Model Reference Adaptive Control

Tansel Yucelen
University of South Florida

***Abstract*—Adaptive control methods present effective system-theoretical tools in order to achieve closed-loop system stability and performance in the presence of exogenous disturbances and system uncertainties, where they are generally classified as either direct or indirect. A well-known class of direct adaptive control methods is model reference adaptive control architectures. In particular, these architectures employ two major components — a reference model and a parameter adjustment mechanism. A desired closed-loop dynamical system response is captured by the reference model for which its response is compared with the response of the uncertain dynamical system. The system error signal resulting from this comparison drives the parameter adjustment mechanism. This mechanism then adjusts the controller parameters in a real-time (i.e., online) fashion for the purpose of driving the trajectories of the uncertain dynamical system to the trajectories of the reference model. The purpose of this article to discuss these components in a basic state feedback setting in order to provide an introduction to the model reference adaptive control design procedure. We also make connections to several other, relatively advanced model reference adaptive control methods for interested readers.**

I. Introduction

Every physical dynamical system, without an exception, is subject to exogenous disturbances and/or system uncertainties. Specifically, the presence of exogenous disturbances arises from, for example, winds and turbulences and the presence of system uncertainties arises from, for example, idealized assumptions, linearization, and degraded modes of operation. Throughout this article, we often refer to exogenous disturbances and system uncertainties as system anomalies, since they generally tend to negatively alter desired stability and performance characteristics of physical systems.

When designing control laws for dynamical systems subject to anomalies, a fundamental problem is to achieve closed-loop system stability and performance. Similar in spirit to robust control methods, in particular, adaptive control theory presents effective system-theoretical tools to address this fundamental problem. In contrast to robust control methods, however, they have the capability to deal with system anomalies in a real-time (i.e., online) fashion. This implies that are not tuned to a worst-case scenario as robust control methods and they can continuously improve their performance in real-time. From a general point of view, adaptive control methods are classified as either direct or indirect. In this article, we focus on model reference adaptive control architectures, a well-known class of direct adaptive control methods.

Tansel Yucelen is an Assistant Professor of the Department of Mechanical Engineering and the Director of the Laboratory for Autonomy, Control, Information, and Systems (LACIS, `http://lacis.eng.usf.edu/`) at the University of South Florida, Tampa, Florida 33620, United States of America (email: `yucelen@usf.edu`).

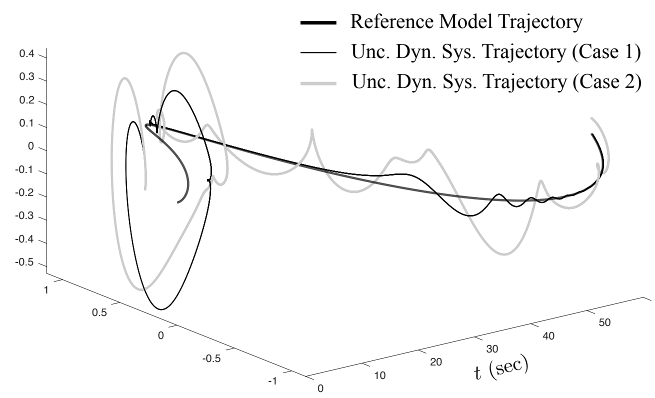

Fig. 1. For a given reference model trajectory, this figure illustrates two responses of a model reference adaptive control architecture for a dynamical system with time-invariant structured system uncertainty (Case 1) and time-varying structured system uncertainty (Case 2). Specifically, in Case 1, observe that the uncertain dynamical system trajectory asymptotically converges to the reference model trajectory. Furthermore, in Case 2, the uncertain dynamical system trajectory converges to a close neighborhood around the reference model trajectory (i.e., this convergence is not exact as the one in Case 1).

While this is not intended to be a survey article (we refer to excellent books [1]–[9] and relatively recent surveys [10]–[13] on this point), one should mention that the authors of [14] and [15] originally proposed the model reference adaptive control concept. Specifically, to control a given uncertain dynamical system (see Section II), this concept has two major components — a reference model (see Section III) and a parameter adjustment mechanism (see Section IV).

A desired closed-loop dynamical system response is captured by the reference model for which its state (respectively, output) is compared with the state (respectively, output) of the uncertain dynamical system. The system error signal resulting from this comparison drives the parameter adjustment mechanism. This mechanism then adjusts the controller parameters in real-time for the purpose of driving the trajectories of the uncertain dynamical system to the trajectories of the reference model. Depending on the nature of the system uncertainties, the difference between these trajectories can asymptotically vanish for time-invariant structured system uncertainties (see Case 1 in Figure 1 as an illustration of the asymptotic convergence notion) or the distance of this difference can remain bounded in time with often an user-adjustable bound for time-varying structured system uncertainties or time-invariant/time-varying unstructured system uncertainties (see Case 2 in Figure 1 as an illustration of the boundedness notion).

In this article, we discuss model reference adaptive control in a basic setting (see Sections II, III, and IV). In particular, for the purpose of providing an introduction to the model reference adaptive control design procedure, we follow a state feedback approach and focus on answering the fundamental question on how a control designer can make the state trajectories of an uncertain dynamical system follow the state trajectories of a given reference model (i.e., state feedback design for state tracking). We also make connections to several other, relatively advanced model reference adaptive control methods for interested readers (see Section V). Finally, a fairly standard notation is used in the following sections, where $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+$ (respectiely, $\overline{\mathbb{R}}_+$) denotes the set of positive (respectively, nonnegative) real numbers, $\mathbb{R}_+^{n\times n}$ (respectively, $\overline{\mathbb{R}}_+^{n\times n}$) denotes the set of $n \times n$ positive-definite (respectively, nonnegative-definite) real matrices, $\mathbb{D}^{n\times n}$ denotes the set of $n \times n$ real matrices with diagonal scalar entries, $\|\cdot\|_2$ denotes the Euclidian norm, $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius matrix norm, $\lambda_{\min}(A)$ (respectively, $\lambda_{\max}(A)$) denotes the minimum (respectively, maximum) eigenvalue of a real and square matrix $A \in \mathbb{R}^{n\times n}$, and "$\triangleq$" denotes the equality by definition.

II. Uncertain Dynamical System

In this article, we consider the dynamics of a physical system represented in the following state-space form

$$\dot{x}_{\mathrm{p}}(t) = A_{\mathrm{p}}x_{\mathrm{p}}(t) + B_{\mathrm{p}}\Lambda u(t) + B_{\mathrm{p}}\delta_{\mathrm{p}}\big(x_{\mathrm{p}}(t)\big), \quad x_{\mathrm{p}}(0) = x_{\mathrm{p}_0}. \quad (1)$$

Here, $x_{\mathrm{p}}(t) \in \mathbb{R}^{n_{\mathrm{p}}}$ stands for a measurable (i.e., accessible) state vector and $u(t) \in \mathbb{R}^m$ stands for a control vector. In addition, $A_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}}\times n_{\mathrm{p}}}$ denotes a known system matrix and $B_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}}\times m}$ denotes a known control matrix. Furthermore, $\delta_{\mathrm{p}} : \mathbb{R}^{n_{\mathrm{p}}} \to \mathbb{R}^m$ represents a time-invariant system uncertainty and $\Lambda \in \mathbb{R}_+^{m\times m} \cap \mathbb{D}^{m\times m}$ represents an unknown control effectiveness matrix. For the well-posedness of the model reference adaptive control problem, we assume that the pair $(A_{\mathrm{p}}, B_{\mathrm{p}})$ is controllable and the system uncertainty is composed of locally Lipschitz functions.

A considerable set of physical systems such as aerial and ground robots are either explicitly (or approximately) satisfy the state-space form in (1). To elucidate this point, consider the mass, spring, and damper system given in Figure 2. In addition, consider that all the physical model parameters α, β, and m are unknown. Letting $x_{\mathrm{p}1}(t) \triangleq p(t)$, $x_{\mathrm{p}2}(t) \triangleq \dot{p}(t)$, and $x_{\mathrm{p}}(t) \triangleq [x_{\mathrm{p}1}(t), x_{\mathrm{p}2}(t)]^{\mathrm{T}}$, the equations of motion of this system, $m\ddot{p}(t) = -\alpha p(t) - \beta\dot{p}^3(t) + u(t)$, can be rewritten as

$$\dot{x}_{\mathrm{p}}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_{\mathrm{p}}} x_{\mathrm{p}}(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_{\mathrm{p}}} \underbrace{[1/m]}_{\Lambda} u(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_{\mathrm{p}}} \underbrace{\begin{bmatrix} -\alpha/m \\ -\beta/m \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} x_{\mathrm{p}1}(t) \\ x_{\mathrm{p}2}^3(t) \end{bmatrix}}_{\delta_{\mathrm{p}}(x_{\mathrm{p}}(t))}, \quad x_{\mathrm{p}}(0) = \underbrace{\begin{bmatrix} p(0) \\ \dot{p}(0) \end{bmatrix}}_{x_{\mathrm{p}0}}, \quad (2)$$

which is clearly in the form given by (1).

Mathematically speaking, considering the way that the system uncertainty appears in (1), it is often said that $\delta_{\mathrm{p}}\big(x_{\mathrm{p}}(t)\big)$ is a matched system uncertainty (i.e., since Λ is nonsingular, the control vector can access to this uncertainty). This state-space form is widely-adopted in the model reference adaptive control literature (see, for example, the books cited above). There are also several works that consider the cases when the nature of the system uncertainties are unmatched (see, for example, relatively recent papers [16]–[20] and references therein).

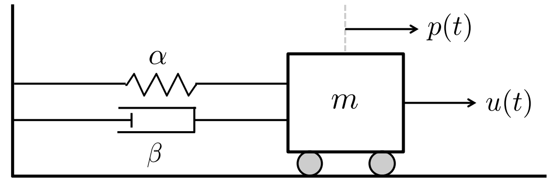

Fig. 2. A mass, spring, and damper system with a linear spring force (i.e., $f_{\mathrm{s}} = -\alpha p(t)$, $\alpha \in \mathbb{R}_+$) and a nonlinear damping force (i.e., $f_{\mathrm{d}} = -\beta\dot{p}^3(t)$, $\beta \in \mathbb{R}_+$) in a frictionless surface ($p(t) \in \mathbb{R}$ denotes the position of the mass with coefficient $m \in \mathbb{R}_+$ and $u(t) \in \mathbb{R}$ denotes the control input).

Next, for model reference adaptive control purposes, it is important to parameterize the system uncertainty in (1). To this end, depending on the nature of the system uncertainty $\delta_{\mathrm{p}}\big(x_{\mathrm{p}}(t)\big)$, one can consider either structured or unstructured parameterizations. Revisiting the mass, spring, and damper problem in (2) as an example, the system uncertainty physically appears in the form that an unknown weight matrix $[-\alpha/m, -\beta/m]$ multiplies a known basis function $[x_{\mathrm{p}1}(t), x_{\mathrm{p}2}^3(t)]^{\mathrm{T}}$. In this case, since the basis function is available to the model reference adaptive control designer, we say that the nature of the system uncertainty is structured (and otherwise we say that it is unstructured). In general, there is a considerable set of physical systems that has structured system uncertainties, specifically the ones that can be modeled by the first principles. Motivated from this standpoint, we here consider that the system uncertainty in (1) satisfies the following structured parameterization (see, for example, the books cited above)

$$\delta_{\mathrm{p}}(x_{\mathrm{p}}) = W_{\mathrm{p}}^{\mathrm{T}}\sigma_{\mathrm{p}}(x_{\mathrm{p}}), \quad x_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}}}, \quad (3)$$

as in the mass, spring, and damper example, where $W_{\mathrm{p}} \in \mathbb{R}^{s\times m}$ denotes an unknown weight matrix and $\sigma_{\mathrm{p}} : \mathbb{R}^{n_{\mathrm{p}}} \to \mathbb{R}^s$ denotes a known basis function of the form

$$\sigma_{\mathrm{p}}(x_{\mathrm{p}}) = \big[\sigma_{\mathrm{p}_1}(x_{\mathrm{p}}), \sigma_{\mathrm{p}_2}(x_{\mathrm{p}}), \ldots, \sigma_{\mathrm{p}_s}(x_{\mathrm{p}})\big]^{\mathrm{T}}. \quad (4)$$

This parameterization also captures exogenous constant disturbances (bias) by setting, for example, the first entry of the basis function to one (i.e., $\sigma_{\mathrm{p}_1}(x_{\mathrm{p}}) = 1$).

Having said that the system uncertainties can be unstructured in certain applications. For example, consider a state-space form of an aerial robot performing an aggressive maneuver at high angles of attack. In this case, the system uncertainty affecting this robot may not be modeled exactly as in (3) and a structured basis function may not be known to the designer. In this case and other similar situations, it is of practice to utilize universal function approximators (e.g., neural networks) to parameterize such unstructured system uncertainties, for example, in the form $\delta_{\rm p}(x_{\rm p})= W_{\rm p}^{\rm T}\sigma_{\rm p}^{\rm app}(x_{\rm p})+\epsilon^{\rm app}(x_{\rm p})$, $x_{\rm p} \in \mathcal{D}_{x_{\rm p}} \subset \mathbb{R}^{n_{\rm p}}$, where $W_{\rm p} \in \mathbb{R}^{s\times m}$ stands for an unknown weight matrix, $\sigma_{\rm p}^{\rm app} : \mathcal{D}_{x_{\rm p}} \to \mathbb{R}^s$ stands for a known basis function (e.g., composed of radial basis functions), and $\epsilon_{\rm p}^{\rm app} : \mathcal{D}_{x_{\rm p}} \to \mathbb{R}^m$ stands for an unknown residual approximation error (see, for example, [9], [21]–[31] and references therein). Note that $\mathcal{D}_{x_{\rm p}}$ is a compact subset of $\mathbb{R}^{n_{\rm p}}$.

Considering the uncertain dynamical system given by (1) subject to the time-invariant structured system uncertainty parameterization given by (3), we now state the model reference adaptive control objective: *In the presence of system anomalies, design a real-time control algorithm in order to drive the trajectories of the uncertain dynamical system to the trajectories of a reference model (i.e., to desired closed-loop dynamical system trajectories determined by the designer).* Based on the discussion related to Figure 1, since we consider a time-invariant structured system uncertainty here, one expects the difference between both trajectories to vanish asymptotically. In order to start addressing the stated model reference adaptive control problem, the next section first discusses the reference model selection.

III. Reference Model

Since a reference model captures a desired closed-loop dynamical system behavior, one needs to define such behavior in order to construct a reference model. In practice, there generally exists a static (see Section III.A) or dynamic (see Section III.B) nominal control law, which achieves a desired closed-loop system behavior in the absence of system anomalies. To this end, it is of practice to select the reference model based on the properties of these nominal control laws. We below discuss the selection of the reference model under two subsections that are aligned with the above discussion.

A. Reference Model Selection in the Presence of a Static Nominal Control Law

Consider the feedback control law given by

$$u(t) \quad = \quad u_{\rm n}(t) + u_{\rm a}(t), \tag{5}$$

where $u_{\rm n}(t) \in \mathbb{R}^m$ and $u_{\rm a}(t) \in \mathbb{R}^m$ respectively stand for the nominal and adaptive control laws. Moreover, consider a static nominal control law in the form given by

$$u_{\rm n}(t) \quad = \quad -K_1 x_{\rm p}(t) + K_2 c(t). \tag{6}$$

In (6), $K_1 \in \mathbb{R}^{m\times n_{\rm p}}$ stands for a feedback gain matrix and $K_2 \in \mathbb{R}^{m\times n_{\rm c}}$ stands for a feedforward gain matrix. In addition, $c(t) \in \mathbb{R}^{n_{\rm c}}$ represents a given uniformly continuous bounded command. Since the static nominal control law given by (6) is designed by a control user in order to achieve a desired closed-loop system behavior in terms of stability and performance in the absence of system anomalies, we now select the reference model based on this insight.

Motivated from the above discussion, we first ignore system anomalies as $(\Lambda, W_{\rm p}) = (I, 0)$ in (1) and set $u_{\rm a}(t) \equiv 0$ in (5). In this case, the following expression follows from (1)

$$\begin{aligned} \dot{x}_{\rm p}(t) &= A_{\rm p} x_{\rm p}(t) + B_{\rm p} u_{\rm n}(t) \\ &= A_{\rm p} x_{\rm p}(t) + B_{\rm p}(-K_1 x_{\rm p}(t) + K_2 c(t)) \\ &= \underbrace{(A_{\rm p} - B_{\rm p}K_1)}_{A_{\rm r}\in\mathbb{R}^{n_{\rm p}\times n_{\rm p}}} x_{\rm p}(t) + \underbrace{B_{\rm p}K_2}_{B_{\rm r}\in\mathbb{R}^{n_{\rm p}\times n_{\rm c}}} c(t) \\ &= A_{\rm r} x_{\rm p}(t) + B_{\rm r} c(t). \end{aligned} \tag{7}$$

Specifically, since system stability is the first requirement in any control system design, one must choose the nominal control law gain K_1 to make $A_{\rm r}$ in (7) is Hurwitz (i.e., the spectrum of $A_{\rm r}$ is composed of eigenvalues with negative real parts). Furthermore, we let $z_{\rm p}(t) \triangleq E_{\rm p} x_{\rm p}(t)$, where $E_{\rm p} \in \mathbb{R}^{n_{\rm c}\times n_{\rm p}}$ allows to select a subset of $x_{\rm p}(t)$ to be followed by $c(t)$. To this end, it is of practice to select K_2 such that

$$-E_{\rm p} A_{\rm r}^{-1} B_{\rm r} \quad = \quad I, \tag{8}$$

holds. With this selection, one can readily show that $z_{\rm p}(t) \to c(t)$ as $t \to \infty$ for constant commands (we refer to the last paragraph of Section III-B for a discussion addressing time-varying commands).

Now, the since selection of the pair (K_1, K_2) yields to a desired level of closed-loop system behavior in the absence of system anomalies, we select the reference model as

$$\dot{x}_{\rm r}(t) \quad = \quad A_{\rm r} x_{\rm r}(t) + B_{\rm r} c(t), \quad x_{\rm r}(0) = x_{\rm r0}, \tag{9}$$

where $x_{\rm r}(t) \in \mathbb{R}^{n_{\rm p}}$ stands for the reference model state vector. Based on this reference model selection subject to an existing static nominal control law, one can now construct the system error between $x_{\rm p}(t)$ and $x_{\rm r}(t)$ as

$$e(t) \quad \triangleq \quad x_{\rm p}(t) - x_{\rm r}(t), \quad e(t) \in \mathbb{R}^{n_{\rm p}}, \tag{10}$$

and develop a parameter adjustment mechanism to drive the trajectories of the uncertain dynamical system to the trajectories of this reference model in the presence of system anomalies (see Section IV).

B. Reference Model Selection in the Presence of a Dynamic Nominal Control Law

Similar to the above subsection, consider the feedback control law in (5). Furthermore, consider a general dynamic nominal control law in the form, for example, given by

$$\dot{x}_{\rm c}(t) \quad = \quad A_{\rm c} x_{\rm c}(t) + B_{\rm c}\phi_{\rm p}(t), \quad x_{\rm c}(0) = x_{\rm c0}, \tag{11}$$

$$z_{\rm c}(t) \quad = \quad C_{\rm c} x_{\rm c}(t) + D_{\rm c}\phi_{\rm p}(t), \tag{12}$$

$$\phi_{\rm p}(t) \quad = \quad E_{\rm p} x_{\rm p}(t) - c(t), \tag{13}$$

$$u_{\rm n}(t) \quad = \quad -K_1^\star x_{\rm p}(t) - K_2^\star z_{\rm c}(t), \tag{14}$$

where $A_{\rm c} \in \mathbb{R}^{p_{\rm a}\times p_{\rm a}}$, $B_{\rm c} \in \mathbb{R}^{p_{\rm a}\times n_{\rm c}}$, $C_{\rm c} \in \mathbb{R}^{p_{\rm b}\times p_{\rm a}}$, and $D_{\rm c} \in \mathbb{R}^{p_{\rm b}\times n_{\rm c}}$ stand for nominal control design matrices, $E_{\rm p} \in \mathbb{R}^{n_{\rm c}\times n_{\rm p}}$ allows to choose a subset of $x_{\rm p}(t)$ to be followed by a given bounded command $c(t)$, and $K_1^\star \in \mathbb{R}^{m\times n_{\rm p}}$ and $K_2^\star \in \mathbb{R}^{m\times p_{\rm b}}$ denote feedback gain matrices. Moreover, $x_{\rm c}(t) \in \mathbb{R}^{p_{\rm a}}$ represents the nominal control state vector, $z_{\rm c}(t) \in \mathbb{R}^{p_{\rm b}}$ represents the nominal control output vector, and $\phi_{\rm p}(t) \in \mathbb{R}^{n_{\rm c}}$ represents the command following error between $z_{\rm p}(t) = E_{\rm p} x_{\rm p}(t)$ and $c(t)$.

Since the dynamic control law given by (11), (12), (13), and (14) is designed by a control user in order to achieve a desired closed-loop system behavior in the absence of system anomalies, in what follows we next set $(\Lambda, W_{\mathrm{p}}) = (I, 0)$ in (1) and $u_{\mathrm{a}}(t) \equiv 0$ in (5) similar to the previous subsection. Specifically, defining

$$x(t) \quad \triangleq \quad [x_{\mathrm{p}}^{\mathrm{T}}(t), x_{\mathrm{c}}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{n_{\mathrm{p}}+p_{\mathrm{a}}}, \tag{15}$$

one can write in this case

$$\begin{aligned} \dot{x}(t) \quad &= \quad \underbrace{\begin{bmatrix} A_{\mathrm{p}} - B_{\mathrm{p}}K_1^\star - B_{\mathrm{p}}K_2^\star D_{\mathrm{c}}E_{\mathrm{p}} & -B_{\mathrm{p}}K_2^\star C_{\mathrm{c}} \\ B_{\mathrm{c}}E_{\mathrm{p}} & A_{\mathrm{c}} \end{bmatrix}}_{A_{\mathrm{r}}^\star \in \mathbb{R}^{(n_{\mathrm{p}}+p_{\mathrm{a}}) \times (n_{\mathrm{p}}+p_{\mathrm{a}})}} x(t) \\ &\quad + \underbrace{\begin{bmatrix} B_{\mathrm{p}}K_2^\star D_{\mathrm{c}} \\ -B_{\mathrm{c}} \end{bmatrix}}_{B_{\mathrm{r}}^\star \in \mathbb{R}^{(n_{\mathrm{p}}+p_{\mathrm{a}}) \times n_{\mathrm{c}}}} c(t) \\ &= \quad A_{\mathrm{r}}^\star x(t) + B_{\mathrm{r}}^\star c(t). \end{aligned} \tag{16}$$

Once again, $A_{\mathrm{r}}^\star$ must be Hurwitz here by the nominal control law design and (16) yields to a desired level of closed-loop system behavior in the absence of system anomalies.

As a consequence, one can now select the reference model in the form given by

$$\dot{x}_{\mathrm{r}}(t) \quad = \quad A_{\mathrm{r}}^\star x_{\mathrm{r}}(t) + B_{\mathrm{r}}^\star c(t), \quad x_{\mathrm{r}}(0) = x_{\mathrm{r}0}, \tag{17}$$

where $x_{\mathrm{r}}(t) \in \mathbb{R}^{n_{\mathrm{p}}+p_{\mathrm{a}}}$ stands for the reference model state vector. Based on this reference model selection subject to an existing dynamic nominal control law, one can also construct the system error as

$$e(t) \quad \triangleq \quad x(t) - x_{\mathrm{r}}(t), \quad e(t) \in \mathbb{R}^{n_{\mathrm{p}}+p_{\mathrm{a}}}, \tag{18}$$

where its role will be clear in Section IV.

Based on the general structure of the dynamic nominal control law given by (11), (12), (13), and (14), we now illustrative how a control designer can select the corresponding matrices to obtain, for example, a proportional-integral nominal control law that is well-adopted in model reference adaptive control architectures. For this purpose, we set $A_{\mathrm{c}} = 0$, $B_{\mathrm{c}} = I$, $C_{\mathrm{c}} = I$, and $D_{\mathrm{c}} = 0$, where this selection also gives $p_{\mathrm{a}} = n_{\mathrm{c}}$ and $p_{\mathrm{b}} = n_{\mathrm{c}}$. The special structure of the proportional-integral nominal control law now follows from the expressions (11), (12), (13), and (14) as

$$\dot{x}_{\mathrm{c}}(t) \quad = \quad E_{\mathrm{p}}x_{\mathrm{p}} - c(t), \tag{19}$$

$$u_{\mathrm{n}}(t) \quad = \quad -Kx(t), \tag{20}$$

where $K \triangleq [K_1^\star, K_2^\star] \in \mathbb{R}^{m \times n}$ and $n \triangleq n_{\mathrm{p}} + n_{\mathrm{c}}$. In this case, the structure of $A_{\mathrm{r}}^\star$ and $B_{\mathrm{r}}^\star$ matrices used in the reference model given by (17) has the form given by

$$A_{\mathrm{r}}^\star \quad = \quad \begin{bmatrix} A_{\mathrm{p}} - B_{\mathrm{p}}K_1^\star & -B_{\mathrm{p}}K_2^\star \\ E_{\mathrm{p}} & 0 \end{bmatrix}, \tag{21}$$

$$B_{\mathrm{r}}^\star \quad = \quad \begin{bmatrix} 0 \\ -I \end{bmatrix}. \tag{22}$$

Furthermore, since $A_{\mathrm{r}}^\star$ is considered to be a Hurwitz matrix (under the rank condition

$$\operatorname{rank}\left(\begin{bmatrix} A_{\mathrm{p}} & B_{\mathrm{p}} \\ E_{\mathrm{p}} & 0 \end{bmatrix}\right) = n_{\mathrm{p}} + m, \tag{23}$$

one can always make $A_{\mathrm{r}}^\star$ Hurwitz in the proportional-integral nominal control law design since the pair $(A_{\mathrm{p}}, B_{\mathrm{p}})$ is assumed to be controllable [Chapter 13.4, 9]), one can readily show that

$$-[E_{\mathrm{p}}, 0]A_{\mathrm{r}}^{\star -1}B_{\mathrm{r}}^\star \quad = \quad I. \tag{24}$$

In other words, once the selection of the pair $(K_1^\star, K_2^\star)$ yields to a Hurwitz $A_{\mathrm{r}}^\star$ matrix, then $z_{\mathrm{p}}(t) \to c(t)$ as $t \to \infty$ for constant commands.

To address time-varying commands, a control designer can pursue one of the following directions. First, the bandwidth of the resulting reference model can be judiciously adjusted to extract a desired level of time-varying command following performance. Second, one can resort to internal model principle to achieve a perfect time-varying command following performance (Chapter 3 in [9], Chapter 1 in [32]), which is standard in the literature. The former method is desired when the commands are not necessarily known before the implementation of the control law (e.g., for cases when an online path planning algorithm generates the command trajectories). The latter method is desired otherwise when the commands are generated by a known exosystem.

IV. Parameter Adjustment Mechanism

Based on two common reference model selections given in Section III.A and Section III.B, we now discuss how to determine the adaptive control law in (5) and derive a parameter adjustment mechanism in order to asymptotically drive the trajectories of the uncertain dynamical system to the trajectories of a selected reference model. We begin our discussion by introducing the system error dynamics, which considers a reference model chosen first based on a static nominal control law (see Section IV.A) and then based on a dynamic nominal control law (see Section IV.B).

A. System Error Dynamics in the Presence of a Static Nominal Control Law

Consider the uncertain dynamical system given by (1) subject to the structured system uncertainty parameterization given by (3). In addition, consider the feedback control law given by (5) subject to the static nominal control law given by (6). Then, one can write

$$\begin{aligned} \dot{x}_{\mathrm{p}}(t) \quad &= \quad A_{\mathrm{p}}x_{\mathrm{p}}(t) + B_{\mathrm{p}}\Lambda(u_{\mathrm{n}}(t) + u_{\mathrm{a}}(t)) + B_{\mathrm{p}}W_{\mathrm{p}}^{\mathrm{T}}\sigma_{\mathrm{p}}\big(x_{\mathrm{p}}\big) \\ &= \quad A_{\mathrm{p}}x_{\mathrm{p}}(t) + B_{\mathrm{p}}u_{\mathrm{n}}(t) + B_{\mathrm{p}}\Lambda(I - \Lambda^{-1})u_{\mathrm{n}}(t) \\ &\quad + B_{\mathrm{p}}\Lambda(u_{\mathrm{a}}(t) + \Lambda^{-1}W_{\mathrm{p}}^{\mathrm{T}}\sigma_{\mathrm{p}}(x_{\mathrm{p}}(t))) \\ &= \quad A_{\mathrm{r}}x_{\mathrm{p}}(t) + B_{\mathrm{r}}c(t) + B_{\mathrm{p}}\Lambda(I - \Lambda^{-1})u_{\mathrm{n}}(t) \\ &\quad + B_{\mathrm{p}}\Lambda\big(u_{\mathrm{a}}(t) + \Lambda^{-1}W_{\mathrm{p}}^{\mathrm{T}}\sigma_{\mathrm{p}}(x_{\mathrm{p}}(t))\big) \\ &= \quad A_{\mathrm{r}}x_{\mathrm{p}}(t) + B_{\mathrm{r}}c(t) + B_{\mathrm{p}}\Lambda\big(u_{\mathrm{a}}(t) + W^{\mathrm{T}}\sigma(\cdot)\big), \end{aligned} \tag{25}$$

where the aggregated system uncertainty matrix is denoted by

$$W \quad \triangleq \quad [\Lambda^{-1}W_{\mathrm{p}}^{\mathrm{T}}, (I - \Lambda^{-1})]^{\mathrm{T}}, \tag{26}$$

and the aggregated basis function composed of known signals is denoted by

$$\sigma(x_{\mathrm{p}}(t), u_{\mathrm{n}}(t)) \quad \triangleq \quad [\sigma_{\mathrm{p}}^{\mathrm{T}}(x_{\mathrm{p}}(t)), u_{\mathrm{n}}^{\mathrm{T}}(t)]^{\mathrm{T}}. \tag{27}$$

In the above aggregated basis function, note that $u_{\mathrm{n}}(t)$ satisfies (6). Based on (25) and the reference model given by (9), the system error dynamics now follows from (10) as

$$\begin{aligned}\dot{e}(t) &= \dot{x}_{\mathrm{p}}(t) - \dot{x}_{\mathrm{r}}(t) \\ &= A_{\mathrm{r}} x_{\mathrm{p}}(t) + B_{\mathrm{r}} c(t) + B_{\mathrm{p}} \Lambda \big(u_{\mathrm{a}}(t) + W^{\mathrm{T}} \sigma(\cdot)\big) \\ &\quad - A_{\mathrm{r}} x_{\mathrm{r}}(t) - B_{\mathrm{r}} c(t) \\ &= A_{\mathrm{r}} e(t) + B_{\mathrm{p}} \Lambda \big(u_{\mathrm{a}}(t) + W^{\mathrm{T}} \sigma(\cdot)\big). \end{aligned} \tag{28}$$

Now, since the goal of the adaptive control law is to suppress the system anomalies, it is of practice to choose it as

$$u_{\mathrm{a}}(t) \quad = \quad -\hat{W}^{\mathrm{T}}(t) \sigma(\cdot), \tag{29}$$

where this selection is motivated by the term "$u_{\mathrm{a}}(t)+W^{\mathrm{T}}\sigma(\cdot)$" in (28). In (29), $\hat{W}(t)$ is an estimate of the aggregated system uncertainty matrix satisfying a parameter adjustment mechanism to be defined below. As a consequence, the system error dynamics given by (28) can be equivalently written as

$$\begin{aligned}\dot{e}(t) &= A_{\mathrm{r}} e(t) - B_{\mathrm{p}} \Lambda \underbrace{\big(\hat{W}(t) - W\big)}_{\tilde{W}(t)}{}^{\mathrm{T}} \sigma(\cdot) \\ &= A_{\mathrm{r}} e(t) - B_{\mathrm{p}} \Lambda \tilde{W}^{\mathrm{T}}(t) \sigma(\cdot). \end{aligned} \tag{30}$$

In practice, there can be situations that both A_{p} and B_{p} in (1) are also unknown. If this is the case, then one may not readily construct a nominal control law since no prior information is available about the physical system; that is, $u_{\mathrm{n}}(t) \equiv 0$ in (5). Yet, it is still possible to construct the system error dynamics as in (30) when the following well-known matching conditions hold: There exist unknown matrices K_1 and K_2 such that $A_{\mathrm{r}} = A_{\mathrm{p}} - B_{\mathrm{p}} K_1$ and $B_{\mathrm{r}} = B_{\mathrm{p}} K_2$ hold. Note that these matching conditions only imply the knowledge of the structure of A_{p} and B_{p} with some unknown parameter entries in the sense that for a judiciously given A_{r} and B_{r} with A_{r} being Hurwitz and B_{r} satisfying (8), one can guarantee the existence of unknown matrices K_1 and K_2 to make both $A_{\mathrm{r}} = A_{\mathrm{p}} - B_{\mathrm{p}} K_1$ and $B_{\mathrm{r}} = B_{\mathrm{p}} K_2$ satisfied. Now, under these matching conditions, one can write

$$\begin{aligned}\dot{e}(t) &= \dot{x}_{\mathrm{p}}(t) - \dot{x}_{\mathrm{r}}(t) \\ &= A_{\mathrm{r}} x_{\mathrm{p}}(t) + B_{\mathrm{r}} c(t) + B_{\mathrm{p}} \Lambda \big(u_{\mathrm{a}}(t) + W^{\mathrm{T}} \sigma(\cdot)\big) \\ &\quad - A_{\mathrm{r}} x_{\mathrm{r}}(t) - B_{\mathrm{r}} c(t) \\ &= A_{\mathrm{r}} e(t) + \big(A_{\mathrm{p}} - A_{\mathrm{r}}\big) x_{\mathrm{p}}(t) + B_{\mathrm{p}} \Lambda \big(u(t) \\ &\quad + \Lambda^{-1} W_{\mathrm{p}}^{\mathrm{T}} \sigma_{\mathrm{p}}(x_{\mathrm{p}}(t)) - \Lambda^{-1} K_2 c(t)\big) \\ &= A_{\mathrm{r}} e(t) + B_{\mathrm{p}} \Lambda \big(u(t) + W^{\mathrm{T}} \sigma(\cdot)\big), \end{aligned} \tag{31}$$

with $W \triangleq \big[\Lambda^{-1} W_{\mathrm{p}}^{\mathrm{T}}, \Lambda^{-1} K_1, -\Lambda^{-1} K_2\big]^{\mathrm{T}}$ being the aggregated system uncertainty that now also includes the unknown matrices K_1 and K_2 and $\sigma(x_{\mathrm{p}}(t), c(t)) \triangleq \big[\sigma_{\mathrm{p}}^{\mathrm{T}}(x_{\mathrm{p}}(t)), x_{\mathrm{p}}^{\mathrm{T}}(t), c^{\mathrm{T}}(t)\big]^{\mathrm{T}}$ being the corresponding aggregated basis function that is again composed of known signals. Therefore, one can choose $u(t) = -\hat{W}^{\mathrm{T}}(t)\sigma(\cdot)$ in order to obtain the error dynamics given by (30). For further details on the content provided in this paragraph, we refer to, for example, Section 9.5 of [9]. To summarize, one can write the system error dynamics given by (30) whether A_{p} and B_{p} in (1) are known or they are unknown when the matching conditions hold for the latter case. For this reason, we focus on the former case in this article without loss of any generality and owing to the fact that there generally exists a nominal control law designed with some modeling knowledge in the feedback control of physical systems subject to system anomalies.

B. System Error Dynamics in the Presence of a Dynamic Nominal Control Law

Consider the uncertain dynamical system given by (1) subject to the structured system uncertainty parameterization given by (3). In addition, consider the feedback control law given by (5) subject to the dynamic nominal control law given by (11), (12), (13), and (14). Then, following similar steps to the ones given in the above subsection, one can write

$$\dot{x}(t) \quad = \quad A_{\mathrm{r}}^{\star} x(t) + B_{\mathrm{r}}^{\star} c(t) + B \Lambda \big(u_{\mathrm{a}}(t) + W^{\mathrm{T}} \sigma(\cdot)\big), \tag{32}$$

where $B \triangleq [B_{\mathrm{p}}^{\mathrm{T}}(t), 0]^{\mathrm{T}}$. In (32), once again, the aggregated system uncertainty matrix is denoted by (26) and the aggregated basis function composed of known signals is denoted by (27). In this aggregated basis function, note that $u_{\mathrm{n}}(t)$ satisfies (14). Based on (32) and the reference model given by (17), the system error dynamics now follows from (18) as

$$\begin{aligned}\dot{e}(t) &= \dot{x}(t) - \dot{x}_{\mathrm{r}}(t) \\ &= A_{\mathrm{r}}^{\star} x(t) + B_{\mathrm{r}}^{\star} c(t) + B \Lambda \big(u_{\mathrm{a}}(t) + W^{\mathrm{T}} \sigma(\cdot)\big) \\ &\quad - A_{\mathrm{r}}^{\star} x_{\mathrm{r}}(t) - B_{\mathrm{r}}^{\star} c(t) \\ &= A_{\mathrm{r}}^{\star} e(t) + B \Lambda \big(u_{\mathrm{a}}(t) + W^{\mathrm{T}} \sigma(\cdot)\big). \end{aligned} \tag{33}$$

Now, since the resulting system error dynamics is identical in spirit to the one derived in the above subsection, we choose the adaptive control law as (29), where this allows one to equivalently write (33) as

$$\begin{aligned}\dot{e}(t) &= A_{\mathrm{r}}^{\star} e(t) - B \Lambda \underbrace{\big(\hat{W}(t) - W\big)}_{\tilde{W}(t)}{}^{\mathrm{T}} \sigma(\cdot) \\ &= A_{\mathrm{r}}^{\star} e(t) - B \Lambda \tilde{W}^{\mathrm{T}}(t) \sigma(\cdot). \end{aligned} \tag{34}$$

Finally, following the discussion provided in the last paragraph of Section IV-A, we would like to highlight that it is also possible here to write the system error dynamics as in (34) for the case when both A_{p} and B_{p} in (1) are also unknown.

C. Derivation of the Parameter Adjustment Mechanism

In this subsection, we now derive a standard parameter adjustment mechanism (see, for example, the books cited earlier in this article) for $\hat{W}(t)$ used in the adaptive control signal given by (29) in order to asymptotically drive the trajectories of the uncertain dynamical system to the trajectories of a selected reference model. We begin by noting that the system error dynamics given by (30) based on a static nominal control law is identical in spirit to the system error dynamics given by (34) based on a dynamic control law. Therefore, in what follows, we consider the system error dynamics given by

$$\dot{e}(t) = Fe(t) - G\Lambda\tilde{W}^{\mathrm{T}}(t)\sigma(\cdot), \quad e(0) = e_0, \quad (35)$$

where the pair $(F, G) = (A_{\mathrm{r}}, B_{\mathrm{p}})$ for the system error dynamics given by (30) and $(F, G) = (A_{\mathrm{r}}^{\star}, B)$ for the system error dynamics given by (34). Note also that F is Hurwitz since either A_{r} or $A_{\mathrm{r}}^{\star}$ is Hurwitz. In addition, F being Hurwitz implies that the Lyapunov equation given by

$$0 = F^{\mathrm{T}}P + PF + I, \quad (36)$$

holds for a unique positive-definite P [33], [34].

Next, we utilize the Lyapunov function candidate in the form given by

$$\mathcal{V}\big(e, \tilde{W}\big) = e^{\mathrm{T}}Pe + \gamma^{-1}\mathrm{tr}\ \big(\tilde{W}\Lambda^{\frac{1}{2}}\big)^{\mathrm{T}}\big(\tilde{W}\Lambda^{\frac{1}{2}}\big). \quad (37)$$

Note that $\mathcal{V}\big(0,0\big) = \big[e^{\mathrm{T}}Pe + \gamma^{-1}\mathrm{tr}\ \big(\tilde{W}\Lambda^{\frac{1}{2}}\big)^{\mathrm{T}} \cdot\big(\tilde{W}\Lambda^{\frac{1}{2}}\big)\big]_{(e,\tilde{W})=(0,0)} = 0$, $\mathcal{V}\big(e, \tilde{W}\big) > 0$ for all $(e, \tilde{W})$ excluding $(e, \tilde{W}) = (0, 0)$, and $\mathcal{V}\big(e, \tilde{W}\big)$ is radially unbounded. Moreover, $\gamma \in \mathbb{R}_+$ used in the last term of (37) is discussed in the next subsection. The time derivative of this Lyapunov function candidate is now calculated as

$$\begin{aligned}
\dot{\mathcal{V}}(\cdot) &= 2e^{\mathrm{T}}(t)P\dot{e}(t) + 2\gamma^{-1}\mathrm{tr}\ \Lambda\tilde{W}^{\mathrm{T}}(t)\dot{\tilde{W}}(t) \\
&= 2e^{\mathrm{T}}(t)PFe(t) - 2e^{\mathrm{T}}(t)PG\Lambda\tilde{W}^{\mathrm{T}}(t)\sigma(\cdot) \\
&\quad +2\gamma^{-1}\mathrm{tr}\ \Lambda\tilde{W}^{\mathrm{T}}(t)\dot{\hat{W}}(t) \\
&= e^{\mathrm{T}}(t)(F^{\mathrm{T}}P + PF)e(t) - 2e^{\mathrm{T}}(t)PG\Lambda\tilde{W}^{\mathrm{T}}(t)\sigma(\cdot) \\
&\quad +2\gamma^{-1}\mathrm{tr}\ \Lambda\tilde{W}^{\mathrm{T}}(t)\dot{\hat{W}}(t) \\
&= -e^{\mathrm{T}}(t)e(t) - 2\mathrm{tr}\ \Lambda\tilde{W}^{\mathrm{T}}(t)\sigma(\cdot)e^{\mathrm{T}}(t)PG \\
&\quad +2\gamma^{-1}\mathrm{tr}\ \Lambda\tilde{W}^{\mathrm{T}}(t)\dot{\hat{W}}(t) \\
&= -\|e(t)\|_2^2 - 2\gamma^{-1}\mathrm{tr}\ \Lambda\tilde{W}^{\mathrm{T}}(t)\big(\gamma\sigma(\cdot)e^{\mathrm{T}}(t)PG \\
&\quad -\dot{\hat{W}}(t)\big). \quad (38)
\end{aligned}$$

Motivated by the term "$\gamma\sigma(\cdot)e^{\mathrm{T}}(t)PG - \dot{\hat{W}}(t)$" in (38), select the parameter adjustment mechanism as

$$\dot{\hat{W}}(t) = \gamma\sigma(\cdot)e^{\mathrm{T}}(t)PG, \quad \hat{W}(0) = \hat{W}_0. \quad (39)$$

Using (39) in (38), one can now write

$$\dot{\mathcal{V}}(\cdot) = -\|e(t)\|_2^2 \leq 0, \quad (40)$$

which guarantees that the system error $e(t)$ and the weight estimation error $\tilde{W}(t)$ are Lyapunov stable. This result further means that the pair $(e(t), \tilde{W}(t))$ are bounded for all $t \in \overline{\mathbb{R}}_+$. Since $\sigma(\cdot)$ is bounded for all $t \in \overline{\mathbb{R}}_+$ in this case, it follows from (35) that $\dot{e}(t)$ is bounded. As a consequence, $\ddot{\mathcal{V}}(\cdot)$ is bounded for all $t \in \overline{\mathbb{R}}_+$. It now follows from the Barbalat's lemma [9], [33], [34] that

$$\lim_{t\to\infty} \dot{\mathcal{V}}\big(e(t), \tilde{W}(t)\big) = 0. \quad (41)$$

Therefore,

$$\lim_{t\to\infty} e(t) = 0, \quad (42)$$

which shows that the parameter adjustment mechanism given by (39) asymptotically drives the trajectories of the uncertain dynamical system to the trajectories of the selected reference model. This conclusion addresses the model reference adaptive control problem.

Finally, the above result does not necessarily imply

$$\lim_{t\to\infty} \tilde{W}(t) = 0, \quad (43)$$

unless the closed-loop dynamical system is persistently exciting (we refer to the books cited earlier on this point). From a practical standpoint, (43) may not be necessary since (42) is sufficient to conclude that an uncertain dynamical system of interest can be asymptotically made to behave as a reference model capturing a desired closed-loop dynamical system performance.

D. Role of the Learning Rate

The positive scalar γ used in the parameter adjustment mechanism given by (39) is called as the learning rate (adaptation gain) in the literature. Note that this scalar can be replaced by its positive-definite diagonal matrix version provided that this matrix version is also properly used in the Lyapunov function candidate given by (37). Specifically, the role of the learning rate in the model reference adaptive control design is to control system error transients.

To elucidate this point, note that

$$\begin{aligned}
\mathcal{V}(e(t), \tilde{W}(t)) &\leq \mathcal{V}(e_0, \tilde{W}_0) \\
&= e_0^{\mathrm{T}}Pe_0 + \gamma^{-1}\mathrm{tr}\ \big(\tilde{W}_0\Lambda^{\frac{1}{2}}\big)^{\mathrm{T}}\big(\tilde{W}_0\Lambda^{\frac{1}{2}}\big),
\end{aligned} \quad (44)$$

which follows from (40). For the ease of the following discussion, consider that $e_0 = 0$ and $\tilde{W}_0 = W$, where this initialization of the model reference adaptive controller yields

$$\mathcal{V}(e(t), \tilde{W}(t)) \leq \gamma^{-1}\mathrm{tr}\ \big(W\Lambda^{\frac{1}{2}}\big)^{\mathrm{T}}\big(W\Lambda^{\frac{1}{2}}\big). \quad (45)$$

Now, using $\mathcal{V}(e(t), \tilde{W}(t)) \geq e^{\mathrm{T}}(t)Pe(t) \geq \lambda_{\min}(P)\|e(t)\|_2^2$ in (45) gives

$$\|e(t)\|_2 \leq \gamma^{-\frac{1}{2}}\lambda_{\min}^{-\frac{1}{2}}(P)\|W\Lambda^{\frac{1}{2}}\|_{\mathrm{F}}, \quad (46)$$

which shows that the system error is inversely proportional to the learning rate. As a consequence, it can be made small during the transient time subject to a large learning rate.

On the expression given by (46), one should also make the following two observations for interested readers. In particular, the first one is that the upper bound on the system error given by (46) may get overly conservative as the matrix dimension of the product $W\Lambda^{\frac{1}{2}}$ increases. The second one is that this upper bound includes the unknown pair (Λ, W), where this implies that it may not be a-priori verifiable at the design stage especially in the absence of excessive numerical (e.g., Monte-Carlo) studies and vehicle testing. Therefore, it is of practical interest to study how model reference adaptive control architectures can achieve a-priori, user-defined performance guarantees for safety-critical applications. To this end, the authors of, for example, [35]–[38] (see also references therein) use a system error dependent learning rate $\gamma(e(t))$ instead of the constant one to address this problem.

E. Time-Varying System Uncertainties

In the presence of time-varying system uncertainties, that is, when $\delta_{\rm p}\big(x_{\rm p}(t)\big)$ is replaced with $\delta_{\rm p}\big(t, x_{\rm p}(t)\big)$ in (1), we consider the structured parameterization given by

$$\delta_{\rm p}\big(t, x_{\rm p}\big) \quad = \quad W_{\rm p}^{\rm T}(t)\sigma_{\rm p}\big(x_{\rm p}\big), \quad x_{\rm p} \in \mathbb{R}^{n_{\rm p}}, \tag{47}$$

instead of the one in (3), where it is generally assumed that $\|W_{\rm p}(t)\|_{\rm F} \leq w_{\rm p}$ and $\|\dot{W}_{\rm p}(t)\|_{\rm F} \leq \dot{w}_{\rm p}$ with $w_{\rm p} \in \mathbb{R}_+$ and $\dot{w}_{\rm p} \in \mathbb{R}_+$ (it is possible to remove the latter assumption for applications involving sudden change in dynamics; see, for example, [28], [39]). This parameterization now captures exogenous (time-varying) disturbances by setting, for example, the first entry of the basis function to one (i.e., $\sigma_{{\rm p}_1}\big(x_{\rm p}\big)= 1$). In this case, only the definition of the aggregated system uncertainty matrix in (26) changes to

$$W(t) \quad \triangleq \quad [\Lambda^{-1}W_{\rm p}^{\rm T}(t), (I - \Lambda^{-1})]^{\rm T}, \tag{48}$$

and the system error dynamics given by (35) remains the same. Note here that $\|W_{\rm p}(t)\|_{\rm F} \leq w_{\rm p}$ and $\|\dot{W}_{\rm p}(t)\|_{\rm F} \leq \dot{w}_{\rm p}$ imply $\|W(t)\|_{\rm F} \leq w$ and $\|\dot{W}(t)\|_{\rm F} \leq \dot{w}$ with $w \in \mathbb{R}_+$ and $\dot{w} \in \mathbb{R}_+$.

For ensuring robustness of model reference adaptive control designs subject to time-varying system uncertainties, it is of practice to modify the parameter adjustment mechanism in (39) with a leakage term $-\alpha\hat{W}(t)$, $\alpha \in \mathbb{R}_+$ (called σ-modification term [40]), with a system error dependent leakage term $-\beta\|e(t)\|_2\hat{W}(t)$, $\beta \in \mathbb{R}_+$ (called e-modification term [41]), or by resorting to a projection operator [42]. For example, with the leakage term $-\alpha\hat{W}(t)$ added to (39), the modified parameter adjustment mechanism becomes

$$\dot{\hat{W}}(t) \quad = \quad \gamma\big(\sigma(\cdot)e^{\rm T}(t)PG - \alpha\hat{W}(t)\big). \tag{49}$$

Note that all these modification terms are also effective tools for ensuring robustness of model reference adaptive control designs subject to not only time-varying structured system uncertainties but also time-invariant/time-varying unstructured system uncertainties.

Based on the discussion related to Figure 1, one now expects the boundedness of the system error dynamics given by (35) subject to the modified parameter adjustment mechanism given by (49) in the case of time-varying system uncertainties. To elucidate this point, the time derivative of the Lyapunov function candidate in (37) can be calculated as

$$\begin{aligned}\dot{\mathcal{V}}(\cdot) \quad &= \quad -\|e(t)\|_2^2 - 2\alpha\ {\rm tr}\Lambda\tilde{W}^{\rm T}(t)\hat{W}(t) \\ &\qquad -2\gamma^{-1}\ {\rm tr}\Lambda\tilde{W}^{\rm T}(t)\dot{W}(t) \\ &= \quad -\|e(t)\|_2^2 - 2\alpha\ {\rm tr}\Lambda\tilde{W}^{\rm T}(t)(\tilde{W}(t) + W(t)) \\ &\qquad -2\gamma^{-1}\ {\rm tr}\Lambda\tilde{W}^{\rm T}(t)\dot{W}(t) \\ &\leq \quad -\|e(t)\|_2^2 - 2\alpha\|\tilde{W}(t)\Lambda^{\frac{1}{2}}\|_{\rm F}^2 + 2\theta\|\tilde{W}(t)\|_{\rm F},\end{aligned} \tag{50}$$

where

$$\theta \quad \triangleq \quad \|\Lambda\|_{\rm F}(\alpha w + \gamma^{-1}\dot{w}). \tag{51}$$

In (50), note that

$$-2\alpha\|\tilde{W}(t)\Lambda^{\frac{1}{2}}\|_{\rm F}^2 \quad \leq \quad -2\alpha\lambda_{\min}(\Lambda)\|\tilde{W}(t)\|_{\rm F}^2, \tag{52}$$

$$2\theta\|\tilde{W}(t)\|_{\rm F} \quad \leq \quad k^\star\|\tilde{W}(t)\|_{\rm F}^2 + \frac{1}{k^\star}\theta^2, \tag{53}$$

where the latter is an application of Young's inequality with an arbitrary scalar $k^\star \in \mathbb{R}_+$. For mathematical convenience, we here set this scalar to $k^\star = \alpha\lambda_{\min}(\Lambda)$. As a consequence, the following expression now follows from (50)

$$\dot{\mathcal{V}}(\cdot) \ \leq \ -\|e(t)\|_2^2 - \alpha\lambda_{\min}(\Lambda)\|\tilde{W}(t)\|_{\rm F}^2 + \frac{\theta^2}{\alpha\lambda_{\min}(\Lambda)}. \tag{54}$$

Note that (54) implies $\dot{\mathcal{V}}(\cdot) \leq 0$ outside the compact set

$$\mathcal{S} \quad \triangleq \quad \Big\{\big(e(t), \tilde{W}(t)\big)\colon \|e(t)\|_2 \leq \underbrace{\frac{\theta}{(\alpha\lambda_{\min}(\Lambda))^{\frac{1}{2}}}}_{\zeta_1}\Big\} \cap \Big\{\big(e(t), \tilde{W}(t)\big)\colon \|\tilde{W}(t)\|_{\rm F} \leq \underbrace{\frac{\theta}{\alpha\lambda_{\min}(\Lambda)}}_{\zeta_2}\Big\}. \tag{55}$$

Therefore, the evolution of $\mathcal{V}(\cdot)$ is upper bounded by

$$\begin{aligned}\mathcal{V}(\cdot) \quad &\leq \quad \max_{(e(t),\tilde{W}(t))\in\mathcal{S}} \mathcal{V}(\cdot) \\ &= \quad \lambda_{\max}(P)\zeta_1^2 + \gamma^{-1}\lambda_{\max}(\Lambda)\zeta_2^2 \\ &= \quad \theta^2\bigg(\frac{\lambda_{\max}(P)}{\alpha\lambda_{\min}(\Lambda)} + \frac{\gamma^{-1}\lambda_{\max}(\Lambda)}{\alpha^2\lambda_{\min}^2(\Lambda)}\bigg),\end{aligned} \tag{56}$$

since $\mathcal{V}(\cdot)$ cannot grow outside $\mathcal{S}$. This establishes the boundedness of the pair $\big(e(t), \tilde{W}(t)\big)$.

Using $\lambda_{\min}(P)\|e(t)\|_2^2 \leq \mathcal{V}(\cdot)$ in (56), one can obtain the bound of the system error for $t \geq T$ as

$$\begin{aligned}\|e(t)\|_2 \quad &\leq \quad \frac{\big(\max_{(e(t),\tilde{W}(t))\in\mathcal{S}}\mathcal{V}(\cdot)\big)^{\frac{1}{2}}}{(\lambda_{\min}(P))^{\frac{1}{2}}} \\ &= \quad \frac{\theta}{(\lambda_{\min}(P))^{\frac{1}{2}}}\bigg(\frac{\lambda_{\max}(P)}{\alpha\lambda_{\min}(\Lambda)} + \frac{\gamma^{-1}\lambda_{\max}(\Lambda)}{\alpha^2\lambda_{\min}^2(\Lambda)}\bigg)^{\frac{1}{2}} \\ &= \quad \frac{\|\Lambda\|_{\rm F}(\alpha w + \gamma^{-1}\dot{w})}{(\lambda_{\min}(P))^{\frac{1}{2}}}\bigg(\frac{\lambda_{\max}(P)}{\alpha\lambda_{\min}(\Lambda)} \\ &\qquad + \frac{\gamma^{-1}\lambda_{\max}(\Lambda)}{\alpha^2\lambda_{\min}^2(\Lambda)}\bigg)^{\frac{1}{2}} = \zeta^\star,\end{aligned} \tag{57}$$

where this bound $\zeta^\star$ can be made small by the model reference adaptive control designer through judiciously tuning the parameter pair (γ, α) (e.g., see Figure 3). The above analysis can be also used for the system error dependent leakage term and the projection operator with appropriate changes.

Finally, we note that the above modification terms are also popular system-theoretical tools to achieve closed-loop system stability when the uncertain dynamical system given by (1) has added perturbation terms. To elucidate this point in a simple setting, consider, for example, the dynamics of a physical system represented in the following state-space form

$$\begin{aligned}\dot{x}_{\rm p}(t) \quad &= \quad A_{\rm p}x_{\rm p}(t) + B_{\rm p}\Lambda u(t) + B_{\rm p}\delta_{\rm p}\big(t, x_{\rm p}(t)\big) + d(t), \\ &\qquad x_{\rm p}(0) = x_{{\rm p}_0},\end{aligned} \tag{58}$$

where the perturbation term $d(t) \in \mathbb{R}^{n_{\rm p}}$ is added under the assumption $\|d(t)\|_2 \leq d^\star$, $d^\star \in \mathbb{R}_+$. Here, it is considered that this perturbation term is unmatched (i.e., $B_{\rm p}^{\rm T}d(t) = 0$). Otherwise, it can be included in $\delta_{\rm p}\big(t, x_{\rm p}(t)\big)$ based on the

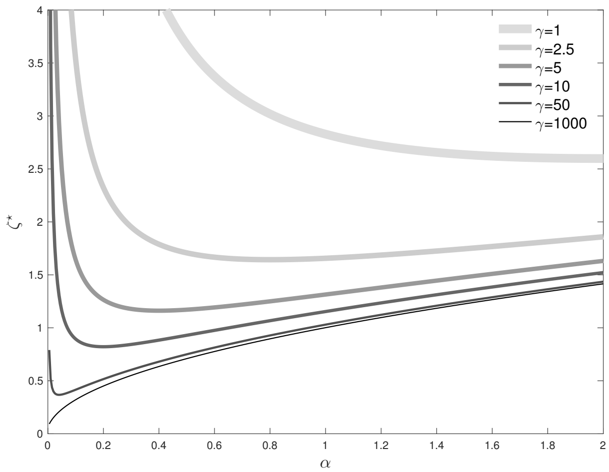

Fig. 3. An illustration of the system error bound $\zeta^\star$ given by (57) as a function of the parameter pair (γ, α) with $w = 1$, $\dot{w} = 1$, $\lambda_{\min}(P) = 1$, $\lambda_{\max}(P) = 1$, $\|\Lambda\|_{\mathrm{F}} = 1$, $\lambda_{\min}(\Lambda) = 1$, and $\lambda_{\max}(\Lambda) = 1$.

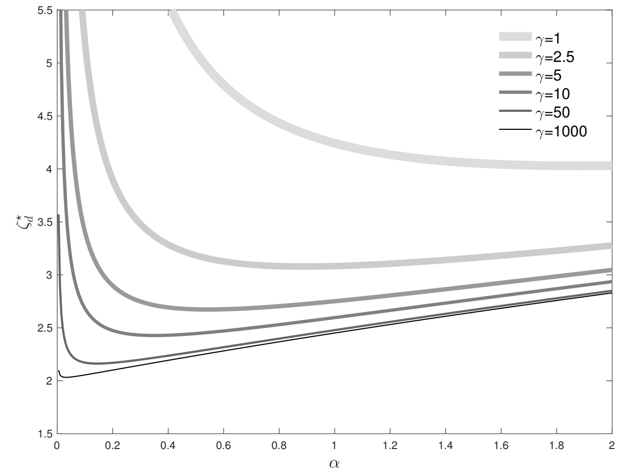

Fig. 4. An illustration of the system error bound $\zeta_d^\star$ given by (64) as a function of the parameter pair (γ, α) with $d^\star = 1$, $w = 1$, $\dot{w} = 1$, $\lambda_{\min}(P) = 1$, $\lambda_{\max}(P) = 1$, $\|\Lambda\|_{\mathrm{F}} = 1$, $\lambda_{\min}(\Lambda) = 1$, and $\lambda_{\max}(\Lambda) = 1$.

above discussion. In this case, one now has the system error dynamics given by

$$\dot{e}(t) = Fe(t) - G\Lambda\tilde{W}^{\mathrm{T}}(t)\sigma(\cdot) + d_{\mathrm{u}}(t), \quad e(0) = e_0, \tag{59}$$

where $(F, G, d_{\mathrm{u}}(t)) = (A_{\mathrm{r}}, B_{\mathrm{p}}, d(t))$ in the presence of the static nominal control law and $(F, G, d_{\mathrm{u}}(t)) = (A_{\mathrm{r}}, B_{\mathrm{p}}, Dd(t))$ in the presence of the dynamic nominal control law with $D \triangleq [I, 0]^{\mathrm{T}}$. For both cases, we note that $\|d_{\mathrm{u}}(t)\|_2 \leq d^\star$, which follows from $\|d(t)\|_2 \leq d^\star$.

Based on the modified parameter adjustment mechanism given by (49), the expression in (54) now becomes

$$\begin{aligned}\dot{\mathcal{V}}(\cdot) \;\leq\; & -\|e(t)\|_2^2 - \alpha\lambda_{\min}(\Lambda)\|\tilde{W}(t)\|_{\mathrm{F}}^2 + \frac{\theta^2}{\alpha\lambda_{\min}(\Lambda)} \\ & +2e^{\mathrm{T}}(t)Pd_{\mathrm{u}}(t).\end{aligned} \tag{60}$$

For the last term on the right hand side of (60), one can now obtain an upper bound given by

$$\begin{aligned}2e^{\mathrm{T}}(t)Pd_{\mathrm{u}}(t) \;&\leq\; 2\lambda_{\max}(P)d^\star\|e(t)\|_2 \\ &\leq\; 0.5\|e(t)\|_2^2 + 2\lambda_{\max}^2(P)(d^\star)^2,\end{aligned} \tag{61}$$

where Young's inequality is used. Using (61) in (60), we can then write

$$\begin{aligned}\dot{\mathcal{V}}(\cdot) \;\leq\; & -0.5\|e(t)\|_2^2 - \alpha\lambda_{\min}(\Lambda)\|\tilde{W}(t)\|_{\mathrm{F}}^2 + \frac{\theta^2}{\alpha\lambda_{\min}(\Lambda)} \\ & +2\lambda_{\max}^2(P)(d^\star)^2,\end{aligned} \tag{62}$$

where $\dot{\mathcal{V}}(\cdot) \leq 0$ outside the compact set

$$\begin{aligned}\mathcal{S}' \;\triangleq\; & \Big\{\big(e(t), \tilde{W}(t)\big) \colon \|e(t)\|_2 \leq \zeta_1'\Big\} \cap \Big\{\big(e(t), \tilde{W}(t)\big) \colon \\ & \|\tilde{W}(t)\|_{\mathrm{F}} \leq \zeta_2'\Big\}.\end{aligned} \tag{63}$$

with $\zeta_1' \triangleq \big(\frac{2\theta^2}{\alpha\lambda_{\min}(\Lambda)} + 4\lambda_{\max}^2(P)(d^\star)^2\big)^{\frac{1}{2}}$ and $\zeta_2' \triangleq \big(\frac{\theta^2}{\alpha\lambda_{\min}(\Lambda)} + 2\lambda_{\max}^2(P)(d^\star)^2\big)^{\frac{1}{2}} / \big(\alpha\lambda_{\min}(\Lambda)\big)^{\frac{1}{2}}$. Following the previous analysis steps identically, one can now establish the boundedness of the pair $\big(e(t), \tilde{W}(t)\big)$, where the bound of the system error for $t \geq T$ is given by

$$\|e(t)\|_2 \;\leq\; \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}(\zeta_1')^2 + \gamma^{-1}\frac{\lambda_{\max}(\Lambda)}{\lambda_{\min}(P)}(\zeta_2')^2\right)^{\frac{1}{2}} = \zeta_d^\star. \tag{64}$$

Unlike the previous bound $\zeta^\star$ given by (57), which is derived in the absence of the considered perturbation term $d(t)$, the new bound $\zeta_d^\star$ given by (64) may not be made small by the model reference adaptive control designer (e.g., see Figure 4) since the term $d(t)$ is considered to be unmatched. To address this issue especially when $d^\star$ is not sufficiently small, we refer to, for example, [20]. Once again, the above analysis guaranteeing closed-loop system stability can be also used for the system error dependent leakage term and the projection operator with appropriate changes.

V. Discussions

In this section, we first present a model reference adaptive control design example on the mass, spring, and damper system shown in Figure 2 (Section V.A) and then make connections to several other available model reference adaptive control methods for interested readers (Section V.B).

A. Illustrative Numerical Example

Consider the mass, spring, and damper system given by (2) with a linear spring force $f_{\mathrm{s}} = -\alpha p(t)$ and a nonlinear damping force $f_{\mathrm{d}} = -\beta\dot{p}^3(t)$ in a frictionless surface; see Figure 2. In addition, consider that all the physical model parameters α, β, and m are unknown (we set these unknown parameters to $\alpha = 1$, $\beta = 1$, and $m = 10$ in this numerical study). Based on the known pair $(A_{\mathrm{p}}, B_{\mathrm{p}})$ of this physical system, we design a static nominal control law in the form given by (6), where we set $K_1 = [0.250, 1.000]$ and $K_2 = [0.250]$. Note that this selection of the static nominal control law gains yield to a reference model response captured by (9) in the form

$$\dot{x}_{\rm r}(t) \;=\; \underbrace{\begin{bmatrix} 0 & 1 \\ -0.250 & -1.000 \end{bmatrix}}_{A_{\rm r}} x_{\rm r}(t) + \underbrace{\begin{bmatrix} 0 \\ 0.250 \end{bmatrix}}_{B_{\rm r}} c(t), \quad (65)$$

which has a natural frequency of 0.5 radians per second and a damping ratio of 1.0. Furthermore, with $E_{\rm p} = [1, 0]$ in (65), this reference model design guarantees (8). We also choose a filtered square wave reference command $c(t)$.

For executing the parameter adjustment mechanism given by (39) for the adaptive control law in (29), we solved the Lyapunov equation given by (36) as

$$P \;=\; \begin{bmatrix} 2.625 & 2.000 \\ 2.000 & 2.500 \end{bmatrix}. \quad (66)$$

Moreover, in this numerical study, all initial conditions are set to zero. Figure 5 presents the above model reference adaptive control design for three learning rates. Specifically, when $\gamma = 0$, a desirable command following performance could not be obtained, since this case corresponds to $u_{\rm a}(t) \equiv 0$ here. When we set $\gamma = 0.5$, a desirable command following performance is achieved around $t = 70$ seconds. When this learning rate is increased to $\gamma = 5.0$, a desirable command following performance is achieved around $t = 25$ seconds. For cases when $\gamma \neq 0$, the designed parameter adjustment mechanism asymptotically drives the trajectories of the considered uncertain dynamical system to the trajectories of the selected reference model. While we consider here a static nominal control law based adaptive control design, the same conclusion can be easily illustrated with an adaptive control design utilizing a dynamic nominal control law.

B. Further Reading

We begin by noting that several topics presented here in detail partially build on the model reference adaptive control overview sections of the author documented in, for example, [20], [28], [37], [38], [43]–[51]. In addition, we would like to refer to the recent book [9] as well as other books [1]–[8] for reading related to the topics covered in this article on model reference adaptive control theory. For the nonlinear stability analysis tools and methods adopted in this article, we also refer to the excellent books [33], [34].

While the goal of this article is to provide an introduction to the model reference adaptive control design procedure in a basic setting, we also would like to make connections to several other, relatively advanced methods for interested readers. Once again, this is not intended to be a survey article and the following list only presents a fraction of available related literature from the viewpoint of the author (see, for example, relatively recent surveys [10]–[13] for a detailed literature review as well as the aforementioned books). In particular, for the purpose of achieving stringent transient performance characteristics, a model reference adaptive control designer may resort to high learning rates as noted in Section IV.D. However, as it is known, parameter adjustment mechanisms with high learning rates may yield to signals with high-frequency content (e.g., compare the oscillations between $\gamma = 0.5$ and $\gamma = 5.0$ cases in Figure 5), which can result in system instability due to, for example, the presence of unmodeled system dynamics [46], [52], [53]. To this end, the authors of [5], [43], [44], [46], [54]–[64] (also see references therein) make notable contributions to the field of model reference adaptive control theory, where their algorithms have the ability to suppress high-frequency oscillations.

From the robustness point of view, important practical developments in model reference adaptive control theory is the consideration of the presence of actuator (amplitude saturation, rate saturation, and/or dynamics) limitations and unmodeled dynamics in feedback loops. We refer to the contributions documented in [48], [65]–[82] on the former topic and [7], [39], [50], [52], [83]–[97] on the latter topic (also see references therein). Furthermore, for applications where parameter convergence (43) is desired under a relaxed persistency of excitation condition, we refer to relatively recent contributions in [30], [98]–[101] and references therein. For adaptive control architectures that employ a gain scheduled, time varying, and/or nonlinear reference models, we also refer to, for example, notable research results in [102]–[112].

As noted earlier, Sections II, III, and IV of this article focuses on a state feedback model reference adaptive control design for state tracking. For important research contributions that focuses on state feedback model reference adaptive control design for output tracking and output feedback model reference adaptive control design for output tracking, we refer

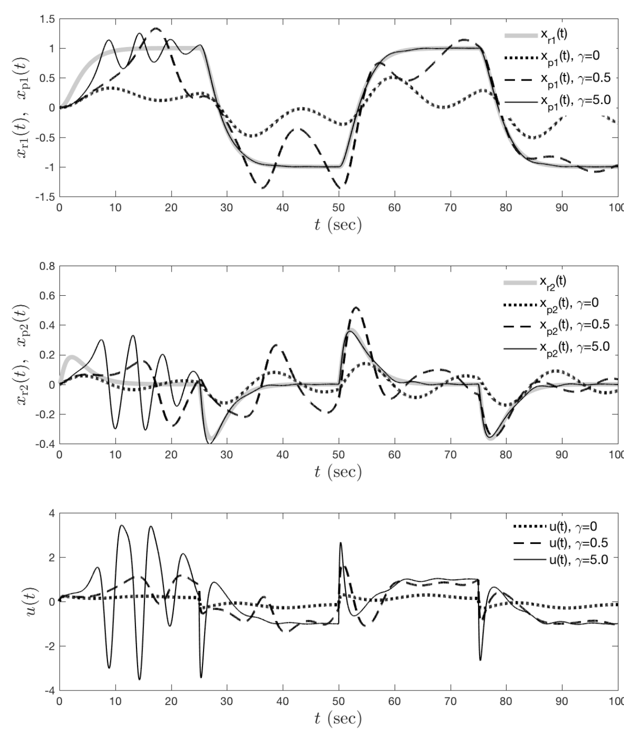

Fig. 5. Command following performance with the model reference adaptive control design in Section V.A for the mass, spring, and damper system given by (2), where it is considered that all the physical model parameters α, β, and m are unknown.

to not only to the books cited above but also, for example, the relatively recent contributions in [9], [17], [63], [113]–[121]. Finally, with regard to emerging developments in model reference adaptive control theory, we refer to [122]–[124] for secure cyber-physical systems applications, [125]–[132] for control applications over wireless networks with reduced communications, [51], [133] for applications involving human-control interactions, and [134]–[144] for cooperative control of multiagent systems (also see references therein).

VI. Acknowledgment

The author wish to thank his related research collaborators (Sivasubramanya Balakrishnan, Eric A. Butcher, Anthony J. Calise, Animesh Chakravarthy, Girish Chowdhary, Gerardo De La Torre, Eric Feron, Eric N. Johnson, Mario L. Fravolini, Frank Fresconi, Wassim M. Haddad, Ali T. Kutay, Kilsoo Kim, Frank Lewis, Jonathan A. Muse, Morad Nazari, Nhan Nguyen, Mehrdad Pakmehr, Farzad Pourboghrat, Simon Schatz, Jeff Shamma, Rifat Sipahi, James Steck, Bong-Jun Yang, and Yildiray Yildiz) and graduate students (Ali Albattat, Ehsan Arabi, K. Merve Dogan, Benjamin C. Gruenwald, S. Burak Sarsilmaz, Daniel Wagner, and Kevin Wilcher) for many fruitful discussions on model reference adaptive control theory as well as its applications.

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