GAMES 101 向量与线性代数

图形学依赖：

数学、物理学（光学和力学）、信号处理、数值分析、美学

Vector 向量/矢量

Usually written as $\vec{a}$ or in bold **a**.

Using start and end point $\vec{AB} = B - A$.

表示方向和长度

并不关心绝对开始的位置

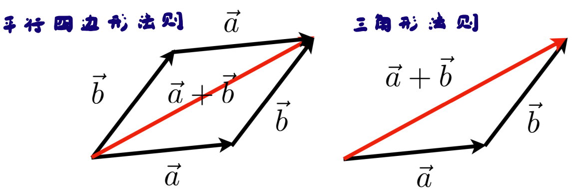
Vector Normalization

Magnitude (length) of a vector written as $\|\vec{a}\|$.

Unit vector 单位向量：$\hat{a} = \vec{a} / \|\vec{a}\|$

used to represent directions.

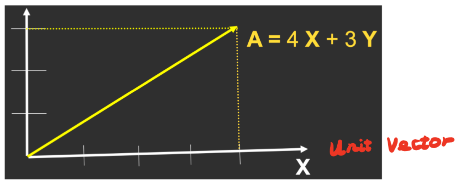
Vector Addition



- Geometrically: Parallelogram law & Triangle law
- Algebraically: Simply add coordinates

Cartesian Coordinates

笛卡尔坐标系



- X and Y can be any (usually orthogonal unit) vectors

列向量

$$\mathbf{A} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{A}^T = (x, y) \qquad \|\mathbf{A}\| = \sqrt{x^2 + y^2}$$

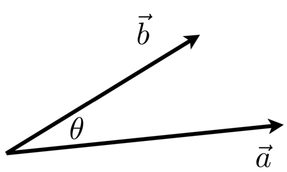
Transpose

Vector Multiplication

Dot product 点乘

Cross product 叉乘

Dot (scalar) Product



vector

$$\vec{a} \cdot \vec{b} = \underbrace{\|\vec{a}\|\|\vec{b}\| \cos\theta}_{\text{number}}$$

vector

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

- For unit vectors

$$\cos\theta = \hat{a} \cdot \hat{b}$$

Dot (scalar) Product

- Properties

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$(k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b}) = k(\vec{a} \cdot \vec{b})$$

Dot Product in Cartesian Coordinates

- Component-wise multiplication, then adding up
  - In 2D

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = x_a x_b + y_a y_b$$

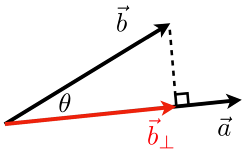
  - In 3D

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = x_a x_b + y_a y_b + z_a z_b$$

17/42

Dot Product for Projection

- $\vec{b}_\perp$: projection of $\vec{b}$ onto $\vec{a}$
  - $\vec{b}_\perp$ must be along $\vec{a}$ (or along $\hat{a}$)
    - $\vec{b}_\perp = k\hat{a}$
  - What's its magnitude k?
    - $k = \|\vec{b}_\perp\| = \|\vec{b}\| \cos\theta$



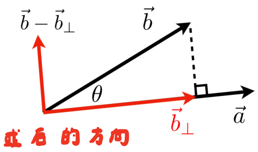
1. 两个向量之间的夹角
2. 一个向量在另一个向量上的投影

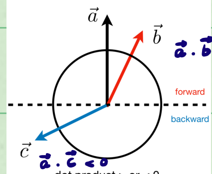
Dot Product in Graphics

把向量进行垂直
or 平行的分解

- Measure how close two directions are
- Decompose a vector
- Determine forward / backward

决定前或后的方向
两个向量有多么接近



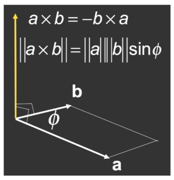


$\vec{a} \cdot \vec{b} > 0$

$\vec{a} \cdot \vec{c} < 0$

Cross (vector) Product

向量叉乘



叉乘的结果仍是一个向量且:
1. 相互垂直
2. 不在同一平面

- Cross product is orthogonal to two initial vectors
- Direction determined by right-hand rule

  a×b: 右手由 $\vec{a}$ 旋转向 $\vec{b}$

- Useful in constructing coordinate systems (later) 用作定义三维坐标系

Cross product: Properties

$\vec{x} \times \vec{y} = +\vec{z}$ 右手坐标系

$\vec{y} \times \vec{x} = -\vec{z}$

$\vec{y} \times \vec{z} = +\vec{x}$

$\vec{z} \times \vec{y} = -\vec{x}$

$\vec{z} \times \vec{x} = +\vec{y}$

$\vec{x} \times \vec{z} = -\vec{y}$

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\vec{a} \times \vec{a} = \vec{0}$

$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$\vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b})$

Cross Product: Cartesian Formula?

$$\vec{a} \times \vec{b} = \begin{pmatrix} y_a z_b - y_b z_a \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix}$$

$\vec{a} = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix}$

$\vec{b} = \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$

- Later in this lecture

$$\vec{a} \times \vec{b} = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

dual matrix of vector a

Cross Product in Graphics

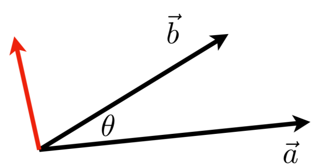
- Determine left / right

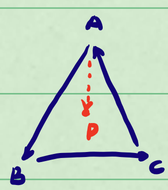
  判定左/右

- Determine **inside / outside**

  判定内/外

  光栅化的基础





$\vec{AB} \times \vec{AP}$

$\vec{BC} \times \vec{BP}$

$\vec{CA} \times \vec{CP}$

P点在 $\triangle ABC$ 内部

P点只在三条边的左边/右边

Orthonormal Coordinate Frames

向量乘法的应用：定义直角坐标系

- Any set of 3 vectors (in 3D) that

$$\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| = 1$$

cosθ $\quad \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{w} = \vec{u} \cdot \vec{w} = 0$

$$\vec{w} = \vec{u} \times \vec{v} \quad \text{(right-handed)}$$

$$\vec{p} = (\vec{p} \cdot \vec{u})\vec{u} + (\vec{p} \cdot \vec{v})\vec{v} + (\vec{p} \cdot \vec{w})\vec{w}$$

(projection)

Matrices 矩阵 计算机图形学利用矩阵表示变换(transform)

What is a matrix

- Array of numbers (m × n = m rows, n columns)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

- Addition and multiplication by a scalar are trivial: element by element

Matrix-Matrix Multiplication

- # (number of) columns in A must = # rows in B
  (M x N) (N x P) = (M x P)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix} = \begin{pmatrix} 1\times3 + 2\times3 & 1\times6 + 3\times7 & 33 & 13 \\ 5\times3 + 2\times2 & 5\times6 + 2\times7 & 61 & 26 \\ 0\times3 + 4\times2 & 0\times6 + 4\times7 & 32 & 12 \end{pmatrix}$$

Element (i,j) in the product is
the dot product of row i from A and column j from B

Matrix-Matrix Multiplication

- Properties
  - **Non-commutative** 没有交换律
    (AB and BA are different in general)
  - Associative and distributive
    - (AB)C=A(BC) 结合律
    - A(B+C) = AB + AC 分配律
    - (A+B)C = AC + BC

Matrix-Vector Multiplication

变换矩阵

- Treat vector as a column matrix (m×1)
- Key for transforming points (next lecture)
- Official spoiler: 2D reflection about y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

Transpose of a Matrix

矩阵的转置

- Switch rows and columns (ij -> ji)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

- Property

$$(AB)^T = B^T A^T$$

Identity Matrix and Inverses

单位矩阵和矩阵的逆

$$I_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AA^{-1} = A^{-1}A = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Vector multiplication in Matrix form

用矩阵表示向量点乘和叉乘

- Dot product?

$$\vec{a}\cdot\vec{b} = \vec{a}^T\vec{b}$$

$$= \underbrace{\overbrace{\begin{pmatrix} x_a & y_a & z_a \end{pmatrix}}^{\text{vector}}}_{1\times N} \underbrace{\overbrace{\begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}}^{\text{vector}}}_{N\times1} = \underbrace{\overbrace{\begin{pmatrix} x_a x_b + y_a y_b + z_a z_b \end{pmatrix}}^{\text{number}}}_{1\times1}$$

- Cross product?

$$\vec{a}\times\vec{b} = A^*b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

dual matrix of vector a