CS240 Lecture Notes

Haocen Jiang
haocen.jiang@edu.uwaterloo.ca
Prof. Arne Storjohann

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Contents

Introduction and Asymptotic Analysis

Random Access Machine (RAM) Model

- The `random access machine` has a set of memory cells, each of which stores one item of data.
- Any `access` to a memory location takes constant time
- Any `primitive operation` takes constant time.
- The `running time` of a program can be computed to be the number of memory accesses plus the number of primitive operations

Order Notation

O-notation:

- $f(n) \in O(g(n))$ if there exist constant $c > 0$ and $n_o > 0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_o$
- $f(n)$ grows "no faster than" $g(n)$

Example 1

Prove that $(n+1)^5 \in O(n^5)$
we need to prove that $\exists c > 0, n_o > 0$ s.t. $0 \leq f(n) \leq cg(n)\ \forall n \geq n_o$
Proof. Note that $n + 1 \leq 2n\ \forall n \geq 1$ Raise both side the the power of 5 gives:

$$(n+1)^5 \leq 32n^5$$

Thus we have found $c = 32$ and $n_o = 1$

- **Properties**: Assume that $f(n)$ and $g(n)$ are both ***asymptotically non-negative***
  1. $f(n) \in O(af(n))$ for any constant a
     pf. $0 \leq f(n) \leq \frac{1}{a}af(n)$ for all $n \geq n_o := N$
  2. if $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$, then $f(n) \in O(h(n))$
     pf. $f(n) \in O(g(n)) \Rightarrow \exists c_1, n_1 > 0$ s.t. $f(n) \leq c_1 g(n)\ \forall n \geq n_1$
     $g(n) \in O(h(n)) \Rightarrow \exists c_2, n_2 > 0$ s.t. $g(n) \leq c_2 h(n)\ \forall n \geq n_2$
     $\therefore f(n) \leq c_1 c_2 h(n)$ for all $n \geq \max(\mathrm{n_1}, \mathrm{n_2})$
  3. a) $\max(f(n), g(n)) \in O(f(n) + g(n))$
        p.f. $0 \leq max(f(n), g(n)) \leq 1 \cdot [f(n) + g(n)]\ \forall n \geq N$
     b) $f(n) + g(n) \in O(max(f(n), g(n)))$
        p.f. $0 \leq f(n) + g(n) \leq 2 \cdot [max(f(n), g(n))]\ \forall n \geq N$
  4. a) $a_0 + a_1 n + \cdots + a_d n^d \in O(n^d)$ if $a_d > 0$
     b) $n^d \in O(a_0 + a_1 n + \cdots + a_d n^d)$

Ω-notation:

- $f(n) \in O(g(n))$ if there exist constant $c > 0$ and $n_o > 0$ such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_o$
- $f(n)$ grows "no slower than" $g(n)$

Example 2

$n^3 \log(n) \in \Omega(n^3)$ since $\log(n) \geq 1$ for all $n \geq 3$

Θ-notation:

- $f(n) \in O(g(n))$ if there exist constant $c_1, c_2 > 0$ and $n_o > 0$ such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_o$
- $f(n)$ grows "at the same rate as" $g(n)$
- **Fact:** $f(n) \in \Theta(g(n))$ if and only if $f(n) \in O(g(n))$ adn $f(n) \in \Omega(g(n))$

Example 3

$2n^3 - n^2 \in \Theta(n^3)$

o-notation:

- $f(n) \in o(g(n))$ if **for all** constants $c > 0$, there exist $n_o > 0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_o$
- $f(n)$ grows "slower than" $g(n)$

Example 4

Claim: $2010n^2 + 1388n \in o(n^3)$ ***proof.***
let $c > 0$ be given, then

$$\begin{aligned} 2010n^2 + 1388n &< 5000n^2 \\ &= \left(\frac{5000}{n}\right) n^3 \\ &\leq cn^3 \quad \forall n \geq \frac{5000}{c} \end{aligned}$$

ω-notation:

- $f(n) \in \omega(g(n))$ if **for all** constants $c > 0$, there exist $n_o > 0$ such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_o$
- $f(n)$ grows "faster than" $g(n)$
- $f(n) \in \omega(g(n)) \Leftrightarrow g(n) \in o(f(n))$

Complexity of Algorithm

Common growth rate

- $\Theta(1)$(*constant complexity*)
- $\Theta(\log n)$(*logarithmic complexity*) e.g. binary search
- $\Theta(n)$(*linear complexity*)
- $\Theta(n \log n))$(*linearithmetic complexity*)e.g. merge sort
- $\Theta(n^2)$(*quadratic complexity*)
- $\Theta(n^3)$(*cubic complexity*)e.g. matrix multiplication
- $\Theta(2^n)$(*quadratic complexity*)

Techniques for Order Notation

Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_o$. Suppose that

$$L = \lim_{n \to \infty} \frac{f(n)}{g(n)}$$

Then

$$f(n) \in \begin{cases} o(g(n)) & \text{if } L = 0 \\ \Theta(g(n)) & \text{if } 0 < L < \infty \\ \omega(g(n)) & \text{if } L = \infty \end{cases}$$

Example 5 A1P3

Prove of disprove the following statements

(a) $f(n) \notin o(g(n))$ and $f(n) \notin \omega(g(n)) \Rightarrow f(n) \in \Theta(g(n))$

disprove: Counter example, consider $f(n) := n$ and $g(n) := \begin{cases} 1 & \text{n is odd} \\ n^2 & \text{n is even} \end{cases}$.

- For any odd number $n_1 > c$, we have $f(n_1) = n_1 > c = cg(n_1)$, showing that $f(n) \notin O(g(n))$, and therefore, $f(n) \notin o(g(n))$ - Similarly, for any even number $n_1 > 1/c$ we have $cg(n_1) = cn_1^2 > n_1 = f(n_1)$, showing that $f(n) \notin \Omega(g(n))$ and therefore, $f(n) \notin \omega(g(n))$ - However, since $f(n) \notin \Omega(g(n))$, it has to be the case that $f(n) \notin \Theta(g(n))$

(b) $min(f(n), g(n) \in \Theta\left(\frac{f(n)g(n)}{f(n)+g(n)}\right)$

Proof. We will show that $\frac{f(n)g(n)}{f(n)+g(n)} \leq min(f(n), g(n)) \leq 2\frac{f(n)g(n)}{f(n)+g(n)}$ for all $n \geq 1$. The desired result will then follow from the definition of Θ using $c_1 = 1, c_2 = 2$ and $n_0 = 1//$ By assumption, f and g are positive, so $fg/(f+g) = \min(f,g)\max(f,g)/(f+g)$, which is less than $\min(f,g)$ since $\max(f,g)/(f+g) < 1$. Similarly, $\min(f,g) = 2fg/(2\max(f,g)) \leq 2fg/(f+g)$

Example 6

Prove that $n(2 + \sin(n\pi/2)$ is $\Theta(n)$. Note that $\lim_{n\to\infty}(2 + \sin n\pi/2)$ does not exist

Proof. $n \leq n(2 + \sin\frac{n\pi}{2}) \leq 3n$

Example 7

Compare the growth rates of $\log n$ and n^i (where $i > 0$ is a real number).

$$\lim_{n \to \infty} \frac{\log n}{n^i} = \lim_{n \to \infty} \frac{1/n}{in^{i-1}} = \lim_{n \to \infty} \frac{1}{in^i} = 0$$

This implies that $\log n \in o(n^i)$

Relationships between Order Notations

- $f(n) \in \Theta(g(n)) \Leftrightarrow g(n) \in \Theta(f(n))$
- $f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$
- $f(n) \in o(g(n)) \Leftrightarrow g(n) \in \omega(f(n))$

- $f(n) \in \Theta(g(n)) \Leftrightarrow f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$
- $f(n) \in o(g(n)) \Rightarrow f(n) \in O(g(n))$
- $f(n) \in o(g(n)) \Rightarrow f(n) \notin \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \in \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \notin O(g(n))$

"Maximum" rules

- $O(f(n) + g(n)) = O(max\{f(n), g(n)\})$
- $\Theta(f(n) + g(n)) = \Theta(max\{f(n), g(n)\})$
- $\Omega(f(n) + g(n)) = \Omega(max\{f(n), g(n)\})$

Transitivity

If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$, then $f(n) \in O(h(n))$ If $f(n) \in \Omega(g(n))$ and $g(n) \in O(h(n))$, then $f(n) \in \Omega(h(n))$

Techniques for Algorithm Analysis

Two general strategies are as follows.

- Use Θ-bounds throughout the analysis and obtain a Θ-bound for the complexity of the algorithm.
- Prove a O-bound and a matching Ω-bound separately. Use upper bounds (for O-bounds) and lower bounds (for Ω-bound) early and frequently. This may be easier because upper/lower bounds are easier to sum.

Worst-case complexity of an algorithms:

The worst-case running time of an algorithm A is a function $f : \mathbb{Z}^+ \to \mathbb{R}$ mapping n (the input size) to the ***longest*** running time for any input instance of size n:

$$T_A(n) = \max\{T_A(I) : Size(I) = n\}.$$

Average-case complexity of an algorithm:

The average-case running time of an algprith A is a function $f : \mathbb{Z}^+ \to \mathbb{R}$ mapping n (the input size) to the ***average*** running time over all instances of size n:

$$T_A^{avg}(n) = \frac{1}{|\{I : Size(I) = n\}|} \sum_{\{I:Size(I)=n\}} T_A(I).$$

Notes on O-notation

- It is important not to try to make comparisons between algorithms using O-notations.
- For example, suppose algorithm A_1 and A_2 both solve the same problem, A_1 has worst-case run-time $O(n^3)$ and A_2 has worst-case run-time $O(n^2)$. We **cannot** conclude that A_2 is more efficient

> **NOTE**
> 1.The worst-case run-time may only be achieved on some instances.
> 2. O-notation is an upper bound. A_1 may well have worst-case run-time $O(n)$. If we want to be able to compare algorithms, we should always use Θ-notation.

Example 8

Goal: Use asymptotic notation to simplify run-time analysis.

```
Test1(n)
1.    sum ← 0
2.    for i ← 1 to n do
3.        for j ← i to n do
4.            sum ← sum + (i − j)^2
5.    return sum
```

- size of instance is n
- line1 and line5 execute only once: $\Theta(1)$
- running time proportional to: number of iterations if the j-loop

Direct Method:

$$\#\text{ of iteration} = \sum_{i=1}^{n}(n-i+1) = \sum_{i=1}^{n} i = \frac{n(n+1}{2}$$

$\Rightarrow$ # of iterations of j-loop is $\Theta(n^2)$
$\Rightarrow$ Complexity of Test 1 is $\Theta(n^2)$

Sloppy Method:

$$\#\text{ of iteration} = \sum_{i=1}^{n}(n-i+1) \leq \sum_{i=1}^{n} n = n^2$$

$\Rightarrow$ Complexity of Test 1 is $O(n^2)$

Merge Sort

```
MergeSort(A, ℓ ← 0, r ← n − 1)
A: array of size n, 0 ≤ ℓ ≤ r ≤ n − 1
    if (r ≤ ℓ) then
        return
    else
        m = (r + ℓ)/2
        MergeSort(A, ℓ, m)
        MergeSort(A, m + 1, r)
        Merge(A, ℓ, m, r)
```

$$\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + cn \\
&= 2\left(2T\left(\frac{n}{4}\right) + c\left(\frac{n}{2}\right)\right) + cn \\
&= 4T\left(\frac{n}{4}\right) + c\left(2\left(\frac{n}{2}\right) + n\right) \\
&= 4\left(2T\left(\frac{n}{8}\right) + c\left(\frac{n}{4}\right)\right) + c\left(2\left(\frac{n}{2}\right) + n\right) \\
&= 8T\left(\frac{n}{8}\right) + c\left(4\left(\frac{n}{4}\right) + 2\left(\frac{n}{2}\right) + n\right) \\
&= \dots \\
&= nc + c\left(n + 2\left(\frac{n}{2}\right) + 4\left(\frac{n}{4}\right) + \cdots + \left(\frac{n}{2}\right)\left(\frac{n}{\frac{n}{2}}\right)\right) \\
&= nc + cn\log(n)
\end{aligned}$$

Some Recurrence Relations

Recursion	resolves to	example
$T(n) = T(n/2) + \Theta(1)$	$T(n) \in \Theta(\log n)$	Binary search
$T(n) = 2T(n/2) + \Theta(n)$	$T(n) \in \Theta(n \log n)$	Mergesort
$T(n) = 2T(n/2) + \Theta(\log n)$	$T(n) \in \Theta(n)$	Heapify ($\rightarrow$ later)
$T(n) = T(cn) + \Theta(n)$ for some $0 < c < 1$	$T(n) \in \Theta(n)$	Selection ($\rightarrow$ later)
$T(n) = 2T(n/4) + \Theta(1)$	$T(n) \in \Theta(\sqrt{n})$	Range Search ($\rightarrow$ later)
$T(n) = T(\sqrt{n}) + \Theta(1)$	$T(n) \in \Theta(\log \log n)$	Interpolation Search ($\rightarrow$ later)

Helpful Formulas

Arithmetic Sequence

$$\sum_{i=0}^{n-1}(a + di) = na + \frac{dn(n-1)}{2} \in \Theta(n^2)$$

Geometric Sequence

$$\sum_{i=0}^{n-1} ar^i = \begin{cases} a\frac{r^n-1}{r-1} \in \Theta(r^n) & \text{if } r > 1 \\ na \in \Theta(n) & \text{if } r = 1 \\ a\frac{1-r^n}{1-r} \in \Theta(1) & \text{if } 0 < r < 1 \end{cases}$$

A few more

$\sum_{i=1}^{n} \frac{1}{i^2} = \frac{\pi^2}{6}$ $\qquad$ $\sum_{i=1}^{n} i^k \in \Theta(n^{k+1})$

Priority Queues

Abstract Data types

Abstract Data Type(ADT): A description of information and a collection of operations on that information.

- We can say **what is stored**
- We can say **what can be done with it**
- We **Do not** say how it is implemented

Possible Properties of the data

- can check $a = b$ or $a \neq b$
- sets of items may be *totally ordered*
- items may be elements of a ring, e.g. $\{+, -, \times\}$ make sense

Stack ADT

- Stack: an ADT consisting of a collection of items with operations:
  - *push:* inserting an item
  - *pop:* removing the most recently inserted item
- Items are removed in *last-in first-out* order **(LIFO)**. We need no assumptions on items

Realization of Stack ADT: Arrays

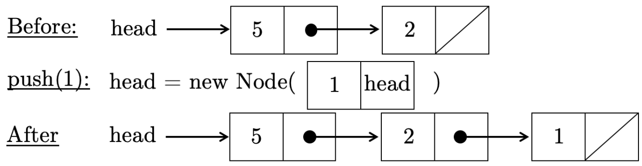
Store the data in an array and keep track of the size of the array. Add the new data to the end of the array every time we insert. Delete the last item in the array when we need to pop an item.

```
pop() //size>0
  temp = A[size-1]
  size--
  return temp
```

Overflow Handling: if the array is full

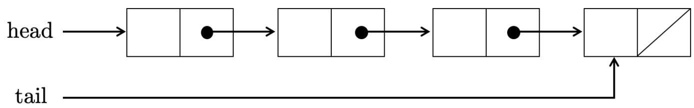
- create new array twice the size
- copy items over
- takes time $\Theta(n)$, but happens rarely
- average over operation costs $\Theta(1)$ overhead.
- In CS240, always assume array has enough space.

Realization of Stack ADT: Linked List



Queue ADT

- Queue: an ADT consisting of a collection of items with operations:
  - *enqueue*: inserting an item
  - *dequeue:* removing the least recent inserted item
- Items are removed in first-in first-out (**FIFO**) order.
- Items enter the queue at the *rear* and are removed from the *front*
- Realizations of Queue ADT
  - using (circular) arrays(partially filled)
  - using linked lists



Priority Queue ADT

Priority Queue ADT

- Priority Queue: An ADT consists of items (each having a *priority*) with operations:
  - *insert*: inserting an item tagged with a priority
  - *deleteMax*: removing the item of *highest priority*
- the priority is also called *key*
- the above definition is for a **maximum-oriented** priority queue. A **minimum-oriented** priority queue is defined in the natural way, by replacing the operation *deleteMax* by *deleteMin*

PQ-sort

```
PQ-Sort(A[0..n − 1])
1.   initialize PQ to an empty priority queue
2.   for k ← 0 to n − 1 do
3.        PQ.insert(A[k])
4.   for k ← n − 1 down to 0 do
5.        A[k] ← PQ.deleteMax()
```

Realizations of Priority Queue: Unsorted Array

- *insert*: $\Theta(1)$
  - insert at position n, increment n
- *deleteMax:* $\Theta(n)$
  - find maximum priority of element $A[i]$
  - swap $A[i]$ with $A[n-1]$
  - return $A[n-1]$ and decrement n

Realizations of Priority Queue: Sorted Array

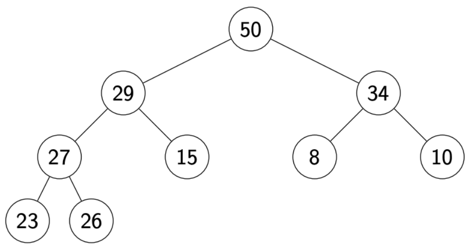
- *insert*: $\Theta(n)$
  - find the correct position to insert
  - shift all the elements after to make room
- *deleteMax:* $O(1)$
  - delete the last element and decrement n

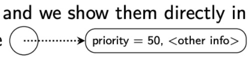
Goal: achieve $O(\log(n))$ run-time for both *insert* and *deleteMax*
Solution: use heap: max stores two possible candidates for the next biggest item

Binary Heaps

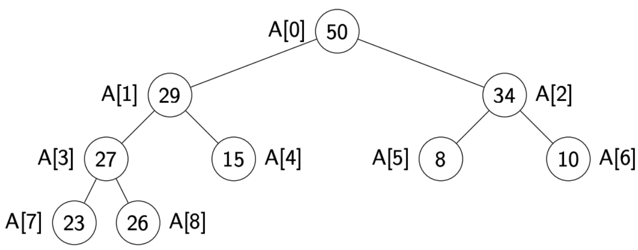
A *(binary)heap* is a certain type of binary tree such that

1) **Structural properties**
   - all levels are full except for the last level
   - last level is left justified
2) **heap-order properties**
   - for any node i, key of the parent of i is larger than to equal to the key of i



(In our examples we only show the priorities, and we show them directly in the node. A more accurate picture would be )

Lemma: The height of a heap with n nodes is $\Theta(\log n$.
Heap in Array: Heaps should not be stored as binary trees.
Let H be a heap of n items and let$A$ be an array of size n. Store root in $A[0]$ and continue with elements level-by-level from top to bottom, in each level left-to-right.



It is easy to navigate the heap using this array representation:

- the *root* node is $A[0]$,
- the *left child* of $A[i]$ (if it exists) is $A[2i+1]$,
- the *right child* of $A[i]$ (if it exists) is $A[2i+2]$,
- the *parent* of $A[i]$ ($i \neq 0$) is $A[\lfloor \frac{i-1}{2} \rfloor]$,
- the *last* node is $A[n-1]$

Operations in Binary Heaps

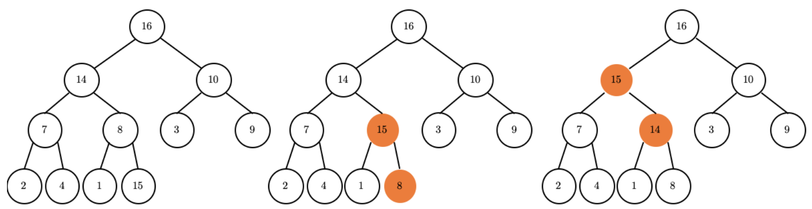
Insertion in Heaps

- Place the new key at the first free leaf
- since we have a array-representation, increase the *size* of the array and insert the new `last(size)`: (bottom level, left most free spot)
- the **heap order property** may be violated: perform a `fix-up`
  - compare the key of the node with its parent
  - if the key bigger than its parent, swap with the parent
  - The new item bubbles up until it reaches its correct place in the heap.
  - **Time:** $O(\text{height of heap}) = O(\log n)$

```
fix-up(A, k)
k: an index corresponding to a node of the heap
    while parent(k) exists and A[parent(k)] < A[k] do
        swap A[k] and A[parent(k)]
        k ← parent(k)
```

Example 9 fix-up

insert(15) on $A = [16, 14, 10, 7, 8, 3, 9, 2, 4, 1]$



$A = [16, 15, 10, 7, 14, 3, 9, 2, 4, 1, 8]$

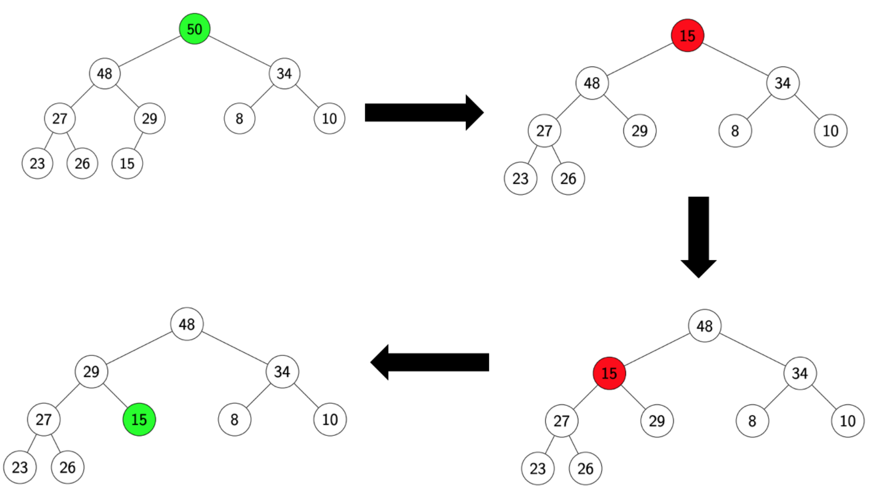
DeleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf, decrease the size
- the **heap order property** may be violated: perform a `fix-down` on the root:
  - compare the node with its children
  - if it is larger than both children, we are done
  - otherwise swap the node with the larger children, then perform `fix-down` on it again
  - **Time:** $O(\text{height of heap}) = O(\log n)$

```
fix-down(A, n, k)
A: an array that stores a heap of size n
k: an index corresponding to a node of the heap
    while k is not a leaf do
        // Find the child with the larger key
        j ← left child of k
        if (j is not last(n) and A[j + 1] > A[j])
            j ← j + 1
        if A[k] ≥ A[j] break
        swap A[j] and A[k]
        k ← j
```

Example 10 fix-down

perform `deleteMax` on $A = [16, 15, 10, 7, 14, 3, 9, 2, 4, 1, 8]$



Proprity queue using max heaps

- Use a partially filled array
- keep track of size of the heap
- Keep heap-order satisfied using
  - `fix-up` on `insert`
  - `fix-done` on `deleteMax`
- Goal achieved: $O(\log n)$ for `insert` and `deleteMax`

PQ-sort and HeapSort

Recall Priority Queue Sort

1) for $i = 0$ to $n - 1$, insert $A[i]$ into the priority queue
2) for $i = n - 1$ to 0, `deleteMax()` from the PQ and insert the key into $A[i]$

```
PQ-SortWithHeaps(A)
    initialize H to an empty heap
    for k ← 0 to n − 1 do
        H.insert(A[k])
    for k ← n − 1 down to 0 do
        A[k] ← H.deleteMax()
```

$\Rightarrow$ using heap for PQ: PQ sort takes:

- $O(n \log n)$ time
- $O(n)$ auxiliary space

Improvements

1: Use the same array for input/output
$\Rightarrow$ need only $O(1)$ auxiliary space

2: ***Heapify:*** Do line one faster

- We know all items to insert beforehand
- can atually build heap in $O(n)$ time!

Heapofy: Bottom up creation of heap

Put items in nearly complete binary tree, for $i = n - 1$ down to 1, do `fix-down` position i

```
heapify(A)
A: an array
    n ← A.size()
    for i ← parent(last(n)) downto 0 do
        fix-down(A, n, i)
```

Example 11 Heapify

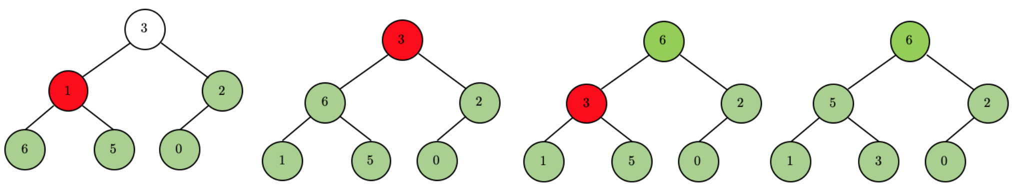
Run `Heapify` on $A = [3, 1, 2, 6, 5, 0]$

for $i = 5, 4, 3$, nothing to do

for $i = 2$, key is OK

for $i = 1$, sway with 6, we are done

for $i = 0$, swap with 6, swap with 5, we are done



Note:

- `fix-down` does nothing for keys at the last level h
- code could start at the key $A[\lfloor \frac{n-1}{2} \rfloor]$
- Heapify using `fix-up` also works, but this might take more time when the heap gets large, because we need to call `fix-up` on all the nodes at level h, which will iterate through its branch to all the way to the root.

HeapSort

- Idea: *PQ-Sort* with heaps
- But use the same input array A for (in-place) storing the heap.

```
HeapSort(A, n)
    // heapify
    n ← A.size()
    for i ← parent(last(n)) downto 0 do
        fix-down(A, n, i)
    // repeatedly find maximum
    while n > 1
        // do deleteMax
        swap items at A[root()] and A[last(n)])
        decrease n
        fix-down(A, n, root())
```

- The `for-loop` takes $\Theta(n)$ time and the `while-loop` tkaes $O(n \log n)$ time.
- number of swaps is bounded by

$$\sum_{i=0}^{h} i2^{h-i} \leq \sum_{i=0}^{h} \frac{in}{2^i} \leq n \sum_{i=0}^{\infty} \frac{i}{2^i} = 2n$$

Intro for the Selection Problem

Problem

- **Given:** array $A[0, 1, \ldots, n-1]$, index k with $0 \leq k \leq n-1$
- **Want:** item that would be at $A[k]$ if A were sorted

Possible solutions

1. Make k passes through the array, deleting the minimum number each time. $\Rightarrow \Theta(kn)$
2. Sort the array first, the return $A[k] \Rightarrow \Theta(n \log n)$
3. Build a maxHeap from A and call `deleteMax` $(n-k+1)$ times $\Rightarrow \Theta(n + (n-k+1)\log n)$
4. Build a minHeap from A and call `deleteMin` k times $\Rightarrow \Theta(n + k \log n)$

Sorting and Randomized Algorithms

QuictSelect

Selection and sorting

The *selection problem*: Given an array A of n numbers, and $0 \leq k < n$, find the element that would be at position k of the sorted array.

- The best heap-based algorithm had running time $\Theta(n + k \log n)$, and for *median finding*, this is $\Theta(n \log n)$
- Question: Can we do selection in linear time?
- The **quick-select** answers this question in the affirmative

Partition and choose-pivot

quick-select and related algorithm *quick-sort* rely on two subroutines

- `choose-pivot`(A): Choose an index p. We will use the *pivot value* $v < -A[p]$ to rearrange the array. The simplest idea is to **return the last element**
- `partition`(A, p): Rearrange A and return *pivot-index* i so that

- the pivot value v is in $A[i]$,
- all items in $A[0, \ldots, i-1]$ are $\leq v$,
- all items in $A[i+1, \ldots, n-1]$ are $\geq v$.

0				p			n - 1
?	$\cdots$	$\cdots$	?	v	$\cdots$	$\cdots$	?

$\leq v$	$\leq v$	$\cdots$	$\leq v$	v	$\geq v$	$\cdots$	$\geq v$
0				i			n - 1

Implementations

```
partition(A, p)
A: array of size n,   p: integer s.t. 0 ≤ p < n
      Create empty lists small and large.
      v ← A[p]
      for each element x in A[0, ..., p−1] or A[p+1 ... n−1]
            if x < v append x to small
            else append x to large
      i ← size(small)
      Overwrite A[0 ... i−1] by elements in small
      Overwrite A[i] by v
      Overwrite A[i+1 ... n−1] by elements in large
      return i
```

```
partition(A, p)
A: array of size n,   p: integer s.t. 0 ≤ p < n
      swap(A[n − 1], A[p])
      i ← −1,   j ← n − 1,   v ← A[n − 1]
      loop
            do i ← i + 1 while i < n and A[i] < v
            do j ← j − 1 while j > 0 and A[j] > v
            if i ≥ j then break    (goto 9)
            else swap(A[i], A[j])
      end loop
      swap(A[n − 1], A[i])
      return i
```

Quick-Select1

```
quick-select1(A, k)
A: array of size n,   k: integer s.t. 0 ≤ k < n
      p ← choose-pivot1(A)
      i ← partition(A, p)
      if i = k then
            return A[i]
      else if i > k then
            return quick-select1(A[0, 1, ..., i − 1], k)
      else if i < k then
            return quick-select1(A[i + 1, i + 2, ..., n − 1], k − i − 1)
```

- Recall that i is returned by partition, thus we have no control
- if $i = k$, then return v as the solution
- if $i > k$
  - There are $i > k$ items in A that are $\leq v$
  - Therefore the desired return value m is $\leq v$
  - Find m by searching recursively on the left
- if $i < k$
  - There are $i < k$ items in A that are $\leq v$
  - Therefore the desired return value m is $\geq v$
  - Find m by searching recursively on the right

Runtime of Quick-Select1

Let $T(n)$ be the run time, if we select from n elements. Since partition takes $O(n)$ if $n \geq 2$, for some constant c, we have

$$T(n) = \begin{cases} c & n = 1 \\ T(\text{size of the subarray}) + cn & n \geq 2 \end{cases}$$

Worse-case: size of the subarray is $n-1$, therefore

$$
\begin{aligned}
T^{\text{worst}}(n) &\le cn + T^{\text{worst}}(n-1) \\
&\le cn + c(n-1) + T^{\text{worst}}(n-2) \\
&\le cn + c(n-1) + c(n-2) + T^{\text{worst}}(n-3) \\
&\le \ldots \\
&\le cn + c(n-1) + \cdots + c(2) + c(1) \\
&= c\frac{n(n+1)}{2} \in O(n^2)
\end{aligned}
$$

Best-case: One single partition: $\Theta(n)$

Average-case: There are infinitely many instances of size n, how to calculate the average? ***Sorting Permutation***

Sorting Permutation

Observation: `quickSelect` is comparison based: It doesn't care what actual input numbers are: it only cares if $A[i] \le A[j]$

For example: It will act the same on inputs $A = [4, 8, 2]$ and $A = [5, 7, 3]$

Simplifying assumption: All input numbers are distinct, since only their relative order matter, we can characterize type of inputs by sorting permutation π

- We assume all $n!$ permutations are *equally likely*
- Average cost is the sum of costs for all permutations, divided by $n!$

Therefore, we an analyze the average case of `quick-select1`:

- If we know the pivot-index i, then the subarrays have sizes i and $n-i-1$
- $T(n) \le cn + \max(T(i), T(n-i-1))$
- How many *sorting permutations* π lead to index i?
  * Let's sat $p = n-1$
  * Pivot-index = i $\iff$ the pivot element $A[p]$ is $i'th$ smallest $\iff$ $\pi(i) = p$
  * All other $n-1$ elements of π could be arbitrary
- Thus there are $(n-1)!$ sorting permutations has pivot index i

$$
T^{avg}(n) \le \frac{1}{n!} \underbrace{\sum_{i=0}^{n-1}}_{\text{split by i}} \underbrace{(n-1)!}_{\text{have pivot-index}i} \underbrace{(cn + \max(T^{avg}(i), T^{avg}(n-i-1)))}_{\text{run-time bound if pivot index is}i}
$$

- **Lemma:** $\sum_{i=0}^{n-1} \max(i, n-i-1) \le \frac{3}{4}n^2$

  Proof. **If n is even:**

$$
\begin{aligned}
\sum_{i=0}^{n-1} \max(i, n-i-1) &= 2\sum_{i=\frac{n}{2}}^{n-1} i \\
&= \frac{3}{4}n^2 - \frac{1}{2}n \\
&\le \frac{3}{4}n^2
\end{aligned}
$$

  if n is odd:

$$
\begin{aligned}
\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1} \max(i, n-i-1) &= \lfloor\frac{n}{2}\rfloor + 2\sum_{i=0}^{n-1} i \\
&= \lfloor\frac{n}{2}\rfloor + n^2 - \lceil\frac{n}{2}\rceil^2 + \lceil\frac{n}{2}\rceil - n \\
&< \frac{3}{4}n^2
\end{aligned}
$$

□

- **Theorem:** $T^{avg}(n) \leq 4cn$ (prove by Induction)
  * *Base case:* n = 1

$$T^{avg}(1) = c \leq 4c$$

  * *Induction hypothesis*: Assume that $T^{avg}(N) \leq 4cn$ for all $N < n, n > 1$
  * *Inductive Step*

    Proof.

$$\begin{aligned} T^{avg}(n) &\leq cn + \frac{1}{n}\sum_{i=0}^{n-1} \max(T(i), T(n-i-1)) \\ &\leq cn + \frac{1}{n}\sum_{i=0}^{n-1} \max(4ci, 4c(n-i-1)) \\ &= cn + \frac{4c}{n}\sum_{i=0}^{n-1} \max(i, n-i-1) \\ &\leq cn + \frac{4c}{n}\frac{3}{4}n^2, \text{ by Lemma} \\ &= cn + 3cn \\ &= 4cn \end{aligned}$$

□

- i.e. $T^{avg}(n) \in O(n)$, which is a tight upper bound

Randomized Algorithms

Expected Running Time

- A *randomized algorithm* is one which relies on some random numbers in addition to the input
- The cost will depend on the input and the random numbers used
- Define $T(I, R)$ to be the running time of the randomized algorithm for instance I and the sequence of random numbers R.
- The *expected running time* $T^{exp}(I)$ for instance I is the *expected value* for $T(I, R)$:

$$T^{exp}(I) = E[T(I, R)] = \sum_R T(I, R) \cdot P(R)$$

- The *worse-case expected running time* is

$$T^{exp}_{avg}(n) = \max_{\{I: size(I)=n\}} T^{exp}(I)$$

-
- The *average-case expected running time* is

$$T^{exp}_{avg}(n) = \frac{1}{|\{I : size(I) = n\}|} \sum_{\{I:size(I)=n\}} T^{exp}(I)$$

Randomized Quick Select

```
choose-pivot2(A)
1.    return random(n)
```

```
quick-select2(A, k)
1.    p ← choose-pivot2(A)
2.    . . .
```

- To achieve average-case run-time, we randomly permute inputs.
- **simple Idea:** pick pivot-index p randomly in $\{0, \dots, n-1\}$
- **key insight:** $P(\text{index of pivot is } i) = \frac{1}{n}$
- **Detour 1:** How to choose random index?
  Use language provided `random(max)`
- **Detour 2:** How to analyze an algorithm that use `random`?
  - Measure expected running time of randomized algorithm A,
  - For one instance I

$$T^{exp}(I) = \sum_{r} T(A \text{ on } I \text{ with } r \text{ chosen}) \cdot P(\text{r was chosen})$$

  - For `quick-select`

$$\begin{aligned} T^{exp}(n) &= cn + \sum_{i=0}^{n-1} P(\text{pivot index is i}) \cdot (\text{run-time if index is i}) \\ &\leq cn + \sum_{i=0}^{n-1} \frac{1}{n} \max(T^{exp}(i), T^{exp}(n-i-1)) \end{aligned}$$

> **Note**
> 1. The message is that randomized `quick-select` has $O(n)$ expected run time.
> 2. This expression is the same is the running time of non-randomized quick-select. For average-case of non-randomized quick-select, $\frac{1}{n}$ represents the proportion of the permutation with i chosen as pivot index the over all the possible permutation. While here , $\frac{1}{n}$ is a probability

QuickSort

Hoare developed *partition* and *quick-select* in 1960; together with a *sorting* method based on partitioning:

```
quick-sort1(A)
A: array of size n
    if n ≤ 1 then return
    p ← choose-pivot1(A)
    i ← partition(A, p)
    quick-sort1(A[0, 1, ..., i − 1])
    quick-sort1(A[i + 1, ..., n − 1])
```

Worst case

$T^{\text{worst}}(n) = T^{\text{worst}}(n-1) + \Theta(n)$ Same as quick-select: $T^{\text{worst}}(n) \in \Theta(n^2)$

Best case

$T^{\text{best}}(n) = T^{\text{best}}(\lfloor \frac{n-1}{2} \rfloor) + T^{\text{best}}(\lfloor \frac{n-1}{2} \rfloor) + \Theta(n)$

Average case

- Rather than analyze run-time, can simply count comparisons.
- Observe: partition uses $\leq n$ comparisons.

- Recurrence relation (if we know pivot-index)

$$T(n) \leq \begin{cases} 0 & n \leq 1 \\ n + T(i) + T(n-i-1) & n > 1, \text{pivot index } i \end{cases}$$

$$\begin{aligned}
T^{\text{avg}}(n) &= \frac{1}{n!} \sum_{\text{perm } \pi} (\#\text{ of comparisons if input has sorting permutation } \pi) \\
&= \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\text{perm } \pi} (\#\text{ of comparisons if input has sorting permutation } \pi) \\
&= \frac{1}{n!} \sum_{i=0}^{n-1} (\#\text{of permutations with pivot index } i)(n + T^{\text{avg}}(i) + T^{\text{avg}}(n-i-1)) \\
&\leq \frac{(n-1)!}{n!} \sum_{i=0}^{n-1} (n + T^{\text{avg}}(i) + T^{\text{avg}}(n-i-1)) \\
&= n + \frac{1}{n} \sum_{i=1}^{n-1} T^{\text{avg}}(i) + \frac{1}{n} \sum_{i=0}^{n-1} T^{\text{avg}}(n-i-1) \\
&= n + \frac{1}{n} \sum_{i=0}^{n-1} T^{\text{avg}}(i) \text{ (and can forget } i=0 \text{ since } T(0)=0)
\end{aligned}$$

Theorem: $T^{\text{avg}}(n) \leq 2n \log_{4/3} n$ for all $n \geq 1$.

Proof. Proof by induction on n.

Base case: $n = 1$, ok, since $T(1) = 0 \leq 2 \cdot 1 \cdot \log 1$.

Inductive Hypothesis: Assume $T^{\text{avg}}(N) \leq 2N \log_{4/3} N$ for all $N < n, n \geq 2$.

Step ($n \geq 2$):

$$\begin{aligned}
T^{\text{avg}}(n) &\leq n + \frac{2}{n} \sum_{i=1}^{n-1} T^{\text{avg}}(i) \\
&\leq n + \frac{2}{n} \sum_{i=1}^{n-1} (2 \cdot i \cdot \log_{4/3} i) \\
&\leq n + \frac{4}{n} \sum_{i=1}^{\frac{3}{4}n} i \underbrace{\log_{4/3} i}_{\leq \log_{4/3} \frac{3}{4}n} + \frac{4}{n} \sum_{i=\frac{3}{4}n+1}^{n-1} i \underbrace{\log_{4/3} i}_{\leq \log_{4/3} n}
\end{aligned}$$

Recall:

$$\begin{aligned}
\log_{4/3} \frac{3}{4} n &= \log_{4/3} \frac{3}{4} + \log_{4/3} n \\
&= (\log_{4/3} n) - 1 \\
&\leq n + \frac{4}{n} \sum_{i=1}^{\frac{3}{4}n} i(\log_{4/3} n - 1) + \frac{4}{n} \sum_{i=\frac{3}{4}n+1}^{n-1} i \log_{4/3} n \\
&\leq n + \frac{4}{n} \sum i = 1^{n-1} i \log_{4/3} n - \frac{4}{n} \underbrace{\sum i = 1^{\frac{3}{4}n} i}_{\leq \frac{1}{2} \frac{9}{16} n^2} \\
&\leq n + \frac{4}{n} \frac{(n-1)n}{2} \log_{4/3} n - n \\
&\leq 2n \log_{4/3} n
\end{aligned}$$

□

Message: Quicksort is fast ($\Theta(n \log n)$) on average, but not in worst case.

Tips and tricks for Quick Sort

- **Choosing pivots**
  - Simplest idea: use $A[n-1]$ as pivot.
  - Better idea: pick middle element $A\left[\lfloor \frac{n}{2} \rfloor\right]$
  - Even better idea: median of 3. Look at $A[0]$, $A\left[\lfloor \frac{n}{2} \rfloor\right]$, $A[n-1]$. Sort them, put min/max at $A[0]$, $A[n-1]$. Use middle as pivot.
  - Weird idea: use the median. Use faster version of Quick Select. Theoretically good runtime, but horribly slow in practice.
  - Another good idea: use a random pivot. Can argue: get same recurrence as for average case, so expected runtime $\Theta(n \log n)$
- **Reduce auxilliary space**
  - QuickSort uses auxilliary space for recursion stack, this could be $\Theta(n)$
  - Improve to $\Theta(\log n)$ by recursing on smaller side first
  - Do not recurse on bigger side. Instead, keep markers of what needs sorting and loop.
- **End recursion early**
  - Orginal code had if $(n \leq 1)$...
  - Replace by if $(n \leq 20)$
  - Find array not sorted, but items close to correct position.
  - On this input, insertion sort takes $O(n)$ time.

Lower bound for comparison sorting

- Have seen: sorting can be done in $\Theta(n \log n)$ time.
- Can we sort in $o(n \log n)$?
- Answer depends on what we allow.

We have seen many sorting algorithms:

Sort	Running time	Analysis
Selection Sort	$\Theta(n^2)$	worst-case
Insertion Sort	$\Theta(n^2)$	worst-case
Merge Sort	$\Theta(n \log n)$	worst-case
Heap Sort	$\Theta(n \log n)$	worst-case
quick-sort1	$\Theta(n \log n)$	average-case
quick-sort2	$\Theta(n \log n)$	expected
quick-sort3	$\Theta(n \log n)$	worst-case

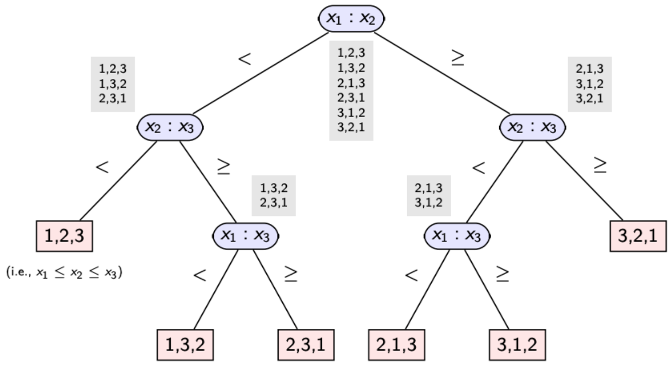
Theorem

Any comparison based sorting algorithm A use $\Omega(n \log n)$ comparisons in worst case.

"Comparison based" uses key comparisons. (i.e., questions like $A[i] \leq A[j]$ and nothing else)

We study the decision tree of A

Comparison-based algorithms can be expressed as *decision tree*.
To sort $\{x_1, x_2, x_3\}$:



- interior nodes: comparisons
- children labeled by outcome
- leaves: result returned
- depth of leaf $\equiv$ number of comparisons to get there
- worst case number of comparisons $\equiv$ length of tree

Proof of the theorem:

- there are $n!$ permutations, each gives a different result
- so at least $n!$ leaves in tree
- at least $n!$ nodes
- height $\geq \log n! \in \Omega(n \log n)$

Non-Comparison-Based Sorting

Previously, we looked at comparison based sorting that needs $\Omega(n \log n)$ comparisons. We will non look at **digits sorting**

Assumptions

- Given numbers with digits in $\{0, 1, 2, \ldots, R-1\}$
  - R is called the radix. $R = 2, 10, 16, 128, \ldots$ are most common
  - Example: $R = 4,\ A = [123, 230, 21, 320, 210, 232, 101]$
- All keys have the same number of m digits
  - In computer, $m = 32$ or $m = 64$
  - can achieve after padding with leading 0s.
  - Example :$R = 4,\ A = [123, 230, 021, 320, 210, 232, 101]$
- Therefore, all numbers are in range $\{0, 1, \ldots, R^m - 1\}$

Bucket Sort

- We sort the numbers by a single digit
- Create a "bucket" for each possible digit. Array $B[0 \ldots R-1]$ of the lists
- Copy item with digit i into bucket $B[i]$
- At the end, copy buckets in order into A

```
Bucket-sort(A, d)
A: array of size n, contains numbers with digits in {0, ..., R − 1}
d: index of digit by which we wish to sort
    Initialize an array B[0...R − 1] of empty lists
    for i ← 0 to n − 1 do
        Append A[i] at end of B[d^th digit of A[i]]
    i ← 0
    for j ← 0 to R − 1 do
        while B[j] is non-empty do
            move first element of B[j] to A[i++]
```

Example 12

Sort array A by last digit:

A		B					A
123		$B[0]$	$\rightarrow$ 230	$\rightarrow$ 320	$\rightarrow$ 210		230
230		$B[1]$	$\rightarrow$ 021	$\rightarrow$ 101			320
021		$B[2]$	$\rightarrow$ 232				210
320	$\Longrightarrow$	$B[3]$	$\rightarrow$ 123			$\Longrightarrow$	021
210							101
232							232
101							123

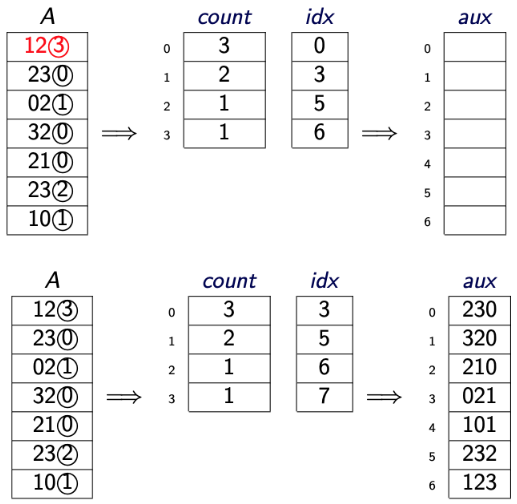
- This is **Stable**: equal items stay in original order.
- Run-time of sorting one digit is $\Theta(n + R)$, space $\Theta(n)$

Count Sort

- Bucket sort wastes space for `linked lists`
- **Observe:** we know exactly where numbers in $B[j]$ goes!
  - The first of them is at index $|B[0]| + \cdots + |B[j-1]|$
  - The others follows
- So compute $|B[j]|$ then copy A directly to the new array.
- count $C[j] = |B[j]|$, index $idx[j]$ = first index to put $B[j]$ into.

```
key-indexed-count-sort(A, d)
A: array of size n, contains numbers with digits in {0, ..., R − 1}
d: index of digit by which we wish to sort
// count how many of each kind there are
    count ← array of size R, filled with zeros
    for i ← 0 to n − 1 do
        increment count[d^th digit of A[i]]
// find left boundary for each kind
    idx ← array of size R, idx[0] = 0
    for i ← 1 to R − 1 do
        idx[i] ← idx[i − 1] + count[i − 1]
// move to new array in sorted order, then copy back
    aux ← array of size n
    for i ← 0 to n − 1 do
        aux[idx[A[i]]] ← A[i]
        increment idx[A[i]]
    A ← copy(aux)
```

Example 13



Sorting multidigit numbers

- **MSD-Radix-Sort**
  - To sort large numbers, we compare leading digit, then each group by next digit, etc.

```
MSD-Radix-sort(A, l, r, d)
A: array of size n, contains m-digit radix-R numbers
l, r, d: integers, 0 ≤ l, r ≤ n − 1, 1 ≤ d ≤ m
    if l < r
        partition A[l..r] into bins according to dth digit
        if d < m
            for i ← 0 to R − 1 do
                let l_i and r_i be boundaries of ith bin
                MSD-Radix-sort(A, l_i, r_i, d + 1)
```

  - Partition using count-sort
  - **Drawback:** Too many recursions
  - Runtime: $O(m \cdot (n + R))$
- **LSD-Radix-Sort**
  - Key Insight: when $d = i$, the array is sorted w.r.t. the last $m - i$ digits
  - for $i < m$, we change order of 2 items $A[k]$ and $A[j]$ only if they have different i^{th} digit

```
LSD-radix-sort(A)
A: array of size n, contains m-digit radix-R numbers
    for d ← m down to 1 do
        key-indexed-count-sort(A, d)
```

  - Run-time for both MSD and LSD are $O(m(n + R))$
  - But LSD has cleaner code, no recursion
  - LSD looks at all digits, MSD only looks at those it needs to.

Summary

Sort	Run-time	Analysis	Comments
Insertion Sort	$\Theta(n^2)$	worst-case	good if mostly sorted; **stable**
Merge Sort	$\Theta(n \log n)$	worst-case	flexible; merge runtime useful; **stable**
Heap Sort	$\Theta(n \log n)$	worst-case	clean code; **in-place**
Quick Sort	$\Theta(n \log n)$	worst-case	**in-place**; fastest in practice
Randomized QuickSort	$\Theta(n \log n)$ $\Theta(n^2)$ $\Theta(n \log n)$	average-case worse-case expected-case	
Key-Indexed	$\Theta(n+R)$	worst-case	**stable**; need integers in $[0, R)$
Radix Sort	$\Theta(m(n+R))$	worst-case	**stable**; needs m-digit radix-R numbers

Dictionaries

ADT Dictionaries

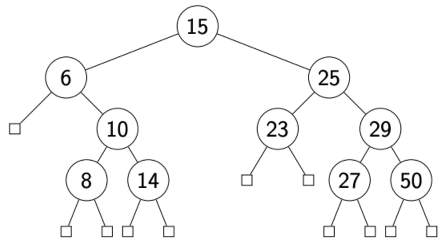
Dictionary

- A `dictionary` is a collection of items, each of which contains a `key` and some data, and is called a `key-value pair` (KVP). Keys can be compared and are (typically) unique.
- **Operations**
  - `insert(key, value)`: inserts a KVP
  - `search(key)`: returns the KVP with this `key`
  - `delete(key)`: delete the KVP from dictionary
- **Common Assumptions:**
  - All keys are distinct
  - keys can be compared in $O(1)$ time
  - KVP takes $O(1)$ space.
- **Implementations we may have seen:**
  - **Unsorted array or linked list:**
    * $\Theta(n)$ search
    * $\Theta(1)$ insert
    * $\Theta(n)$ delete
  - **Sorted array:**
    * $\Theta(\log n)$ binary search
    * $\Theta(n)$ insert
    * $\Theta(n)$ delete

Review: BST

- Either empty
- of KVP at root with left, right subtrees
  - keys in left subtree are smaller than the key at root
  - keys in right subtree are larger than the key at root
- **Insert and Search**
  - run time O(max number of level)=O(height)
  - unfortunately, height $\in \Omega(n)$ for some BSTs
- **Delete**: run time is O(height)
  - if x is a leaf, just delete it
  - if x has one child, delete it and move the child up
  - Else, swap key at x with the key at successor node and then delete that node (i.e. go right once and then go all the way left)

Example 14 BST



AVL Trees

Balanced BST

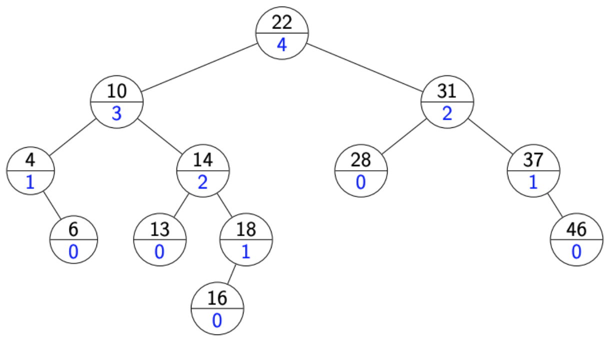
- impose some conditions on BST
- show that these guarantee the height of $O(\log n)$
- modify `insert/delete` so that they maintain these conditions

AVL Trees

- The AVL conditions: The heights of the left subtree L and right subtree R differ by at most 1.
- i.e. every node has **balance** $\in \{-1, 0, 1\}$, where balance:=height(R)-height(L)
- Note that the height of a tree is the length of the longest path from the root to any leaf, and the height of an empty tree is defined to be -1

Example 15 AVL Tree

The lower numbers indicate the height of the subtree



Theorem

Any AVL Tree has height $O(\log n)$

Proof. It's enough to show that In any AVL tree wit height h and n nodes, $h \leq \log_c n$ for some c

- **rephrase:** In any AVL tree with height h and n nodes: $c^h \leq n$
- or equivalently, It the height is h, then there must be at least c^h nodes
- Define $N(h)$ = smallest number of nodes in an AVL tree of height h. The by induction $N(h) \geq (\sqrt{2})^h$

Base case: $N(0) = 1, (\sqrt{2})^0 = 1$, $N(1) = \sqrt{2} \geq \sqrt{2}$
Inductive step

$$\begin{aligned} N(h) &= N(h-1) + N(h-2) + 1 \\ &\geq 2N(h-2) + 1 \\ &\geq (\sqrt{2})^2 \cdot (\sqrt{2})^{h-2} \\ &= (\sqrt{2})^h \end{aligned}$$

□

Insertion in AVL Trees

Insert

- do a BST insert
- move up the tree from the new node, updating heights
- as soon as we find a unbalanced node, fix via **Rotation**

```
AVL-insert(r, k, v)
    z ← BST-insert(r, k, v)
    z.height ← 0
    while (z is not null)
        setHeightFromChildren(z)
        if (|z.left.height − z.right.height| = 2) then
            AVL-fix(z) // see later
            break      // can argue that we are done
        else
            z ← parent of z
```

```
setHeightFromChildren(u)
    u.height ← 1 + max{u.left.height, u.right.height}
```

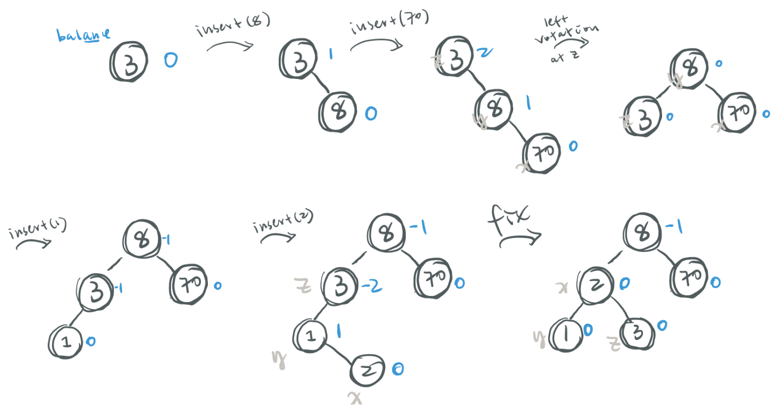
```
AVL-fix(z)
// Find child and grand-child that go deepest.
    if (z.right.height > z.left.height) then
        y ← z.right
        if (y.left.height > y.right.height) then
            x ← y.left
        else x ← y.right
    else
        y ← z.left
        if (y.right.height > y.left.height) then
            x ← y.right
        else x ← y.left
    Apply appropriate rotation to restructure at x, y, z
```

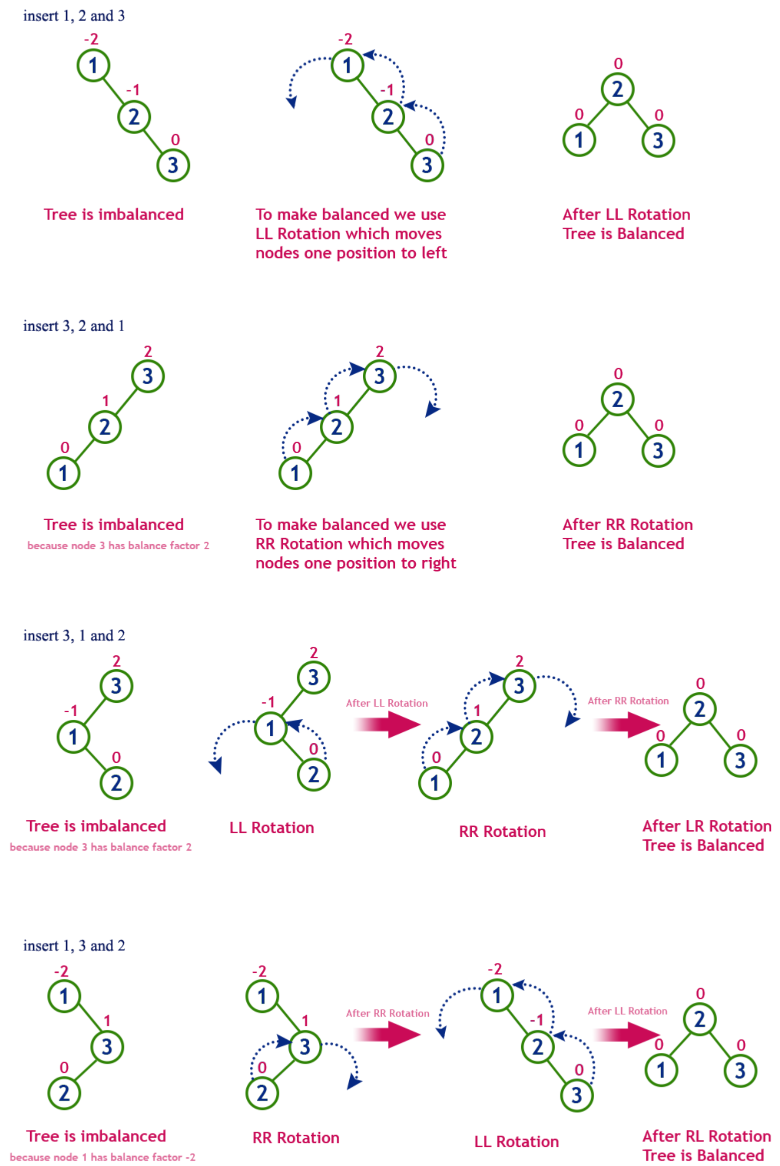
Rotations in BST

- Observe: There are many BSTs with the same set of keys
- Goad: rearrange the tree so that
  - keep ordering-property intact
  - move “bigger subtree” up
  - do only local changes $O(1)$

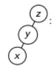
AVL Rotations

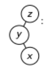
Example 16

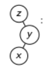


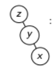


$http://btechsmartclass.com/DS/U5_T2.html$, All balance factor in above pictures are inverse of the ones we define

```
AVL-fix(z)
    ... // identify y and x as before
    case
```


```
        : // Right rotation
          rotate-right(z)
```


```
        : // Double-right rotation
          rotate-left(y)
          rotate-right(z)
```


```
        : // Double-left rotation
          rotate-right(y)
          rotate-left(z)
```


```
        : // Left rotation
          rotate-left(z)
```

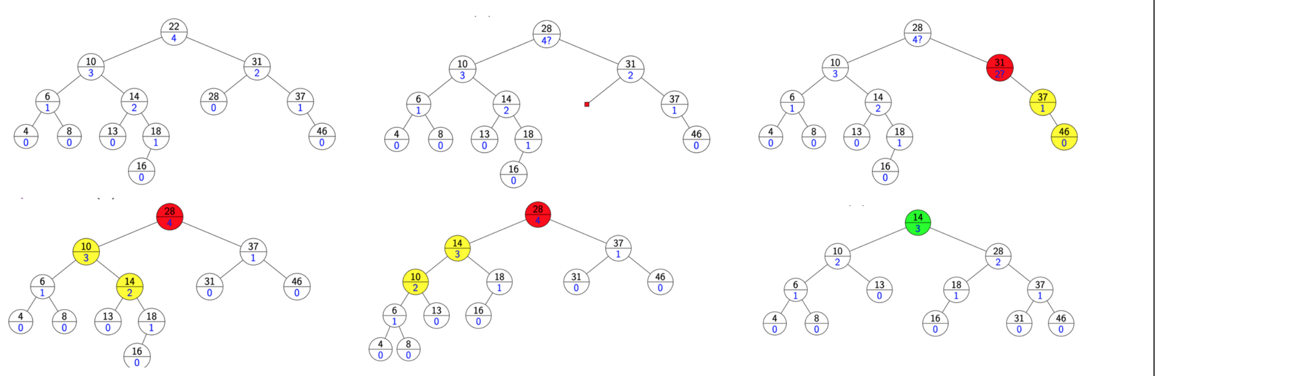
Deletion in AVL Trees

Remove the key k with `BST-delete`. We assume that `BST-delete` returns the place where structural change happened, i.e., the parent z of the node that got deleted. (This is not necessarily near the one that had k.) Now go back up to root, update heights, and rotate if needed

```
AVL-delete(r, k)
    z ← BST-delete(r, k)
    while (z is not null)
        setHeightFromChildren(z)
        if (|z.left.height − z.right.height| = 2) then
            AVL-fix(z)
        // Always continue up the path and fix if needed.
        z ← parent of z
```

Example 17 AVL-Delete

$T.Delete(22)$



AVL Tree Operations Runtime

- All of BST operations take $O(height)$
- It takes $O(height)$ to trace back up to the root updating balances
- **Calling `AVL-fix`**
  - `insert`: $O(1)$ rotations, in fact *at most once*
  - `delete`: $O(height)$ rotations

Other Dictionary Implementations

Skip List

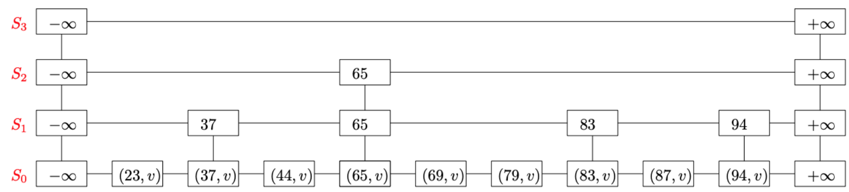
Skip List

- discovered in 1987
- randomized data structure for dictionary ADT
- competes with and always beats AVL Trees

A hierarchy S of ordered linked lists (levels) S_0, S_1, ůůů, S_h:

- Each list S_i contains the special keys $-\infty$ and $+\infty$ (sentinels)
- List S_0 contains the KVPs of S in non-decreasing order. (The other lists store only keys, or links to nodes in S_0.)
- Each list is a subsequence of the previous one, i.e., $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_h$
- List S_h contains only the sentinels

Example 18 Skip List



- The skip list consists of a reference to the topmost left node.
- Each node p has a reference to *after*(p), *below*(p),
- Each KVP belongs to a *tower* of nodes
- **Intuition**: $|S_i| \cong 2|S_{i+1}| \Rightarrow$ height $\in O(\log n)$
- also, we use randomization to satisfy with high probabilities

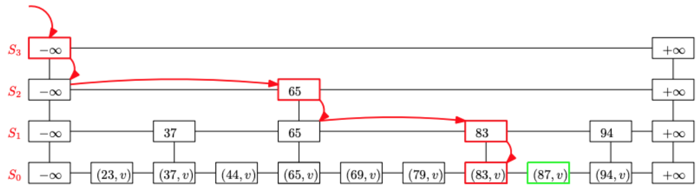
Search

- Start at the top left and move right/down as needed
- keep track of nodes at drop down location
- return a stack s
- next key of $top(s)$ is the searched key if in dictionary

```
skip-search(L, k)
    p ← topmost left node of L
    P ← stack of nodes, initially containing p
    while below(p) ≠ null do
        p ← below(p)
        while key(after(p)) < k do
            p ← after(p)
        push p onto P
    return P
```

Example 19 Skip List Search

`Skip-Search(S,87)`



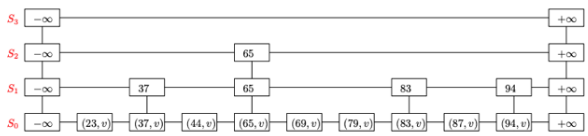
The stack contains $s = [(S_0, 83), (S_1, 83), (S_2, 65), (S_3, -\infty)]$, and the key 87 is $next(top(s))$

Insert

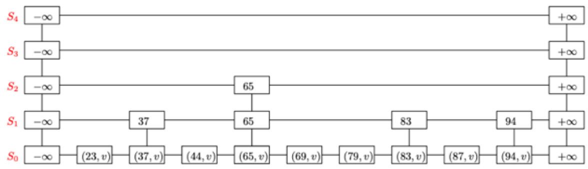
- Determine the tower height by randomly flipping a coin until get a tails
- Increase the height of the skip list if needed
- Search for key, which returns the stack of predecessors $s = [(S_0, p_0), (S_1, p_1), \ldots, (S_i, p_i)]$
- Insert the KVP(k, v) after p_0 in S_0 and insert the key k after p_j in S_j for $1 \le j \le i$

Example 20

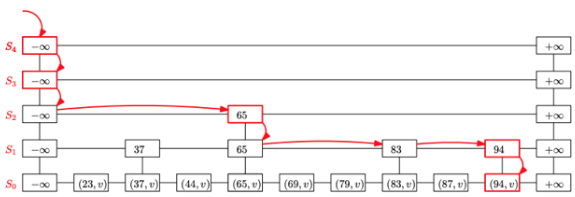
`insert(`S`,100,`v`)` with Coin tosses: $H, H, H, T \Rightarrow i = 3$



Height Increase



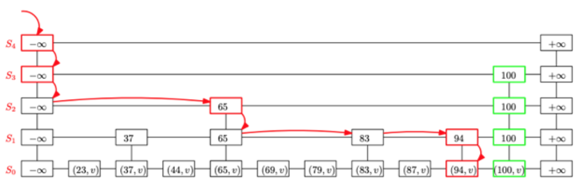
Search(S, 100)



returns

$(S_0, -\infty)$
$(S_1, -\infty)$
$(S_2, 65)$
$(S_3, 94)$
$(S_4, 94)$

Insert the KVP



Delete

- Search for the key, which returns the stack of predecessors
- Remove the items after predecessors, if they store the key
- remove duplicated layers that only have sentinels

Analysis

- Questions to ask
  1. What is the expected height?
  2. What is the expected space?
  3. How long does `search` take?
- Here, only do height bound
- Let x_k = height of the tower k = the max level that contains k, we have

$$P(x_k \geq 0) = 1, P(x_k \geq 1) = 1/2, P(x_k \geq 2) = 1/4, \ldots, P(x_k \geq i) = 1/2^i$$

$$P(\text{height } \geq i = P(\max_k\{x_k\} \geq i) \leq \sum_k P(x_k \geq i) = n\frac{1}{2^i}$$

- Therefore, we have $P(h \geq 3\log n) \leq \frac{n}{2^{3\log n}} = \frac{n}{n^3} = \frac{1}{n^2}$
- So, $P(h \leq 3\log n) \geq 1 - 1/n^2$

Summary

- Expected space usage: $O(n)$
- Expected height: $O(\log n)$
- A skip list with n items has height at most $3\log n$
- Skip-Search: $O(\log n)$ expected time
- Skip-Insert: $O(\log n)$ expected time
- Skip-Delete: $O(\log n)$ expected time
- Skip lists are fast and simple to implement in practice