```
Text(0.5, 1.0, 'Simulation of solutions (v) for the multi factor model with
Euler scheme')
```




## 0.3 Milstein Scheme

Higher order correction scheme to the Euler scheme.

**Milstein Approximation for** $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$

$$\tilde{X}_{t_j} = \tilde{X}_{t_{j-1}} + \mu(t_{j-1}, \tilde{X}_{t_{j-1}})\Delta t + \sigma(t_{j-1}, \tilde{X}_{t_{j-1}})\sqrt{\Delta t}Z_j + \frac{1}{2}\sigma(t_{j-1}, \tilde{X}_{t_{j-1}})\sigma_x(t_{j-1}, \tilde{X}_{t_{j-1}})\Delta t(Z_j^2 - 1), \tag{7}$$

where $\{Z_j\}_{j=1}^N$ are i.i.d $\mathcal{N}(0, 1)$ random variables.

**Example: Milstein Approximation for Geometric Brownian Motion**

Idea: $\Delta X_t \approx \mu X_t \Delta t + \sigma X_t \Delta W_t + 0.5\sigma^2 X_t \Delta t(Z_j^2 - 1)$

Steps: 1. $\tilde{X}_0 = X_0$ 2. $\tilde{X}_{t_j} = \tilde{X}_{t_{j-1}} + \mu\tilde{X}_{t_{j-1}}\Delta t + \sigma\tilde{X}_{t_{j-1}}\sqrt{\Delta t}Z_j + 0.5\sigma^2\tilde{X}_{t_{j-1}}\Delta t(Z_j^2 - 1)$ where $\{Z_j, j = 1, \ldots, N\}$ are i.i.d. $\mathcal{N}(0, 1)$. 3. For $t \in [0, T]$, let $\tilde{X}_t = \tilde{X}_{t_{j-1}}$ for $t_{j-1} \leq t < t_j$.

[46]:
```
# SDE model parameters
mu, sigma, X0 = 1, 1, 2

# Simulation parameters
T, N = 6, 2**8
dt = T / N
t = np.arange(dt, T + dt, dt)  # Start at dt because Y = X0 at t = 0
```

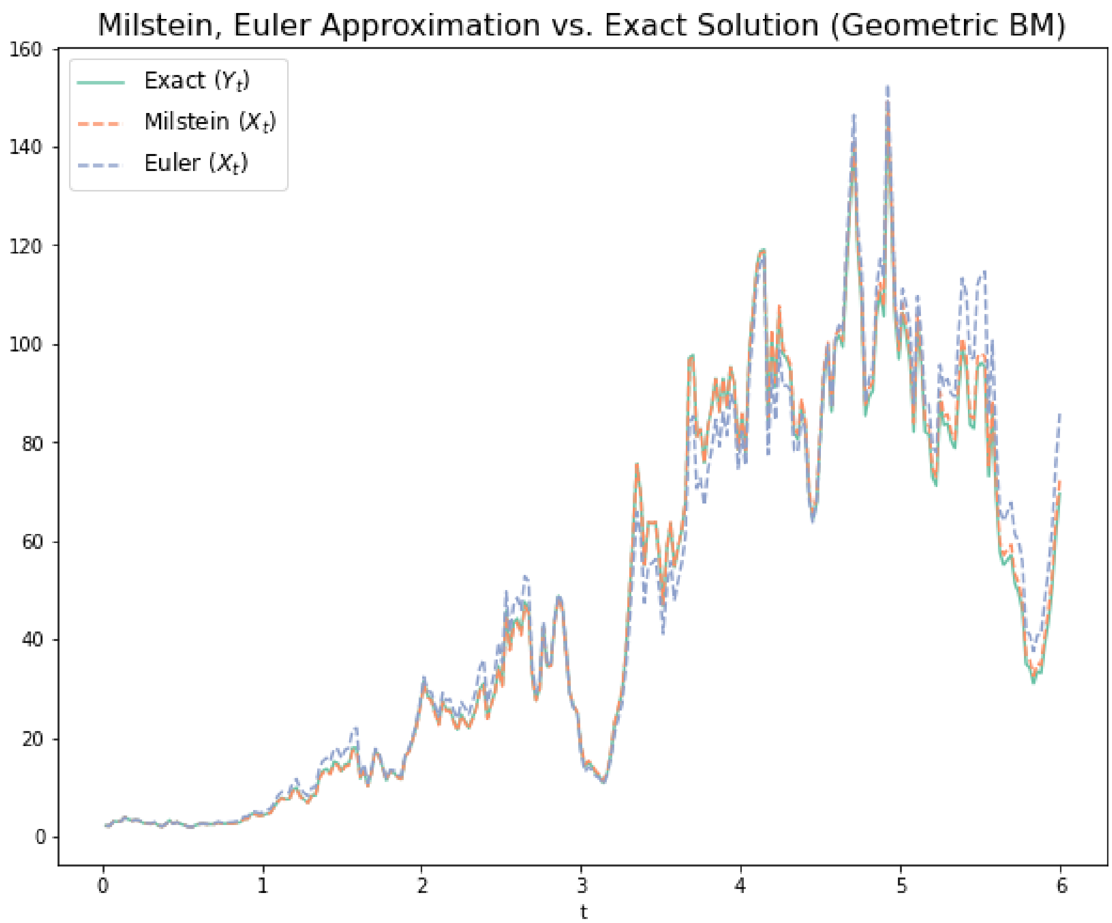
[47]:
```
# Create Brownian Motion
np.random.seed(5)
Z = np.random.normal(0, 1, N)
B  = np.cumsum(np.sqrt(dt) * Z)
```

[48]:
```
# Exact Solution
Y = X0 * np.exp((mu - 0.5*sigma**2) * t + (sigma * B))
```

[49]:
```
# Euler Scheme
X_eu, X = [], X0
for j in range(N):
    X += mu * X * dt + sigma * X * np.sqrt(dt) * Z[j]
    X_eu.append(X)
```

[50]:
```
# Milstein Scheme
X_mil, X = [], X0
for j in range(N):
    X += mu * X * dt + sigma * X * np.sqrt(dt) * Z[j] + 0.5 * sigma ** 2 * X *␣
 ↪dt * (Z[j] ** 2 - 1)
    X_mil.append(X)
```

[51]:
```
# create a color palette
palette = plt.get_cmap('Set2')
plt.figure(figsize=(10, 8))
# Plot
plt.plot(t, Y, label="Exact ($Y_t$)",color=palette(0))
plt.plot(t, X_mil, label="Milstein ($X_t$)",color=palette(1), ls='--')
plt.plot(t, X_eu, label="Euler ($X_t$)",color=palette(2), ls='--')
plt.title('Milstein, Euler Approximation vs. Exact Solution (Geometric BM)',␣
 ↪fontsize=16)
plt.xlabel('t'); plt.legend(loc=2, prop={'size': 12});
```




### 0.3.1 Appendix

**Convergence**

- Weak Convergence: As $\Delta t \to 0$, $error^w(\Delta t) = \sup_{t_j} |E(X(t_j) - E(Y(t_j))|$, goes to zero.
- Strong Convergence: As $\Delta t \to 0$, $error^s(\Delta t) = E(\sup_{t_j} |X(t_j) - Y(t_j)|)$, goes to zero.

We now compute the above error terms for the Euler Scheme and Milstein Scheme for a range of $\Delta t$ values. Specifically, we simulate 10000 sample paths for each value of $\Delta t$, compute the errors and plot the weak and strong error terms for each approximation against $\Delta t$ values.

[ ]:
```
# Initiate dt grid and lists to store errors
strong_err_eu, strong_err_mil, weak_err_eu, weak_err_mil = [], [], [], []
dt_grid = [2 ** (R-10) for R in range(6, -1, -1)]
M = 1000

# Look through values of dt
for dt in dt_grid:
    # Given dt
```

```
    # Setup discretized grid
    t = np.arange(dt, T + dt, dt)
    N = len(t) # N steps in a sample path

    # Initiate vectors to store errors and time series (along N steps)
    err_eu, err_mil = [], []
    Y_sum, X_eu_sum, X_mil_sum = np.zeros(N), np.zeros(N), np.zeros(N)

    # Generate sample paths (M in total)
    for i in range(M):
        # Create Brownian Motion
        np.random.seed(i)
        Z = np.random.normal(0, 1, N)
        B  = np.cumsum(np.sqrt(dt) * Z)

        # Exact solution
        Y = X0 * np.exp((mu - 0.5*sigma**2) * t + (sigma * B))

        # Simulate stochastic processes
        Xeut, Xmilt, X_eu, X_mil = X0, X0, [], []
        for j in range(N):

            # Euler Scheme
            Xeut += mu * Xeut * dt + sigma * Xeut * np.sqrt(dt) * Z[j]
            X_eu.append(Xeut)

            # Milstein Scheme
            Xmilt += mu * Xmilt * dt + sigma * Xmilt * np.sqrt(dt) * Z[j] + 0.5 * sigma ** 2 * Xmilt * dt * (Z[j] ** 2 - 1)
            X_mil.append(Xmilt)

        # Compute strong errors of a sample path and add to those across from other sample paths
        err_eu.append(max(abs(Y - X_eu)))
        err_mil.append(max(abs(Y - X_mil)))

        # Add Y and X values to previous sample paths
        Y_sum += Y
        X_eu_sum += X_eu
        X_mil_sum += X_mil

    # Compute mean of absolute errors and find maximum (strong error)
    strong_err_eu.append(np.mean(err_eu))
    strong_err_mil.append(np.mean(err_mil))

    # Compute error of means and find maximum (weak error)
    weak_err_eu.append(max(abs(Y_sum - X_eu_sum)/M))
```

```
    weak_err_mil.append(max(abs(Y_sum - X_mil_sum)/M))
```

[ ]:
```
# plt.figure(figsize=(10, 8))
plt.plot([T/d for d in dt_grid], strong_err_eu, label="Euler Scheme - Strong␣
 ↪Error",color=palette(0))
plt.plot([T/d for d in dt_grid], weak_err_eu, label="Euler Scheme - Weak␣
 ↪Error",color=palette(0),ls='--')
plt.plot([T/d for d in dt_grid], strong_err_mil, label="Milstein Scheme -␣
 ↪Strong Error",color=palette(1))
plt.plot([T/d for d in dt_grid], weak_err_mil, label="Milstein Scheme - Weak␣
 ↪Error",color=palette(1),ls='--')
plt.title('Convergence of SDE Approximations')
plt.xlabel('$T/\Delta t$'); plt.ylabel('Error (e($\Delta t$))'); plt.
 ↪legend(loc=1);
```

[ ]:
```
# plt.figure(figsize=(10, 8))
plt.loglog([T/d for d in dt_grid], strong_err_eu, label="Euler Scheme - Strong␣
 ↪Error",color=palette(0))
plt.loglog([T/d for d in dt_grid], weak_err_eu, label="Euler Scheme - Weak␣
 ↪Error",color=palette(0),ls='--')
plt.loglog([T/d for d in dt_grid], strong_err_mil, label="Milstein Scheme -␣
 ↪Strong Error",color=palette(1))
plt.loglog([T/d for d in dt_grid], weak_err_mil, label="Milstein Scheme - Weak␣
 ↪Error",color=palette(1),ls='--')
plt.title('Convergence of SDE Approximations')
plt.xlabel('$T/\Delta t$'); plt.ylabel('Error (e($\Delta t$))'); plt.
 ↪legend(loc=3);
```

[ ]:
```

```