Lecture 3

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Lecture 3

Roadmap of week 1(May 16 - May 24):

- Probability (lecture 1)
- Permutation and combination (lecture 2)
- Conditional probability (lecture 2)
- Discrete random variable, PMF and CDF (lecture 3)
- Continuous random variable, PDF and CDF (lecture 4)

Review

- Permutation, combination
- Conditional probability
- Independence

Permutation

- **Theorem**: The number of permutations of n distinct objects is $n!$
- **Theorem**: The number of permutations of n distinct objects taken r at a time is $nPr = \frac{n!}{(n-r)!}$

Combination

- A **combination** is a selection of r objects taken from n distinct objects without regard to the order of selection.
- The number of ordered samples equals the number of unordered samples multiplied by the number of ways to order each sample.
- $C(n, r) = \frac{n!}{(n-r)! * r!}$

Conditional probability

- If A and B are any two events in a sample space Ω and $\mathbb{P}(A) \neq 0$, the conditional probability of B given A is

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

Independence

- Informally speaking, two events are independent if the occurrence or no-occurrence of either one does not affect the probability of the occurrence of the other.
- Definition: two events A and B are **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A)$$

Roadmap of lecture 3

- Discrete random variable
- Probability mass function (PMF)
- Cumulative distribution function (CDF)

Discrete random variable

- Recall the sample space we mentioned earlier
- Definition: the set of all possible outcomes of an experiment is called the **sample space** (denoted as Ω), and each outcome (denoted as ω) in a sample space is called an **element** of the sample space.
- Consider the experiment of flipping 3 fair coins, then our sample space will be $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

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Discrete random variable

- $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.
- Based on elements of the sample space, we may be interested in some 'numbers'
- For example

$$X = \{\text{number of H}\} \quad Y = \begin{cases} 1 \text{ if } HHH \\ 0 \text{ otherwise} \end{cases} \quad K = \begin{cases} 5 \text{ if } HHH \\ 7 \text{ if } TTT \\ 0 \text{ otherwise} \end{cases}$$

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Discrete random variable

- Definition: If Ω is a sample space with a probability measure and X is a real-valued function defined over the elements of Ω, then X is called a **random variable**.
- a random variable maps random outcomes to random values (which will be useful in the future)

Discrete random variable

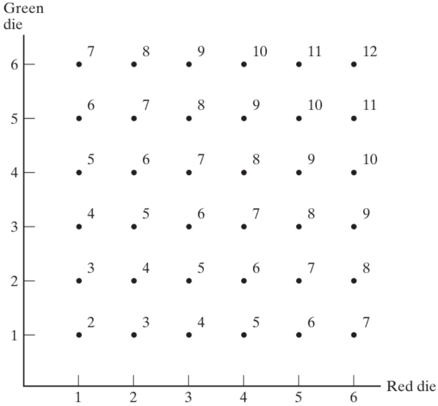
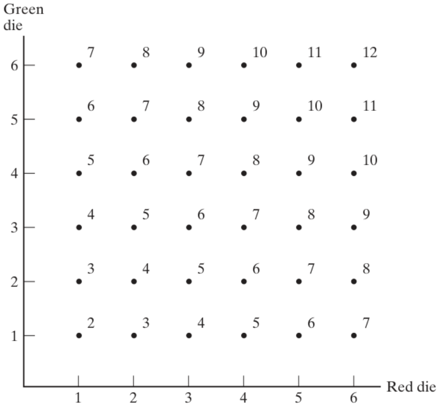


Figure 1. The total number of points rolled with a pair of dice.

Note: it is possible to map different random outcomes to the same number, but not the reverse.

Discrete random variable



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An exercise

- A fair coin is tossed 4 times. Consider the random variable $X \equiv \{\text{the total number of heads}\}$. List the elements of the sample space, the corresponding values x of the random variable X, and the corresponding probabilities of each value.
- We have limited our discussion to discrete sample spaces, and hence to **discrete random variables**, namely, random variables whose range is finite or countably infinite (e.g. integer, rational).

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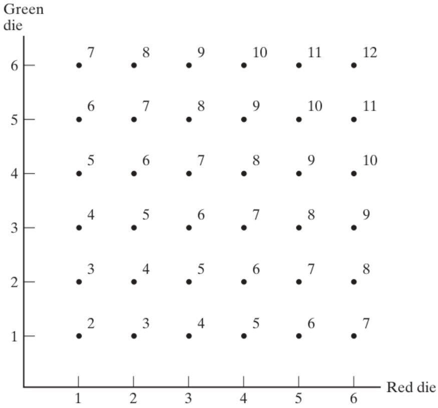
Probability mass function

Are there better ways to describe this random variable?



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Probability mass function

x	$P(X = x)$
2	$\frac{1}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$
5	$\frac{4}{36}$
6	$\frac{5}{36}$
7	$\frac{6}{36}$
8	$\frac{5}{36}$
9	$\frac{4}{36}$
10	$\frac{3}{36}$
11	$\frac{2}{36}$
12	$\frac{1}{36}$

Probability mass function

- Definition: If X is a discrete random variable, the function given by $f(x) = \mathbb{P}(X = x)$ for each x within the range of X is called the **probability mass function** of X.
- Given the above definition, we need two properties for this function p
  - $f(x) \geq 0$ for each value within its domain
  - $\Sigma_x f(x) = 1$, where the summation extends over all the values within its domain.

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Probability mass function

- Consider flipping a fair coin 4 times, what will be the PMF of random variable $X \equiv \{\text{the total number of heads}\}$?

$$f(x) = \begin{cases} \frac{1}{16} \text{ if } x = 0 \\ \frac{4}{16} \text{ if } x = 1 \\ \frac{6}{16} \text{ if } x = 2 \\ \frac{4}{16} \text{ if } x = 3 \\ \frac{1}{16} \text{ if } x = 4 \end{cases}$$

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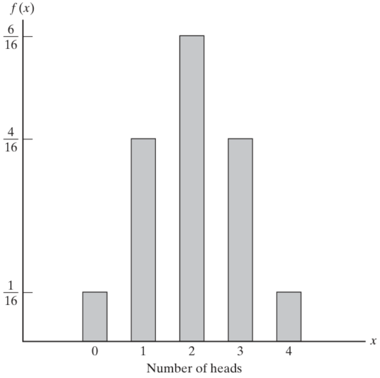

Cumulative distribution function

- Besides the probability of realizing an exact value, it is also useful in many cases to know the probability that the value of a random variable is less than or equal to some real number x. Some examples can be "the probability of getting a grade lower than A^-", "the probability of Duke basketball team getting less than 30 scores in the next game"
- Definition: If X is a discrete random variable, the function given by $F(x) = \mathbb{P}(X \leq x) = \Sigma_{t \leq x} f(t)$, for $-\infty < x < \infty$, where $f(t)$ is the PMF of X at t, is called the **cumulative distribution function** (CDF) of X

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Cumulative distribution function

- Again, consider flipping a fair coin 4 times, try to find the CDF of random variable $X \equiv \{\text{the total number of heads}\}$.

$$
\begin{aligned}
F(0) &= f(0) = \frac{1}{16} \\
F(1) &= f(0) + f(1) = \frac{5}{16} \\
F(2) &= f(0) + f(1) + f(2) = \frac{11}{16} \\
F(3) &= f(0) + f(1) + f(2) + f(3) = \frac{15}{16} \\
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Cumulative distribution function

Hence, the CDF is given by

$$F(x) = \begin{cases} 0 \text{ if } x < 0 \\ \frac{1}{16} \text{ if } x \in [0, 1) \\ \frac{5}{16} \text{ if } x \in [1, 2) \\ \frac{11}{16} \text{ if } x \in [2, 3) \\ \frac{15}{16} \text{ if } x \in [3, 4) \\ 1 \text{ if } x \geq 4 \end{cases}$$

Note: this distribution function is defined not only for the values taken on by the given random variable, but for all real numbers.

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Bernoulli RV

- There are some well-known random variables, the first one we will discuss is the Bernoulli random variable.
- This random variable can be described by the following PMF: $f(1) = p, f(0) = 1 - p, p \in [0, 1]$, which can be considered as a generalization of a coin (can be unfair) toss.

The binomial distribution

- Given the Bernoulli random variable, we can study the following problem now: we would like to run n independent times of an experiment (or trial), there is a probability of p to get a success and a probability of $1 - p$ to get a failure. Consider random variable $X \equiv \{\text{total number of successes}\}$, what is $f(x) = \mathbb{P}(X = x)$?
- first consider an example of toss 3 coins (using what we have learned about probability and combinations)
- and then derive the general result: the binomial distribution

$$f(x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

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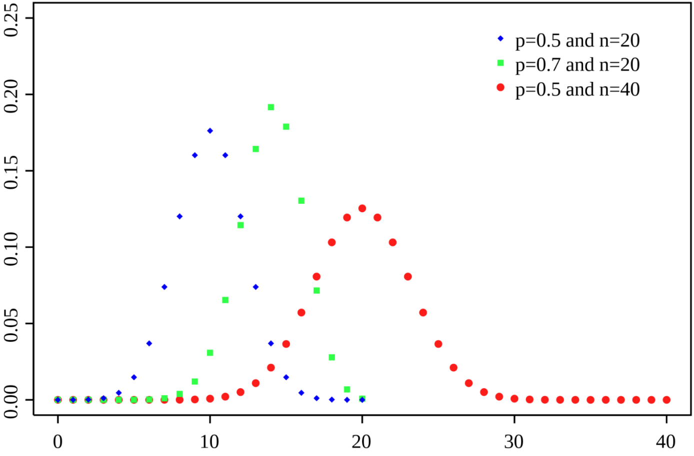

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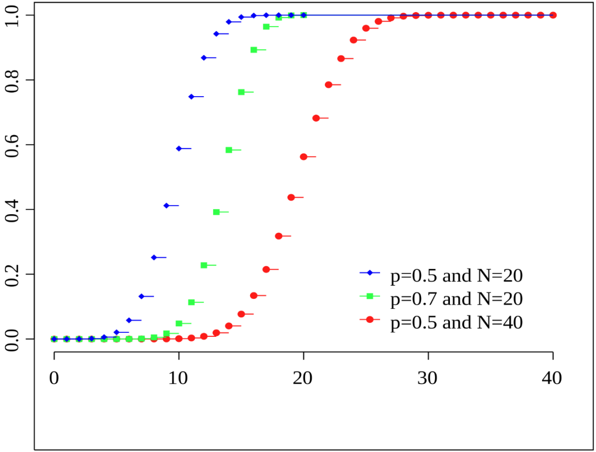

Poisson distribution

- Consider a call center which receives, randomly, an average of $= 3$ calls per minute at all times of day. If the calls are independent, receiving one does not change the probability of when the next one will arrive. We want to learn the probability of receiving k calls in a minute.
- the Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time if these events occur with a known constant mean rate and independently of the time since the last event. The prob is given by $\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$
- Note: this is a limit case of the binomial distribution (where $n \to \infty, p \to 0, np \to \lambda$)

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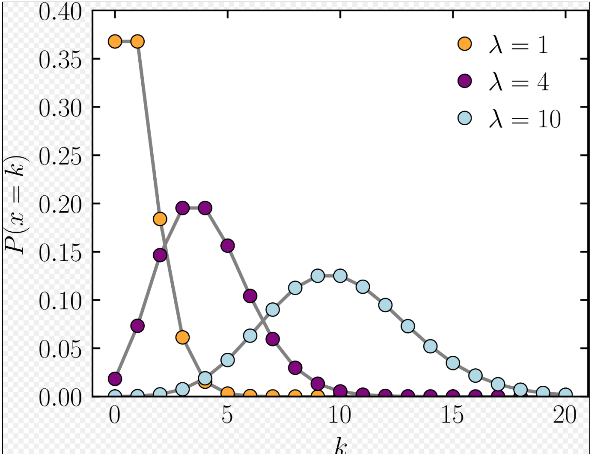

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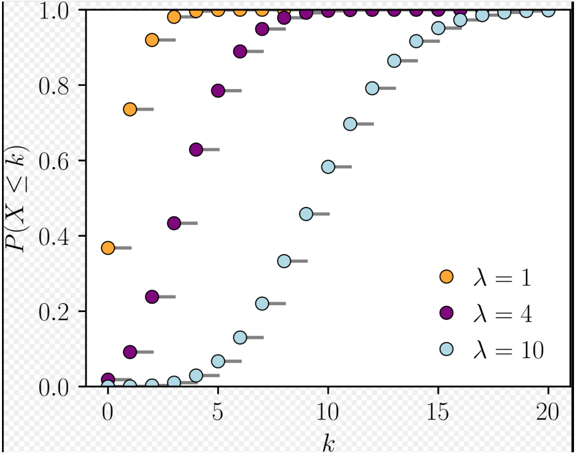

Preview of next lecture

- Continuous random variable
- Uniform RV
- The exponential density
- The normal distribution