Subsidy Design in Budget-Constrained Matching

Haoqi Tong
Duke University

October 13, 2025

Abstract

We study optimal subsidy design in one-to-one matching markets with budget constraints. Under complete information about match values, we provide an algorithm that computes the *minimal* subsidies required to implement the welfare-maximizing matching. Under incomplete information, the planner observes only the status quo stable matching (but not the underlying match values). Departing from a mechanism-design approach, we consider *robust* subsidy policies that improve outcomes without any preference elicitation. The central result delivers conditions under which a subsidy allocation *guarantees* a weakly better stable matching for *every* profile of match values consistent with the observed outcome. Intuitively, subsidies induce both *direct* and *spillover* effects; any subsidy policy that fails to adequately cover these effects can potentially generate new blocking and thereby reduce welfare relative to the observed outcome.

1 Introduction

Budget constraints are pervasive in matching markets with transfers: public schools hire teachers under fixed annual funding,[1] and hospitals finance residency positions from limited program resources. When these constraints bind, the transfers required to support the efficient assignment can be infeasible, yielding suboptimal allocations. Empirically, high-value teachers disproportionately sort into better-resourced schools, leaving low-income communities underserved;[2] likewise, high-quality physicians concentrate in urban hospitals, aggravating shortages of medical care in rural areas.[3] Taken together, the evidence indicates that budget constraints weaken resource-limited institutions' ability to compete for talent, yielding systematically suboptimal matching from a social perspective.

Targeted subsidies can alleviate such inefficiencies by enabling otherwise unaffordable high-value matches.[4] However, poorly designed financial interventions can be counterproductive: unconditional salary increases have limited effects on performance (De Ree et al., 2018); pay programs that are

[1]This is not merely a wage-rigidity issue: many public systems use flexible pay, including Wisconsin's Act 10 (Biasi (2021)), Texas's Teacher Excellence Initiative (TEI; Hanushek et al. (2023)), and Florida's SB 736 (Jones (2013)). Stipends for high-need positions and performance-based bonuses are also common.

[2]See, for example, Lankford et al. (2002), Clotfelter et al. (2006), Sass et al. (2012) and Ajzenman et al. (2024) for evidence that more qualified or effective teachers tend to sort into higher-income or better-funded schools, while high-poverty schools are often staffed with less-experienced or lower-qualified teachers.

[3]See Rosenblatt and Hart (2000) and Machado et al. (2021) on persistent rural physician shortages and systematically lower physician density in rural counties.

[4]See, for example, Clotfelter et al. (2008) for reduced turnover in North Carolina, Hendricks (2014) for reduced turnover and improved student achievement in Texas, and Baron (2022) and Biasi (2021) for improved teacher quality and student achievement in Wisconsin.

mis-specified or weakly targeted may fail to raise teacher effectiveness (Fryer, 2013; Goodman and Turner, 2013); and uneven subsidy allocation can aid targeted schools while imposing negative spillovers on nearby non-subsidized schools, yielding ambiguous aggregate welfare (Castro and Esposito, 2022; Kho et al., 2023). These considerations motivate our central question: how should a social planner optimally allocate subsidies in a budget-constrained matching market?

We address this question from two complementary perspectives. In the complete-information setting, where the planner observes all match values, we develop an algorithmic procedure that computes the *minimal* total subsidy required to implement the efficient matching. The procedure augments school budgets only when subsidies are necessary to implement the efficient matching. The algorithm terminates in finitely many rounds and delivers a stable allocation that coincides with the first best.

The incomplete-information setting is more challenging and, importantly, more relevant for policy. Here the planner observes only the status quo—namely, the stable matching and its supporting wages—but not the underlying match values. Multiple match-value profiles can rationalize the observed outcome. We call a subsidy allocation *non-distortive* if, for every match-value profile consistent with the observed data, it guarantees the existence of a stable outcome that is weakly better than the status quo. Our central result gives a tight characterization of when a subsidy policy is non-distortive, via two *coverage* conditions:

- **Direct-effect coverage.** Schools receiving a subsidy may newly afford teachers they could not afford at the status quo. When a teacher becomes newly affordable, the subsidized school can make an offer, generating direct upward wage pressure on the incumbent school. For each such approach, the incumbent school must have sufficient post-subsidy budget to defend its match. Thus, the subsidy must preserve the outbidding relations revealed by the status quo stable outcome.
- **Spillover coverage.** The direct effect described above can propagate along chains of bid–counterbid moves. Consequently, schools not directly challenged by a subsidized school may still face indirect wage pressure. The subsidy policy must therefore ensure that each school can absorb the maximal upstream indirect wage pressure that can reach it along such spillover chains.

These two requirements are jointly necessary and sufficient: they characterize exactly the set of non-distortive subsidy allocations under incomplete information. Equivalently, a subsidy policy is non-distortive if and only if it satisfies direct-effect and spillover coverage; if either condition fails, then there exists a match-value profile consistent with the observed outcome under which every stable allocation induced by the policy is strictly worse than the status quo.

The remainder of the paper is organized as follows. Section 2 reviews the related literature. Section 3 presents the model. Section 4.1 discusses classical benchmark results without budget constraints. Section 4.2 analyzes the complete-information case and introduces the *Minimum Subsidy Implementation (MSI)* algorithm. Section 4.3 considers the incomplete-information case and characterizes non-distortive subsidies. Section 4.4 studies how different informational environments shape the set of non-distortive policies. Section 5 concludes. All proofs are collected in the Appendix.

2 Related Literature

We build on a large and influential theoretical literature that forms the classical foundations of matching theory. In one-to-one matching without transfers, Gale and Shapley (1962) establish the existence of stable matchings and introduce the deferred acceptance (DA) algorithm.[5] In

[5]For a comparison of mechanisms in school choice, see Abdulkadiroğlu and Sönmez (2003).

transferable-utility (TU) environments, Shapley and Shubik (1971) show that the set of stable outcomes coincides with the core of the assignment game and attains efficiency[6]. Dynamic price (wage) adjustment and lattice structure of stable outcomes are then developed in Crawford and Knoer (1981) and Kelso and Crawford (1982), with the gross-substitutes condition guaranteeing existence and a lattice structure of stable allocations[7]. Matching with contracts unifies many environments under substitutable choice, preserving existence and lattice properties (Hatfield and Milgrom, 2005)[8]. This paper adopts this canonical TU framework and studies the optimal subsidy allocation with budget constraints.

The next closely related branch is matching with constraints. A large literature studies how various constraints alter classical matching results. When employers face hard budget caps on salaries, stable matchings may fail to exist (Mongell and Roth, 1986; Abizada, 2016). To deal with the nonexistence, recent research "approximates" stable matchings (Kawase and Iwasaki, 2017, 2018) or identifies weaker conditions to restore the existence of stable and efficient outcomes (Jagadeesan and Teytelboym, 2021). Complementarities in preferences—such as couples applying jointly in a match—can likewise disrupt existence and incentive properties. In practice, the medical "resident match" algorithm was modified to accommodate couples (Roth and Peranson, 1999). In theory, researchers identify conditions on preferences (Klaus and Klijn, 2005); consider large markets (Kojima et al., 2013); or slightly perturb capacities (Nguyen and Vohra, 2018) to recover the existence of stable matchings. Distributional constraints (e.g. diversity quotas or regional caps) similarly undermine stability or efficiency. Simply hard-coding such quotas into Deferred Acceptance can result in no stable outcome or in efficiency losses (Hafalir et al., 2013). In response, new allocation mechanisms (Kamada and Kojima, 2015; Kominers and Sönmez, 2016) and stability notions (Kamada and Kojima, 2017) have been developed for such settings. Recent works also characterize constraints that preserve the substitutes condition (Kojima et al., 2020). Relative to this literature, our paper does not propose new stability concepts or constrained mechanisms. Instead, it characterizes *minimal* relaxations of constraints that restore the efficient matching under complete information and provides a *robust* way to relax constraints under incomplete information.

Another branch of theoretical work has studied matching with incomplete information. Roth (1989) initiates the study, showing that stable matching mechanisms retain certain incentive compatibility properties even when agents lack full knowledge of others' preferences. Building on this foundation, Ehlers and Massó (2007) identify conditions under which truth-telling is indeed a Bayesian Nash equilibrium in a stable matching mechanism. More recently, Fernandez et al. (2022) demonstrate that many classical matching results are fragile to even small amounts of uncertainty about others' preferences. Other than incentives, the welfare implications of incomplete information are mixed: on one hand, Li and Rosen (1998) find that uncertainty can lead to inefficient early contracting (unraveling) in matching markets; on the other hand, Coles et al. (2013) show that adding a simple signaling mechanism increases worker welfare and the total number of matches. Other researchers have extended the notion of stability itself to Bayesian settings, proposing new solution concepts for matching under incomplete information. For example, Liu et al. (2014) and Liu (2020) introduce stability notions for markets with one-sided uncertainty, while Chen and Hu (2023) develop a stability concept for environments with two-sided incomplete information. In contrast to these agent-centric approaches, we adopt a planner's perspective. The planner observes an existing stable matching but does not know the agents' true preferences. Rather than eliciting additional information or

[6] Also, see Becker's analysis of assortative matching, under complete information and without frictions, maximizing aggregate output (Becker, 1973).

[7] A complementary auction interpretation is provided by Demange et al. (1986).

[8] See Fleiner (2003) for a fixed-point approach.

relying on beliefs, we characterize a robust policy intervention based solely on observable features of the status quo allocation – specifically, the outbidding and spillover relations inferred from the current matching. This approach departs from the prior literature by focusing on a policy design that guarantees a weak welfare improvement for all preference profiles consistent with the observed outcome, without requiring any further information about agents' private valuations.

The theoretical literature on optimal subsidies and other interventions in two-sided matching markets remains relatively scarce, but a few recent works have begun to address this gap. Kojima et al. (2024) demonstrate that naive subsidy or tax policies can violate the gross substitutes condition, which is essential for the existence of stable matchings, and they provide a complete characterization of intervention policies that preserve substitutability (and hence stability). Dupuy et al. (2020) show that match-specific taxes reshape the set of stable matchings by altering effective match values, potentially generating allocative inefficiency. A key theoretical insight in Dupuy et al. (2020) is that any taxed matching market can be reinterpreted as an untaxed market with adjusted match values, implying that policymakers can, in principle, design a tax/subsidy scheme to implement a targeted matching. Beyond equilibrium characterizations, Yokote (2020) take an algorithmic approach: they prove that an optimal tax/subsidy policy exists to achieve a policy target (e.g., the number of matches or a distributional requirement) while maximizing social welfare, which in turn allows one to identify and compute the minimal subsidies needed to satisfy quota constraints. However, all of the above assume complete information; to the best of our knowledge, we are the first to study optimal interventions in an incomplete-information environment. Our paper contributes by providing a unifying framework that (i) computes the *minimum* subsidies needed to implement the efficient matching under complete information and (ii) characterizes the full set of *non-distortive* subsidies under incomplete information only using knowledge of observed outcomes.

3 Model

The matching–with–transfers framework has broad applications. Throughout, we refer to the two sides as "schools" and "teachers," but the interpretation is flexible: one may read them as firms and workers in Kelso and Crawford (1982) or as hospitals and physicians in Hatfield and Milgrom (2005).

Let $S = \{s_1, \ldots, s_n\}$ denote the set of schools and $T = \{t_1, \ldots, t_n\}$ the set of teachers, with $|S| = |T| = n$. For each pair $(s,t) \in S \times T$ there is a match value $v_{s,t} \geq 0$ and a match-specific cost $c_{s,t} \geq 0$. Each school $s \in S$ has a budget $b_s \geq 0$. We impose two standard assumptions: (i) nonnegative surplus, $v_{s,t} \geq c_{s,t}$ for all (s,t); and (ii) no intrinsically infeasible pairs, $b_s \geq c_{s,t}$ for all (s,t). This is a one-to-one, quasilinear matching environment with transfers.

An *allocation* specifies who matches with whom and the wages paid on matched pairs.

Definition 1 (Allocation)**.** An allocation is a pair (μ, w) where $\mu : S \to T \cup \{\emptyset\}$ is a matching (injective on the set of matched schools) and $w = (w_{s,\mu(s)})_{s \in S^\mu}$ is a vector of nonnegative wages on matched pairs. Here $S^\mu = \{s \in S : \mu(s) \in T\}$ and $T^\mu = \{t \in T : \mu^{-1}(t) \in S\}$; we write $\mu^{-1}(t) = \emptyset$ if t is unmatched.

Utilities are quasilinear with a zero outside option. If $s \in S^\mu$ matches with $t = \mu(s)$ at wage $w_{s,t}$, then

$$u_s(\mu, w) = v_{s,t} - w_{s,t}, \qquad u_t(\mu, w) = w_{s,t} - c_{s,t}.$$

Unmatched agents receive utility 0.

Following the standard approach, we define feasibility, individual rationality, and (budget-constrained) stability as follows.

Definition 2 (Feasibility)**.** An allocation (μ, w) is *feasible* if wages are defined only on matched pairs and, for every matched school $s \in S^\mu$ with $t = \mu(s)$,

$$w_{s,t} \;\leq\; b_s.$$

That is, the wage does not exceed the school's budget.

Definition 3 (Individual rationality (IR))**.** An allocation (μ, w) is *individually rational* if every matched agent obtains nonnegative utility; i.e., for every $s \in S^\mu$ with $t = \mu(s)$,

$$v_{s,t} - w_{s,t} \;\geq\; 0 \quad \text{and} \quad w_{s,t} - c_{s,t} \;\geq\; 0.$$

Equivalently, both the school and the teacher weakly prefer their match to remaining unmatched.

Definition 4 (Stability)**.** An allocation (μ, w) is *stable* if it is feasible and individually rational, and there is no *blocking pair* $(s, t) \in S \times T$ together with an *affordable* wage $\widehat{w}$ such that

$$\widehat{w} \;\leq\; b_s, \qquad v_{s,t} - \widehat{w} \;>\; u_s(\mu, w), \qquad \widehat{w} - c_{s,t} \;>\; u_t(\mu, w).$$

In words, no school–teacher pair can profitably deviate at a wage that respects the school's budget and makes both the school and the teacher strictly better off than under (μ, w).

We study optimal subsidy design when budget constraints generate inefficiencies. Before turning to properties and results, we formalize the subsidy object.

Definition 5 (Subsidy allocation)**.** Given baseline budgets $b = (b_s)_{s \in S}$, a *subsidy allocation* is a nonnegative vector $\tau = (\tau_s)_{s \in S}$, and the post-subsidy budgets are

$$b_s^\tau \;=\; b_s + \tau_s \qquad \text{for all } s \in S.$$

Thus each school's budget weakly increases (the social planner cannot tax schools), and the increment for s is τ_s.

4 Results

This section first reviews classical benchmark results and existence theorems. We then examine how binding budget caps introduce novel considerations, leading to the subsidy design problem.

4.1 Existing results

Benchmark without budgets.

If every school has an effectively unbounded budget ($b_s = +\infty$ for all s), stable allocations coincide with the utilitarian optimum (Shapley and Shubik, 1971). Formally, let

$$\mu^* \in \arg\max_{\mu} \sum_{s \in S} \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big).$$

Then there exist wages $w = (w_{s,\mu^*(s)})_{s \in S}$ such that (μ^*, w) is stable (Definition 4). Moreover, every stable allocation (μ, w) attains the same maximal total surplus. Equilibrium wages can be computed by well-known constructive procedures: with discrete wages, the algorithms of Crawford and Knoer (1981) and Kelso and Crawford (1982); with continuous wages, the progressive ascending auction of Demange et al. (1986).

This benchmark is the classical assignment game: the set of core payoffs forms a lattice that supports all efficient matchings, with extremal points yielding the teacher- and school-optimal divisions of surplus. Accordingly, any inefficiency in our setting is driven by binding budget caps; when caps are absent or slack, efficiency obtains automatically.

Existence of stable allocations and lattice structure.

To establish existence, fix any deterministic tie-breaking rule to resolve indifferences in agents' choices; this induces single-valued choice functions on both sides in our one-to-one, quasilinear environment (unit-demand schools; teachers preferring higher net wages). By Fleiner (2003, Theorem 10), the set of stable allocations is then nonempty and forms a complete lattice.[9] Moreover, these stable allocations coincide with stability in our model (Definition 4). Hence we need not restrict tie-breaking ex ante: for any deterministic tie-breaking rule, a stable allocation exists in our setup. In addition, Fleiner (2003, Theorem 11) guarantees extremal elements—namely, the teacher-optimal and school-optimal stable allocations—with every other stable allocation lying between them.

Two observations will be useful below. First, introducing budgets in this one-to-one, quasilinear environment does not affect the existence or lattice structure of stable allocations. Second, budget caps *do* restrict the feasible contract set by truncating affordable wages. Our analysis therefore focuses on how targeted subsidies expand feasibility to recover efficient outcomes.

4.2 Complete information

In this subsection we assume complete information: the planner observes the entire value profile $v = (v_{s,t})$, cost profile $c = (c_{s,t})$, and baseline budgets $b = (b_s)$. The planner's objective is to implement the utilitarian-optimal matching μ^*.

If transfers are effectively unbounded, the planner can implement μ^* by granting sufficiently large subsidies to all schools, thereby reverting to the unconstrained case. One could also guarantee μ^* by raising each school's budget to the wage it pays in the school–optimal stable outcome of the unconstrained problem. Both approaches, however, may entail more financial assistance than necessary. We therefore seek the *minimal* subsidy vector that implements μ^*.

Formally, let $b = (b_s)_{s\in S}$ denote baseline budgets. Choose a subsidy vector $\tau = (\tau_s)_{s\in S}$ and wages w such that (μ^*, w) is stable under the augmented budgets b^τ, where $b^\tau_s = b_s + \tau_s$. The planner's problem is

$$\min_{\tau\in\mathbb{R}^n_{\geq 0}} \sum_{s\in S} \tau_s \qquad \text{subject to} \qquad (\mu^*, w) \text{ is stable under budgets } b^\tau.$$

Minimum Subsidy Implementation (MSI) algorithm.

We propose a constructive procedure to compute the smallest necessary subsidies: the *Minimum Subsidy Implementation (MSI)* algorithm. The idea is straightforward. Start from the lowest individually rational wages for the target matching μ^*. If some rival schools can profitably poach

[9] The induced choice functions satisfy *substitutability* in the sense of Kelso and Crawford (1982) and Hatfield and Milgrom (2005): when some contracts become unavailable, no agent's choice switches to a contract that was previously rejected.

a teacher at an affordable wage, raise that teacher's wage to the *lowest* level that deters all such deviations. Then subsidize only those schools that cannot afford the required wages. In this way, subsidies are introduced gradually and only where needed, keeping μ^* stable while minimizing total transfers. The algorithm proceeds in rounds, each with two stages.

Round 1 (Initialization). Set each matched teacher's wage to her cost, and raise any school's budget that lies below this cost:

$$w^1_{s,\mu^*(s)} \;=\; c_{s,\mu^*(s)}, \qquad b^1_s \;=\; \max\{\, b_s,\; c_{s,\mu^*(s)} \,\} \quad \text{for each } s \in S.$$

If $b^1_s > b_s$, the initial subsidy $\tau_s = b^1_s - b_s$ makes it feasible to pay $\mu^*(s)$ the lowest individually rational wage.

Rounds $r = 2, 3, \ldots$ (Updating). Each subsequent round consists of the following two stages:

(i) **Wage stage.** For each school–teacher pair (s,t), compute the lowest *deterrence wage* $d^r_{s,t}$ that prevents school s from profitably deviating to t at the current budgets:

$$d^r_{s,t} \;=\; \min\Big\{\, b^{r-1}_s,\;\; v_{s,t} - v_{s,\mu^*(s)} + w^{r-1}_{s,\mu^*(s)} \Big\}.$$

(Here the second term makes s just indifferent between $(s,\mu^*(s))$ at $w^{r-1}_{s,\mu^*(s)}$ and (s,t); the $\min\{\cdot\}$ enforces affordability.) Then, for each matched pair $(s,\mu^*(s))$, raise that teacher's wage (if needed) to defeat all direct poaches:

$$w^r_{s,\mu^*(s)} \;=\; \max_{s' \in S} \Big\{\, d^r_{s',\,\mu^*(s)} - c_{s',\,\mu^*(s)} + c_{s,\,\mu^*(s)} \Big\}.$$

Equivalently, $w^r_{s,\mu^*(s)} - c_{s,\mu^*(s)} = \max_{s' \in S}\{\, d^r_{s',\,\mu^*(s)} - c_{s',\,\mu^*(s)} \}$.

(ii) **Subsidy stage.** Lift any school's budget that is now below its required wage:

$$b^r_s \;=\; \max\{\, b^{r-1}_s,\;\; w^r_{s,\mu^*(s)} \,\} \quad \text{for each } s \in S.$$

Termination. Stop at the first round $R \geq 1$ such that $w^R_{s,\mu^*(s)} = w^{R-1}_{s,\mu^*(s)}$ for all $s \in S$ (no wage increases). The resulting subsidy vector is

$$\tau_s \;=\; b^R_s - b_s, \qquad s \in S.$$

An illustrative example.

To illustrate how the algorithm works, we consider a 3×3 market with schools $S = s_1, s_2, s_3$ and teachers $T = t_1, t_2, t_3$. Costs are zero for all pairs ($c_{s,t} = 0$), and the match values ($v_{s,t}$) are given in the table below. The unique optimal matching is the diagonal μ^* with $\mu^*(s_i) = t_i$, which yields total welfare $3 + 6 + 6 = 15$.

	s_1	s_2	s_3
t_1	3	1	0
t_2	4	6	4
t_3	8	7	6

To visualize values at the pair level, we display for each school a bar chart whose bar heights are the corresponding match values.

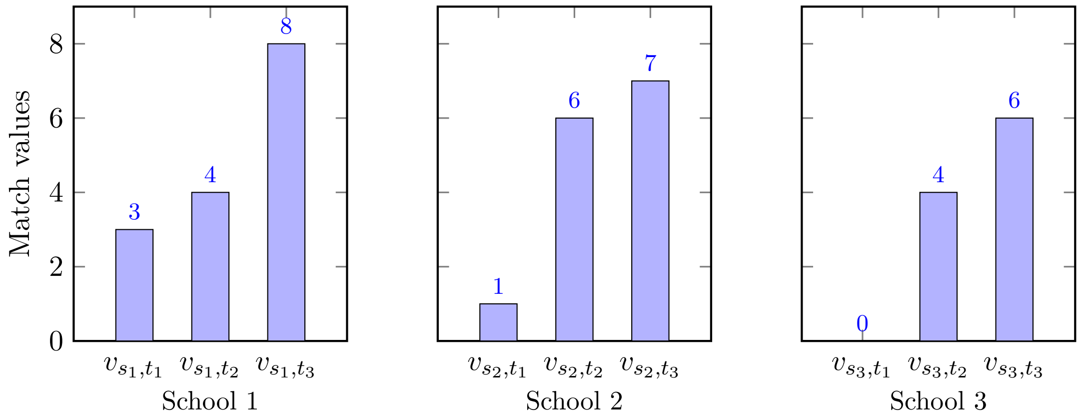

Figure 1: Match values for the 3×3 example.

When budgets are effectively unbounded, μ^* can be supported by a stable wage vector. Moreover, the school-optimal allocation is

$$\mu(s_i) = t_i, \quad w_{s_1,t_1} = 0, \quad w_{s_2,t_2} = 3, \quad w_{s_3,t_3} = 5$$

The following figure reports, for every (s_i, t_j), the decomposition of the total match value into the teacher's (yellow) and school's (blue) utilities under these supporting wages.

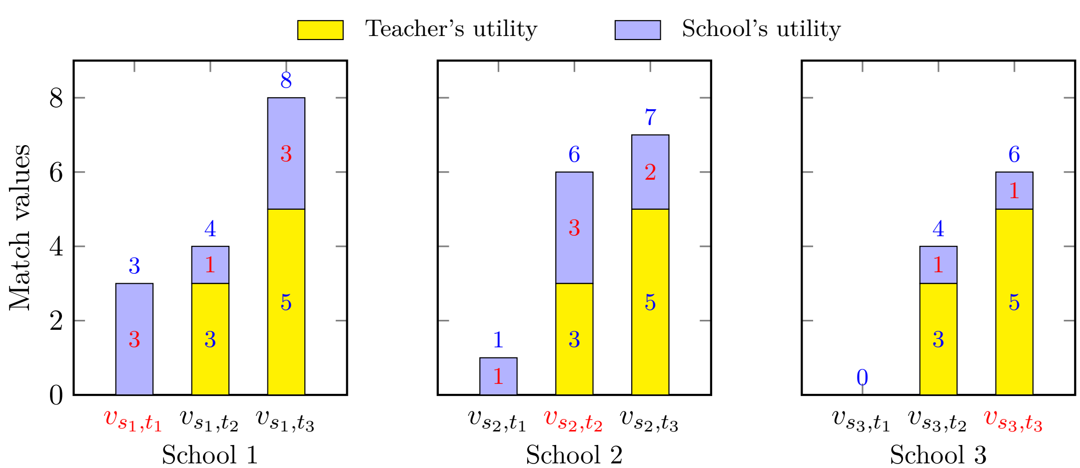

Figure 2: The school-optimal allocation without budget constraints.

Now impose budgets $b_{s_1} = 4$, $b_{s_2} = 1$, and $b_{s_3} = 2$, where budgets are depicted by red dashed lines in the following figure:

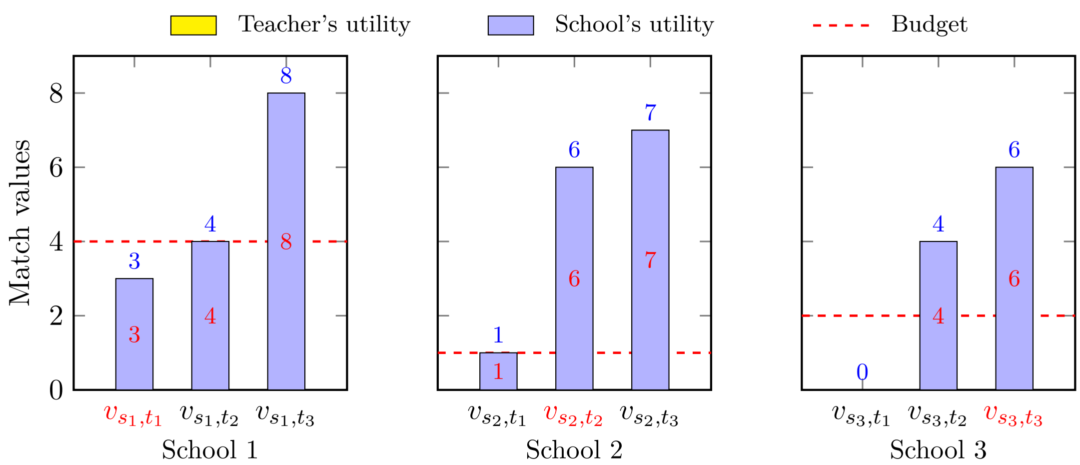

Figure 3: $b = \{4, 1, 2\}$ for the 3×3 example.

Any allocation that implements μ^* must pay teacher t_3 at a wage $w_{s_3,t_3} \leq b_{s_3} = 2$. Teacher t_3 then obtains utility at most 2. Also, s_1 obtains utility at most 3 (the total match value) from matching t_1.

Consider the pair (s_1, t_3). School s_1 can afford any $\widehat{w} \in (2, 4]$ to t_3, and at such a wage,

$$u_{t_3} = \widehat{w} \; > \; 2 \qquad \text{(strict gain for teacher } t_3\text{)},$$

$$u_{s_1} = v_{s_1,t_3} - \widehat{w} \; \geq \; 8 - 4 = 4 \; > \; 3 \geq u_{s_1}(\mu^*, w) \qquad \text{(strict gain for school } s_1\text{)}.$$

Hence $(s_1, t_3, \widehat{w})$ is a profitable deviation that respects s_1's budget and strictly benefits both agents. Therefore, no stable allocation can implement μ^* under the stated budget caps.

Now let us consider the MSI algorithm. In the initialization step, each school is matched to its μ^* partner at the lowest individually rational wage. Since $c_{s,t} = 0$ for all (s,t), the initial wages are all zero and are affordable for every school, so no subsidies are required in Round 1. In Round 2, we compute the lowest deterrence wages $\{d^2_{s,t}\}$ that prevent profitable poaching at current budgets. This yields

$$\begin{aligned} d^2_{s_1,t_1} &= 0, & d^2_{s_1,t_2} &= 1, & d^2_{s_1,t_3} &= 4, \\ d^2_{s_2,t_1} &= -5, & d^2_{s_2,t_2} &= 0, & d^2_{s_2,t_3} &= 1, \\ d^2_{s_3,t_1} &= -6, & d^2_{s_3,t_2} &= -2, & d^2_{s_3,t_3} &= 0, \end{aligned}$$

and the required wages on the matched pairs update to

$$w^2_{s_1,t_1} = 0, \qquad w^2_{s_2,t_2} = 1, \qquad w^2_{s_3,t_3} = 4.$$

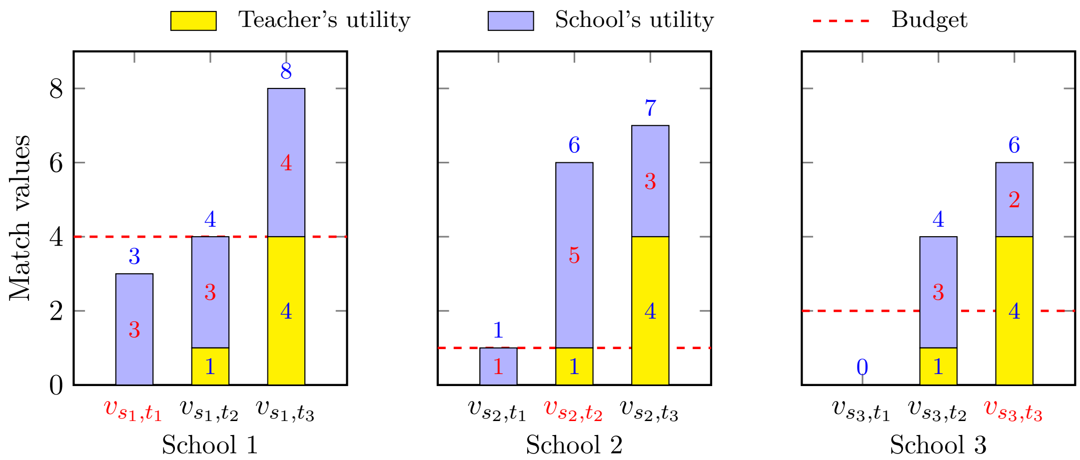

Figure 4: Round 2 wage stage of the MSI algorithm.

At these required wages, school s_3 cannot afford $w^2_{s_3,t_3} = 4$ given $b_{s_3} = 2$. The MSI subsidy step therefore augments s_3's budget by 2 (from 2 to 4). No other school requires additional funds at this stage.

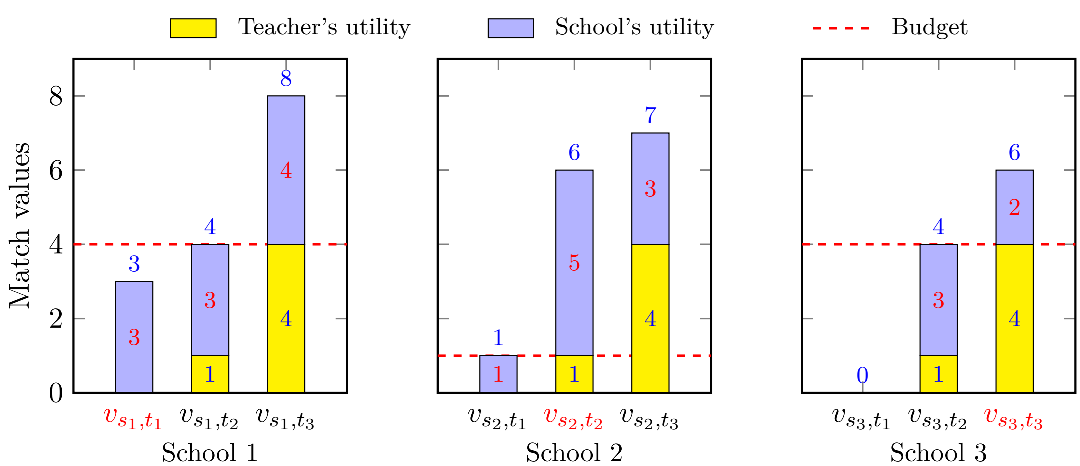

Figure 5: Round 2 subsidy stage of the MSI algorithm.

With b_{s_3} raised, the next wage update increases the deterrence requirement for t_2 to $w^3_{s_2,t_2} = 2$.

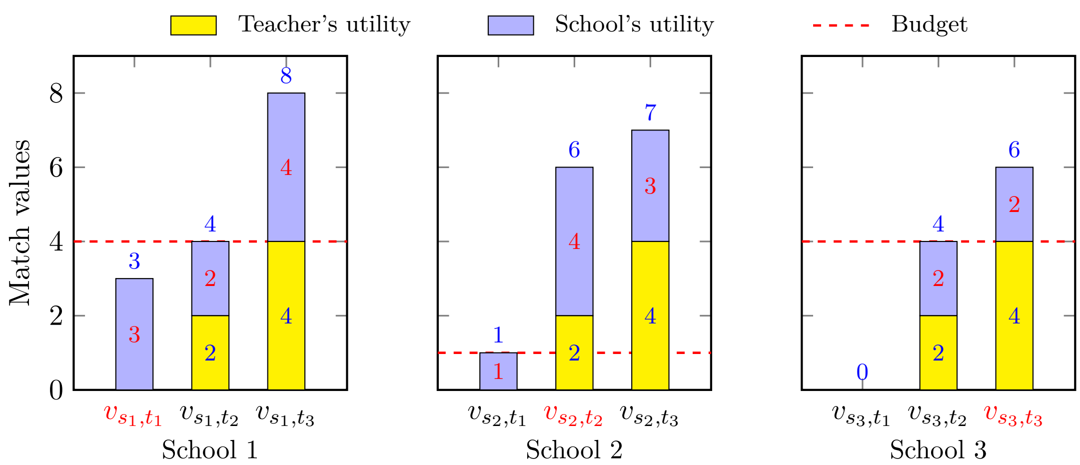

Figure 6: Round 3 wage stage of the MSI algorithm.

This makes s_2's budget insufficient ($b_{s_2} = 1$), so the subsidy step grants 1 unit to s_2 (raising b_{s_2} to 2).

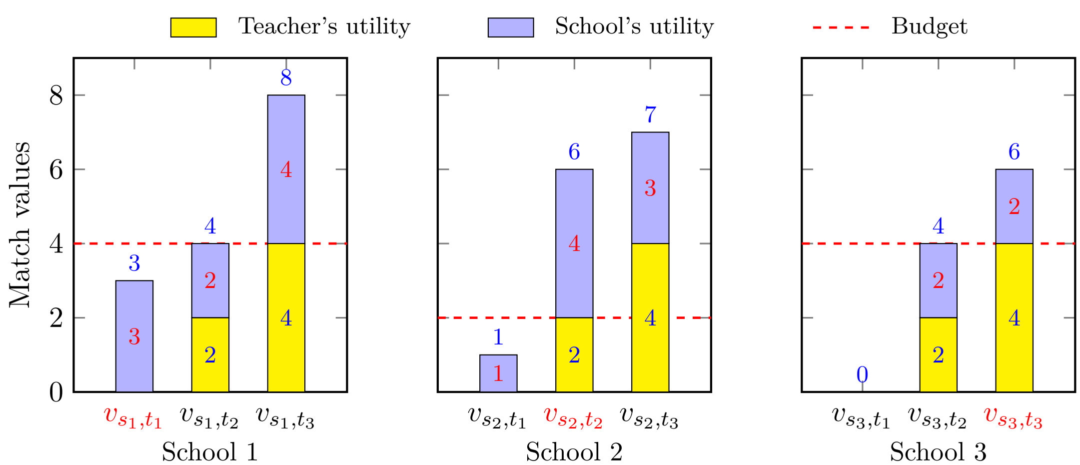

Figure 7: Termination of the MSI algorithm.

No further wage increases occur thereafter, so MSI terminates. The final outcome implements μ^* with the minimal subsidy vector $\tau = (0, 1, 2)$ and the total subsidies are 3 units.

Interpretation and discussion.

- *Economic meaning.* The wage stage computes the *lowest* deterrence wages needed to prevent deviations against μ^*. The subsidy stage injects funds only where a school's current budget is insufficient to cover its required wage. Thus, subsidies are used only when unavoidable.
- *Convergence.* The MSI process converges in finitely many rounds: wages are nondecreasing and bounded above by match values, hence a fixed point is reached.
- *Optimality at termination.* Upon termination at round R, (μ^*, w^R) is stable under b^R. Therefore, MSI implements μ^*.

- *Monotonicity and minimality.* Wages and budgets are weakly increasing across rounds. No subsidy is wasted: b_s^r rises only when needed to sustain the updated deterrence wage. At termination, each budget has been raised just enough to support μ^*.

These properties are summarized formally below.

Proposition 1 (Properties of MSI)**.** *To implement the socially optimal matching μ^*, the MSI algorithm satisfies:*

(i) MSI terminates in finitely many rounds.

(ii) If MSI terminates at round R with wage profile w^R and budget vector b^R, then (μ^*, w^R) is stable under budgets b^R.

(iii) If $(\mu^*, \tilde{w})$ is stable under budgets $\tilde{b} = b + \tilde{\tau}$, then $\tilde{b}_s \geq b_s^R$ for all $s \in S$. In particular, b^R is componentwise minimal among all budget vectors that admit a stable outcome implementing μ^*.

4.3 Incomplete information

The preceding analysis assumes complete information: the planner observes the full environment (v, c, b). This benchmark is analytically useful—it identifies the *minimum* subsidies required to implement μ^*—but its policy relevance is limited, since the complete–information assumption rarely holds in practice. In practice, the planner typically observes an existing allocation, but does not have full knowledge of schools' and teachers' preferences over potential partners, and baseline budgets may also be unknown.

We therefore turn to an incomplete–information setting. Throughout this subsection, the planner observes an existing allocation (μ, w) and knows that it is stable under the true (but partially unobserved) primitives. We begin with the case in which match values are unobserved but costs and baseline budgets are known: the planner knows $c = (c_{s,t})_{s,t}$ and $b = (b_s)_s$, but not $v = (v_{s,t})_{s,t}$. Our main result in this environment is Theorem 1. We then extend the analysis to two progressively weaker informational environments: first, when both values and costs (v, c) are unobserved (Theorem 2); and finally, when none of (v, c, b) are observed (Theorem 3).

Before introducing new definitions and analysis, we first illustrate the incomplete–information logic with a 3×3 example that will recur throughout Section 4.3. The planner observes the stable outcome (μ, w) and budgets b but not the value profile v; costs are zero for all pairs.

$$\begin{cases} \mu(s_1) = t_1, & w_{s_1,t_1} = 1, \quad b_{s_1} = 2 \\ \mu(s_2) = t_2, & w_{s_2,t_2} = 2, \quad b_{s_2} = 3 \\ \mu(s_3) = t_3, & w_{s_3,t_3} = 3, \quad b_{s_3} = 6 \end{cases}$$

Figure 8 depicts the observed wages (yellow), unknown schools' utilities (gray), and budget caps (red dashed lines).

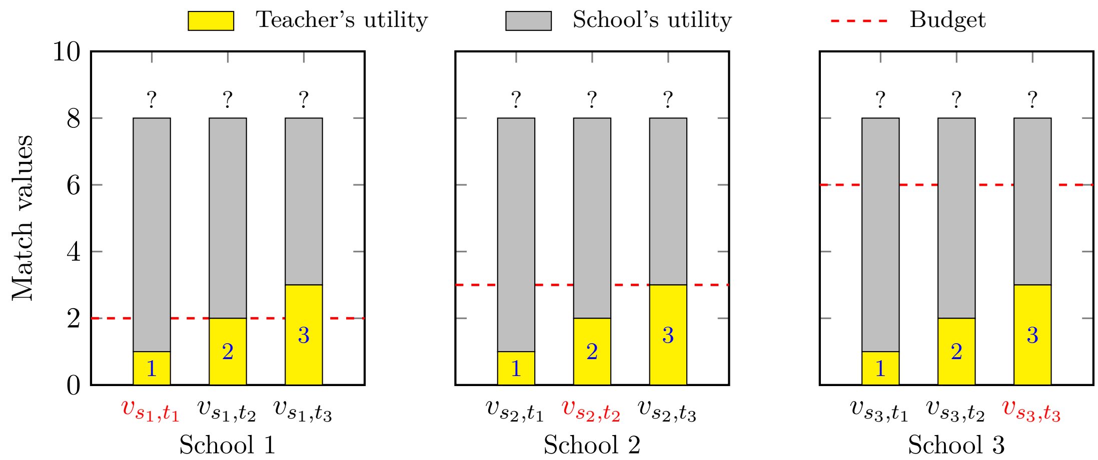

Figure 8: 3×3 example with incomplete information.

Even though v is unobserved, stability and affordability already imply informative inequalities. For school s_1, stability at (s_1, t_1) yields

$$v_{s_1,t_1} \;\geq\; w_{s_1,t_1} \;=\; 1.$$

Beyond this, the data are silent about v_{s_1,t_2} and v_{s_1,t_3}. Indeed, s_1's maximum affordable offer is $b_{s_1} = 2$, while the incumbent effective wages for t_2 and t_3 are $w_{s_2,t_2} = 2$ and $w_{s_3,t_3} = 3$, respectively. Since s_1 cannot make a strictly higher effective offer to either t_2 or t_3, the observed stability places no additional restriction on s_1's valuations for those teachers.

The planner can infer more from the observations for s_2. Stability of (s_2, t_2) implies

$$v_{s_2,t_2} \;\geq\; w_{s_2,t_2} \;=\; 2.$$

Moreover, s_2 can afford to poach t_1 because $b_{s_2} = 3 \geq w_{s_1,t_1} = 1$. Stability of (μ, w) therefore requires that t_2 be at least as attractive to s_2 as t_1 at the current wages:

$$D_{21} \;\equiv\; v_{s_2,t_2} - v_{s_2,t_1} \;\geq\; w_{s_2,t_2} - w_{s_1,t_1} \;=\; 1.$$

Analogously for s_3, stability of (s_3, t_3) gives

$$v_{s_3,t_3} \;\geq\; w_{s_3,t_3} \;=\; 3.$$

Since $b_{s_3} = 6$ exceeds both incumbent wages $w_{s_1,t_1} = 1$ and $w_{s_2,t_2} = 2$, school s_3 could afford to make offers to t_1 and t_2. Thus stability imposes two revealed-preference inequalities:

$$D_{31} \;\equiv\; v_{s_3,t_3} - v_{s_3,t_1} \;\geq\; w_{s_3,t_3} - w_{s_1,t_1} \;=\; 2,$$
$$D_{32} \;\equiv\; v_{s_3,t_3} - v_{s_3,t_2} \;\geq\; w_{s_3,t_3} - w_{s_2,t_2} \;=\; 1.$$

Figure 9 depicts these inferences: D_{21}, D_{31}, and D_{32} highlight the revealed lower bounds implied by stability and affordability.

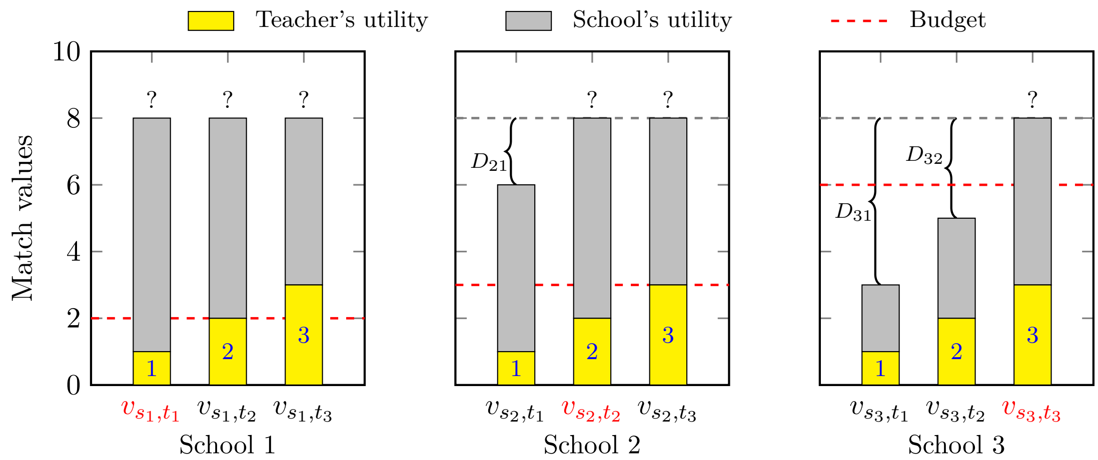

Figure 9: Inference from the observed stable allocation.

4.3.1 Setup and notation

Equipped with the reasoning from the illustrative example, there are many value profiles consistent with the observed allocation. Formally, given an observed stable outcome (μ, w) under costs c and budgets b, define the set of *consistent* value profiles

$$\mathcal{V}(\mu, w; b, c) \;=\; \{\, v : \; (\mu, w) \text{ is stable under } (v, c, b) \,\}.$$

To compare matchings across such profiles, we measure welfare for any matching μ by

$$W(\mu; v, c) \;=\; \sum_{s \in S} \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big).$$

When v is unobserved, the planner cannot identify the welfare–maximizing matching. We therefore pursue a *robust* policy: choose subsidies that (weakly) improve outcomes for *every* value profile consistent with the observed data. To formalize this objective, we introduce two definitions.

Definition 6 (Post–subsidy optimal welfare)**.** Fix v, c, b. For a subsidy allocation $\tau = (\tau_s)_{s \in S}$ (so $b_s^\tau = b_s + \tau_s$), define

$$\overline{W}(v, c, b, \tau) \;\equiv\; \max_{\mu} \Big\{\, W(\mu; v, c) \;:\; \exists\, w \text{ with } (\mu, w) \text{ stable under } (v, c, b^\tau) \Big\}.$$

Thus $\overline{W}(v, c, b, \tau)$ is the highest welfare attainable by any stable allocation after applying τ.

Definition 7 (Non–distortive subsidy)**.** In an allocation (μ, w) with budgets b and costs c, a subsidy allocation τ is *non–distortive* for $(\mu, w; b, c)$ if, for all $v \in \mathcal{V}(\mu, w; b, c)$,

$$\overline{W}(v, c, b, \tau) \;\geq\; W(\mu; v, c).$$

Equivalently, under the post–subsidy budgets b^τ, there exists a stable allocation whose welfare is weakly higher than that of the observed matching μ for every value profile consistent with $(\mu, w; b, c)$.

Non–distortiveness requires the subsidy to be robust across all value profiles consistent with what the social planner observes: regardless of which $v \in \mathcal{V}(\mu, w; b, c)$ is true, the best stable outcome after the subsidy does not reduce welfare relative to the status quo (μ, w). If the inequality in Definition 7 fails, then there exists a consistent v for which every post–subsidy stable outcome is strictly worse than μ, in which case the subsidy is *distortive*. Our goal in what follows is to characterize the set of non–distortive subsidies.

4.3.2 Direct effect

Providing subsidies relaxes previously binding budget constraints and can thereby change the set of stable allocations. In particular, schools that were constrained in the pre-subsidy market may become able to compete for teachers they could not previously afford. To capture such constraints at the observed outcome, we introduce an outbidding relation.

Definition 8 (Outbid)**.** Given an observed stable allocation (μ, w) under budgets b and costs c, we say that school s_i is *outbid by* school s_j if

$$b_{s_i} - c_{s_i,\mu(s_j)} \ \leq \ w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}.$$

Conversely, we say that s_i *is not outbid by* s_j, denoted $s_i \xrightarrow{\mu} s_j$, if

$$b_{s_i} - c_{s_i,\mu(s_j)} \ > \ w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}.$$

Equivalently, s_i is outbid by s_j if s_i cannot offer teacher $\mu(s_j)$ any effective wage (wage minus cost) that strictly exceeds her current effective wage at s_j, and $s_i \xrightarrow{\mu} s_j$ if it can.

A subsidy to a previously constrained school may enable it to approach a teacher it could not afford before, creating upward wage pressure on that teacher's current employer. We quantify this *direct* wage pressure as follows.

Definition 9 (Direct effect)**.** Given an observed stable outcome (μ, w) under budgets b and costs c, and a subsidy allocation τ (so $b^\tau_s = b_s + \tau_s$), define for each school s_j:

$$\Delta^d_j \ \equiv \ \max_{i:\ s_i \text{ is outbid by } s_j \text{ in } (\mu,w)} \Big[\big(b^\tau_{s_i} - c_{s_i,\mu(s_j)}\big) - \big(w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}\big) \Big]_+,$$

where $[x]_+ \coloneqq \max\{x, 0\}$, with the convention $\Delta^d_j = 0$ if the index set is empty.

Intuition (direct effect). Consider any rival s_i that was previously outbid by s_j. After subsidies, s_i may be able to exceed the incumbent effective wage of $\mu(s_j)$, forcing s_j to raise that wage to defend its match. The quantity Δ^d_j is the maximal *direct* wage-lifting pressure that s_j may face from its previously outbid rivals.

Such direct effects can generate efficiency losses. Consider our illustrative example and the following subsidy allocation:

$$\tau_{s_1} = 2, \qquad \tau_{s_2} = 0, \qquad \tau_{s_3} = 0.$$

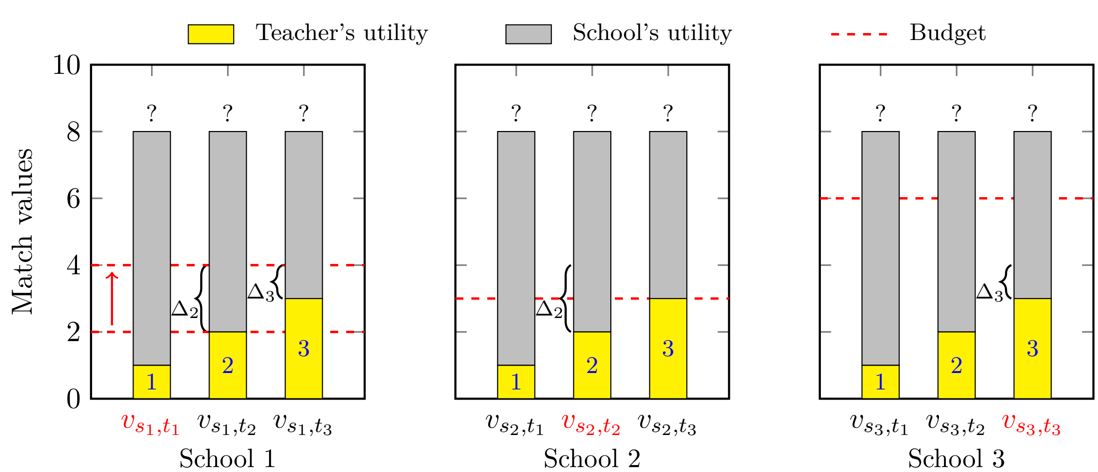

Figure 10: Direct effect illustration.

As depicted in Figure 10, the subsidy to s_1 increases its budget, creating wage-lifting pressures of 2 on s_2 and 1 on s_3. Because s_3 has a sufficiently large budget ($b_{s_3} = 6$), it can afford to defend its current match with t_3 against potential poaching from s_1. In contrast, s_2 has a budget of only 3, which is insufficient to defend its partner t_2 from the new offer that s_1 can now make. Consequently, even when s_2 and t_2 generate a high match value, s_2 may be forced to leave t_2, resulting in a less efficient allocation.

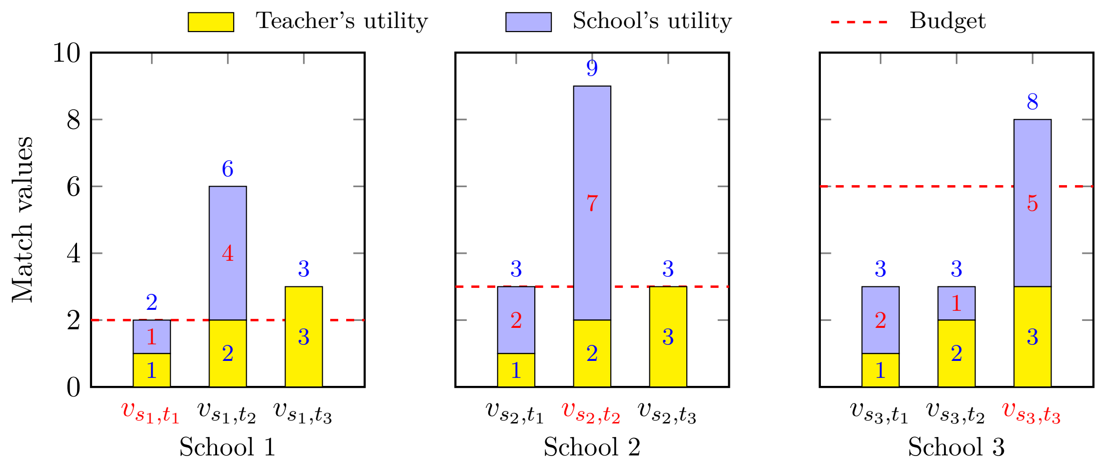

Figure 11: Efficiency loss generated by direct effect.

The value profile illustrated in Figure 11 is *consistent* with the observed pre-subsidy allocation (μ, w). After the subsidy $\tau_{s_1} = 2$, school s_1 can make an offer to t_2 that strictly dominates the highest affordable offer of s_2, thereby forcing s_2 to relinquish t_2. Being unable to match the new effective wage, s_2 loses t_2 even though the match (s_2, t_2) generates a high underlying value. The resulting post-subsidy allocation is therefore strictly less efficient than the original stable outcome.

We refer to $b_s^\tau - w_{s,\mu(s)}$ as school s's *budget reserve*: the unspent budget remaining after paying its incumbent teacher under post-subsidy budgets b^τ. The preceding example shows that relaxing one school's constraint can unintentionally destabilize other matches, generating welfare losses via direct poaching. To prevent such distortions, a subsidy allocation must ensure that every school retains sufficient budget reserve to defend its incumbent match against the strongest potential challenge from previously constrained rivals. This requirement is captured by the following condition.

Condition 1 (Direct-effect coverage (DC))**.** *In an allocation (μ, w) with budgets b and costs c, a subsidy allocation τ satisfies* DC *if, for every s_j,*

$$b_{s_j}^\tau - w_{s_j,\mu(s_j)} \;\geq\; \Delta_j^d.$$

Equivalently, for any s_i that is outbid by s_j before the subsidy,

$$b_{s_j}^\tau - c_{s_j,\mu(s_j)} \;\geq\; b_{s_i}^\tau - c_{s_i,\mu(s_j)}.$$

Intuition. DC requires each school s_j to hold enough post-subsidy budget to absorb the largest direct wage-lifting pressure Δ_j^d generated by its previously outbid rivals. If DC fails for some s_j, a formerly constrained rival s_i may offer a higher effective wage to $\mu(s_j)$ after receiving the subsidy, inducing $\mu(s_j)$ to switch and thereby lowering total welfare. In this case, the subsidy is *distortive* via the direct-poaching channel. We formalize this observation below:

Lemma 1 (Non-distortion implies DC)**.** *In an allocation (μ, w) with budgets b and costs c, if a subsidy allocation τ is non-distortive, then τ satisfies* DC*.*

4.3.3 Spillover effect

Direct wage pressure at one school can propagate through the market. When a subsidized school challenges an incumbent for its teacher, the incumbent may have to raise that teacher's wage to block the deviation; the higher wage lowers the incumbent's own payoff at its current match and can make some *other* teacher comparatively more attractive. Therefore, direct wage pressure can travel along a chain of bid–counterbid moves that is determined by the pre–subsidy structure of the observed allocation. We illustrate this transmission with the following example, where the subsidy allocation is

$$\tau_{s_1} = 2, \qquad \tau_{s_2} = 0, \qquad \tau_{s_3} = 0.$$

so only s_2 receives 2 units of funds.

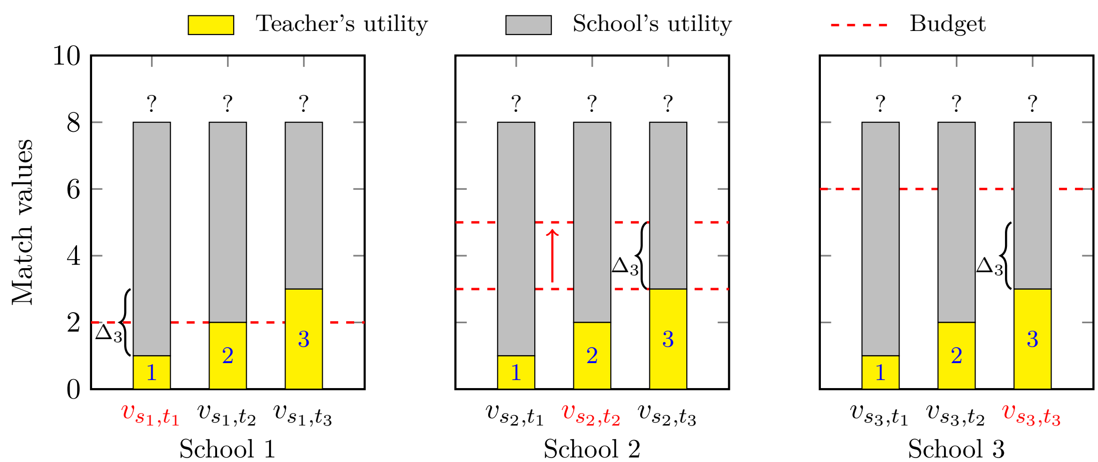

Figure 12: Spillover effect illustration.

In the configuration of Figure 12, revealed preference at the observed outcome implies that s_2 does *not* wish to pursue t_1 after receiving the subsidy. However, with $\tau_{s_2} = 2$ (so $b^{\tau}_{s_2} = 5$), school s_2 can feasibly approach t_3, whose incumbent wage is $w_{s_3,t_3} = 3$. School s_3 can defend t_3 (its budget is $b_{s_3} = 6$) by raising t_3's wage, which reduces s_3's payoff at the match (s_3, t_3). This reduction makes t_1 relatively more attractive to s_3, so s_3 may next bid for t_1. Hence the initial direct pressure faced by s_3 is *transmitted* to s_1 through the chain $s_2 \to s_3 \to s_1$.

To formalize this propagation, we now introduce two notions: (i) a *spillover chain*, which describes the pre-subsidy paths along which direct pressure can travel, and (ii) the *spillover effect* of a subsidy, which quantifies the induced wage pressure transmitted along such chains.

Definition 10 (Spillover chain)**.** Given an observed stable outcome (μ, w) under budgets b and costs c, we say there exists a *spillover chain* from s_i to s_j if there is a sequence of schools

$$C = (s_{k_1}, s_{k_2}, \ldots, s_{k_m}) \quad \text{with } k_1 = i,\ k_m = j,\ m \geq 2,$$

such that

$$s_{k_1} \xrightarrow{\mu} s_{k_2} \xrightarrow{\mu} s_{k_3} \cdots \xrightarrow{\mu} s_{k_m}.$$

At each step, before the subsidy, the predecessor s_{k_ℓ} can feasibly approach the teacher employed by its successor $s_{k_{\ell+1}}$, so wage-lifting pressure can propagate along the chain from s_i to s_j.

Once a direct effect arises at some school, the resulting wage pressure may propagate along the spillover chains defined above. For example, if s_1 raises its wage in response to pressure, it may then find another teacher relatively more attractive, thereby transmitting pressure further along the chain. To formalize this propagation, we define the spillover effects below.

Definition 11 (Spillover effect)**.** Given an observed stable outcome (μ, w) under budgets b and costs c, and a subsidy allocation τ (so $b_s^\tau = b_s + \tau_s$), consider a spillover chain

$$C: \; s_1 \xrightarrow{\mu} s_2 \xrightarrow{\mu} \cdots \xrightarrow{\mu} s_j.$$

Let Δ_1^d denote the direct effect at s_1 (Definition 9). Define the chain–specific spillover effects $\{\delta_i^C\}_{i=2}^j$ recursively by

$$\delta_2^C \;\equiv\; \min\Big\{\Delta_1^d,\; b_{s_1}^\tau - c_{s_1,\mu(s_2)} - \big(w_{s_2,\mu(s_2)} - c_{s_2,\mu(s_2)}\big)\Big\},$$

and for $i = 3, \ldots, j$,

$$\delta_i^C \;\equiv\; \min\Big\{\delta_{i-1}^C,\; b_{s_{i-1}}^\tau - c_{s_{i-1},\mu(s_i)} - \big(w_{s_i,\mu(s_i)} - c_{s_i,\mu(s_i)}\big)\Big\}.$$

We refer to δ_j^C as the *spillover effect* reaching s_j along C. The total spillover effect faced by s_j is

$$\Delta_j^s \;\equiv\; \sup_{C\text{: spillover chains ending at } s_j} \delta_j^C,$$

with the convention that the supremum over an empty family equals 0.

Remark (budget-capped propagation). The $\min\{\cdot,\cdot\}$ in the recursion for δ reflects that, at each link of a spillover chain, the pressure that can be transmitted is capped both by the incoming pressure and by the successor's affordability gap (determined by its post-subsidy budget and costs). Consequently, spillover pressure is weakly decreasing along any chain: $\delta_1^C \geq \delta_2^C \geq \cdots \geq \delta_j^C$.

Remark (cycles). A spillover chain may, in principle, include a cycle. However, because the transmitted pressure weakly declines at each step, traversing a cycle cannot amplify the effect. It therefore suffices to consider *simple* (cycle-free) chains, which already capture the maximal spillover effect transmitted through the network of schools:

$$\Delta_j^s \;=\; \max_{C\text{: simple spillover chains ending at } s_j} \delta_j^C.$$

Equipped with the above definitions, we return to Figure 12. School s_3 faces a direct wage pressure of 2. Because there is a spillover chain from s_3 to s_1, this pressure is transmitted downstream: s_3's defense of t_3 can prompt it to seek t_1, thereby imposing a spillover pressure of 2 on s_1. If s_1's post-subsidy budget reserve $b_{s_1}^\tau - w_{s_1,\mu(s_1)}$ is insufficient to absorb this pressure, s_1 may be unable to defend t_1, potentially generating an efficiency loss. Figure 13 exhibits a value profile consistent with the observed outcome in which s_1 is indeed forced to relinquish t_1 after the subsidy, leading to a welfare loss.

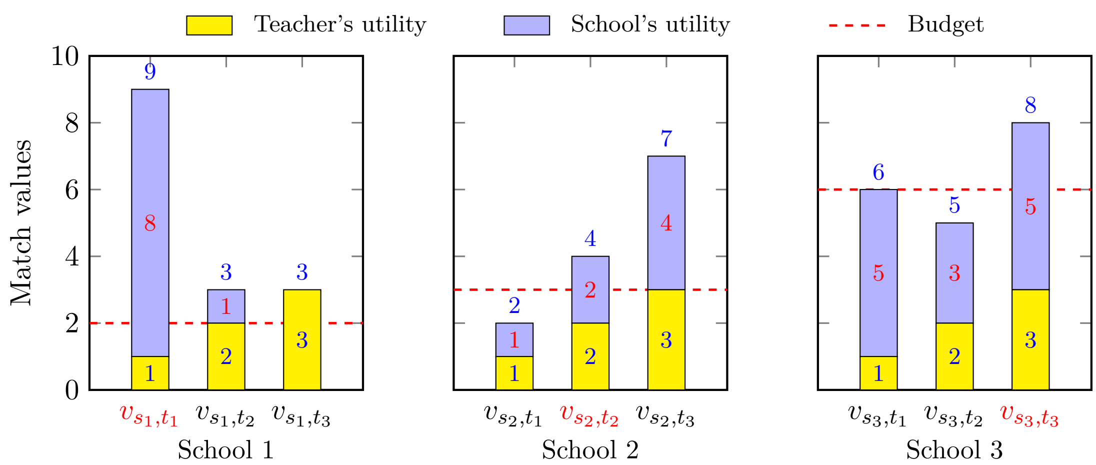

Figure 13: Efficiency loss generated by spillover effect.

Motivated by this transmission logic, we require subsidies to cover not only each school's *direct* pressure but also the *spillover* that can reach it through any spillover chain.

Condition 2 (Spillover coverage (SC))**.** *In an allocation* (μ, w) *with budgets* b *and costs* c*, a subsidy allocation* τ *satisfies* SC *if, for every school* s_j*,*

$$b^{\tau}_{s_j} - w_{s_j,\mu(s_j)} \;\geq\; \Delta^s_j.$$

Intuition. SC guarantees that each school has enough post–subsidy budget reserve to absorb the *largest* wage pressure that can arrive via spillover chains originating anywhere in the market. If SC fails at some s_j, then—even if direct-effect coverage holds—pressure propagated from upstream schools can still induce a profitable deviation at $\mu(s_j)$, making every post–subsidy stable outcome strictly worse than the status quo. In that case, the subsidy is *distortive* via spillovers. We formalize this necessity below.

Lemma 2 (Non-distortion implies SC)**.** *In an allocation* (μ, w) *with budgets* b *and costs* c*, if a subsidy allocation* τ *is non-distortive, then* τ *satisfies* SC*.*

4.3.4 Main results

Lemmas 1 and 2 establish that both *Direct-effect coverage (DC)* and *Spillover coverage (SC)* are *necessary* for a subsidy allocation to be non-distortive. However, these results do not yet tell us how to design such a subsidy. The next theorem provides a complete answer by showing that the two conditions are not only necessary but also sufficient.

Theorem 1 (Characterization of non-distortive subsidies)**.** *In an allocation* (μ, w) *with budgets* b *and costs* c*, a subsidy allocation* τ *is non-distortive if and only if* τ *satisfies both* DC *and* SC*. Equivalently, for all* s_j*,*

$$b^{\tau}_{s_j} - w_{s_j,\mu(s_j)} \;\geq\; \Lambda_j \;\equiv\; \max\{\Delta^d_j,\; \Delta^s_j\}.$$

Characterization. Theorem 1 closes the loop: the two coverage conditions are jointly necessary and sufficient for non-distortion. Together, they provide a *full characterization* of all subsidy allocations that guarantee, for every value profile consistent with the observed outcome, the *existence* of a post-subsidy stable allocation whose welfare is weakly higher than that of the status quo.

Implementation. Although Theorem 1 is an existence result and does not prescribe how to select among post-subsidy stable outcomes, its proof is *constructive*: under budgets b^{τ}, the school–optimal stable allocation (with a tie-breaking rule favoring the status-quo matching) achieves weakly higher welfare than the status quo. Therefore, if the planner can operate a centralized clearinghouse, a welfare-improving outcome can be *implemented* by running a school–proposing extended Deferred Acceptance (DA) algorithm.

Practical implications. Operationally, the theorem reduces design or verification of a non-distortive policy to a system of simple inequalities:

$$b^{\tau}_{s_j} \;\geq\; w_{s_j,\mu(s_j)} + \Lambda_j \qquad \text{for all } s_j.$$

Hence the planner can test any candidate subsidy allocation—or design one—without solving for unobserved preferences or recomputing the full set of stable outcomes.

Economic interpretation. Economically, the characterization bundles two distinct conditions that every robust subsidy must satisfy. First, *DC* ensures the subsidy *respects the observed outbidding structure*, preventing new direct poaching that would overturn pre-subsidy dominance relations among schools. Second, *SC* accounts for indirect exposure through *spillover chains*: even a school that faces no direct poaching may still be affected by wage pressure transmitted from others. Together, DC and SC ensure that post-subsidy budgets are sufficient to absorb both direct and propagated wage pressures.

Methodological implications. Methodologically, Theorem 1 offers a robust policy prescription without a mechanism-design layer. The planner need not elicit private information about schools' preferences: the observable features (μ, w, b, c) and the outbidding and spillover relations inferred from them suffice to design a non-distortive subsidy across all consistent value profiles.

4.4 The value of information under incomplete information

This subsection studies how the availability of information affects the set of non-distortive subsidies. In the incomplete-information environment, the planner may lack full knowledge of match values, costs, or even budgets. We will show that information constraints have a significant impact on policy design: as the planner's information set becomes more limited, the set of non-distortive subsidies contracts, and the planner loses flexibility to target assistance across schools. Conversely, when richer information is available, the planner can design targeted interventions.

4.4.1 When the planner has rich information

When the planner observes the observed allocation (μ, w) together with costs c and budgets b—but not the underlying match values v—the set of feasible non-distortive subsidies is characterized by Theorem 1. In this setting, the planner possesses enough information to design *targeted* subsidies that respect the revealed outbidding and spillover structures. A uniform transfer across schools,

$$\tau_{s_i} = \tau_{s_j} > 0 \qquad \forall i, j,$$

is therefore not necessarily optimal. Before stating this result formally, we define a notion of dominance to compare different subsidy allocations.

Definition 12 (Dominance)**.** In an allocation (μ, w) with budgets b and costs c, a subsidy allocation τ is said to be *dominated* if there exists another subsidy τ' such that

$$\overline{W}(v, c, b, \tau') \;\geq\; \overline{W}(v, c, b, \tau) \quad \text{for all } v \in \mathcal{V}(\mu, w; b, c),$$

and

$$\sum_{s \in S} \tau'_s \;<\; \sum_{s \in S} \tau_s.$$

Intuitively, a subsidy allocation τ is dominated if another policy τ' can achieve at least the same welfare for all value profiles consistent with the observed environment while requiring a strictly smaller total transfer.

Proposition 2 (Uniform subsidies can be dominated)**.** *A positive uniform subsidy allocation is dominated if there exists s_j such that*

$$\begin{cases} b_{s_i} - c_{s_i,t_j} \leq w_{s_j,t_j} - c_{s_j,t_j}, \\ b_{s_i} - c_{s_i,t_j} < b_{s_j} - c_{s_j,t_j}, \end{cases} \qquad \textit{for all } i \neq j,$$

and

$$s_j \xrightarrow{\mu} s_i \quad \textit{for all } i \neq j.$$

Proposition 2 shows that when the planner knows costs and budgets, but not valuations, a uniform policy, though non-distortive, may still be inefficient. With this intermediate level of information, the planner can often achieve at least the same welfare for all consistent value profiles while using fewer total resources. For instance, recall Figure 8: under a uniform transfer $\tau = (1, 1, 1)$, school s_3 already has enough budget reserve, so the 1 unit allocated to s_3 does not relax any binding constraint and is wasted. A different subsidy $\tau' = (1, 1, 0)$ achieves the same level of improvement while strictly reducing total transfers, thereby dominating the uniform policy.

In short, richer information enables *targeted* and more cost-effective subsidy design, whereas informational constraints progressively limit this flexibility.

4.4.2 When the planner has limited information

We next consider an environment in which the planner does not observe match values or costs but still knows the observed allocation (μ, w) and the baseline budgets b. In this case, the definition of consistency and non-distortion must be adjusted to account for the unobserved parameters.

Define the set of value and cost profiles *consistent* with the observed environment $(\mu, w; b)$:

$$\mathcal{V}, \mathcal{C}(\mu, w; b) \;\equiv\; \{\, v, c : \; (\mu, w) \text{ is stable under } (v, c, b) \,\}.$$

Definition 13 (Non-distortive subsidy with unknown v and c)**.** In an allocation (μ, w) with budgets b, we say a subsidy allocation $\tau = (\tau_s)_{s \in S}$ is *non-distortive* for $(\mu, w; b)$ if, for every value and cost profile $v, c \in \mathcal{V}, \mathcal{C}(\mu, w; b)$,

$$\overline{W}(v, c, b, \tau) \;\geq\; W(\mu; v, c).$$

Theorem 2 (Characterization with unknown v and c)**.** *In an allocation (μ, w) with budgets b, a subsidy allocation τ is non-distortive if and only if*

$$b^{\tau}_{s_j} - w_{s_j, \mu(s_j)} \;\geq\; \max_{s_i} \tau_{s_i} \qquad \textit{for all } s_j.$$

Theorem 2 shows that when the planner cannot observe the underlying values or costs, the flexibility to differentiate subsidies is sharply reduced. The non-distortive set is uniformly constrained by the largest subsidy granted to any school, limiting the scope for targeted assistance. Equivalently, each school's post-subsidy budget reserve must cover a *uniform* lower bound, which is the greatest potential direct effect that any single subsidized school can generate.

4.4.3 When the planner has minimal information

Finally, consider the extreme case in which the planner cannot observe match values, costs, or budgets and only knows the existing allocation (μ, w). With such minimal information, any differentiation across schools may generate distortions for some consistent environment.

Define the set of value, cost, and budget profiles *consistent* with the observed allocation (μ, w):

$$\mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w) \equiv \{ v, c, b : (\mu, w) \text{ is stable under } (v, c, b) \}.$$

Definition 14 (Non-distortive subsidy with unknown v, c, and b)**.** In an allocation (μ, w), a subsidy allocation $\tau = (\tau_s)_{s \in S}$ is *non-distortive* for (μ, w) if, for every value, cost, and budget profile $v, c, b \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$,

$$\overline{W}(v, c, b, \tau) \geq W(\mu; v, c).$$

Theorem 3 (Characterization with unknown v, c, and b)**.** *In an observed allocation (μ, w), a subsidy allocation τ is non-distortive if and only if*

$$\tau_{s_i} = \tau_{s_j} \qquad \textit{for all } i \neq j.$$

When information is severely limited, the planner observes neither schools' preferences v, nor teachers' preferences c, nor schools' constraints b. Deprived of this information, the planner loses all flexibility to target assistance across schools. Uniform assistance is therefore the only policy that guarantees robustness across all environments consistent with the observed outcome.

Discussion. Taken together, Proposition 2 and Theorems 1–3 reveal how information constraints shape the design of non-distortive subsidies. As the planner's information set becomes coarser, the feasible set of policies narrows: *with rich information, targeted subsidies can dominate a uniform policy; with partial information, the scope for differentiation shrinks; and with minimal information, uniform subsidies are the only robust choice.* Hence, the degree of information available to the planner governs the flexibility of policy design and determines whether the planner should implement targeted interventions or uniform transfers in practice.

5 Conclusion

This paper analyzes optimal subsidy design in budget-constrained one-to-one matching markets. We deliver two main contributions. First, under complete information, we develop a constructive algorithm that computes the smallest transfers needed to implement the welfare-maximizing matching. This procedure yields the first-best allocation with minimal transfers, demonstrating that efficiency can be restored with appropriately targeted subsidies.

Second, under incomplete information, we develop a robust policy framework in which the planner observes only the status quo stable outcome but not the underlying match values. Our central theoretical result fully characterizes *non-distortive* subsidies in this setting. Specifically, Theorem 1 identifies two simple and testable requirements—*direct-effect coverage* and *spillover coverage*—that are jointly necessary and sufficient to ensure that, for every value profile consistent with the observed allocation, there exists a post-subsidy stable matching with (weakly) higher welfare than the status quo.

This incomplete-information approach is the core innovation of the paper: it delivers a policy prescription that does not require any preference elicitation, in contrast to traditional mechanism design. Our results show that, even under incomplete information, the planner can improve welfare in a robust manner. In addition, by comparing different informational limitations, we show how the planner's information set governs the choice between targeted and uniform transfers—yielding a clear value-of-information message for policymakers: richer information admits finer targeting, whereas minimal information leaves uniform assistance as the only robust option.

To conclude, our analysis deepens the connection between matching theory and policy implementation, clarifying how incomplete information shapes subsidy design. The results provide a useful tool for market designers and policymakers operating under informational constraints, enabling the design and implementation of more effective allocations in real-world matching markets.

Appendix

Proof of proposition 1 (i)

Step 1: View schools as a directed graph.

Consider the set of schools as a directed graph, and define fixed edge weights on the school graph:

$$\delta_{s\to s'} = \big(v_{s,\mu^*(s')} - c_{s,\mu^*(s')}\big) - \big(v_{s,\mu^*(s)} - c_{s,\mu^*(s)}\big).$$

Step 2: MSI is equivalent to Bellman–Ford algorithm.

Using MSI's offer and wage update, we have

$$d^r_{s,\mu^*(s')} = \min\{b^{r-1}_s,\ w^{r-1}_s + v_{s,\mu^*(s')} - v_{s,\mu^*(s)}\}$$

$$w^r_{s',\mu^*(s')} - c_{s',\mu^*(s')} = \max_s\{d^r_{s,\mu^*(s')} - c_{s,\mu^*(s')}\}$$

which is equivalent to

$$w^r_{s',\mu^*(s')} = \max_s\big\{\min\{b^{r-1}_s - c_{s,\mu^*(s')} + c_{s',\mu^*(s')}, w^{r-1}_s + \delta_{s\to s'}\}\big\}$$

Then each round of the MSI extends node distance by one edge, adds the fixed edge weight, and applies a cap (the min function) on the edge weight. This is the Bellman–Ford algorithm with path relaxation.

Step 3: No positive cycles.

Then, the wages and budgets in the MSI algorithm can be considered as finding the longest-path (ending at school s) weights. Since μ^* maximizes total surplus, then for every directed cycle C

$$\sum_{(s\to s')\in C} \delta_{s\to s'} = \sum_{(s\to s')\in C} \big(s_{s,\mu^*(s')} - s_{s,\mu^*(s)}\big) \le 0.$$

That is, there are no positive cycles in the graph.

Conclusion: finite termination.

There are no positive cycles, so the Bellman–Ford algorithm with path relaxation terminates in n rounds[10], which finishes the proof.

[10] This is the longest-path with no positive cycles analogue of Bellman–Ford for shortest-path with no negative cycles. See Cormen et al. (2009) §24.1.

Proof of proposition 1 (ii)

Individual rationality.

By initialization, $w^1_{s,\mu^*(s)} = c_{s,\mu^*(s)}$ for all s. By construction of MSI, matched-edge wages are nondecreasing across rounds, so $w^R_{s,\mu^*(s)} \geq c_{s,\mu^*(s)}$ for all s. Thus every matched teacher earns at least cost, and every matched school's payoff is $v_{s,\mu^*(s)} - w^R_{s,\mu^*(s)} \geq 0$ by the construction of the wage updates and optimality of μ^*. Hence (μ^*, w^R) is IR.

Budget feasibility.

Budgets are updated by $b^r_s = \max\{b^{r-1}_s, w^r_{s,\mu^*(s)}\}$. Therefore $b^R_s \geq w^R_{s,\mu^*(s)}$ for all s, so wages are affordable at R.

No blocking pair.

At the termination round R, fix any school teacher pair (s,t). By construction of the MSI algorithm, we have

$$w^R_{\mu^{*-1}(t),t} - c_{\mu^{*-1}(t),t} = \max_{s' \in S} \; \{d^R_{s',t} - c_{s',t}\} \geq d^R_{s,t} - c_{s,t}$$

where the LHS is the effective wage received by teacher t, and the RHS is the highest effective wage school s is willing to offer. Therefore, there is no profitable deviation.

Proof of Proposition 1 (iii)

We use an inductive argument to show the minimality of MSI subsidies.

Base step

Let $(\widetilde{w}, \widetilde{b})$ be any wages and post-subsidy budgets such that $(\mu^*, \widetilde{w})$ is stable under $\widetilde{b} = b + \widetilde{\tau}$ with $\widetilde{\tau} \geq 0$. Then stability implies IR, which gives

$$\widetilde{w}_{s,\mu^*(s)} \geq c_{s,\mu^*(s)} = w^1_{s,\mu^*(s)}.$$

Also $\widetilde{b}_s \geq b_s$ (since $\widetilde{b} = b + \widetilde{\tau}$ with $\widetilde{\tau} \geq 0$) and budget feasibility gives $\widetilde{b}_s \geq \widetilde{w}_{s,\mu^*(s)} \geq c_{s,\mu^*(s)}$, hence

$$\widetilde{b}_s \geq \max\{b_s, c_{s,\mu^*(s)}\} = b^1_s.$$

Induction step

Fix $r \geq 2$. If $(\mu^*, \widetilde{w})$ is stable under $\widetilde{b}$ and

$$\widetilde{w}_{s,\mu^*(s)} \geq w^{r-1}_{s,\mu^*(s)} \quad \text{and} \quad \widetilde{b}_s \geq b^{r-1}_s \quad \text{for all } s,$$

then we want to show

$$\widetilde{w}_{s,\mu^*(s)} \geq w^r_{s,\mu^*(s)} \quad \text{for all } s \qquad \text{and} \qquad \widetilde{b}_s \geq \max\{b^{r-1}_s, w^r_{s,\mu^*(s)}\} = b^r_s \quad \text{for all } s.$$

Proof. For any pair (s, t), stability of $(\mu^*, \widetilde{w})$ under $\widetilde{b}$ gives

$$\widetilde{w}_{\mu^{*-1}(t),t} - c_{\mu^{*-1}(t),t} \;\geq\; \min\{\widetilde{b}_s,\, v_{s,t} - v_{s,\mu^*(s)} + \widetilde{w}_{s,\mu^*(s)}\} - c_{s,t},$$

where the LHS is the effective wage of teacher t in $(\mu^*, \widetilde{w})$, and the RHS is the highest effective wage s would offer to t without lowering its profit and subject to its budget.

By the induction hypothesis $\widetilde{b}_s \geq b^{r-1}_s$ and $\widetilde{w}_{s,\mu^*(s)} \geq w^{r-1}_{s,\mu^*(s)}$, hence

$$\min\{\widetilde{b}_s,\, v_{s,t} - v_{s,\mu^*(s)} + \widetilde{w}_{s,\mu^*(s)}\} - c_{s,t} \;\geq\; \min\{b^{r-1}_s,\, v_{s,t} - v_{s,\mu^*(s)} + w^{r-1}_{s,\mu^*(s)}\} - c_{s,t} = d^r_{s,t} - c_{s,t}.$$

Taking the maximum over schools,

$$\widetilde{w}_{\mu^{*-1}(t),t} - c_{\mu^{*-1}(t),t} \;\geq\; \max_s\{d^r_{s,t} - c_{s,t}\} = w^r_{\mu^{*-1}(t),t} - c_{\mu^{*-1}(t),t},$$

which proves $\widetilde{w} \geq w^r$. Budget feasibility then yields

$$\widetilde{b}_s \;\geq\; \widetilde{w}_{s,\mu^*(s)} \;\geq\; w^r_{s,\mu^*(s)}.$$

Together with $\widetilde{b}_s \geq b^{r-1}_s$ (induction hypothesis) we obtain

$$\widetilde{b}_s \;\geq\; \max\{b^{r-1}_s,\, w^r_{s,\mu^*(s)}\} = b^r_s.$$

Given Proposition 1 (i), MSI terminates in finite rounds. The above induction shows that the MSI termination budget b^R_s is the componentwise minimum among all post-subsidy budgets that implement μ^*, which completes the proof. □

Proof of Lemma 1

Proof. We prove Lemma 1 by contradiction.

Suppose we have a non-distortive subsidy allocation τ but DC fails. Then there exists s_j such that

$$b^{\tau}_{s_j} - w_{s_j,\mu(s_j)} < \Delta^d_j \tag{L1-1}$$

By definition of Δ^d_j

$$\Delta^d_j \equiv \max_{i:\ s_i \text{ is outbid by } s_j \text{ in } (\mu,w)} \left[(b^{\tau}_{s_i} - c_{s_i,\mu(s_j)}) - (w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)})\right]_+$$

Then there exists s_i such that

$$b_{s_i} - c_{s_i,\mu(s_j)} \leq w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)} \tag{L1-2}$$

And

$$\Delta^d_j = (b^{\tau}_{s_i} - c_{s_i,\mu(s_j)}) - (w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}) \tag{L1-3}$$

Combining (L1-1) and (L1-3), we have

$$b^{\tau}_{s_j} - c_{s_j,\mu(s_j)} < b^{\tau}_{s_i} - c_{s_i,\mu(s_j)} \tag{L1-4}$$

Construct the following value profile v

$$v_{s_i,\mu(s_i)} = w_{s_i,\mu(s_i)}, \quad v_{s_i,\mu(s_j)} = c_{s_i,\mu(s_j)} + V, \quad v_{s_j,\mu(s_j)} = c_{s_j,\mu(s_j)} + 2V$$

For all remaining matched pairs $(s, \mu(s))$ set

$$v_{s,\mu(s)} = w_{s,\mu(s)}$$

For all remaining unmatched pairs (s,t) set

$$v_{s,t} = c_{s,t}$$

Here $V > 0$ is a large enough value such that

$$V > \Sigma_s(w_{s,\mu(s)} - c_{s,\mu(s)}) + \Sigma_s b^{\tau}_s$$

We claim that (μ, w) is stable under budgets b given (v, c), so $v \in \mathcal{V}(\mu, w; b, c)$.

To show (μ, w) is stable, we need to show the following:

- (μ, w) is feasible:
  This is directly implied by the observed market structure $(\mu, w; b, c)$, so we have $b_s \geq w_{s,\mu(s)}$.
- IR holds for all matched pairs:
  Teachers' IR are also directly implied by the observed market structure $(\mu, w; b, c)$, so we have $w_{s,\mu(s)} \geq c_{s,\mu(s)}$.
  Now we need to check schools' IR. By construction of v, we have

  $$2V > V > \Sigma_s(w_{s,\mu(s)} - c_{s,\mu(s)}) + \Sigma_s b^{\tau}_s \geq w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}$$

  Therefore, for school s_j, we have

  $$v_{s_j,\mu(s_j)} - w_{s_j,\mu(s_j)} = 2V - w_{s_j,\mu(s_j)} + c_{s_j,\mu(s_j)} > 0$$

  So IR holds for school s_j.
  For any school $s \neq s_j$, by construction of v, we have

  $$v_{s,\mu(s)} - w_{s,\mu(s)} = 0$$

  So IR holds for school $s \neq s_j$. Therefore, IR hold for all schools.

- There is no blocking pairs:
  For any unmatched pair $(s,t) \neq (s_i, \mu(s_j))$, by construction of v, we have $v_{s,t} = c_{s,t}$. Then any deviating wage $w' > c_{s,t}$ will lead to a payoff $v_{s,t} - w' < 0$, which cannot block the current matching.
  For $(s_i, \mu(s_j))$, any deviating wage w' need to satisfy

$$w' - c_{s_i,\mu(s_j)} > w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)}$$

  By (L1-2), we have

$$w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)} \geq b_{s_i} - c_{s_i,\mu(s_j)}$$

  Then we have

$$w' > b_{s_i}$$

  So any deviating wage is not affordable, and $(s_i, \mu(s_j))$ cannot block the current matching.

Therefore, the observed allocation (μ, w) is stable, and we have

$$v \in \mathcal{V}(\mu, w; b, c)$$

Now we claim that, for any matching μ' where $\mu'(s_j) \neq \mu(s_j)$ we have

$$W(\mu'; v, c) < W(\mu; v, c)$$

To show the above, by definition of $W(\mu'; v, c)$, we have

$$W(\mu'; v, c) = \Sigma_s \big(v_{s,\mu'(s)} - c_{s,\mu'(s)}\big)$$

Then by construction of v, we have

$$v_{s,\mu'(s)} - c_{s,\mu'(s)} = \begin{cases} 0 & \text{if} \quad \mu'(s) \neq \mu(s) \\ w_{s,\mu(s)} - c_{s,\mu(s)} & \text{if} \quad \mu'(s) = \mu(s) \end{cases} \quad \text{for } s \neq s_i, s_j$$

Also, we know

$$v_{s,\mu(s)} - c_{s,\mu(s)} = w_{s,\mu(s)} - c_{s,\mu(s)} \quad \text{for } s \neq s_i, s_j$$

Then we have

$$v_{s,\mu'(s)} - c_{s,\mu'(s)} \leq v_{s,\mu(s)} - c_{s,\mu(s)} \quad \text{for } s \neq s_i, s_j$$

Now consider s_i. By construction of v, the best match we can form is matching s_i with $\mu(s_j)$, which gives

$$v_{s_i,\mu'(s_i)} - c_{s_i,\mu'(s_i)} \leq v_{s_i,\mu(s_j)} - c_{s_i,\mu(s_j)} = V$$

Now consider s_j. Since $\mu'(s_j) \neq \mu(s_j)$, by construction of v, we have

$$v_{s_j,\mu'(s_j)} - c_{s_j,\mu'(s_j)} = 0$$

Summing the above will give

$$\begin{aligned} W(\mu'; v, c) &= \Sigma_s \big(v_{s,\mu'(s)} - c_{s,\mu'(s)}\big) \\ &\leq \Sigma_{s \neq s_i, s_j} \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big) + V + 0 \\ &< \Sigma_{s \neq s_i, s_j} \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big) + 2V + w_{s_i,\mu(s_i)} - c_{s_i,\mu(s_i)} \\ &= \Sigma_s \big(v_{s,\mu(s)} - c_{s,\mu(s)}\big) \\ &= W(\mu; v, c) \end{aligned}$$

Therefore, we have

$$W(\mu'; v, c) < W(\mu; v, c) \quad \text{if} \quad \mu'(s_j) \neq \mu(s_j) \tag{L1-5}$$

Recall τ is non-distortive, so we have

$$\overline{W}(v, c, b, \tau) \geq W(\mu; v, c)$$

Since $W(\mu'; v, c) < W(\mu; v, c)$ for any matching μ' where $\mu'(s_j) \neq \mu(s_j)$, there must be at least one stable allocation $(\tilde{\mu}, \tilde{w})$ where $\tilde{\mu}(s_j) = \mu(s_j)$ after the subsidy.

Consider the above $(\tilde{\mu}, \tilde{w})$, we have

$$\tilde{\mu}(s_j) = \mu(s_j)$$

By feasibility of $(\tilde{\mu}, \tilde{w})$, we have

$$\tilde{w}_{s_j, \tilde{\mu}(s_j)} \leq b^{\tau}_{s_j}$$

Recall (L1-4), we then have

$$b^{\tau}_{s_i} - c_{s_i, \mu(s_j)} > b^{\tau}_{s_j} - c_{s_j, \mu(s_j)} \geq \tilde{w}_{s_j, \tilde{\mu}(s_j)} - c_{s_j, \mu(s_j)}$$

Then, by $\tilde{\mu}(s_j) = \mu(s_j)$, we have

$$b^{\tau}_{s_i} - c_{s_i, \tilde{\mu}(s_j)} > \tilde{w}_{s_j, \tilde{\mu}(s_j)} - c_{s_j, \tilde{\mu}(s_j)}$$

which means a wage of $b^{\tau}_{s_i}$ from school s_i is strictly better for $\tilde{\mu}(s_j)$.

Now consider s_i. Since we know $\tilde{\mu}(s_j) = \mu(s_j)$, so s_i cannot be matched with $\mu(s_j)$ after the subsidy. Then by construction of v, the highest utility s_i can enjoy in $(\tilde{\mu}, \tilde{w})$ is matching $\mu(s_i)$ with a wage of $c_{s_i, \mu(s_i)}$. So the highest utility of s_i in $(\tilde{\mu}, \tilde{w})$ is

$$w_{s_i, \mu(s_i)} - c_{s_i, \mu(s_i)}$$

However, by construction of v, matching $\tilde{\mu}(s_j)$ with a wage of $b^{\tau}_{s_i}$ will lead to a utility of

$$V + c_{s_i, \mu(s_j)} - b^{\tau}_{s_i}$$

Recall V is large enough

$$V > \Sigma_s (w_{s, \mu(s)} - c_{s, \mu(s)}) + \Sigma_s b^{\tau}_s$$

So we have

$$V + c_{s_i, \mu(s_j)} - b^{\tau}_{s_i} > [w_{s_i, \mu(s_i)} - c_{s_i, \mu(s_i)} + b^{\tau}_{s_i}] + c_{s_i, \mu(s_j)} - b^{\tau}_{s_i} \geq w_{s_i, \mu(s_i)} - c_{s_i, \mu(s_i)}$$

Therefore, matching $\tilde{\mu}(s_j)$ with a wage of $b^{\tau}_{s_i}$ is also strictly better for s_i, and $(s_i, \tilde{\mu}(s_j))$ will block $(\tilde{\mu}, \tilde{w})$, contradicting the stability of $(\tilde{\mu}, \tilde{w})$. Therefore, s_j cannot match $\mu(s_j)$ after the subsidy, and (L1-5) then contradicts τ being non-distortive, which finishes the proof. $\square$

Proof of Lemma 2

Proof. We prove Lemma 2 by contradiction.

WOLG, let $\mu(s_i) = t_i$ for notational convenience. Suppose we have a non-distortive subsidy allocation τ but SC fails. Then there exists s_j such that

$$b^{\tau}_{s_j} - w_{s_j,t_j} < \Delta^s_j \tag{L2-1}$$

By definition of Δ^s_j, there exists a spillover chain (after reindexing)

$$C: \; s_1 \xrightarrow{\mu} s_2 \xrightarrow{\mu} \cdots \xrightarrow{\mu} s_j$$

such that

$$\Delta^s_j = \delta^C_j$$

Choose the shortest violating chain, then SC holds at smaller indices, which is

$$b^{\tau}_{s_k} - w_{s_k,t_k} \geq \delta^C_k \quad \text{for } k = 2, \ldots, j-1 \tag{L2-2}$$

By Lemma 1, DC holds at s_1, which is

$$b^{\tau}_{s_1} - w_{s_1,t_1} \geq \Delta^d_1 \tag{L2-3}$$

By definition of Δ^d_1, there exists a school s_0 (after reindexing) such that

$$b_{s_0} - c_{s_0,t_1} \leq w_{s_1,t_1} - c_{s_1,t_1} \tag{L2-4}$$

And

$$\Delta^d_1 = (b^{\tau}_{s_0} - c_{s_0,t_1}) - (w_{s_1,t_1} - c_{s_1,t_1}) \tag{L2-5}$$

Construct the following value profile v

$$v_{s_0,t_0} = w_{s_0,t_0}, \quad v_{s_k,t_k} = w_{s_k,t_k} + V \;\; \text{for } k = 1, \ldots, j-1, \quad v_{s_j,t_j} = w_{s_j,t_j} + 2V$$

For all other matched pairs

$$v_{s_k,t_k} = w_{s_k,t_k} \;\; \text{for } k \notin \{0, 1, \ldots, j\}$$

For the following unmatched pairs

$$v_{s_k,t_{k+1}} = c_{s_k,t_{k+1}} + \left(w_{s_{k+1},t_{k+1}} - c_{s_{k+1},t_{k+1}}\right) + V \;\; \text{for } k = 0, \ldots, j-1$$

For all other unmatched pairs

$$v_{s,t} = c_{s,t} \;\; \text{for all remaining } (s,t)$$

Here $V > 0$ is a large enough value such that

$$V > \Sigma_i (w_{s_i,t_i} - c_{s_i,t_i}) + \Sigma_i b^{\tau}_{s_i}$$

We claim that (μ, w) is stable under budgets b given (v, c), so $v \in \mathcal{V}(\mu, w; b, c)$.

To show (μ, w) is stable, we need to show the following:

- (μ, w) is feasible:
  This is directly implied by the observed market structure $(\mu, w; b, c)$, so we have $b_{s_i} \geq w_{s_i,t_i}$.
- IR holds for all matched pairs:
  Teachers' IR are also directly implied by the observed market structure $(\mu, w; b, c)$, so we have $w_{s_i,t_i} \geq c_{s_i,t_i}$.
  Now we need to check schools' IR. By construction of V, we have

$$V > \Sigma_i(w_{s_i,t_i} - c_{s_i,t_i}) + \Sigma_i b^{\tau}_{s_i} \geq w_{s_i,t_i} - c_{s_i,t_i} \text{ for } \forall i$$

  Therefore, for school s_i, where $i \in \{1, \ldots, j\}$, by construction of v, we have

$$v_{s_i,t_i} - w_{s_i,t_i} \geq V - w_{s_i,t_i} + c_{s_i,t_i} > 0$$

  So IR holds for all school s_i where $i \in \{1, \ldots, j\}$.
  For any school s_i where $i \notin \{1, \ldots, j\}$, by construction of v, we have

$$v_{s_i,t_i} - w_{s_i,t_i} = 0$$

  So IR holds for any school s_i where $i \notin \{1, \ldots, j\}$. Therefore, IR holds for all schools.
- There is no blocking pairs:
  For any unmatched pair $(s, t) \neq (s_i, t_{i+1})$, where $i \notin \{0, 1, \ldots, j-1\}$, by construction of v, we have $v_{s,t} = c_{s,t}$. Then any deviating wage $w' > c_{s,t}$ will lead to a payoff $v_{s,t} - w' < 0$, which cannot block the current matching.
  For any unmatched pair (s_i, t_{i+1}), where $i \in \{1, \ldots, j-1\}$, any deviating wage w' need to satisfy

$$w' - c_{s_i,t_{i+1}} > w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}$$

  By construction of v, we have

$$v_{s_i,t_{i+1}} - w' = c_{s_i,t_{i+1}} + \left(w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}\right) + V - w' < V$$

  And s_i is matching with t_i with wage w_{s_i,t_i}, which gives a utility of

$$v_{s_i,t_i} - w_{s_i,t_i} = V > v_{s_i,t_{i+1}} - w'$$

  Therefore, (s_i, t_{i+1}), where $i \in \{1, \ldots, j-1\}$, cannot block the current matching.
  For (s_0, t_1), any deviating wage w' need to satisfy

$$w' - c_{s_0,t_1} > w_{s_1,t_1} - c_{s_1,t_1}$$

  By (L2-4), we have

$$w_{s_1,t_1} - c_{s_1,t_1} \geq b_{s_0} - c_{s_0,t_1}$$

  Then we have

$$w' > b_{s_0}$$

  So any deviating wage is not affordable, and (s_0, t_1) cannot block the current matching.

Therefore, the observed allocation (μ, w) is stable, and we have

$$v \in \mathcal{V}(\mu, w; b, c)$$

Now we claim

$$W(\mu'; v, c) < W(\mu; v, c) \quad \text{if} \quad \mu'(s_j) \neq t_j \tag{L2-6}$$

To show the above, by definition of $W(\mu'; v, c)$, we have

$$W(\mu'; v, c) = \Sigma_s (v_{s,\mu'(s)} - c_{s,\mu'(s)})$$

Then by construction of v, we have

$$v_{s,\mu'(s)} - c_{s,\mu'(s)} = \begin{cases} 0 & \text{if} \quad \mu'(s) \neq \mu(s) \\ w_{s,\mu(s)} - c_{s,\mu(s)} & \text{if} \quad \mu'(s) = \mu(s) \end{cases} \quad \text{for } s \notin \{s_0, s_1, ...s_j\}$$

Also, we know

$$v_{s,\mu(s)} - c_{s,\mu(s)} = w_{s,\mu(s)} - c_{s,\mu(s)} \quad \text{for } s \notin \{s_0, s_1, ...s_j\}$$

Then we have

$$v_{s,\mu'(s)} - c_{s,\mu'(s)} \leq v_{s,\mu(s)} - c_{s,\mu(s)} \quad \text{for } s \notin \{s_0, s_1, ...s_j\} \tag{L2-7}$$

Now consider $s_i \in \{s_0, s_1, ...s_{j-1}\}$. By construction of v, the best match we can form is matching s_i with t_{i+1}, which gives

$$v_{s_i,\mu'(s_i)} - c_{s_i,\mu'(s_i)} \leq v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} = (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) + V$$

Now consider s_j. Since $\mu'(s_j) \neq t_j$, by construction of v, we have

$$v_{s_j,\mu'(s_j)} - c_{s_j,\mu'(s_j)} = 0$$

Then we have

$$\Sigma_{s \in \{s_0, s_1 ... s_j\}} (v_{s,\mu'(s)} - c_{s,\mu'(s)}) \leq \Sigma_{s \in \{s_1 ... s_j\}} (w_{s,\mu(s)} - c_{s,\mu(s)}) + jV + 0$$

Now consider the welfare generated in μ for $s \in \{s_0, s_1 ... s_j\}$. By construction of v, we have

$$\Sigma_{s \in \{s_0, s_1 ... s_j\}} (v_{s,\mu(s)} - c_{s,\mu(s)}) = \Sigma_{s \in \{s_0, s_1 ... s_j\}} (w_{s,\mu(s)} - c_{s,\mu(s)}) + (j+1)V$$

By $V > 0$, and $w_{s_0,\mu(s_0)} - c_{s_0,\mu(s_0)} \geq 0$, we have

$$\Sigma_{s \in \{s_0, s_1 ... s_j\}} (v_{s,\mu'(s)} - c_{s,\mu'(s)}) < \Sigma_{s \in \{s_0, s_1 ... s_j\}} (v_{s,\mu(s)} - c_{s,\mu(s)}) \tag{L2-8}$$

Combining (L2-7) and (L2-8), we have

$$\begin{aligned} W(\mu'; v, c) &= \Sigma_s (v_{s,\mu'(s)} - c_{s,\mu'(s)}) \\ &= \Sigma_{s \notin \{s_0, s_1, ... s_j\}} (v_{s,\mu'(s)} - c_{s,\mu'(s)}) + \Sigma_{s \in \{s_0, s_1, ... s_j\}} (v_{s,\mu'(s)} - c_{s,\mu'(s)}) \\ &< \Sigma_{s \notin \{s_0, s_1, ... s_j\}} (v_{s,\mu(s)} - c_{s,\mu(s)}) + \Sigma_{s \in \{s_0, s_1, ... s_j\}} (v_{s,\mu(s)} - c_{s,\mu(s)}) \\ &= \Sigma_s (v_{s,\mu(s)} - c_{s,\mu(s)}) \\ &= W(\mu; v, c) \end{aligned}$$

Therefore, we have

$$W(\mu'; v, c) < W(\mu; v, c) \quad \text{if} \quad \mu'(s_j) \neq t_j \tag{L2-6}$$

Recall τ is non-distortive, so we have

$$\overline{W}(v, c, b, \tau) \geq W(\mu; v, c)$$

Since $W(\mu'; v, c) < W(\mu; v, c)$ for any matching μ' where $\mu'(s_j) \neq t_j$, there must be at least one stable allocation $(\tilde{\mu}, \tilde{w})$ where $\tilde{\mu}(s_j) = t_j$ after the subsidy.

Consider the above $(\tilde{\mu}, \tilde{w})$, we have

$$\tilde{\mu}(s_j) = t_j$$

By failure of SC at s_j, we have

$$b^{\tau}_{s_j} - w_{s_j,t_j} < \delta^C_j$$

which is equivalent to

$$b^{\tau}_{s_j} - c_{s_j,t_j} < (w_{s_j,t_j} - c_{s_j,t_j}) + \delta^C_j$$

Then the effective wage received by t_j satisfies

$$\tilde{w}_{s_j,t_j} - c_{s_j,t_j} \leq b^{\tau}_{s_j} - c_{s_j,t_j} < (w_{s_j,t_j} - c_{s_j,t_j}) + \delta^C_j$$

By definition of δ^C_j

$$\delta^C_j = \min\left\{\delta^C_{j-1}, b^{\tau}_{s_{j-1}} - c_{s_{j-1},t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\}$$

we have

$$b^{\tau}_{s_{j-1}} - c_{s_{j-1},t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) \geq \delta^C_j$$

which is equivalent to

$$b^{\tau}_{s_{j-1}} - c_{s_{j-1},t_j} \geq (w_{s_j,t_j} - c_{s_j,t_j}) + \delta^C_j$$

Hence s_{j-1} can afford to offer t_j an effective wage of $(w_{s_j,t_j} - c_{s_j,t_j}) + \delta^C_j$, strictly exceeding t_j's current effective wage at s_j. By our construction of v, matching t_j with the above effective wage will give s_{j-1} a utility of

$$c_{s_{j-1},t_j} + (w_{s_j,t_j} - c_{s_j,t_j}) + V - \left([w_{s_j,t_j} - c_{s_j,t_j}] + \delta^C_j + c_{s_{j-1},t_j}\right) = V - \delta^C_j$$

Therefore the utility of s_{j-1} in $(\tilde{\mu}, \tilde{w})$ must be strictly higher than $V - \delta^C_j$ (it is strict because $b^{\tau}_{s_j} - w_{s_j,t_j} < \delta^C_j$ is strict). Recall our construction of V, we then have

$$V > \Sigma_i (w_{s_i,t_i} - c_{s_i,t_i}) + \Sigma_i b^{\tau}_{s_i} \geq b^{\tau}_{s_1}$$

By Lemma 1, DC holds for s_1, we have

$$b^{\tau}_{s_1} - w_{s_1,t_1} \geq \Delta^d_1 \quad \Rightarrow \quad b^{\tau}_{s_1} \geq \Delta^d_1 \tag{L2-9}$$

By definition of δ^C_j along the spillover chain C, we have

$$\Delta^d_1 \geq \delta^C_2 \geq \delta^C_3 ... \geq \delta^C_j \quad \Rightarrow \quad b^{\tau}_{s_1} \geq \delta^C_j$$

So we have

$$V - \delta^C_j > 0$$

meaning that s_{j-1} is enjoying a strictly positive utility in $(\tilde{\mu}, \tilde{w})$. Recall our construction of v, for any teacher $t \neq t_{j-1}, t_j$, we have

$$v_{s_{j-1},t} = c_{s_{j-1},t}$$

Then teacher's IR will leave s_{j-1} a non-positive utility. Also, we know $\tilde{\mu}(s_j) = t_j$, so the only possible matching giving s_{j-1} a positive utility is $\tilde{\mu}(s_{j-1}) = t_{j-1}$. Since utility of s_{j-1} in $(\tilde{\mu}, \tilde{w})$ is strictly higher than $V - \delta^C_j$, we have

$$v_{s_{j-1},t_{j-1}} - \tilde{w}_{s_{j-1},t_{j-1}} > V - \delta^C_j$$

Recall our construction of v, and $\delta_j^C \leq \delta_{j-1}^C$ by construction, we have

$$\tilde{w}_{s_{j-1},t_{j-1}} < v_{s_{j-1},t_{j-1}} - V + \delta_j^C = w_{s_{j-1},t_{j-1}} + \delta_j^C \leq w_{s_{j-1},t_{j-1}} + \delta_{j-1}^C$$

Thus the effective wage received by t_{j-1} satisfies

$$\tilde{w}_{s_{j-1},t_{j-1}} - c_{s_{j-1},t_{j-1}} < w_{s_{j-1},t_{j-1}} - c_{s_{j-1},t_{j-1}} + \delta_{j-1}^C$$

By the same logic, s_{j-2} can afford the RHS for t_{j-1} to enjoy a positive utility, meaning that s_{j-2} is matching with t_{j-2} in $(\tilde{\mu}, \tilde{w})$. And the wage of t_{j-2} satisfies

$$\tilde{w}_{s_{j-2},t_{j-2}} - c_{s_{j-2},t_{j-2}} < w_{s_{j-2},t_{j-2}} - c_{s_{j-2},t_{j-2}} + \delta_{j-2}^C$$

Continuing this induction yields $\tilde{\mu}(s_1) = t_1$, and

$$\tilde{w}_{s_1,t_1} - c_{s_1,t_1} < w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d$$

Recall (L2-5), s_0 can afford the RHS effective wage to t_1. By our construction of v,

$$v_{s_0,t_0} = w_{s_0,t_0}, \qquad v_{s_0,t_1} = c_{s_0,t_1} + V, \qquad v_{s_0,t_i} = c_{s_0,t_i}, \quad \text{for } i \neq 0, 1$$

The highest utility s_0 can enjoy (when not matching t_1) in $(\tilde{\mu}, \tilde{w})$ will be

$$w_{s_0,t_0} - v_{s_0,t_0}$$

However, matching t_1 with a wage of $c_{s_0,t_1} + w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d$ will give a utility of

$$v_{s_0,t_1} - (c_{s_0,t_1} + w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d)$$

Recall our construction of v, the above quantity is

$$v_{s_0,t_1} - (c_{s_0,t_1} + w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d) = V - (w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d)$$

Recall V is large enough and (L2-9)

$$V > \Sigma_i(w_{s_i,t_i} - c_{s_i,t_i}) + \Sigma_i b_{s_i}^\tau \geq w_{s_0,t_0} - v_{s_0,t_0} + w_{s_1,t_1} - c_{s_1,t_1} + b_{s_1}^\tau \geq w_{s_0,t_0} - v_{s_0,t_0} + w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d$$

Then we have

$$V - (w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d) > w_{s_0,t_0} - v_{s_0,t_0}$$

Therefore, deviating to t_1 with a wage of $c_{s_0,t_1} + w_{s_1,t_1} - c_{s_1,t_1} + \Delta_1^d$ will be strictly profitable for s_0 and t_1, contradicting the stability of $(\tilde{\mu}, \tilde{w})$. Therefore, s_j cannot be matched with t_j after the subsidy, and (L2-6) will contradict τ being non-distortive, which finishes the proof. □

Proof of Theorem 1.

Proof. Lemma 1 and Lemma 2 establish necessity. It remains to prove sufficiency.

Let (μ, w), b, and c denote the pre-subsidy allocation, budgets, and costs. For notational convenience, relabel so that $\mu(s_i) = t_i$ for all indices that appear below.

Fix any value profile $v \in \mathcal{V}(\mu, w; b, c)$ and a subsidy allocation τ that satisfies DC and SC. Consider the post-subsidy school-optimal stable allocation under tie-breaking in favor of μ (schools and teachers weakly prefer their μ-partners when indifferent). Let (μ', w') be the resulting allocation under budgets b^τ and costs c.

It suffices to show $W(\mu'; v, c) \geq W(\mu; v, c)$ for every $v \in \mathcal{V}(\mu, w; b, c)$, because then

$$\overline{W}(v, c, b, \tau) \;\geq\; W(\mu'; v, c) \;\geq\; W(\mu; v, c)$$

so τ is non-distortive.

We now show

$$W(\mu'; v, c) \geq W(\mu; v, c)$$

by contradiction. Suppose

$$W(\mu'; v, c) < W(\mu; v, c)$$

Step 1:

In this step, we show there exists a welfare-decreasing cycle $\mathcal{C}$.

Since

$$W(\mu'; v, c) < W(\mu; v, c)$$

Then we have $\mu \neq \mu'$, otherwise the welfare of μ, μ' should be the same. Given $\mu \neq \mu'$, we need the following definition

Definition 15 (Cycle and fixed points)**.** For two different matchings μ and μ'. Define the permutation operator

$$\sigma(s) = \mu^{-1}(\mu'(s))$$

A set of distinct schools $\{s_1, s_2 \ldots, s_m\}$ (after reindexing) forms a **cycle** of length m if

$$\sigma(s_i) = s_{i+1} \text{ for } i = 1, \ldots, m-1 \quad \text{and} \quad \sigma(s_m) = s_1$$

When $m = 1$ we have $\sigma(s) = s$, equivalently $\mu'(s) = \mu(s)$. We call such s a **fixed point**.

Since we have $\mu(s_i) = t_i$, we can WOLG let each cycle include both schools and teachers because they share the same set of indices.

Given the definition above, since the market is finite, we can partition S uniquely into different cycles with length $m \geq 2$ and fixed points.

Now we claim there must be at least one cycle with length $m \geq 2$

$$\mathcal{C} = \{s_1, \ldots, s_m, t_1, \ldots, t_m\}$$

where the welfare change is negative, that is

$$\sum_{i=1}^{m}\left(v_{s_i,t_{i+1}}-c_{s_i,t_{i+1}}\right)<\sum_{i=1}^{m}\left(v_{s_i,t_i}-c_{s_i,t_i}\right)\quad\text{with}\quad t_{m+1}\equiv t_1$$

We show the above claim by contradiction. Suppose for all cycles with length $m\geq 2$, we have

$$\sum_{i=1}^{m}\left(v_{s_i,t_{i+1}}-c_{s_i,t_{i+1}}\right)\geq\sum_{i=1}^{m}\left(v_{s_i,t_i}-c_{s_i,t_i}\right)$$

We also know the welfare of a fixed point does not change

$$v_{s,\mu(s)}-c_{s,\mu(s)}=v_{s,\mu'(s)}-c_{s,\mu'(s)}\quad\text{if}\quad\mu(s)=\mu'(s)$$

Summing over all cycles, we have

$$\Sigma_s(v_{s,\mu(s)}-c_{s,\mu(s)})\geq\Sigma_s(v_{s,\mu'(s)}-c_{s,\mu'(s)})\quad\Rightarrow\quad W(\mu';v,c)\geq W(\mu;v,c)$$

contradicting our hypothesis. Therefore, there exists a cycle with length $m\geq 2$

$$\mathcal{C}=\{s_1,\ldots,s_m,t_1,\ldots,t_m\}$$

and

$$\sum_{i=1}^{m}\left(v_{s_i,t_{i+1}}-c_{s_i,t_{i+1}}\right)<\sum_{i=1}^{m}\left(v_{s_i,t_i}-c_{s_i,t_i}\right)\quad\text{with}\quad t_{m+1}\equiv t_1\tag{T1-1}$$

And the matching pairs are

$$\begin{cases}\mu:\ (s_1,t_1),(s_2,t_2),\ldots,(s_m,t_m)\\ \mu':\ (s_1,t_2),(s_2,t_3),\ldots,(s_m,t_1)\end{cases}$$

Step 2:

In this step, we show the following cannot be true in $\mathcal{C}$:

$$\exists\ i\text{ such that}\quad b_{s_i}-c_{s_i,t_i i+1}\leq w_{s_{i+1},t_{i+1}}-c_{s_{i+1},t_{i+1}}\quad\text{where}\quad s_i\in\mathcal{C}$$

that is, there is no $s_i\in\mathcal{C}$ such that, s_i is outbid by s_{i+1} in (μ,w).

We show the above by contradiction. Suppose there is an index i such that

$$b_{s_i}-c_{s_i,t_i i+1}\leq w_{s_{i+1},t_{i+1}}-c_{s_{i+1},t_{i+1}}$$

WOLG, let $i=1$ by relabeling, we then have

$$b_{s_1}-c_{s_1,t_2}\leq w_{s_2,t_2}-c_{s_2,t_2}$$

Now we want to show the following affordability conditions

$$b^{\tau}_{s_{i+1}}-c_{s_{i+1},t_{i+1}}\geq w'_{s_i,t_{i+1}}-c_{s_i,t_{i+1}}\quad\text{for}\quad i=1,2,...m\quad\text{with}\quad t_{m+1}\equiv t_1\tag{AFF}$$

Proof. We show (AFF) by induction in three steps.

- Step 2.1: We first show $b^\tau_{s_2} - c_{s_2,t_2} \;\geq\; w'_{s_1,t_2} - c_{s_1,t_2}$.
  Since s_1 was outbid by s_2 in (μ, w), DC gives

$$b^\tau_{s_2} - w_{s_2,t_2} \geq \Delta_2^d \geq (b^\tau_{s_1} - c_{s_1,t_2}) - (w_{s_2,t_2} - c_{s_2,t_2}) \geq (w'_{s_1,t_2} - c_{s_1,t_2}) - (w_{s_2,t_2} - c_{s_2,t_2})$$

  where the last inequality is by feasibility of (s_1, t_2) in (μ', w'). Rearranging gives

$$b^\tau_{s_2} - c_{s_2,t_2} \geq w'_{s_1,t_2} - c_{s_1,t_2}$$

  which is the desired inequality.
- Step 2.2: Now we show $b^\tau_{s_3} - c_{s_3,t_3} \geq w'_{s_2,t_3} - c_{s_2,t_3}$.
  Split by whether s_2 is outbid by s_3 in μ.
  - Case 2.2.1: If s_2 is outbid by s_3 in (μ, w), we have

$$b_{s_2} - c_{s_2,t_3} \leq w_{s_3,t_3} - c_{s_3,t_3}$$

    then DC gives

$$b^\tau_{s_3} - w_{s_3,t_3} \geq \Delta_3^d \geq (b^\tau_{s_2} - c_{s_2,t_3}) - (w_{s_3,t_3} - c_{s_3,t_3}) \geq (w'_{s_2,t_3} - c_{s_2,t_3}) - (w_{s_3,t_3} - c_{s_3,t_3})$$

    hence

$$b^\tau_{s_3} - c_{s_3,t_3} \geq w'_{s_2,t_3} - c_{s_2,t_3}$$

  - Case 2.2.2: If s_2 is not outbid by s_3 in (μ, w), then we have

$$b_{s_2} - c_{s_2,t_3} > w_{s_3,t_3} - c_{s_3,t_3}$$

    Then $s_2 \xrightarrow{\mu} s_3$ forms a spillover chain C. Because τ satisfies SC,

$$b^\tau_{s_3} - w_{s_3,t_3} \geq \delta_3^C \tag{T1-2.1}$$

    where

$$\delta_3^C = \min\left\{\Delta_2, b^\tau_{s_2} - c_{s_2,t_3} - (w_{s_3,t_3} - c_{s_3,t_3})\right\}$$

    If δ_3^C takes the latter value, then by feasibility $b^\tau_{s_2} \geq w'_{s_2,t_3}$ and the above, we have

$$b^\tau_{s_3} - w_{s_3,t_3} \geq \delta_3^C = b^\tau_{s_2} - c_{s_2,t_3} - (w_{s_3,t_3} - c_{s_3,t_3}) \geq w'_{s_2,t_3} - c_{s_2,t_3} - (w_{s_3,t_3} - c_{s_3,t_3})$$

    which rearranges to the desired inequality

$$b^\tau_{s_3} - c_{s_3,t_3} \geq w'_{s_2,t_3} - c_{s_2,t_3}$$

    If δ_3^C takes the former value Δ_2, then by definition of Δ_2 and feasibility $b^\tau_{s_1} \geq w'_{s_1,t_2}$, we have

$$\delta_3^C = \Delta_2 \geq \left[(b^\tau_{s_1} - c_{s_1,t_2}) - (w_{s_2,t_2} - c_{s_2,t_2})\right] \geq (w'_{s_1,t_2} - c_{s_1,t_2}) - (w_{s_2,t_2} - c_{s_2,t_2}) \tag{T1-2.2}$$

    Pre-subsidy stability of (s_2, t_2) vs. (s_2, t_3) implies

$$v_{s_2,t_2} - w_{s_2,t_2} \;\geq\; v_{s_2,t_3} - w_{s_3,t_3} + c_{s_3,t_3} - c_{s_2,t_3}$$

    By step 2.1 and tie-breaking in favor of μ, s_2 can match w'_{s_1,t_2}. So post-subsidy stability of (s_2, t_3) vs. (s_2, t_2) requires

$$v_{s_2,t_3} - w'_{s_2,t_3} \geq v_{s_2,t_2} - w'_{s_1,t_2} + c_{s_1,t_2} - c_{s_2,t_2}$$

Combining yields

$$(w'_{s_1,t_2} - c_{s_1,t_2}) - (w_{s_2,t_2} - c_{s_2,t_2}) \;\geq\; (w'_{s_2,t_3} - c_{s_2,t_3}) - (w_{s_3,t_3} - c_{s_3,t_3}) \tag{$*_{2\to3}$}$$

Using $(*_{2\to3})$, (T1-2.1) and (T1-2.2), we have

$$b^{\tau}_{s_3} - w_{s_3,t_3} \geq \delta^C_3 \geq (w'_{s_1,t_2} - c_{s_1,t_2}) - (w_{s_2,t_2} - c_{s_2,t_2}) \geq w'_{s_2,t_3} - c_{s_2,t_3} - (w_{s_3,t_3} - c_{s_3,t_3})$$

which rearranges to the desired inequality

$$b^{\tau}_{s_3} - c_{s_3,t_3} \;\geq\; w'_{s_2,t_3} - c_{s_2,t_3}$$

- Step 2.3: Now we show $b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \;\geq\; w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}$ for $i = 3, \ldots, m$.
  Assume the inequality holds for $i-1$.
  If s_i is outbid by s_{i+1} in (μ, w), case 2.2.1 will give the desired inequality.
  If s_i is not outbid by s_{i+1} in (μ, w), then find the largest index q such that s_{q-1} is outbid by s_q, and $s_q \xrightarrow{\mu} s_{q+1} \ldots \xrightarrow{\mu} s_{i+1}$ forms a spillover chain. Such q exists because we know

  $$b_{s_1} - c_{s_1,t_2} \leq w_{s_2,t_2} - c_{s_2,t_2}$$

  Given the spillover chain $s_q \xrightarrow{\mu} s_{q+1} \ldots \xrightarrow{\mu} s_{i+1}$, SC gives

  $$b^{\tau}_{s_{i+1}} - w_{s_{i+1},t_{i+1}} \geq \delta^C_{i+1} \tag{T1-2.3}$$

  Then check δ^C_{i+1}, we have

  $$\delta^C_{i+1} = \min\left\{\delta^C_i, b^{\tau}_{s_i} - c_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}})\right\}$$

  If δ^C_{i+1} takes the latter value, (T1-2.3) and feasibility give the following:

  $$b^{\tau}_{s_{i+1}} - w_{s_{i+1},t_{i+1}} \geq \delta^C_{i+1} = b^{\tau}_{s_i} - c_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) \geq w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}})$$

  and rearranging gives the desired inequality

  $$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \;\geq\; w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}$$

  If δ^C_{i+1} takes the former value, then

  $$\delta^C_{i+1} = \delta^C_i$$

  By definition of δ^C_i, we have

  $$\delta^C_i = \min\left\{\delta^C_{i-1}, b^{\tau}_{s_{i-1}} - c_{s_{i-1},t_i} - (w_{s_i,t_i} - c_{s_i,t_i})\right\}$$

  There will be two cases:
  - Case 2.3.1: If δ^C_i takes the latter value, then

    $$\delta^C_i = b^{\tau}_{s_{i-1}} - c_{s_{i-1},t_i} - (w_{s_i,t_i} - c_{s_i,t_i}) \geq w'_{s_{i-1},t_i} - c_{s_{i-1},t_i} - (w_{s_i,t_i} - c_{s_i,t_i}) \tag{T1-2.4}$$

    Now recall the induction hypothesis, pre-subsidy stability of (s_i, t_i) vs. (s_i, t_{i+1}), and post-subsidy stability of (s_i, t_{i+1}) vs. (s_i, t_i), we can derive the following similar to case 2.2.2

    $$(w'_{s_{i-1},t_i} - c_{s_{i-1},t_i}) - (w_{s_i,t_i} - c_{s_i,t_i}) \geq (w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) \tag{$*_{i\to i+1}$}$$

    (T1-2.4) and $(*_{i\to i+1})$ will give

    $$b^{\tau}_{s_{i+1}} - w_{s_{i+1},t_{i+1}} \geq \delta^C_{i+1} = \delta^C_i \geq (w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}})$$

    which rearranges to the desired inequality

    $$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \;\geq\; w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}$$

– Case 2.3.2: If δ_i^C takes the former value, then check δ_{i-1}^C. We either combine $(*_{i-1\to i})$ and $(*_{i\to i+1})$ to get the desired inequality or move to δ_{i-2}^C. This backward induction terminates at index q. DC of Δ_q^d and telescoping $(*)$ will give the desired inequality.

This completes the proof of (AFF). $\square$

By (AFF), for each i, the contract $(s_{i+1}, t_{i+1}, w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} + c_{s_{i+1},t_{i+1}})$ is feasible. At that wage, t_{i+1} enjoys the same effective wage. Tie–breaking in favor of μ implies t_{i+1} will accept the offer. Therefore, in the post–subsidy allocation (μ', w'), stability requires that s_{i+1} does not want to switch from t_{i+2} to t_{i+1} at wage $(w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} + c_{s_{i+1},t_{i+1}})$, which gives

$$v_{s_{i+1},t_{i+2}} - w'_{s_{i+1},t_{i+2}} > v_{s_{i+1},t_{i+1}} - w'_{s_i,t_{i+1}} + c_{s_i,t_{i+1}} - c_{s_{i+1},t_{i+1}} \qquad (i = 1, \ldots, m)$$

Summing these m inequalities gives the following

$$\sum_{i=1}^{m} (v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) > \sum_{i=1}^{m} (v_{s_i,t_i} - c_{s_i,t_i})$$

But by step 1, we know

$$\sum_{i=1}^{m} (v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) < \sum_{i=1}^{m} (v_{s_i,t_i} - c_{s_i,t_i})$$

which is a contradiction. Therefore, the hypothesis at the beginning of step 2 cannot be true, which gives

$$b_{s_i} - c_{s_i,t_{i+1}} > w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} \quad \text{for all} \quad s_i \in \mathcal{C}$$

Step 3:

In this step, we want to show there exists i such that

$$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \leq w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}$$

meaning that s_{i+1} is outbid by s_i in (μ', w').

We again prove by contradiction. Suppose the following

$$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} > w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} \quad \text{for } i = 1, 2, ..., m$$

Then every school s_i can match t_i with her current effective wage. Stability of (μ', w') then requires

$$v_{s_{i+1},t_{i+2}} - w'_{s_{i+1},t_{i+2}} > v_{s_{i+1},t_{i+1}} - w'_{s_i,t_{i+1}} + c_{s_i,t_{i+1}} - c_{s_{i+1},t_{i+1}} \quad \text{for } i = 1, 2, ..., m$$

Summing these m inequalities gives

$$\sum_{i=1}^{m} (v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) > \sum_{i=1}^{m} (v_{s_i,t_i} - c_{s_i,t_i})$$

But by step 1, we know

$$\sum_{i=1}^{m} (v_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) < \sum_{i=1}^{m} (v_{s_i,t_i} - c_{s_i,t_i})$$

a contradiction. Therefore there exists i with

$$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \leq w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}$$

Step 4: Utility drop of s_i

In this step, we derive the utility drop of s_i, where i is the index at the end of step 3.

Let i be the index at the end of step 3, we then have

$$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \leq w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} \tag{T1-4.1}$$

Recall τ satisfies DC and SC, which implies

$$b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \geq (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) + \Lambda_{i+1} \tag{T1-4.2}$$

where by definition, we have

$$\Lambda_{i+1} \equiv \max\{\Delta^d_{i+1}, \Delta^s_{i+1}\}$$

Combining (T1-4.1), (T1-4.2) and feasibility of $w'_{s_i,t_{i+1}}$, we have

$$b^{\tau}_{s_i} - c_{s_i,t_{i+1}} \geq w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} \geq b^{\tau}_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \geq (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) + \Lambda_{i+1} \tag{T1-4.3}$$

Now we define

$$\Lambda_{i\to j} \equiv \min\Big\{\Lambda_i,\ b^{\tau}_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\}$$

which denotes the spillover effect transmitted at $s_i \xrightarrow{\mu} s_j$. And we develop the following lemma

Lemma 3. *If $s_i \xrightarrow{\mu} s_j$, then*

$$\Lambda_j \geq \Lambda_{i\to j}$$

Proof. For any spillover chain $C : s_1 \xrightarrow{\mu} s_2 ... \xrightarrow{\mu} s_i$ ending at s_i, there is another spillover chain $C' : s_1 \xrightarrow{\mu} s_2 ... \xrightarrow{\mu} s_i \xrightarrow{\mu} s_j$. Therefore, by definition of spillover effects

$$\begin{aligned}
\Delta^s_j &\equiv \sup_{C\text{: spillover chains ending at } s_j} \delta^C_j \\
&\geq \sup_{C\text{: spillover chains ending at } s_i \text{ then } s_j} \delta^C_j \\
&= \sup_{C\text{: spillover chains ending at } s_i} \min\Big\{\delta^C_i,\ b^{\tau}_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\}
\end{aligned}$$

Since $b^{\tau}_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)$ is a constant, after changing the order of min and sup, we have

$$\begin{aligned}
\Delta^s_j &\geq \sup_{C\text{: spillover chains ending at } s_i} \min\Big\{\delta^C_i,\ b^{\tau}_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\} \\
&= \min\Big\{\sup_{C\text{: spillover chains ending at } s_i} \delta^C_i,\ b^{\tau}_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\} \\
&= \min\Big\{\Delta^s_i,\ b^{\tau}_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\} \qquad \text{(L3-1)}
\end{aligned}$$

Also, since $s_i \xrightarrow{\mu} s_j$, definition of Δ^s_j gives

$$\Delta^s_j \geq \min\Big\{\Delta^d_i,\ b^{\tau}_{s_i} - c_{s_i,t_j} - \big(w_{s_j,t_j} - c_{s_j,t_j}\big)\Big\} \tag{L3-2}$$

There will be two cases:

- $b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) \geq \max\{\Delta^d_i, \Delta^s_i\}$
  Then we have
$$\Delta^s_j \geq \min\left\{\Delta^d_i,\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\} = \Delta^d_i$$
  and
$$\Delta^s_j \geq \min\left\{\Delta^s_i,\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\} = \Delta^s_i$$
  Then we have
$$\Delta^s_j \geq \max\{\Delta^d_i, \Delta^s_i\} \geq \min\left\{\max\{\Delta^d_i, \Delta^s_i\},\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\}$$
  where the last term is
$$\min\left\{\max\{\Delta^d_i, \Delta^s_i\},\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\} = \min\left\{\Lambda_i,\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\} = \Lambda_{i\to j}$$
  So we have
$$\Delta^s_j \geq \Lambda_{i\to j}$$
- $b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) < \max\{\Delta^d_i, \Delta^s_i\}$
  Then one of (L3-1), (L3-2) will imply
$$\Delta^s_j \geq b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) = \min\left\{\max\{\Delta^d_i, \Delta^s_i\},\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\}$$
  The last term is again
$$\min\left\{\max\{\Delta^d_i, \Delta^s_i\},\ b^\tau_{s_i} - c_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\} = \Lambda_{i\to j}$$
  Then we have
$$\Delta^s_j \geq \Lambda_{i\to j}$$

So in both cases, we have $\Delta^s_j \geq \Lambda_{i\to j}$. Recall definition of Λ_j, we have
$$\Lambda_j \equiv \max\{\Delta^d_j, \Delta^s_j\} \geq \Delta^s_j \geq \Lambda_{i\to j} \tag{L3-3}$$
which finishes the proof of Lemma 3. □

Recall step 2, we have $s_i \xrightarrow{\mu} s_{i+1}$. Then by Lemma 3, we have
$$\Lambda_{i+1} \geq \Lambda_{i\to i+1} \equiv \min\left\{\Lambda_i,\ b^\tau_{s_i} - c_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}})\right\}$$
where $\Lambda_{i\to i+1}$ denotes the spillover effect transmitted at $s_i \xrightarrow{\mu} s_{i+1}$.

There will be two cases:

- Case 4.1: If $\Lambda_{i\to i+1}$ takes the latter value, meaning that spillover effects from s_i to s_{i+1} is capped by post-subsidy budgets, then we have
$$\Lambda_{i+1} \geq \Lambda_{i\to i+1} = b^\tau_{s_i} - c_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) \tag{T1-4.4}$$
  combine (T1-4.3), (T1-4.4) and feasibility of (s_i, t_{i+1}, w'), we have
$$b^\tau_{s_{i+1}} - c_{s_{i+1},t_{i+1}} \geq (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) + \Lambda_{i+1} \geq b^\tau_{s_i} - c_{s_i,t_{i+1}} \geq w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}$$

meaning that s_{i+1} can afford the effective wage of t_{i+1} in (μ', w'). Therefore, by tie-breaking in favor of μ, s_{i+1} must strictly prefer his current matching (s_{i+1}, t_{i+2}). This post-subsidy stability gives

$$v_{s_{i+1},t_{i+2}} - w'_{s_{i+1},t_{i+2}} > v_{s_{i+1},t_{i+1}} - (w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} + c_{s_{i+1},t_{i+1}}) \tag{T1-4.5}$$

By step 2, we have $s_{i+1} \xrightarrow{\mu} s_{i+2}$, pre-subsidy stability requires

$$v_{s_{i+1},t_{i+1}} - w_{s_{i+1},t_{i+1}} \geq v_{s_{i+1},t_{i+2}} - (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}} + c_{s_{i+1},t_{i+2}}) \tag{T1-4.6}$$

combine (T1-4.5) and (T1-4.6) and rearrange, we have

$$(w'_{s_{i+1},t_{i+2}} - c_{s_{i+1},t_{i+2}}) - (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}}) < (w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}})$$

where the LHS is the effective wage change of t_{i+2}, the RHS is the effective wage change of t_{i+1}. Recall (T1-4.4) and feasibility $b^\tau_{s_i} \geq w'_{s_i,t_{i+1}}$, we have

$$RHS \leq b^\tau_{s_i} - c_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) = \Lambda_{i\to i+1}$$

which gives

$$(w'_{s_{i+1},t_{i+2}} - c_{s_{i+1},t_{i+2}}) - (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}}) < \Lambda_{i\to i+1} \tag{T1-4.7}$$

We now show s_{i+2} can afford this effective wage $(w'_{s_{i+1},t_{i+2}} - c_{s_{i+1},t_{i+2}})$ to t_{i+2}. Consider the post-subsidy budget of school s_{i+2}, by DC and SC, we have

$$b^\tau_{s_{i+2}} - c_{s_{i+2},t_{i+2}} \geq (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}}) + \Lambda_{i+2} \tag{T1-4.8}$$

and by $s_{i+1} \xrightarrow{\mu} s_{i+2}$ and Lemma 3, we have

$$\Lambda_{i+2} \geq \Lambda_{i+1\to i+2} \equiv \min\left\{\Lambda_{i+1},\ b^\tau_{s_{i+1}} - c_{s_{i+1},t_{i+2}} - (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}})\right\}$$

If $\Lambda_{i+1\to i+2}$ takes the former value, $\Lambda_{i+2} \geq \Lambda_{i+1\to i+2} = \Lambda_{i+1} \geq \Lambda_{i\to i+1}$, then (T1-4.7), (T1-4.8) will give the following

$$b^\tau_{s_{i+2}} - c_{s_{i+2},t_{i+2}} \geq (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}}) + \Lambda_{i+2} \geq (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}}) + \Lambda_{i\to i+1} > w'_{s_{i+1},t_{i+2}} - c_{s_{i+1},t_{i+2}}$$

If $\Lambda_{i+1\to i+2}$ takes the latter value, then (T1-4.8) will give

$$b^\tau_{s_{i+2}} - c_{s_{i+2},t_{i+2}} \geq (w_{s_{i+2},t_{i+2}} - c_{s_{i+2},t_{i+2}}) + \Lambda_{i+2} \geq b^\tau_{s_{i+1}} - c_{s_{i+1},t_{i+2}} \geq w'_{s_{i+1},t_{i+2}} - c_{s_{i+1},t_{i+2}}$$

Therefore, in both cases, s_{i+2} can afford an effective wage of $w'_{s_{i+1},t_{i+2}} - c_{s_{i+1},t_{i+2}}$ to t_{i+2}. Then post-subsidy stability requires that s_{i+2} must strictly prefer his current matching (s_{i+2}, t_{i+3}). Similar to the logic in (T1-4.5), (T1-4.6) and (T1-4.7). We can get

$$(w'_{s_{i+2},t_{i+3}} - c_{s_{i+2},t_{i+3}}) - (w_{s_{i+3},t_{i+3}} - c_{s_{i+3},t_{i+3}}) < \Lambda_{i\to i+1}$$

Keep the above induction along the cycle $\mathcal{C}$, we finally circle back to index $i-1$ and have

$$(w'_{s_{i-1},t_i} - c_{s_{i-1},t_i}) - (w_{s_i,t_i} - c_{s_i,t_i}) < \Lambda_{i\to i+1} \tag{T1-4.9}$$

Similarly, by DC and SC, we have

$$b^\tau_{s_i} - w_{s_i,t_i} \geq \Lambda_i \tag{T1-4.10}$$

Recall definition of $\Lambda_{i\to i+1}$, we have

$$\Lambda_{i\to i+1} \equiv \min\Big\{\Lambda_i,\ b^{\tau}_{s_i} - c_{s_i,t_{i+1}} - \big(w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}\big)\Big\} \leq \Lambda_i \tag{T1-4.11}$$

By (T1-4.9), (T1-4.10), (T1-4.11), we have

$$b^{\tau}_{s_i} - w_{s_i,t_i} \geq \Lambda_i \geq \Lambda_{i\to i+1} > (w'_{s_{i-1},t_i} - c_{s_{i-1},t_i}) - (w_{s_i,t_i} - c_{s_i,t_i})$$

Therefore, s_i can afford an effective wage of $(w'_{s_{i-1},t_i} - c_{s_{i-1},t_i})$ to t_i. Post-subsidy stability requires

$$v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}} > v_{s_i,t_i} - (w'_{s_{i-1},t_i} - c_{s_{i-1},t_i} + c_{s_i,t_i}) \tag{T1-4.12}$$

Also, by step 2, $s_i \xrightarrow{\mu} s_{i+1}$, pre-subsidy stability requires

$$v_{s_i,t_i} - w_{s_i,t_i} \geq v_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} + c_{s_i,t_{i+1}}) \tag{T1-4.13}$$

combine (T1-4.12), (T1-4.13) and rearrange, we have

$$(w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}}) - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}) < (w'_{s_{i-1},t_i} - c_{s_{i-1},t_i}) - (w_{s_i,t_i} - c_{s_i,t_i})$$

where the LHS is the effective wage change of t_{i+1}, the RHS is the effective wage change of t_i. Recall (T1-4.1) and (T1-4.2) for the LHS and (T1-4.9) for the RHS, we have

$$\Lambda_{i+1} \leq LHS < RHS \leq \Lambda_{i\to i+1} \leq \Lambda_{i+1}$$

which is a contradiction. Therefore, $\Lambda_{i\to i+1}$ takes the former value

$$\Lambda_{i\to i+1} = \Lambda_i$$

- Case 4.2: Given case 4.1, we have

  $$\Lambda_{i\to i+1} = \Lambda_i$$

  By Lemma 3

  $$\Lambda_{i+1} \geq \Lambda_{i\to i+1} = \Lambda_i$$

  then recall (T1-4.1), (T1-4.2), we have

  $$w'_{s_i,t_{i+1}} - c_{s_i,t_{i+1}} \geq \big(w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}\big) + \Lambda_{i+1} \geq \big(w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}}\big) + \Lambda_i \tag{T1-4.14}$$

(T1-4.14) shows that the effective wage of t_{i+1} increases by at least Λ_i. Recall $s_i \xrightarrow{\mu} s_{i+1}$, pre-subsidy stability requires

$$v_{s_i,t_i} - w_{s_i,t_i} \geq v_{s_i,t_{i+1}} - (w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} + c_{s_i,t_{i+1}}) \tag{T1-4.15}$$

Combine (T1-4.14) and (T1-4.15), we have

$$v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}} \leq v_{s_i,t_i} - w_{s_i,t_i} - \Lambda_i \tag{T1-4.16}$$

(T1-4.16) is the key observation in step 4. It shows that s_i suffers at least a utility drop of its own wage-lifting pressure. Intuitively, it means that s_i is paying s_{i+1} an unnecessarily high wage in (μ', w').

Step 5: Utility drop of all schools in $\mathcal{C}$

In step 5, our target is to show each school in cycle $\mathcal{C}$ suffers a huge utility loss, which is the following

$$v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}} \le v_{s_i,t_i} - w_{s_i,t_i} - \Lambda_i \qquad i = 1, \ldots, m \tag{T1-5.1}$$

We show (T1-5.1) by induction, step 4 shows that (T1-5.1) is true for index i. Now consider school s_{i-1}, and we first show the following

$$\Lambda_{i-1\to i} \equiv \min\Big\{\Lambda_{i-1},\ b^{\tau}_{s_{i-1}} - c_{s_{i-1},t_i} - \big(w_{s_i,t_i} - c_{s_i,t_i}\big)\Big\} = \Lambda_{i-1} \tag{T1-5.2}$$

We show the above by contradiction, that is, assume

$$\Lambda_{i-1\to i} = b^{\tau}_{s_{i-1}} - c_{s_{i-1},t_i} - \big(w_{s_i,t_i} - c_{s_i,t_i}\big) < \Lambda_{i-1}$$

Then consider the effective wage of t_i in (μ', w'), we have

$$w'_{s_{i-1},t_i} - c_{s_{i-1},t_i} \le b^{\tau}_{s_{i-1}} - c_{s_{i-1},t_i} = w_{s_i,t_i} - c_{s_i,t_i} + \Lambda_{i-1\to i} \tag{T1-5.3}$$

meaning that the effective wage of s_i increases at most $\Lambda_{i-1\to i}$.

By τ satisfying DC and SC, we have

$$b^{\tau}_{s_i} - w_{s_i,t_i} \ge \Lambda_i$$

By step 2, $s_{i-1} \xrightarrow{\mu} s_i$. Then by Lemma 3, we have

$$b^{\tau}_{s_i} - w_{s_i,t_i} \ge \Lambda_i \ge \Lambda_{i-1\to i} \tag{T1-5.4}$$

Combine (T1-5.3), (T1-5.4) and rearrange

$$b^{\tau}_{s_i} - c_{s_i,t_i} \ge w'_{s_{i-1},t_i} - c_{s_{i-1},t_i}$$

meaning that s_i can afford the effective wage of t_i in (μ', w').

Recall our induction hypothesis (T1-4.16), s_i suffers at least a utility drop of Λ_i in (μ', w'). However, by (T1-5.3), matching t_i back with the same effective wage suffers at most $\Lambda_{i-1\to i}$, where $\Lambda_{i-1\to i} \le \Lambda_i$. Therefore, s_i should weakly prefer matching t_i. Also, μ' is generated with a tie-breaking rule favoring μ, s_i should strictly prefer matching t_i back, contradicting the stability of (μ', w').

Hence, we have

$$\Lambda_{i-1\to i} = \Lambda_{i-1}$$

Now consider the utility of s_{i-1}, there will be two cases

- Case 5.1: s_i is outbid by s_{i-1} in (μ', w'). Then we have

$$b^{\tau}_{s_i} - c_{s_i,t_i} \le w'_{s_{i-1},t_i} - c_{s_{i-1},t_i} \tag{T1-5.5}$$

  by DC and SC, we know

$$b^{\tau}_{s_i} - w_{s_i,t_i} \ge \Lambda_i \ge \Lambda_{i-1\to i} = \Lambda_{i-1}$$

  then we have

$$b^{\tau}_{s_i} - c_{s_i,t_i} = b^{\tau}_{s_i} - w_{s_i,t_i} + w_{s_i,t_i} - c_{s_i,t_i} \ge w_{s_i,t_i} - c_{s_i,t_i} + \Lambda_{i-1} \tag{T1-5.6}$$

(T1-5.5) and (T1-5.6) then give

$$w'_{s_{i-1},t_i} - c_{s_{i-1},t_i} \geq w_{s_i,t_i} - c_{s_i,t_i} + \Lambda_{i-1} \tag{T1-5.7}$$

We also know $s_{i-1} \xrightarrow{\mu} s_i$, pre-subsidy stability between (s_{i-1}, t_{i-1}) and (s_{i-1}, t_i) gives

$$v_{s_{i-1},t_{i-1}} - w_{s_{i-1},t_{i-1}} \geq v_{s_{i-1},t_i} - w_{s_i,t_i} + c_{s_i,t_i} - c_{s_{i-1},t_i} \tag{T1-5.8}$$

(T1-5.7) and (T1-5.8) then give the following

$$v_{s_{i-1},t_i} - w'_{s_{i-1},t_i} \leq v_{s_{i-1},t_{i-1}} - w_{s_{i-1},t_{i-1}} - \Lambda_{i-1}$$

which is (T1-5.1) for index $i-1$.

- Case 5.2: s_i is not outbid by s_{i-1} in (μ', w'). Then s_i can afford the effective wage of t_i after the subsidy. Post-subsidy stability requires

$$v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}} \geq v_{s_i,t_i} - w'_{s_{i-1},t_i} + c_{s_{i-1},t_i} - c_{s_i,t_i}$$

Since $s_{i-1} \xrightarrow{\mu} s_i$, pre-subsidy stability requires

$$v_{s_{i-1},t_{i-1}} - w_{s_{i-1},t_{i-1}} \geq v_{s_{i-1},t_i} - w_{s_i,t_i} + c_{s_i,t_i} - c_{s_{i-1},t_i}$$

Combining the above two inequalities,

$$(v_{s_{i-1},t_i} - w'_{s_{i-1},t_i}) - (v_{s_{i-1},t_{i-1}} - w_{s_{i-1},t_{i-1}}) \leq (v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}}) - (v_{s_i,t_i} - w_{s_i,t_i}) \tag{T1-5.9}$$

meaning that s_{i-1} suffers a larger utility drop than s_i. Combine our induction hypothesis (T1-4.16) and (T1-5.9), we have

$$(v_{s_{i-1},t_i} - w'_{s_{i-1},t_i}) - (v_{s_{i-1},t_{i-1}} - w_{s_{i-1},t_{i-1}}) \leq -\Lambda_i \leq -\Lambda_{i-1\to i} = -\Lambda_{i-1}$$

which rearranges to (T1-5.1) for index $i-1$

$$v_{s_{i-1},t_i} - w'_{s_{i-1},t_i} \leq v_{s_{i-1},t_{i-1}} - w_{s_{i-1},t_{i-1}} - \Lambda_{i-1}$$

Given the induction step and (T1-4.16) at index i, we conclude that

$$v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}} \leq v_{s_i,t_i} - w_{s_i,t_i} - \Lambda_i \qquad i = 1, \ldots, m$$

that is, every school s_i in $\mathcal{C}$ suffers a utility loss at least Λ_i.

Step 6: A dominant and feasible reallocation

In this step, we are trying to construct a feasible and better allocation for all schools in $\mathcal{C}$.

Construct a hypothetical allocation $(\widetilde{\mu}, \widetilde{w})$ as follows

$$\widetilde{\mu}(s_i) = t_i, \qquad \widetilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \quad (s_i \in \mathcal{C})$$

and leave all matches/wages outside $\mathcal{C}$ as in (μ', w'). By DC and SC, $b^{\tau}_{s_i} - w_{s_i,t_i} \geq \Lambda_i$ for every $i \in \mathcal{C}$, hence $(\widetilde{\mu}, \widetilde{w})$ is feasible under b^{τ}.

- IR hold for all schools and teachers in $\mathcal{C}$.

  For each teacher, she is receiving her wage in (μ, w) plus a nonnegative wage buffer, IR in $(\widetilde{\mu}, \widetilde{w})$ is implied by IR in (μ, w). For each school, it is weakly better than (μ', w'), so IR in $(\widetilde{\mu}, \widetilde{w})$ is implied by IR in (μ', w').
- Schools in $\mathcal{C}$ are weakly better, so no blocks with outside teachers.

  For each $s_i \in \mathcal{C}$, step 5 established

$$v_{s_i,t_{i+1}} - w'_{s_i,t_{i+1}} \le v_{s_i,t_i} - w_{s_i,t_i} - \Lambda_i \qquad i = 1, \ldots, m$$

  Thus every school in $\mathcal{C}$ weakly improves. If (s_i, t) with $t \notin \mathcal{C}$ can block $(\widetilde{\mu}, \widetilde{w})$, then it would also block (μ', w'), contradicting the stability of (μ', w'). Therefore there is no block between a school in $\mathcal{C}$ and an outside teacher.
- No deviations within the cycle.

To show the last piece, we first develop the following lemma

Lemma 4. *For any school s_i, s_j, consider an allocation $(\tilde{\mu}, \tilde{w})$ after the subsidy where*

$$\begin{cases} \tilde{\mu}(s_i) = t_i \\ \tilde{\mu}(s_j) = t_j \end{cases} \qquad \begin{cases} \tilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \\ \tilde{w}_{s_j,t_j} = w_{s_j,t_j} + \Lambda_j \end{cases}$$

then (s_i, t_j) does not form a blocking pair in $(\tilde{\mu}, \tilde{w})$.

Proof. We consider the following two cases:
- Case 1: s_i is not outbid by s_j in μ. So we have $s_i \xrightarrow{\mu} s_j$. Pre-subsidy stability gives

$$v_{s_i,t_i} - w_{s_i,t_i} \ge v_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) + c_{s_i,t_j} \tag{L4-1}$$

  Recall Lemma 3 and $s_i \xrightarrow{\mu} s_j$, we have

$$\Lambda_j \ge \Lambda_{i\to j} \equiv \min\left\{\Lambda_i,\ b^{\tau}_{s_i} - c_{s_i,t_j} - \left(w_{s_j,t_j} - c_{s_j,t_j}\right)\right\}$$

  If $\Lambda_{i\to j} = \Lambda_i$, then we have

$$\Lambda_j \ge \Lambda_i$$

  Recall (L4-1), we then have

$$v_{s_i,t_i} - w_{s_i,t_i} - \Lambda_i \ge v_{s_i,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) + c_{s_i,t_j} - \Lambda_j$$

  where the LHS is the utility of s_i in $(\tilde{\mu}, \tilde{w})$. And the RHS is the utility of deviating to t_j with t_j getting the same effective wage. Therefore, s_i still does not gain by deviating to t_j, and no block arises. If $\Lambda_{i\to j} = b^{\tau}_{s_i} - c_{s_i,t_j} - \left(w_{s_j,t_j} - c_{s_j,t_j}\right)$, then we have

$$\Lambda_j \ge b^{\tau}_{s_i} - c_{s_i,t_j} - \left(w_{s_j,t_j} - c_{s_j,t_j}\right)$$

  which rearranges to

$$b^{\tau}_{s_i} - c_{s_i,t_j} \le w_{s_j,t_j} - c_{s_j,t_j} + \Lambda_j = \tilde{w}_{s_j,t_j} - c_{s_j,t_j}$$

  meaning that s_i is outbid by s_j in $(\tilde{\mu}, \tilde{w})$, and no block arises.

- Case 2: s_i is outbid by s_j in μ.
  By Definition 9

  $$\Delta_j^d \geq (b_{s_i}^\tau - c_{s_i,t_j}) - (w_{s_j,t_j} - c_{s_j,t_j})$$

  Since $\Lambda_j \geq \Delta_j^d$, we have

  $$b_{s_i}^\tau - c_{s_i,t_j} \leq (w_{s_j,t_j} - c_{s_j,t_j}) + \Lambda_j = \widetilde{w}_{s_j,t_j} - c_{s_j,t_j}$$

  So s_i is still outbid by s_j in $(\widetilde{\mu}, \widetilde{w})$, and no block arises.

The above has shown that (s_i, t_j) cannot block $(\widetilde{\mu}, \widetilde{w})$, which finishes the proof of Lemma 4. $\square$

Now we are ready to show there are no deviations within the cycle. Recall our construction of $(\widetilde{\mu}, \widetilde{w})$. For any $s_i, s_j \in \mathcal{C}$, we have

$$\begin{cases} \tilde{\mu}(s_i) = t_i \\ \tilde{\mu}(s_j) = t_j \end{cases} \qquad \begin{cases} \tilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \\ \tilde{w}_{s_j,t_j} = w_{s_j,t_j} + \Lambda_j \end{cases}$$

Then Lemma 4 directly implies that (s_i, t_j) cannot block $(\widetilde{\mu}, \widetilde{w})$.

If $(\widetilde{\mu}, \widetilde{w})$ is stable, then every school in $\mathcal{C}$ would be weakly better than in (μ', w'), and at least one strictly so. Because teachers' effective wages weakly decrease, and the total welfare generated by $\widetilde{\mu}$ on $\mathcal{C}$ is strictly larger by (T1-1).

But (μ', w') is the school-optimal stable allocation, hence $(\widetilde{\mu}, \widetilde{w})$ cannot be stable. We have shown that there is no block within the cycle, and there is no block between a school in $\mathcal{C}$ and an outside teacher. Therefore, any blocking pair must involve an outside school and a teacher in $\mathcal{C}$. Hence, there exist $s_k \notin \mathcal{C}$, $t_j \in \mathcal{C}$, and a feasible wage $\widehat{w}$ such that $(s_k, t_j, \widehat{w})$ blocks $(\widetilde{\mu}, \widetilde{w})$.

Step 7:

Let $(s_k, t_j, \widehat{w})$ be a blocking pair for $(\widetilde{\mu}, \widetilde{w})$ with $s_k \notin \mathcal{C}$ and $t_j \in \mathcal{C}$. In this step, we are trying to show

$$s_k \xrightarrow{\mu} s_j$$

and

$$v_{s_k,\mu'(s_k)} - w'_{s_k,\mu'(s_k)} < v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_k$$

Because t_j accepts the deviation, teacher's incentive implies

$$\widehat{w} - c_{s_k,t_j} > \widetilde{w}_{s_j,t_j} - c_{s_j,t_j} = (w_{s_j,t_j} - c_{s_j,t_j}) + \Lambda_j$$

hence

$$b^{\tau}_{s_k} - c_{s_k,t_j} \geq \widehat{w} - c_{s_k,t_j} > (w_{s_j,t_j} - c_{s_j,t_j}) + \Lambda_j \tag{T1-7.1}$$

We now show $s_k \xrightarrow{\mu} s_j$ pre-subsidy. Suppose, to the contrary, that s_k is outbid by s_j in (μ, w), we then have

$$b_{s_k} - c_{s_k,t_j} \leq w_{s_j,t_j} - c_{s_j,t_j}$$

By DC, we have

$$\Delta^d_j \geq (b^{\tau}_{s_k} - c_{s_k,t_j}) - (w_{s_j,t_j} - c_{s_j,t_j})$$

By construction of Λ_j, we have

$$\Lambda_j \geq \Delta^d_j$$

Then we have

$$\Lambda_j \geq \Delta^d_j \geq (b^{\tau}_{s_k} - c_{s_k,t_j}) - (w_{s_j,t_j} - c_{s_j,t_j}) \tag{T1-7.2}$$

(T1-7.1) and (T1-7.2) imply

$$(b^{\tau}_{s_k} - c_{s_k,t_j}) - (w_{s_j,t_j} - c_{s_j,t_j}) > \Lambda_j \geq \Delta^d_j \geq (b^{\tau}_{s_k} - c_{s_k,t_j}) - (w_{s_j,t_j} - c_{s_j,t_j})$$

which is a contradiction.

Therefore, we have $s_k \xrightarrow{\mu} s_j$, which means

$$b_{s_k} - c_{s_k,t_j} > w_{s_j,t_j} - c_{s_j,t_j}$$

By Lemma 3, we have

$$\Lambda_j \geq \Lambda_{k\to j} \tag{T1-7.3}$$

Using (T1-7.1), we have

$$b^{\tau}_{s_k} - c_{s_k,t_j} > (w_{s_j,t_j} - c_{s_j,t_j}) + \Lambda_j \geq (w_{s_j,t_j} - c_{s_j,t_j}) + \Lambda_{k\to j}$$

By definition of $\Lambda_{k\to j}$, we have

$$b^{\tau}_{s_k} - c_{s_k,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) > \Lambda_{k\to j} \equiv \min\left\{\Lambda_k,\ b^{\tau}_{s_k} - c_{s_k,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})\right\}$$

If it takes the latter value

$$\Lambda_{k\to j} = b^{\tau}_{s_k} - c_{s_k,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})$$

Then we have

$$b^{\tau}_{s_k} - c_{s_k,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) > \Lambda_{k\to j} = b^{\tau}_{s_k} - c_{s_k,t_j} - (w_{s_j,t_j} - c_{s_j,t_j})$$

which is a contradiction. So we have

$$b^{\tau}_{s_k} - c_{s_k,t_j} - (w_{s_j,t_j} - c_{s_j,t_j}) > \Lambda_{k\to j}$$

and

$$\Lambda_{k\to j} = \Lambda_k$$

Recall (T1-7.3), we have

$$\Lambda_j \geq \Lambda_{k\to j} = \Lambda_k \tag{T1-7.4}$$

We now show the outside blocker loses at least Λ_k. Since $(s_k, t_j, \widehat{w})$ blocks $(\widetilde{\mu}, \widetilde{w})$, school s_k's deviation condition and (T1-7.1) gives

$$v_{s_k,\mu'(s_k)} - w'_{s_k,\mu'(s_k)} < v_{s_k,t_j} - \hat{w} < \Big(v_{s_k,t_j} - c_{s_k,t_j}\Big) - \Big(w_{s_j,t_j} - c_{s_j,t_j}\Big) - \Lambda_j \tag{T1-7.5}$$

By $s_k \xrightarrow{\mu} s_j$, pre-subsidy stability between (s_k, t_k) and (s_k, t_j) implies

$$v_{s_k,t_k} - w_{s_k,t_k} \geq \Big(v_{s_k,t_j} - c_{s_k,t_j}\Big) - \Big(w_{s_j,t_j} - c_{s_j,t_j}\Big) \tag{T1-7.6}$$

Combining (T1-7.4), (T1-7.5) and (T1-7.6) yields

$$v_{s_k,\mu'(s_k)} - w'_{s_k,\mu'(s_k)} < v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_j \leq v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_k \tag{T1-7.7}$$

Thus s_k suffers a utility drop of at least Λ_k.

Case 7.A: Outside blocker is a fixed point

Let $(s_k, t_j, \widehat{w})$ be the blocking pair for $(\widetilde{\mu}, \widetilde{w})$ from Step 6, and suppose s_k is a fixed point, that is

$$\mu'(s_k) = \mu(s_k) = t_k$$

By (T1-7.7), s_k loses at least Λ_k, we have

$$v_{s_k,t_k} - w'_{s_k,t_k} < v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_k$$

which implies

$$w'_{s_k,t_k} > w_{s_k,t_k} + \Lambda_k$$

Update the hypothetical allocation $(\widetilde{\mu}, \widetilde{w})$ as follows:

$$\widetilde{\mu}(s_i) = t_i \quad (s_i \in \mathcal{C} \cup s_k \cup t_k), \qquad \widetilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \quad (s_i \in \mathcal{C} \cup s_k \cup t_k)$$

and keep all other matches/wages as in (μ', w').

This hypothetical allocation is feasible by DC and SC. We now show it is also stable within $\mathcal{C} \cup s_k \cup t_k$.

We can use the same reasoning in Step 6 to show

- IR hold for all schools and teachers in $\mathcal{C} \cup s_k \cup t_k$.
- Schools in $\mathcal{C} \cup s_k \cup t_k$ are weakly better, no blocks with outside teachers.
- Lemma 4 shows that there are no deviations within $\mathcal{C} \cup s_k \cup t_k$.

Case 7.B: Outside blocker belongs to a cycle without outbid

Step B.1: Utility drop at the outside blocker on $\mathcal{D}$

Let $(s_k, t_j, \widehat{w})$ be the blocking pair for $(\widetilde{\mu}, \widetilde{w})$ from Step 6, and suppose s_k belongs to a cycle $\mathcal{D}$ (we adopt similar index as in $\mathcal{C}$, so school s_i is matching teacher t_{i+1}) with the following:

$$b_{s_i} - c_{s_i,t_{i+1}} > w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} \qquad \text{for all } s_i \in \mathcal{D} \tag{T1-7.7}$$

Repeating the argument of Step 7, we have $s_k \xrightarrow{\mu} s_j$ and

$$v_{s_k,t_{k+1}} - w'_{s_k,t_{k+1}} < v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_k \tag{T1-7.8}$$

that is, s_k's utility drops by at least Λ_k.

Step B.2: Utility drop on the entire cycle $\mathcal{D}$

We now show that every school in $\mathcal{D}$ loses at least its own Λ.

By DC and SC, we have

$$b^{\tau}_{s_k} - w_{s_k,t_k} \geq \Lambda_k$$

meaning that s_k can afford a wage of $w_{s_k,t_k} + \Lambda_k$ to t_k. If t_k accepts this offer, then s_k is getting strictly better because it loses more than Λ_k in (μ', w'). And matching t_k with a wage of $w_{s_k,t_k} + \Lambda_k$ only loses Λ_k. Therefore, by stability of (μ', w'), the effective wage of t_k must be strictly higher than $w_{s_k,t_k} - c_{s_k,t_k} + \Lambda_k$. Formally,

$$w'_{s_{k-1},t_k} - c_{s_{k-1},t_k} > (w_{s_k,t_k} - c_{s_k,t_k}) + \Lambda_k \tag{T1-7.9}$$

Given the feasibility of (μ', w'), we have

$$b^{\tau}_{s_{k-1}} - c_{s_{k-1},t_k} \geq w'_{s_{k-1},t_k} - c_{s_{k-1},t_k} > (w_{s_k,t_k} - c_{s_k,t_k}) + \Lambda_k$$

Repeat the argument in step 7 from (T1-7.1) to (T1-7.4), we have

$$\Lambda_k \geq \Lambda_{k-1\to k} = \Lambda_{k-1} \tag{T1-7.10}$$

Now consider the utility of s_{k-1}. By our assumption (T1-7.7) , we have $s_{k-1} \xrightarrow{\mu} s_k$. Pre-subsidy stability gives

$$v_{s_{k-1},t_{k-1}} - w_{s_{k-1},t_{k-1}} \geq (v_{s_{k-1},t_k} - c_{s_{k-1},t_k}) - (w_{s_k,t_k} - c_{s_k,t_k}) \tag{T1-7.11}$$

(T1-7.9), (T1-7.10) and (T1-7.11) give the following:

$$v_{s_{k-1},t_k} - w'_{s_{k-1},t_k} < v_{s_{k-1},t_{k-1}} - w_{s_{k-1},t_{k-1}} - \Lambda_k \leq v_{s_{k-1},t_{k-1}} - w_{s_{k-1},t_{k-1}} - \Lambda_{k-1} \tag{T1-7.12}$$

Iterating the same one-step argument successively around $\mathcal{D}$ implies that

$$v_{s_k,t_{k+1}} - w'_{s_k,t_{k+1}} < v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_k \qquad \text{for all } s_k \in \mathcal{D} \tag{T1-7.13}$$

which means every school in $\mathcal{D}$ loses at least its own Λ.

Step B.3: Expand the hypothetical allocation to $\mathcal{C} \cup \mathcal{D}$

Similar to step A, we now update the hypothetical allocation $(\widetilde{\mu}, \widetilde{w})$ as follows:

$$\widetilde{\mu}(s_i) = t_i \quad (s_i \in \mathcal{C} \cup \mathcal{D})$$

$$\widetilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \quad (s_i \in \mathcal{C} \cup \mathcal{D})$$

and keep all other matches/wages as in (μ', w'). Similarly, this updated $(\widetilde{\mu}, \widetilde{w})$ is feasible by DC and SC.

To show the stability within $\mathcal{C} \cup \mathcal{D}$. We can use the same reasoning in step 6 to show

- IR hold for all schools and teachers in $\mathcal{C} \cup \mathcal{D}$.
- Schools in $\mathcal{C} \cup \mathcal{D}$ are weakly better, no blocks with outside teachers.
- Lemma 4 shows that there are no deviations within $\mathcal{C} \cup \mathcal{D}$.

Case 7.C: Outside blocker belongs to a cycle with outbid

Let $(s_k, t_j, \widehat{w})$ be the blocking pair for $(\widetilde{\mu}, \widetilde{w})$ from Step 6, and suppose s_k belongs to a cycle $\mathcal{E}$ (we adopt similar index as in $\mathcal{C}$, so school s_i is matching teacher t_{i+1}) with the following:

$$\exists\ i \text{ such that} \qquad b_{s_i} - c_{s_i,t_{i+1}} \leq w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} \qquad \text{where } s_i \in \mathcal{E} \tag{T1-7.14}$$

Repeating the argument of Step 7, we have $s_k \xrightarrow{\mu} s_j$ and

$$v_{s_k,t_{k+1}} - w'_{s_k,t_{k+1}} < v_{s_k,t_k} - w_{s_k,t_k} - \Lambda_k \tag{T1-7.8'}$$

that is, s_k's utility drops by at least Λ_k.

Repeat the argument of Step B.2. For s_{k-1}, we still get the following

$$w'_{s_{k-1},t_k} - c_{s_{k-1},t_k} > \left(w_{s_k,t_k} - c_{s_k,t_k}\right) + \Lambda_k \tag{T1-7.9'}$$

Then by feasibility of (μ', w'), we have

$$b^{\tau}_{s_{k-1}} - c_{s_{k-1},t_k} \geq w'_{s_{k-1},t_k} - c_{s_{k-1},t_k} > \left(w_{s_k,t_k} - c_{s_k,t_k}\right) + \Lambda_k \tag{T1-7.15}$$

The above rules out the possibility that s_{k-1} is outbid by s_k in (μ, w), because otherwise DC and SC requires

$$\Lambda_k \geq b^{\tau}_{s_{k-1}} - c_{s_{k-1},t_k} - \left(w_{s_k,t_k} - c_{s_k,t_k}\right) \tag{T1-7.16}$$

and (T1-7.15), (T1-7.16) lead to a contradiction.

Therefore, we now have $s_{k-1} \xrightarrow{\mu} s_k$ and (T1-7.15). Repeating the logic from (T1-7.1) to (T1-7.4), we have

$$\Lambda_{k-1} \leq \Lambda_k \tag{T1-7.17}$$

Combine (T1-7.9) and (T1-7.17), we have

$$w'_{s_{k-1},t_k} - c_{s_{k-1},t_k} > \left(w_{s_k,t_k} - c_{s_k,t_k}\right) + \Lambda_{k-1} \tag{T1-7.18}$$

Recall $s_{k-1} \xrightarrow{\mu} s_k$, pre-subsidy stability requires

$$v_{s_{k-1},t_{k-1}} - w_{s_{k-1},t_{k-1}} \geq \left(v_{s_{k-1},t_k} - c_{s_{k-1},t_k}\right) - \left(w_{s_k,t_k} - c_{s_k,t_k}\right) \tag{T1-7.19}$$

Combining (T1-7.18) and (T1-7.19) gives

$$v_{s_{k-1},t_k} - w'_{s_{k-1},t_k} < v_{s_{k-1},t_{k-1}} - w_{s_{k-1},t_{k-1}} - \Lambda_{k-1} \tag{T1-7.20}$$

Then we can repeat the logic from (T1-7.8') to (T1-20) to get

$$s_{k-2} \xrightarrow{\mu} s_{k-1}$$

and

$$v_{s_{k-2},t_{k-1}} - w'_{s_{k-2},t_{k-1}} < v_{s_{k-2},t_{k-2}} - w_{s_{k-2},t_{k-2}} - \Lambda_{k-2} \tag{T1-7.21}$$

However, different from Case 7.B, we have (T1-7.14)

$$\exists\ i \text{ such that} \qquad b_{s_i} - c_{s_i,t_{i+1}} \leq w_{s_{i+1},t_{i+1}} - c_{s_{i+1},t_{i+1}} \qquad \text{where } s_i \in \mathcal{E} \tag{T1-7.14}$$

and the above one-step propagation (T1-7.21) will finally give $s_i \xrightarrow{\mu} s_{i+1}$ leading to a contradiction of (T1-7.14).

Step 8: Exhaustion and final contradiction

We first find a welfare-decreasing cycle $\mathcal{C}$ in step 1. Then follow step 2 to step 6, we initialize reallocation on $\mathcal{C}$ and let

$$\widetilde{\mu}(s_i) = t_i, \qquad \widetilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \quad (s_i \in \mathcal{C})$$

Let $\mathcal{R}$ denote the set of schools and teachers that have been reallocated. So $\mathcal{R} = \mathcal{C}$ at the beginning. Then we iteratively apply the expansion rule: whenever there exists a blocking pair $(s_k, t_j, \widehat{w})$ for $(\widetilde{\mu}, \widetilde{w})$ with $t_j \in \mathcal{R}$ and $s_k \notin \mathcal{R}$, expand the reallocation set and update $(\widetilde{\mu}, \widetilde{w})$ as follows

- If s_k is a fixed point, apply 7.A and add s_k, t_k into $\mathcal{R}$ and let

$$\widetilde{\mu}(s_i) = t_i \quad (s_i \in \mathcal{R} \cup s_k \cup t_k), \qquad \widetilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \quad (s_i \in \mathcal{R} \cup s_k \cup t_k)$$

- If s_k belongs to a cycle $\mathcal{D}$ without outbid, apply 7.B and add $\mathcal{D}$ into $\mathcal{R}$ and let

$$\widetilde{\mu}(s_i) = t_i \quad (s_i \in \mathcal{R} \cup \mathcal{D}), \qquad \widetilde{w}_{s_i,t_i} = w_{s_i,t_i} + \Lambda_i \quad (s_i \in \mathcal{R} \cup \mathcal{D})$$

Keep the above expansion, there are two possibilities.

(1) All outside schools are Case 7.A or Case 7.B.

Since the market is finite, the expansion terminates when it exhausts the set of all schools. By internal stability of $\mathcal{R}$ and the absence of remaining outside schools, $(\widetilde{\mu}, \widetilde{w})$ is stable. Every school is weakly better than in (μ', w'), and at least one is strictly better, contradicting the school–optimality of (μ', w').

(2) Some outside schools are Case 7.C.

By Case 7.C, when the expansion reaches a cycle with outbid, we directly have a contradiction.

In either branch, we reach a contradiction. Therefore, the assumption in step 1

$$W_{v,c}(\mu') < W_{v,c}(\mu)$$

is false. Hence, for every value profile $v \in \mathcal{V}(\mu, w; b, c)$, we have

$$\overline{W}(v, c, b, \tau) \geq W_{v,c}(\mu') \geq W_{v,c}(\mu)$$

which completes the proof.

□

Proof of Proposition 2.

Proof. WLOG, we again let $\mu(s_i) = t_i$ for all i.

Let τ be the positive uniform subsidy allocation and let ∇ be the quantity of subsidy allocated to each school. Then we have

$$\nabla = \tau_s > 0 \quad \text{for all } s \in S$$

We claim τ is dominated by the following τ', where

$$\begin{cases} \tau'(s) = \tau_s = \nabla \quad \text{for all } s \neq s_j \\ \tau'(s_j) = \max\left\{ \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla - b_{s_j}, \max_{i \neq j}\{w_{s_i,t_i} - c_{s_i,t_i} + c_{s_j,t_i}\} + \nabla - b_{s_j}, 0\right\} \end{cases}$$

We first verify $\tau'(s_j) < \tau(s_j)$. Recall conditions of Proposition 2

$$b_{s_i} - c_{s_i,t_j} < b_{s_j} - c_{s_j,t_j} \quad \text{for all } i \neq j$$

so we have

$$\max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} - b_{s_j} + \nabla < \nabla$$

Then recall conditions of Proposition 2, we have $s_j \xrightarrow{\mu} s_i$ for any $i \neq j$, so we have

$$b_{s_j} - c_{s_j,t_i} > w_{s_i,t_i} - c_{s_i,t_i} \quad \text{for all } i \neq j$$

Th gives

$$\max_{i \neq j}\{w_{s_i,t_i} - c_{s_i,t_i} + c_{s_j,t_i}\} + \nabla - b_{s_j} < \nabla$$

also, we know $\nabla > 0$, which gives

$$\tau'(s_j) < \nabla = \tau(s_j) \quad \Rightarrow \quad \sum_{s \in S} \tau'_s < \sum_{s \in S} \tau_s$$

Recall the definition of $\overline{W}(v, c, b, \tau)$, we have

$$\overline{W}(v, c, b, \tau) \equiv \max_{\mu} \left\{ W(\mu; v, c) : \ \exists\ w \text{ such that } (\mu, w) \text{ is stable under budgets } b^\tau \text{ given } (v, c) \right\}$$

Then for $\forall\ v \in \mathcal{V}(\mu, w; b, c)$, there exists a stable allocation (μ', w') after the subsidy such that

$$\overline{W}(v, c, b, \tau) = W(\mu', v, c)$$

Fix v, and consider the corresponding (μ', w'), there will be two cases

- $\mu'(s_j) = t_j$
  That is, s_j does not change its teacher from μ to μ'. There will be three more subcases:
  - $w'_{s_j,t_j} \leq \max\left\{w_{s_j,t_j}, \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla\right\}$
    Now we replace τ with τ'. By construction of τ', we have $\tau'(s_j) \geq 0$, so

    $$b^{\tau'}_{s_j} \geq b_{s_j} \geq w_{s_j,t_j} \geq w'_{s_j,t_j}$$

    Also, by construction of τ', we have

    $$b^{\tau'}_{s_j} \geq \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla$$

The above two inequalities give

$$b_{s_j}^{\tau'} \geq \max\left\{w_{s_j,t_j}, \max_{i\neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla\right\} \geq w'_{s_j,t_j}$$

Meaning that (μ', w') is still feasible. Now we verify it is stable:

Since the only thing changed is the budget of s_j, so IR still hold. And we only need to check if (s_j, t_k) forms a blocking pair where $k \neq j$.

If s_j is outbid by $\mu'^{-1}(t_k)$ in (μ', w') with subsidy τ, we have

$$b_{s_j}^{\tau} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

Since $\tau_{s_j} > \tau'_{s_j}$, we have

$$b_{s_j}^{\tau'} - c_{s_j,t_k} < b_{s_j}^{\tau} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

So s_j is still outbid by $\mu'^{-1}(t_k)$ in (μ', w') with subsidy τ', and (s_j, t_k) does not form a blocking pair.

If $s_j \xrightarrow{\mu'} \mu'^{-1}(t_k)$ with τ, then stability requires

$$v_{s_j,\mu'(s_j)} - w'_{s_j,\mu'(s_j)} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k})$$

Since the allocation does not change, there is still no incentive for s_j to deviate to t_k with subsidy τ', and (s_j, t_k) does not form a blocking pair.

Therefore, (μ', w') is still feasible and stable when the subsidy is τ'. Then we have

$$\overline{W}(\tau', v, b, c) \geq W(\mu', v, c) = \overline{W}(v, c, b, \tau)$$

– $w'_{s_j,t_j} > w_{s_j,t_j} > \max_{i\neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla$

Then we can construct the following allocation $(\mu', \tilde{w})$, where

$$\begin{cases} \tilde{w}_{s,\mu'(s)} = w'_{s,\mu'(s)} & \text{for all } s \neq s_j \\ \tilde{w}_{s_j,t_j} = w_{s_j,t_j} \end{cases}$$

Now we show $(\mu', \tilde{w})$ is feasible and stable after the subsidy.

Recall $\tau'(s_j) \geq 0$, so

$$b_{s_j}^{\tau'} \geq b_{s_j} \geq w_{s_j,t_j} \geq w'_{s_j,t_j}$$

meaning that $(\mu', \tilde{w})$ is feasible.

Now we need to verify there are no blocking pairs in $(\mu', \tilde{w})$ when the subsidy is τ'. Comparing to (μ', w') with subsidy τ, we only changed $\tau'(s_j)$ and $\tilde{w}_{s_j,t_j}$. Therefore, we only need to check if (s_j, t_k) or (s_k, t_j) form a blocking pair where $k \neq j$.

First consider (s_j, t_k). If s_j is outbid by $\mu'^{-1}(t_k)$ in (μ', w') with subsidy τ, we have

$$b_{s_j}^{\tau} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

Since $\tau_{s_j} > \tau'_{s_j}$, we have

$$b_{s_j}^{\tau'} - c_{s_j,t_k} < b_{s_j}^{\tau} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k} = \tilde{w}_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

So s_j is still outbid by $\mu'^{-1}(t_k)$ in $(\mu', \tilde{w})$ with subsidy τ', and (s_j, t_k) does not form a blocking pair.

If $s_j \xrightarrow{\mu'} \mu'^{-1}(t_k)$ when the subsidy is τ, then stability requires

$$v_{s_j,t_j} - w'_{s_j,t_j} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k})$$

By construction of $\tilde{w}$ and the case assumption, we have

$$\tilde{w}_{s_j,t_j} = w_{s_j,t_j} < w'_{s_j,t_j}$$

which gives

$$v_{s_j,t_j} - \tilde{w}_{s_j,t_j} > v_{s_j,t_j} - w'_{s_j,t_j} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k})$$

meaning that there is still no incentive for s_j to deviate to t_k, and (s_j, t_k) does not form a blocking pair.

Now consider consider (s_k, t_j). By construction of $\tilde{w}$ and the case assumption, we have

$$\tilde{w}_{s_j,t_j} = w_{s_j,t_j} > \max_{i \neq j}(b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}) + \nabla \geq b_{s_k} - c_{s_k,t_j} + c_{s_j,t_j} + \nabla$$

By construction of τ', we then have

$$b^{\tau'}_{s_k} - c_{s_k,t_j} = b_{s_k} - c_{s_k,t_j} + \nabla < \tilde{w}_{s_j,t_j} - c_{s_j,t_j}$$

meaning that s_k is outbid by s_j in $(\mu', \tilde{w})$ with subsidy τ', and (s_k, t_j) does not form a blocking pair.

Lastly, we check IR for s_k and t_k. By construction of $\tilde{w}$, we have

$$\tilde{w}_{s_j,t_j} = w_{s_j,t_j}$$

so IR of t_k is implied by IR of (μ, w) before the subsidy. For s_k, we have

$$v_{s_j,t_j} - \tilde{w}_{s_j,t_j} > v_{s_j,t_j} - w'_{s_j,t_j}$$

so IR of s_k is implied by IR of (μ', w') with subsidy τ.

To conclude, $(\mu', \tilde{w})$ is feasible and stable with subsidy τ', so we have

$$\overline{W}(\tau', v, b, c) \geq W(\mu', v, c) = \overline{W}(v, c, b, \tau)$$

– $w'_{s_j,t_j} > \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla \geq w_{s_j,t_j}$
 Then we can construct the following allocation $(\mu', \tilde{w})$, where

$$\begin{cases} \tilde{w}_{s,\mu'(s)} = w'_{s,\mu'(s)} & \text{for all } s \neq s_j \\ \tilde{w}_{s_j,t_j} = \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla \end{cases}$$

Now we show $(\mu', \tilde{w})$ is feasible and stable after the subsidy.

By construction of τ', we have

$$b^{\tau'}_{s_j} = b_{s_j} + \tau'_{s_j} \geq \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla \geq \tilde{w}_{s_j,t_j}$$

meaning that $(\mu', \tilde{w})$ is feasible.

Now we need to verify there are no blocking pairs in $(\mu', \tilde{w})$ when the subsidy is τ'. Comparing to (μ', w') with subsidy τ, we only changed $\tau'(s_j)$ and $\tilde{w}_{s_j,t_j}$. Therefore, we only need to check if (s_j, t_k) or (s_k, t_j) form a blocking pair where $k \neq j$.

First consider (s_j, t_k). If s_j is outbid by $\mu'^{-1}(t_k)$ in (μ', w') with subsidy τ, we have

$$b^{\tau}_{s_j} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

Since $\tau_{s_j} > \tau'_{s_j}$, we have

$$b^{\tau'}_{s_j} - c_{s_j,t_k} < b^{\tau}_{s_j} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k} = \tilde{w}_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

So s_j is still outbid by $\mu'^{-1}(t_k)$ in $(\mu', \tilde{w})$ with subsidy τ', and (s_j, t_k) does not form a blocking pair.

If $s_j \xrightarrow{\mu'} \mu'^{-1}(t_k)$ when the subsidy is τ, then stability requires

$$v_{s_j,t_j} - w'_{s_j,t_j} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k})$$

By construction of $\tilde{w}$ and the case assumption we have

$$\tilde{w}_{s_j,t_j} = \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla < w'_{s_j,t_j}$$

which gives

$$v_{s_j,t_j} - \tilde{w}_{s_j,t_j} > v_{s_j,t_j} - w'_{s_j,t_j} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k})$$

meaning that there is still no incentive for s_j to deviate to t_k, and (s_j, t_k) does not form a blocking pair.

Now consider consider (s_k, t_j). By construction of $\tilde{w}$, we have

$$\tilde{w}_{s_j,t_j} = \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla \geq b_{s_k} - c_{s_k,t_j} + c_{s_j,t_j} + \nabla$$

which gives

$$b^{\tau'}_{s_k} - c_{s_k,t_j} \leq \tilde{w}_{s_j,t_j} - c_{s_j,t_j}$$

meaning that s_k is outbid by s_j in $(\mu', \tilde{w})$ with subsidy τ', and (s_k, t_j) does not form a blocking pair.

Lastly, we check IR for s_k and t_k. By construction of $\tilde{w}$ and the case assumption, we have

$$\tilde{w}_{s_j,t_j} = \max_{i \neq j}\{b_{s_i} - c_{s_i,t_j} + c_{s_j,t_j}\} + \nabla \geq w_{s_j,t_j}$$

so IR of t_k is implied by IR of (μ, w) before the subsidy. For s_k, we have

$$v_{s_j,t_j} - \tilde{w}_{s_j,t_j} > v_{s_j,t_j} - w'_{s_j,t_j}$$

so IR of s_k is implied by IR of (μ', w') with subsidy τ.

To conclude, $(\mu', \tilde{w})$ is feasible and stable with subsidy τ', so we have

$$\overline{W}(\tau', v, b, c) \geq W(\mu', v, c) = \overline{W}(v, c, b, \tau)$$

- $\mu'(s_j) = t_k \neq t_j$
  That is, s_j changes its teacher from μ to μ'. Now we want to show

$$w'_{s_j,t_k} \leq b^{\tau'}_{s_j}$$

Recall conditions of Proposition 2, we have $s_j \xrightarrow{\mu} s_k$. Pre-subsidy stability then requires

$$v_{s_j,t_j} - w_{s_j,t_j} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w_{s_k,t_k} - c_{s_k,t_k}) \tag{P2-1}$$

Now consider (μ', w') with subsidy τ. Let s_m be the school matching t_j in μ', then we have

$$\mu'^{-1}(t_j) = s_m$$

Recall conditions in Proposition 2, we have

$$\begin{cases} b_{s_m} - c_{s_m,t_j} \leq w_{s_j,t_j} - c_{s_j,t_j} \\ b_{s_m} - c_{s_m,t_j} < b_{s_j} - c_{s_j,t_j} \end{cases}$$

We then have

$$\begin{cases} b^{\tau}_{s_m} - c_{s_m,t_j} = b_{s_m} + \nabla - c_{s_m,t_j} \leq w_{s_j,t_j} - c_{s_j,t_j} + \nabla \\ b^{\tau}_{s_m} - c_{s_m,t_j} = b_{s_m} + \nabla - c_{s_m,t_j} < b_{s_j} - c_{s_j,t_j} + \nabla \end{cases}$$

By feasibility of (μ', w'), we then have

$$w_{s_j,t_j} - c_{s_j,t_j} + \nabla \geq b^{\tau}_{s_m} - c_{s_m,t_j} \geq w'_{s_m,t_j} - c_{s_m,t_j} \tag{P2-2}$$

and

$$b^{\tau}_{s_j} - c_{s_j,t_j} = b_{s_j} - c_{s_j,t_j} + \nabla > b^{\tau}_{s_m} - c_{s_m,t_j} \geq w'_{s_m,t_j} - c_{s_m,t_j} \tag{P2-3}$$

(P2-3) implies that s_j can offer a strictly higher effective wage to t_j. Then the post-subsidy stability requires

$$v_{s_j,t_k} - w'_{s_j,t_k} \geq v_{s_j,t_j} - c_{s_j,t_j} - (w'_{s_m,t_j} - c_{s_m,t_j}) \tag{P2-4}$$

Combining (P2-1) and (P2-4), we have

$$(w'_{s_j,t_k} - c_{s_j,t_k}) - (w_{s_k,t_k} - c_{s_k,t_k}) \leq (w'_{s_m,t_j} - c_{s_m,t_j}) - (w_{s_j,t_j} - c_{s_j,t_j}) \tag{P2-5}$$

Recall (P2-2), we then have

$$(w'_{s_j,t_k} - c_{s_j,t_k}) - (w_{s_k,t_k} - c_{s_k,t_k}) \leq \nabla$$

which is equivalent to

$$w'_{s_j,t_k} \leq w_{s_k,t_k} - c_{s_k,t_k} + c_{s_j,t_k} + \nabla \tag{P2-6}$$

Recall construction of τ', we have

$$b^{\tau'}_{s_j} = b_{s_j} + \tau'_{s_j} \geq \max_{i \neq j}\{w_{s_i,t_i} - c_{s_i,t_i} + c_{s_j,t_i}\} + \nabla \geq w_{s_k,t_k} - c_{s_k,t_k} + c_{s_j,t_k} + \nabla \tag{P2-7}$$

(P2-6) and (P2-7) then give

$$w'_{s_j,t_k} \leq b^{\tau'}_{s_j} \tag{P2-8}$$

Now we replace τ with τ'. (P2-8) shows that (μ', w') is feasible with subsidy τ'. It remains to show it is stable.

Since the only thing changed is the budget of s_j, so IR still hold. And we only need to check if (s_j, t_k) forms a blocking pair where $k \neq j$.

If s_j is outbid by $\mu'^{-1}(t_k)$ in (μ', w') with subsidy τ, we have

$$b_{s_j}^{\tau} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

Since $\tau_{s_j} > \tau'_{s_j}$, we have

$$b_{s_j}^{\tau'} - c_{s_j,t_k} < b_{s_j}^{\tau} - c_{s_j,t_k} \leq w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k}$$

So s_j is still outbid by $\mu'^{-1}(t_k)$ in (μ', w') with subsidy τ', and (s_j, t_k) does not form a blocking pair.

If $s_j \xrightarrow{\mu'} \mu'^{-1}(t_k)$ with τ, then stability requires

$$v_{s_j,\mu'(s_j)} - w'_{s_j,\mu'(s_j)} \geq v_{s_j,t_k} - c_{s_j,t_k} - (w'_{\mu'^{-1}(t_k),t_k} - c_{\mu'^{-1}(t_k),t_k})$$

Since the allocation does not change, there is still no incentive for s_j to deviate to t_k with subsidy τ', and (s_j, t_k) does not form a blocking pair.

Therefore, (μ', w') is still feasible and stable when the subsidy is τ'. Then we have

$$\overline{W}(\tau', v, b, c) \geq W(\mu', v, c) = \overline{W}(v, c, b, \tau)$$

Therefore, in all cases, we have

$$\overline{W}(\tau', v, b, c) \geq W(\mu', v, c) = \overline{W}(v, c, b, \tau)$$

and recall

$$\tau'(s_j) < \nabla = \tau(s_j) \quad \Rightarrow \quad \sum_{s \in S} \tau'_s < \sum_{s \in S} \tau_s$$

The above analysis works for all $v \in \mathcal{V}(\mu, w; b, c)$, meaning that τ is dominated by τ', which finishes the proof of Proposition 2. $\square$

Proof of Theorem 2.

Proof. We first show the necessity. That is, if τ is non-distortive, then we have

$$b^{\tau}_{s_j} - w_{s_j,\mu(s_j)} \geq \max_{s_i} \tau_{s_i} \quad \text{for } \forall j \tag{T2-1}$$

WLOG, we again let $\mu(s_i) = t_i$ by reindexing.

Fix two distinct indices $k \neq j$. Consider the following cost and value profile (v^*, c^*), where for all matched pairs, we have

$$\begin{cases} v^*_{s_i,t_i} = w_{s_i,t_i} \\ c^*_{s_i,t_i} = w_{s_i,t_i} \end{cases} \quad \text{for all } i$$

and for all unmatched pairs (s,t) where $t \neq \mu(s)$, we have

$$\begin{cases} v^*_{s_k,t_j} = b_{s_k} \\ c^*_{s_k,t_j} = b_{s_k} \end{cases} \quad \text{and} \quad \begin{cases} v^*_{s,t} = 0 \\ c^*_{s,t} = 0 \end{cases} \quad \text{for all } (s,t) \neq (s_k, t_j)$$

We now show the above (v^*, c^*) is consistent with (μ, w) and b, that is

$$(v^*, c^*) \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

To show the above, we need to verify the following:

- (μ, w) is individually rational.
  By construction of (v^*, c^*), each school s_i is enjoying a utility of
  $$v^*_{s_i,t_i} - w_{s_i,t_i} = 0 \geq 0 \quad \text{for all } i$$
  each teacher t_i is enjoying a utility of
  $$w_{s_i,t_i} - c^*_{s_i,t_i} = 0 \geq 0 \quad \text{for all } i$$
  so IR hold for all schools and teachers.
- (μ, w) is feasible.
  This is directly implied by the observed (μ, w) and b and we have
  $$b_{s_i} \geq w_{s_i,t_i} \quad \text{for all } i$$
- There are no blocking pairs.
  By construction of (v^*, c^*), we have
  $$v^*_{s,t} = c^*_{s,t} \quad \text{for all } s,t \text{ where } t \neq \mu(s)$$
  therefore, for any unmatched pair (s,t), any deviating wage $\hat{w}$ strictly profitable for teacher t will require
  $$\hat{w} - c^*_{s,t} > 0$$
  which gives
  $$v^*_{s,t} - \hat{w} < 0$$
  so $\hat{w}$ cannot be profitable for school s, so there are no blocking pairs.

Given the above, we have shown that (μ, w) is stable, which gives

$$(v^*, c^*) \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

Fix the above c^*, by definition of $\mathcal{V}, \mathcal{C}(\mu, w; b)$, and $\mathcal{V}(\mu, w; b, c)$, we have

$$v^* \in \mathcal{V}(\mu, w; b, c^*)$$

therefore, we know that $\mathcal{V}(\mu, w; b, c^*) \neq \emptyset$.

Recall definition of $\mathcal{V}, \mathcal{C}(\mu, w; b)$, and $\mathcal{V}(\mu, w; b, c)$, we have

$$(v', c^*) \in \mathcal{V}, \mathcal{C}(\mu, w; b) \quad \text{for all } v' \in \mathcal{V}(\mu, w; b, c^*)$$

Since τ is non-distortive for $(\mu, w; b)$, we have

$$\overline{W}(v, c, b, \tau) \;\geq\; W(\mu; v, c) \quad \text{for all } (v, c) \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

which implies

$$\overline{W}(\tau, v', b, c^*) \;\geq\; W(\mu; v', c^*) \quad \text{for all } v' \in \mathcal{V}(\mu, w; b, c^*)$$

which means τ is also non-distortive for $(\mu, w; b, c^*)$.

Recall Theorem 1, we know τ satisfies DC. And by our construction of c^*, we have

$$b_{s_k} - c^*_{s_k,t_j} = b_{s_k} - b_{s_k} = 0 \leq w_{s_j,t_j} - w_{s_j,t_j} = w_{s_j,t_j} - c^*_{s_j,t_j}$$

Therefore, s_k is outbid by s_j in $(\mu, w; b, c^*)$, and DC then requires

$$b^\tau_{s_j} - w_{s_j,t_j} \geq b^\tau_{s_k} - c^*_{s_k,t_j} = \tau_{s_k} + b_{s_k} - b_{s_k} = \tau_{s_k} \tag{T2-2}$$

The above analysis works for any indices $k \neq j$, so (T2-2) is true for any $k \neq j$, which gives

$$b^\tau_{s_j} - w_{s_j,t_j} \geq \max_{s_i} \tau_{s_i} \quad \text{for } \forall j$$

which is the desired inequality (T2-1), finishing the proof of necessity.

Now we show sufficiency, that is, if τ satisfies

$$b^\tau_{s_j} - w_{s_j,t_j} \geq \max_{s_i} \tau_{s_i} \quad \text{for } \forall j \tag{T2-1}$$

then τ is non-distortive for $(\mu, w; b)$.

For any

$$(v, c) \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

fix the above c, and we want to show τ is non-distortive for $(\mu, w; b, c)$.

We first consider DC. For any school s_j, by definition, we have

$$\Delta^d_j \equiv \max_{i:\ s_i \text{ is outbid by } s_j \text{ in } (\mu, w)} \left[(b^\tau_{s_i} - c_{s_i,\mu(s_j)}) - (w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)})\right]_+ \quad \text{for } \forall j$$

which gives

$$
\begin{aligned}
\Delta_j^d &= \max_{i:\ s_i \text{ is outbid by } s_j \text{ in } (\mu,w)} \Big[(b_{s_i}^\tau - c_{s_i,\mu(s_j)}) - (w_{s_j,\mu(s_j)} - c_{s_j,\mu(s_j)})\Big]_+ \\
&\le \max_{i:\ s_i \text{ is outbid by } s_j \text{ in } (\mu,w)} \tau_{s_i} \\
&\le \max_{s_i} \tau_{s_i} \quad \text{for } \forall j
\end{aligned}
\tag{T2-3}
$$

We then consider SC, and by definition, we have

$$
\Delta_j^s \equiv \sup_{C:\ \text{spillover chains ending at } s_j} \delta_j^C \tag{T2-4}
$$

For any spillover chain C ending at s_j, let C be the following by reindexing:

$$
C:\ s_1 \xrightarrow{\mu} s_2 \xrightarrow{\mu} \cdots \xrightarrow{\mu} s_j
$$

Then by definition of δ_j^C, we have

$$
\delta_j^C \le \delta_{j_1}^C ... \delta_2^C \le \Delta_1^d
$$

By (T2-3), we then have

$$
\delta_j^C \le \Delta_1^d \le \max_{s_i} \tau_{s_i} \tag{T2-5}
$$

Since (T2-5) holds for any spillover chain C ending at s_j, we recall (T2-4) and get the following

$$
\Delta_j^s \le \max_{s_i} \tau_{s_i} \quad \text{for } \forall j \tag{T2-6}
$$

(T2-1), (T2-3) and (T2-6) then give

$$
b_{s_j}^\tau - w_{s_j,\mu(s_j)} \ge \max_{s_i} \tau_{s_i} \ge \max\{\Delta_j^d, \Delta_j^s\} \quad \text{for } \forall j
$$

then by Theorem 1, τ is non-distortive for $(\mu, w; b, c)$.

Now consider any (v, c) such that

$$
(v, c) \in \mathcal{V}, \mathcal{C}(\mu, w; b)
$$

by definition of $\mathcal{V}, \mathcal{C}(\mu, w; b)$, and $\mathcal{V}(\mu, w; b, c)$, we have

$$
v \in \mathcal{V}(\mu, w; b, c)
$$

by τ being non-distortive for $(\mu, w; b, c)$, we have

$$
\overline{W}(\tau, v', b, c) \ \ge\ W(\mu; v', c) \quad \text{for all } v' \in \mathcal{V}(\mu, w; b, c)
$$

which implies

$$
\overline{W}(v, c, b, \tau) \ \ge\ W(\mu; v, c)
$$

Therefore, τ is non-distortive for $(\mu, w; b)$, which proves the sufficiency. $\square$

Proof of Theorem 3.

Proof. We first show the necessity. That is, if τ is non-distortive, then we have

$$\tau_{s_i} = \tau_{s_j} \quad \text{for } \forall i \neq j \tag{T3-1}$$

WLOG, we again let $\mu(s_i) = t_i$ by reindexing.

Fix two distinct indices $k \neq j$. Consider the following cost, value and budget profile (v^*, c^*, b^*), where for all matched pairs, we have

$$\begin{cases} v^*_{s_i,t_i} = w_{s_i,t_i} \\ c^*_{s_i,t_i} = w_{s_i,t_i} \\ b^*_{s_i} = w_{s_i,t_i} \end{cases} \qquad \text{for all } i$$

and for all unmatched pairs (s,t) where $t \neq \mu(s)$, we have

$$\begin{cases} v^*_{s,t} = 0 \\ c^*_{s,t} = 0 \end{cases} \qquad \text{for all } (s,t) \text{ where } t \neq \mu(s)$$

We now show the above (v^*, c^*, b^*) is consistent with (μ, w), that is

$$(v^*, c^*, b^*) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$$

To show the above, we need to verify the following:

- (μ, w) is individually rational.
  By construction of (v^*, c^*), each school s_i is enjoying a utility of

  $$v^*_{s_i,t_i} - w_{s_i,t_i} = 0 \geq 0 \quad \text{for all } i$$

  each teacher t_i is enjoying a utility of

  $$w_{s_i,t_i} - c^*_{s_i,t_i} = 0 \geq 0 \quad \text{for all } i$$

  so IR hold for all schools and teachers.
- (μ, w) is feasible.
  This is directly implied by the construction of (c^*, b^*) and we have

  $$b^*_{s_i} \geq w_{s_i,t_i} \quad \text{for all } i$$

- There are no blocking pairs.
  By construction of (v^*, c^*), we have

  $$v^*_{s,t} = c^*_{s,t} \quad \text{for all } s, t \text{ where } t \neq \mu(s)$$

  therefore, for any unmatched pair (s,t), any deviating wage $\hat{w}$ strictly profitable for teacher t will require

  $$\hat{w} - c^*_{s,t} > 0$$

  which gives

  $$v^*_{s,t} - \hat{w} < 0$$

  so $\hat{w}$ cannot be profitable for school s, so there are no blocking pairs.

Given the above, we have shown that (μ, w) is stable, which gives

$$(v^*, c^*, b^*) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$$

Fix the above b^*, by definition of $\mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$ and $\mathcal{V}, \mathcal{C}(\mu, w; b)$, we have

$$(v^*, c^*) \in \mathcal{V}, \mathcal{C}(\mu, w; b^*)$$

therefore, we know that $\mathcal{V}, \mathcal{C}(\mu, w; b^*) \neq \emptyset$.

Recall definition of $\mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$ and $\mathcal{V}, \mathcal{C}(\mu, w; b)$, we have

$$(v', c', b^*) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w) \quad \text{for all } (v', c') \in \mathcal{V}, \mathcal{C}(\mu, w; b^*)$$

Since τ is non-distortive for $(\mu, w; b)$, we have

$$\overline{W}(v, c, b, \tau) \geq W(\mu; v, c) \quad \text{for all } (v, c, b) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$$

which implies

$$\overline{W}(\tau, v', b^*, c') \geq W(\mu; v', c') \quad \text{for all } (v', c') \in \mathcal{V}, \mathcal{C}(\mu, w; b^*)$$

which means τ is also non-distortive for $(\mu, w; b^*)$.

Recall Theorem 2, we know τ satisfies (T2-1), which is

$$b^{\tau}_{s_j} - w_{s_j, t_j} \geq \max_{s_i} \tau_{s_i}$$

which gives

$$b^{\tau}_{s_j} - w_{s_j, t_j} \geq \tau_{s_k}$$

Recall construction of b^*, we have

$$b^{\tau}_{s_j} - w_{s_j, t_j} = \tau_{s_j} + b^*_{s_j} - w_{s_j, t_j} = \tau_{s_j} + w_{s_j, t_j} - w_{s_j, t_j} = \tau_{s_j} \geq \tau_{s_k}$$

The above analysis works for any indices $k \neq j$, so we have

$$\tau_{s_j} = \tau_{s_k} \quad \text{for } \forall j \neq k$$

which is the desired (T3-1), finishing the proof of necessity.

Now we show sufficiency, that is, if τ satisfies

$$\tau_{s_i} = \tau_{s_j} \quad \text{for } \forall i \neq j \tag{T3-1}$$

then τ is non-distortive for (μ, w).

For any

$$(v, c, b) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$$

fix the above b, and we want to show τ is non-distortive for $(\mu, w; b)$.

By definition, (μ, w) is stable for any $(v, c, b) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$. Feasibility then gives

$$b_{s_j} \geq w_{s_j, t_j} \quad \text{for all } j \tag{T3-2}$$

Then by (T3-1) and (T3-2), we have

$$b^{\tau}_{s_j} - w_{s_j, t_j} = \tau_{s_j} + b_{s_j} - w_{s_j, t_j} \geq \tau_{s_j} = \tau_{s_i} \quad \text{for } \forall i \neq j$$

which then gives

$$b^{\tau}_{s_j} - w_{s_j,t_j} \geq \max_{s_i} \tau_{s_i} \quad \text{for } \forall j$$

Then by Theorem 2, τ is non-distortive for $(\mu, w; b)$.

Now consider any (v, c, b) such that

$$(v, c, b) \in \mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$$

Then fix b. By definition of $\mathcal{V}, \mathcal{C}, \mathcal{B}(\mu, w)$ and $\mathcal{V}, \mathcal{C}(\mu, w; b)$, we have

$$(v, c) \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

by τ being non-distortive for $(\mu, w; b)$, we have

$$\overline{W}(\tau, v', b, c') \;\geq\; W(\mu; v', c') \quad \text{for all } (v', c') \in \mathcal{V}, \mathcal{C}(\mu, w; b)$$

which implies

$$\overline{W}(v, c, b, \tau) \;\geq\; W(\mu; v, c)$$

Therefore, τ is non-distortive for (μ, w), which proves the sufficiency. $\square$

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