# Subsidy Design in Budget-Constrained Matching

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## Abstract

How should a social planner optimally allocate subsidies in a budget-constrained matching market? We study this question across settings that differ in the information available to the planner. Under complete information, the planner fully observes match values and budget constraints. We provide an algorithm that computes the minimal total subsidy required to implement the welfare-maximizing matching. Under incomplete information, the planner observes the status-quo stable matching and agents' budget constraints, but not the underlying match values. We characterize robust subsidy policies that improve aggregate welfare relative to the observed outcome. Our central result provides necessary and sufficient conditions for a subsidy to be *non-distortive*: for every profile of match values consistent with the observed outcome, there exists a stable matching that is weakly welfare-improving relative to the status quo. We then examine how the planner's information set shapes the flexibility of designing targeted subsidies. In the most restrictive environment with no information about match values or budget constraints, a uniform subsidy emerges as the only non-distortive policy. Rather than eliciting preferences, our approach infers the underlying information structure from the observed outcome to guide policy design, providing a new method for addressing informational constraints.

## 1 Introduction

Budget constraints are pervasive features of matching markets with transfers. Public schools hire teachers under fixed annual funding,[1] and hospitals fund residency positions with limited program resources.

With these constraints, institutions may be unable to offer transfers sufficient to implement efficient allocations, leading to inefficient matching. Empirically, high-value teachers disproportionately sort into better-resourced schools, leaving low-income communities underserved.[2] Likewise, physicians concentrate in urban hospitals, contributing to physician shortages in rural areas.[3] Taken together, these patterns indicate a policy problem: resource-limited institutions cannot compete for talent at prevailing budgets, generating systematic inefficiencies.

Policymakers have implemented various financial interventions to alleviate such inefficiencies, but the evidence is mixed. On the one hand, targeted transfers have delivered meaningful gains: in

[1]This is not merely a wage-rigidity issue: many public systems use flexible pay, including Wisconsin's Act 10 (Biasi (2021)), Texas's Teacher Excellence Initiative (TEI; Hanushek et al. (2023)), and Florida's SB 736 (Jones (2013)). Stipends for high-need positions and performance-based bonuses are also common.

[2]See, for example, Lankford et al. (2002), Clotfelter et al. (2006), Sass et al. (2012), and Ajzenman et al. (2024).

[3]See, for example, Rosenblatt and Hart (2000) and Machado et al. (2021).

North Carolina, a \$1,800 annual bonus reduced teacher turnover by about 17%, helping retain teachers in high-poverty schools (Clotfelter et al., 2008); in Wisconsin, public schools increased spending on teacher compensation after receiving additional state funding, which improved teacher quality and student outcomes (Baron, 2022). On the other hand, targeted subsidies can improve outcomes at recipient schools but may do so by poaching teachers from nearby non-subsidized schools, generating negative spillovers and ambiguous aggregate welfare effects (Castro and Esposito, 2022; Kho et al., 2023).

These considerations motivate our central question: how should a social planner allocate subsidies in a budget-constrained matching market? The key challenge is informational: in practice, the planner often observes only the existing market outcome but not the underlying match values, so the efficient matching is not identified ex ante.

We study this question in a model of one-to-one matching with transfers in which the social planner has incomplete information. Our analysis spans a spectrum of informational environments: from complete knowledge of match values and budgets to cases where only the observed outcome is available. Throughout, we use schools and teachers as the running example, but the framework applies broadly to other two-sided markets with transfers.

At one extreme, the planner has complete information and observes schools' budget constraints, each teacher's cost of working at each school, and the match values for all school–teacher pairs. We develop an algorithmic procedure—analogous to the Bellman–Ford algorithm—that computes the minimal total subsidy required to implement the welfare-maximizing matching. The procedure augments school budgets only when such transfers are necessary to attain the efficient outcome; it terminates in finitely many steps and yields a stable allocation that coincides with the first-best matching.

Our main focus is the intermediate case, in which the planner faces incomplete information. The planner observes a status-quo stable matching, the wages paid from schools to teachers, schools' budget constraints, and each teacher's cost of working at each school, but not the underlying match values. Because multiple match-value profiles can rationalize the observed outcome, the welfare-maximizing matching is no longer identified.

To address these informational constraints, we develop a novel inference-based approach to policy design: rather than eliciting preferences through mechanism design, we infer the set of match-value profiles consistent with the observed outcome. We then characterize subsidy policies that are robust, meaning they weakly improve efficiency relative to the status quo for all match-value profiles consistent with the planner's observations.

We call a subsidy allocation *non-distortive* if, under the post-subsidy budgets, there exists a stable matching that is weakly welfare-improving relative to the status quo for every match-value profile consistent with the planner's observations. Our central result (Theorem 1) provides a complete characterization: a subsidy allocation is non-distortive if and only if it satisfies both of the following conditions.

- **Direct-effect coverage.** A subsidized school may newly be able to afford a teacher it could not hire at the status quo. When such a teacher becomes affordable, the subsidized school's offer imposes direct upward wage pressure on that teacher's incumbent school. If the incumbent school lacks sufficient budget to retain its current teacher, it may be forced to give up the match even when the match value is high, resulting in potential efficiency loss. Therefore, each school's post-subsidy budget must be sufficient to cover the direct wage pressure imposed by newly subsidized rivals.

- **Spillover coverage.** Beyond the direct effects, subsidies can generate spillover effects through wage competition. A school directly challenged by a subsidized rival may need to raise the wage of its current teacher, which can make other teachers in the market more attractive. As a result, schools not directly challenged by a subsidized rival may still face spillover wage pressure through indirect chains of wage competition. To prevent potential efficiency loss, the subsidy policy must ensure that each school has sufficient post-subsidy budget to cover the maximum spillover wage pressure it may face.

Our results make two main contributions. On the practical side, we provide an exact characterization of the robust subsidy policies that improve aggregate welfare across all match-value profiles consistent with the observed outcome. This characterization offers actionable guidance for policy design under planner-side informational constraints. To the best of our knowledge, this is the first paper to study robust policy intervention in matching markets when the planner faces incomplete information.

More importantly, our contribution is methodological: we develop a new way to address informational constraints in economic design. Rather than relying on mechanism design to elicit private information, we infer the underlying information structure from the observed market outcome and use this inference to guide policy. We show that the observed data, together with the inferred structure, are sufficient to construct a robust policy prescription that guarantees improvement across all match-value profiles consistent with the planner's observations.

Building on this characterization, we analyze how the planner's information set shapes the set of non-distortive policies. Theorems 2 and 3 show that as the planner's information about budgets and costs becomes more limited, her ability to design targeted subsidies correspondingly diminishes. When only match values are unobservable, the planner retains substantial flexibility to design targeted subsidies. When both match values and costs of working are unobservable, this flexibility is significantly reduced. When match values, costs, and budgets are all unobservable, only uniform subsidies remain non-distortive. This analysis shows that the degree of information available to the planner governs the flexibility of policy design and determines whether targeted subsidies or uniform transfers are appropriate in practice.

Beyond its theoretical contributions, our findings contribute to the longstanding debate between efficiency and fairness in policy design. Policymakers often adopt uniform subsidies for reasons of fairness or political feasibility. Our framework offers an alternative efficiency-based rationale: when the planner faces severe informational constraints, uniform transfers emerge as the unique robust efficiency-improving option. Thus, efficiency and fairness need not conflict—under limited information, they may coincide.

The remainder of the paper is organized as follows. Section 2 reviews related literature. Section 3 presents the model. Section 4.1 discusses classical results without and with budget constraints. Section 4.2 covers the complete-information analysis and introduces the *Minimum Subsidy Implementation (MSI)* algorithm. Section 4.3 considers the incomplete-information case and characterizes non-distortive subsidies. Section 4.4 explores how different informational environments shape the set of non-distortive policies. Section 5 concludes. All proofs are collected in the Appendix.

## 2 Related Literature

We adopt a one-to-one matching framework with transferable utility (TU) to study the optimal design of subsidies. This framework belongs to a class of two-sided matching models in which utility can be freely transferred between agents, typically via monetary payments.[4] The TU framework is well established in the matching literature, including foundational models such as the assignment game (Shapley and Shubik, 1971) and job matching models (Crawford and Knoer, 1981; Kelso and Crawford, 1982). Hatfield and Milgrom (2005) later unified many of these models within a broader matching-with-contracts framework. For comprehensive surveys of the matching literature, see Roth and Sotomayor (1990) and Abdulkadiroğlu and Sönmez (2013).

Although numerous empirical studies examine the effects of various financial interventions,[5] the theoretical literature on subsidy design remains relatively scarce. Our paper contributes to this line of research by providing a unifying framework that (i) computes the minimum subsidies needed to implement the efficient matching under complete information, and (ii) characterizes the exact set of non-distortive subsidies under incomplete information. The focus on incomplete information distinguishes our analysis from prior work. Existing research has addressed related questions such as whether a financial intervention policy preserves substitutability and stability (Kojima et al., 2024), how match-specific taxes or subsidies reshape the set of stable matchings (Dupuy et al., 2020), and how to design taxes or subsidies to achieve policy goals like a distributional requirement (Yokote, 2020). However, all of these studies assume that the social planner has complete information. To the best of our knowledge, our paper is the first to examine robust policy intervention in matching markets when the planner faces informational constraints.

Our approach to addressing informational constraints is novel in two dimensions. First, we rely on inference rather than elicitation, which is a fundamental departure from much of the existing literature. Traditionally, economists use mechanism design to overcome information gaps (see, for example, Holmström, 1979; Myerson, 1981). In this framework, the planner offers agents a menu of options and elicits their private information based on the choices they make. In contrast, we infer the underlying information structure from the observed market outcome and identify robust policies that improve aggregate welfare across all scenarios consistent with that inference. Our findings show that inference from observed data can provide valuable guidance for policy intervention.

Second, our notion of robustness is arguably stronger. In the literature on robust mechanism design (see, for example, Bergemann and Morris, 2005; Chung and Ely, 2007; Carroll, 2015), robustness typically refers to a non-Bayesian or prior-free approach.[6] Researchers do not assume the planner knows the distribution of agents' private information and focus on non-Bayesian objectives such as dominant strategy implementation or minimax regret. Robustness is thus defined at the level of the mechanism. In contrast, our framework defines robustness at the policy level: each non-distortive subsidy must improve welfare across all match-value profiles consistent with the planner's observations. By drawing inference from realized outcomes, our approach offers a more targeted and arguably stronger notion of robustness than is standard in the literature.

Perhaps surprisingly, our use of market data for inference shares a common spirit with the partial identification literature in econometrics. Econometricians have applied partial identification techniques to study treatment effects (Manski, 1990; Manski and Pepper, 2000), uncover market structure (Ciliberto and Tamer, 2009), recover underlying preferences (Haile and Tamer, 2003; Agarwal and Somaini, 2018), and more broadly set-identify parameters of interest (Chesher and

[4]Matching models with non-transferable utility (NTU) have also had broad theoretical and practical impact, including applications such as the marriage model, school choice, and kidney exchange. See Gale and Shapley (1962); Becker (1973); Abdulkadiroğlu and Sönmez (2003); Roth et al. (2004).

[5]See, for example, Clotfelter et al. (2008); Fryer (2013); Goodman and Turner (2013); Hendricks (2014); De Ree et al. (2018); Biasi (2021); Baron (2022); Castro and Esposito (2022); Kho et al. (2023)

[6]See Carroll (2019) for a survey of robustness in mechanism design and contracting.

Rosen, 2017), all without relying on strong assumptions.[7] In our setting, all match-value profiles consistent with the planner's observations can be interpreted as the sharp identification region of schools' preferences over teachers. If we view the change in aggregate welfare as the treatment effect of a financial intervention, then our characterization of non-distortive subsidies corresponds to the set of policies that yield a non-negative treatment effect across the entire identification region. In this sense, our work bridges two seemingly distinct branches of literature: partial identification and policy design, and our findings suggest broader potential for combining methods across fields to improve economic design through interdisciplinary research.

Another related branch of the literature examines matching under constraints. A large body of work studies how different types of constraints affect the existence of stable matchings.[8] In contrast, existence is guaranteed in our one-to-one TU framework, and we do not focus on restoring it. Instead, we ask a complementary question: how can efficiency be restored when budget constraints lead to suboptimal outcomes? We characterize the minimal relaxations of budget constraints that restore efficiency under complete information, and develop a robust approach to relaxing constraints under incomplete information.

Our work is also connected to the literature on matching with incomplete information, although most existing studies focus on environments in which the agents, rather than the planner, face informational limitations. Early contributions such as Roth (1989) and Ehlers and Massó (2007) examine the incentive properties of matching mechanisms when agents lack full knowledge of others' preferences. Beyond incentives, researchers have also explored welfare implications: Li and Rosen (1998) shows that incomplete information can lead to inefficient early contracting, while Coles et al. (2013) finds that introducing a simple signaling mechanism can convey credible information and improve both match rates and welfare.[9] Different from these agent-centric perspectives, our paper adopts a planner-centric view: here, it is the planner who lacks full information about agents' preferences. We focus on the implications of informational constraints for policy design.

## 3 Model

The matching–with–transfers framework has broad applications. Throughout, we refer to the two sides as "schools" and "teachers," but the interpretation is flexible: one may read them as firms and workers in Kelso and Crawford (1982) or as hospitals and physicians in Hatfield and Milgrom (2005).

Let $S = \{s_1, \dots, s_n\}$ denote the set of schools and $T = \{t_1, \dots, t_n\}$ the set of teachers, with $|S| = |T| = n$.[10] For each pair $(s,t) \in S \times T$ there is a match value $v_{s,t} \geq 0$ and a match-specific cost $c_{s,t} \geq 0$. Each school $s \in S$ has a budget $b_s \geq 0$. We impose two standard assumptions: (i) nonnegative surplus, $v_{s,t} \geq c_{s,t}$ for all $(s,t)$; and (ii) no intrinsically infeasible pairs, $b_s \geq c_{s,t}$ for all $(s,t)$. This is a one-to-one, quasilinear matching environment with transfers.

[7] For surveys, see Tamer (2010); Ho and Rosen (2017).

[8] See Mongell and Roth (1986); Abizada (2016) for matching with budget constraints; Roth and Peranson (1999); Klaus and Klijn (2005); Kojima et al. (2013); Nguyen and Vohra (2018) for matching with couples; and Kamada and Kojima (2015, 2017) for distributional constraints. In job matching, Kojima et al. (2020) characterizes all constraints that preserve the substitutes condition, which guarantees the existence of stable matchings.

[9] Recent work, including Liu et al. (2014), Liu (2020), and Chen and Hu (2023), proposes new stability concepts for settings with asymmetric information among agents.

[10] The framework can be extended to allow unequal numbers of schools and teachers. If some agents remain unmatched, we can introduce dummy schools or teachers with zero match value and zero cost to pair with the unassigned agents. The analysis then extends without modification.

Let $S = \{s_1, \dots, s_n\}$ denote the set of schools and $T = \{t_1, \dots, t_n\}$ the set of teachers, with $|S| = |T| = n$.[11] For each pair $(s,t) \in S \times T$ there is a match value $v_{s,t} \geq 0$ enjoyed by the school and a match-specific cost $c_{s,t} \geq 0$ borne by the teacher. Each school $s \in S$ has a budget $b_s \geq 0$. We impose two standard assumptions: (i) nonnegative surplus, $v_{s,t} \geq c_{s,t}$ for all $(s,t)$; and (ii) no intrinsically infeasible pairs, $b_s \geq c_{s,t}$ for all $(s,t)$. This is a one-to-one, quasilinear matching environment with transfers.

An *allocation* specifies who matches with whom and the wages paid on matched pairs.

**Definition 1** (Allocation)**.** An allocation is a pair $(\mu, w)$, where $\mu : S \to T \cup \{\emptyset\}$ is a matching. We assume each school hires at most one teacher and each teacher works at most at one school.[12] Let $S^\mu = \{s \in S : \mu(s) \in T\}$ and $T^\mu = \{t \in T : \mu^{-1}(t) \in S\}$. The wage vector is $w = \big(w_{s,\mu(s)}\big)_{s \in S^\mu}$ and assigns a nonnegative wage to each matched pair. We write $\mu(s) = \emptyset$ if school $s$ is unmatched and $\mu^{-1}(t) = \emptyset$ if teacher $t$ is unmatched.

Utilities are quasilinear with a zero outside option. If a school $s$ is matched with a teacher $t$ at wage $w_{s,t}$, then

$$u_s = v_{s,t} - w_{s,t} \qquad \text{and} \qquad u_t = w_{s,t} - c_{s,t}.$$

Unmatched agents receive utility 0. We denote by $u_s(\mu, w)$ and $u_t(\mu, w)$ the utilities of school $s$ and teacher $t$ under allocation $(\mu, w)$.

Following the standard approach, we define feasibility, individual rationality, and (budget-constrained) stability as follows.

**Definition 2** (Feasibility)**.** An allocation $(\mu, w)$ is *feasible* if wages are defined only on matched pairs and, for every $s \in S^\mu$ with $t = \mu(s)$,

$$w_{s,t} \;\leq\; b_s.$$

That is, the wage does not exceed the school's budget.

**Definition 3** (Individual rationality (IR))**.** An allocation $(\mu, w)$ is *individually rational* if every matched agent obtains nonnegative utility; i.e., for every $s \in S^\mu$ with $t = \mu(s)$,

$$v_{s,t} - w_{s,t} \;\geq\; 0 \quad \text{and} \quad w_{s,t} - c_{s,t} \;\geq\; 0.$$

Equivalently, both the school and the teacher weakly prefer their match to remaining unmatched.

**Definition 4** (Stability)**.** An allocation $(\mu, w)$ is *stable* if it is feasible and individually rational, and there is no *blocking pair* $(s,t) \in S \times T$ together with an *affordable* wage $\widehat{w}$ such that

$$\widehat{w} \;\leq\; b_s, \qquad v_{s,t} - \widehat{w} \;>\; u_s(\mu, w), \qquad \widehat{w} - c_{s,t} \;>\; u_t(\mu, w).$$

In words, no school–teacher pair can profitably deviate at a wage that respects the school's budget and makes both the school and the teacher strictly better off than under $(\mu, w)$.

[11]The framework can be extended to allow unequal numbers of schools and teachers. If some agents remain unmatched, we can introduce dummy schools or teachers with zero match value and zero cost to pair with the unassigned agents. The analysis then extends without modification.

[12]If one wishes to allow a school to hire multiple teachers, interpret each position as a separate "school–position" that hires one teacher; budgets are then position-specific.

We adopt the notion of core rather than the strict core of Crawford and Knoer (1981) and Kelso and Crawford (1982). When schools face no binding budget constraints, the two notions coincide. By contrast, with binding budgets, the strict core can be empty even in very small markets.[13] For this reason, we study the core throughout the paper.

We study optimal subsidy design when budget constraints generate inefficiencies. Before turning to the section of results, we formalize the subsidy object.

**Definition 5** (Subsidy allocation)**.** Given baseline budgets $b = (b_s)_{s\in S}$, a *subsidy allocation* is a nonnegative vector $\tau = (\tau_s)_{s\in S}$, and the post-subsidy budgets are

$$b_s^\tau \;=\; b_s + \tau_s \qquad \text{for all } s \in S.$$

Thus each school's budget weakly increases, and the increment for $s$ is $\tau_s$.

# 4 Results

This section first reviews classical results. We then examine how binding budget caps introduce novel considerations, leading to the subsidy design problem.

## 4.1 Existing results

**Existence of stable allocations and lattice structure with no budget constraints.**

If every school has an effectively unbounded budget ($b_s = +\infty$ for all $s$), stable allocations coincide with the utilitarian optimum (Shapley and Shubik, 1971). Formally, let

$$\mu^* \in \arg\max_{\mu} \sum_{s\in S} \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big).$$

Then there exist wages $w = (w_{s,\mu^*(s)})_{s\in S}$ such that $(\mu^*, w)$ is stable. Moreover, every stable allocation $(\mu, w)$ attains the same maximal total surplus. Equilibrium wages can be computed by the salary adjustment procedures in Crawford and Knoer (1981) and Kelso and Crawford (1982).

This benchmark is the classical assignment game: the set of core payoffs forms a lattice that supports all efficient matchings, with extremal points yielding the teacher- and school-optimal divisions of surplus. Accordingly, any inefficiency in our setting is driven by binding budgets. When budget constraints are absent or slack, efficiency obtains automatically.

**Existence of stable allocations and lattice structure with budget constraints.**

We now consider the case in which schools face budget constraints. When the wage set is finite, any deterministic tie-breaking rule induces single-valued choice functions over contracts. In our one-to-one, quasilinear environment, these choice functions satisfy substitutability and the law of aggregate demand in Hatfield and Milgrom (2005), which guarantees a nonempty set of stable allocations in this discrete market. Moreover, by Hatfield and Milgrom (2005, Theorem 3), the set of stable allocations forms a lattice with extremal elements—namely, the school-optimal and teacher-optimal stable allocations—and every other stable allocation lies between them.

[13] For example, with two schools and two teachers, zero costs, zero budgets, and identical strict school preferences for teacher 1 over teacher 2, any allocation employing teacher 1 can be weakly blocked by moving teacher 1 to the other school at the same wage of zero: the teacher is indifferent while the receiving school strictly gains. Hence no allocation lies in the strict core.

In the continuous market, the feasible wage set is compact and utilities are continuous in wages. Fisher (2020) establishes the existence of a stable allocation.[14] Moreover, Schlegel (2018) shows that, for any deterministic tie-breaking rule, the school-optimal and teacher-optimal core allocations in the continuous market arise as limits of their discrete counterparts as the wage grid goes to zero.[15]

Two observations will be useful below. First, introducing budgets in this one-to-one, quasilinear environment does not affect the existence or lattice structure of stable allocations. Second, budget caps *do* restrict the feasible contract set by truncating affordable wages, which can lead to inefficiencies. Our analysis, therefore, focuses on how targeted subsidies expand feasibility to recover efficient outcomes.

## 4.2 Complete information benchmark

In this subsection we first study the setup of complete information as a benchmark: we assume the planner observes the entire value profile $v = (v_{s,t})$, cost profile $c = (c_{s,t})$, and baseline budgets $b = (b_s)$. The planner's objective is to implement the utilitarian-optimal matching $\mu^*$.

If transfers are effectively unbounded, the planner can implement $\mu^*$ by granting sufficiently large subsidies to all schools, thereby reverting to the unconstrained case. One could also guarantee $\mu^*$ by raising each school's budget to the wage it pays in the school–optimal stable outcome of the unconstrained problem. Both approaches, however, may entail more financial assistance than necessary. We therefore seek the *minimal* subsidy vector that implements $\mu^*$.

Formally, let $b = (b_s)_{s\in S}$ denote baseline budgets. Choose a subsidy vector $\tau = (\tau_s)_{s\in S}$ and wages $w$ such that $(\mu^*, w)$ is stable under the post-subsidy budgets $b^\tau$, where $b^\tau_s = b_s + \tau_s$. The planner's problem is

$$\min_{\tau\in\mathbb{R}^n_+} \sum_{s\in S} \tau_s \qquad \text{subject to} \qquad (\mu^*, w) \text{ is stable under budgets } b^\tau.$$

**Minimum Subsidy Implementation (MSI) algorithm.**

We propose a constructive procedure to compute the smallest necessary subsidies: the *Minimum Subsidy Implementation (MSI)* algorithm. The idea is straightforward and mirrors a Bellman–Ford–type relaxation scheme to find the minimum subsidies needed. Concretely, starting from the lowest individually rational wages for the target matching $\mu^*$, if some rival school can profitably poach a teacher at an affordable wage, we raise that teacher's wage to the *lowest* level that deters all such deviations. Then subsidize only those schools that cannot afford the required wages. In this way, subsidies are introduced gradually and only where needed, keeping $\mu^*$ stable while minimizing total transfers. The algorithm proceeds in rounds, each with two stages.

**Round 1 (Initialization).** Fix the welfare-maximizing assignment $\mu^*$ and initialize at that matching: each school $s$ is paired with $\mu^*(s)$, and each teacher $t$ with $\mu^{*-1}(t)$. Set each matched teacher's wage to her cost, and set each school's budget to its initial budget:

$$w^1_{s,\mu^*(s)} \;=\; c_{s,\mu^*(s)}, \qquad b^1_s \;=\; b_s \quad \text{for each } s \in S.$$

By our assumption $b_s \geq c_{s,t}$ for all $(s,t)$, $w^1$ is affordable and we do not need to subsidize any school in this round.

[14] This contrasts with the many-to-one setting in Mongell and Roth (1986), where contracts fail substitutability when firms face budget constraints. In one-to-one matching with quasilinear utilities, substitutability holds.

[15] This result requires that teachers are gross substitutes for schools, which holds in our one-to-one, quasilinear framework.

**Rounds $r = 2, 3, \ldots$ (Updating).** Each subsequent round consists of the following two stages:

(i) **Wage stage.** For each school–teacher pair $(s,t)$, compute the lowest *deterrence wage* $d^r_{s,t}$ that prevents school $s$ from profitably deviating to $t$ at the current budgets:

$$d^r_{s,t} \;=\; \min\Big\{\, b^{r-1}_s,\;\; v_{s,t} - v_{s,\mu^*(s)} + w^{r-1}_{s,\mu^*(s)} \Big\}.$$

(Here the second term makes $s$ indifferent between $(s,\mu^*(s))$ at $w^{r-1}_{s,\mu^*(s)}$ and $(s,t)$; the $\min\{\cdot\}$ captures affordability.) Then, for each matched pair $(s,\mu^*(s))$, raise that teacher's wage (if needed) to defend all such deviations:

$$w^r_{s,\mu^*(s)} \;=\; \max_{s' \in S} \Big\{\, d^r_{s',\,\mu^*(s)} - c_{s',\,\mu^*(s)} + c_{s,\,\mu^*(s)} \Big\}.$$

Equivalently, $w^r_{s,\mu^*(s)} - c_{s,\mu^*(s)} = \max_{s' \in S}\{\, d^r_{s',\,\mu^*(s)} - c_{s',\,\mu^*(s)} \}$.

(ii) **Subsidy stage.** Lift any school's budget that is now below its required wage:

$$b^r_s \;=\; \max\{\, b^{r-1}_s,\;\; w^r_{s,\mu^*(s)} \,\} \quad \text{for each } s \in S.$$

**Termination.** Stop at the first round $r^* \geq 1$ such that $w^{r^*}_{s,\mu^*(s)} = w^{r^*-1}_{s,\mu^*(s)}$ for all $s \in S$ (no wage increases). The resulting subsidy vector is

$$\tau_s \;=\; b^{r^*}_s - b_s, \qquad s \in S.$$

**An illustrative example.**

To illustrate the algorithm, consider a $3 \times 3$ market with schools $S = \{s_1, s_2, s_3\}$ and teachers $T = \{t_1, t_2, t_3\}$. Costs are zero for all pairs ($c_{s,t} = 0$). The match values ($v_{s,t}$) are given in the table below. In this instance, the unique optimal matching is the diagonal $\mu^*$ with $\mu^*(s_i) = t_i$, yielding total welfare $3 + 6 + 6 = 15$.[16]

| | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|
| $t_1$ | 3 | 1 | 0 |
| $t_2$ | 4 | 6 | 4 |
| $t_3$ | 8 | 7 | 6 |

To visualize values at the pair level, we display for each school a bar chart whose bar heights are the corresponding match values.

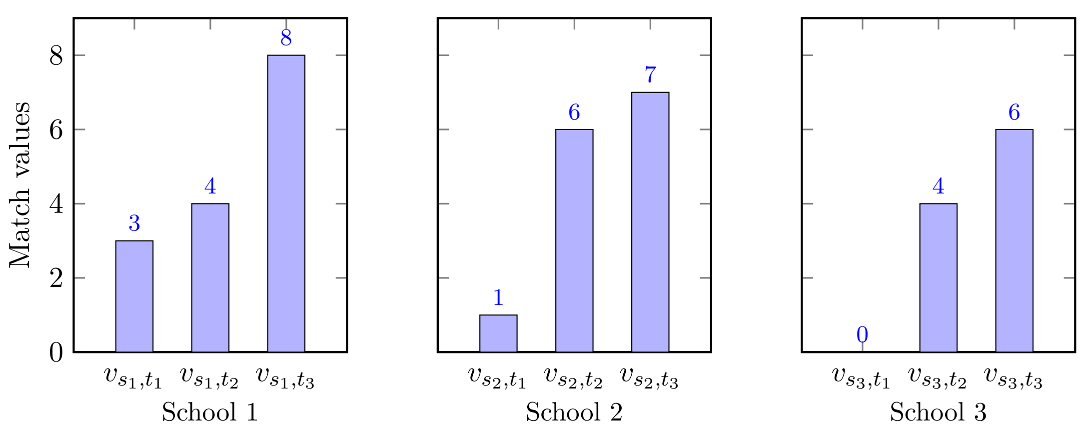


Figure 1: Match values for the $3 \times 3$ example.

[16] When the market is large, one can use the Hungarian algorithm to identify a welfare-maximizing assignment.

We first consider the case where budgets are effectively unbounded. Then $\mu^*$ can be supported in a stable allocation. Moreover, the school-optimal allocation is

$$\mu^*(s_i) = t_i, \quad w_{s_1,t_1} = 0, \quad w_{s_2,t_2} = 3, \quad w_{s_3,t_3} = 5$$

The following figure reports, for every $(s_i, t_j)$, the decomposition of the total match value into the teacher's (yellow) and school's (blue) utilities in the school-optimal allocation.

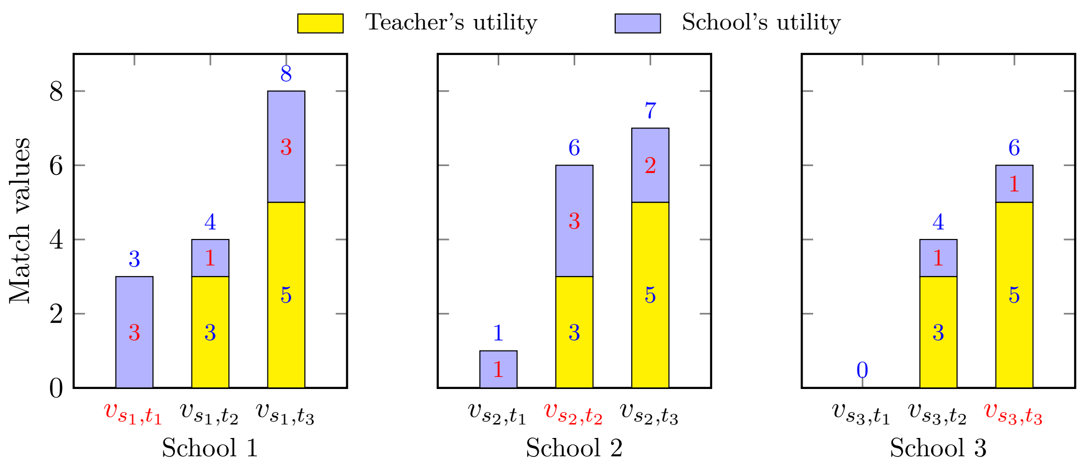


Figure 2: The school-optimal allocation without budget constraints.

We now examine how budget constraints can lead to inefficiency by considering the following illustrative budget configuration: $b_{s_1} = 4$, $b_{s_2} = 1$, and $b_{s_3} = 2$. Budgets are depicted by red dashed lines in the figure below.

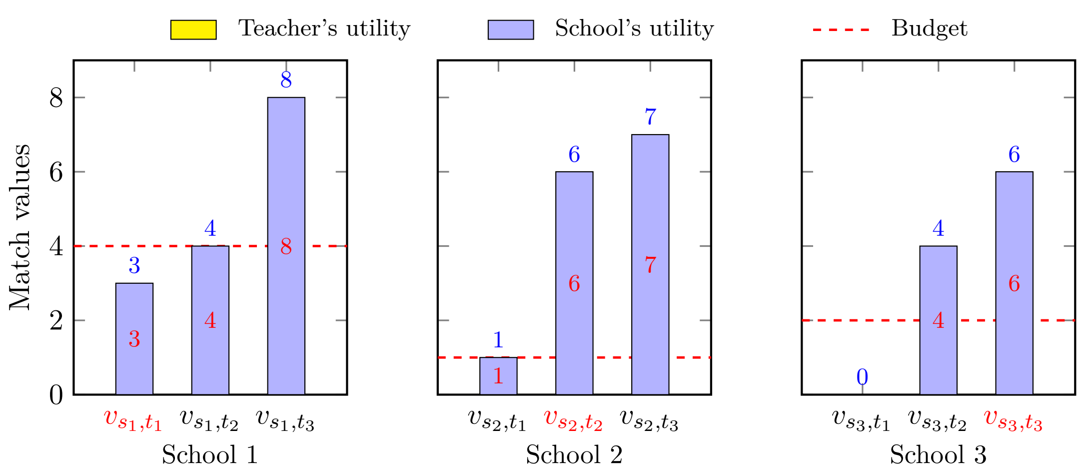


Figure 3: $b = \{4, 1, 2\}$ for the $3 \times 3$ example.

Any allocation that implements $\mu^*$ must match $s_3$ with $t_3$, and therefore must pay $t_3$ a wage $w_{s_3,t_3} \leq b_{s_3} = 2$, since the wage must be covered by $s_3$'s budget. It follows that teacher $t_3$ obtains utility at most 2. Likewise, in $\mu^*$, school $s_1$ is matched with $t_1$. Since wages are nonnegative, the utility of $s_1$ is at most the total match value, $v_{s_1,t_1} = 3$.

Consider the pair $(s_1, t_3)$. School $s_1$ can afford any $\widehat{w} \in (2, 4]$ to $t_3$, and at such a wage,

$$\begin{aligned} u_{t_3} &= \widehat{w} \ > \ 2 && \text{(strict gain for teacher } t_3\text{)},\\ u_{s_1} &= v_{s_1,t_3} - \widehat{w} \ \geq \ 8 - 4 = 4 \ > \ 3 \geq u_{s_1}(\mu^*, w) && \text{(strict gain for school } s_1\text{)}. \end{aligned}$$

Hence $(s_1, t_3, \widehat{w})$ is a profitable deviation that respects $s_1$'s budget and strictly benefits both agents. Therefore, no stable allocation can implement $\mu^*$ under the stated budget caps.

Now let us consider the MSI algorithm. In the initialization step, each school is matched to its $\mu^*$ partner at the lowest individually rational wage. Since $c_{s,t} = 0$ for all $(s, t)$, the initial wages are all zero in Round 1. In Round 2, we compute the lowest deterrence wages $\{d^2_{s,t}\}$ that prevent profitable poaching at current budgets. This yields the following deterrence wages

$$\begin{aligned} d^2_{s_1,t_1} &= \ \ 0, & d^2_{s_1,t_2} &= 1, & d^2_{s_1,t_3} &= 4,\\ d^2_{s_2,t_1} &= -5, & d^2_{s_2,t_2} &= 0, & d^2_{s_2,t_3} &= 1,\\ d^2_{s_3,t_1} &= -6, & d^2_{s_3,t_2} &= -2, & d^2_{s_3,t_3} &= 0, \end{aligned}$$

To interpret these deterrence wages, consider $d^2_{s_1,t_2} = 1$ as an example: this means that, to prevent school $s_1$ from profitably poaching teacher $t_2$, the wage of $t_2$ must be set to at least 1. Similarly, $d^2_{s_1,t_3} = 4$ indicates that $s_1$ must face a wage of at least 4 on $t_3$ to deter a profitable deviation. Each entry captures the minimum wage required to make poaching unprofitable at the current round's budget levels.

After computing these deterrence wages, the MSI algorithm updates the wages on the matched pairs accordingly. By construction, the updated wage on each matched pair equals the highest deterrence wage imposed by potential poachers. This yields:

$$w^2_{s_1,t_1} = 0, \qquad w^2_{s_2,t_2} = 1, \qquad w^2_{s_3,t_3} = 4.$$

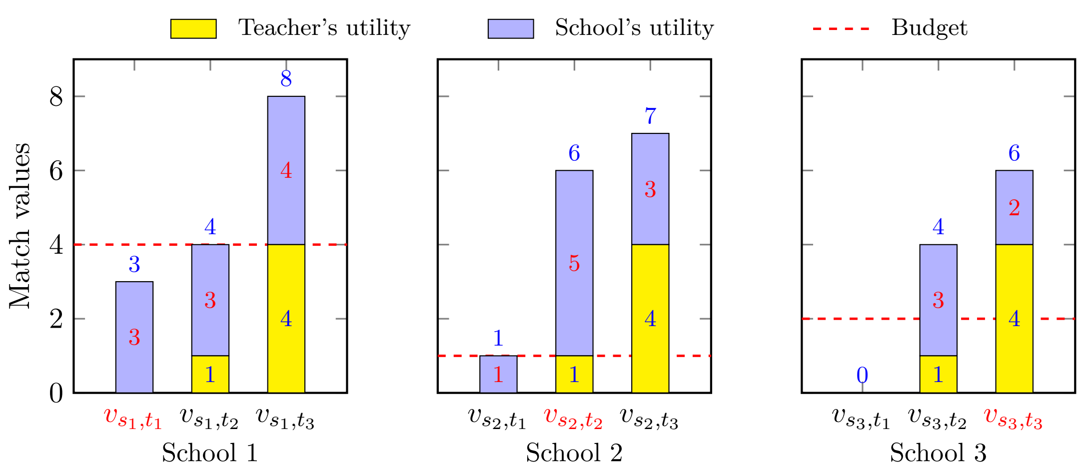


Figure 4: Round 2 wage stage of the MSI algorithm.

At these required wages, school $s_3$ cannot afford $w^2_{s_3,t_3} = 4$ given $b_{s_3} = 2$. The MSI subsidy step therefore augments $s_3$'s budget by 2 (from 2 to 4). No other school requires additional funds at this stage.

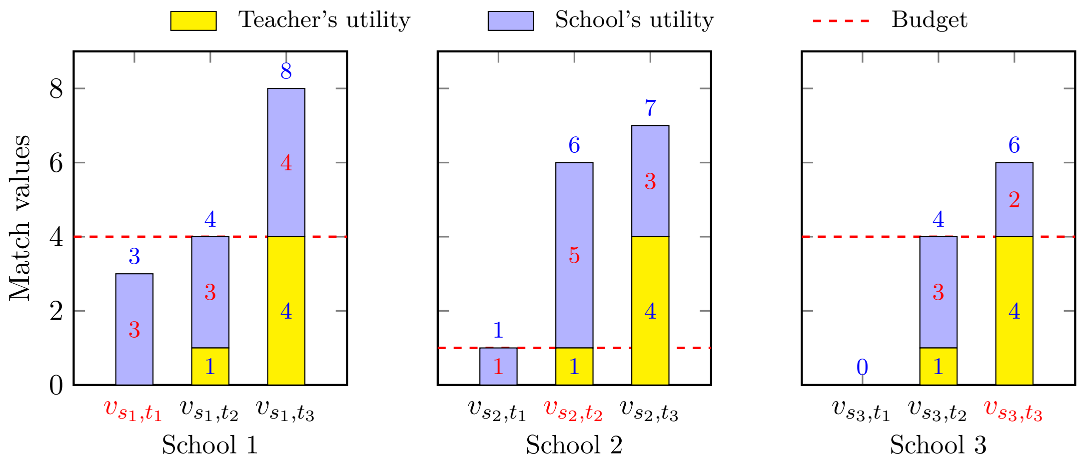


Figure 5: Round 2 subsidy stage of the MSI algorithm.

With $b_{s_3}$ raised, the next wage update increases the deterrence requirement for $t_2$ to $w^3_{s_2,t_2} = 2$.

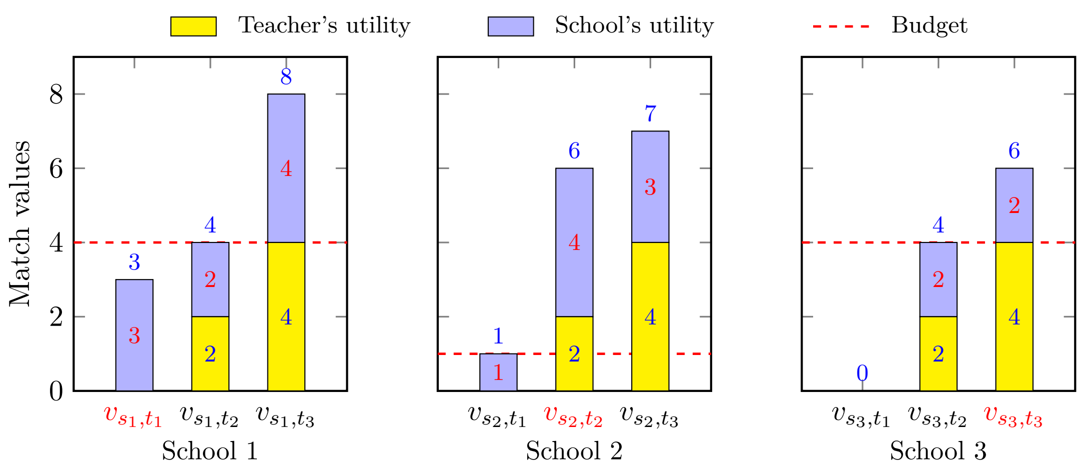


Figure 6: Round 3 wage stage of the MSI algorithm.

This makes $s_2$'s budget insufficient ($b_{s_2} = 1$), so the subsidy step grants 1 unit to $s_2$ (raising $b_{s_2}$ to 2).

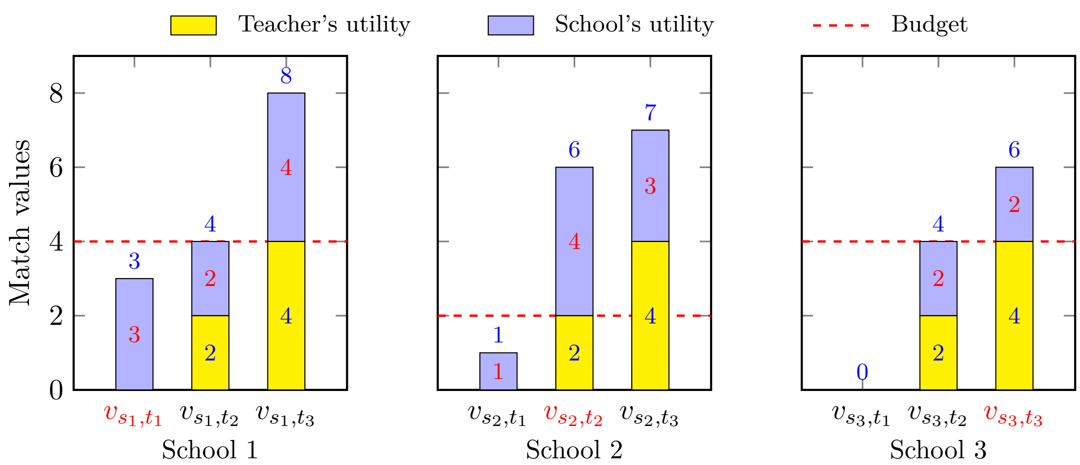


Figure 7: Termination of the MSI algorithm.

No further wage increases occur thereafter, so MSI terminates. The final outcome implements $\mu^*$ with the minimal subsidy vector $\tau = (0, 1, 2)$ and the total subsidies are 3 units.

**Interpretation and discussion.**

- *Economic meaning.* The wage stage computes the *lowest* deterrence wages needed to prevent deviations from $\mu^*$. The subsidy stage injects funds only where a school's current budget is insufficient to cover its required wage. Thus, subsidies are used only when necessary to preserve stability.
- *Convergence.* The MSI process terminates in finitely many rounds. Since $\mu^*$ is socially optimal, the underlying graph of required wage adjustments contains no positive-weight cycles; otherwise, there will be a matching strictly better than $\mu^*$. This guarantees that the sequence of wage updates—analogous to shortest-path computations in the Bellman-Ford algorithm—stabilizes after finitely many steps.
- *Optimality at termination.* Upon termination at round $r^*$, the outcome $(\mu^*, w^{r^*})$ is stable under budgets $b^{r^*}$. Hence, MSI implements $\mu^*$ using the computed wage and budget profiles.
- *Monotonicity and minimality.* Wages and budgets are weakly increasing across rounds. No subsidy is wasted: the budget of a school is raised only when needed to meet its updated deterrence wage. At termination, each school's budget has been raised just enough to support $\mu^*$, yielding a componentwise minimal implementation.

These properties are summarized formally below.

**Proposition 1** (Properties of MSI)**.** *To implement the socially optimal matching $\mu^*$, the MSI algorithm satisfies:*

*(i) MSI terminates in finitely many rounds.*

*(ii) If MSI terminates at round $r^*$ with wage profile $w^{r^*}$ and budget vector $b^{r^*}$, then $(\mu^*, w^{r^*})$ is stable under budgets $b^{r^*}$.*

*(iii) If $(\mu^*, \tilde{w})$ is stable under some budget vector $\tilde{b} = b + \tilde{\tau}$, then $\tilde{b}_s \geq b_s^{r^*}$ for all $s \in S$. In particular, $b^{r^*}$ is componentwise minimal among all budget vectors that admit a stable outcome implementing $\mu^*$.*

### 4.3 Incomplete information

The preceding analysis assumes complete information: the planner observes the full environment $(v, c, b)$. This benchmark is analytically useful—it identifies the *minimum* subsidies required to implement $\mu^*$—but its policy relevance is limited, since the complete–information assumption rarely holds in practice. In practice, the planner typically observes an existing allocation, but does not have full knowledge of schools' and teachers' preferences over potential partners, and baseline budgets may also be unknown.

We therefore turn to an incomplete–information setting. Throughout this subsection, the planner observes an existing allocation $(\mu, w)$ and knows that it is stable under the true (but partially unobserved) primitives. We begin with the case in which match values are unobserved, but costs and baseline budgets are known: the planner knows $c = (c_{s,t})_{s,t}$ and $b = (b_s)_s$, but not $v = (v_{s,t})_{s,t}$. We start with this case because it is empirically plausible: many components of $c$ (e.g., commuting costs or the quality and condition of school facilities) can be measured or credibly estimated from administrative and census data, whereas match values reflect school–teacher fit and productivity that are typically unobserved to the planner. Importantly, we continue to assume that match values are known to the agents themselves—i.e., schools and teachers make informed decisions—but the planner lacks access to this information. Our main result in this environment is Theorem 1. We then extend the analysis to two progressively weaker informational environments: first, when both values and costs $(v, c)$ are unobserved by the planner (Theorem 2); and finally, when none of $(v, c, b)$ are known to the planner (Theorem 3).

Before introducing new definitions and analysis, we first illustrate the incomplete–information logic with a $3 \times 3$ example that will recur throughout Section 4.3. The planner observes the following stable outcome $(\mu, w)$ and budgets $b$ but not the value profile $v$, and costs are known to be zero for all pairs.

$$\begin{cases} \mu(s_1) = t_1, & w_{s_1,t_1} = 1, & b_{s_1} = 2 \\ \mu(s_2) = t_2, & w_{s_2,t_2} = 2, & b_{s_2} = 3 \\ \mu(s_3) = t_3, & w_{s_3,t_3} = 3, & b_{s_3} = 6 \end{cases}$$

Figure 8 depicts the observed wages (yellow), unknown schools' utilities (gray), and budget caps (red dashed lines).

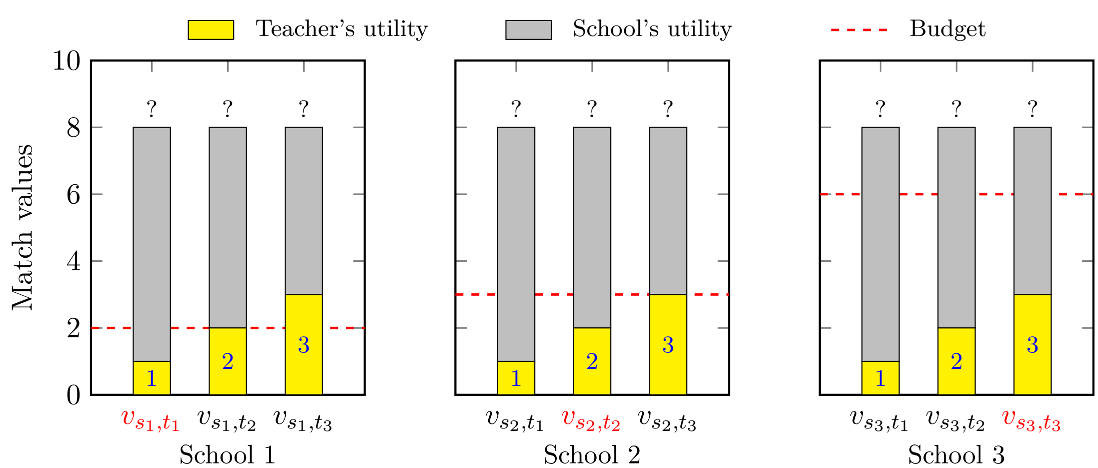


Figure 8: $3 \times 3$ example with incomplete information.

Even though $v$ is unobserved, stability and affordability already imply informative inequalities. For school $s_1$, stability at $(s_1, t_1)$ yields

$$v_{s_1,t_1} \;\geq\; w_{s_1,t_1} \;=\; 1.$$

Beyond this, the data are silent about $v_{s_1,t_2}$ and $v_{s_1,t_3}$. Indeed, $s_1$'s maximum affordable offer is $b_{s_1} = 2$, while the incumbent effective wages for $t_2$ and $t_3$ are $w_{s_2,t_2} = 2$ and $w_{s_3,t_3} = 3$, respectively. Since $s_1$ cannot make a strictly higher effective offer to either $t_2$ or $t_3$, the observed stability places no additional restriction on $s_1$'s valuations for those teachers.

The planner can infer more from the observations for $s_2$. Stability of $(s_2, t_2)$ implies

$$v_{s_2,t_2} \;\geq\; w_{s_2,t_2} \;=\; 2.$$

Moreover, $s_2$ can afford to poach $t_1$ because $b_{s_2} = 3 > w_{s_1,t_1} = 1$. Stability of $(\mu, w)$ therefore requires that $t_2$ be at least as attractive to $s_2$ as $t_1$ at the current wages:

$$D_{21} \;\equiv\; v_{s_2,t_2} - v_{s_2,t_1} \;\geq\; w_{s_2,t_2} - w_{s_1,t_1} \;=\; 1.$$

Analogously for $s_3$, stability of $(s_3, t_3)$ gives

$$v_{s_3,t_3} \;\geq\; w_{s_3,t_3} \;=\; 3.$$

Since $b_{s_3} = 6$ exceeds both incumbent wages $w_{s_1,t_1} = 1$ and $w_{s_2,t_2} = 2$, school $s_3$ could afford to make offers to $t_1$ and $t_2$. Thus stability imposes two revealed-preference inequalities:

$$D_{31} \;\equiv\; v_{s_3,t_3} - v_{s_3,t_1} \;\geq\; w_{s_3,t_3} - w_{s_1,t_1} \;=\; 2,$$
$$D_{32} \;\equiv\; v_{s_3,t_3} - v_{s_3,t_2} \;\geq\; w_{s_3,t_3} - w_{s_2,t_2} \;=\; 1.$$

Figure 9 depicts these values in the graph: $D_{21}$, $D_{31}$, and $D_{32}$.

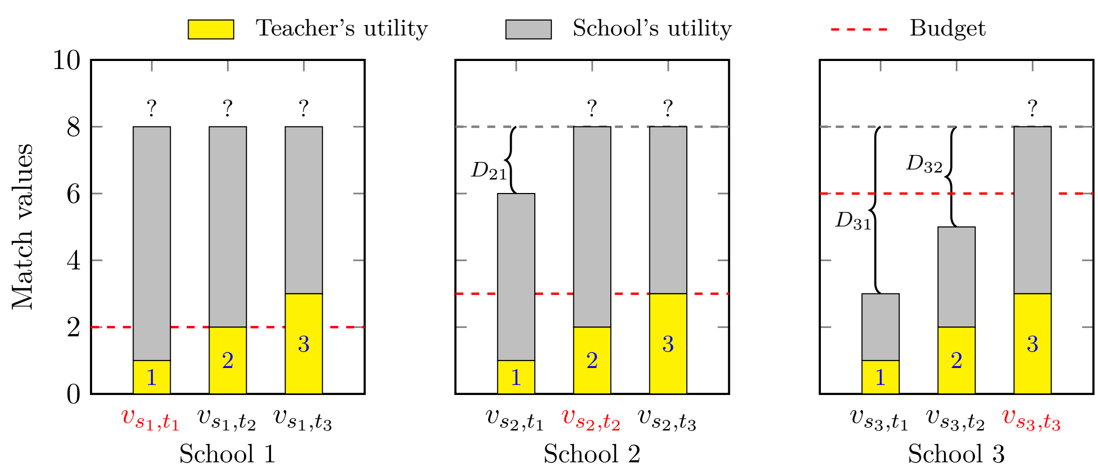


Figure 9: Inference from the observed stable allocation.

### 4.3.1 Setup and notation

Equipped with the reasoning from the illustrative example, there are many value profiles consistent with the observed allocation. Formally, given an observed stable outcome $(\mu, w)$ under costs $c$ and budgets $b$, define the set of *consistent* value profiles

$$\mathcal{V}(\mu, w; b, c) \;=\; \{\, v : \; (\mu, w) \text{ is stable under } (v, c, b) \,\}.$$

To compare matchings across such profiles, we measure welfare for any matching $\mu$ by

$$W(\mu; v, c) \;=\; \sum_{s \in S} \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big).$$

When $v$ is unobserved, the planner cannot identify the welfare–maximizing matching. We therefore pursue a *robust* policy: choose subsidies that (weakly) improve outcomes for *every* value profile consistent with the observed data. To formalize this objective, we introduce two definitions.

**Definition 6** (Post–subsidy optimal welfare)**.** Fix $v, c, b$. For a subsidy allocation $\tau = (\tau_s)_{s \in S}$ (so $b_s^\tau = b_s + \tau_s$), define

$$\overline{W}(v, c, b, \tau) \;\equiv\; \max_{\mu} \Big\{ W(\mu; v, c) \;:\; \exists\, w \text{ with } (\mu, w) \text{ stable under } (v, c, b^\tau) \Big\}.$$

Thus $\overline{W}(v, c, b, \tau)$ is the highest welfare attainable after applying $\tau$.

**Definition 7** (Non–distortive subsidy)**.** In an allocation $(\mu, w)$ with budgets $b$ and costs $c$, a subsidy allocation $\tau$ is *non–distortive* for $(\mu, w; b, c)$ if, for all $v \in \mathcal{V}(\mu, w; b, c)$,

$$\overline{W}(v, c, b, \tau) \;\geq\; W(\mu; v, c).$$

Equivalently, under the post–subsidy budgets $b^\tau$, there exists a stable allocation whose welfare is weakly higher than that of the observed matching $\mu$ for every value profile consistent with $(\mu, w; b, c)$.

Non–distortiveness requires the subsidy to be robust across all value profiles consistent with what the social planner observes: regardless of which $v \in \mathcal{V}(\mu, w; b, c)$ is true, the best stable outcome after the subsidy does not reduce welfare relative to the status quo $(\mu, w)$. If the inequality in Definition 7 fails, then there exists a consistent $v$ for which every post–subsidy stable outcome is strictly worse than $\mu$, in which case the subsidy is *distortive*. Our goal in what follows is to characterize the set of non–distortive subsidies.

### 4.3.2 Direct effect

Providing subsidies relaxes previously binding budget constraints and can thereby change the set of stable allocations. In particular, schools that were constrained in the pre-subsidy market may become able to compete for teachers they could not previously afford. To capture such constraints at the observed outcome, we introduce an outbidding relation.

**Definition 8** (Outbid)**.** Given an observed stable allocation $(\mu, w)$ under budgets $b$ and costs $c$, we say that school $s_i$ is *outbid* by school $s_j$ if

$$b_{s_i} - c_{s_i,\mu(s_j)} \;\leq\; w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}.$$

Conversely, we say that $s_i$ *is not outbid by* $s_j$, denoted $s_i \xrightarrow{\mu} s_j$, if

$$b_{s_i} - c_{s_i,\mu(s_j)} \;>\; w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}.$$

Equivalently, $s_i$ is outbid by $s_j$ if $s_i$ cannot offer teacher $\mu(s_j)$ any effective wage (wage minus cost) that strictly exceeds her current effective wage at $s_j$, and $s_i \xrightarrow{\mu} s_j$ if it can.

A subsidy to a previously constrained school may enable it to approach a teacher it could not afford before, creating upward wage pressure on that teacher's current employer. We quantify this *direct* wage pressure as follows.

**Definition 9** (Direct effect)**.** Given an observed stable outcome $(\mu, w)$ under budgets $b$ and costs $c$, and a subsidy allocation $\tau$ (so $b^\tau_s = b_s + \tau_s$), define for each school $s_j$:

$$\Delta^d_j \;\equiv\; \max_{i:\ s_i \text{ is outbid by } s_j \text{ in } (\mu,w)} \Big[\big(b^\tau_{s_i} - c_{s_i,\mu(s_j)}\big) - \big(w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}\big)\Big]_+,$$

where $[x]_+ \coloneqq \max\{x, 0\}$, with the convention $\Delta^d_j = 0$ if the index set is empty.

*Intuition (direct effect).* Consider any rival $s_i$ that was previously outbid by $s_j$. After subsidies, $s_i$ may be able to exceed the incumbent effective wage of $\mu(s_j)$, forcing $s_j$ to raise that wage to defend its match. The quantity $\Delta^d_j$ is the maximal *direct* wage-lifting pressure that $s_j$ may face from its previously outbid rivals.

Such direct effects can generate efficiency losses. Consider our illustrative example and the following subsidy allocation:

$$\tau_{s_1} = 2, \qquad \tau_{s_2} = 0, \qquad \tau_{s_3} = 0.$$

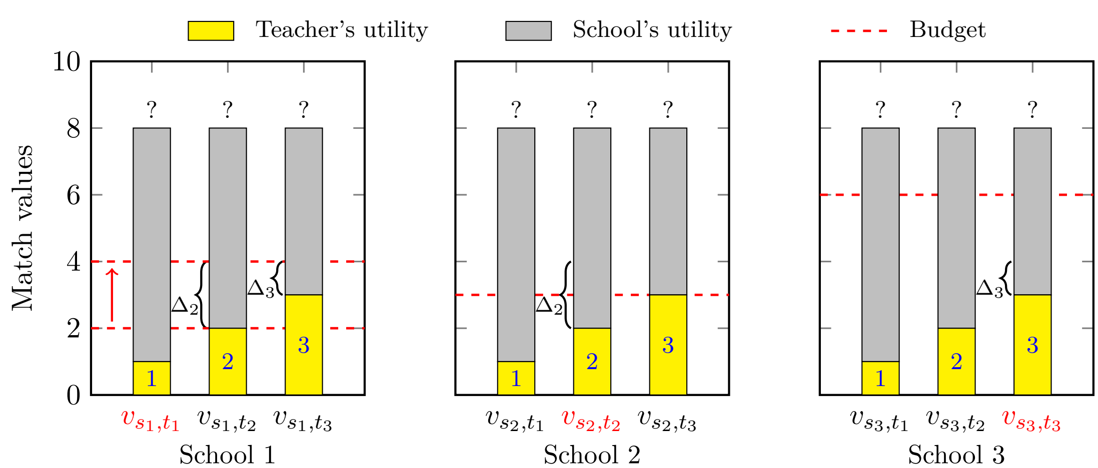


Figure 10: Direct effect illustration.

As depicted in Figure 10, the subsidy to $s_1$ increases its budget, creating wage-lifting pressures of 2 on $s_2$ and 1 on $s_3$. Because $s_3$ has a sufficiently large budget ($b_{s_3} = 6$), it can afford to defend its current match with $t_3$ against potential poaching from $s_1$. In contrast, $s_2$ has a budget of only 3, which is insufficient to defend its partner $t_2$ from the new offer that $s_1$ can now make. Consequently, even when $s_2$ and $t_2$ generate a high match value, $s_2$ may be forced to leave $t_2$, resulting in a less efficient allocation.

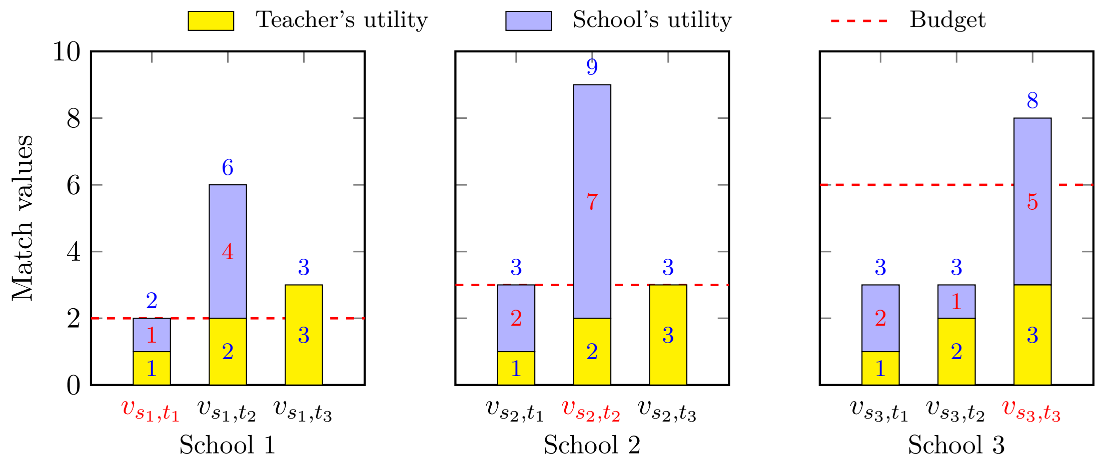


Figure 11: Efficiency loss generated by direct effect.

The value profile illustrated in Figure 11 is *consistent* with the observed pre-subsidy allocation $(\mu, w)$. After the subsidy $\tau_{s_1} = 2$, school $s_1$ can make an offer to $t_2$ that strictly dominates the highest affordable offer of $s_2$, thereby forcing $s_2$ to relinquish $t_2$. Being unable to match the new effective wage, $s_2$ loses $t_2$ even though the match $(s_2, t_2)$ generates a high underlying value. The resulting post-subsidy allocation is therefore strictly less efficient than the original stable outcome.

We refer to $b^\tau_s - w_{s,\mu(s)}$ as school $s$'s *budget reserve*: the unspent budget remaining after paying its incumbent teacher under post-subsidy budgets $b^\tau$. The preceding example shows that relaxing one school's constraint can unintentionally destabilize other matches, generating welfare losses via direct poaching. To prevent such distortions, a subsidy allocation must ensure that every school retains sufficient budget reserve to defend its incumbent match against the strongest potential challenge from previously constrained rivals. This requirement is captured by the following condition.

**Condition 1** (Direct-effect coverage (DC))**.** *In an allocation $(\mu, w)$ with budgets $b$ and costs $c$, a subsidy allocation $\tau$ satisfies* DC *if, for every $s_j$,*

$$b^\tau_{s_j} - w_{s_j,\mu(s_j)} \;\geq\; \Delta^d_j.$$

*Equivalently, for any $s_i$ that is outbid by $s_j$ before the subsidy,*

$$b^\tau_{s_j} - c_{s_j,\mu(s_j)} \;\geq\; b^\tau_{s_i} - c_{s_i,\mu(s_j)}.$$

*Intuition.* DC requires each school $s_j$ to hold enough post-subsidy budget reserve to absorb the largest direct wage-lifting pressure $\Delta^d_j$ generated by its previously outbid rivals. If DC fails for some $s_j$, a formerly constrained rival $s_i$ may offer a higher effective wage to $\mu(s_j)$ after receiving the subsidy, inducing $\mu(s_j)$ to switch and thereby lowering total welfare. In this case, the subsidy is distortive via the direct-poaching channel. We formalize this observation below:

**Lemma 1** (Non-distortion implies DC)**.** *In an allocation $(\mu, w)$ with budgets $b$ and costs $c$, if a subsidy allocation $\tau$ is non-distortive, then $\tau$ satisfies* DC*.*

### 4.3.3 Spillover effect

Direct wage pressure at one school can propagate through the market. When a subsidized school challenges an incumbent for its teacher, the incumbent may have to raise that teacher's wage to block the deviation; the higher wage lowers the incumbent's own payoff at its current match and can make some *other* teacher comparatively more attractive. Therefore, direct wage pressure can travel along a chain of bid–counterbid moves that is determined by the pre–subsidy structure of the observed allocation. We illustrate this transmission with the following example, where the subsidy allocation is

$$\tau_{s_1} = 0, \qquad \tau_{s_2} = 2, \qquad \tau_{s_3} = 0.$$

so only $s_2$ receives 2 units of funds.

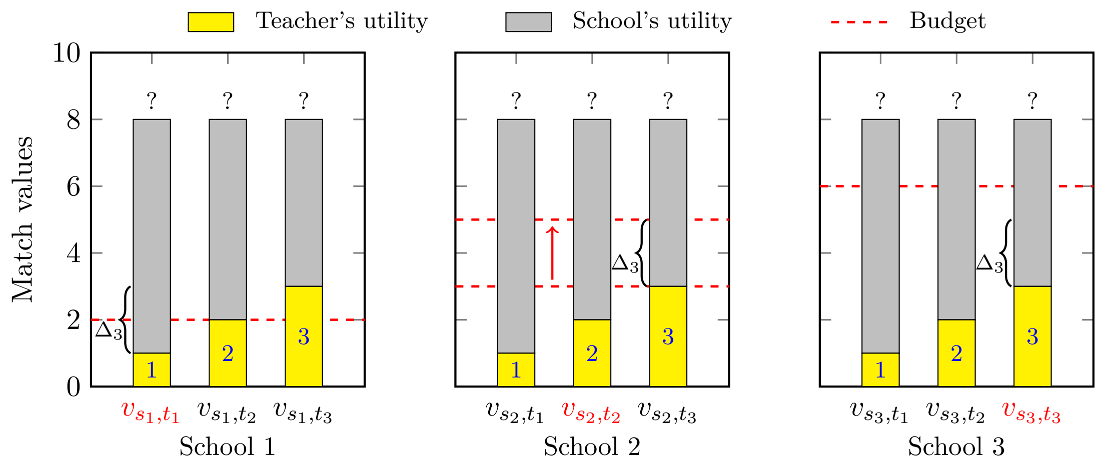


Figure 12: Spillover effect illustration.

In the configuration of Figure 12, revealed preference at the observed outcome implies that $s_2$ does *not* wish to pursue $t_1$ after receiving the subsidy. However, with $\tau_{s_2} = 2$ (so $b^{\tau}_{s_2} = 5$), school $s_2$ can feasibly approach $t_3$, whose incumbent wage is $w_{s_3,t_3} = 3$. School $s_3$ can defend $t_3$ (its budget is $b_{s_3} = 6$) by raising $t_3$'s wage, which reduces $s_3$'s payoff at the match $(s_3, t_3)$. This reduction makes $t_1$ relatively more attractive to $s_3$, so $s_3$ may next bid for $t_1$. Hence the initial direct pressure faced by $s_3$ is *transmitted* to $s_1$ through the chain $s_2 \to s_3 \to s_1$.

To formalize this propagation, we now introduce two notions: (i) a *spillover chain*, which describes the pre-subsidy paths along which direct pressure can travel, and (ii) the *spillover effect* of a subsidy, which quantifies the induced wage pressure transmitted along such chains.

**Definition 10** (Spillover chain)**.** Given an observed stable outcome $(\mu, w)$ under budgets $b$ and costs $c$, we say there exists a *spillover chain* from $s_i$ to $s_j$ if there is a sequence of schools

$$C = (s_{k_1}, s_{k_2}, \dots, s_{k_m}) \quad \text{with } k_1 = i,\ k_m = j,\ m \geq 2,$$

such that

$$s_{k_1} \xrightarrow{\mu} s_{k_2} \xrightarrow{\mu} s_{k_3} \cdots \xrightarrow{\mu} s_{k_m}.$$

At each step, before the subsidy, the predecessor $s_{k_\ell}$ can feasibly approach the teacher employed by its successor $s_{k_{\ell+1}}$, so wage-lifting pressure can propagate along the chain from $s_i$ to $s_j$.

Once a direct effect arises at some school, the resulting wage pressure may propagate along the spillover chains defined above. For example, if $s_1$ raises its wage in response to pressure, it may then find another teacher relatively more attractive, thereby transmitting pressure further along the chain. To formalize this propagation, we define the spillover effects below.

**Definition 11** (Spillover effect)**.** Given an observed stable outcome $(\mu, w)$ under budgets $b$ and costs $c$, and a subsidy allocation $\tau$ (so $b^\tau_s = b_s + \tau_s$), consider a spillover chain

$$C:\ s_1 \xrightarrow{\mu} s_2 \xrightarrow{\mu} \cdots \xrightarrow{\mu} s_j.$$

Let $\Delta^d_1$ denote the direct effect at $s_1$ (Definition 9). Define the chain–specific spillover effects $\{\delta^C_i\}_{i=2}^j$ recursively by

$$\delta^C_2 \ \equiv\ \min\Big\{\Delta^d_1,\ b^\tau_{s_1} - c_{s_1,\mu(s_2)} - \big(w_{s_2,\mu(s_2)} - c_{s_2,\mu(s_2)}\big)\Big\},$$

and for $i = 3, \ldots, j$,

$$\delta^C_i \ \equiv\ \min\Big\{\delta^C_{i-1},\ b^\tau_{s_{i-1}} - c_{s_{i-1},\mu(s_i)} - \big(w_{s_i,\mu(s_i)} - c_{s_i,\mu(s_i)}\big)\Big\}.$$

We refer to $\delta^C_j$ as the *spillover effect* reaching $s_j$ along $C$. The total spillover effect faced by $s_j$ is

$$\Delta^s_j \ \equiv \sup_{C:\,\text{spillover chains ending at } s_j} \delta^C_j,$$

with the convention that the supremum over an empty family equals 0.

*Remark (budget-capped propagation).* The $\min\{\cdot,\cdot\}$ in the recursion for $\delta$ reflects that, at each link of a spillover chain, the pressure that can be transmitted is capped both by the incoming pressure and by the successor's affordability gap (determined by its post-subsidy budget and costs). Consequently, spillover pressure is weakly decreasing along any chain: $\delta^C_1 \geq \delta^C_2 \geq \cdots \geq \delta^C_j$.

*Remark (cycles).* A spillover chain may, in principle, include a cycle. However, because the transmitted pressure weakly declines at each step, traversing a cycle cannot amplify the effect. It therefore suffices to consider *simple* (cycle-free) chains, which already capture the maximal spillover effect transmitted through the network of schools:

$$\Delta^s_j \ = \max_{C:\,\text{simple spillover chains ending at } s_j} \delta^C_j.$$

Equipped with the above definitions, we return to Figure 12. School $s_3$ faces a direct wage pressure of 2. Because there is a spillover chain from $s_3$ to $s_1$, this pressure is transmitted downstream: $s_3$'s defense of $t_3$ can prompt it to seek $t_1$, thereby imposing a spillover pressure of 2 on $s_1$. If $s_1$'s post-subsidy budget reserve $b^\tau_{s_1} - w_{s_1,\mu(s_1)}$ is insufficient to absorb this pressure, $s_1$ may be unable to defend $t_1$, potentially generating an efficiency loss. Figure 13 exhibits a value profile consistent with the observed outcome in which $s_1$ is indeed forced to relinquish $t_1$ after the subsidy, leading to a welfare loss.

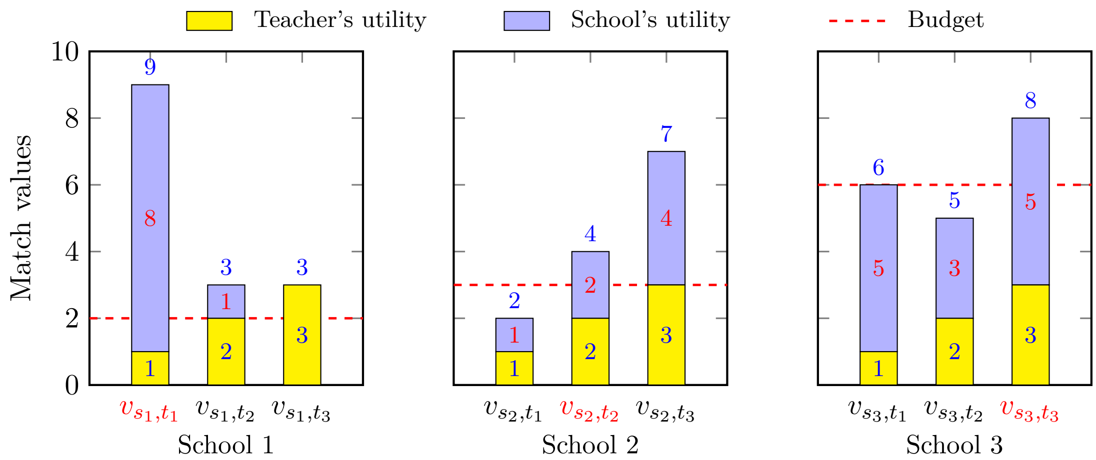


Figure 13: Efficiency loss generated by spillover effect.

Motivated by this transmission logic, we require subsidies to cover not only each school's *direct* pressure but also the *spillover* that can reach it through any spillover chain.

**Condition 2** (Spillover coverage (SC))**.** *In an allocation* $(\mu, w)$ *with budgets* $b$ *and costs* $c$*, a subsidy allocation* $\tau$ *satisfies* SC *if, for every school* $s_j$*,*

$$b^{\tau}_{s_j} - w_{s_j,\mu(s_j)} \ \geq \ \Delta^s_j.$$

*Intuition.* SC guarantees that each school has enough post–subsidy budget reserve to absorb the largest wage pressure that can arrive via spillover chains originating anywhere in the market. If SC fails at some $s_j$, then—even if direct-effect coverage holds—pressure propagated from upstream schools can still induce a profitable deviation at $\mu(s_j)$, making every post–subsidy stable outcome strictly worse than the status quo. In that case, the subsidy is distortive via spillovers. We formalize this necessity below.

**Lemma 2** (Non-distortion implies SC)**.** *In an allocation* $(\mu, w)$ *with budgets* $b$ *and costs* $c$*, if a subsidy allocation* $\tau$ *is non-distortive, then* $\tau$ *satisfies* SC*.*

### 4.3.4 Main result

Lemmas 1 and 2 establish that both *Direct-effect coverage (DC)* and *Spillover coverage (SC)* are necessary for a subsidy allocation to be non-distortive. However, these results do not yet tell us how to design such a subsidy. The next theorem provides a complete answer by showing that the two conditions are not only necessary but also sufficient.

**Theorem 1** (Characterization of non-distortive subsidies)**.** *In an allocation* $(\mu, w)$ *with budgets* $b$ *and costs* $c$*, a subsidy allocation* $\tau$ *is non-distortive if and only if* $\tau$ *satisfies both* DC *and* SC*. Equivalently, for all* $s_j$*,*

$$b^{\tau}_{s_j} - w_{s_j,\mu(s_j)} \ \geq \ \Lambda_j \ \equiv \ \max\{\Delta^d_j, \ \Delta^s_j\}.$$

Theorem 1 closes the loop: the two coverage conditions are jointly necessary and sufficient for non-distortion. Together, they provide a full characterization of all subsidy allocations that guarantee the existence of a post-subsidy stable allocation whose welfare is weakly higher than that of the status quo, for every value profile consistent with the observed outcome.

Although Theorem 1 is an existence result and does not prescribe how to select among post-subsidy stable outcomes, its proof is constructive: under budgets $b^\tau$, the school–optimal stable allocation (with a tie-breaking rule favoring the status-quo matching) achieves weakly higher welfare than the status quo. Therefore, if the planner can operate a centralized clearinghouse, a welfare-improving outcome can be implemented by running a school–proposing extended Deferred Acceptance (DA) algorithm in reality.

Operationally, the theorem reduces design or verification of a non-distortive policy to a system of simple inequalities:

$$b^{\tau}_{s_j} \;\geq\; w_{s_j,\mu(s_j)} + \Lambda_j \qquad \text{for all } s_j.$$

Hence the planner can test any candidate subsidy allocation—or design one—without solving for unobserved preferences or recomputing the full set of stable outcomes.

Economically, the characterization bundles two distinct conditions that every robust subsidy must satisfy. First, DC ensures the subsidy respects the observed outbidding structure, preventing new direct poaching that would generate efficiency loss. Second, SC accounts for indirect wage pressure through spillover chains: even a school that faces no direct poaching may still be affected by wage pressure transmitted from others. Together, DC and SC ensure that post-subsidy budgets are sufficient to absorb both direct and propagated wage pressures.

Methodologically, Theorem 1 offers a robust policy prescription without a mechanism-design layer. The planner need not elicit private information about schools' preferences: the observable features $(\mu, w, b, c)$ and the outbidding and spillover relations inferred from them suffice to design a non-distortive subsidy across all consistent value profiles.

## 4.4 The value of information under incomplete information

This subsection studies how the availability of information affects the set of non-distortive subsidies. In the incomplete-information environment, the planner may lack full knowledge of match values, costs, or even budgets. We will show that information constraints have a significant impact on policy design: as the planner's information set becomes more limited, the set of non-distortive subsidies contracts, and the planner loses flexibility to target assistance across schools. Conversely, when richer information is available, the planner can design targeted interventions.

### 4.4.1 When the planner has rich information

When the planner observes the status quo allocation $(\mu, w)$ together with costs $c$ and budgets $b$—but not the underlying match values $v$—the set of non-distortive subsidies is characterized by Theorem 1. In this setting, the planner possesses enough information to design targeted subsidies that respect the revealed outbidding and spillover structures. A uniform transfer across schools,

$$\tau_{s_i} = \tau_{s_j} > 0 \qquad \forall i, j,$$

can be dominated by a targeted subsidy allocation. Before stating this result formally, we first define a notion of dominance to compare different subsidy allocations.

**Definition 12** (Dominance)**.** In an allocation $(\mu, w)$ with budgets $b$ and costs $c$, a subsidy allocation $\tau$ is said to be *dominated* if there exists another subsidy $\tau'$ such that

$$\overline{W}(v, c, b, \tau') \;\geq\; \overline{W}(v, c, b, \tau) \quad \text{for all } v \in \mathcal{V}(\mu, w; b, c),$$

and

$$\sum_{s \in S} \tau'_s \; < \; \sum_{s \in S} \tau_s.$$

Intuitively, a subsidy allocation $\tau$ is dominated if another policy $\tau'$ can achieve at least the same welfare for all value profiles consistent with the observed environment while requiring a strictly smaller total transfer.

**Proposition 2** (Uniform subsidies can be dominated)**.** *A positive uniform subsidy allocation is dominated if there exists $s_j$ such that*

$$\begin{cases} b_{s_i} - c_{s_i,\mu(s_j)} \leq w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}, \\ b_{s_i} - c_{s_i,\mu(s_j)} < b_{s_j} - c_{s_j,\mu(s_j)}, \end{cases} \qquad \textit{for all } i \neq j,$$

*and*

$$s_j \xrightarrow{\mu} s_i \quad \textit{for all } i \neq j.$$

Proposition 2 shows that when the planner knows costs and budgets, but not valuations, a uniform policy, though non-distortive, may still be dominated. With this intermediate level of information, the planner can often achieve at least the same welfare for all consistent value profiles while using fewer total resources. For instance, recall Figure 8: under a uniform transfer $\tau = (1, 1, 1)$, school $s_3$ already has enough budget reserve, so the 1 unit allocated to $s_3$ does not relax any binding constraint and is wasted. A different subsidy $\tau' = (1, 1, 0)$ achieves the same level of improvement while strictly reducing total transfers, thereby dominating the uniform policy.

In short, richer information enables targeted and more cost-effective subsidy design, whereas informational constraints progressively limit this flexibility.

#### 4.4.2 When the planner has limited information

We next consider an environment in which the planner does not observe match values or costs but still knows the observed allocation $(\mu, w)$ and the baseline budgets $b$. In this case, the definition of consistency and non-distortion must be adjusted to account for the unobserved parameters.

Define the set of value and cost profiles *consistent* with the observed environment $(\mu, w; b)$:

$$\mathcal{V}, \mathcal{C}(\mu, w; b) \; \equiv \; \{\, v, c : \; (\mu, w) \text{ is stable under } (v, c, b) \,\}.$$

**Definition 13** (Non-distortive subsidy with unknown $v$ and $c$)**.** In an allocation $(\mu, w)$ with budgets $b$, we say a subsidy allocation $\tau = (\tau_s)_{s \in S}$ is *non-distortive* for $(\mu, w; b)$ if, for every value and cost profile $v, c \in \mathcal{V}, \mathcal{C}(\mu, w; b)$,

$$\overline{W}(v, c, b, \tau) \; \geq \; W(\mu; v, c).$$

**Theorem 2** (Characterization with unknown $v$ and $c$)**.** *In an allocation $(\mu, w)$ with budgets $b$, a subsidy allocation $\tau$ is non-distortive if and only if*

$$b^{\tau}_{s_j} - w_{s_j,\mu(s_j)} \; \geq \; \max_{s_i} \tau_{s_i} \qquad \textit{for all } s_j.$$

Theorem 2 shows that when the planner cannot observe the underlying values or costs, the flexibility to differentiate subsidies is sharply reduced. The non-distortive set is uniformly constrained by the largest subsidy granted to any school, limiting the scope for targeted assistance. Equivalently, each school's post-subsidy budget reserve must cover a *uniform* lower bound, which is the greatest potential direct effect that any single subsidized school can generate.

### 4.4.3 When the planner has minimal information

Finally, consider the extreme case in which the planner cannot observe match values, costs, or budgets and only knows the existing allocation $(\mu, w)$. With such minimal information, any differentiation across schools may generate distortions for some consistent environment.

Define the set of value, cost, and budget profiles *consistent* with the observed allocation $(\mu, w)$:

$$\mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w) \ \equiv \ \{\, v, c, b : \ (\mu, w) \text{ is stable under } (v, c, b) \,\}.$$

**Definition 14** (Non-distortive subsidy with unknown $v$, $c$, and $b$)**.** In an allocation $(\mu, w)$, a subsidy allocation $\tau = (\tau_s)_{s \in S}$ is *non-distortive* for $(\mu, w)$ if, for every value, cost, and budget profile $v, c, b \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$,

$$\overline{W}(v, c, b, \tau) \ \geq \ W(\mu; v, c).$$

**Theorem 3** (Characterization with unknown $v$, $c$, and $b$)**.** *In an observed allocation $(\mu, w)$, a subsidy allocation $\tau$ is non-distortive if and only if*

$$\tau_{s_i} = \tau_{s_j} \qquad \textit{for all } i \neq j.$$

When information is severely limited, the planner observes neither schools' preferences $v$, nor teachers' preferences $c$, nor schools' constraints $b$. Deprived of this information, the planner loses all flexibility to target assistance across schools. Uniform assistance is therefore the only policy that guarantees robustness across all environments consistent with the observed outcome.

### 4.4.4 Discussion

Taken together, Proposition 2 and Theorems 1–3 reveal how information constraints shape the design of non-distortive subsidies. As the planner's information set becomes coarser, the feasible set of policies narrows: (i) with rich information, targeted subsidies can dominate a uniform policy; (ii) with partial information, the scope for differentiation shrinks; and (iii) with minimal information, uniform subsidies are the only robust choice. Hence, the degree of information available to the planner governs the flexibility of policy design and determines whether the planner should implement targeted interventions or uniform transfers in practice.

It is worth noting that uniform subsidies are often adopted in practice due to fairness or political considerations. However, the notion of non-distortion employed here is purely efficiency-based, focusing on preserving the total welfare generated by the existing matching. The results above therefore provide an alternative rationale for uniform policies: even when efficiency is the sole objective, information constraints alone may render uniform subsidies the only non-distortive—and thus efficiency-preserving—option available to the planner.

## 5 Conclusion

This paper studies subsidy design in budget-constrained matching markets when the social planner operates under informational constraints. It offers two contributions: one policy-oriented and one methodological.

For policy implications, our results show that, even under incomplete information, the planner can robustly improve welfare. We provide a complete characterization of subsidy allocations that guarantee welfare improvements. In particular, Theorem 1 identifies two intuitive and testable requirements: *direct-effect coverage* and *spillover coverage*. These two conditions are necessary and sufficient for a subsidy allocation to (weakly) increase aggregate welfare across all environments consistent with the planner's observations. In addition, we show how the planner's information set governs the choice between targeted and uniform transfers, thereby offering a clear value-of-information message for policymakers: richer information enables finer targeting, whereas minimal information leaves uniform transfers as the only robust option.

The methodological contribution is more important: we develop a new approach for researchers and policymakers to address informational constraints in economic design. Rather than relying on mechanism design to elicit private information, we infer the underlying information structure from observed market outcomes and use that inference to guide policy design. Our findings show that observed data, together with this inference, are sufficient to construct a robust policy prescription. While we apply this method to subsidy design in matching markets, this inference-based approach can be broadly applied to other settings with incomplete information.

To conclude, our analysis strengthens the connection between matching theory and policy implementation by characterizing how a social planner can operate under incomplete information, thereby enabling the design and implementation of more effective policies in real-world matching markets. We also develop a novel approach for researchers and policymakers to address informational constraints, paving the way for improved economic design.

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# Appendix

## Proof of proposition 1 (i)

**Step 1: View schools as a directed graph.**

Consider the set of schools as a directed graph, and define fixed edge weights on the school graph:

$$\delta_{s\to s'} = \big(v_{s,\mu^*(s')} - c_{s,\mu^*(s')}\big) - \big(v_{s,\mu^*(s)} - c_{s,\mu^*(s)}\big).$$

**Step 2: MSI is equivalent to Bellman–Ford algorithm.**

Using MSI's wage and budget update, we have

$$d^r_{s,\mu^*(s')} = \min\{b^{r-1}_s,\ w^{r-1}_s + v_{s,\mu^*(s')} - v_{s,\mu^*(s)}\}$$

$$w^r_{s',\mu^*(s')} - c_{s',\mu^*(s')} = \max_s\{d^r_{s,\mu^*(s')} - c_{s,\mu^*(s')}\}$$

which is equivalent to

$$w^r_{s',\mu^*(s')} = \max_s\big\{\min\{b^{r-1}_s - c_{s,\mu^*(s')} + c_{s',\mu^*(s')}, w^{r-1}_s + \delta_{s\to s'}\}\big\}$$

Then each round of the MSI extends node distance (the wage) by one edge, adds the fixed edge weight, and applies a cap (the min function) on the edge weight. This is similar to the Bellman–Ford algorithm with path relaxation. In other words, determining the terminal wage and budget for each school $s$ is equivalent to solving a longest-path problem ending at node $s$. The terminal values correspond to the maximum total edge weight along any path terminating at $s$.

**Step 3: No positive cycles.**

Since $\mu^*$ maximizes total surplus, then for every directed cycle $C$

$$\sum_{(s\to s')\in C}\delta_{s\to s'} = \sum_{(s\to s')\in C}\big\{\big(v_{s,\mu^*(s')} - c_{s,\mu^*(s')}\big) - \big(v_{s,\mu^*(s)} - c_{s,\mu^*(s)}\big)\big\} \le 0.$$

That is, there are no positive cycles in the graph.

**Step 4: Longest paths are cycle-free.**

By Step 3, there are no positive cycles in the school graph. By construction of the MSI algorithm, traversing any cycle (i) does not change budgets and (ii) cannot increase the total path weight. Hence the maximum path weight (the termination wage and budget) is achieved by a simple (cycle-free) path. Consequently, the number of nodes of the longest path is bounded by $n$, the number of schools in the market.

**Conclusion: finite termination.**

Since longest paths are cycle-free and have at most $n$ nodes, the Bellman–Ford–type relaxation in MSI terminates after at most $n$ rounds (including the initialization).[17] This completes the proof.

[17] This is the longest-path, no–positive-cycle analogue of Bellman–Ford for shortest paths with no negative cycles; see Cormen et al. (2009), §24.1.

## Proof of proposition 1 (ii)

### Individual rationality.

By initialization, $w^1_{s,\mu^*(s)} = c_{s,\mu^*(s)}$ for all $s$. By construction of MSI, matched-edge wages are nondecreasing across rounds, so $w^{r^*}_{s,\mu^*(s)} \geq c_{s,\mu^*(s)}$ for all $s$. Thus every matched teacher earns at least cost, and every matched school's payoff is $v_{s,\mu^*(s)} - w^{r^*}_{s,\mu^*(s)} \geq 0$ by the construction of the wage updates and optimality of $\mu^*$. Hence $(\mu^*, w^{r^*})$ is IR.

### Budget feasibility.

Budgets are updated by $b^r_s = \max\{b^{r-1}_s, w^r_{s,\mu^*(s)}\}$. Therefore $b^{r^*}_s \geq w^{r^*}_{s,\mu^*(s)}$ for all $s$, so wages are affordable at $r^*$.

### No blocking pair.

At the termination round $r^*$, fix any school-teacher pair $(s,t)$. By construction of the MSI algorithm, we have

$$w^{r^*}_{\mu^{*-1}(t),t} - c_{\mu^{*-1}(t),t} = \max_{s' \in S} \{d^{r^*}_{s',t} - c_{s',t}\} \geq d^{r^*}_{s,t} - c_{s,t}$$

where the LHS is the effective wage received by teacher $t$, and the RHS is the highest effective wage school $s$ is willing to offer. Therefore, there is no profitable deviation.

### Proof of Proposition 1 (iii)

We use an inductive argument to show the minimality of MSI subsidies.

#### Base step

Let $(\widetilde{w}, \widetilde{b})$ be any wages and post-subsidy budgets such that $(\mu^*, \widetilde{w})$ is stable under $\widetilde{b} = b + \widetilde{\tau}$ with $\widetilde{\tau} \geq 0$. Then stability implies IR, which gives

$$\widetilde{w}_{s,\mu^*(s)} \geq c_{s,\mu^*(s)} = w^1_{s,\mu^*(s)}.$$

Also $\widetilde{b}_s \geq b_s$ (since $\widetilde{b} = b + \widetilde{\tau}$ with $\widetilde{\tau} \geq 0$) and budget feasibility gives $\widetilde{b}_s \geq \widetilde{w}_{s,\mu^*(s)} \geq c_{s,\mu^*(s)}$, hence

$$\widetilde{b}_s \geq \max\{b_s, c_{s,\mu^*(s)}\} = b^1_s.$$

#### Induction step

Fix $r \geq 2$. If $(\mu^*, \widetilde{w})$ is stable under $\widetilde{b}$ and

$$\widetilde{w}_{s,\mu^*(s)} \geq w^{r-1}_{s,\mu^*(s)} \quad \text{and} \quad \widetilde{b}_s \geq b^{r-1}_s \quad \text{for all } s,$$

then we want to show

$$\widetilde{w}_{s,\mu^*(s)} \geq w^{r}_{s,\mu^*(s)} \quad \text{for all } s \qquad \text{and} \qquad \widetilde{b}_s \geq \max\{b^{r-1}_s, w^r_{s,\mu^*(s)}\} = b^r_s \quad \text{for all } s.$$

*Proof.* For any pair $(s,t)$, stability of $(\mu^*, \widetilde{w})$ under $\widetilde{b}$ gives

$$\widetilde{w}_{\mu^{*-1}(t),t} - c_{\mu^{*-1}(t),t} \;\geq\; \min\{\widetilde{b}_s,\, v_{s,t} - v_{s,\mu^*(s)} + \widetilde{w}_{s,\mu^*(s)}\} - c_{s,t},$$

where the LHS is the effective wage of teacher $t$ in $(\mu^*, \widetilde{w})$, and the RHS is the highest effective wage $s$ would offer to $t$ without lowering its profit and subject to its budget.

By the induction hypothesis $\widetilde{b}_s \geq b^{r-1}_s$ and $\widetilde{w}_{s,\mu^*(s)} \geq w^{r-1}_{s,\mu^*(s)}$, hence

$$\min\{\widetilde{b}_s,\, v_{s,t} - v_{s,\mu^*(s)} + \widetilde{w}_{s,\mu^*(s)}\} - c_{s,t} \;\geq\; \min\{b^{r-1}_s,\, v_{s,t} - v_{s,\mu^*(s)} + w^{r-1}_{s,\mu^*(s)}\} - c_{s,t} = d^r_{s,t} - c_{s,t}.$$

Taking the maximum over schools,

$$\widetilde{w}_{\mu^{*-1}(t),t} - c_{\mu^{*-1}(t),t} \;\geq\; \max_s\{d^r_{s,t} - c_{s,t}\} = w^r_{\mu^{*-1}(t),t} - c_{\mu^{*-1}(t),t},$$

which proves $\widetilde{w} \geq w^r$. Budget feasibility then yields

$$\widetilde{b}_s \;\geq\; \widetilde{w}_{s,\mu^*(s)} \;\geq\; w^r_{s,\mu^*(s)}.$$

Together with $\widetilde{b}_s \geq b^{r-1}_s$ (induction hypothesis) we obtain

$$\widetilde{b}_s \;\geq\; \max\{b^{r-1}_s,\, w^r_{s,\mu^*(s)}\} = b^r_s.$$

Given Proposition 1 (i), MSI terminates in finite rounds. The above induction shows that the MSI termination budget $b^{r^*}_s$ is the componentwise minimum among all post-subsidy budgets that implement $\mu^*$, which completes the proof. $\square$

## Proof of Lemma 1

*Proof.* We prove Lemma 1 by contradiction.

Suppose we have a non-distortive subsidy allocation $\tau$ but DC fails. Then there exists $s_j$ such that

$$b^\tau_{s_j} - w_{s_j,\mu(s_j)} < \Delta^d_j \tag{L1-1}$$

By definition of $\Delta^d_j$

$$\Delta^d_j \equiv \max_{i:\ s_i \text{ is outbid by } s_j \text{ in } (\mu,w)} \Big[(b^\tau_{s_i} - c_{s_i,\mu(s_j)}) - (w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)})\Big]_+$$

Then there exists $s_i$ such that

$$b_{s_i} - c_{s_i,\mu(s_j)} \le w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)} \tag{L1-2}$$

And

$$\Delta^d_j = (b^\tau_{s_i} - c_{s_i,\mu(s_j)}) - (w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}) \tag{L1-3}$$

Combining (L1-1) and (L1-3), we have

$$b^\tau_{s_j} - c_{s_j,\mu(s_j)} < b^\tau_{s_i} - c_{s_i,\mu(s_j)} \tag{L1-4}$$

Construct the following value profile $v$

$$v_{s_i,\mu(s_i)} = w_{s_i,\mu(s_i)}, \quad v_{s_i,\mu(s_j)} = c_{s_i,\mu(s_j)} + V, \quad v_{s_j,\mu(s_j)} = c_{s_j,\mu(s_j)} + 2V$$

For all remaining matched pairs $(s, \mu(s))$ set

$$v_{s,\mu(s)} = w_{s,\mu(s)}$$

For all remaining unmatched pairs $(s, t)$ set

$$v_{s,t} = c_{s,t}$$

Here $V > 0$ is a large enough value such that

$$V > \Sigma_s(w_{s,\mu(s)} - c_{s,\mu(s)}) + \Sigma_s b^\tau_s$$

We claim that $(\mu, w)$ is stable under budgets $b$ given $(v, c)$, so $v \in \mathcal{V}(\mu, w; b, c)$.

To show $(\mu, w)$ is stable, we need to show the following:

- $(\mu, w)$ is feasible:
  This is directly implied by the observed market structure $(\mu, w; b, c)$, so we have $b_s \ge w_{s,\mu(s)}$.
- IR holds for all matched pairs:
  Teachers' IR are also directly implied by the observed market structure $(\mu, w; b, c)$, so we have $w_{s,\mu(s)} \ge c_{s,\mu(s)}$.
  Now we need to check schools' IR. By construction of $v$, we have

  $$2V > V > \Sigma_s(w_{s,\mu(s)} - c_{s,\mu(s)}) + \Sigma_s b^\tau_s \ge w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}$$

  Therefore, for school $s_j$, we have

  $$v_{s_j,\mu(s_j)} - w_{s_j,\mu(s_j)} = 2V - w_{s_j,\mu(s_j)} + c_{s_j,\mu(s_j)} > 0$$

  So IR holds for school $s_j$.
  For any school $s \ne s_j$, by construction of $v$, we have

  $$v_{s,\mu(s)} - w_{s,\mu(s)} = 0$$

  So IR holds for school $s \ne s_j$. Therefore, IR hold for all schools.

- There is no blocking pairs:
  For any unmatched pair $(s,t) \neq (s_i, \mu(s_j))$, by construction of $v$, we have $v_{s,t} = c_{s,t}$. Then any deviating wage $w' > c_{s,t}$ will lead to a payoff $v_{s,t} - w' < 0$, which cannot block the current matching.
  For $(s_i, \mu(s_j))$, any deviating wage $w'$ need to satisfy

$$w' - c_{s_i,\mu(s_j)} > w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}$$

  By (L1-2), we have

$$w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)} \geq b_{s_i} - c_{s_i,\mu(s_j)}$$

  Then we have

$$w' > b_{s_i}$$

  So any deviating wage is not affordable, and $(s_i, \mu(s_j))$ cannot block the current matching.

Therefore, the observed allocation $(\mu, w)$ is stable, and we have

$$v \in \mathcal{V}(\mu, w; b, c)$$

Now we claim that, for any matching $\mu'$ where $\mu'(s_j) \neq \mu(s_j)$ we have

$$W(\mu'; v, c) < W(\mu; v, c)$$

To show the above, by definition of $W(\mu'; v, c)$, we have

$$W(\mu'; v, c) = \Sigma_s \big(v_{s,\mu'(s)} - c_{s,\mu'(s)}\big)$$

Then by construction of $v$, we have

$$v_{s,\mu'(s)} - c_{s,\mu'(s)} = \begin{cases} 0 & \text{if} \quad \mu'(s) \neq \mu(s) \\ w_{s,\mu(s)} - c_{s,\mu(s)} & \text{if} \quad \mu'(s) = \mu(s) \end{cases} \quad \text{for } s \neq s_i, s_j$$

Also, we know

$$v_{s,\mu(s)} - c_{s,\mu(s)} = w_{s,\mu(s)} - c_{s,\mu(s)} \quad \text{for } s \neq s_i, s_j$$

Then we have

$$v_{s,\mu'(s)} - c_{s,\mu'(s)} \leq v_{s,\mu(s)} - c_{s,\mu(s)} \quad \text{for } s \neq s_i, s_j$$

Now consider $s_i$. By construction of $v$, the best match we can form is matching $s_i$ with $\mu(s_j)$, which gives

$$v_{s_i,\mu'(s_i)} - c_{s_i,\mu'(s_i)} \leq v_{s_i,\mu(s_j)} - c_{s_i,\mu(s_j)} = V$$

Now consider $s_j$. Since $\mu'(s_j) \neq \mu(s_j)$, by construction of $v$, we have

$$v_{s_j,\mu'(s_j)} - c_{s_j,\mu'(s_j)} = 0$$

Summing the above will give

$$\begin{aligned} W(\mu'; v, c) &= \Sigma_s \big(v_{s,\mu'(s)} - c_{s,\mu'(s)}\big) \\ &\leq \Sigma_{s \neq s_i, s_j} \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big) + V + 0 \\ &< \Sigma_{s \neq s_i, s_j} \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big) + 2V + w_{s_i,\mu(s_i)} - c_{s_i,\mu(s_i)} \\ &= \Sigma_s \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big) \\ &= W(\mu; v, c) \end{aligned}$$

Therefore, we have

$$W(\mu'; v, c) < W(\mu; v, c) \quad \text{if} \quad \mu'(s_j) \neq \mu(s_j) \tag{L1-5}$$

Recall $\tau$ is non-distortive, so we have

$$\overline{W}(v, c, b, \tau) \geq W(\mu; v, c)$$

Since $W(\mu'; v, c) < W(\mu; v, c)$ for any matching $\mu'$ where $\mu'(s_j) \neq \mu(s_j)$, there must be at least one stable allocation $(\tilde{\mu}, \tilde{w})$ where $\tilde{\mu}(s_j) = \mu(s_j)$ after the subsidy.

Consider the above $(\tilde{\mu}, \tilde{w})$, we have

$$\tilde{\mu}(s_j) = \mu(s_j)$$

By feasibility of $(\tilde{\mu}, \tilde{w})$, we have

$$\tilde{w}_{s_j,\tilde{\mu}(s_j)} \leq b^{\tau}_{s_j}$$

Recall (L1-4), we then have

$$b^{\tau}_{s_i} - c_{s_i,\mu(s_j)} > b^{\tau}_{s_j} - c_{s_j,\mu(s_j)} \geq \tilde{w}_{s_j,\tilde{\mu}(s_j)} - c_{s_j,\mu(s_j)}$$

Then, by $\tilde{\mu}(s_j) = \mu(s_j)$, we have

$$b^{\tau}_{s_i} - c_{s_i,\tilde{\mu}(s_j)} > \tilde{w}_{s_j,\tilde{\mu}(s_j)} - c_{s_j,\tilde{\mu}(s_j)}$$

which means a wage of $b^{\tau}_{s_i}$ from school $s_i$ is strictly better for $\tilde{\mu}(s_j)$.

Now consider $s_i$. Since we know $\tilde{\mu}(s_j) = \mu(s_j)$, so $s_i$ cannot be matched with $\mu(s_j)$ after the subsidy. Then by construction of $v$, the highest utility $s_i$ can enjoy in $(\tilde{\mu}, \tilde{w})$ is matching $\mu(s_i)$ with a wage of $c_{s_i,\mu(s_i)}$. So the highest utility of $s_i$ in $(\tilde{\mu}, \tilde{w})$ is

$$w_{s_i,\mu(s_i)} - c_{s_i,\mu(s_i)}$$

However, by construction of $v$, matching $\tilde{\mu}(s_j)$ with a wage of $b^{\tau}_{s_i}$ will lead to a utility of

$$V + c_{s_i,\mu(s_j)} - b^{\tau}_{s_i}$$

Recall $V$ is large enough

$$V > \Sigma_s(w_{s,\mu(s)} - c_{s,\mu(s)}) + \Sigma_s b^{\tau}_s$$

So we have

$$V + c_{s_i,\mu(s_j)} - b^{\tau}_{s_i} > [w_{s_i,\mu(s_i)} - c_{s_i,\mu(s_i)} + b^{\tau}_{s_i}] + c_{s_i,\mu(s_j)} - b^{\tau}_{s_i} \geq w_{s_i,\mu(s_i)} - c_{s_i,\mu(s_i)}$$

Therefore, matching $\tilde{\mu}(s_j)$ with a wage of $b^{\tau}_{s_i}$ is also strictly better for $s_i$, and $(s_i, \tilde{\mu}(s_j))$ will block $(\tilde{\mu}, \tilde{w})$, contradicting the stability of $(\tilde{\mu}, \tilde{w})$. Therefore, $s_j$ cannot match $\mu(s_j)$ after the subsidy, and (L1-5) then contradicts $\tau$ being non-distortive, which finishes the proof. $\square$

## Proof of Lemma 2

*Proof.* We prove Lemma 2 by contradiction.

WOLG, let $\mu(s_i) = t_i$ for notational convenience. Suppose we have a non-distortive subsidy allocation $\tau$ but SC fails. Then there exists $s_j$ such that

$$b^\tau_{s_j} - w_{s_j,t_j} < \Delta^s_j \tag{L2-1}$$

By definition of $\Delta^s_j$, there exists a spillover chain (after reindexing)

$$C: \; s_1 \xrightarrow{\mu} s_2 \xrightarrow{\mu} \cdots \xrightarrow{\mu} s_j$$

such that

$$\Delta^s_j = \delta^C_j$$

Choose the shortest violating chain, then SC holds at smaller indices, which is

$$b^\tau_{s_k} - w_{s_k,t_k} \geq \delta^C_k \quad \text{for } k = 2, \dots, j-1 \tag{L2-2}$$

By Lemma 1, DC holds at $s_1$, which is

$$b^\tau_{s_1} - w_{s_1,t_1} \geq \Delta^d_1 \tag{L2-3}$$

By definition of $\Delta^d_1$, there exists a school $s_0$ (after reindexing) such that

$$b_{s_0} - c_{s_0,t_1} \leq w_{s_1,t_1} - c_{s_1,t_1} \tag{L2-4}$$

And

$$\Delta^d_1 = (b^\tau_{s_0} - c_{s_0,t_1}) - (w_{s_1,t_1} - c_{s_1,t_1}) \tag{L2-5}$$

Construct the following value profile $v$

$$v_{s_0,t_0} = w_{s_0,t_0}, \quad v_{s_k,t_k} = w_{s_k,t_k} + V \;\text{ for } k = 1, \dots, j-1, \quad v_{s_j,t_j} = w_{s_j,t_j} + 2V$$

For all other matched pairs

$$v_{s_k,t_k} = w_{s_k,t_k} \;\text{ for } k \notin \{0, 1, \dots, j\}$$

For the following unmatched pairs

$$v_{s_k,t_{k+1}} = c_{s_k,t_{k+1}} + \left(w_{s_{k+1},t_{k+1}} - c_{s_{k+1},t_{k+1}}\right) + V \;\text{ for } k = 0, \dots, j-1$$

For all other unmatched pairs

$$v_{s,t} = c_{s,t} \;\text{ for all remaining } (s,t)$$

Here $V > 0$ is a large enough value such that

$$V > \Sigma_i (w_{s_i,t_i} - c_{s_i,t_i}) + \Sigma_i b^\tau_{s_i}$$

We claim that $(\mu, w)$ is stable under budgets $b$ given $(v, c)$, so $v \in \mathcal{V}(\mu, w; b, c)$.

To show $(\mu, w)$ is stable, we need to show the following:

- $(\mu, w)$ is feasible:
  This is directly implied by the observed market structure $(\mu, w; b, c)$, so we have $b_{s_i} \geq w_{s_i,t_i}$.
- IR holds for all matched pairs:
  Teachers' IR are also directly implied by the observed market structure $(\mu, w; b, c)$, so we have $w_{s_i,t_i} \geq c_{s_i,t_i}$.
  Now we need to check schools' IR. By construction of $V$, we have

  $$V > \Sigma_i(w_{s_i,t_i} - c_{s_i,t_i}) + \Sigma_i b^{\tau}_{s_i} \geq w_{s_i,t_i} - c_{s_i,t_i} \text{ for } \forall i$$

  Therefore, for school $s_i$, where $i \in \{1, \ldots, j\}$, by construction of $v$, we have

  $$v_{s_i,t_i} - w_{s_i,t_i} \geq V - w_{s_i,t_i} + c_{s_i,t_i} > 0$$

  So IR holds for all school $s_i$ where $i \in \{1, \ldots, j\}$.
  For any school $s_i$ where $i \notin \{1, \ldots, j\}$, by construction of $v$, we have

  $$v_{s_i,t_i} - w_{s_i,t_i} = 0$$

  So IR holds for any school $s_i$ where $i \notin \{1, \ldots, j\}$. Therefore, IR holds for all schools.
- There is no blocking pairs:
  For any unmatched pair $(s,t) \neq (s_i, t_{i+1})$, where $i \notin \{0, 1, \ldots, j-1\}$, by construction of $v$, we have $v_{s,t} = c_{s,t}$. Then any deviating wage $w' > c_{s,t}$ will lead to a payoff $v_{s,t} - w' < 0$, which cannot block the current matching.
  For any unmatched pair $(s_i, t_{i+1})$, where $i \in \{1, \ldots, j-1\}$, any deviating wage $w'$ need to satisfy

  $$w' - c_{s_i,t_{i+1}} > w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}$$

  By construction of $v$, we have

  $$v_{s_i,t_{i+1}} - w' = c_{s_i,t_{i+1}} + (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) + V - w' < V$$

  And $s_i$ is matching with $t_i$ with wage $w_{s_i,t_i}$, which gives a utility of

  $$v_{s_i,t_i} - w_{s_i,t_i} = V > v_{s_i,t_{i+1}} - w'$$

  Therefore, $(s_i, t_{i+1})$, where $i \in \{1, \ldots, j-1\}$, cannot block the current matching.
  For $(s_0, t_1)$, any deviating wage $w'$ need to satisfy

  $$w' - c_{s_0,t_1} > w_{s_1,t_1} - c_{s_1,t_1}$$

  By (L2-4), we have

  $$w_{s_1,t_1} - c_{s_1,t_1} \geq b_{s_0} - c_{s_0,t_1}$$

  Then we have

  $$w' > b_{s_0}$$

  So any deviating wage is not affordable, and $(s_0, t_1)$ cannot block the current matching.

Therefore, the observed allocation $(\mu, w)$ is stable, and we have

$$v \in \mathcal{V}(\mu, w; b, c)$$

Now we claim

$$W(\mu'; v, c) < W(\mu; v, c) \quad \text{if} \quad \mu'(s_j) \neq t_j \tag{L2-6}$$

To show the above, by definition of $W(\mu'; v, c)$, we have

$$W(\mu'; v, c) = \Sigma_s \big(v_{s,\mu'(s)} - c_{s,\mu'(s)}\big)$$

Then by construction of $v$, we have

$$v_{s,\mu'(s)} - c_{s,\mu'(s)} = \begin{cases} 0 & \text{if} \quad \mu'(s) \neq \mu(s) \\ w_{s,\mu(s)} - c_{s,\mu(s)} & \text{if} \quad \mu'(s) = \mu(s) \end{cases} \quad \text{for } s \notin \{s_0, s_1, ...s_j\}$$

Also, we know

$$v_{s,\mu(s)} - c_{s,\mu(s)} = w_{s,\mu(s)} - c_{s,\mu(s)} \quad \text{for } s \notin \{s_0, s_1, ...s_j\}$$

Then we have

$$v_{s,\mu'(s)} - c_{s,\mu'(s)} \leq v_{s,\mu(s)} - c_{s,\mu(s)} \quad \text{for } s \notin \{s_0, s_1, ...s_j\} \tag{L2-7}$$

Now consider $s_i \in \{s_0, s_1, ...s_{j-1}\}$. By construction of $v$, the best match we can form is matching $s_i$ with $t_{i+1}$, which gives

$$v_{s_i,\mu'(s_i)} - c_{s_i,\mu'(s_i)} \leq v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} = (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) + V$$

Now consider $s_j$. Since $\mu'(s_j) \neq t_j$, by construction of $v$, we have

$$v_{s_j,\mu'(s_j)} - c_{s_j,\mu'(s_j)} = 0$$

Then we have

$$\Sigma_{s\in\{s_0,s_1...s_j\}}(v_{s,\mu'(s)} - c_{s,\mu'(s)}) \leq \Sigma_{s\in\{s_1...s_j\}}(w_{s,\mu(s)} - c_{s,\mu(s)}) + jV + 0$$

Now consider the welfare generated in $\mu$ for $s \in \{s_0, s_1...s_j\}$. By construction of $v$, we have

$$\Sigma_{s\in\{s_0,s_1...s_j\}}(v_{s,\mu(s)} - c_{s,\mu(s)}) = \Sigma_{s\in\{s_0,s_1...s_j\}}(w_{s,\mu(s)} - c_{s,\mu(s)}) + (j+1)V$$

By $V > 0$, and $w_{s_0,\mu(s_0)} - c_{s_0,\mu(s_0)} \geq 0$, we have

$$\Sigma_{s\in\{s_0,s_1...s_j\}}(v_{s,\mu'(s)} - c_{s,\mu'(s)}) < \Sigma_{s\in\{s_0,s_1...s_j\}}(v_{s,\mu(s)} - c_{s,\mu(s)}) \tag{L2-8}$$

Combining (L2-7) and (L2-8), we have

$$\begin{aligned} W(\mu'; v, c) &= \Sigma_s \big(v_{s,\mu'(s)} - c_{s,\mu'(s)}\big) \\ &= \Sigma_{s\notin\{s_0,s_1,...s_j\}} \big(v_{s,\mu'(s)} - c_{s,\mu'(s)}\big) + \Sigma_{s\in\{s_0,s_1,...s_j\}} \big(v_{s,\mu'(s)} - c_{s,\mu'(s)}\big) \\ &< \Sigma_{s\notin\{s_0,s_1,...s_j\}} \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big) + \Sigma_{s\in\{s_0,s_1,...s_j\}} \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big) \\ &= \Sigma_s \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big) \\ &= W(\mu; v, c) \end{aligned}$$

Therefore, we have

$$W(\mu'; v, c) < W(\mu; v, c) \quad \text{if} \quad \mu'(s_j) \neq t_j \tag{L2-6}$$

Recall $\tau$ is non-distortive, so we have

$$\overline{W}(v, c, b, \tau) \geq W(\mu; v, c)$$

Since $W(\mu'; v, c) < W(\mu; v, c)$ for any matching $\mu'$ where $\mu'(s_j) \neq t_j$, there must be at least one stable allocation $(\tilde{\mu}, \tilde{w})$ where $\tilde{\mu}(s_j) = t_j$ after the subsidy.

Consider the above $(\tilde{\mu}, \tilde{w})$, we have

$$\tilde{\mu}(s_j) = t_j$$

By failure of SC at $s_j$, we have

$$b^\tau_{s_j} - w_{s_j,t_j} < \delta^C_j$$

which is equivalent to

$$b^\tau_{s_j} - c_{s_j,t_j} < (w_{s_j,t_j} - c_{s_j,t_j}) + \delta^C_j$$

Then the effective wage received by $t_j$ satisfies

$$\tilde{w}_{s_j,t_j} - c_{s_j,t_j} \leq b^\tau_{s_j} - c_{s_j,t_j} < (w_{s_j,t_j} - c_{s_j,t_j}) + \delta^C_j$$

By definition of $\delta^C_j$

$$\delta^C_j = \min\left\{\delta^C_{j-1}, b^\tau_{s_{j-1}} - c_{s_{j-1},t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\}$$

we have

$$b^\tau_{s_{j-1}} - c_{s_{j-1},t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) \geq \delta^C_j$$

which is equivalent to

$$b^\tau_{s_{j-1}} - c_{s_{j-1},t_j} \geq (w_{s_j,t_j} - c_{s_j,t_j}) + \delta^C_j$$

Hence $s_{j-1}$ can afford to offer $t_j$ an effective wage of $(w_{s_j,t_j} - c_{s_j,t_j}) + \delta^C_j$, strictly exceeding $t_j$'s current effective wage at $s_j$. By our construction of $v$, matching $t_j$ with the above effective wage will give $s_{j-1}$ a utility of

$$c_{s_{j-1},t_j} + (w_{s_j,t_j} - c_{s_j,t_j}) + V - \left([w_{s_j,t_j} - c_{s_j,t_j}] + \delta^C_j + c_{s_{j-1},t_j}\right) = V - \delta^C_j$$

Therefore the utility of $s_{j-1}$ in $(\tilde{\mu}, \tilde{w})$ must be strictly higher than $V - \delta^C_j$ (it is strict because $b^\tau_{s_j} - w_{s_j,t_j} < \delta^C_j$ is strict). Recall our construction of $V$, we then have

$$V > \Sigma_i(w_{s_i,t_i} - c_{s_i,t_i}) + \Sigma_i b^\tau_{s_i} \geq b^\tau_{s_1}$$

By Lemma 1, DC holds for $s_1$, we have

$$b^\tau_{s_1} - w_{s_1,t_1} \geq \Delta^d_1 \quad \Rightarrow \quad b^\tau_{s_1} \geq \Delta^d_1 \tag{L2-9}$$

By definition of $\delta^C_j$ along the spillover chain $C$, we have

$$\Delta^d_1 \geq \delta^C_2 \geq \delta^C_3 ... \geq \delta^C_j \quad \Rightarrow \quad b^\tau_{s_1} \geq \delta^C_j$$

So we have

$$V - \delta^C_j > 0$$

meaning that $s_{j-1}$ is enjoying a strictly positive utility in $(\tilde{\mu}, \tilde{w})$. Recall our construction of $v$, for any teacher $t \neq t_{j-1}, t_j$, we have

$$v_{s_{j-1},t} = c_{s_{j-1},t}$$

Then teacher's IR will leave $s_{j-1}$ a non-positive utility. Also, we know $\tilde{\mu}(s_j) = t_j$, so the only possible matching giving $s_{j-1}$ a positive utility is $\tilde{\mu}(s_{j-1}) = t_{j-1}$. Since utility of $s_{j-1}$ in $(\tilde{\mu}, \tilde{w})$ is strictly higher than $V - \delta^C_j$, we have

$$v_{s_{j-1},t_{j-1}} - \tilde{w}_{s_{j-1},t_{j-1}} > V - \delta^C_j$$

Recall our construction of $v$, and $\delta_j^C \le \delta_{j-1}^C$ by construction, we have

$$\tilde{w}_{s_{j-1},t_{j-1}} < v_{s_{j-1},t_{j-1}} - V + \delta_j^C = w_{s_{j-1},t_{j-1}} + \delta_j^C \le w_{s_{j-1},t_{j-1}} + \delta_{j-1}^C$$

Thus the effective wage received by $t_{j-1}$ satisfies

$$\tilde{w}_{s_{j-1},t_{j-1}} - c_{s_{j-1},t_{j-1}} < w_{s_{j-1},t_{j-1}} - c_{s_{j-1},t_{j-1}} + \delta_{j-1}^C$$

By the same logic, $s_{j-2}$ can afford the RHS for $t_{j-1}$ to enjoy a positive utility, meaning that $s_{j-2}$ is matching with $t_{j-2}$ in $(\tilde{\mu}, \tilde{w})$. And the wage of $t_{j-2}$ satisfies

$$\tilde{w}_{s_{j-2},t_{j-2}} - c_{s_{j-2},t_{j-2}} < w_{s_{j-2},t_{j-2}} - c_{s_{j-2},t_{j-2}} + \delta_{j-2}^C$$

Continuing this induction yields $\tilde{\mu}(s_1) = t_1$, and

$$\tilde{w}_{s_1,t_1} - c_{s_1,t_1} < w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d$$

Recall (L2-5), $s_0$ can afford the RHS effective wage to $t_1$. By our construction of $v$,

$$v_{s_0,t_0} = w_{s_0,t_0}, \qquad v_{s_0,t_1} = c_{s_0,t_1} + V, \qquad v_{s_0,t_i} = c_{s_0,t_i}, \quad \text{for } i \neq 0, 1$$

The highest utility $s_0$ can enjoy (when not matching $t_1$) in $(\tilde{\mu}, \tilde{w})$ will be

$$w_{s_0,t_0} - v_{s_0,t_0}$$

However, matching $t_1$ with a wage of $c_{s_0,t_1} + w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d$ will give a utility of

$$v_{s_0,t_1} - (c_{s_0,t_1} + w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d)$$

Recall our construction of $v$, the above quantity is

$$v_{s_0,t_1} - (c_{s_0,t_1} + w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d) = V - (w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d)$$

Recall $V$ is large enough and (L2-9)

$$V > \Sigma_i (w_{s_i,t_i} - c_{s_i,t_i}) + \Sigma_i b_{s_i}^\tau \ge w_{s_0,t_0} - v_{s_0,t_0} + w_{s_1,t_1} - c_{s_1,t_1} + b_{s_1}^\tau \ge w_{s_0,t_0} - v_{s_0,t_0} + w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d$$

Then we have

$$V - (w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d) > w_{s_0,t_0} - v_{s_0,t_0}$$

Therefore, deviating to $t_1$ with a wage of $c_{s_0,t_1} + w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d$ will be strictly profitable for $s_0$ and $t_1$, contradicting the stability of $(\tilde{\mu}, \tilde{w})$. Therefore, $s_j$ cannot be matched with $t_j$ after the subsidy, and (L2-6) will contradict $\tau$ being non-distortive, which finishes the proof. □

## Proof of Theorem 1.

*Proof.* Lemma 1 and Lemma 2 establish necessity. It remains to prove sufficiency.

Let $(\mu, w)$, $b$, and $c$ denote the pre-subsidy allocation, budgets, and costs. For notational convenience, relabel so that $\mu(s_i) = t_i$ for all indices that appear below.

Fix any value profile $v \in \mathcal{V}(\mu, w; b, c)$ and a subsidy allocation $\tau$ that satisfies DC and SC. Consider the post-subsidy school-optimal stable allocation under tie-breaking in favor of $\mu$ (schools and teachers weakly prefer their $\mu$-partners when indifferent). Let $(\mu', w')$ be the resulting allocation under budgets $b^\tau$ and costs $c$.

It suffices to show $W(\mu'; v, c) \geq W(\mu; v, c)$ for every $v \in \mathcal{V}(\mu, w; b, c)$, because then

$$\overline{W}(v, c, b, \tau) \ \geq \ W(\mu'; v, c) \ \geq \ W(\mu; v, c)$$

so $\tau$ is non-distortive.

We now show

$$W(\mu'; v, c) \geq W(\mu; v, c)$$

by contradiction. Suppose

$$W(\mu'; v, c) < W(\mu; v, c)$$

**Step 1:**

In this step, we show there exists a welfare-decreasing cycle $\mathcal{C}$.

Since

$$W(\mu'; v, c) < W(\mu; v, c)$$

Then we have $\mu \neq \mu'$, otherwise the welfare of $\mu, \mu'$ should be the same. Given $\mu \neq \mu'$, we need the following definition

**Definition 15** (Cycle and fixed points)**.** For two different matchings $\mu$ and $\mu'$. Define the permutation operator

$$\sigma(s) = \mu^{-1}(\mu'(s))$$

A set of distinct schools $\{s_1, s_2 \ldots, s_m\}$ (after reindexing) forms a **cycle** of length $m$ if

$$\sigma(s_i) = s_{i+1} \text{ for } i = 1, \ldots, m-1 \quad \text{and} \quad \sigma(s_m) = s_1$$

When $m = 1$ we have $\sigma(s) = s$, equivalently $\mu'(s) = \mu(s)$. We call such $s$ a **fixed point**.

Since we have $\mu(s_i) = t_i$, we can WLOG let each cycle include both schools and teachers because they share the same set of indices.

Given the definition above, since the market is finite, we can partition $S$ uniquely into different cycles with length $m \geq 2$ and fixed points.

Now we claim there must be at least one cycle with length $m \geq 2$

$$\mathcal{C} = \{s_1, \ldots, s_m, t_1, \ldots, t_m\}$$

where the welfare change is negative, that is

$$\sum_{i=1}^{m}\left(v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}\right) < \sum_{i=1}^{m}\left(v_{s_i,t_i} - c_{s_i,t_i}\right) \quad \text{with} \quad t_{m+1} \equiv t_1$$

We show the above claim by contradiction. Suppose for all cycles with length $m \geq 2$, we have

$$\sum_{i=1}^{m}\left(v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}\right) \geq \sum_{i=1}^{m}\left(v_{s_i,t_i} - c_{s_i,t_i}\right)$$

We also know the welfare of a fixed point does not change

$$v_{s,\mu(s)} - c_{s,\mu(s)} = v_{s,\mu'(s)} - c_{s,\mu'(s)} \quad \text{if} \quad \mu(s) = \mu'(s)$$

Summing over all cycles, we have

$$\Sigma_s(v_{s,\mu(s)} - c_{s,\mu(s)}) \geq \Sigma_s(v_{s,\mu'(s)} - c_{s,\mu'(s)}) \quad \Rightarrow \quad W(\mu'; v, c) \geq W(\mu; v, c)$$

contradicting our hypothesis. Therefore, there exists a cycle with length $m \geq 2$

$$\mathcal{C} = \{s_1, \ldots, s_m, t_1, \ldots, t_m\}$$

and

$$\sum_{i=1}^{m}\left(v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}\right) < \sum_{i=1}^{m}\left(v_{s_i,t_i} - c_{s_i,t_i}\right) \quad \text{with} \quad t_{m+1} \equiv t_1 \tag{T1-1}$$

And the matching pairs are

$$\begin{cases} \mu: \ (s_1,t_1),(s_2,t_2),\ldots,(s_m,t_m) \\ \mu': \ (s_1,t_2),(s_2,t_3),\ldots,(s_m,t_1) \end{cases}$$

**Step 2:**

In this step, we show the following cannot be true in $\mathcal{C}$:

$$\exists\ i \text{ such that} \quad b_{s_i} - c_{s_i,t_i i+1} \leq w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} \quad \text{where} \quad s_i \in \mathcal{C}$$

that is, there is no $s_i \in \mathcal{C}$ such that, $s_i$ is outbid by $s_{i+1}$ in $(\mu, w)$.

We show the above by contradiction. Suppose there is an index $i$ such that

$$b_{s_i} - c_{s_i,t_i i+1} \leq w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}$$

WOLG, let $i = 1$ by relabeling, we then have

$$b_{s_1} - c_{s_1,t_2} \leq w_{s_2,t_2} - c_{s_2,t_2}$$

Now we want to show the following affordability conditions

$$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \geq w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} \quad \text{for} \quad i = 1,2,...m \quad \text{with} \quad t_{m+1} \equiv t_1 \tag{AFF}$$

*Proof.* We show (AFF) by induction in three steps.

- Step 2.1: We first show $b^\tau_{s_2} - c_{s_2,t_2} \;\geq\; w'_{s_1,t_2} - c_{s_1,t_2}$.
Since $s_1$ was outbid by $s_2$ in $(\mu, w)$, DC gives

$$b^\tau_{s_2} - w_{s_2,t_2} \geq \Delta^d_2 \geq (b^\tau_{s_1} - c_{s_1,t_2}) - (w_{s_2,t_2} - c_{s_2,t_2}) \geq (w'_{s_1,t_2} - c_{s_1,t_2}) - (w_{s_2,t_2} - c_{s_2,t_2})$$

where the last inequality is by feasibility of $(s_1, t_2)$ in $(\mu', w')$. Rearranging gives

$$b^\tau_{s_2} - c_{s_2,t_2} \geq w'_{s_1,t_2} - c_{s_1,t_2}$$

which is the desired inequality.
- Step 2.2: Now we show $b^\tau_{s_3} - c_{s_3,t_3} \geq w'_{s_2,t_3} - c_{s_2,t_3}$.
Split by whether $s_2$ is outbid by $s_3$ in $\mu$.
  - Case 2.2.1: If $s_2$ is outbid by $s_3$ in $(\mu, w)$, we have

    $$b_{s_2} - c_{s_2,t_3} \leq w_{s_3,t_3} - c_{s_3,t_3}$$

    then DC gives

    $$b^\tau_{s_3} - w_{s_3,t_3} \geq \Delta^d_3 \geq (b^\tau_{s_2} - c_{s_2,t_3}) - (w_{s_3,t_3} - c_{s_3,t_3}) \geq (w'_{s_2,t_3} - c_{s_2,t_3}) - (w_{s_3,t_3} - c_{s_3,t_3})$$

    hence

    $$b^\tau_{s_3} - c_{s_3,t_3} \geq w'_{s_2,t_3} - c_{s_2,t_3}$$
  - Case 2.2.2: If $s_2$ is not outbid by $s_3$ in $(\mu, w)$, then we have

    $$b_{s_2} - c_{s_2,t_3} > w_{s_3,t_3} - c_{s_3,t_3}$$

    Then $s_2 \xrightarrow{\mu} s_3$ forms a spillover chain $C$. Because $\tau$ satisfies SC,

    $$b^\tau_{s_3} - w_{s_3,t_3} \geq \delta^C_3 \tag{T1-2.1}$$

    where

    $$\delta^C_3 = \min\left\{\Delta_2, b^\tau_{s_2} - c_{s_2,t_3} - (w_{s_3,t_3} - c_{s_3,t_3})\right\}$$

    If $\delta^C_3$ takes the latter value, then by feasibility $b^\tau_{s_2} \geq w'_{s_2,t_3}$ and the above, we have

    $$b^\tau_{s_3} - w_{s_3,t_3} \geq \delta^C_3 = b^\tau_{s_2} - c_{s_2,t_3} - (w_{s_3,t_3} - c_{s_3,t_3}) \geq w'_{s_2,t_3} - c_{s_2,t_3} - (w_{s_3,t_3} - c_{s_3,t_3})$$

    which rearranges to the desired inequality

    $$b^\tau_{s_3} - c_{s_3,t_3} \geq w'_{s_2,t_3} - c_{s_2,t_3}$$

    If $\delta^C_3$ takes the former value $\Delta_2$, then by definition of $\Delta_2$ and feasibility $b^\tau_{s_1} \geq w'_{s_1,t_2}$, we have

    $$\delta^C_3 = \Delta_2 \geq \left[(b^\tau_{s_1} - c_{s_1,t_2}) - (w_{s_2,t_2} - c_{s_2,t_2})\right] \geq (w'_{s_1,t_2} - c_{s_1,t_2}) - (w_{s_2,t_2} - c_{s_2,t_2}) \tag{T1-2.2}$$

    Pre-subsidy stability of $(s_2, t_2)$ vs. $(s_2, t_3)$ implies

    $$v_{s_2,t_2} - w_{s_2,t_2} \;\geq\; v_{s_2,t_3} - w_{s_3,t_3} + c_{s_3,t_3} - c_{s_2,t_3}$$

    By step 2.1 and tie-breaking in favor of $\mu$, $s_2$ can match $w'_{s_1,t_2}$. So post-subsidy stability of $(s_2, t_3)$ vs. $(s_2, t_2)$ requires

    $$v_{s_2,t_3} - w'_{s_2,t_3} \geq v_{s_2,t_2} - w'_{s_1,t_2} + c_{s_1,t_2} - c_{s_2,t_2}$$

Combining yields

$$(w'_{s_1,t_2} - c_{s_1,t_2}) - (w_{s_2,t_2} - c_{s_2,t_2}) \;\geq\; (w'_{s_2,t_3} - c_{s_2,t_3}) - (w_{s_3,t_3} - c_{s_3,t_3}) \qquad (*_{2\to 3})$$

Using $(*_{2\to 3})$, (T1-2.1) and (T1-2.2), we have

$$b^{\tau}_{s_3} - w_{s_3,t_3} \geq \delta^C_3 \geq (w'_{s_1,t_2} - c_{s_1,t_2}) - (w_{s_2,t_2} - c_{s_2,t_2}) \geq w'_{s_2,t_3} - c_{s_2,t_3} - (w_{s_3,t_3} - c_{s_3,t_3})$$

which rearranges to the desired inequality

$$b^{\tau}_{s_3} - c_{s_3,t_3} \;\geq\; w'_{s_2,t_3} - c_{s_2,t_3}$$

- Step 2.3: Now we show $b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \;\geq\; w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}$ for $i = 3, \ldots, m$.
Assume the inequality holds for $i-1$.
If $s_i$ is outbid by $s_{i+1}$ in $(\mu, w)$, case 2.2.1 will give the desired inequality.
If $s_i$ is not outbid by $s_{i+1}$ in $(\mu, w)$, then find the largest index $q$ such that $s_{q-1}$ is outbid by $s_q$, and $s_q \xrightarrow{\mu} s_{q+1} ... \xrightarrow{\mu} s_{i+1}$ forms a spillover chain. Such $q$ exists because we know

$$b_{s_1} - c_{s_1,t_2} \leq w_{s_2,t_2} - c_{s_2,t_2}$$

Given the spillover chain $s_q \xrightarrow{\mu} s_{q+1} ... \xrightarrow{\mu} s_{i+1}$, SC gives

$$b^{\tau}_{s_{i+1}} - w_{s_{i+1},t_{i+1}} \geq \delta^C_{i+1} \qquad \text{(T1-2.3)}$$

Then check $\delta^C_{i+1}$, we have

$$\delta^C_{i+1} = \min\left\{\delta^C_i, b^{\tau}_{s_i} - c_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}})\right\}$$

If $\delta^C_{i+1}$ takes the latter value, (T1-2.3) and feasibility give the following:

$$b^{\tau}_{s_{i+1}} - w_{s_{i+1},t_{i+1}} \geq \delta^C_{i+1} = b^{\tau}_{s_i} - c_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) \geq w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}})$$

and rearranging gives the desired inequality

$$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \;\geq\; w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}$$

If $\delta^C_{i+1}$ takes the former value, then

$$\delta^C_{i+1} = \delta^C_i$$

By definition of $\delta^C_i$, we have

$$\delta^C_i = \min\left\{\delta^C_{i-1}, b^{\tau}_{s_{i-1}} - c_{s_{i-1},t_i} - (w_{s_i,t_i} - c_{s_i,t_i})\right\}$$

There will be two cases:
  - Case 2.3.1: If $\delta^C_i$ takes the latter value, then

$$\delta^C_i = b^{\tau}_{s_{i-1}} - c_{s_{i-1},t_i} - (w_{s_i,t_i} - c_{s_i,t_i}) \geq w'_{s_{i-1},t_i} - c_{s_{i-1},t_i} - (w_{s_i,t_i} - c_{s_i,t_i}) \qquad \text{(T1-2.4)}$$

Now recall the induction hypothesis, pre-subsidy stability of $(s_i, t_i)$ vs. $(s_i, t_{i+1})$, and post-subsidy stability of $(s_i, t_{i+1})$ vs. $(s_i, t_i)$, we can derive the following similar to case 2.2.2

$$(w'_{s_{i-1},t_i} - c_{s_{i-1},t_i}) - (w_{s_i,t_i} - c_{s_i,t_i}) \geq (w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) \quad (*_{i\to i+1})$$

(T1-2.4) and $(*_{i\to i+1})$ will give

$$b^{\tau}_{s_{i+1}} - w_{s_{i+1},t_{i+1}} \geq \delta^C_{i+1} = \delta^C_i \geq (w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}})$$

which rearranges to the desired inequality

$$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \;\geq\; w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}$$

– Case 2.3.2: If $\delta_i^C$ takes the former value, then check $\delta_{i-1}^C$. We either combine $(*_{i-1\to i})$ and $(*_{i\to i+1})$ to get the desired inequality or move to $\delta_{i-2}^C$. This backward induction terminates at index $q$. DC of $\Delta_q^d$ and telescoping $(*)$ will give the desired inequality.

This completes the proof of (AFF). $\square$

By (AFF), for each $i$, the contract $(s_{i+1}, t_{i+1}, w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} + c_{s_{i+1},t_{i+1}})$ is feasible. At that wage, $t_{i+1}$ enjoys the same effective wage. Tie–breaking in favor of $\mu$ implies $t_{i+1}$ will accept the offer. Therefore, in the post–subsidy allocation $(\mu', w')$, stability requires that $s_{i+1}$ does not want to switch from $t_{i+2}$ to $t_{i+1}$ at wage $(w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} + c_{s_{i+1},t_{i+1}})$, which gives

$$v_{s_{i+1},t_{i+2}} - w'_{s_{i+1},t_{i+2}} > v_{s_{i+1},t_{i+1}} - w'_{s_i,t_{i+1}} + c_{s_i,t_{i+1}} - c_{s_{i+1},t_{i+1}} \qquad (i = 1, \ldots, m)$$

Summing these $m$ inequalities gives the following

$$\sum_{i=1}^{m} (v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) > \sum_{i=1}^{m} (v_{s_i,t_i} - c_{s_i,t_i})$$

But by step 1, we know

$$\sum_{i=1}^{m} (v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) < \sum_{i=1}^{m} (v_{s_i,t_i} - c_{s_i,t_i})$$

which is a contradiction. Therefore, the hypothesis at the beginning of step 2 cannot be true, which gives

$$b_{s_i} - c_{s_i,t_{i+1}} > w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} \quad \text{for all} \quad s_i \in \mathcal{C}$$

### Step 3:

In this step, we want to show there exists $i$ such that

$$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \le w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}$$

meaning that $s_{i+1}$ is outbid by $s_i$ in $(\mu', w')$.

We again prove by contradiction. Suppose the following

$$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} > w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} \quad \text{for } i = 1, 2, ..., m$$

Then every school $s_i$ can match $t_i$ with her current effective wage. Stability of $(\mu', w')$ then requires

$$v_{s_{i+1},t_{i+2}} - w'_{s_{i+1},t_{i+2}} > v_{s_{i+1},t_{i+1}} - w'_{s_i,t_{i+1}} + c_{s_i,t_{i+1}} - c_{s_{i+1},t_{i+1}} \quad \text{for } i = 1, 2, ..., m$$

Summing these $m$ inequalities gives

$$\sum_{i=1}^{m} (v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) > \sum_{i=1}^{m} (v_{s_i,t_i} - c_{s_i,t_i})$$

But by step 1, we know

$$\sum_{i=1}^{m} (v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) < \sum_{i=1}^{m} (v_{s_i,t_i} - c_{s_i,t_i})$$

a contradiction. Therefore there exists $i$ with

$$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \le w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}$$

**Step 4: Utility drop of $s_i$**

In this step, we derive the utility drop of $s_i$, where $i$ is the index at the end of step 3.

Let $i$ be the index at the end of step 3, we then have

$$b^\tau_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \leq w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} \tag{T1-4.1}$$

Recall $\tau$ satisfies DC and SC, which implies

$$b^\tau_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \geq (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) + \Lambda_{i+1} \tag{T1-4.2}$$

where by definition, we have

$$\Lambda_{i+1} \equiv \max\{\Delta^d_{i+1}, \Delta^s_{i+1}\}$$

Combining (T1-4.1), (T1-4.2) and feasibility of $w'_{s_i,t_{i+1}}$, we have

$$b^\tau_{s_i} - c_{s_i,t_{i+1}} \geq w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} \geq b^\tau_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \geq (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) + \Lambda_{i+1} \tag{T1-4.3}$$

Now we define

$$\Lambda_{i\to j} \equiv \min\Big\{\Lambda_i,\ b^\tau_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\}$$

which denotes the spillover effect transmitted at $s_i \xrightarrow{\mu} s_j$. And we develop the following lemma

**Lemma 3.** *If $s_i \xrightarrow{\mu} s_j$, then*

$$\Lambda_j \geq \Lambda_{i\to j}$$

*Proof.* For any spillover chain $C : s_1 \xrightarrow{\mu} s_2 ... \xrightarrow{\mu} s_i$ ending at $s_i$, there is another spillover chain $C' : s_1 \xrightarrow{\mu} s_2 ... \xrightarrow{\mu} s_i \xrightarrow{\mu} s_j$. Therefore, by definition of spillover effects

$$\begin{aligned}
\Delta^s_j &\equiv \sup_{C\text{: spillover chains ending at } s_j} \delta^C_j \\
&\geq \sup_{C\text{: spillover chains ending at } s_i \text{ then } s_j} \delta^C_j \\
&= \sup_{C\text{: spillover chains ending at } s_i} \min\Big\{\delta^C_i,\ b^\tau_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\}
\end{aligned}$$

Since $b^\tau_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)$ is a constant, after changing the order of min and sup, we have

$$\begin{aligned}
\Delta^s_j &\geq \sup_{C\text{: spillover chains ending at } s_i} \min\Big\{\delta^C_i,\ b^\tau_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\} \\
&= \min\Big\{\sup_{C\text{: spillover chains ending at } s_i} \delta^C_i,\ b^\tau_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\} \\
&= \min\Big\{\Delta^s_i,\ b^\tau_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\} \qquad \text{(L3-1)}
\end{aligned}$$

Also, since $s_i \xrightarrow{\mu} s_j$, definition of $\Delta^s_j$ gives

$$\Delta^s_j \geq \min\Big\{\Delta^d_i,\ b^\tau_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\} \tag{L3-2}$$

There will be two cases:

- $b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) \geq \max\{\Delta^d_i, \Delta^s_i\}$
  Then we have
  $$\Delta^s_j \geq \min\left\{\Delta^d_i,\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\} = \Delta^d_i$$
  and
  $$\Delta^s_j \geq \min\left\{\Delta^s_i,\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\} = \Delta^s_i$$
  Then we have
  $$\Delta^s_j \geq \max\{\Delta^d_i, \Delta^s_i\} \geq \min\left\{\max\{\Delta^d_i, \Delta^s_i\},\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\}$$
  where the last term is
  $$\min\left\{\max\{\Delta^d_i, \Delta^s_i\},\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\} = \min\left\{\Lambda_i,\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\} = \Lambda_{i\to j}$$
  So we have
  $$\Delta^s_j \geq \Lambda_{i\to j}$$
- $b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) < \max\{\Delta^d_i, \Delta^s_i\}$
  Then one of (L3-1), (L3-2) will imply
  $$\Delta^s_j \geq b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) = \min\left\{\max\{\Delta^d_i, \Delta^s_i\},\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\}$$
  The last term is again
  $$\min\left\{\max\{\Delta^d_i, \Delta^s_i\},\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\} = \Lambda_{i\to j}$$
  Then we have
  $$\Delta^s_j \geq \Lambda_{i\to j}$$

So in both cases, we have $\Delta^s_j \geq \Lambda_{i\to j}$. Recall definition of $\Lambda_j$, we have

$$\Lambda_j \equiv \max\{\Delta^d_j, \Delta^s_j\} \geq \Delta^s_j \geq \Lambda_{i\to j} \tag{L3-3}$$

which finishes the proof of Lemma 3. □

Recall step 2, we have $s_i \xrightarrow{\mu} s_{i+1}$. Then by Lemma 3, we have

$$\Lambda_{i+1} \geq \Lambda_{i\to i+1} \equiv \min\left\{\Lambda_i,\ b^\tau_{s_i} - c_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}})\right\}$$

where $\Lambda_{i\to i+1}$ denotes the spillover effect transmitted at $s_i \xrightarrow{\mu} s_{i+1}$.

There will be two cases:

- Case 4.1: If $\Lambda_{i\to i+1}$ takes the latter value, meaning that spillover effects from $s_i$ to $s_{i+1}$ is capped by post-subsidy budgets, then we have
  $$\Lambda_{i+1} \geq \Lambda_{i\to i+1} = b^\tau_{s_i} - c_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) \tag{T1-4.4}$$
  combine (T1-4.3), (T1-4.4) and feasibility of $(s_i, t_{i+1}, w')$, we have
  $$b^\tau_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \geq (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) + \Lambda_{i+1} \geq b^\tau_{s_i} - c_{s_i,t_{i+1}} \geq w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}$$

meaning that $s_{i+1}$ can afford the effective wage of $t_{i+1}$ in $(\mu', w')$. Therefore, by tie-breaking in favor of $\mu$, $s_{i+1}$ must strictly prefer his current matching $(s_{i+1}, t_{i+2})$. This post-subsidy stability gives

$$v_{s_{i+1},t_{i+2}} - w'_{s_{i+1},t_{i+2}} > v_{s_{i+1},t_{i+1}} - (w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} + c_{s_{i+1},t_{i+1}}) \tag{T1-4.5}$$

By step 2, we have $s_{i+1} \xrightarrow{\mu} s_{i+2}$, pre-subsidy stability requires

$$v_{s_{i+1},t_{i+1}} - w_{s_{i+1},t_{i+1}} \geq v_{s_{i+1},t_{i+2}} - (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}} + c_{s_{i+1},t_{i+2}}) \tag{T1-4.6}$$

combine (T1-4.5) and (T1-4.6) and rearrange, we have

$$(w'_{s_{i+1},t_{i+2}} - c_{s_{i+1},t_{i+2}}) - (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}}) < (w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}})$$

where the LHS is the effective wage change of $t_{i+2}$, the RHS is the effective wage change of $t_{i+1}$. Recall (T1-4.4) and feasibility $b^\tau_{s_i} \geq w'_{s_i,t_{i+1}}$, we have

$$RHS \leq b^\tau_{s_i} - c_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) = \Lambda_{i\to i+1}$$

which gives

$$(w'_{s_{i+1},t_{i+2}} - c_{s_{i+1},t_{i+2}}) - (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}}) < \Lambda_{i\to i+1} \tag{T1-4.7}$$

We now show $s_{i+2}$ can afford this effective wage $(w'_{s_{i+1},t_{i+2}} - c_{s_{i+1},t_{i+2}})$ to $t_{i+2}$. Consider the post-subsidy budget of school $s_{i+2}$, by DC and SC, we have

$$b^\tau_{s_{i+2}} - c_{s_{i+2},t_{i+2}} \geq (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}}) + \Lambda_{i+2} \tag{T1-4.8}$$

and by $s_{i+1} \xrightarrow{\mu} s_{i+2}$ and Lemma 3, we have

$$\Lambda_{i+2} \geq \Lambda_{i+1\to i+2} \equiv \min\left\{\Lambda_{i+1},\ b^\tau_{s_{i+1}} - c_{s_{i+1},t_{i+2}} - (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}})\right\}$$

If $\Lambda_{i+1\to i+2}$ takes the former value, $\Lambda_{i+2} \geq \Lambda_{i+1\to i+2} = \Lambda_{i+1} \geq \Lambda_{i\to i+1}$, then (T1-4.7), (T1-4.8) will give the following

$$b^\tau_{s_{i+2}} - c_{s_{i+2},t_{i+2}} \geq (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}}) + \Lambda_{i+2} \geq (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}}) + \Lambda_{i\to i+1} > w'_{s_{i+1},t_{i+2}} - c_{s_{i+1},t_{i+2}}$$

If $\Lambda_{i+1\to i+2}$ takes the latter value, then (T1-4.8) will give

$$b^\tau_{s_{i+2}} - c_{s_{i+2},t_{i+2}} \geq (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}}) + \Lambda_{i+2} \geq b^\tau_{s_{i+1}} - c_{s_{i+1},t_{i+2}} \geq w'_{s_{i+1},t_{i+2}} - c_{s_{i+1},t_{i+2}}$$

Therefore, in both cases, $s_{i+2}$ can afford an effective wage of $w'_{s_{i+1},t_{i+2}} - c_{s_{i+1},t_{i+2}}$ to $t_{i+2}$. Then post-subsidy stability requires that $s_{i+2}$ must strictly prefer his current matching $(s_{i+2}, t_{i+3})$. Similar to the logic in (T1-4.5), (T1-4.6) and (T1-4.7). We can get

$$(w'_{s_{i+2},t_{i+3}} - c_{s_{i+2},t_{i+3}}) - (w_{s_{i+3},t_{i+3}} - c_{s_{i+3},t_{i+3}}) < \Lambda_{i\to i+1}$$

Keep the above induction along the cycle $\mathcal{C}$, we finally circle back to index $i-1$ and have

$$(w'_{s_{i-1},t_i} - c_{s_{i-1},t_i}) - (w_{s_i,t_i} - c_{s_i,t_i}) < \Lambda_{i\to i+1} \tag{T1-4.9}$$

Similarly, by DC and SC, we have

$$b^\tau_{s_i} - w_{s_i,t_i} \geq \Lambda_i \tag{T1-4.10}$$

Recall definition of $\Lambda_{i\to i+1}$, we have

$$\Lambda_{i\to i+1} \equiv \min\left\{\Lambda_i,\ b^{\tau}_{s_i} - c_{s_i,t_{i+1}} - \left(w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}\right)\right\} \le \Lambda_i \tag{T1-4.11}$$

By (T1-4.9), (T1-4.10), (T1-4.11), we have

$$b^{\tau}_{s_i} - w_{s_i,t_i} \ge \Lambda_i \ge \Lambda_{i\to i+1} > (w'_{s_{i-1},t_i} - c_{s_{i-1},t_i}) - (w_{s_i,t_i} - c_{s_i,t_i})$$

Therefore, $s_i$ can afford an effective wage of $(w'_{s_{i-1},t_i} - c_{s_{i-1},t_i})$ to $t_i$. Post-subsidy stability requires

$$v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}} > v_{s_i,t_i} - (w'_{s_{i-1},t_i} - c_{s_{i-1},t_i} + c_{s_i,t_i}) \tag{T1-4.12}$$

Also, by step 2, $s_i \xrightarrow{\mu} s_{i+1}$, pre-subsidy stability requires

$$v_{s_i,t_i} - w_{s_i,t_i} \ge v_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} + c_{s_i,t_{i+1}}) \tag{T1-4.13}$$

combine (T1-4.12), (T1-4.13) and rearrange, we have

$$(w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) < (w'_{s_{i-1},t_i} - c_{s_{i-1},t_i}) - (w_{s_i,t_i} - c_{s_i,t_i})$$

where the LHS is the effective wage change of $t_{i+1}$, the RHS is the effective wage change of $t_i$. Recall (T1-4.1) and (T1-4.2) for the LHS and (T1-4.9) for the RHS, we have

$$\Lambda_{i+1} \le LHS < RHS \le \Lambda_{i\to i+1} \le \Lambda_{i+1}$$

which is a contradiction. Therefore, $\Lambda_{i\to i+1}$ takes the former value

$$\Lambda_{i\to i+1} = \Lambda_i$$

- Case 4.2: Given case 4.1, we have

  $$\Lambda_{i\to i+1} = \Lambda_i$$

  By Lemma 3

  $$\Lambda_{i+1} \ge \Lambda_{i\to i+1} = \Lambda_i$$

  then recall (T1-4.1), (T1-4.2), we have

  $$w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} \ge \left(w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}\right) + \Lambda_{i+1} \ge \left(w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}\right) + \Lambda_i \tag{T1-4.14}$$

(T1-4.14) shows that the effective wage of $t_{i+1}$ increases by at least $\Lambda_i$. Recall $s_i \xrightarrow{\mu} s_{i+1}$, pre-subsidy stability requires

$$v_{s_i,t_i} - w_{s_i,t_i} \ge v_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} + c_{s_i,t_{i+1}}) \tag{T1-4.15}$$

Combine (T1-4.14) and (T1-4.15), we have

$$v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}} \le v_{s_i,t_i} - w_{s_i,t_i} - \Lambda_i \tag{T1-4.16}$$

(T1-4.16) is the key observation in step 4. It shows that $s_i$ suffers at least a utility drop of its own wage-lifting pressure. Intuitively, it means that $s_i$ is paying $s_{i+1}$ an unnecessarily high wage in $(\mu', w')$.

**Step 5: Utility drop of all schools in $\mathcal{C}$**

In step 5, our target is to show each school in cycle $\mathcal{C}$ suffers a huge utility loss, which is the following

$$v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}} \leq v_{s_i,t_i} - w_{s_i,t_i} - \Lambda_i \qquad i = 1, \dots, m \tag{T1-5.1}$$

We show (T1-5.1) by induction, step 4 shows that (T1-5.1) is true for index $i$. Now consider school $s_{i-1}$, and we first show the following

$$\Lambda_{i-1\to i} \equiv \min\Big\{\Lambda_{i-1},\ b^{\tau}_{s_{i-1}} - c_{s_{i-1},t_i} - \big(w_{s_i,t_i} - c_{s_i,t_i}\big)\Big\} = \Lambda_{i-1} \tag{T1-5.2}$$

We show the above by contradiction, that is, assume

$$\Lambda_{i-1\to i} = b^{\tau}_{s_{i-1}} - c_{s_{i-1},t_i} - \big(w_{s_i,t_i} - c_{s_i,t_i}\big) < \Lambda_{i-1}$$

Then consider the effective wage of $t_i$ in $(\mu', w')$, we have

$$w'_{s_{i-1},t_i} - c_{s_{i-1},t_i} \leq b^{\tau}_{s_{i-1}} - c_{s_{i-1},t_i} = w_{s_i,t_i} - c_{s_i,t_i} + \Lambda_{i-1\to i} \tag{T1-5.3}$$

meaning that the effective wage of $s_i$ increases at most $\Lambda_{i-1\to i}$.

By $\tau$ satisfying DC and SC, we have

$$b^{\tau}_{s_i} - w_{s_i,t_i} \geq \Lambda_i$$

By step 2, $s_{i-1} \xrightarrow{\mu} s_i$. Then by Lemma 3, we have

$$b^{\tau}_{s_i} - w_{s_i,t_i} \geq \Lambda_i \geq \Lambda_{i-1\to i} \tag{T1-5.4}$$

Combine (T1-5.3), (T1-5.4) and rearrange

$$b^{\tau}_{s_i} - c_{s_i,t_i} \geq w'_{s_{i-1},t_i} - c_{s_{i-1},t_i}$$

meaning that $s_i$ can afford the effective wage of $t_i$ in $(\mu', w')$.

Recall our induction hypothesis (T1-4.16), $s_i$ suffers at least a utility drop of $\Lambda_i$ in $(\mu', w')$. However, by (T1-5.3), matching $t_i$ back with the same effective wage suffers at most $\Lambda_{i-1\to i}$, where $\Lambda_{i-1\to i} \leq \Lambda_i$. Therefore, $s_i$ should weakly prefer matching $t_i$. Also, $\mu'$ is generated with a tie-breaking rule favoring $\mu$, $s_i$ should strictly prefer matching $t_i$ back, contradicting the stability of $(\mu', w')$.

Hence, we have

$$\Lambda_{i-1\to i} = \Lambda_{i-1}$$

Now consider the utility of $s_{i-1}$, there will be two cases

- Case 5.1: $s_i$ is outbid by $s_{i-1}$ in $(\mu', w')$. Then we have

  $$b^{\tau}_{s_i} - c_{s_i,t_i} \leq w'_{s_{i-1},t_i} - c_{s_{i-1},t_i} \tag{T1-5.5}$$

  by DC and SC, we know

  $$b^{\tau}_{s_i} - w_{s_i,t_i} \geq \Lambda_i \geq \Lambda_{i-1\to i} = \Lambda_{i-1}$$

  then we have

  $$b^{\tau}_{s_i} - c_{s_i,t_i} = b^{\tau}_{s_i} - w_{s_i,t_i} + w_{s_i,t_i} - c_{s_i,t_i} \geq w_{s_i,t_i} - c_{s_i,t_i} + \Lambda_{i-1} \tag{T1-5.6}$$

(T1-5.5) and (T1-5.6) then give

$$w'_{s_{i-1},t_i} - c_{s_{i-1},t_i} \geq w_{s_i,t_i} - c_{s_i,t_i} + \Lambda_{i-1} \tag{T1-5.7}$$

We also know $s_{i-1} \xrightarrow{\mu} s_i$, pre-subsidy stability between $(s_{i-1}, t_{i-1})$ and $(s_{i-1}, t_i)$ gives

$$v_{s_{i-1},t_{i-1}} - w_{s_{i-1},t_{i-1}} \geq v_{s_{i-1},t_i} - w_{s_i,t_i} + c_{s_i,t_i} - c_{s_{i-1},t_i} \tag{T1-5.8}$$

(T1-5.7) and (T1-5.8) then give the following

$$v_{s_{i-1},t_i} - w'_{s_{i-1},t_i} \leq v_{s_{i-1},t_{i-1}} - w_{s_{i-1},t_{i-1}} - \Lambda_{i-1}$$

which is (T1-5.1) for index $i-1$.

- Case 5.2: $s_i$ is not outbid by $s_{i-1}$ in $(\mu', w')$. Then $s_i$ can afford the effective wage of $t_i$ after the subsidy. Post-subsidy stability requires

$$v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}} \geq v_{s_i,t_i} - w'_{s_{i-1},t_i} + c_{s_{i-1},t_i} - c_{s_i,t_i}$$

Since $s_{i-1} \xrightarrow{\mu} s_i$, pre-subsidy stability requires

$$v_{s_{i-1},t_{i-1}} - w_{s_{i-1},t_{i-1}} \geq v_{s_{i-1},t_i} - w_{s_i,t_i} + c_{s_i,t_i} - c_{s_{i-1},t_i}$$

Combining the above two inequalities,

$$(v_{s_{i-1},t_i} - w'_{s_{i-1},t_i}) - (v_{s_{i-1},t_{i-1}} - w_{s_{i-1},t_{i-1}}) \leq (v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}}) - (v_{s_i,t_i} - w_{s_i,t_i}) \tag{T1-5.9}$$

meaning that $s_{i-1}$ suffers a larger utility drop than $s_i$. Combine our induction hypothesis (T1-4.16) and (T1-5.9), we have

$$(v_{s_{i-1},t_i} - w'_{s_{i-1},t_i}) - (v_{s_{i-1},t_{i-1}} - w_{s_{i-1},t_{i-1}}) \leq -\Lambda_i \leq -\Lambda_{i-1\to i} = -\Lambda_{i-1}$$

which rearranges to (T1-5.1) for index $i-1$

$$v_{s_{i-1},t_i} - w'_{s_{i-1},t_i} \leq v_{s_{i-1},t_{i-1}} - w_{s_{i-1},t_{i-1}} - \Lambda_{i-1}$$

Given the induction step and (T1-4.16) at index $i$, we conclude that

$$v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}} \leq v_{s_i,t_i} - w_{s_i,t_i} - \Lambda_i \qquad i = 1, \ldots, m$$

that is, every school $s_i$ in $\mathcal{C}$ suffers a utility loss at least $\Lambda_i$.

### Step 6: A dominant and feasible reallocation

In this step, we are trying to construct a feasible and better allocation for all schools in $\mathcal{C}$.

Construct a hypothetical allocation $(\widetilde{\mu}, \widetilde{w})$ as follows

$$\widetilde{\mu}(s_i) = t_i, \qquad \widetilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \quad (s_i \in \mathcal{C})$$

and leave all matches/wages outside $\mathcal{C}$ as in $(\mu', w')$. By DC and SC, $b^\tau_{s_i} - w_{s_i,t_i} \geq \Lambda_i$ for every $i \in \mathcal{C}$, hence $(\widetilde{\mu}, \widetilde{w})$ is feasible under $b^\tau$.

- IR hold for all schools and teachers in $\mathcal{C}$.

  For each teacher, she is receiving her wage in $(\mu, w)$ plus a nonnegative wage buffer, IR in $(\widetilde{\mu}, \widetilde{w})$ is implied by IR in $(\mu, w)$. For each school, it is weakly better than $(\mu', w')$, so IR in $(\widetilde{\mu}, \widetilde{w})$ is implied by IR in $(\mu', w')$.
- Schools in $\mathcal{C}$ are weakly better, so no blocks with outside teachers.

  For each $s_i \in \mathcal{C}$, step 5 established

$$v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}} \leq v_{s_i,t_i} - w_{s_i,t_i} - \Lambda_i \qquad i = 1, \ldots, m$$

  Thus every school in $\mathcal{C}$ weakly improves. If $(s_i, t)$ with $t \notin \mathcal{C}$ can block $(\widetilde{\mu}, \widetilde{w})$, then it would also block $(\mu', w')$, contradicting the stability of $(\mu', w')$. Therefore there is no block between a school in $\mathcal{C}$ and an outside teacher.
- No deviations within the cycle.

To show the last piece, we first develop the following lemma

**Lemma 4.** *For any school $s_i, s_j$, consider an allocation $(\tilde{\mu}, \tilde{w})$ after the subsidy where*

$$\begin{cases} \tilde{\mu}(s_i) = t_i \\ \tilde{\mu}(s_j) = t_j \end{cases} \qquad \begin{cases} \tilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \\ \tilde{w}_{s_j,t_j} = w_{s_j,t_j} + \Lambda_j \end{cases}$$

*then $(s_i, t_j)$ does not form a blocking pair in $(\tilde{\mu}, \tilde{w})$.*

*Proof.* We consider the following two cases:

- Case 1: $s_i$ is not outbid by $s_j$ in $\mu$. So we have $s_i \xrightarrow{\mu} s_j$. Pre-subsidy stability gives

$$v_{s_i,t_i} - w_{s_i,t_i} \geq v_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) + c_{s_i,t_j} \tag{L4-1}$$

  Recall Lemma 3 and $s_i \xrightarrow{\mu} s_j$, we have

$$\Lambda_j \geq \Lambda_{i \to j} \equiv \min\Big\{\Lambda_i,\ b^{\tau}_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\}$$

  If $\Lambda_{i \to j} = \Lambda_i$, then we have

$$\Lambda_j \geq \Lambda_i$$

  Recall (L4-1), we then have

$$v_{s_i,t_i} - w_{s_i,t_i} - \Lambda_i \geq v_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) + c_{s_i,t_j} - \Lambda_j$$

  where the LHS is the utility of $s_i$ in $(\tilde{\mu}, \tilde{w})$. And the RHS is the utility of deviating to $t_j$ with $t_j$ getting the same effective wage. Therefore, $s_i$ still does not gain by deviating to $t_j$, and no block arises. If $\Lambda_{i \to j} = b^{\tau}_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)$, then we have

$$\Lambda_j \geq b^{\tau}_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)$$

  which rearranges to

$$b^{\tau}_{s_i} - c_{s_i,t_j} \leq w_{s_j,t_j} - c_{s_j,t_j} + \Lambda_j = \tilde{w}_{s_j,t_j} - c_{s_j,t_j}$$

  meaning that $s_i$ is outbid by $s_j$ in $(\tilde{\mu}, \tilde{w})$, and no block arises.

- Case 2: $s_i$ is outbid by $s_j$ in $\mu$.
  By Definition 9
  $$\Delta_j^d \geq (b_{s_i}^{\tau} - c_{s_i,t_j}) - (w_{s_j,t_j} - c_{s_j,t_j})$$
  Since $\Lambda_j \geq \Delta_j^d$, we have
  $$b_{s_i}^{\tau} - c_{s_i,t_j} \leq (w_{s_j,t_j} - c_{s_j,t_j}) + \Lambda_j = \widetilde{w}_{s_j,t_j} - c_{s_j,t_j}$$
  So $s_i$ is still outbid by $s_j$ in $(\widetilde{\mu}, \widetilde{w})$, and no block arises.

The above has shown that $(s_i, t_j)$ cannot block $(\widetilde{\mu}, \widetilde{w})$, which finishes the proof of Lemma 4. $\square$

Now we are ready to show there are no deviations within the cycle. Recall our construction of $(\widetilde{\mu}, \widetilde{w})$. For any $s_i, s_j \in \mathcal{C}$, we have

$$\begin{cases} \tilde{\mu}(s_i) = t_i \\ \tilde{\mu}(s_j) = t_j \end{cases} \qquad \begin{cases} \tilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \\ \tilde{w}_{s_j,t_j} = w_{s_j,t_j} + \Lambda_j \end{cases}$$

Then Lemma 4 directly implies that $(s_i, t_j)$ cannot block $(\widetilde{\mu}, \widetilde{w})$.

If $(\widetilde{\mu}, \widetilde{w})$ is stable, then every school in $\mathcal{C}$ would be weakly better than in $(\mu', w')$, and at least one strictly so. Because teachers' effective wages weakly decrease, and the total welfare generated by $\widetilde{\mu}$ on $\mathcal{C}$ is strictly larger by (T1-1).

But $(\mu', w')$ is the school-optimal stable allocation, hence $(\widetilde{\mu}, \widetilde{w})$ cannot be stable. We have shown that there is no block within the cycle, and there is no block between a school in $\mathcal{C}$ and an outside teacher. Therefore, any blocking pair must involve an outside school and a teacher in $\mathcal{C}$. Hence, there exist $s_k \notin \mathcal{C}$, $t_j \in \mathcal{C}$, and a feasible wage $\widehat{w}$ such that $(s_k, t_j, \widehat{w})$ blocks $(\widetilde{\mu}, \widetilde{w})$.

**Step 7:**

Let $(s_k, t_j, \widehat{w})$ be a blocking pair for $(\widetilde{\mu}, \widetilde{w})$ with $s_k \notin \mathcal{C}$ and $t_j \in \mathcal{C}$. In this step, we are trying to show

$$s_k \xrightarrow{\mu} s_j$$

and

$$v_{s_k,\mu'(s_k)} - w'_{s_k,\mu'(s_k)} < v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_k$$

Because $t_j$ accepts the deviation, teacher's incentive implies

$$\widehat{w} - c_{s_k,t_j} > \widetilde{w}_{s_j,t_j} - c_{s_j,t_j} = (w_{s_j,t_j} - c_{s_j,t_j}) + \Lambda_j$$

hence

$$b^{\tau}_{s_k} - c_{s_k,t_j} \geq \widehat{w} - c_{s_k,t_j} > (w_{s_j,t_j} - c_{s_j,t_j}) + \Lambda_j \tag{T1-7.1}$$

We now show $s_k \xrightarrow{\mu} s_j$ pre-subsidy. Suppose, to the contrary, that $s_k$ is outbid by $s_j$ in $(\mu, w)$, we then have

$$b_{s_k} - c_{s_k,t_j} \leq w_{s_j,t_j} - c_{s_j,t_j}$$

By DC, we have

$$\Delta^d_j \geq (b^{\tau}_{s_k} - c_{s_k,t_j}) - (w_{s_j,t_j} - c_{s_j,t_j})$$

By construction of $\Lambda_j$, we have

$$\Lambda_j \geq \Delta^d_j$$

Then we have

$$\Lambda_j \geq \Delta^d_j \geq (b^{\tau}_{s_k} - c_{s_k,t_j}) - (w_{s_j,t_j} - c_{s_j,t_j}) \tag{T1-7.2}$$

(T1-7.1) and (T1-7.2) imply

$$(b^{\tau}_{s_k} - c_{s_k,t_j}) - (w_{s_j,t_j} - c_{s_j,t_j}) > \Lambda_j \geq \Delta^d_j \geq (b^{\tau}_{s_k} - c_{s_k,t_j}) - (w_{s_j,t_j} - c_{s_j,t_j})$$

which is a contradiction.

Therefore, we have $s_k \xrightarrow{\mu} s_j$, which means

$$b_{s_k} - c_{s_k,t_j} > w_{s_j,t_j} - c_{s_j,t_j}$$

By Lemma 3, we have

$$\Lambda_j \geq \Lambda_{k\to j} \tag{T1-7.3}$$

Using (T1-7.1), we have

$$b^{\tau}_{s_k} - c_{s_k,t_j} > (w_{s_j,t_j} - c_{s_j,t_j}) + \Lambda_j \geq (w_{s_j,t_j} - c_{s_j,t_j}) + \Lambda_{k\to j}$$

By definition of $\Lambda_{k\to j}$, we have

$$b^{\tau}_{s_k} - c_{s_k,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) > \Lambda_{k\to j} \equiv \min\left\{\Lambda_k,\ b^{\tau}_{s_k} - c_{s_k,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\}$$

If it takes the latter value

$$\Lambda_{k\to j} = b^{\tau}_{s_k} - c_{s_k,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})$$

Then we have

$$b^{\tau}_{s_k} - c_{s_k,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) > \Lambda_{k\to j} = b^{\tau}_{s_k} - c_{s_k,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})$$

which is a contradiction. So we have

$$b^{\tau}_{s_k} - c_{s_k,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) > \Lambda_{k\to j}$$

and

$$\Lambda_{k\to j} = \Lambda_k$$

Recall (T1-7.3), we have

$$\Lambda_j \geq \Lambda_{k\to j} = \Lambda_k \tag{T1-7.4}$$

We now show the outside blocker loses at least $\Lambda_k$. Since $(s_k, t_j, \widehat{w})$ blocks $(\widetilde{\mu}, \widetilde{w})$, school $s_k$'s deviation condition and (T1-7.1) gives

$$v_{s_k,\mu'(s_k)} - w'_{s_k,\mu'(s_k)} < v_{s_k,t_j} - \hat{w} < \Big(v_{s_k,t_j} - c_{s_k,t_j}\Big) - \Big(w_{s_j,t_j} - c_{s_j,t_j}\Big) - \Lambda_j \tag{T1-7.5}$$

By $s_k \xrightarrow{\mu} s_j$, pre-subsidy stability between $(s_k, t_k)$ and $(s_k, t_j)$ implies

$$v_{s_k,t_k} - w_{s_k,t_k} \geq \Big(v_{s_k,t_j} - c_{s_k,t_j}\Big) - \Big(w_{s_j,t_j} - c_{s_j,t_j}\Big) \tag{T1-7.6}$$

Combining (T1-7.4), (T1-7.5) and (T1-7.6) yields

$$v_{s_k,\mu'(s_k)} - w'_{s_k,\mu'(s_k)} < v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_j \leq v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_k \tag{T1-7.7}$$

Thus $s_k$ suffers a utility drop of at least $\Lambda_k$.

### Case 7.A: Outside blocker is a fixed point

Let $(s_k, t_j, \widehat{w})$ be the blocking pair for $(\widetilde{\mu}, \widetilde{w})$ from Step 6, and suppose $s_k$ is a fixed point, that is

$$\mu'(s_k) = \mu(s_k) = t_k$$

By (T1-7.7), $s_k$ loses at least $\Lambda_k$, we have

$$v_{s_k,t_k} - w'_{s_k,t_k} < v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_k$$

which implies

$$w'_{s_k,t_k} > w_{s_k,t_k} + \Lambda_k$$

Update the hypothetical allocation $(\widetilde{\mu}, \widetilde{w})$ as follows:

$$\widetilde{\mu}(s_i) = t_i \quad (s_i \in \mathcal{C} \cup s_k \cup t_k), \qquad \widetilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \quad (s_i \in \mathcal{C} \cup s_k \cup t_k)$$

and keep all other matches/wages as in $(\mu', w')$.

This hypothetical allocation is feasible by DC and SC. We now show it is also stable within $\mathcal{C} \cup s_k \cup t_k$.

We can use the same reasoning in Step 6 to show

- IR hold for all schools and teachers in $\mathcal{C} \cup s_k \cup t_k$.
- Schools in $\mathcal{C} \cup s_k \cup t_k$ are weakly better, no blocks with outside teachers.
- Lemma 4 shows that there are no deviations within $\mathcal{C} \cup s_k \cup t_k$.

### Case 7.B: Outside blocker belongs to a cycle without outbid

#### Step B.1: Utility drop at the outside blocker on $\mathcal{D}$

Let $(s_k, t_j, \widehat{w})$ be the blocking pair for $(\widetilde{\mu}, \widetilde{w})$ from Step 6, and suppose $s_k$ belongs to a cycle $\mathcal{D}$ (we adopt similar index as in $\mathcal{C}$, so school $s_i$ is matching teacher $t_{i+1}$) with the following:

$$b_{s_i} - c_{s_i,t_{i+1}} > w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} \qquad \text{for all } s_i \in \mathcal{D} \tag{T1-7.7}$$

Repeating the argument of Step 7, we have $s_k \xrightarrow{\mu} s_j$ and

$$v_{s_k,t_{k+1}} - w'_{s_k,t_{k+1}} < v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_k \tag{T1-7.8}$$

that is, $s_k$'s utility drops by at least $\Lambda_k$.

#### Step B.2: Utility drop on the entire cycle $\mathcal{D}$

We now show that every school in $\mathcal{D}$ loses at least its own $\Lambda$.

By DC and SC, we have

$$b^{\tau}_{s_k} - w_{s_k,t_k} \geq \Lambda_k$$

meaning that $s_k$ can afford a wage of $w_{s_k,t_k} + \Lambda_k$ to $t_k$. If $t_k$ accepts this offer, then $s_k$ is getting strictly better because it loses more than $\Lambda_k$ in $(\mu', w')$. And matching $t_k$ with a wage of $w_{s_k,t_k} + \Lambda_k$ only loses $\Lambda_k$. Therefore, by stability of $(\mu', w')$, the effective wage of $t_k$ must be strictly higher than $w_{s_k,t_k} - c_{s_k,t_k} + \Lambda_k$. Formally,

$$w'_{s_{k-1},t_k} - c_{s_{k-1},t_k} > (w_{s_k,t_k} - c_{s_k,t_k}) + \Lambda_k \tag{T1-7.9}$$

Given the feasibility of $(\mu', w')$, we have

$$b^{\tau}_{s_{k-1}} - c_{s_{k-1},t_k} \geq w'_{s_{k-1},t_k} - c_{s_{k-1},t_k} > (w_{s_k,t_k} - c_{s_k,t_k}) + \Lambda_k$$

Repeat the argument in step 7 from (T1-7.1) to (T1-7.4), we have

$$\Lambda_k \geq \Lambda_{k-1\to k} = \Lambda_{k-1} \tag{T1-7.10}$$

Now consider the utility of $s_{k-1}$. By our assumption (T1-7.7) , we have $s_{k-1} \xrightarrow{\mu} s_k$. Pre-subsidy stability gives

$$v_{s_{k-1},t_{k-1}} - w_{s_{k-1},t_{k-1}} \geq (v_{s_{k-1},t_k} - c_{s_{k-1},t_k}) - (w_{s_k,t_k} - c_{s_k,t_k}) \tag{T1-7.11}$$

(T1-7.9), (T1-7.10) and (T1-7.11) give the following:

$$v_{s_{k-1},t_k} - w'_{s_{k-1},t_k} < v_{s_{k-1},t_{k-1}} - w_{s_{k-1},t_{k-1}} - \Lambda_k \leq v_{s_{k-1},t_{k-1}} - w_{s_{k-1},t_{k-1}} - \Lambda_{k-1} \tag{T1-7.12}$$

Iterating the same one-step argument successively around $\mathcal{D}$ implies that

$$v_{s_k,t_{k+1}} - w'_{s_k,t_{k+1}} < v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_k \qquad \text{for all } s_k \in \mathcal{D} \tag{T1-7.13}$$

which means every school in $\mathcal{D}$ loses at least its own $\Lambda$.

**Step B.3: Expand the hypothetical allocation to $\mathcal{C} \cup \mathcal{D}$**

Similar to step A, we now update the hypothetical allocation $(\widetilde{\mu}, \widetilde{w})$ as follows:

$$\widetilde{\mu}(s_i) = t_i \quad (s_i \in \mathcal{C} \cup \mathcal{D})$$

$$\widetilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \quad (s_i \in \mathcal{C} \cup \mathcal{D})$$

and keep all other matches/wages as in $(\mu', w')$. Similarly, this updated $(\widetilde{\mu}, \widetilde{w})$ is feasible by DC and SC.

To show the stability within $\mathcal{C} \cup \mathcal{D}$. We can use the same reasoning in step 6 to show

- IR hold for all schools and teachers in $\mathcal{C} \cup \mathcal{D}$.
- Schools in $\mathcal{C} \cup \mathcal{D}$ are weakly better, no blocks with outside teachers.
- Lemma 4 shows that there are no deviations within $\mathcal{C} \cup \mathcal{D}$.

### Case 7.C: Outside blocker belongs to a cycle with outbid

Let $(s_k, t_j, \widehat{w})$ be the blocking pair for $(\widetilde{\mu}, \widetilde{w})$ from Step 6, and suppose $s_k$ belongs to a cycle $\mathcal{E}$ (we adopt similar index as in $\mathcal{C}$, so school $s_i$ is matching teacher $t_{i+1}$) with the following:

$$\exists\ i \text{ such that} \qquad b_{s_i} - c_{s_i,t_{i+1}} \leq w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} \qquad \text{where } s_i \in \mathcal{E} \tag{T1-7.14}$$

Repeating the argument of Step 7, we have $s_k \xrightarrow{\mu} s_j$ and

$$v_{s_k,t_{k+1}} - w'_{s_k,t_{k+1}} < v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_k \tag{T1-7.8'}$$

that is, $s_k$'s utility drops by at least $\Lambda_k$.

Repeat the argument of Step B.2. For $s_{k-1}$, we still get the following

$$w'_{s_{k-1},t_k} - c_{s_{k-1},t_k} > (w_{s_k,t_k} - c_{s_k,t_k}) + \Lambda_k \tag{T1-7.9'}$$

Then by feasibility of $(\mu', w')$, we have

$$b^{\tau}_{s_{k-1}} - c_{s_{k-1},t_k} \geq w'_{s_{k-1},t_k} - c_{s_{k-1},t_k} > (w_{s_k,t_k} - c_{s_k,t_k}) + \Lambda_k \tag{T1-7.15}$$

The above rules out the possibility that $s_{k-1}$ is outbid by $s_k$ in $(\mu, w)$, because otherwise DC and SC requires

$$\Lambda_k \geq b^{\tau}_{s_{k-1}} - c_{s_{k-1},t_k} - (w_{s_k,t_k} - c_{s_k,t_k}) \tag{T1-7.16}$$

and (T1-7.15), (T1-7.16) lead to a contradiction.

Therefore, we now have $s_{k-1} \xrightarrow{\mu} s_k$ and (T1-7.15). Repeating the logic from (T1-7.1) to (T1-7.4), we have

$$\Lambda_{k-1} \leq \Lambda_k \tag{T1-7.17}$$

Combine (T1-7.9) and (T1-7.17), we have

$$w'_{s_{k-1},t_k} - c_{s_{k-1},t_k} > (w_{s_k,t_k} - c_{s_k,t_k}) + \Lambda_{k-1} \tag{T1-7.18}$$

Recall $s_{k-1} \xrightarrow{\mu} s_k$, pre-subsidy stability requires

$$v_{s_{k-1},t_{k-1}} - w_{s_{k-1},t_{k-1}} \geq (v_{s_{k-1},t_k} - c_{s_{k-1},t_k}) - (w_{s_k,t_k} - c_{s_k,t_k}) \tag{T1-7.19}$$

Combining (T1-7.18) and (T1-7.19) gives

$$v_{s_{k-1},t_k} - w'_{s_{k-1},t_k} < v_{s_{k-1},t_{k-1}} - w_{s_{k-1},t_{k-1}} - \Lambda_{k-1} \tag{T1-7.20}$$

Then we can repeat the logic from (T1-7.8') to (T1-20) to get

$$s_{k-2} \xrightarrow{\mu} s_{k-1}$$

and

$$v_{s_{k-2},t_{k-1}} - w'_{s_{k-2},t_{k-1}} < v_{s_{k-2},t_{k-2}} - w_{s_{k-2},t_{k-2}} - \Lambda_{k-2} \tag{T1-7.21}$$

However, different from Case 7.B, we have (T1-7.14)

$$\exists\ i \text{ such that} \qquad b_{s_i} - c_{s_i,t_{i+1}} \leq w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} \qquad \text{where } s_i \in \mathcal{E} \tag{T1-7.14}$$

and the above one-step propagation (T1-7.21) will finally give $s_i \xrightarrow{\mu} s_{i+1}$ leading to a contradiction of (T1-7.14).

### Step 8: Exhaustion and final contradiction

We first find a welfare-decreasing cycle $\mathcal{C}$ in step 1. Then follow step 2 to step 6, we initialize reallocation on $\mathcal{C}$ and let

$$\widetilde{\mu}(s_i) = t_i, \qquad \widetilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \quad (s_i \in \mathcal{C})$$

Let $\mathcal{R}$ denote the set of schools and teachers that have been reallocated. So $\mathcal{R} = \mathcal{C}$ at the beginning. Then we iteratively apply the expansion rule: whenever there exists a blocking pair $(s_k, t_j, \widehat{w})$ for $(\widetilde{\mu}, \widetilde{w})$ with $t_j \in \mathcal{R}$ and $s_k \notin \mathcal{R}$, expand the reallocation set and update $(\widetilde{\mu}, \widetilde{w})$ as follows

- If $s_k$ is a fixed point, apply 7.A and add $s_k, t_k$ into $\mathcal{R}$ and let

$$\widetilde{\mu}(s_i) = t_i \quad (s_i \in \mathcal{R} \cup s_k \cup t_k), \qquad \widetilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \quad (s_i \in \mathcal{R} \cup s_k \cup t_k)$$

- If $s_k$ belongs to a cycle $\mathcal{D}$ without outbid, apply 7.B and add $\mathcal{D}$ into $\mathcal{R}$ and let

$$\widetilde{\mu}(s_i) = t_i \quad (s_i \in \mathcal{R} \cup \mathcal{D}), \qquad \widetilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \quad (s_i \in \mathcal{R} \cup \mathcal{D})$$

Keep the above expansion, there are two possibilities.

**(1) All outside schools are Case 7.A or Case 7.B.**

Since the market is finite, the expansion terminates when it exhausts the set of all schools. By internal stability of $\mathcal{R}$ and the absence of remaining outside schools, $(\widetilde{\mu}, \widetilde{w})$ is stable. Every school is weakly better than in $(\mu', w')$, and at least one is strictly better, contradicting the school–optimality of $(\mu', w')$.

**(2) Some outside schools are Case 7.C.**

By Case 7.C, when the expansion reaches a cycle with outbid, we directly have a contradiction.

In either branch, we reach a contradiction. Therefore, the assumption in step 1

$$W_{v,c}(\mu') < W_{v,c}(\mu)$$

is false. Hence, for every value profile $v \in \mathcal{V}(\mu, w; b, c)$, we have

$$\overline{W}(v, c, b, \tau) \geq W_{v,c}(\mu') \geq W_{v,c}(\mu)$$

which completes the proof.

$\square$

**Proof of Proposition 2.**

*Proof.* WLOG, we again let $\mu(s_i) = t_i$ for all $i$.

Let $\tau$ be the positive uniform subsidy allocation and let $\nabla$ be the quantity of subsidy allocated to each school. Then we have

$$\nabla = \tau_s > 0 \quad \text{for all } s \in S$$

We claim $\tau$ is dominated by the following $\tau'$, where

$$\begin{cases} \tau'(s) = \tau_s = \nabla \quad \text{for all } s \neq s_j \\ \tau'(s_j) = \max\left\{ \max_{i\neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla - b_{s_j}, \max_{i\neq j}\{w_{s_i,t_i} - c_{s_i,t_i} + c_{s_j,t_i}\} + \nabla - b_{s_j}, 0\right\} \end{cases}$$

We first verify $\tau'(s_j) < \tau(s_j)$. Recall conditions of Proposition 2

$$b_{s_i} - c_{s_i,t_j} < b_{s_j} - c_{s_j,t_j} \quad \text{for all } i \neq j$$

so we have

$$\max_{i\neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} - b_{s_j} + \nabla < \nabla$$

Then recall conditions of Proposition 2, we have $s_j \xrightarrow{\mu} s_i$ for any $i \neq j$, so we have

$$b_{s_j} - c_{s_j,t_i} > w_{s_i,t_i} - c_{s_i,t_i} \quad \text{for all } i \neq j$$

Th gives

$$\max_{i\neq j}\{w_{s_i,t_i} - c_{s_i,t_i} + c_{s_j,t_i}\} + \nabla - b_{s_j} < \nabla$$

also, we know $\nabla > 0$, which gives

$$\tau'(s_j) < \nabla = \tau(s_j) \quad \Rightarrow \quad \sum_{s\in S}\tau'_s < \sum_{s\in S}\tau_s$$

Recall the definition of $\overline{W}(v, c, b, \tau)$, we have

$$\overline{W}(v, c, b, \tau) \equiv \max_{\mu}\Big\{W(\mu; v, c) : \ \exists\ w \text{ such that } (\mu, w) \text{ is stable under budgets } b^{\tau} \text{ given } (v, c)\Big\}$$

Then for $\forall\ v \in \mathcal{V}(\mu, w; b, c)$, there exists a stable allocation $(\mu', w')$ after the subsidy such that

$$\overline{W}(v, c, b, \tau) = W(\mu', v, c)$$

Fix $v$, and consider the corresponding $(\mu', w')$, there will be two cases

- $\mu'(s_j) = t_j$
  That is, $s_j$ does not change its teacher from $\mu$ to $\mu'$. There will be three more subcases:
  - $w'_{s_j,t_j} \leq \max\left\{w_{s_j,t_j}, \max_{i\neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla\right\}$
    Now we replace $\tau$ with $\tau'$. By construction of $\tau'$, we have $\tau'(s_j) \geq 0$, so

    $$b^{\tau'}_{s_j} \geq b_{s_j} \geq w_{s_j,t_j} \geq w'_{s_j,t_j}$$

    Also, by construction of $\tau'$, we have

    $$b^{\tau'}_{s_j} \geq \max_{i\neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla$$

The above two inequalities give

$$b_{s_j}^{\tau'} \geq \max\left\{w_{s_j,t_j}, \max_{i\neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla\right\} \geq w'_{s_j,t_j}$$

Meaning that $(\mu', w')$ is still feasible. Now we verify it is stable:

Since the only thing changed is the budget of $s_j$, so IR still hold. And we only need to check if $(s_j, t_k)$ forms a blocking pair where $k \neq j$.

If $s_j$ is outbid by $\mu'^{-1}(t_k)$ in $(\mu', w')$ with subsidy $\tau$, we have

$$b_{s_j}^{\tau} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

Since $\tau_{s_j} > \tau'_{s_j}$, we have

$$b_{s_j}^{\tau'} - c_{s_j,t_k} < b_{s_j}^{\tau} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

So $s_j$ is still outbid by $\mu'^{-1}(t_k)$ in $(\mu', w')$ with subsidy $\tau'$, and $(s_j, t_k)$ does not form a blocking pair.

If $s_j \xrightarrow{\mu'} \mu'^{-1}(t_k)$ with $\tau$, then stability requires

$$v_{s_j,\mu'(s_j)} - w'_{s_j,\mu'(s_j)} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k})$$

Since the allocation does not change, there is still no incentive for $s_j$ to deviate to $t_k$ with subsidy $\tau'$, and $(s_j, t_k)$ does not form a blocking pair.

Therefore, $(\mu', w')$ is still feasible and stable when the subsidy is $\tau'$. Then we have

$$\overline{W}(\tau', v, b, c) \geq W(\mu', v, c) = \overline{W}(v, c, b, \tau)$$

– $w'_{s_j,t_j} > w_{s_j,t_j} > \max_{i\neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla$

Then we can construct the following allocation $(\mu', \tilde{w})$, where

$$\begin{cases} \tilde{w}_{s,\mu'(s)} = w'_{s,\mu'(s)} & \text{for all } s \neq s_j \\ \tilde{w}_{s_j,t_j} = w_{s_j,t_j} \end{cases}$$

Now we show $(\mu', \tilde{w})$ is feasible and stable after the subsidy.

Recall $\tau'(s_j) \geq 0$, so

$$b_{s_j}^{\tau'} \geq b_{s_j} \geq w_{s_j,t_j} \geq w'_{s_j,t_j}$$

meaning that $(\mu', \tilde{w})$ is feasible.

Now we need to verify there are no blocking pairs in $(\mu', \tilde{w})$ when the subsidy is $\tau'$. Comparing to $(\mu', w')$ with subsidy $\tau$, we only changed $\tau'(s_j)$ and $\tilde{w}_{s_j,t_j}$. Therefore, we only need to check if $(s_j, t_k)$ or $(s_k, t_j)$ form a blocking pair where $k \neq j$.

First consider $(s_j, t_k)$. If $s_j$ is outbid by $\mu'^{-1}(t_k)$ in $(\mu', w')$ with subsidy $\tau$, we have

$$b_{s_j}^{\tau} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

Since $\tau_{s_j} > \tau'_{s_j}$, we have

$$b_{s_j}^{\tau'} - c_{s_j,t_k} < b_{s_j}^{\tau} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k} = \tilde{w}_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

So $s_j$ is still outbid by $\mu'^{-1}(t_k)$ in $(\mu', \tilde{w})$ with subsidy $\tau'$, and $(s_j, t_k)$ does not form a blocking pair.

If $s_j \xrightarrow{\mu'} \mu'^{-1}(t_k)$ when the subsidy is $\tau$, then stability requires

$$v_{s_j,t_j} - w'_{s_j,t_j} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k})$$

By construction of $\tilde{w}$ and the case assumption, we have

$$\tilde{w}_{s_j,t_j} = w_{s_j,t_j} < w'_{s_j,t_j}$$

which gives

$$v_{s_j,t_j} - \tilde{w}_{s_j,t_j} > v_{s_j,t_j} - w'_{s_j,t_j} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k})$$

meaning that there is still no incentive for $s_j$ to deviate to $t_k$, and $(s_j, t_k)$ does not form a blocking pair.

Now consider consider $(s_k, t_j)$. By construction of $\tilde{w}$ and the case assumption, we have

$$\tilde{w}_{s_j,t_j} = w_{s_j,t_j} > \max_{i \neq j}(b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}) + \nabla \geq b_{s_k} - c_{s_k,t_j} + c_{s_j,t_j} + \nabla$$

By construction of $\tau'$, we then have

$$b^{\tau'}_{s_k} - c_{s_k,t_j} = b_{s_k} - c_{s_k,t_j} + \nabla < \tilde{w}_{s_j,t_j} - c_{s_j,t_j}$$

meaning that $s_k$ is outbid by $s_j$ in $(\mu', \tilde{w})$ with subsidy $\tau'$, and $(s_k, t_j)$ does not form a blocking pair.

Lastly, we check IR for $s_k$ and $t_k$. By construction of $\tilde{w}$, we have

$$\tilde{w}_{s_j,t_j} = w_{s_j,t_j}$$

so IR of $t_k$ is implied by IR of $(\mu, w)$ before the subsidy. For $s_k$, we have

$$v_{s_j,t_j} - \tilde{w}_{s_j,t_j} > v_{s_j,t_j} - w'_{s_j,t_j}$$

so IR of $s_k$ is implied by IR of $(\mu', w')$ with subsidy $\tau$.

To conclude, $(\mu', \tilde{w})$ is feasible and stable with subsidy $\tau'$, so we have

$$\overline{W}(\tau', v, b, c) \geq W(\mu', v, c) = \overline{W}(v, c, b, \tau)$$

– $w'_{s_j,t_j} > \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla \geq w_{s_j,t_j}$
Then we can construct the following allocation $(\mu', \tilde{w})$, where

$$\begin{cases} \tilde{w}_{s,\mu'(s)} = w'_{s,\mu'(s)} & \text{for all } s \neq s_j \\ \tilde{w}_{s_j,t_j} = \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla \end{cases}$$

Now we show $(\mu', \tilde{w})$ is feasible and stable after the subsidy.

By construction of $\tau'$, we have

$$b^{\tau'}_{s_j} = b_{s_j} + \tau'_{s_j} \geq \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla \geq \tilde{w}_{s_j,t_j}$$

meaning that $(\mu', \tilde{w})$ is feasible.

Now we need to verify there are no blocking pairs in $(\mu', \tilde{w})$ when the subsidy is $\tau'$. Comparing to $(\mu', w')$ with subsidy $\tau$, we only changed $\tau'(s_j)$ and $\tilde{w}_{s_j,t_j}$. Therefore, we only need to check if $(s_j, t_k)$ or $(s_k, t_j)$ form a blocking pair where $k \neq j$.

First consider $(s_j, t_k)$. If $s_j$ is outbid by $\mu'^{-1}(t_k)$ in $(\mu', w')$ with subsidy $\tau$, we have

$$b^{\tau}_{s_j} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

Since $\tau_{s_j} > \tau'_{s_j}$, we have

$$b^{\tau'}_{s_j} - c_{s_j,t_k} < b^{\tau}_{s_j} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k} = \tilde{w}_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

So $s_j$ is still outbid by $\mu'^{-1}(t_k)$ in $(\mu', \tilde{w})$ with subsidy $\tau'$, and $(s_j, t_k)$ does not form a blocking pair.

If $s_j \xrightarrow{\mu'} \mu'^{-1}(t_k)$ when the subsidy is $\tau$, then stability requires

$$v_{s_j,t_j} - w'_{s_j,t_j} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k})$$

By construction of $\tilde{w}$ and the case assumption we have

$$\tilde{w}_{s_j,t_j} = \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla < w'_{s_j,t_j}$$

which gives

$$v_{s_j,t_j} - \tilde{w}_{s_j,t_j} > v_{s_j,t_j} - w'_{s_j,t_j} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k})$$

meaning that there is still no incentive for $s_j$ to deviate to $t_k$, and $(s_j, t_k)$ does not form a blocking pair.

Now consider consider $(s_k, t_j)$. By construction of $\tilde{w}$, we have

$$\tilde{w}_{s_j,t_j} = \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla \geq b_{s_k} - c_{s_k,t_j} + c_{s_j,t_j} + \nabla$$

which gives

$$b^{\tau'}_{s_k} - c_{s_k,t_j} \leq \tilde{w}_{s_j,t_j} - c_{s_j,t_j}$$

meaning that $s_k$ is outbid by $s_j$ in $(\mu', \tilde{w})$ with subsidy $\tau'$, and $(s_k, t_j)$ does not form a blocking pair.

Lastly, we check IR for $s_k$ and $t_k$. By construction of $\tilde{w}$ and the case assumption, we have

$$\tilde{w}_{s_j,t_j} = \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla \geq w_{s_j,t_j}$$

so IR of $t_k$ is implied by IR of $(\mu, w)$ before the subsidy. For $s_k$, we have

$$v_{s_j,t_j} - \tilde{w}_{s_j,t_j} > v_{s_j,t_j} - w'_{s_j,t_j}$$

so IR of $s_k$ is implied by IR of $(\mu', w')$ with subsidy $\tau$.

To conclude, $(\mu', \tilde{w})$ is feasible and stable with subsidy $\tau'$, so we have

$$\overline{W}(\tau', v, b, c) \geq W(\mu', v, c) = \overline{W}(v, c, b, \tau)$$

- $\mu'(s_j) = t_k \neq t_j$

  That is, $s_j$ changes its teacher from $\mu$ to $\mu'$. Now we want to show

$$w'_{s_j,t_k} \leq b^{\tau'}_{s_j}$$

Recall conditions of Proposition 2, we have $s_j \xrightarrow{\mu} s_k$. Pre-subsidy stability then requires

$$v_{s_j,t_j} - w_{s_j,t_j} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w_{s_k,t_k} - c_{s_k,t_k}) \tag{P2-1}$$

Now consider $(\mu', w')$ with subsidy $\tau$. Let $s_m$ be the school matching $t_j$ in $\mu'$, then we have

$$\mu'^{-1}(t_j) = s_m$$

Recall conditions in Proposition 2, we have

$$\begin{cases} b_{s_m} - c_{s_m,t_j} \leq w_{s_j,t_j} - c_{s_j,t_j} \\ b_{s_m} - c_{s_m,t_j} < b_{s_j} - c_{s_j,t_j} \end{cases}$$

We then have

$$\begin{cases} b^{\tau}_{s_m} - c_{s_m,t_j} = b_{s_m} + \nabla - c_{s_m,t_j} \leq w_{s_j,t_j} - c_{s_j,t_j} + \nabla \\ b^{\tau}_{s_m} - c_{s_m,t_j} = b_{s_m} + \nabla - c_{s_m,t_j} < b_{s_j} - c_{s_j,t_j} + \nabla \end{cases}$$

By feasibility of $(\mu', w')$, we then have

$$w_{s_j,t_j} - c_{s_j,t_j} + \nabla \geq b^{\tau}_{s_m} - c_{s_m,t_j} \geq w'_{s_m,t_j} - c_{s_m,t_j} \tag{P2-2}$$

and

$$b^{\tau}_{s_j} - c_{s_j,t_j} = b_{s_j} - c_{s_j,t_j} + \nabla > b^{\tau}_{s_m} - c_{s_m,t_j} \geq w'_{s_m,t_j} - c_{s_m,t_j} \tag{P2-3}$$

(P2-3) implies that $s_j$ can offer a strictly higher effective wage to $t_j$. Then the post-subsidy stability requires

$$v_{s_j,t_k} - w'_{s_j,t_k} \geq v_{s_j,t_j} - c_{s_j,t_j} - (w'_{s_m,t_j} - c_{s_m,t_j}) \tag{P2-4}$$

Combining (P2-1) and (P2-4), we have

$$(w'_{s_j,t_k} - c_{s_j,t_k}) - (w_{s_k,t_k} - c_{s_k,t_k}) \leq (w'_{s_m,t_j} - c_{s_m,t_j}) - (w_{s_j,t_j} - c_{s_j,t_j}) \tag{P2-5}$$

Recall (P2-2), we then have

$$(w'_{s_j,t_k} - c_{s_j,t_k}) - (w_{s_k,t_k} - c_{s_k,t_k}) \leq \nabla$$

which is equivalent to

$$w'_{s_j,t_k} \leq w_{s_k,t_k} - c_{s_k,t_k} + c_{s_j,t_k} + \nabla \tag{P2-6}$$

Recall construction of $\tau'$, we have

$$b^{\tau'}_{s_j} = b_{s_j} + \tau'_{s_j} \geq \max_{i \neq j}\{w_{s_i,t_i} - c_{s_i,t_i} + c_{s_j,t_i}\} + \nabla \geq w_{s_k,t_k} - c_{s_k,t_k} + c_{s_j,t_k} + \nabla \tag{P2-7}$$

(P2-6) and (P2-7) then give

$$w'_{s_j,t_k} \leq b^{\tau'}_{s_j} \tag{P2-8}$$

Now we replace $\tau$ with $\tau'$. (P2-8) shows that $(\mu', w')$ is feasible with subsidy $\tau'$. It remains to show it is stable.

Since the only thing changed is the budget of $s_j$, so IR still hold. And we only need to check if $(s_j, t_k)$ forms a blocking pair where $k \neq j$.

If $s_j$ is outbid by $\mu'^{-1}(t_k)$ in $(\mu', w')$ with subsidy $\tau$, we have

$$b^{\tau}_{s_j} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

Since $\tau_{s_j} > \tau'_{s_j}$, we have

$$b^{\tau'}_{s_j} - c_{s_j,t_k} < b^{\tau}_{s_j} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

So $s_j$ is still outbid by $\mu'^{-1}(t_k)$ in $(\mu', w')$ with subsidy $\tau'$, and $(s_j, t_k)$ does not form a blocking pair.

If $s_j \xrightarrow{\mu'} \mu'^{-1}(t_k)$ with $\tau$, then stability requires

$$v_{s_j,\mu'(s_j)} - w'_{s_j,\mu'(s_j)} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k})$$

Since the allocation does not change, there is still no incentive for $s_j$ to deviate to $t_k$ with subsidy $\tau'$, and $(s_j, t_k)$ does not form a blocking pair.

Therefore, $(\mu', w')$ is still feasible and stable when the subsidy is $\tau'$. Then we have

$$\overline{W}(\tau', v, b, c) \geq W(\mu', v, c) = \overline{W}(v, c, b, \tau)$$

Therefore, in all cases, we have

$$\overline{W}(\tau', v, b, c) \geq W(\mu', v, c) = \overline{W}(v, c, b, \tau)$$

and recall

$$\tau'(s_j) < \nabla = \tau(s_j) \quad \Rightarrow \quad \sum_{s \in S} \tau'_s < \sum_{s \in S} \tau_s$$

The above analysis works for all $v \in \mathcal{V}(\mu, w; b, c)$, meaning that $\tau$ is dominated by $\tau'$, which finishes the proof of Proposition 2. $\square$

**Proof of Theorem 2.**

*Proof.* We first show the necessity. That is, if $\tau$ is non-distortive, then we have

$$b^{\tau}_{s_j} - w_{s_j,\mu(s_j)} \geq \max_{s_i} \tau_{s_i} \quad \text{for } \forall j \tag{T2-1}$$

WLOG, we again let $\mu(s_i) = t_i$ by reindexing.

Fix two distinct indices $k \neq j$. Consider the following cost and value profile $(v^*, c^*)$, where for all matched pairs, we have

$$\begin{cases} v^*_{s_i,t_i} = w_{s_i,t_i} \\ c^*_{s_i,t_i} = w_{s_i,t_i} \end{cases} \qquad \text{for all } i$$

and for all unmatched pairs $(s,t)$ where $t \neq \mu(s)$, we have

$$\begin{cases} v^*_{s_k,t_j} = b_{s_k} \\ c^*_{s_k,t_j} = b_{s_k} \end{cases} \qquad \text{and} \qquad \begin{cases} v^*_{s,t} = 0 \\ c^*_{s,t} = 0 \end{cases} \qquad \text{for all } (s,t) \neq (s_k, t_j)$$

We now show the above $(v^*, c^*)$ is consistent with $(\mu, w)$ and $b$, that is

$$(v^*, c^*) \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

To show the above, we need to verify the following:

- $(\mu, w)$ is individually rational.
  By construction of $(v^*, c^*)$, each school $s_i$ is enjoying a utility of

  $$v^*_{s_i,t_i} - w_{s_i,t_i} = 0 \geq 0 \quad \text{for all } i$$

  each teacher $t_i$ is enjoying a utility of

  $$w_{s_i,t_i} - c^*_{s_i,t_i} = 0 \geq 0 \quad \text{for all } i$$

  so IR hold for all schools and teachers.
- $(\mu, w)$ is feasible.
  This is directly implied by the observed $(\mu, w)$ and $b$ and we have

  $$b_{s_i} \geq w_{s_i,t_i} \quad \text{for all } i$$

- There are no blocking pairs.
  By construction of $(v^*, c^*)$, we have

  $$v^*_{s,t} = c^*_{s,t} \quad \text{for all } s,t \text{ where } t \neq \mu(s)$$

  therefore, for any unmatched pair $(s,t)$, any deviating wage $\hat{w}$ strictly profitable for teacher $t$ will require

  $$\hat{w} - c^*_{s,t} > 0$$

  which gives

  $$v^*_{s,t} - \hat{w} < 0$$

  so $\hat{w}$ cannot be profitable for school $s$, so there are no blocking pairs.

Given the above, we have shown that $(\mu, w)$ is stable, which gives

$$(v^*, c^*) \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

Fix the above $c^*$, by definition of $\mathcal{V}, \mathcal{C}(\mu, w; b)$, and $\mathcal{V}(\mu, w; b, c)$, we have

$$v^* \in \mathcal{V}(\mu, w; b, c^*)$$

therefore, we know that $\mathcal{V}(\mu, w; b, c^*) \neq \emptyset$.

Recall definition of $\mathcal{V}, \mathcal{C}(\mu, w; b)$, and $\mathcal{V}(\mu, w; b, c)$, we have

$$(v', c^*) \in \mathcal{V}, \mathcal{C}(\mu, w; b) \quad \text{for all } v' \in \mathcal{V}(\mu, w; b, c^*)$$

Since $\tau$ is non-distortive for $(\mu, w; b)$, we have

$$\overline{W}(v, c, b, \tau) \ \geq \ W(\mu; v, c) \quad \text{for all } (v, c) \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

which implies

$$\overline{W}(\tau, v', b, c^*) \ \geq \ W(\mu; v', c^*) \quad \text{for all } v' \in \mathcal{V}(\mu, w; b, c^*)$$

which means $\tau$ is also non-distortive for $(\mu, w; b, c^*)$.

Recall Theorem 1, we know $\tau$ satisfies DC. And by our construction of $c^*$, we have

$$b_{s_k} - c^*_{s_k,t_j} = b_{s_k} - b_{s_k} = 0 \leq w_{s_j,t_j} - w_{s_j,t_j} = w_{s_j,t_j} - c^*_{s_j,t_j}$$

Therefore, $s_k$ is outbid by $s_j$ in $(\mu, w; b, c^*)$, and DC then requires

$$b^\tau_{s_j} - w_{s_j,t_j} \geq b^\tau_{s_k} - c^*_{s_k,t_j} = \tau_{s_k} + b_{s_k} - b_{s_k} = \tau_{s_k} \tag{T2-2}$$

The above analysis works for any indices $k \neq j$, so (T2-2) is true for any $k \neq j$, which gives

$$b^\tau_{s_j} - w_{s_j,t_j} \geq \max_{s_i} \tau_{s_i} \quad \text{for } \forall j$$

which is the desired inequality (T2-1), finishing the proof of necessity.

Now we show sufficiency, that is, if $\tau$ satisfies

$$b^\tau_{s_j} - w_{s_j,t_j} \geq \max_{s_i} \tau_{s_i} \quad \text{for } \forall j \tag{T2-1}$$

then $\tau$ is non-distortive for $(\mu, w; b)$.

For any

$$(v, c) \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

fix the above $c$, and we want to show $\tau$ is non-distortive for $(\mu, w; b, c)$.

We first consider DC. For any school $s_j$, by definition, we have

$$\Delta^d_j \equiv \max_{i:\ s_i \text{ is outbid by } s_j \text{ in } (\mu, w)} \Big[ (b^\tau_{s_i} - c_{s_i,\mu(s_j)}) - (w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}) \Big]_+ \quad \text{for } \forall j$$

which gives

$$\Delta_j^d = \max_{i:\ s_i \text{ is outbid by } s_j \text{ in } (\mu,w)} \left[(b^\tau_{s_i} - c_{s_i,\mu(s_j)}) - (w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)})\right]_+$$
$$\leq \max_{i:\ s_i \text{ is outbid by } s_j \text{ in } (\mu,w)} \tau_{s_i}$$
$$\leq \max_{s_i} \tau_{s_i} \quad \text{for } \forall j \tag{T2-3}$$

We then consider SC, and by definition, we have

$$\Delta_j^s \equiv \sup_{C:\ \text{spillover chains ending at } s_j} \delta_j^C \tag{T2-4}$$

For any spillover chain $C$ ending at $s_j$, let $C$ be the following by reindexing:

$$C:\ s_1 \xrightarrow{\mu} s_2 \xrightarrow{\mu} \cdots \xrightarrow{\mu} s_j$$

Then by definition of $\delta_j^C$, we have

$$\delta_j^C \leq \delta_{j_1}^C ... \delta_2^C \leq \Delta_1^d$$

By (T2-3), we then have

$$\delta_j^C \leq \Delta_1^d \leq \max_{s_i} \tau_{s_i} \tag{T2-5}$$

Since (T2-5) holds for any spillover chain $C$ ending at $s_j$, we recall (T2-4) and get the following

$$\Delta_j^s \leq \max_{s_i} \tau_{s_i} \quad \text{for } \forall j \tag{T2-6}$$

(T2-1), (T2-3) and (T2-6) then give

$$b^\tau_{s_j} - w_{s_j,\mu(s_j)} \geq \max_{s_i} \tau_{s_i} \geq \max\{\Delta_j^d, \Delta_j^s\} \quad \text{for } \forall j$$

then by Theorem 1, $\tau$ is non-distortive for $(\mu, w; b, c)$.

Now consider any $(v, c)$ such that

$$(v, c) \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

by definition of $\mathcal{V}, \mathcal{C}(\mu, w; b)$, and $\mathcal{V}(\mu, w; b, c)$, we have

$$v \in \mathcal{V}(\mu, w; b, c)$$

by $\tau$ being non-distortive for $(\mu, w; b, c)$, we have

$$\overline{W}(\tau, v', b, c) \ \geq \ W(\mu; v', c) \quad \text{for all } v' \in \mathcal{V}(\mu, w; b, c)$$

which implies

$$\overline{W}(v, c, b, \tau) \ \geq \ W(\mu; v, c)$$

Therefore, $\tau$ is non-distortive for $(\mu, w; b)$, which proves the sufficiency. $\square$

## Proof of Theorem 3.

*Proof.* We first show the necessity. That is, if $\tau$ is non-distortive, then we have

$$\tau_{s_i} = \tau_{s_j} \quad \text{for } \forall i \neq j \tag{T3-1}$$

WLOG, we again let $\mu(s_i) = t_i$ by reindexing.

Fix two distinct indices $k \neq j$. Consider the following cost, value and budget profile $(v^*, c^*, b^*)$, where for all matched pairs, we have

$$\begin{cases} v^*_{s_i,t_i} = w_{s_i,t_i} \\ c^*_{s_i,t_i} = w_{s_i,t_i} \\ b^*_{s_i} = w_{s_i,t_i} \end{cases} \qquad \text{for all } i$$

and for all unmatched pairs $(s,t)$ where $t \neq \mu(s)$, we have

$$\begin{cases} v^*_{s,t} = 0 \\ c^*_{s,t} = 0 \end{cases} \qquad \text{for all } (s,t) \text{ where } t \neq \mu(s)$$

We now show the above $(v^*, c^*, b^*)$ is consistent with $(\mu, w)$, that is

$$(v^*, c^*, b^*) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$$

To show the above, we need to verify the following:

- $(\mu, w)$ is individually rational.
  By construction of $(v^*, c^*)$, each school $s_i$ is enjoying a utility of
  $$v^*_{s_i,t_i} - w_{s_i,t_i} = 0 \geq 0 \quad \text{for all } i$$
  each teacher $t_i$ is enjoying a utility of
  $$w_{s_i,t_i} - c^*_{s_i,t_i} = 0 \geq 0 \quad \text{for all } i$$
  so IR hold for all schools and teachers.
- $(\mu, w)$ is feasible.
  This is directly implied by the construction of $(c^*, b^*)$ and we have
  $$b^*_{s_i} \geq w_{s_i,t_i} \quad \text{for all } i$$
- There are no blocking pairs.
  By construction of $(v^*, c^*)$, we have
  $$v^*_{s,t} = c^*_{s,t} \quad \text{for all } s, t \text{ where } t \neq \mu(s)$$
  therefore, for any unmatched pair $(s,t)$, any deviating wage $\hat{w}$ strictly profitable for teacher $t$ will require
  $$\hat{w} - c^*_{s,t} > 0$$
  which gives
  $$v^*_{s,t} - \hat{w} < 0$$
  so $\hat{w}$ cannot be profitable for school $s$, so there are no blocking pairs.

Given the above, we have shown that $(\mu, w)$ is stable, which gives

$$(v^*, c^*, b^*) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$$

Fix the above $b^*$, by definition of $\mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$ and $\mathcal{V}, \mathcal{C}(\mu, w; b)$, we have

$$(v^*, c^*) \in \mathcal{V}, \mathcal{C}(\mu, w; b^*)$$

therefore, we know that $\mathcal{V}, \mathcal{C}(\mu, w; b^*) \neq \emptyset$.

Recall definition of $\mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$ and $\mathcal{V}, \mathcal{C}(\mu, w; b)$, we have

$$(v', c', b^*) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w) \quad \text{for all } (v', c') \in \mathcal{V}, \mathcal{C}(\mu, w; b^*)$$

Since $\tau$ is non-distortive for $(\mu, w; b)$, we have

$$\overline{W}(v, c, b, \tau) \geq W(\mu; v, c) \quad \text{for all } (v, c, b) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$$

which implies

$$\overline{W}(\tau, v', b^*, c') \geq W(\mu; v', c') \quad \text{for all } (v', c') \in \mathcal{V}, \mathcal{C}(\mu, w; b^*)$$

which means $\tau$ is also non-distortive for $(\mu, w; b^*)$.

Recall Theorem 2, we know $\tau$ satisfies (T2-1), which is

$$b^{\tau}_{s_j} - w_{s_j, t_j} \geq \max_{s_i} \tau_{s_i}$$

which gives

$$b^{\tau}_{s_j} - w_{s_j, t_j} \geq \tau_{s_k}$$

Recall construction of $b^*$, we have

$$b^{\tau}_{s_j} - w_{s_j, t_j} = \tau_{s_j} + b^*_{s_j} - w_{s_j, t_j} = \tau_{s_j} + w_{s_j, t_j} - w_{s_j, t_j} = \tau_{s_j} \geq \tau_{s_k}$$

The above analysis works for any indices $k \neq j$, so we have

$$\tau_{s_j} = \tau_{s_k} \quad \text{for } \forall j \neq k$$

which is the desired (T3-1), finishing the proof of necessity.

Now we show sufficiency, that is, if $\tau$ satisfies

$$\tau_{s_i} = \tau_{s_j} \quad \text{for } \forall i \neq j \tag{T3-1}$$

then $\tau$ is non-distortive for $(\mu, w)$.

For any

$$(v, c, b) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$$

fix the above $b$, and we want to show $\tau$ is non-distortive for $(\mu, w; b)$.

By definition, $(\mu, w)$ is stable for any $(v, c, b) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$. Feasibility then gives

$$b_{s_j} \geq w_{s_j, t_j} \quad \text{for all } j \tag{T3-2}$$

Then by (T3-1) and (T3-2), we have

$$b^{\tau}_{s_j} - w_{s_j, t_j} = \tau_{s_j} + b_{s_j} - w_{s_j, t_j} \geq \tau_{s_j} = \tau_{s_i} \quad \text{for } \forall i \neq j$$

which then gives

$$b^{\tau}_{s_j} - w_{s_j,t_j} \geq \max_{s_i} \tau_{s_i} \quad \text{for } \forall j$$

Then by Theorem 2, $\tau$ is non-distortive for $(\mu, w; b)$.

Now consider any $(v, c, b)$ such that

$$(v, c, b) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$$

Then fix $b$. By definition of $\mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$ and $\mathcal{V}, \mathcal{C}(\mu, w; b)$, we have

$$(v, c) \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

by $\tau$ being non-distortive for $(\mu, w; b)$, we have

$$\overline{W}(\tau, v', b, c') \;\geq\; W(\mu; v', c') \quad \text{for all } (v', c') \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

which implies

$$\overline{W}(v, c, b, \tau) \;\geq\; W(\mu; v, c)$$

Therefore, $\tau$ is non-distortive for $(\mu, w)$, which proves the sufficiency. □