Appendix A
Equations of Motion in the Configuration and State Spaces

A.1 Discrete Linear Systems

A.1.1 Configuration Space

Consider a system with a single degree of freedom and assume that the equation expressing its dynamic equilibrium is a second order ordinary differential equation (ODE) in the generalized coordinate x. Assume also that the forces entering the dynamic equilibrium equation are

- a force depending on the acceleration (inertia force),
- a force depending on velocity (damping force),
- a force depending on displacement (elastic force),
- a force, usually applied from outside the system, that depends neither on coordinate x nor on its derivatives, but is a generic function of time (external forcing function).

If the dependence of the first three forces from acceleration, velocity and displacement, respectively, is linear, the system is linear. Moreover, if the constants of such a linear combination, usually referred to as mass m, damping coefficient c and stiffness k do not depend on time, the system is time-invariant.

The dynamic equilibrium equation is thus

$$m\ddot{x} + c\dot{x} + kx = f(t). \tag{A.1}$$

If the system has a number n of degrees of freedom, the most general form for a linear, time-invariant set of second order ordinary differential equations is

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{f}(t), \tag{A.2}$$

where:

- $\mathbf{x}$ is a vector of order n (n is the number of degrees of freedom of the system) where the generalized coordinates are listed.

G. Genta, *Introduction to the Mechanics of Space Robots*, Space Technology Library 26,
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- $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are matrices, whose order is $n \times n$; they contain the characteristics (independent from time) of the system.
- $\mathbf{f}$ is a vector function of time containing the forcing functions acting on the system.

Matrix $\mathbf{A}_1$ is usually symmetrical but, even if it is not, it is possible to rearrange the equations of motion in such a way that $\mathbf{A}_1$ becomes symmetrical. The other two matrices in general are not such and can be written as the sum of a symmetrical and a skew–symmetrical matrix

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{C} + \mathbf{G})\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{H})\mathbf{x} = \mathbf{f}(t), \tag{A.3}$$

where:

- $\mathbf{M}$, the *mass matrix* of the system, is a symmetrical matrix of order $n \times n$ (coincides with $\mathbf{A}_1$). Usually it is not singular.
- $\mathbf{C}$ is the real symmetric *viscous damping matrix* (it is the symmetric part of $\mathbf{A}_2$).
- $\mathbf{K}$ is the real symmetric *stiffness matrix* (it is the symmetric part of $\mathbf{A}_3$).
- $\mathbf{G}$ is the real skew–symmetric *gyroscopic matrix* (it is the skew–symmetric part of $\mathbf{A}_2$).
- $\mathbf{H}$ is the real skew–symmetric *circulatory matrix* (it is the skew–symmetric part of $\mathbf{A}_3$).

Remark A.1 The same form of (A.2) may result from mathematical modeling of physical systems whose equations of motion are obtained by means of space discretization techniques, such as the well-known finite elements method.

$\mathbf{x}$ is a vector in the sense it is column matrix. However, a set of n numbers can be interpreted as a vector in a n-dimensional space. This space containing vector $\mathbf{x}$ is usually referred to as *configuration space*, since any point in this space can be associated to a configuration of the system. Actually, not all points of the configuration space, intended as an infinite n-dimensional space, correspond to configurations that are physically possible for the system: it is thus possible to define a subspace containing all possible configurations. Moreover, even system that are dealt with using linear equations of motion are linear only for configurations not much displaced from a reference configuration (usually the equilibrium configuration) and the linear equation (A.2) applies only in an even smaller subspace of the configuration space.

A simple system with two degrees of freedom is shown in Fig. A.1a; it consists of two masses and two springs whose behavior is linear in a zone around the equilibrium configuration with $x_1 = x_2 = 0$ but then behave in a nonlinear way to fail at a certain elongation. In the configuration space, which in the case of a two degrees of freedom system has two dimensions and thus is a plane, there is a linearity zone, surrounded by a zone where the system behaves in nonlinear way. Around the latter there is another zone where the system loses its structural integrity.

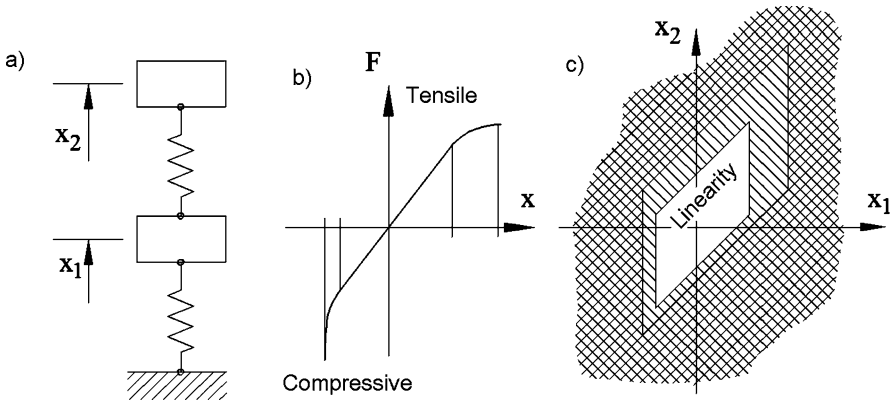

Fig. A.1 Sketch of a system with two degrees of freedom (**a**) made by two masses and two springs, whose characteristics (**b**) are linear only in a zone about the equilibrium position. Three zones can be identified in the configuration space (**c**): in the inner zone the system behaves linearly while in a second zone the system is nonlinear. The latter is surrounded by a 'forbidden' zone (from G. Genta, L. Morello, *The Automotive Chassis*, Springer, New York, 2009)

Remark A.2 In the figure x_1 and x_2 are inertial coordinates, but they are assumed to be zero in the static equilibrium configuration. Moreover, gravitational acceleration has been neglected. This is quite common when dealing with linear systems: the static problem (finding the equilibrium configuration) is separated from the dynamic problem (studying the dynamics about the equilibrium configuration).

A.1.2 State Space

A set of n second order differential equations is a set of order $2n$ and can be expressed in the form of a set of $2n$ first order equations.

In a way similar to what seen above, a generic linear differential equation with constant coefficients can be written in the form of a set of first order differential equations

$$\mathbf{A}_1\dot{\mathbf{x}} + \mathbf{A}_2\mathbf{x} = \mathbf{f}(t). \tag{A.4}$$

In system dynamics this set of equations is usually solved in the first derivatives (monic form) and the forcing function is written as the linear combination of the minimum number of functions expressing the *inputs* of the system. The independent variables are said to be the *state variables* and the equation is written as

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}, \tag{A.5}$$

where

- $\mathbf{z}$ is a vector of order m, in which the state variables are listed (m is the number of the state variables). If (A.5) comes from (A.2), $m = 2n$.

- **A** is a matrix of order $m \times m$, independent from time, called the dynamic matrix,
- **u** is a vector function of time, where the inputs acting on the system are listed (if r is the number of inputs, its size is $r \times 1$),
- **B** is a matrix independent from time that states how the various inputs act in the various equations. It is called the input gain matrix and its size is $m \times r$.

As was seen for vector **x**, also **z** is a column matrix that may be considered as a vector in a m-dimensional space. This space is usually referred to as the *state space*, since each point of this space corresponds to a given state of the system.

The configuration space is a subspace of the space state.

If (A.5) comes from (A.2), a set of n auxiliary variables must be introduced to transform the system from the configuration to the state space. Even if other choices are possible (see Sect. A.6), the simplest alternative is using as auxiliary variables the derivatives of the generalized coordinates (generalized velocities). Half of the state variables are then generalized coordinates and the other half are generalized velocities.

If the state variables are ordered with velocities first and then coordinates, it follows that

$$\mathbf{z} = \begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{Bmatrix}.$$

A number n of equations expressing the link between coordinates and velocities must be added to the n equations (A.2). By using symbol **v** for the generalized velocities $\dot{\mathbf{x}}$, and solving the equations in the derivatives of the state variables, the set of $2n$ equations corresponding to (A.3) is thus

$$\begin{cases} \dot{\mathbf{v}} = -\mathbf{M}^{-1}(\mathbf{C}+\mathbf{G})\mathbf{v} - \mathbf{M}^{-1}(\mathbf{K}+\mathbf{H})\mathbf{x} + \mathbf{M}^{-1}\mathbf{f}(t), \\ \dot{\mathbf{x}} = \mathbf{v}. \end{cases} \tag{A.6}$$

Assuming that inputs **u** coincide with the forcing functions **f**, matrices **A** and **B** are then linked to **M**, **C**, **K**, **G** and **H** by the following relationships:

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}(\mathbf{C}+\mathbf{G}) & -\mathbf{M}^{-1}(\mathbf{K}+\mathbf{H}) \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \tag{A.7}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{M}^{-1} \\ \mathbf{0} \end{bmatrix}. \tag{A.8}$$

The first n out of the $m = 2n$ equations constituting the state equation (A.5) are the dynamic equilibrium equations and are usually referred to as dynamic equations. The other n express the relationship between the position and the velocity variables and are usually referred to as kinematic equations.

Often, more than the state vector **z**, what is interesting is a given linear combination of states **z** and inputs **u**, usually referred to as *output vector*. The state equation (A.5) is thus associated to an *output equation*:

$$\mathbf{y} = \mathbf{C}\mathbf{z} + \mathbf{D}\mathbf{u}, \tag{A.9}$$

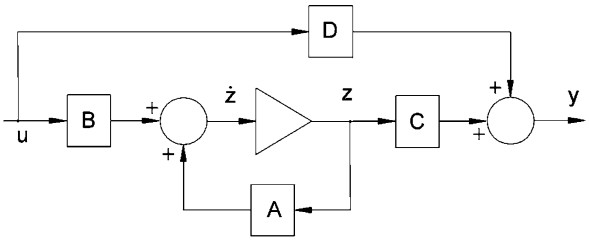

Fig. A.2 Block diagram corresponding to (A.10)

where

- **y** is a vector where the output variables of the system are listed (if the number of outputs is s, its size is $s \times 1$).
- **C** is a matrix of order $s \times m$, independent from time, called the *output gain matrix*.
- **D** is a matrix independent from time that states how the inputs enters into the linear combination yielding the output of the system. It is called *direct link matrix* and its size is $s \times r$. In many cases the inputs do not enter the linear combination yielding the outputs, and **D** is nil.

The four matrices **A**, **B**, **C** and **D** are usually referred to as the quadruple of the dynamic system.

Summarizing, the equations that define the dynamic behavior of the system, from input to output, are

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} = \mathbf{C}\mathbf{z} + \mathbf{D}\mathbf{u}. \end{cases} \tag{A.10}$$

Remark A.3 While the state equations are differential equations, the output equations are algebraic. The dynamics of the system is then concentrated in the former.

The input–output relationship described by (A.10) can be described by the block diagram shown in Fig. A.2.

The linearity of a set of equations allows to state that a solution exists and is unique. The general solution of the equation of motion is the sum of the general solution of the homogeneous equation associated to it and a particular solution of the complete equation. This is true for any differential linear set of equations, even if it is not time-invariant.

The former is the free response of the system, the latter the response to the forcing function.

A.1.3 Free Motion

Consider the equation of motion written in the configuration space (A.2). As already stated, matrix $\mathbf{A}_1$ is symmetrical, while the other two may not be such.

The homogeneous equation

$$\mathbf{A}_1\ddot{\mathbf{x}}(t) + \mathbf{A}_2\dot{\mathbf{x}}(t) + \mathbf{A}_3\mathbf{x}(t) = 0 \tag{A.11}$$

describes the free motion of the system and allows us to study its stability.

The solution of (A.11) can be written as

$$\mathbf{x}(t) = \mathbf{x}_0\, e^{st}, \tag{A.12}$$

where $\mathbf{x}_0$ and s are a vector and a scalar, respectively, both complex and constant. To state the time history of the solution allows to transform the differential equation in an algebraic equation

$$\left(\mathbf{A}_1 s^2 + \mathbf{A}_2 s + \mathbf{A}_3\right)\mathbf{x}_0 = \mathbf{0}. \tag{A.13}$$

It is a set of linear algebraic homogeneous equations. The coefficients matrix is a second order *lambda matrix*;[1] it is square and, since the mass matrix $\mathbf{A}_1 = \mathbf{M}$ is not singular, the lambda matrix is said to be *regular*.

The equation of motion (A.11) has solutions different from the trivial one

$$\mathbf{x}_0 = \mathbf{0} \tag{A.14}$$

if and only if the determinant of the matrix of the coefficients vanishes.

$$\det\left(\mathbf{A}_1 s^2 + \mathbf{A}_2 s + \mathbf{A}_3\right) = 0. \tag{A.15}$$

Equation (A.15) is the characteristic equation of a generalized eigenproblem. Its solutions s_i are the eigenvalues of the system and the corresponding vectors $\mathbf{x}_{0_i}$ are its eigenvectors $\mathbf{q}_i$. The rank of the matrix of the coefficients obtained in correspondence to each eigenvalue s_i defines its multiplicity: if the rank is $n - \alpha_i$, the multiplicity is α_i. The eigenvalues are $2n$ and, correspondingly, there are $2n$ eigenvectors.

Remark A.4 If the multiplicity of some eigenvalues is larger than 1, the eigenvectors corresponding to identical eigenvalues are different from each other. Moreover, any linear combination of these eigenvectors is itself an eigenvector.

Remark A.5 Since the matrices of the system $\mathbf{A}_i$ are real, the characteristic equation (A.15) has real coefficients. Its solutions, the eigenvalues, are thus either real numbers or complex conjugate pairs.

[1] The term *lambda matrix* comes from the habit of using symbol λ for the coefficient appearing into the solution $\mathbf{q}(t) = \mathbf{q}_0\, e^{\lambda t}$. Here symbol s has been used instead of λ, following a more modern habit.

A.1.4 Conservative Natural Systems

If the gyroscopic matrix $\mathbf{G}$ is not present the system is said to be *natural*. If the damping and circulatory matrices $\mathbf{C}$ and $\mathbf{H}$ also vanish, the system is *conservative*. A system with $\mathbf{G} = \mathbf{C} = \mathbf{H} = \mathbf{0}$ (or, as is usually referred to, a MK system) is thus both natural and conservative. The characteristic equation reduces to the algebraic equation

$$\det\left(\mathbf{M}s_i^2 + \mathbf{K}\right) = 0. \tag{A.16}$$

The eigenproblem can be reduced in canonical form

$$\mathbf{D}\mathbf{q}_i = \mu_i \mathbf{q}_i, \tag{A.17}$$

where the dynamic matrix in the configuration space $\mathbf{D}$ (not to be confused with the dynamic matrix in the state space $\mathbf{A}$) is

$$\mathbf{D} = \mathbf{M}^{-1}\mathbf{K}, \tag{A.18}$$

and the parameter in which the eigenproblem is written is

$$\mu_i = -s_i^2. \tag{A.19}$$

Since matrices $\mathbf{M}$ and $\mathbf{K}$ are positive defined ($\mathbf{K}$ may be positive semi-defined), the n eigenvalues μ_i are all real and positive (or zero) and the eigenvalues in terms of s_i are $2n$ imaginary numbers in pairs with opposite sign

$$(s_i, \overline{s}_i) = \pm i\sqrt{\mu_i}. \tag{A.20}$$

The n eigenvectors $\mathbf{q}_i$ of size n are real vectors.

Since all the eigenvalues s_i are imaginary, the solutions (A.12) reduce to undamped harmonic oscillations

$$\mathbf{x}(t) = \mathbf{x}_0\, e^{i\omega t}, \tag{A.21}$$

where

$$\omega = is = \sqrt{\mu} \tag{A.22}$$

is the (circular) frequency.

The n values of ω_i, computed in correspondence of the eigenvalues μ_i, are the natural frequencies or eigenfrequencies of the system, usually written as ω_{n_i}.

If $\mathbf{M}$ or $\mathbf{K}$ are not positive defined or semi-defined, at least one of the eigenvalues μ_i is negative, and thus one of the pair of solutions in s is real, made of a positive and a negative value. As it will be seen below, the real negative solution corresponds to a time history that decays in time in a nonoscillatory way, while the positive one to a time history that increases in time in an unbounded way. The system is thus unstable.

A.1.5 Properties of the Eigenvectors

The eigenvectors are orthogonal with respect to both the stiffness and mass matrices. This propriety can be demonstrated simply by writing the dynamic equilibrium equation in harmonic oscillations for the ith mode

$$\mathbf{K}\mathbf{q}_i = \omega_i^2 \mathbf{M}\mathbf{q}_i. \tag{A.23}$$

Equation (A.23) can be premultiplied by the transpose of the jth eigenvector

$$\mathbf{q}_j^T \mathbf{K}\mathbf{q}_i = \omega_i^2 \mathbf{q}_j^T \mathbf{M}\mathbf{q}_i. \tag{A.24}$$

The same can be done for the equation written for the jth mode and premultiplied by the transpose of the ith eigenvector

$$\mathbf{q}_i^T \mathbf{K}\mathbf{q}_j = \omega_j^2 \mathbf{q}_i^T \mathbf{M}\mathbf{q}_j. \tag{A.25}$$

By subtracting (A.25) from (A.24) it follows that

$$\mathbf{q}_j^T \mathbf{K}\mathbf{q}_i - \mathbf{q}_i^T \mathbf{K}\mathbf{q}_j = \omega_i^2 \mathbf{q}_j^T \mathbf{M}\mathbf{q}_i - \omega_j^2 \mathbf{q}_i^T \mathbf{M}\mathbf{q}_j. \tag{A.26}$$

Remembering that, owing to the symmetry of matrices $\mathbf{K}$ and $\mathbf{M}$,

$$\mathbf{q}_j^T \mathbf{K}\mathbf{q}_i = \mathbf{q}_i^T \mathbf{K}\mathbf{q}_j \quad \text{and} \quad \mathbf{q}_j^T \mathbf{M}\mathbf{q}_i = \mathbf{q}_i^T \mathbf{M}\mathbf{q}_j,$$

it follows that

$$\left(\omega_i^2 - \omega_j^2\right) \mathbf{q}_j^T \mathbf{M}\mathbf{q}_i = 0. \tag{A.27}$$

In the same way, it can be shown that

$$\left(\frac{1}{\omega_i^2} - \frac{1}{\omega_j^2}\right) \mathbf{q}_j^T \mathbf{K}\mathbf{q}_i = 0. \tag{A.28}$$

From (A.28) and (A.27), assuming that the natural frequencies are all different from each other, it follows that, if $i \neq j$,

$$\mathbf{q}_i^T \mathbf{M}\mathbf{q}_j = 0, \qquad \mathbf{q}_i^T \mathbf{K}\mathbf{q}_j = 0, \tag{A.29}$$

which are the relationships defining the orthogonality properties of the eigenvectors with respect to the mass and stiffness matrices, respectively.

If $i = j$, the results of the same products are not zero:

$$\mathbf{q}_i^T \mathbf{M}\mathbf{q}_i = \overline{M_i}, \qquad \mathbf{q}_i^T \mathbf{K}\mathbf{q}_j = \overline{K_i}. \tag{A.30}$$

Constants $\overline{M_i}$ and $\overline{K_i}$ are the modal mass and modal stiffness of the ith mode, respectively. They are linked to the natural frequencies by the relationship

$$\omega_i = \sqrt{\frac{\overline{K_i}}{\overline{M_i}}}, \tag{A.31}$$

stating that the ith natural frequency coincides with the natural frequency of a system with a single degree of freedom whose mass is the ith modal mass and whose stiffness is the ith modal stiffness.

A.1.6 Uncoupling of the Equations of Motion

Any vector in the configuration space (i.e. any configuration of the system) can be considered as a linear combination of the eigenvectors

$$\mathbf{x} = \sum_{i=1}^{n} \eta_i \mathbf{x}_i, \tag{A.32}$$

where the coefficients of the linear combination η_i are the modal coordinates of the system.

This is possible because the eigenvectors are linearly independent and form a possible reference frame in the space of the configurations of the system. It must be explicitly stated that the eigenvectors are orthogonal with respect to the mass and stiffness matrices (they are said to be *m-orthogonal* and *k-orthogonal*), but they are not orthogonal with each other. This means that, in general,

$$\mathbf{q}_i^T \mathbf{q}_j \neq 0. \tag{A.33}$$

In the space of the configurations, the eigenvectors are n vectors that can be taken as a reference frame. However, they are not orthogonal with respect to each other.

By defining a matrix of the eigenvectors $\mathbf{\Phi}$ whose columns are the eigenvectors, the modal transformation and the corresponding inverse transformation can be written as

$$\mathbf{x} = \mathbf{\Phi}\boldsymbol{\eta}, \qquad \boldsymbol{\eta} = \mathbf{\Phi}^{-1}\mathbf{x}. \tag{A.34}$$

Remark A.6 Since the eigenvectors are linearly independent, $\mathbf{\Phi}$ is not singular; the inverse modal transformation always exists.

By introducing such a transformation into the equation of motion it follows that

$$\mathbf{M}\mathbf{\Phi}\ddot{\boldsymbol{\eta}} + \mathbf{K}\mathbf{\Phi}\boldsymbol{\eta} = \mathbf{f}(t), \tag{A.35}$$

and, by premultiplying each term by $\mathbf{\Phi}^T$,

$$\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}\ddot{\boldsymbol{\eta}} + \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi}\boldsymbol{\eta} = \mathbf{\Phi}^T\mathbf{f}(t). \tag{A.36}$$

Matrices $\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}$ and $\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi}$ are the modal mass matrix and the modal stiffness matrix. Owing to the properties of the eigenvectors they are diagonal:

$$\begin{cases} \mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \mathrm{diag}[\overline{M_i}] = \overline{\mathbf{M}}, \\ \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \mathrm{diag}[\overline{K_i}] = \overline{\mathbf{K}}. \end{cases} \tag{A.37}$$

Vector $\mathbf{\Phi}^T\mathbf{f}(t)$ is said to be the *modal force vector* $\overline{\mathbf{f}}(t)$.

Since the modal matrices are diagonal, the equations of motion uncouple from each other and, instead of studying a system with n degrees of freedom, it is possible to study n uncoupled systems with a single degree of freedom, whose equations of motion are

$$\overline{\mathbf{M}}\ddot{\boldsymbol{\eta}} + \overline{\mathbf{K}}\boldsymbol{\eta} = \overline{\mathbf{f}}. \tag{A.38}$$

Remark A.7 Equations (A.34) are a coordinate transformation in the space of the configurations. The n values η_i are the n coordinates of the point representing the configuration of the system, with reference to the system of the eigenvectors. They are said to be principal, modal, or normal coordinates.

Remark A.8 Modal uncoupling holds in general only for MK systems.

The eigenvectors are the solutions of a linear set of homogeneous equations and, thus, are not unique: for each mode, an infinity of eigenvectors exists, all proportional to each other. Because the eigenvectors can be seen as a set of n vectors in the n-dimensional space providing a system of reference, the length of such vectors is not determined, but their directions are known. In other words, the scales of the axes are arbitrary.

There are many ways of normalizing the eigenvectors. The simplest is by stating that the value of one particular element or of the largest one is set to unity.

As an alternative, each eigenvector can be divided by its Euclidean norm, obtaining unit vectors in the space of the configurations.

Another way is to normalize the eigenvectors in such a way that the modal masses are equal to unity. This can be done by dividing each eigenvector by the square root of the corresponding modal mass. In the latter case, each modal stiffness coincides with the corresponding eigenvalue, i.e., with the square of the natural frequency. Equation (A.38) reduces to

$$\ddot{\boldsymbol{\eta}} + \left[\omega^2\right]\boldsymbol{\eta} = \overline{\mathbf{f}}', \tag{A.39}$$

where $[\omega^2] = \mathrm{diag}\{\omega_i^2\}$ is the matrix of the eigenvalues and the modal forces $\overline{\mathbf{f}}'(t)$ are

$$\overline{f_i'} = \frac{\overline{f_i}}{\overline{M_i}} = \frac{\mathbf{q}_i^T\mathbf{f}}{\mathbf{q}_i\mathbf{M}\mathbf{q}_i}. \tag{A.40}$$

Modal uncoupling has, however, another advantage: since not all modes are equally important in determining the response of the system, a limited number of modes (usually those characterized by the lowest natural frequencies) is often sufficient for obtaining the response with good accuracy.

If only the first m modes are considered,[2] the savings in terms of computation time, and hence cost, are usually noticeable, because only m eigenvalues and eigen-

[2] In the following pages it is assumed that the modes which are retained are those from the first one to the mth.

vectors need to be computed and m systems with one degree of freedom need to be studied. Usually the modes that are more difficult to deal with are those characterized by the highest natural frequencies, particularly if the equations of motion are integrated numerically. The advantage of discarding the higher-order modes is, in this case, great.

When some modes are neglected, the reduced matrix of the eigenvectors

$$\mathbf{\Phi}^* = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m],$$

is not square because has n rows and m columns. The first coordinate transformation (A.34) still holds, and the m values of the modal mass, stiffness and force can be computed as usual. However, the inverse transformation (second equation (A.34)) is not possible, because the inversion of matrix $\mathbf{\Phi}^*$ cannot be performed.

The modal coordinates $\boldsymbol{\eta}$ can be computed from the physical coordinates $\mathbf{x}$ by premultiplying by ${\mathbf{\Phi}^*}^T\mathbf{M}$ both sides of the first equation (A.34) computed using the reduced matrix of the eigenvectors $\mathbf{\Phi}^*$, obtaining

$${\mathbf{\Phi}^*}^T\mathbf{M}\mathbf{x} = {\mathbf{\Phi}^*}^T\mathbf{M}\mathbf{\Phi}^*\boldsymbol{\eta}, \tag{A.41}$$

i.e.

$${\mathbf{\Phi}^*}^T\mathbf{M}\mathbf{x} = \overline{\mathbf{M}}\boldsymbol{\eta}. \tag{A.42}$$

Premultiplying then both sides by the inverse of the matrix of the modal masses, it follows that

$$\boldsymbol{\eta} = \overline{\mathbf{M}}^{-1}{\mathbf{\Phi}^*}^T\mathbf{M}\mathbf{x}. \tag{A.43}$$

Equation (A.43) is the required inverse modal transformation.

Remark A.9 Equation (A.41) and the following ones are approximations, since only a reduced number of modes have been accounted for.

A.1.7 Natural Nonconservative Systems

If matrix $\mathbf{C}$ does not vanish while $\mathbf{G} = \mathbf{H} = \mathbf{0}$, the system is still natural and noncirculatory, but is no more conservative.

The characteristic equation (A.15) cannot be reduced to an eigenproblem in canonical form in the configuration space and the state space formulation must be used.

The general solution of the homogeneous equation associated to (A.5) is of the type

$$\mathbf{z} = \mathbf{z}_0 e^{st}, \tag{A.44}$$

where s is in general a complex number. Its real and imaginary parts are usually indicated with symbols ω and σ

$$\begin{aligned} \omega &= \Im(s), \\ \sigma &= \Re(s) \end{aligned} \tag{A.45}$$

and are the frequency of the free oscillations and the decay rate. Solution (A.44) can thus be written in the form

$$\mathbf{z} = \mathbf{z}_0 e^{\sigma t} e^{i\omega t}, \tag{A.46}$$

or, since both σ and ω are real numbers,

$$\mathbf{z} = \mathbf{z}_0 e^{\sigma t}\big[\cos(\omega t) + i\,\sin(\omega t)\big]. \tag{A.47}$$

By introducing solution (A.44) into the homogeneous equation associated to (A.5), the latter transforms from a set of differential equations into a (homogeneous) set of algebraic equations

$$s\mathbf{z}_0 = \mathbf{A}\mathbf{z}_0, \tag{A.48}$$

i.e.

$$(\mathbf{A} - s\mathbf{I})\mathbf{z}_0 = 0. \tag{A.49}$$

In a way similar to what seen for the equation of motion in the configuration space, this homogeneous equation has solutions other than the trivial solution $\mathbf{z}_0 = 0$ only if the determinant of the coefficients matrix vanishes,

$$\det(\mathbf{A} - s\mathbf{I}) = 0. \tag{A.50}$$

Equation (A.50) can be interpreted as an algebraic equation in s, i.e. the characteristic equation of the dynamic systems. It is an equation of power $2n$, yielding the $2n$ values of s. The $2n$ values of s are the eigenvalues of the system and the corresponding $2n$ values of $\mathbf{z}_0$ are the eigenvectors. In general, both eigenvalues and eigenvectors are complex.

If matrix $\mathbf{A}$ is real, as it is usually the case, the solutions are either real or complex conjugate. The corresponding time histories are (Fig. A.3):

- Real solutions ($\omega = 0$, $\sigma \neq 0$): exponential time histories, either with monotonic decay of the amplitude if the solution is negative (stable, nonoscillatory behavior), or with monotonic increase of the amplitude if the solution is positive (unstable, nonoscillatory behavior).
- Complex conjugate solutions ($\omega \neq 0$, $\sigma \neq 0$): oscillating time histories, expressed by (A.47) with amplitude decay if the real part of the solution is negative (stable, oscillatory behavior) or amplitude increase in time if the real part of the solution is positive (unstable, oscillatory behavior).

If the system is stable, stability is asymptotic.

A third case is that seen previously for conservative systems:

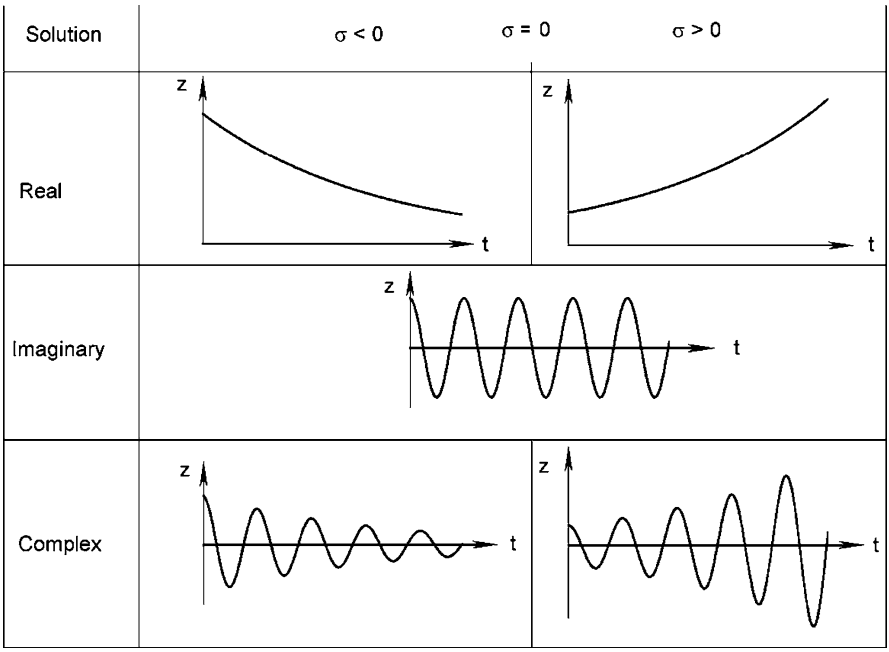

Fig. A.3 Time history of the free motion for the various types of the eigenvalues of the system (from G. Genta, L. Morello, *The Automotive Chassis*, Springer, New York, 2009)

- Imaginary solutions ($\omega \neq 0$, $\sigma = 0$): harmonic time histories (sine or cosine waves, undamped oscillatory behavior). In this case stability is non-asymptotic.

The necessary and sufficient condition for an asymptotically stable behavior is thus that the real part of all eigenvalues is negative.

If any one of the real parts of the eigenvalues is zero, the behavior is still stable, since the amplitude does not grow in an uncontrolled way in time, but not asymptotically stable.

If at least one of them is positive, the system is unstable.

If the system is little damped, i.e. the eigenvalues are conjugate and the decay rates σ are small, the values of the natural frequencies ω are close to those of the corresponding undamped system, that is, to those of the MK system obtained by neglecting the damping matrix $\mathbf{C}$. In this case the oscillation frequencies ω_i are close to those of the corresponding undamped system.

Remark A.10 The term *stable* is here used in the sense that the amplitude of the motion does not grow in an unbounded way and *asymptotically stable* in the sense that the system returns asymptotically to the static equilibrium position. More detailed definitions of stability, like the Liapunov definition, can be found in the literature.

The general solution of the homogeneous equation is a linear combination of the $2n$ solutions

$$\mathbf{z} = \sum_{i=1}^{2n} C_i \mathbf{z}_{0i} e^{s_i t}, \tag{A.51}$$

where the $2n$ constants C_i must be obtained from the initial conditions, i.e. from vector $\mathbf{z}(0)$.

The equation allowing one to compute constants C_i can be written as

$$\mathbf{z}(0) = [\, \mathbf{z}_{01} \; \mathbf{z}_{02} \; \dots \; \mathbf{z}_{02n} \,] \begin{Bmatrix} C_1 \\ C_2 \\ \dots \\ C_{2n} \end{Bmatrix} = \mathbf{\Phi C}, \tag{A.52}$$

where $\mathbf{\Phi}$ is the matrix of the complex eigenvectors in the state space.

A real and negative eigenvalue corresponds to an *overdamped* behavior, which is nonoscillatory, of the relevant mode. If the eigenvalue is complex (with negative real part) the mode has an *underdamped* behavior, that is, it has a damped oscillatory time history. A system with all underdamped modes is said to be underdamped, while if only one of the modes is overdamped, the system is said to be overdamped. If all modes are overdamped, the system cannot have free oscillations, but can oscillate if forced to do so.

Remark A.11 If all matrices $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ are positive defined (or at least semi-defined), as in the case of a structure with viscous damping with positive stiffness and damping, there is no eigenvalue with positive real part and hence the system is stable. If all matrices are strictly positive defined, there no eigenvalue with vanishing real part and thus the system is asymptotically stable.

A.1.8 Systems with Singular Mass Matrix

If matrix $\mathbf{M}$ is singular, it is impossible to write the dynamic matrix in the usual way. Usually this occurs because a vanishingly small inertia is associated to some degrees of freedom. Clearly the problem may be circumvented by associating a very small mass to the relevant degrees of freedom: a new very high natural frequency, which has no physical meaning, is thus introduced and, if this is done carefully, no numerical instability problem is caused. However, it makes little sense to resort to tricks of this kind when it is possible to overcome the problem in a more correct and essentially simple way.

The degrees of freedom can be subdivided in two sets: a vector $\mathbf{x}_1$ containing those to which a nonvanishing inertia is associated, and a vector $\mathbf{x}_2$, containing the other ones. In a similar way all matrices and the forcing functions can be split. The mass matrix $\mathbf{M}_{22}$ vanishes, and if the mass matrix is diagonal, also $\mathbf{M}_{12}$ and $\mathbf{M}_{21}$ vanish.

Assuming that $\mathbf{M}_{12}$ and $\mathbf{M}_{21}$ are zero, the equations of motion become

$$\begin{cases} \mathbf{M}_{11}\ddot{\mathbf{x}}_1 + \mathbf{C}_{11}\dot{\mathbf{x}}_1 + \mathbf{C}_{12}\dot{\mathbf{x}}_2 + \mathbf{K}_{11}\mathbf{x}_1 + \mathbf{K}_{12}\mathbf{x}_2 = \mathbf{f}_1(t), \\ \mathbf{C}_{21}\dot{\mathbf{x}}_1 + \mathbf{C}_{22}\dot{\mathbf{x}}_2 + \mathbf{K}_{21}\mathbf{x}_1 + \mathbf{K}_{22}\mathbf{x}_2 = \mathbf{f}_2(t). \end{cases} \tag{A.53}$$

To simplify the equations of motion neither the gyroscopic nor the circulator matrices are explicitly written, but in what follows no assumption on the symmetry of the stiffness and damping matrices will be done, and the equations hold also for gyroscopic and circulatory systems.

By introducing as state variables the velocities $\mathbf{v}_1$ together with generalized coordinates $\mathbf{x}_1$ and $\mathbf{x}_2$, the state equation is

$$\mathbf{M}^* \begin{Bmatrix} \dot{\mathbf{v}}_1 \\ \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{Bmatrix} = \mathbf{A}^* \begin{Bmatrix} \mathbf{v}_1 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{Bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{f}_1(t) \\ \mathbf{f}_2(t) \end{Bmatrix}, \tag{A.54}$$

where

$$\mathbf{M}^* = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{0} & \mathbf{C}_{12} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{22} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{A}^* = -\begin{bmatrix} \mathbf{C}_{11} & \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{C}_{21} & \mathbf{K}_{21} & \mathbf{K}_{22} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{A.55}$$

The dynamic matrix and the input gain matrix are

$$\mathbf{A} = \mathbf{M}^{*-1}\mathbf{A}^*, \qquad \mathbf{B} = \mathbf{M}^{*-1} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{A.56}$$

Alternatively, the expressions of $\mathbf{M}^*$ and $\mathbf{A}^*$ can be

$$\mathbf{M}^* = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{0} & \mathbf{C}_{21} & \mathbf{C}_{22} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{A}^* = -\begin{bmatrix} \mathbf{0} & \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{0} & \mathbf{K}_{21} & \mathbf{K}_{22} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{A.57}$$

If vector $\mathbf{x}_1$ contains n_1 elements and $\mathbf{x}_2$ contains n_2 elements, the size of the dynamic matrix $\mathbf{A}$ is $2n_1 + n_2$.

A.1.9 Conservative Gyroscopic Systems

If matrix $\mathbf{G}$ is not zero, while both $\mathbf{C}$ and $\mathbf{H}$ vanish, the dynamic matrix reduces to

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{G} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{A.58}$$

By premultiplying the first n equations by $\mathbf{M}$ and the other n by $\mathbf{K}$, it follows that

$$\mathbf{M}^*\dot{\mathbf{z}} + \mathbf{G}^*\mathbf{z} = \mathbf{0}, \tag{A.59}$$

where

$$\mathbf{M}^* = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix}, \qquad \mathbf{G}^* = \begin{bmatrix} \mathbf{G} & \mathbf{K} \\ -\mathbf{K} & \mathbf{0} \end{bmatrix}. \tag{A.60}$$

The first matrix is symmetrical, while the second one is skew–symmetrical.

By introducing solutions (A.44) into the equation of motion, the homogeneous equation

$$s\mathbf{M}^*\mathbf{z}_0 + \mathbf{G}^*\mathbf{z}_0 = \mathbf{0} \tag{A.61}$$

is obtained.

The corresponding eigenproblem has imaginary solutions like those of an MK system, even if the structure of the eigenvectors is different. At any rate the time history of the free oscillations is harmonic and undamped, since the decay rate $\sigma = \Re(s)$ is zero.

A.1.10 General Dynamic Systems

The situation is similar to that seen for natural nonconservative systems, in the sense that the time histories of the free oscillations are those seen in Fig. A.3 and the stability is dominated by the sign of the real part of s.

In general, the presence of a gyroscopic matrix does not reduce the stability of the system, while the presence of a circulatory matrix has a destabilizing effect.

Consider for instance a two degrees of freedom systems made by two independent MK system, each one with a single degree of freedom, and assume that the two masses are equal. The equations for free motion are

$$\begin{cases} m\ddot{x}_1 + k_1 x_1 = 0, \\ m\ddot{x}_2 + k_2 x_2 = 0. \end{cases} \tag{A.62}$$

Introduce now a coupling term in both equations, for instance introducing a spring with stiffness k_{12} between the two masses. The equations of motion become

$$\begin{cases} m\ddot{x}_1 + (k_1 + k_{12})x_1 - k_{12}x_2 = 0, \\ m\ddot{x}_2 - k_{12}x_1 + (k_1 + k_{12})x_2 = 0. \end{cases} \tag{A.63}$$

By introducing parameters

$$\omega_0^2 = \frac{k_1 + k_2 + 2k_{12}}{2m}, \qquad \alpha = \frac{k_2 - k_1}{2m\Omega_0^2}, \qquad \epsilon = \frac{k_{12}}{m\Omega_0^2}, \tag{A.64}$$

the equation of motion can be written as

$$\begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \omega_0^2 \begin{bmatrix} 1-\alpha & \epsilon \\ \epsilon & 1+\alpha \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \mathbf{0}. \tag{A.65}$$

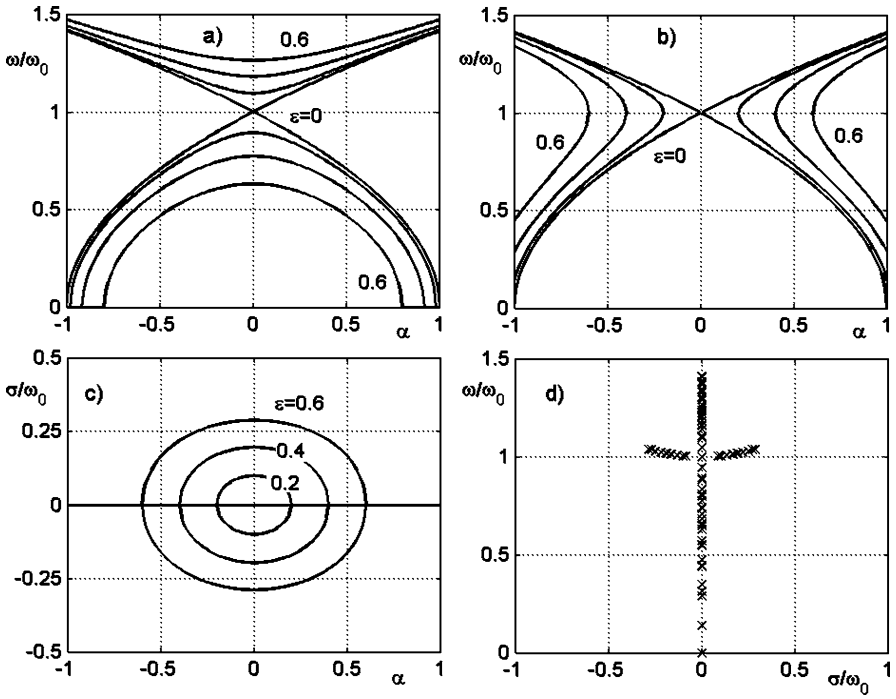

Fig. A.4 Nondimensional natural frequencies as functions of parameters α and ϵ for a system with two degrees of freedom with noncirculatory (**a**) and circulatory (**b**) coupling. Decay rate (**c**) and roots locus (**d**) for the system with circulatory coupling (from G. Genta, L. Morello, *The Automotive Chassis*, Springer, New York, 2009)

Note that

$$-1 \leq \alpha \leq 1. \tag{A.66}$$

The matrix that multiplies the generalized coordinates is symmetrical and thus is a true stiffness matrix. The coupling is said in this case *noncirculatory* or *conservative.* Since there is no damping matrix and the stiffness matrix is positive defined ($-1 \leq \alpha \leq 1$), the eigenvalues are imaginary and the system is stable, even if it is not asymptotically stable as it would be if a positive defined damping matrix were present.

The natural frequencies of the system, made nondimensional by dividing them by ω_0, depend from two parameters, α and ϵ. They are reported in Fig. A.4a as functions of α for some values of ϵ. The distance between the two curves (one for $\omega > \omega_0$ and the other for $\omega < \omega_0$) increases with increasing coupling term ϵ and for this reason this type of coupling is said to be *repulsive*.

Consider now the case with coupling term ϵ in the form

$$\begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \Omega_0^2 \begin{bmatrix} 1-\alpha & \epsilon \\ -\epsilon & 1+\alpha \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \mathbf{0}. \tag{A.67}$$

The terms outside the main diagonal of the stiffness matrix now have the same modulus but opposite sign and the matrix multiplying the displacements is made by a symmetrical part (the stiffness matrix) and a skew–symmetrical part (the circulatory matrix). A coupling of this type is said to be *circulatory* or *non conservative*.

While in the previous case the effect could be caused by the presence of a spring between the two masses, now it cannot be due to springs or similar elements. At any rate there are cases of practical interest where circulatory coupling occurs.

The natural frequencies of the system depend also in this case on the two parameters α and ϵ. They are plotted in nondimensional form, by dividing them by ω_0, in Fig. A.4b as functions of α for some values of ϵ. The two curves now get closer to each other. Starting from the condition with $\alpha = -1$, the two curves meet for a certain value of α in the interval $(-1, 0)$. There is a range, centered in the point with $\alpha = 0$ where the solutions of the eigenproblem (in s) are complex, instead of being imaginary. Beyond this range the two curves separate again.

Since the two curves get closer to each other and finally they meet, this type of coupling is said to be *attractive*.

In the range where the values of s are complex, one of the two solutions has positive real part σ: it follows that an unstable solution exists, as it is possible to see from the decay rate plot in Fig. A.4c and from the roots locus in Fig. A.4d.

As already stated, instability is due to the skew–symmetric matrix due to coupling, i.e. to the fact that a circulatory matrix exists.

A.1.11 Closed Form Solution of the Forced Response

The particular solution of the complete equation depends on the time history of the forcing functions (input) $\mathbf{u}(t)$. In case of harmonic input,

$$\mathbf{u} = \mathbf{u}_0 e^{i\omega t}, \tag{A.68}$$

the response is harmonic as well,

$$\mathbf{z} = \mathbf{z}_0 e^{i\omega t}, \tag{A.69}$$

and has the same frequency ω as the forcing function. As usual, by introducing the time history of the forcing function and of the response into the equation of motion, it transforms into an algebraic equation,

$$(\mathbf{A} - i\omega\mathbf{I})\mathbf{z}_0 + \mathbf{B}\mathbf{u}_0 = 0, \tag{A.70}$$

that allows one to compute the amplitude of the response

$$\mathbf{z}_0 = -(\mathbf{A} - i\omega\mathbf{I})^{-1}\mathbf{B}\mathbf{u}_0. \tag{A.71}$$

If the input is periodic, it may be decomposed in Fourier series and the response to each one of its harmonic components can be computed. The results are then added up. This is possible only since the system is linear.

If the input is not harmonic or at least periodic, it is possible to resort to Laplace transforms or to the Duhamel integral. Also these techniques apply only to linear systems.

A.1.12 Modal Transformation of General Linear Dynamic Systems

Since the eigenvectors of the MK system constitute a reference frame in the configuration space, they can be used to write in modal form the equations of motion of the dynamic system, even if the other matrices do not vanish. More in general, it is possible to say that the eigenvectors of any MK system with n degrees of freedom can be used to write the modal equation of motion of any dynamic system with the same number of degrees of freedom.

The modal equation is thus

$$\overline{\mathbf{M}}\ddot{\boldsymbol{\eta}} + (\overline{\mathbf{C}} + \overline{\mathbf{G}})\dot{\boldsymbol{\eta}} + (\overline{\mathbf{K}} + \overline{\mathbf{H}})\boldsymbol{\eta} = \bar{\mathbf{f}}(t). \tag{A.72}$$

The various modal matrices are all obtained from the corresponding nonmodal matrices by postmultiplying them by the matrix of the eigenvectors of the system and by premultiplying them by its transpose.

Since the eigenvectors are m- and k-orthogonal, but not orthogonal with respect to the other matrices, $\overline{\mathbf{M}}$ and $\overline{\mathbf{K}}$ are diagonal while $\overline{\mathbf{C}}$, $\overline{\mathbf{G}}$ and $\overline{\mathbf{H}}$ are not.

The modal equations of motion are thus not uncoupled.

However, while $\overline{\mathbf{C}}$ may be diagonal in some cases (for example, if $\mathbf{C}$ is a linear combination of $\mathbf{M}$ and $\mathbf{K}$, a situation defined as *proportional damping*), $\overline{\mathbf{G}}$ and $\overline{\mathbf{H}}$, being skew–symmetric, cannot be diagonal.

Nongyroscopic and noncirculatory systems can be uncoupled in case of proportional damping and, if damping is non-proportional but small, it is possible to uncouple in an approximate way the equations of motion by canceling the elements outside the main diagonal of the modal damping matrix $\overline{\mathbf{C}}$.

Neglecting some modes is often possible, but is always an approximation, since all modes, being coupled with each other, affect the response of all other modes. If some of them are neglected, their influence on the other modes is lost.

A.2 Nonlinear Dynamic Systems

Often the state equations of dynamic systems are nonlinear. The reasons of the presence of nonlinearities may be different, like for instance the presence of elements that behave in an intrinsically nonlinear way (e.g. springs producing a force depending in nonlinear way from the displacement), or the presence of trigonometric functions of some of the generalized coordinates in the dynamic or in the kinematic equations. If inertia forces are at any rate linear in the accelerations, the equations of motion can be written in the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{f}_1(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}(t). \tag{A.73}$$

Often function $\mathbf{f}_1$ can be considered as the sum of a linear and a nonlinear part. The equation of motion can thus be written as

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{C}+\mathbf{G})\dot{\mathbf{x}} + (\mathbf{K}+\mathbf{H})\mathbf{x} + \mathbf{f}_2(\mathbf{x},\dot{\mathbf{x}}) = \mathbf{f}(t), \tag{A.74}$$

where function $\mathbf{f}_2$ contains only the nonlinear part of the dynamic system.

Also in the case of nonlinear system, the eigenvectors of the MK linearized system can be used to write the equations of motion in modal form

$$\overline{\mathbf{M}}\ddot{\boldsymbol{\eta}} + (\overline{\mathbf{C}}+\overline{\mathbf{G}})\dot{\boldsymbol{\eta}} + (\overline{\mathbf{K}}+\overline{\mathbf{H}})\boldsymbol{\eta} + \bar{\mathbf{f}}_2(\boldsymbol{\Phi}\boldsymbol{\eta}, \boldsymbol{\Phi}\dot{\boldsymbol{\eta}}) = \bar{\mathbf{f}}(t), \tag{A.75}$$

where vector $\bar{\mathbf{f}}_2$ is obtained by premultiplying vector $\mathbf{f}_2$ by the transpose of the matrix of the eigenvectors. It further couples the equations of motion and makes resorting to a reduced set of modes even more problematic.

The state equations corresponding to (A.73) and (A.74) are

$$\dot{\mathbf{z}} = \mathbf{f}_1(\mathbf{z}) + \mathbf{B}\mathbf{u}, \tag{A.76}$$

or, by separating the linear part from the nonlinear part,

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{f}_2(\mathbf{z}) + \mathbf{B}\mathbf{u}. \tag{A.77}$$

Another way to express the equation of motion or the state equation of a nonlinear system is by writing equations (A.3) or (A.10), where the various matrices are functions of the generalized coordinates and their derivatives, or of the state variables. In the state space it follows

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{A}(\mathbf{z})\mathbf{z} + \mathbf{B}(\mathbf{z})\mathbf{u}, \\ \mathbf{y} = \mathbf{C}(\mathbf{z})\mathbf{z} + \mathbf{D}(\mathbf{z})\mathbf{u}. \end{cases} \tag{A.78}$$

If the system is not time-invariant, the various matrices may also be explicit functions of time

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{A}(\mathbf{z},t)\mathbf{z} + \mathbf{B}(\mathbf{z},t)\mathbf{u}, \\ \mathbf{y} = \mathbf{C}(\mathbf{z},t)\mathbf{z} + \mathbf{D}(\mathbf{z},t)\mathbf{u}. \end{cases} \tag{A.79}$$

It is not possible to obtain a closed form solution of nonlinear systems and concepts like natural frequency or decay rate lose their meaning. It is not even possible to distinguish between free and forced behavior, in the sense that the free oscillations depend from the zone of the state space where the system works. In some zones of the state space the behavior of the system may be stable, while in other ones it may be unstable.

At any rate it is often possible to linearize the equations of motion about any given working conditions, i.e. any given point of the state space, and to use the linearized model so obtained in that area of the space state to study the motion of the system and above all its stability. In this case the motion and the stability are studied *in the small*. It is, however, clear that no general result may be obtained in this way.

If the state equation is written in the form (A.76), its linearization about a point of coordinates $\mathbf{z}_0$ in the state space is

$$\dot{\mathbf{z}} = \left(\frac{\partial \mathbf{f}_1}{\partial \mathbf{z}}\right)_{\mathbf{z}=\mathbf{z}_0} \mathbf{z} + \mathbf{B}\mathbf{u}, \tag{A.80}$$

where $(\frac{\partial \mathbf{f}_1}{\partial \mathbf{z}})_{\mathbf{z}=\mathbf{z}_0}$ is the Jacobian matrix of function $\mathbf{f}_1$ computed in $\mathbf{z}_0$.

If the formulation (A.78) is used, the linearized dynamics of the system about point $\mathbf{z}_0$ can be studied through the linear equation

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{A}(\mathbf{z}_0)\mathbf{z} + \mathbf{B}(\mathbf{z}_0)\mathbf{u}, \\ \mathbf{y} = \mathbf{C}(\mathbf{z}_0)\mathbf{z} + \mathbf{D}(\mathbf{z}_0)\mathbf{u}. \end{cases} \tag{A.81}$$

While the motion and the stability in the small can be studied in closed form, to study the motion *in the large* it is compulsory to resort to the numerical integration of the equations of motion, that is, to resort to numerical simulation.

Remark A.12 Approximate approaches may allow to study in closed form the dynamics of a nonlinear system in some regions of the state space, but the errors they introduce are in general not predictable and they cannot find all the possible solutions.

A.3 Lagrange Equations in the Configuration and State Space

In the case of a relatively simple system it is possible to write directly the equations of motion in the form of (A.3), by writing all forces, internal and external to the system, acting on its various parts. However, if the system is complex, and in particular if the number of degrees of freedom is large, it is expedient to resort to the methods of analytical mechanics.

One of the simplest approaches to write the equations of motion of multi-degrees of freedom systems is by resorting to Lagrange equations that, for a generic mechanical system with n degrees of freedom whose configuration may be expressed using n generalized coordinates x_i, can be written in the form

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{T}}{\partial x_i} + \frac{\partial \mathcal{U}}{\partial x_i} + \frac{\partial \mathcal{F}}{\partial \dot{x}_i} = Q_i \quad (i = 1, \ldots, n), \tag{A.82}$$

where:

- $\mathcal{T}$ is the kinetic energy or the system. It allows us to write in a synthetic way inertia forces. In general,

$$\mathcal{T} = \mathcal{T}(\dot{x}_i, x_i, t).$$

The kinetic energy is in general a quadratic function of the generalized velocities

$$\mathcal{T} = \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2, \tag{A.83}$$

where $\mathcal{T}_0$ does not depend on the velocities, $\mathcal{T}_1$ is linear and $\mathcal{T}_2$ is quadratic.

In the case of linear systems, the kinetic energy must contain terms containing no powers higher than 2 of the velocities and coordinates (or products of more than two of them). As a consequence, $\mathcal{T}_2$ cannot contain displacements

$$\mathcal{T}_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}x_i x_j = \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}}, \tag{A.84}$$

where the terms m_{ij} do not depend either on $\mathbf{x}$ or $\dot{\mathbf{x}}$. If the system is time-invariant, $\mathbf{M}$ is constant.

$\mathcal{T}_1$ is linear in the velocities, and cannot contain terms more than linear in the displacements

$$\mathcal{T}_1 = \frac{1}{2}\dot{\mathbf{x}}^T(\mathbf{M}_1\mathbf{x} + \mathbf{f}_1), \tag{A.85}$$

where matrix $\mathbf{M}_1$ and vector $\mathbf{f}_1$ do not contain the generalized coordinates, even if $\mathbf{f}_1$ may be a function of time even in time-invariant systems.

$\mathcal{T}_0$ does not contain generalized velocities, but only terms of order not greater than 2 in the generalized coordinates:

$$\mathcal{T}_0 = \frac{1}{2}\mathbf{x}^T\mathbf{M}_g\mathbf{x} + \mathbf{x}^T\mathbf{f}_2 + e, \tag{A.86}$$

where matrix $\mathbf{M}_g$, vector $\mathbf{f}_2$ and scalar e are constant. Constant e does not enter the equations of motion. As it will be seen later, the structure of $\mathcal{T}_0$ is similar to that of the potential energy. The term

$$\mathcal{T}_0 - \mathcal{U}$$

is often referred to as *dynamic potential*.

- $\mathcal{U}$ is the potential energy and allows to express in a synthetic form conservative forces. In general,

$$\mathcal{U} = \mathcal{U}(x_i).$$

In the case of linear systems, the potential energy is a quadratic form in the generalized coordinates and, apart from a constant term that does not enter the equations of motion and then has no importance, can be written as

$$\mathcal{U} = \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x} + \mathbf{x}^T\mathbf{f}_0, \tag{A.87}$$

Also in the case of nonlinear systems, by definition the potential energy does not depend on the generalized velocities and its derivatives with respect to the generalized velocities $\dot{x_i}$ vanish. Equation (A.82) is often written with reference to the *Lagrangian function*

$$\mathcal{L} = \mathcal{T} - \mathcal{U}$$

and becomes

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} + \frac{\partial \mathcal{F}}{\partial \dot{x}_i} = Q_i. \tag{A.88}$$

- $\mathcal{F}$ is the Raleigh dissipation function. It allows to express in a synthetic form some types of damping forces. In many cases $\mathcal{F} = \mathcal{F}(\dot{x}_i)$, but it may depend also on the generalized coordinates. In the case of linear systems, the dissipation function is a quadratic form in the generalized velocities and, apart from terms not depending from $\dot{x}_i$ that do not enter the equation of motion and thus have no importance, can be written as

$$\mathcal{F} = \frac{1}{2}\dot{\mathbf{x}}^T \mathbf{C}\dot{\mathbf{x}} + \frac{\mathbf{1}}{\mathbf{2}}\dot{\mathbf{x}}^T(\mathbf{C}_1\mathbf{x} + \mathbf{f}_3). \tag{A.89}$$

- Q_i are generalized forces that cannot be expressed using the above mentioned functions. In general, $Q_i = Q_i(\dot{q}_i, q_i, t)$. In the case of linear systems, they do not depend on the generalized coordinates and velocities, and then

$$Q_i = Q_i(t). \tag{A.90}$$

In the case of linear systems, by performing the relevant derivatives

$$\frac{\partial(\mathcal{T} - \mathcal{U})}{\partial \dot{x}_i} = \mathbf{M}\dot{\mathbf{x}} + \frac{1}{2}(\mathbf{M}_1\mathbf{x} + \mathbf{f}_1), \tag{A.91}$$

$$\frac{d}{dt}\left[\frac{\partial(\mathcal{T} - \mathcal{U})}{\partial \dot{x}_i}\right] = \mathbf{M}\ddot{\mathbf{x}} + \frac{1}{2}\mathbf{M}_1\dot{\mathbf{x}} + \dot{\mathbf{f}}_1, \tag{A.92}$$

$$\frac{\partial(\mathcal{T} - \mathcal{U})}{\partial x_i} = \frac{1}{2}\mathbf{M}_1^T\dot{\mathbf{x}} + \mathbf{M}_g\mathbf{x} - \mathbf{K}\mathbf{x} + \mathbf{f}_2 - \mathbf{f}_0. \tag{A.93}$$

$$\frac{\partial \mathcal{F}}{\partial \dot{x}_i} = \mathbf{C}\dot{\mathbf{x}} + \mathbf{C}_1\mathbf{x} + \mathbf{f}_3, \tag{A.94}$$

the equation of motion becomes

$$\mathbf{M}\ddot{\mathbf{x}} + \frac{1}{2}\left(\mathbf{M}_1 - \mathbf{M}_1^T\right)\dot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + (\mathbf{K} - \mathbf{M}_g + \mathbf{C}_1)\mathbf{x} = -\dot{\mathbf{f}}_1 + \mathbf{f}_2 - \mathbf{f}_3 - \mathbf{f}_0 + \mathbf{Q}. \tag{A.95}$$

Matrix $\mathbf{M}_1$ is normally skew–symmetric. However, even if it is not such, it may be written as the sum of a symmetrical and a skew–symmetrical part

$$\mathbf{M}_1 = \mathbf{M}_{1\text{symm}} + \mathbf{M}_{1\text{skew}}. \tag{A.96}$$

By introducing this form into (A.95), the term

$$\mathbf{M}_1 - \mathbf{M}_1^T$$

becomes

$$\mathbf{M}_{1\text{symm}} + \mathbf{M}_{1\text{skew}} - \mathbf{M}_{1\text{symm}} + \mathbf{M}_{1\text{skew}} = 2\mathbf{M}_{1\text{skew}}.$$

Only the skew–symmetric part of $\mathbf{M}_1$ is included into the equation of motion. Also $\mathbf{C}_1$ is usually skew–symmetrical.

Writing $\mathbf{M}_{1\text{skew}}$ as $1/2\ \mathbf{G}$ and $\mathbf{C}_1$ (or at least its skew–symmetric part; if a symmetric part existed, it could be included into matrix $\mathbf{K}$) as $\mathbf{H}$, and including vectors $\mathbf{f}_0, \dot{\mathbf{f}}_1, \mathbf{f}_2$ and $\mathbf{f}_3$ into forcing functions $\mathbf{Q}$, the equation of motion becomes

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{C} + \mathbf{G})\dot{\mathbf{x}} + (\mathbf{K} - \mathbf{M}_g + \mathbf{H})\mathbf{x} = \mathbf{Q}, \tag{A.97}$$

The mass, stiffness, gyroscopic and circulatory matrices $\mathbf{M}$, $\mathbf{K}$, $\mathbf{G}$ and $\mathbf{H}$ have already been defined. The symmetric matrix $\mathbf{M}_g$ is often defined as *geometric matrix*.[3]

As already said, a system in which $\mathcal{T}_1$ is not present is said to be *natural* and its equation of motion does not contain a gyroscopic matrix. In many cases also $\mathcal{T}_0$ is absent and the kinetic energy is expressed by (A.84).

To write the linearized equation of motion of a nonlinear system two ways are possible. The first is writing the complete expression of the energies, performing the derivatives obtaining the complete equations of motion and then canceling nonlinear terms.

The second one is reducing the expression of the energies to quadratic forms, by developing their expressions in power series and then truncating them after the quadratic terms. The linearized equations of motion are thus directly obtained.

The two ways yield the same result, but the first one is usually computationally much heavier.

To write the state equations, a number n of kinematic equations must be written

$$\dot{x}_i = v_i \quad (i = 1, \ldots, n). \tag{A.98}$$

If the state vector is defined in the usual way

$$\mathbf{z} = \begin{Bmatrix} \mathbf{v} \\ \mathbf{x} \end{Bmatrix},$$

this procedure is straightforward.

A.4 Lagrange Equations for Systems with Constraints

In the previous sections the equations of motion were written in terms of the minimum number of generalized coordinates, i.e. by using as many generalized coordinates as the number of degrees of freedom of the system.

In many cases the system can be modeled as a number of point masses or rigid bodies subjected to constraints: in this case the mentioned approach requires the

[3] Here the symbol $\mathbf{M}_g$ is used instead of the more common $\mathbf{K}_g$ to stress that it comes from the kinetic energy.

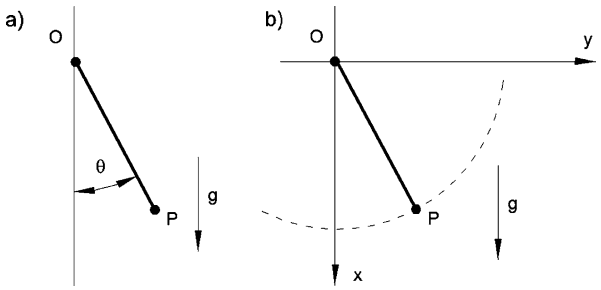

Fig. A.5 Pendulum, modeled using the minimum number of generalized coordinates (**a**), and as a system with two degrees of freedom and subjected to a constraint (**b**)

identifications of the minimum number of generalized coordinates describing all possible configurations of the system that are compatible with the mentioned constraints. An alternative approach is to write the equations of motion of the point masses or the rigid bodies using the same generalized coordinates that would be used if the constraints were not present, and then adding the equations defining the constraints.

For instance, if a pendulum made of a point mass attached to an inextensible massless string is considered, the first approach leads to identifying a single coordinate describing the motion of the pendulum, for instance the swing angle θ in Fig. A.5a. In the second approach the two generalized coordinates describing the motion of the point in a plane (e.g. coordinates x and y) are used, and an equation stating that the distance OP is constant is introduced.

The equations expressing the constraints may be of different types.

A.4.1 Holonomic Constraints

The simplest case is when there is a set of k relationships of the type

$$f_j(\mathbf{x}) = 0 \quad \text{for } j = 1, \ldots, k. \tag{A.99}$$

These constraints are simply geometrical constraint, i.e. define a number of lines or of surfaces to which the various part of the system are constrained. Constraints of this kind are said to be *holonomic*. Also the more general case

$$f_j(\mathbf{x}, t) = 0 \quad \text{for } j = 1, \ldots, k \tag{A.100}$$

deals with holonomic constraints.

In case of holonomic constraints, an augmented Lagrangian function can be defined

$$\mathcal{L}^* = \mathcal{L} + \sum_{j=1}^{k} \lambda_j(t) f_j(\mathbf{x}) = \mathcal{T} - \mathcal{U} + \sum_{j=1}^{k} \lambda_j(t) f_j(\mathbf{x}). \tag{A.101}$$

Functions $\lambda_j(t)$ are said to be the Lagrange multipliers, and are dealt with as additional generalized coordinates of the dynamic system.

Remark A.13 The physical meaning of Lagrange multipliers is not that of coordinates, but of forces exerted by the constraints.

The augmented Lagrangian can be introduced into the Lagrange equations, which become

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{T}}{\partial x_i} + \frac{\partial \mathcal{U}}{\partial x_i} + \frac{\partial \mathcal{F}}{\partial \dot{x}_i} - \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial x_i} = Q_i \quad (i = 1, \ldots, n), \tag{A.102}$$

These n equations of motion must be associated to the k constraint equations (A.99), yielding a set of $n + k$ equations in the $n + k$ unknowns x_i and λ_j.

Remark A.14 The n dynamic equations are ordinary differential equations while the k constraint equations are algebraic.

The simplest example of a system with constraints is the pendulum with length l and mass m of Fig. A.5. If it is studied using the minimum number of coordinates, angle θ can be chosen and the Lagrangian function is

$$\mathcal{L} = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos(\theta). \tag{A.103}$$

The equation of motion is thus

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0. \tag{A.104}$$

In the other approach, based on the explicit statement of the constraint, the generalized coordinates x and y of point P are used as generalized coordinates. The constraint equation states that distance OP is equal to l:

$$\sqrt{x^2 + y^2} - l = 0. \tag{A.105}$$

The augmented Lagrangian function is thus

$$\mathcal{L}^* = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + mgx + \lambda\left(\sqrt{x^2 + y^2} - l\right). \tag{A.106}$$

By performing the relevant derivatives, the equations of motion and the constraint equation are

$$\begin{cases} m\ddot{x} - mgl - \lambda\dfrac{x}{l} = 0, \\ m\ddot{y} - \lambda\dfrac{y}{l} = 0, \\ x^2 + y^2 - l^2 = 0. \end{cases} \tag{A.107}$$

To obtain the same solution as above, the following change of variable can be effected:

$$x = l\cos(\theta). \tag{A.108}$$

From the last equation, it follows that

$$y = l\sin(\theta). \tag{A.109}$$

The accelerations

$$\begin{cases} \ddot{x} = -l\ddot{\theta}\sin(\theta) - l\dot{\theta}^2\cos(\theta), \\ \ddot{y} = l\ddot{\theta}\cos(\theta) - l\dot{\theta}^2\sin(\theta) \end{cases} \tag{A.110}$$

can be introduced into the first two equations, obtaining

$$\begin{cases} -ml\ddot{\theta}\sin(\theta) - ml\dot{\theta}^2\cos(\theta) - mgl - \lambda\cos(\theta) = 0, \\ ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta) - \lambda\sin(\theta) = 0. \end{cases} \tag{A.111}$$

By multiplying the first equation by $-\sin(\theta)$ and the second by $\cos(\theta)$ and adding it follows that

$$l\ddot{\theta} + g\sin(\theta) = 0, \tag{A.112}$$

which coincides with the previous equation. By multiplying the first equation by $\cos(\theta)$ and the second by $\sin(\theta)$ and adding it follows that

$$\lambda = -m\left[l\dot{\theta}^2 + g\cos(\theta)\right]. \tag{A.113}$$

The two terms in the expression for λ are the centripetal force plus the force needed to compensate for the component of the weight in the direction of the wire. λ is thus the force the wire exerts on mass m.

A.4.2 Non-holonomic Constraints

If also the velocities are involved in the constraint equations

$$f_j(\mathbf{x}, \dot{\mathbf{x}}, t) = 0 \quad \text{for } j = 1, \ldots, k \tag{A.114}$$

the constraints are said to be *non-holonomic*. In this case it is impossible to use the augmented Lagrangian function seen for the holonomic constraints.

If the Jacobian matrix

$$\frac{\partial f_j(\mathbf{x}, \dot{\mathbf{x}}, t)}{\partial \dot{\mathbf{x}}}$$

has rank k, i.e. all constraints are non-holonomic and are independent, the equation of motion can be shown to be[4]

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{T}}{\partial x_i} + \frac{\partial \mathcal{U}}{\partial x_i} + \frac{\partial \mathcal{F}}{\partial \dot{x}_i} + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial \dot{x}_i} = Q_i \quad (i = 1, \ldots, n). \tag{A.115}$$

A mixed case is also possible: if there are k_1 holonomic and k_2 non-holonomic constraint that satisfy the above mentioned condition (their Jacobian matrix has rank k_2), the two types of constraints can be dealt with separately as seen in (A.102) and (A.115).

A.5 Hamilton Equations in the Phase Space

If the generalized momenta are used as auxiliary variables instead the generalized velocities, the equations are written with reference to the *phase space* and the *phase vector* instead of the state space and vector.

The generalized momenta are defined, starting from the Lagrangian $\mathcal{L}$, as

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{x}_i}. \tag{A.116}$$

If the system is a natural linear system, this definition reduce to the usual one

$$\mathbf{p} = \mathbf{M}\dot{\mathbf{x}}. \tag{A.117}$$

By including the forces coming from the dissipation function in the generalized forces Q_i, the Lagrange equation simplifies as

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial x_i} + Q_i. \tag{A.118}$$

A function $\mathcal{H}(\dot{x}_i, x_i, t)$, said *Hamiltonian function* is defined as

$$\mathcal{H} = \mathbf{p}^T \dot{\mathbf{x}} - \mathcal{L}. \tag{A.119}$$

Since $\mathcal{H}$ is a function of p_i, x_i and t ($\mathcal{H}(p_i, x_i, t)$), the differential $\delta\mathcal{H}$ is

$$\delta\mathcal{H} = \sum_{i=1}^{n}\left(\frac{\partial \mathcal{H}}{\partial p_i}\delta p_i + \frac{\partial \mathcal{H}}{\partial x_i}\delta x_i\right). \tag{A.120}$$

On the other hand, (A.119) yields

$$\delta\mathcal{H} = \sum_{i=1}^{n}\left(p_i \delta\dot{x}_i + \dot{x}_i \delta p_i - \frac{\partial \mathcal{L}}{\partial x_i}\delta x_i - \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\delta\dot{x}_i\right)$$

[4] M.R. Flannery, The enigma of nonholonomic constraints, Am. J. Phys. 73(3):265–272, 2005; O.M. Moreschi, G. Castellano, Geometric approach to non-holonomic problems satisfying Hamilton's principle, Rev. Unión Mat. Argent. 47(2):125–135, 2005.

$$= \sum_{i=1}^{n} \left(\dot{x}_i \delta p_i - \frac{\partial \mathcal{L}}{\partial x_i} \delta x_i \right), \tag{A.121}$$

and then

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{x}_i, \qquad \frac{\partial \mathcal{H}}{\partial x_i} = -\frac{\partial \mathcal{L}}{\partial x_i}. \tag{A.122}$$

The $2n$ phase space equations are thus

$$\begin{cases} \dot{x}_i = \dfrac{\partial \mathcal{H}}{\partial p_i}, \\ \dot{p}_i = -\dfrac{\partial \mathcal{H}}{\partial x_i} + Q_i. \end{cases} \tag{A.123}$$

A.6 Lagrange Equations in Terms of Pseudo-Coordinates

Often the state equations are written with reference to generalized velocities that are not simply the derivatives of the generalized coordinates. In particular, often it is expedient to use as generalized velocities suitable combinations of the derivatives of the coordinates $v_i = \dot{x}_i$

$$\{w_i\} = \mathbf{A}^T \{\dot{x}_i\}, \tag{A.124}$$

where the coefficients of the linear combinations included into matrix $\mathbf{A}^T$ may be constant, but in general are functions of the generalized coordinates.

Equation (A.124) can in general be inverted, obtaining

$$\{\dot{x}_i\} = \mathbf{B}\{w_i\}, \tag{A.125}$$

where

$$\mathbf{B} = \mathbf{A}^{-T} \tag{A.126}$$

and the symbol $\mathbf{A}^{-T}$ indicates the inverse of the transpose of matrix $\mathbf{A}$.

In some cases matrix $\mathbf{A}^T$ is a rotation matrix and its inverse coincides with its transpose. In those case

$$\mathbf{B} = \mathbf{A}^{-T} = \mathbf{A}.$$

However, in general this does not occur and

$$\mathbf{B} \neq \mathbf{A}.$$

While v_i are the derivatives of the coordinates x_i, in general it is not possible to express w_i as the derivatives of suitable coordinates. Equation (A.124) can be written in the infinitesimal displacements dx_i

$$\{d\theta_i\} = \mathbf{A}^T \{dx_i\}, \tag{A.127}$$

obtaining a set of infinitesimal displacements $d\theta_i$, corresponding to velocities w_i. Equations (A.127) can be integrated, yielding displacements θ_i corresponding to the velocities w_i, only if

$$\frac{\partial a_{js}}{\partial x_k} = \frac{\partial a_{ks}}{\partial x_j}.$$

Otherwise (A.127) cannot be integrated and velocities w_i cannot be considered as the derivatives of true coordinates. In such a case they are said to be the derivatives of *pseudo-coordinates*.

As a first consequence of the non existence of coordinates corresponding to velocities w_i, Lagrange equation (A.82) cannot be written directly using velocities w_i (that cannot be considered as derivatives of the new coordinates), but must be modified to allow the use of velocities and coordinates that are not directly one the derivative of the other.

The use of pseudo-coordinate is fairly common. If, for instance, in the dynamics of a rigid body, the generalized velocities in a reference frame following the body in its motion are used, while the coordinates x_i are the displacements in an inertial frame, matrix $\mathbf{A}^T$ is simply the rotation matrix allowing to pass from the one reference frame to the other. Matrix $\mathbf{B}$ coincides in this case with $\mathbf{A}$, but both are not symmetrical and the velocities in the body-fixed frame cannot be considered as the derivatives of the displacements in that frame. In other words, such a frame rotates continuously and it is not possible to integrate the velocities along the body-fixed axes to obtain the displacements along the same axes. That notwithstanding, it is possible to use the components of the velocity along the body-fixed axes to write the equations of motion.

The kinetic energy can be written in general in the form

$$\mathcal{T} = \mathcal{T}(w_i, x_i, t).$$

The derivatives $\partial\mathcal{T}/\partial\dot{x}_i$ included into the equations of motion are, in matrix form

$$\left\{\frac{\partial \mathcal{T}}{\partial \dot{x}}\right\} = \mathbf{A}\left\{\frac{\partial \mathcal{T}}{\partial w}\right\}, \tag{A.128}$$

where

$$\left\{\frac{\partial \mathcal{T}}{\partial \dot{x}}\right\} = \left[\frac{\partial \mathcal{T}}{\partial \dot{x}_1} \quad \frac{\partial \mathcal{T}}{\partial \dot{x}_2} \quad \dots\right]^T,$$

$$\left\{\frac{\partial \mathcal{T}}{\partial w}\right\} = \left[\frac{\partial \mathcal{T}}{\partial w_1} \quad \frac{\partial \mathcal{T}}{\partial w_2} \quad \dots\right]^T.$$

By differentiating with respect to time, it follows that

$$\frac{\partial}{\partial t}\left(\left\{\frac{\partial \mathcal{T}}{\partial \dot{x}}\right\}\right) = \mathbf{A}\frac{\partial}{\partial t}\left(\left\{\frac{\partial \mathcal{T}}{\partial w}\right\}\right) + \dot{\mathbf{A}}\left\{\frac{\partial \mathcal{T}}{\partial w}\right\}. \tag{A.129}$$

The generic element $\dot{a}_{jk}$ of matrix $\dot{\mathbf{A}}$ is

$$\dot{a}_{jk} = \sum_{i=1}^{n} \frac{\partial a_{jk}}{\partial x_i} \dot{x}_i = \dot{\mathbf{x}}^T \left\{ \frac{\partial a_{jk}}{\partial x} \right\}, \tag{A.130}$$

i.e.

$$\dot{a}_{jk} = \mathbf{w}^T \mathbf{B}^T \left\{ \frac{\partial a_{jk}}{\partial x} \right\}. \tag{A.131}$$

The various $\dot{a}_{jk}$ so computed can be written in matrix form

$$\dot{\mathbf{A}} = \left[\mathbf{w}^T \mathbf{B}^T \left\{ \frac{\partial a_{jk}}{\partial x} \right\} \right]. \tag{A.132}$$

The computation of the derivatives of the generalized coordinates $\partial \mathcal{T} / \partial x$ is usually less straightforward. The generic derivative $\partial \mathcal{T} / \partial x_k$ is

$$\frac{\partial \mathcal{T}^*}{\partial x_k} = \frac{\partial \mathcal{T}}{\partial x_k} + \sum_{i=1}^{n} \frac{\partial \mathcal{T}}{\partial w_i} \frac{\partial w_i}{\partial x_k} = \frac{\partial \mathcal{T}}{\partial x_k} + \sum_{i=1}^{n} \frac{\partial \mathcal{T}}{\partial w_i} \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_k} \dot{x}_j, \tag{A.133}$$

where $\mathcal{T}^*$ is the kinetic energy expressed as a function of the generalized coordinates and their derivatives (the expression to be introduced into the Lagrange equation in its usual form), while $\mathcal{T}$ is expressed as a function of the generalized coordinates and of the velocities in the body-fixed frame. Equation (A.133) can be written as

$$\frac{\partial \mathcal{T}^*}{\partial x_k} = \frac{\partial \mathcal{T}}{\partial x_k} + \mathbf{w}^T \mathbf{B}^T \frac{\partial \mathbf{A}}{\partial x_k} \left\{ \frac{\partial \mathcal{T}}{\partial w} \right\}, \tag{A.134}$$

where the product $\mathbf{w}^T \mathbf{B}^T \frac{\partial \mathbf{A}}{\partial x_k}$ yields a row matrix with n elements that multiplied by the column matrix $\{\frac{\partial \mathcal{T}}{\partial w}\}$ yields the required number.

By combining those row matrices, a square matrix is obtained

$$\left[\mathbf{w}^T \mathbf{B}^T \frac{\partial \mathbf{A}}{\partial x_k} \right], \tag{A.135}$$

and then the column containing the derivatives with respect to the generalized coordinates is

$$\left\{ \frac{\partial \mathcal{T}^*}{\partial x} \right\} = \left\{ \frac{\partial \mathcal{T}}{\partial x} \right\} + \left[\mathbf{w}^T \mathbf{B}^T \frac{\partial \mathbf{A}}{\partial x} \right] \left\{ \frac{\partial \mathcal{T}}{\partial w} \right\}. \tag{A.136}$$

By definition, the potential energy does not depend on the generalized velocities and thus the term $\partial \mathcal{U} / \partial x_i$ is not influenced by the way the generalized velocities are written. Finally, the derivatives of the dissipation function are

$$\left\{ \frac{\partial \mathcal{F}}{\partial \dot{x}} \right\} = \mathbf{A} \left\{ \frac{\partial \mathcal{F}}{\partial w} \right\}. \tag{A.137}$$

The equation of motion (A.82) is thus

$$\mathbf{A}\frac{\partial}{\partial t}\left(\left\{\frac{\partial \mathcal{T}}{\partial w}\right\}\right)+\mathbf{\Gamma}\left\{\frac{\partial \mathcal{T}}{\partial w}\right\}-\left\{\frac{\partial \mathcal{T}}{\partial x}\right\}+\left\{\frac{\partial \mathcal{U}}{\partial x}\right\}+\mathbf{A}\left\{\frac{\partial \mathcal{F}}{\partial w}\right\}=\mathbf{Q}, \tag{A.138}$$

where

$$\mathbf{\Gamma}=\left[\mathbf{w}^T\mathbf{B}^T\left\{\frac{\partial a_{jk}}{\partial x}\right\}\right]-\left[\mathbf{w}^T\mathbf{B}^T\frac{\partial \mathbf{A}}{\partial x_k}\right] \tag{A.139}$$

and $\mathbf{Q}$ is a vector containing the n generalized forces Q_i.

By premultiplying all terms by matrix $\mathbf{B}^T=\mathbf{A}^{-1}$ and attaching the kinematic equations to the dynamic equations, the final form of the state space equations is obtained:

$$\begin{cases}\dfrac{\partial}{\partial t}\left(\left\{\dfrac{\partial \mathcal{T}}{\partial w}\right\}\right)+\mathbf{B}^{\mathbf{T}}\mathbf{\Gamma}\left\{\dfrac{\partial \mathcal{T}}{\partial w}\right\}-\mathbf{B}^T\left\{\dfrac{\partial \mathcal{T}}{\partial x}\right\}+\mathbf{B}^T\left\{\dfrac{\partial \mathcal{U}}{\partial x}\right\}+\left\{\dfrac{\partial \mathcal{F}}{\partial w}\right\}=\mathbf{B}^T\mathbf{Q},\\ \{\dot{x}_i\}=\mathbf{B}\{w_i\}.\end{cases} \tag{A.140}$$

A.7 Motion of a Rigid Body

A.7.1 Generalized Coordinates

A rigid body free in tridimensional space has six degrees of freedom. A possible set of six generalized coordinates defining its pose was stated in Sect. 3.6. Once an inertial reference frame OXYZ and a frame Gxyz fixed to the body and centered in its center of mass are stated, the position of the rigid body is defined by

- the coordinates of point G in the inertial frame OXYZ, i.e. coordinates X_G, Y_G and Z_G, and
- a set of three Euler or Tait–Bryan angles, for instance the yaw (ψ), pitch (θ) and roll (ϕ) angles.

The rotation matrices related to these rotations $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ are defined by (3.3), (3.4) and (3.5) and the total rotation matrix is expressed by (3.7):

$$\begin{aligned}\mathbf{R}&=\mathbf{R}_1\mathbf{R}_2\mathbf{R}_3\\ &=\begin{bmatrix}c(\psi)c(\theta) & c(\psi)s(\theta)s(\phi)-s(\psi)c(\phi) & c(\psi)s(\theta)c(\phi)+s(\psi)s(\phi)\\ s(\psi)c(\theta) & s(\psi)s(\theta)s(\phi)+c(\psi)c(\phi) & s(\psi)s(\theta)c(\phi)-c(\psi)s(\phi)\\ -s(\theta) & c(\theta)s(\phi) & c(\theta)c(\phi)\end{bmatrix},\end{aligned}$$

where symbols cos and sin have been substituted by c and s.

Sometimes roll and pitch angles are small. In this case it is expedient to keep the last two rotations separate from the first ones. The product of the rotation matrices

related to the last two rotations is

$$\mathbf{R}_2\mathbf{R}_3 = \begin{bmatrix} \cos(\theta) & \sin(\theta)\sin(\phi) & \sin(\theta)\cos(\phi) \\ 0 & \cos(\phi) & -\sin(\phi) \\ -\sin(\theta) & \cos(\theta)\sin(\phi) & \cos(\theta)\cos(\phi) \end{bmatrix}, \tag{A.141}$$

which becomes, in the case that the angles are small,

$$\mathbf{R}_2\mathbf{R}_3 \approx \begin{bmatrix} 1 & 0 & \theta \\ 0 & 1 & -\phi \\ -\theta & \phi & 1 \end{bmatrix}. \tag{A.142}$$

The angular velocities $\dot{\psi}$, $\dot{\theta}$ and $\dot{\phi}$ are not applied along x, y and z axes, and thus are not the components Ω_x, Ω_y and Ω_z of the angular velocity in the body-fixed reference frame.[5] Their directions are those of axes Z, y^* and x (see Fig. 3.10), and the angular velocity in the body-fixed frame is

$$\begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} = \dot{\phi}\mathbf{e}_x + \dot{\theta}\mathbf{R}_3^T\mathbf{e}_y + \dot{\psi}[\mathbf{R}_2\mathbf{R}_3]^T\mathbf{e}_z, \tag{A.143}$$

where the unit vectors are obviously

$$\mathbf{e}_x = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \qquad \mathbf{e}_y = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \qquad \mathbf{e}_z = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \tag{A.144}$$

By performing the products, it follows that

$$\begin{cases} \Omega_x = \dot{\phi} - \dot{\psi}\sin(\theta), \\ \Omega_y = \dot{\theta}\cos(\phi) + \dot{\psi}\sin(\phi)\cos(\theta), \\ \Omega_z = \dot{\psi}\cos(\theta)\cos(\phi) - \dot{\theta}\sin(\phi), \end{cases} \tag{A.145}$$

or, in matrix form

$$\begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -\sin(\theta) \\ 0 & \cos(\phi) & \sin(\phi)\cos(\theta) \\ 0 & -\sin(\phi) & \cos(\phi)\cos(\theta) \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix}. \tag{A.146}$$

If the pitch and roll angles are small enough to linearize the relevant trigonometric functions, the components of the angular velocity can be approximated as

$$\begin{cases} \Omega_x = \dot{\phi} - \theta\dot{\psi}, \\ \Omega_y = \dot{\theta} + \phi\dot{\psi}, \\ \Omega_z = \dot{\psi} - \phi\dot{\theta}. \end{cases} \tag{A.147}$$

[5] Often symbols p, q and r are used for the components of the angular velocity in the body-fixed frame.

A.7.2 Equations of Motion—Lagrangian Approach

Assuming that the body axes xyz are principal axes of inertia, the kinetic energy of the rigid body is

$$\begin{aligned}\mathcal{T} = {} & \frac{1}{2}m\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) + \frac{1}{2}J_x\left[\dot{\phi} - \dot{\psi}\sin(\theta)\right]^2 \\ & + \frac{1}{2}J_y\left[\dot{\theta}\cos(\phi) + \dot{\psi}\sin(\phi)\cos(\theta)\right]^2 \\ & + \frac{1}{2}J_z\left[\dot{\psi}\cos(\theta)\cos(\phi) - \dot{\theta}\sin(\phi)\right]^2. \end{aligned} \tag{A.148}$$

Introducing the kinetic energy into the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{T}}{\partial q_i} = Q_i,$$

and performing the relevant derivatives, the six equations of motion are directly obtained. The three equations for translational motion are

$$\begin{cases} m\ddot{X} = Q_X, \\ m\ddot{Y} = Q_Y, \\ m\ddot{Z} = Q_Z. \end{cases} \tag{A.149}$$

The equations for rotational motion are much more complicated

$$\begin{aligned} & \ddot{\psi}\left[J_x\sin^2(\theta) + J_y\sin^2(\phi)\cos^2(\theta) + J_z\cos^2(\phi)\cos^2(\theta)\right] \\ & \quad - \ddot{\phi}J_x\sin(\theta) + \ddot{\theta}(J_y - J_z)\sin(\phi)\cos(\phi)\cos(\theta) \\ & \quad + \dot{\phi}\dot{\theta}\cos(\theta)\left\{\left[1 - 2\sin^2(\phi)\right](J_y - J_z) - J_x\right\} \\ & \quad + 2\dot{\phi}\dot{\psi}(J_y - J_z)\cos(\phi)\cos^2(\theta)\sin(\phi) \\ & \quad + 2\dot{\theta}\dot{\psi}\sin(\theta)\cos(\theta)\left[J_x - \sin^2(\phi)J_y - \cos^2(\phi)J_z\right] \\ & \quad + \dot{\theta}^2(-J_y + J_z)\sin(\phi)\cos(\phi)\sin(\theta) = Q_\psi, \\ & \ddot{\psi}(J_y - J_z)\sin(\phi)\cos(\theta)\cos(\phi) + \ddot{\theta}\left[J_y\cos^2(\phi) + J_z\sin^2(\phi)\right] \\ & \quad + 2\dot{\phi}\dot{\theta}(J_z - J_y)\sin(\phi)\cos(\phi) + \dot{\phi}\dot{\psi}(J_y - J_z)\cos(\theta)\left[1 - 2\sin^2(\phi)\right] \\ & \quad + \dot{\psi}\dot{\phi}J_x\cos(\theta) - \dot{\psi}^2\sin(\theta)\cos(\theta)\left[J_x - J_y\sin^2(\phi) - J_z\cos^2(\phi)\right] = Q_\theta, \\ & + J_x\ddot{\phi} - \sin(\theta)J_x\ddot{\psi} - \dot{\theta}\dot{\psi}J_z\sin^2(\phi)\cos(\theta) \\ & \quad - \dot{\psi}\dot{\theta}\cos(\theta)\left\{J_x + J_y\left[1 - 2\sin^2(\phi)\right] - J_z\cos^2(\phi)\right\} \\ & \quad + \dot{\theta}^2(J_y - J_z)\sin(\phi)\cos(\phi) - \dot{\psi}^2(J_y - J_z)\cos(\phi)\cos^2(\theta)\sin(\phi) = Q_\phi. \end{aligned} \tag{A.150}$$

Remark A.15 Angle ψ does not appear explicitly into the equations of motion and all trigonometric functions can be linearized if the roll and pitch angles are small.

If also the angular velocities are small, the equations of motion for rotations reduce to

$$\begin{cases} J_z\ddot{\psi} = Q_\psi, \\ J_y\ddot{\theta} = Q_\theta, \\ J_x\ddot{\phi} = Q_\phi. \end{cases} \tag{A.151}$$

In this case, the kinetic energy may be directly simplified, by developing the trigonometric functions in Taylor series and neglecting all terms containing products of three or more small quantities. For instance, the term

$$\left[\dot{\phi} - \dot{\psi}\sin(\theta)\right]^2$$

reduces to

$$\left[\dot{\phi} - \dot{\psi}\theta + \dot{\psi}\theta^3/6 + \cdots\right]^2$$

and then to $\dot{\phi}^2$, since all other terms contain products of at least three small quantities. The kinetic energy reduces to

$$\mathcal{T} \approx \frac{1}{2}m\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) + \frac{1}{2}\left(J_x\dot{\phi}^2 + J_y\dot{\theta}^2 + J_z\dot{\psi}^2\right). \tag{A.152}$$

Remark A.16 This approach is simple only if the roll and pitch angles are small. If not, the equations of motion obtained in this way in terms of angular velocities $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ are quite complicated and another approach is more expedient.

A.7.3 Equations of Motion Using Pseudo-Coordinates

Often the forces and moments applied to the rigid body are written with reference to the body-fixed frame. In these cases, the equations of motion are best written with reference to the same frame. The kinetic energy can be written in terms of the components v_x, v_y and v_z (often referred to as u, v and w) of the velocity and Ω_x, Ω_x and Ω_x (often referred to as p, q and r) of the angular velocity.

If the body-fixed frame is a principal frame of inertia, the expression of the kinetic energy is

$$\mathcal{T} = \frac{1}{2}m\left(v_x^2 + v_y^2 + {v_z}^2\right) + \frac{1}{2}\left(J_x\Omega_x^2 + J_y\Omega_y^2 + J_z\Omega_z^2\right).$$

The components of the velocity and of the angular velocity in the body-fixed frame are not the derivatives of coordinate, but are linked to the coordinates by the six kinematic equations

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = \mathbf{R}^T \begin{Bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{Bmatrix}, \tag{A.153}$$

$$\begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -\sin(\theta) \\ 0 & \cos(\phi) & \sin(\phi)\cos(\theta) \\ 0 & -\sin(\phi) & \cos(\theta)\cos(\phi) \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix}, \tag{A.154}$$

that is, in more compact form,

$$\mathbf{w} = \mathbf{A}^T \dot{\mathbf{q}}, \tag{A.155}$$

where the vectors of the generalized velocities and of the derivatives of the generalized coordinates are

$$\mathbf{w} = [\, v_x \; v_y \; v_z \; \Omega_x \; \Omega_y \; \Omega_z \,]^T, \tag{A.156}$$

$$\dot{\mathbf{q}} = [\, \dot{X} \; \dot{Y} \; \dot{Z} \; \dot{\phi} \; \dot{\theta} \; \dot{\psi} \,]^T \tag{A.157}$$

and matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} \mathbf{R} & & \mathbf{0} \\ & \begin{bmatrix} 1 & 0 & -\sin(\theta) \\ 0 & \cos(\phi) & \sin(\phi)\cos(\theta) \\ 0 & -\sin(\phi) & \cos(\theta)\cos(\phi) \end{bmatrix}^T & \\ \mathbf{0} & & \end{bmatrix}. \tag{A.158}$$

Note that the second submatrix is not a rotation matrix (the first one is such), and

$$\mathbf{A}^{-1} \neq \mathbf{A}^T; \qquad \mathbf{B} \neq \mathbf{A}. \tag{A.159}$$

The inverse transformation is (A.125)

$$\dot{\mathbf{q}} = \mathbf{B}\mathbf{w},$$

where $\mathbf{B} = \mathbf{A}^{-T}$.

None of the velocities included in vector $\mathbf{w}$ can be integrated to obtain a set of generalized coordinates, and they must all be considered as derivatives of pseudo-coordinates.

The state space equation, made of the six dynamic and the six kinematic equations, is thus (A.140), simplified since in the present case neither the potential energy nor the dissipation function are present

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\left\{\dfrac{\partial \mathcal{T}}{\partial w}\right\}\right) + \mathbf{B}^T \boldsymbol{\Gamma} \left\{\dfrac{\partial \mathcal{T}}{\partial w}\right\} - \mathbf{B}^T \left\{\dfrac{\partial \mathcal{T}}{\partial q}\right\} = \mathbf{B}^T \mathbf{Q}, \\ \{\dot{q}_i\} = \mathbf{B}\{w_i\}. \end{cases} \tag{A.160}$$

Here $\mathbf{B}^T\mathbf{Q}$ is just a column matrix containing the three components of the force and the three components of the moment applied to the body along the body-fixed axes x, y, z.

The most difficult part of the computation is writing matrix $\mathbf{B}^T\boldsymbol{\Gamma}$. Performing somewhat difficult computations it follows that

$$\mathbf{B}^T \boldsymbol{\Gamma} = \begin{bmatrix} \widetilde{\boldsymbol{\Omega}} & \mathbf{0} \\ \widetilde{\mathbf{V}} & \widetilde{\boldsymbol{\Omega}} \end{bmatrix}, \tag{A.161}$$

where $\widetilde{\boldsymbol{\Omega}}$ and $\widetilde{\mathbf{V}}$ are skew–symmetric matrices containing the components of the angular and linear velocities

$$\widetilde{\boldsymbol{\Omega}} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}, \qquad \widetilde{\mathbf{V}} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}. \tag{A.162}$$

If the body-fixed axes are principal axes of inertia, the dynamic equations are simply

$$\begin{cases} m\dot{v}_x = m\Omega_z v_y - m\Omega_y v_z + F_x, \\ m\dot{v}_y = m\Omega_x v_z - m\Omega_z v_x + F_y, \\ m\dot{v}_z = m\Omega_y v_x - m\Omega_x v_y + F_z, \\ J_x\dot{\Omega}_x = \Omega_y\Omega_z(J_y - J_z) + M_x, \\ J_y\dot{\Omega}_y = \Omega_x\Omega_z(J_z - J_x) + M_y, \\ J_z\dot{\Omega}_z = \Omega_x\Omega_y(J_x - J_y) + M_x. \end{cases} \tag{A.163}$$

Remark A.17 The equations so obtained are much simpler than equations (A.150) and the last three equations are nothing else than Euler equations.

A.8 Multibody Modeling

A robot or a vehicle may be modeled as a first approximation as a system made by a number of rigid bodies connected with each other through joints of different type, springs and dampers.

For instance, the *Apollo* LRV, like any other four wheeled vehicle, can be considered as a rigid body suspended on the ground by its elastic wheels and suspensions. If the suspensions are assumed to constrain all degrees of freedom of the wheel hubs except their motion in vertical direction, the wheel rotation is assumed to be determined by the forward motion of the vehicle (the longitudinal slip is neglected) and the steering is assumed to be an input parameter determined by the will of the driver, such model has 10 degrees of freedom: six determining the position of the frame as a rigid body in tridimensional space plus four, one for each wheel hub.

A free flying spacecraft carrying an arm with three degrees of freedom can be modeled as a system made of a number of rigid bodies, with nine degrees of freedom and so on.

This kind of approach is usually referred to as *multibody* approach.

It can be extended to model the system to a much greater detail. For instance, instead of modeling the suspensions of the LRV as a rigid body (the wheel hub) that can move in a direction parallel to the z-axis (in vertical direction on flat ground) of the vehicle, it is possible to enter into a much greater detail, considering the lower and upper triangles of the suspension and the wheel strut as three rigid bodies connected to each other by cylindrical hinges.

However, if the flexibility of the various elements is neglected, the number of degrees of freedom does not change, since the motion of the various parts of each suspension is determined by the single parameter that is the vertical displacement of the wheel hub.

The fact that the number of degrees of freedom does not change does not mean that the complexity of the model is the same. The mathematical model of a multibody system consists of $6n$ differential equations of dynamic equilibrium, if the motion is studied in the tridimensional space and n is number of rigid bodies and by an adequate number of constraint equations.

For instance, to return to the model of the LRV, the total number of rigid bodies is 13 (the vehicle frame plus four suspensions made by three rigid bodies, the upper and lower triangles and the strut, each) and the dynamic equations are 78. The connections between the various members of each suspensions originate a number of 17 constraint equations for a total of 68 constraint equations. Using the latter to eliminate 68 out of the 78 generalized coordinates describing the motion of all the rigid bodies, the number of dynamic equilibrium equations can be reduced to 10, as it should, since that is the number of the degrees of freedom of the system. The remaining 68 equations can be used to compute the reactions in the constraints after that the motion has been studied.

In general, the equations of motion are nonlinear and the only way to reach a solution is by performing the numerical integration in time of the model or, as it is usually said, by simulating numerically its motion.

The approach usually followed by most *general purpose* multibody computer codes, however, is based on the numerical integration of all equations, the differential dynamic equations plus the algebraic constraint equations. The latter introduce what are usually referred to as algebraic loops, which complicates the numerical integration.

As soon as the complexity of the problem increases, this way of approaching the problem can easily give way to long computer times, owing to the large number of equations, differential and algebraic, involved.

In some cases it is possible to write directly the minimum number of dynamic equations needed to study the motion of the system. The steps to build the model are

- choice of the generalized coordinates
- computation of the kinetic and potential energy, of the dissipation function and of the virtual work of external forces
- application of the Lagrange equations formalism to obtain the equations of motion

If the system is simple, as in the case of the robotic arms described in Example 3.5, this approach yields a compact model that can be used to solve the behavior of the system in a straightforward way. However, even in the case the equations so obtained are simple, they are nevertheless nonlinear and the only way to obtain a result is by numerically integrating them in time, with the advantage of dealing with a smaller number of equations leading to a much shorter computer time.

An intermediate approach can also be followed: the full set of equations, differential and algebraic, is obtained using a full set of generalized coordinates. Then symbolic computer programs are used to eliminate the constraint equations, reducing the set of generalized coordinates by eliminating the constrained ones. The numerical integration is then performed on a smaller set of equations that do not contain algebraic loops.

Appendix B
Equations of Motion for Continuous Systems

B.1 General Considerations

The main feature of the discrete systems studied in the previous appendix is that a finite number of degrees of freedom is sufficient to describe their configuration. Moreover, if the system is linear, the Ordinary Differential Equations (ODEs) of motion can be easily substituted by a set of linear algebraic equations: the natural mathematical tool for the study of linear discrete systems is matrix algebra.

The situation is different when the behavior of a deformable elastic body is studied as a continuous system. Usually a deformable body is modeled as an elastic continuum or, in the case its behavior can be assumed to be linear, as a linear elastic continuum. A continuum can be thought as consisting of an infinity of points.

To describe the undeformed (or initial) configuration of the body, a reference frame is set in space. Many problems can be studied with reference to a two-dimensional frame or even a single coordinate, but there are cases in which a full tridimensional approach is required. The characteristics of the material are defined by functions of the position in all the parts of space (or plane or line) occupied by the continuum. These functions need not, in general, be continuous.

The configuration at any time t can be obtained from the initial configuration once a vector function expressing the displacements of all points is known (Fig. B.1). The displacement of a point is a vector, with a number of components equal to the number of dimensions of the reference frame. Even if in some cases different choices are considered, usually the components of this vector are taken as the generalized coordinates of each point. The number of degrees of freedom of an elastic (or, more generally, deformable) body is thus infinite. The corresponding generalized coordinates can be manipulated as continuous functions of space coordinates and time, and the theory of continuous functions is the natural tool for dealing with deformable continua.

Remark B.1 The function $\overrightarrow{u}(x, y, z, t)$ describing the displacement of the points of the body is differentiable with respect to time at least twice; the first derivative gives the displacement velocity and the second the acceleration. Usually, however, higher-order derivatives also exist.

G. Genta, *Introduction to the Mechanics of Space Robots*, Space Technology Library 26,
DOI 10.1007/978-94-007-1796-1, © Springer Science+Business Media B.V. 2012

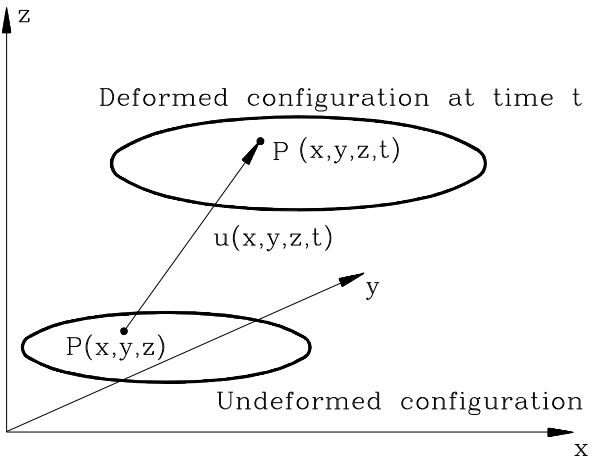

Fig. B.1 Deformation of an elastic continuum; reference frame and displacement vector (from G. Genta, *Vibration Dynamics and Control*, Springer, New York, 2009)

Remark B.2 Reference frame xyz needs not to be an inertial frame. If the flexible body is a 'structure', i.e. a flexible body that does not move except for vibration about its equilibrium position, the reference frame is inertial and $\dot{u}$ can be considered as an absolute velocity, but if it moves as a whole, for instance like a turbine blade or a robot arm, the reference frame may be a moving frame, for instance a frame following the rigid motion of the body and the displacement u and the velocity $\dot{u}$ are relative.

If displacements and rotations can be considered as small quantities, in order not to introduce geometrical nonlinearities, the usual definitions of stresses and strains used in elementary theory of elasticity can be used. When the dynamics of an elastic body can be dealt with as a linear problem, as when the behavior of the material is linear and no geometrical nonlinearity is considered, an infinity of natural frequencies exist as a consequence of the infinity of degrees of freedom of the model.

Assuming that the forces acting on the body are expressed by the function $\mathbf{f}(x, y, z, t)$, the Partial derivatives Differential Equation (PDE) of motion can generally be written as

$$D\big[\mathbf{u}(x, y, z, t)\big] = \mathbf{f}(x, y, z, t), \qquad U\big[\mathbf{u}(x, y, z, t)\big]_B = 0, \tag{B.1}$$

where the differential operator D completely describes the behavior of the body and operator U, defined on the boundary B, states the boundary conditions (only homogeneous boundary conditions are described by (B.1)). The actual form of the differential operator can be obtained by resorting directly to the dynamic equilibrium equations or by writing the kinetic and potential energies and using the Lagrange equations, and the boundary conditions usually follow from geometrical considerations.

The solution of (B.1) exists if an inverse operator D^{-1} can be defined

$$\mathbf{u}(x, y, z, t) = D^{-1}\big[\mathbf{f}(x, y, z, t)\big]. \tag{B.2}$$

Remark B.3 Equation (B.2) is just a formal statement; in most cases the relevant operator cannot be written in explicit form, particularly when the boundary conditions are not the simplest ones.

Because no general approach to the dynamics of an elastic body is feasible, many different models for the study of particular classes of structural elements (beams, plates, shells, etc.) have been developed. Only the bending of beams will be studied here in detail. This choice is only in part due to the fact that many robot elements are studied as beams; it comes also from the need of showing some general properties of continuous systems in the simplest case in order to gain a good insight on the properties of deformable bodies.

The solution of most problems encountered in engineering practice requires dealing with complex structures and the use of continuous models is, consequently, ruled out. For complex shapes the only feasible approach is the discretization of the continuum and then the application of the methods seen for discrete systems. The substitution of a continuous system, characterized by an infinite number of degrees of freedom, with a discrete system, sometimes with a very large but finite number of degrees of freedom, is usually referred to as *discretization*. This step is of primary importance in the solution of practical problems, because the accuracy of the results depends largely on the adequacy of the discrete model to represent the actual system.

B.2 Beams

B.2.1 General Considerations

The simplest continuous system is the prismatic beam. The study of the elastic behavior of beams dates back to Galileo, with important contributions by Daniel Bernoulli, Euler, De Saint Venant, and many others. A *beam* is essentially an elastic solid in which one dimension is prevalent over the others. Often the beam is prismatic (i.e., the cross sections are all equal), homogeneous (i.e., with constant material characteristics), straight (i.e., its axis is a part of a straight line), and untwisted (i.e., the principal axes of elasticity of all sections are equally directed in space). The unidimensional nature of beams allows simplification of the study: each cross section is considered as a rigid body whose thickness in the axial direction is vanishingly small; it has six degrees of freedom, three translational and three rotational. The problem is thus reduced to a unidimensional problem, in the sense that a single coordinate, namely the axial coordinate, is required.

Setting the z-axis of the reference frame along the axis of the beam (Fig. B.2), the six generalized coordinates of each cross section are the axial displacement u_z, the lateral displacements u_x and u_y, the torsional rotation ϕ_z about the z-axis and the flexural rotations ϕ_x and ϕ_y about axes x and y. Displacements and rotations are assumed to be small, so that rotations can be regarded as vector quantities, which

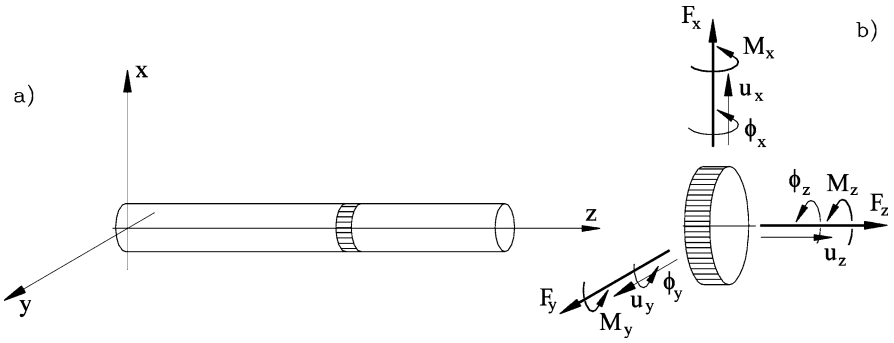

Fig. B.2 Straight beam. (**a**) Sketch and reference frame: (**b**) generalized displacements and forces on a generic cross section (from G. Genta, *Vibration Dynamics and Control*, Springer, New York, 2009)

simplifies all rotation matrices by linearizing trigonometric functions. As a consequence, the three rotations will be considered components of a vector in the same way as the three displacements are components of vector **u**. The generalized forces acting on each cross section and corresponding to the six generalized coordinates defined earlier are the axial force F_z, shear forces F_x and F_y, the torsional moment M_z about the z-axis, and the bending moments M_x and M_y about the x- and y-axes.

From the aforementioned assumptions it follows that all normal stresses in directions other than z (σ_x and σ_y) are assumed to be small enough to be neglected. When geometric and material parameters are not constant along the axis, they must change at a sufficiently slow rate in order not to induce stresses σ_x and σ_y, which could not be considered in this model. If the axis of the beam is assumed to be straight, the axial translation is uncoupled from the other degrees of freedom, at least as a first approximation. A beam loaded only in the axial direction and whose axial behavior is the only one studied is usually referred to as a *bar*. The torsional-rotation degree of freedom is uncoupled from the others only if the area center of all cross sections coincides with their shear center, which happens if all cross sections have two perpendicular planes of symmetry. If the planes of symmetry of all sections are equally oriented (the beam is not twisted) and the x- and y-axes are perpendicular to such planes, the flexural behavior in the xz-plane is uncoupled from that in the yz-plane. The coupling of the degrees of freedom in straight, untwisted beams with cross sections having two planes of symmetry is summarized in Table B.1.

B.2.2 Flexural Vibrations of Straight Beams

Robot arms and legs can easily be modeled as beams. For this reason the bending behavior of straight beams is here dealt with in some detail.

At first assume that the beam is globally at rest in an inertial reference frame.

With the assumptions in Sect. B.2.1, the flexural behavior in each lateral plane can be studied separately from the other degrees of freedom. If bending occurs in the

Table B.1 Generalized coordinates and generalized forces in beams

Type of behavior	Degrees of freedom	Generalized forces
Axial	Displacement u_z	Axial force F_z
Torsional	Rotation ϕ_z	Torsional moment M_z
Flexural (xz-plane)	Displacement u_x Rotation ϕ_y	Shearing force F_x Bending moment M_y
Flexural (yz-plane)	Displacement u_y Rotation ϕ_x	Shearing force F_y Bending moment M_x

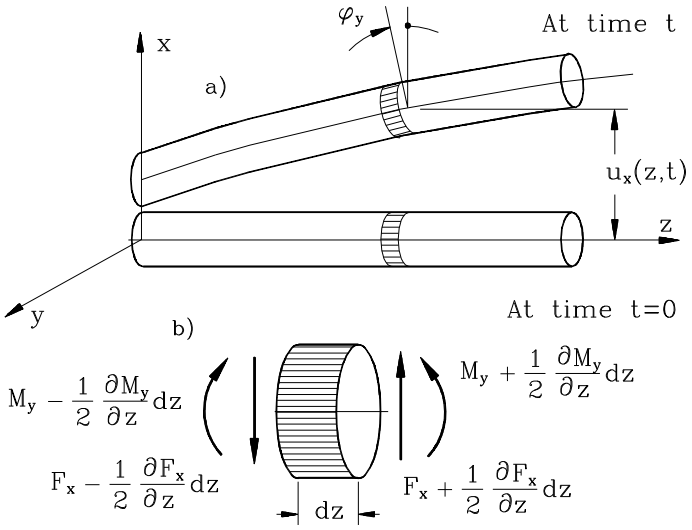

Fig. B.3 Flexural behavior of a straight beam in the xz-plane; (**a**) sketch of the system; (**b**) forces and moments acting on the length dz of the beam (from G. Genta, *Vibration Dynamics and Control*, Springer, New York, 2009)

xz-plane, the relevant generalized coordinates are displacement u_x and rotation ϕ_y. The simplest approach is that often defined as Euler–Bernoulli beam, based on the added assumptions that both shear deformation and rotational inertia of the cross sections are negligible compared with bending deformation and translational inertia, respectively. These assumptions lead to a good approximation if the beam is slender, i.e., if the thickness in the x-direction is much smaller than length l. Note that, at any rate, the thickness in the x-direction must be small enough to use beam theory.

The equilibrium equation for translations in x-direction of the length dz of the beam (Fig. B.3) is

$$\rho A \frac{d^2 u_x}{dt^2} = \frac{\partial F_x}{\partial z} + f_x(z,t). \tag{B.3}$$

If the rotational inertia of the length dz of the beam is neglected, and no distributed bending moment acts on the beam, the equilibrium equation for rotations about y-axis of the length dz of the beam is

$$F_x dz + \frac{\partial M_y}{\partial z} dz = 0. \tag{B.4}$$

By introducing (B.4) into (B.3), it follows that

$$\rho A \frac{d^2 u_x}{dt^2} = -\frac{\partial^2 M_y}{\partial z^2} + f_x(z,t). \tag{B.5}$$

The bending moment is proportional to the curvature of the inflected shape of the beam; neglecting shear deformation and using elementary beam theory the latter coincides with the second derivative of the displacement u_x,

$$M_y = E I_y \frac{\partial^2 u_x}{\partial z^2}, \tag{B.6}$$

where I_y is the area moment of inertia of the cross section of the beam about its y axis. The following equilibrium equation can thus be obtained

$$m(z) \frac{d^2 u_x}{dt^2} + \frac{\partial^2}{\partial z^2} \left[k(z) \frac{\partial^2 u_x}{\partial z^2} \right] = f_x(z,t), \tag{B.7}$$

where the mass and the bending stiffness per unit length are, respectively,

$$m(z) = \rho(z) A(z), \qquad k(z) = E(z) I_y(z). \tag{B.8}$$

Once the lateral displacement u_x has been obtained, the second generalized coordinate ϕ_y is readily obtained: since the cross section remains perpendicular to the deflected shape of the beam owing to neglecting shear deformation, the rotation of the cross section is equal to the slope of the inflected shape,

$$\phi_y = \frac{\partial u_x}{\partial z}. \tag{B.9}$$

Equation (B.7) defines the differential operator D introduced into equation (B.1)

$$D(u_z) = m(z) \frac{d^2 u_x}{dt^2} + \frac{\partial^2}{\partial z^2} \left[k(z) \frac{\partial^2 u_x}{\partial z^2} \right]. \tag{B.10}$$

The boundary conditions $U[u_x(z,t)]_B = 0$ must be stated following the actual conditions at the ends of the beam: if, for instance, they are clamped, both the displacement u_x and the rotation $\partial u_z / \partial z$ must be equated to zero for $z = 0$ and $z = l$ (where l is the length of the beam and the origin is set at the left end).

In the case of a prismatic homogeneous beam, (B.7) reduces to

$$\rho A \frac{d^2 u_x}{dt^2} + E I_y \frac{\partial^4 u_x}{\partial z^4} = f_x(z,t). \tag{B.11}$$

Free Behavior

The solution of the homogeneous equation associated with (B.7) can be expressed as the product of a function of time and a function of the space coordinate

$$u_x(z,t) = q(z)\eta(t). \tag{B.12}$$

Introducing (B.12) into the homogeneous equation associated with (B.7) and separating the variables, it follows that

$$\frac{1}{\eta(t)}\frac{d^2\eta(t)}{dt^2} = \frac{1}{m(z)q(z)}\frac{\partial^2}{\partial z^2}\left[k(z)\frac{\partial^2 q(z)}{\partial z^2}\right]. \tag{B.13}$$

The function on the left-hand side depends on time but not on the space coordinate z. Conversely, the function on the right-hand side is a function of z but not of t. The only possibility of satisfying equation (B.13) for all values of time and of coordinate z is to state that both sides are constant and that the two constants are equal. This constant can be indicated as $-\omega^2$. The condition on the function of time on the left-hand side is thus

$$\frac{1}{\eta(t)}\frac{d^2\eta(t)}{dt^2} = \text{constant} = -\omega^2. \tag{B.14}$$

Neglecting a proportionality constant that will be introduced into function $q(z)$ later, this equation yields a harmonic oscillation with frequency ω

$$\eta(t) = \sin(\omega t + \phi). \tag{B.15}$$

The solution of the equation of motion for free oscillations of the beam is

$$u_x(z,t) = q(z)\sin(\omega t + \phi). \tag{B.16}$$

Function $q(z)$ is said to be the *principal function*. Each point of the bar performs a harmonic motion with frequency ω, while the amplitude is given by the function $q(z)$.

Remark B.4 The resultant motion is a standing wave, with all points of the beam vibrating in phase.

By introducing (B.16) into (B.7), it follows that

$$-\omega^2 m(z)q(z) = \frac{d^2}{dz}\left[k(z)\frac{d^2q(z)}{dz^2}\right], \tag{B.17}$$

or, in the case of a prismatic homogeneous beam,

$$-\omega^2 q(z) = \frac{EI_y}{\rho A}\frac{d^4q(z)}{dz^4}. \tag{B.18}$$

Equations (B.17) and (B.18) are eigenproblems. The second, for example, states that the fourth derivative of function $q(z)$ (with respect to the space coordinate z) is proportional to the function itself, the constant of proportionality being $-\omega^2 \rho A / E I_y$. The values of such a constant allowing the equation to be satisfied by a solution other than the trivial solution $q(z) = 0$ are the eigenvalues, and the corresponding functions $q(z)$ are the eigenfunctions. Equation (B.17), although more complex, has a similar meaning.

Remark B.5 The eigenvalues are infinite in number, and the general solution of the equation of motion (B.17) can be obtained as the sum of an infinity of terms of the type of (B.16).

Remark B.6 The eigenfunctions $q_i(z)$ are defined only as far as their shape is concerned, exactly like the eigenvectors in discrete systems. The amplitude of the various modes can be computed only after the initial conditions have been stated.

Remark B.7 Although the number of eigenfunctions, and hence of modes, is infinite, a small number of principal functions is often sufficient to describe the behavior of an elastic body with the required precision, in a way that is similar to what has already been said for eigenvectors.

Remark B.8 Eigenfunctions have some of the properties seen for eigenvectors, particularly that of orthogonality with respect to the mass $m(z)$ and to the stiffness $k(z)$. As a general rule they are not orthogonal to each other, except when the function $m(z)$ is constant: the eigenfunctions of a prismatic homogeneous beam are thus orthogonal to each other.

The general solution of (B.18) is

$$q(z) = C_1 \sin(az) + C_2 \cos(az) + C_3 \sinh(az) + C_4 \cosh(az), \tag{B.19}$$

where

$$a = \sqrt{\omega} \sqrt[4]{\frac{\rho A}{E I_y}}. \tag{B.20}$$

The rotation is the derivative of the displacement

$$\frac{dq}{dz} = C_1 a \cos(az) - C_2 a \sin(az) + C_3 a \cosh(az) + C_4 a \sinh(az). \tag{B.21}$$

Constants C_i can be computed from the boundary conditions. In the present case four boundary conditions must be stated, which is consistent both with the order of the differential equation and with the number of degrees of freedom involved. Each end of the beam may be free, clamped, simply supported or, a condition seldom accounted for, constrained in such a way to restrain rotations but not displacements.

At a free end the displacement and the rotation are free, but both the bending moment and the shear force must vanish. This can be expressed by the relationships

$$\frac{d^2q}{dz^2} = 0, \qquad \frac{d^3q}{dz^3} = 0. \tag{B.22}$$

If on the contrary an end is clamped, both its displacement and its rotation vanish

$$q = 0, \qquad \frac{dq}{dz} = 0. \tag{B.23}$$

A supported end is free to rotate, and hence the bending moment must vanish, but its displacement is constrained

$$q = 0, \qquad \frac{d^2q}{dz^2} = 0. \tag{B.24}$$

A further condition is the case where the end is free to move, and hence the shear force vanishes, but its rotation is constrained

$$\frac{dq}{dz} = 0, \qquad \frac{d^3q}{dz^3} = 0. \tag{B.25}$$

As an example, consider a prismatic beam clamped at $z = 0$ and free at $z = l$.
At the 'left' end ($z = 0$) both displacement and rotation vanish:

$$\begin{aligned} q(0) &= C_2 + C_4 = 0, \\ \left(\frac{dq}{dz}\right)_{z=0} &= C_1 + C_3 = 0. \end{aligned} \tag{B.26}$$

The second and third derivatives of function $q(z)$ are

$$\begin{aligned} \frac{d^2q}{dz^2} &= -a^2C_1\sin(az) - a^2C_2\cos(az) + a^2C_3\sinh(az) + a^2C_4\cosh(az), \\ \frac{d^3q}{dz^3} &= -a^3C_1\cos(az) + a^3C_2\sin(az) + a^3C_3\cosh(az) + a^3C_4\sinh(az), \end{aligned} \tag{B.27}$$

and then the conditions stating that the 'right' end ($z = l$) is free yield

$$\begin{aligned} -C_1\sin(al) - C_2\cos(al) + C_3\sinh(al) + C_4\cosh(al) &= 0, \\ -C_1\cos(al) + C_2\sin(al) + C_3\cosh(al) + C_4\sinh(al) &= 0. \end{aligned} \tag{B.28}$$

By solving the conditions at the left end ($z = 0$) in C_3 and C_4 and introducing their values in the conditions at the right end, it follows that

$$\begin{bmatrix} \sin(al) + \sinh(al) & \cos(al) + \cosh(al) \\ \cos(al) + \cosh(al) & -\sin(al) + \sinh(al) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \mathbf{0}. \tag{B.29}$$

To obtain a solution other than the trivial solution $C_1 = 0$, $C_2 = 0$, the determinant of the matrix of the coefficients of the set of linear equations in C_1 and C_2 must vanish

$$\sinh^2(al) - \sin^2(al) - \left[\cos(al) + \cosh(al)\right]^2 = 0. \tag{B.30}$$

This equation cannot be solved in closed form in al, but it is easy to obtain numerical solutions. The first four are

$$al = 1.875, 4.694, 7.855, 10.996.$$

For high values of al, approximate solutions can be found, by remembering that

$$\sinh(ax) \approx \cosh(ax) \approx \frac{e^{ax}}{2}$$

and that

$$\sin(ax) \ll \frac{e^{ax}}{2}, \qquad \cos(ax) \ll \frac{e^{ax}}{2}.$$

In this case, the characteristic equation reduces to

$$\cos(al) = 0$$

that yields

$$al = \left(i - \frac{1}{2}\right)\pi.$$

For $i = 3$ this approximated solution yields $al = 7.854$, i.e. an error of only 0.013% with respect to the numerical solution. For larger values of i, the error is negligible.

The natural frequencies can be computed from (B.20), obtaining

$$\omega = \frac{\beta_i^2}{l^2}\sqrt{\frac{EI_y}{\rho A}}, \tag{B.31}$$

where β_i are the values of al obtained above.

To compute the eigenfunctions, it is possible to state the value of one of the constants, for instance $C_1 = 1$. From (B.29) it follows that

$$C_2 = \frac{\sin(al) + \sinh(al)}{\cos(al) + \cosh(al)}, \tag{B.32}$$

and the ith eigenvector, expressed in terms of the nondimensional coordinate

$$\zeta = \frac{z}{l} \tag{B.33}$$

is

$$q_i(\zeta) = \sin(\beta_i\zeta) - \sinh(\beta_i\zeta) - C_2\left[\cos(\beta_i\zeta) - \cosh(\beta_i\zeta)\right]. \tag{B.34}$$

Table B.2 Values of constants $\beta_i = a_i l$ for the various modes with different boundary conditions

Boundary condition	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i>4$
Free–free	0	4.730	7.853	10.996	14.137	$\approx(i+1/2)\pi$
Supported–free	0	1.25π	2.25π	3.25π	4.25π	$(i+1/4)\pi$
Clamped–free	–	1.875	4.694	7.855	10.996	$\approx(i-1/2)\pi$
Supported–supported	–	π	2π	3π	4π	$i\pi$
Supported–clamped	–	3.926	7.069	10.210	13.352	$\approx(i+1/4)\pi$
Clamped–clamped	–	4.730	7.853	10.996	14.137	$\approx(i+1/2)\pi$

The same procedure can be applied also for other boundary conditions.

Expression (B.31) for the natural frequencies holds for any boundary condition. The values of constants $\beta_i = a_i l$ are reported in Table B.2 for the most common boundary conditions.

The eigenfunctions, normalized in such a way that the maximum value of the displacement is equal to unity, are, for different boundary conditions, as follows:

1. *Free–free*. Rigid-body modes:

$$q_0^I(\zeta) = 1, \qquad q_0^{II}(\zeta) = 1 - 2\zeta.$$

Other modes:

$$q_i(\zeta) = \frac{1}{2N}\big\{\sin(\beta_i\zeta) + \sinh(\beta_i\zeta) + N\big[\cos(\beta_i\zeta) + \cosh(\beta_i\zeta)\big]\big\},$$

where

$$N = \frac{\sin(\beta_i) - \sinh(\beta_i)}{-\cos(\beta_i) + \cosh(\beta_i)}.$$

2. *Supported–free*. Rigid-body mode:

$$q_0(\zeta) = \zeta.$$

Other modes:

$$q_i(\zeta) = \frac{1}{2\sin(\beta_i)}\left[\sin(\beta_i\zeta) + \frac{\sin(\beta_i)}{\sinh(\beta_i)}\sinh(\beta_i\zeta)\right].$$

3. *Clamped–free*.

$$q_i(\zeta) = \frac{1}{N_2}\big\{\sin(\beta_i\zeta) - \sinh(\beta_i\zeta) - N_1\big[\cos(\beta_i\zeta) - \cosh(\beta_i\zeta)\big]\big\},$$

where

$$N_1 = \frac{\sin(\beta_i) + \sinh(\beta_i)}{\cos(\beta_i) + \cosh(\beta_i)},$$

$$N_2 = \sin(\beta_i) - \sinh(\beta_i) - N_1\big[\cos(\beta_i) - \cosh(\beta_i)\big].$$

4. *Supported–supported.*

$$q_i(\zeta) = \sin(i\pi\zeta).$$

5. *Supported–clamped.*

$$q_i(\zeta) = \frac{1}{N}\left[\sin(\beta_i\zeta) - \frac{\sin(\beta_i)}{\sinh(\beta_i)}\sinh(\beta_i\zeta)\right],$$

where N is the maximum value of the expression within brackets and must be computed numerically.

6. *Clamped–clamped.*

$$q_i(\zeta) = \frac{1}{N_2}\left\{\sin(\beta_i\zeta) - \sinh(\beta_i\zeta) - N_1\left[\cos(\beta_i\zeta) - \cosh(\beta_i\zeta)\right]\right\},$$

where

$$N_1 = \frac{\sin(\beta_i) - \sinh(\beta_i)}{\cos(\beta_i) - \cosh(\beta_i)}$$

and N_2 is the maximum value of the expression between braces and must be computed numerically.

The first four mode shapes (plus the rigid-body modes where they do exist) for each boundary condition are plotted in Fig. B.4.

Modal Analysis

The property of orthogonality with respect to mass and stiffness means that if $q_i(z)$ and $q_j(z)$ are two distinct eigenfunctions and $(i \neq j)$, it follows that

$$\int_0^l m(z)q_i(z)q_j(z)\,dz = 0, \qquad \int_0^l k(z)\frac{d^2q_i(z)}{dz^2}\frac{d^2q_j(z)}{dz^2}\,dz = 0. \tag{B.35}$$

As already stated, eigenfunctions are not directly orthogonal, except for the case when the mass $m(z)$ is constant along the beam.

If $i = j$, the integrals of (B.35) do not vanish:

$$\int_0^l m(z)\left[q_i(z)\right]^2 dz = \overline{M}_i \neq 0, \qquad \int_0^l k(z)\left[\frac{dq_i(z)}{dz}\right]^2 dz = \overline{K}_i \neq 0. \tag{B.36}$$

These two relationships define the modal masses and stiffness

Remark B.9 The meaning of the modal mass and stiffness in the case of continuous systems is exactly the same as for discrete systems; the only difference is that in the current case the number of modes, and then of modal masses and stiffnesses, is infinite.

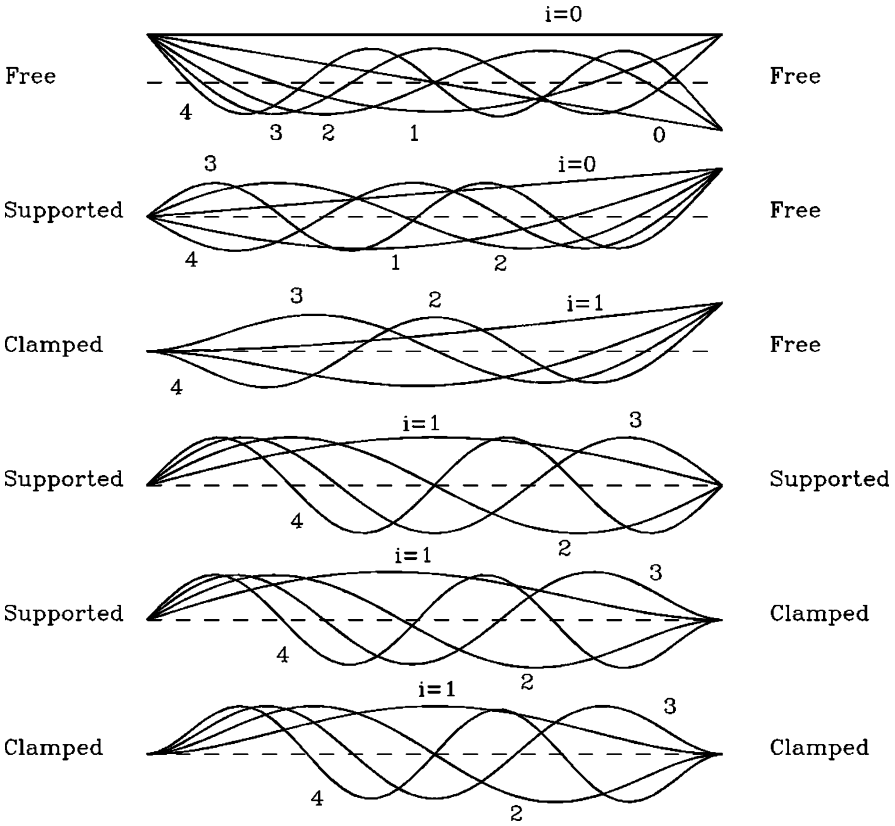

Fig. B.4 Normal modes of a straight beam with different end conditions. The first four modes plus the rigid-body modes, where they exist, are shown (from G. Genta, *Vibration Dynamics and Control*, Springer, New York, 2009)

Any deformed configuration of the system $u_x(z,t)$ can be expressed as a linear combination of the eigenfunctions. The coefficients of this linear combination, which are functions of time, are the modal coordinates $\eta_i(t)$:

$$u_x(z,t) = \sum_{i=0}^{\infty} \eta_i(t) q_i(z). \tag{B.37}$$

Equation (B.37) expresses the modal transformation for continuous systems and is equivalent to the similar relationship seen for discrete systems.

The inverse transformation, needed to compute the modal coordinates $\eta_i(t_k)$ corresponding to any given deformed configuration $u(z,t_k)$ occurring at time t_k, can be obtained through a simple procedure. Multiplying equation (B.37) by the jth eigenfunction and by the mass distribution $m(z)$ and integrating on the whole length of

the bar, it follows that

$$\int_0^l \left[m(z)q_j(z)u(z,t_0)\right]dz = \sum_{i=0}^{\infty} \eta_i(t_0) \int_0^l \left[m(z)q_j(z)q_i(z)\right]dz. \tag{B.38}$$

Of the infinity of terms on the right-hand side, only the term with $i = j$ does not vanish and the integral yields the jth modal mass. Remembering the definition of the modal masses, it follows that

$$\eta_i(t_0) = \frac{1}{\overline{M}_j} \int_0^l \left[m(z)q_j(z)u(z,t_0)\right]dz. \tag{B.39}$$

This relationship can be used to perform the inverse modal transformation, i.e., to compute the modal coordinates corresponding to any deformed shape of the system.

Eigenfunctions can be normalized in several ways, one being that leading to unit values of the modal masses. This is achieved simply by dividing each eigenfunction by the square root of the corresponding modal mass.

The vibration of the beam under the effect of the forcing function $f_x(z,t)$ can be obtained by solving the complete equation (B.7), whose general solution can be expressed as the sum of the complementary function obtained earlier, and a particular integral of the complete equation. Owing to the orthogonality properties of the normal modes $q_i(z)$, the latter can be expressed as a linear combination of the eigenfunctions. Equation (B.37) then also holds in the case of forced motion of the system.

Forced Response

By introducing the modal transformation (B.37) into the equation of motion (B.7), the latter can be transformed into a set of an infinite number of equations in the modal coordinates η_i

$$\overline{M}_i \ddot{\eta}_i(t) + \overline{K}_i \eta_i(t) = \overline{f}_i(t), \tag{B.40}$$

where the ith modal force is defined by the following formulas:

$$\overline{f}_i(t) = \int_0^l q_i(z) f(z,t)\, dz, \qquad \overline{f}_i(t) = \sum_{k=1}^{m} q_i(z_k) f_k(t), \tag{B.41}$$

holding in the cases of a continuous force distribution and m concentrated bending forces $f_k(t)$ acting on points of coordinates z_k, respectively.

Remark B.10 Equations (B.41) correspond exactly to the definition of the modal forces for concentrated systems.

Equations (B.40) can be used to study the forced response of a continuous system to external excitations of any type by reducing it to a number of systems with a single degree of freedom. In the case of continuous systems, their number is infinite, but usually a small number of them is enough to obtain the results with the required precision.

If the excitation is provided by the motion of the structure to which the beam is connected, it is expedient to resort to a coordinate system that moves with the supporting points. In the case of the bending behavior of a beam, only the motion of the supporting structure in x-direction is coupled with its dynamic behavior in the xz-plane. If the origin of the coordinates is displaced by the quantity x_A, the absolute displacement in x-direction $u_{x_{\text{iner}}}(z,t)$ is linked to the relative displacement $u_x(z,t)$ by the obvious relationship

$$u_{x_{\text{iner}}}(z,t) = u_x(z,t) + x_A(t).$$

By writing (B.7) using the relative displacement, it follows that

$$m(z)\frac{d^2u_z}{dt^2} - \frac{\partial}{\partial z}\left[k(z)\frac{\partial u_z}{\partial z}\right] = -m(z)\ddot{x}_A. \tag{B.42}$$

The excitation due to the motion of the constraints can be dealt with simply by using relative coordinates and applying an external force distribution equal to $-m(z)\ddot{x}_A$. The modal forces can be readily computed through (B.41):

$$\overline{f}_i(t) = -r_i\ddot{x}_A, \tag{B.43}$$

where

$$r_i = \int_0^l q_i(z)m(z)\,dz$$

are the modal participation factors related to the lateral motion of the bar.

Remark B.11 The motion of the system has been studied in an inertial reference frame. However, the inflected shape can be expressed as a linear combination of the modes in any reference frame, be it inertial or not, and in particular can be used to study the vibration of a moving beam, like a robot arm or leg.

B.2.3 Effect of Shear Deformation

The rotational inertia of the cross section and the shear deformation were not taken into account in the preceding section. In this section this assumption will be dropped, with reference only to a prismatic homogeneous beam. A beam in which these effects are not neglected is usually referred to as a *Timoshenko beam*. Shear deformation can be accounted for as a deviation of the direction of the deflected shape of the beam not accompanied by a rotation of the cross section (Fig. B.5).

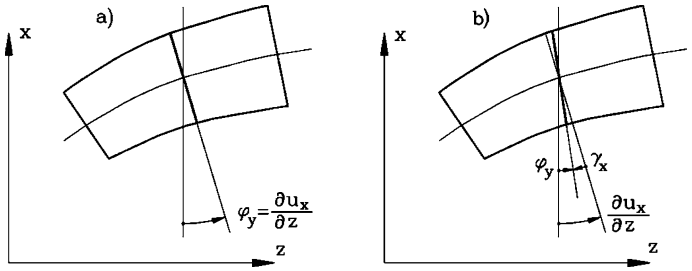

Fig. B.5 Effect of shear deformation on beam bending. (**a**) Euler–Bernoulli beam; (**b**) Timoshenko beam (from G. Genta, *Vibration Dynamics and Control*, Springer, New York, 2009)

The latter is thus no more perpendicular to the deformed shape of the beam, and the rotation of the cross section can be expressed as

$$\phi_y = \frac{\partial u_x}{\partial z} - \gamma_x. \tag{B.44}$$

The shear strain γ_x is linked to the shear force by the relationship

$$\gamma_x = \frac{\chi F_x}{GA}, \tag{B.45}$$

where the shear factor χ depends on the shape of the cross section, even if there is not complete accord on its value. For a circular beam, a value of $10/9$ is usually assumed; for other shapes the expressions reported in Table B.3 can be used.

Equation (B.44) can thus be written in the form

$$\phi_y = \frac{\partial u_x}{\partial z} - \frac{\chi F_x}{GA}. \tag{B.46}$$

The bending moment has no effect on shear deformation: If the latter is accounted for, the relationship linking the bending moment to the inflected shape of the beam becomes

$$M_y = E I_y \frac{\partial \phi_y}{\partial z}. \tag{B.47}$$

The rotational inertia of the cross section is no more neglected, and the dynamic equilibrium equations for displacement and rotation of the length dz of the beam are

$$\begin{cases} \rho A \dfrac{d^2 u_x}{dt^2} = \dfrac{\partial F_x}{\partial z} + f_x(z,t), \\ \rho I_y \dfrac{d^2 \phi_y}{dt^2} = F_x + \dfrac{\partial M_y}{\partial z}. \end{cases} \tag{B.48}$$

Table B.3 Shear factors for some different cross sections (from G.R. Cowper, The shear coefficient in Timoshenko's beam theory, *J. Appl. Mech.*, 1966, 335–340)

$\chi = \frac{7+6\nu}{6(1+\nu)}$	
$\chi = \frac{(7+6\nu)(1+m)^2+4m(5+3\nu)}{6(1+\nu)(1+m)^2}$ where $m = \left(\frac{d_i}{d_o}\right)^2$	
$\chi = \frac{12+11\nu}{10(1+\nu)}$	
$\chi = \frac{40+37\nu+m(16+10\nu)+\nu m^2}{12(1+\nu)(3+m)}$ where $m = \left(\frac{b}{a}\right)^2$	
$\chi = \frac{1.305+1.273\nu}{1+\nu}$	
$\chi = \frac{4+3\nu}{2(1+\nu)}$	
$\chi = \frac{48+39\nu}{20(1+\nu)}$	
$\chi = \frac{p+q\nu+30n^2m(1+m)+5\nu n^2m(8+9m)}{10(1+\nu)(1+3m)^2}$ where $m = \frac{bt_1}{at_a}$, $n = \frac{b}{h}$	
$\chi = \frac{p+q\nu+10n^2\left[m(3+\nu)+3m^2\right]}{10(1+\nu)(1+3m)^2}$ where $m = \frac{bt_1}{at_a}$, $n = \frac{b}{h}$	
$\chi = \frac{p+q\nu}{10(1+\nu)(1+3m)^2}$ where $m = \frac{2A}{ht}$, $A =$ flange area	
$\chi = \frac{p'+q'\nu+30n^2m(1+m)+10\nu n^2m\left(4+5m+m^2\right)}{10(1+\nu)(1+4m)^2}$ $m = \frac{bt_1}{ht_a}$, $n = \frac{b}{h}$	

$p = 12 + 72m + 150m^2 + 90m^3$; $q = 11 + 66m + 135m^2 + 90m^3$
$p' = 12 + 96m + 276m^2 + 192m^3$; $q' = 11 + 88m + 248m^2 + 216m^3$

By solving equation (B.46) in F_x and introducing it into the homogeneous equation (B.48), it follows that

$$\begin{cases} \rho A \dfrac{d^2u_x}{dt^2} = \dfrac{GA}{\chi}\left(\dfrac{\partial^2 u_x}{\partial z^2} - \dfrac{\partial \phi_y}{\partial z}\right), \\ \rho I_y \dfrac{d^2\phi_y}{dt^2} = \dfrac{GA}{\chi}\left(\dfrac{\partial u_x}{\partial z} - \phi_y\right) + EI_y \dfrac{\partial^2 \phi_y}{\partial z^2}. \end{cases} \tag{B.49}$$

By differentiating the second equation (B.49) with respect to z and eliminating ϕ_y, the following equation can be obtained:

$$EI_y \frac{\partial^4 u_x}{\partial z^4} - \rho I_y \left(1 + \frac{E\chi}{G}\right) \frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 u_x}{\partial t^2}\right) + \frac{\rho^2 I_y \chi}{G} \frac{d^4 u_x}{dt^4} + \rho A \frac{d^2 u_x}{dt^2} = 0 \tag{B.50}$$

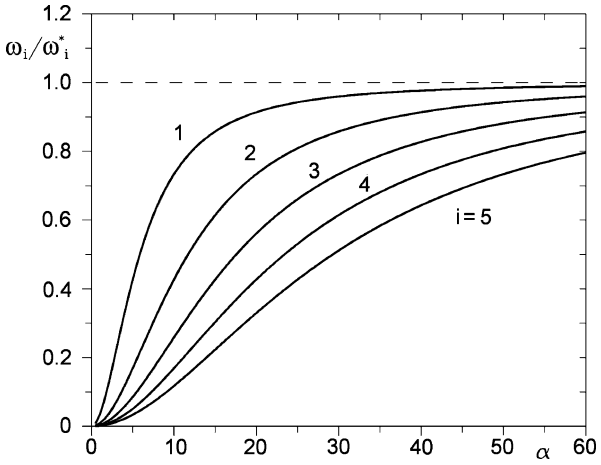

Fig. B.6 Effect of shear deformation on the first five natural frequencies of a simply supported beam. Ratio between the natural frequency computed taking into account rotational inertia and shear deformation and that computed using the Euler–Bernoulli model (from G. Genta, *Vibration Dynamics and Control*, Springer, New York, 2009)

and thus, in the case of free oscillations with harmonic time history,

$$EI_y \frac{d^4 q(z)}{dz^4} + \rho\omega^2 I_y \left(1 + \frac{E\chi}{G}\right) \frac{d^2 q(z)}{dz^2} - \rho\omega^2 \left(A - \omega^2 \frac{\rho I_y \chi}{G}\right) q(z) = 0. \tag{B.51}$$

The same considerations regarding the form of the eigenfunctions (seen in the preceding section) also hold in this case. If the beam is simply supported at both ends, the same eigenfunctions seen in the case of the Euler–Bernoulli beam still hold, and (B.51) can be expressed in nondimensional form as

$$\left(\frac{\omega}{\omega^*}\right)^4 - \left(\frac{\omega}{\omega^*}\right)^2 \frac{\alpha^2}{i^2\pi^2\chi^*} \left(1 + \chi^* + \frac{\alpha^2}{i^2\pi^2}\right) + \frac{\alpha^4}{i^4\pi^4\chi^*} = 0, \tag{B.52}$$

where ω_i^* is the ith natural frequency computed using the Euler–Bernoulli assumptions, the slenderness α of the beam is defined as

$$\alpha = l\sqrt{\frac{A}{I_y}} = \frac{l}{r} \tag{B.53}$$

(r is the radius of inertia of the cross section) and $\chi^* = \chi E/G$.

The results obtained from (B.52) for a beam with circular cross section and material with $\nu = 0.3$, are reported as functions of the slenderness in Fig. B.6.

Remark B.12 The effects of both shear deformation and rotational inertia tend to lower the value of the natural frequencies, the first being stronger than the second by a factor of about 3, as shown by Timoshenko.[1]

Remark B.13 It must be remembered that even the so-called Timoshenko beam model is an approximation, because it is based on the usual approximations of the

[1] S.P. Timoshenko et al., *Vibration Problems in Engineering*, Wiley, New York, 1974.

beam theory, and the very model of a one-dimensional solid is no more satisfactory when the slenderness is very low.

B.3 Discretization of Continuous Systems: The FEM

Many discretization techniques have been developed with the aim of substituting the equation of motion consisting of a partial derivative differential equation (with derivatives with respect to time and space coordinates) with a set of linear ordinary differential equations containing only derivatives with respect to time. The resulting set of equations, generally of the second order, is of the same type seen for discrete systems (hence, the term *discretization*).

The finite element method (FEM) is at present the most common discretization method, mostly because many computer codes based on it are available. In the FEM, the body is divided into a number of regions, called *finite elements*, as opposed to the vanishingly small regions used in writing the differential equations for continuous systems. The deformed shape of each finite element is assumed to be a linear combination of a set of functions of space coordinates through a certain number of parameters, considered the generalized coordinates of the element. Usually such functions of the space coordinates (called *shape functions*) are quite simple and the generalized coordinates have a direct physical meaning, namely generalized displacements at selected points of the element, usually referred to as *nodes*. The analysis then proceeds to writing a set of differential equations of the same type as those obtained for discrete systems.

The finite element method is a general discretization method for the solution of partial derivative differential equations and, consequently, it finds its application in many other fields beyond structural dynamics and structural analysis. The aim of this section is not to provide a complete survey of the method, which can be dealt with only in a specialized text, but simply to describe its main features and to show how it can be used to model the dynamic behavior of robot elements.

The component-mode synthesis method can be used with advantage to reduce the number of degrees of freedom of the model obtained through the FEM, particularly when the structure is made by components that can take different relative position, like robot arms or legs.

B.3.1 Element Characterization

Many different element formulations have been developed, depending on their shape and characteristics: beam elements, shell elements, plate elements, solid elements, and many others. A structure can be built by assembling elements of the same or different types, as dictated by the nature of the problem and by the capabilities of the computer code used.

Because the FEM is usually developed using matrix notation, in order to obtain formulas readily transferable to computer codes, the displacement is written as a

vector of order 3 in the tridimensional space (sometimes of higher order, if rotations are also considered), and the equation expressing the displacement of the points inside each element is

$$\mathbf{u}(x, y, z, t) = \mathbf{N}(x, y, z)\mathbf{q}(t), \tag{B.54}$$

where $\mathbf{q}$ is a vector in which the n generalized coordinates of the element are listed and $\mathbf{N}$ is the matrix containing the shape functions. There are as many rows in $\mathbf{N}$ as in $\mathbf{u}$ and as many columns as the number n of degrees of freedom.

As already stated, usually the degrees of freedom of the elements are the displacements at given points, which are referred to as *nodes*. In this case, (B.54) usually reduces to the simpler form,

$$\begin{Bmatrix} u_x(x, y, z, t) \\ u_y(x, y, z, t) \\ u_z(x, y, z, t) \end{Bmatrix} = \begin{bmatrix} \mathbf{N}(x, y, z) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}(x, y, z) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}(x, y, z) \end{bmatrix} \begin{Bmatrix} \mathbf{q}_x(t) \\ \mathbf{q}_y(t) \\ \mathbf{q}_z(t) \end{Bmatrix}, \tag{B.55}$$

where the displacements in each direction are functions of the nodal displacements in the same direction only. Matrix $\mathbf{N}$, in this case, has only one row and as many columns as the number of nodes of the element. Equation (B.55) has been written for a three-dimensional element; a similar formulation can also be easily obtained for one- or two-dimensional elements.

Each element is essentially the model of a small deformable solid. The behavior of the element is studied using an assumed-modes approach, i.e. assuming that the displacement is a linear combination of the already mentioned arbitrarily assumed shape functions. A limited number, usually quite small, of degrees of freedom is then substituted to the infinity of degrees of freedom of each element.

The freedom in the choice of the shape functions is, however, limited, because they must satisfy several conditions. A first requirement is a simple mathematical formulation, which is needed to lead to developments that are not too complex.

Usually a set of polynomials in the space coordinates is assumed. To get results that are closer to the exact solution of the differential equations, which constitute the continuous model discretized by the FEM, while reducing the size of the elements, the shape functions must

- be continuous and differentiable up to the required order, which depends on the type of element;
- be able to describe rigid-body motions of the element leading to vanishing elastic potential energy;
- lead to a constant strain field when the overall deformation of the element dictates so; and
- lead to a deflected shape of each element that matches the shape of the neighboring elements.

The last condition means that when the nodes of two neighboring elements displace in a compatible way, all the interface between the elements must displace in a compatible way.

Another condition, which is not always satisfied, is that the shape functions are isotropic, i.e., do not show particular geometrical properties that depend on the orientation of the reference frame. Sometimes not all these conditions are completely met; in particular, there are elements that fail to completely satisfy the matching of the deflected shapes of neighboring elements.

The nodes are usually located at the vertices or on the sides of the elements and are common to two or more of them, but points that are internal to an element are sometimes also used.

To write the equations of motion of the element the strains can be expressed as functions of the derivatives of the displacements with respect to space coordinates. In general, it is possible to write a relationship of the type

$$\boldsymbol{\epsilon}(x, y, z, t) = \mathbf{B}(x, y, z)\mathbf{q}(t), \tag{B.56}$$

where $\boldsymbol{\epsilon}$ is a column matrix in which the various elements of the strain tensor are listed (it is commonly referred to as a *strain vector* but it is such only in the sense that it is a column matrix) and $\mathbf{B}$ is a matrix containing appropriate derivatives of the shape functions. $\mathbf{B}$ has as many rows as the number of components of the strain vector and as many columns as the number of degrees of freedom of the element.

If the element is free from initial stresses and strains and the behavior of the material is linear, the stresses can be directly expressed from the strains

$$\boldsymbol{\sigma}(x, y, z, t) = \mathbf{E}\boldsymbol{\epsilon} = \mathbf{E}(x, y, z)\mathbf{B}(x, y, z)\mathbf{q}(t), \tag{B.57}$$

where $\mathbf{E}$ is the stiffness matrix of the material. It is a symmetric square matrix whose elements can theoretically be functions of the space coordinates but are usually constant within the element. The potential energy of the element can be easily expressed as

$$\mathcal{U} = \frac{1}{2}\int_V \boldsymbol{\epsilon}^T \boldsymbol{\sigma}\, dV = \frac{1}{2}\mathbf{q}^T \left(\int_V \mathbf{B}^T \mathbf{E}\mathbf{B}\, dV\right)\mathbf{q}. \tag{B.58}$$

The integral in (B.58) is the stiffness matrix of the element

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{E}\mathbf{B}\, dV. \tag{B.59}$$

Because the shape functions do not depend on time, the generalized velocities can be expressed as

$$\dot{\mathbf{u}}(x, y, z, t) = \mathbf{N}(x, y, z)\dot{\mathbf{q}}(t).$$

In the case where all generalized coordinates are related to displacements, the kinetic energy and the mass matrix of the element can be expressed as

$$\begin{aligned} \mathcal{T} &= \frac{1}{2}\int_V \dot{\mathbf{u}}^T \dot{\mathbf{u}}\rho\, dV = \frac{1}{2}\dot{\mathbf{q}}^T \left(\int_V \rho \mathbf{N}^T \mathbf{N}\, dV\right)\dot{\mathbf{q}}, \\ \mathbf{M} &= \int_V \rho \mathbf{N}^T \mathbf{N}\, dV. \end{aligned} \tag{B.60}$$

In the case that some generalized displacements are physically rotations, (B.60) must be changed in order to introduce moments of inertia, but its basic structure remains the same.

The FEM is often used just to compute the stiffness matrix to be used in the context of the lumped-parameters approach. In this case, the consistent mass matrix (B.60) is not computed and a diagonal matrix obtained by lumping the mass at the nodes is used. The advantage is that of dealing with a diagonal mass matrix, whose inversion to compute the dynamic matrix is far simpler than that of the consistent mass matrix. The accuracy is, however, reduced or, better, a greater number of elements is needed to reach the same accuracy, and thus the convenience of using a particular formulation must be assessed in each case.

Remark B.14 In general, the consistent approach leads to values of the natural frequencies that are in excess with respect to those computed using the elastic continuum model, while those obtained using the lumped-parameters approach are smaller.

If a force distribution $\mathbf{f}(x, y, z, t)$ acts on the body, the virtual work linked with the virtual displacement $\delta\mathbf{u} = \mathbf{N}\delta\mathbf{q}$ and the nodal force vector can be expressed in the form

$$\begin{aligned} \delta\mathcal{L} &= \int_V \delta\mathbf{q}^T \mathbf{N}^T \mathbf{f}(x, y, z, t)\, dV, \\ \mathbf{f}(t) &= \int_V \mathbf{N}^T \mathbf{f}(x, y, z, t)\, dV. \end{aligned} \tag{B.61}$$

In a similar way, it is possible to obtain the nodal force vectors corresponding to surface force distributions or to concentrated forces acting on any point of the element.

The equation of motion of the element is thus the usual one for discrete undamped systems

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t), \tag{B.62}$$

where vector $\mathbf{f}$ contains all forces acting on the element.

Remark B.15 The equations of motion and the relevant matrices have been obtained here by using Lagrange equations; this approach is neither the only one nor the most common.

B.3.2 Timoshenko Beam Element

The beam element is one of the most common elements and is generally available in all computer codes. Several beam formulations have been developed that differ owing to the theoretical formulation (some of them are Euler–Bernoulli element, i.e.,

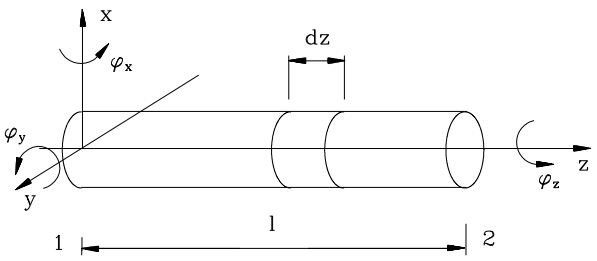

Fig. B.7 Beam element: geometrical definitions and reference frame (from G. Genta, *Vibration Dynamics and Control*, Springer, New York, 2009)

do not take into account shear deformation, while others are Timoshenko elements) and the number of nodes and degrees of freedom per node. The element that will be studied here is often referred to as the *simple Timoshenko beam*. It has two nodes at the ends of the beam and six degrees of freedom per node, and it consists of a prismatic homogeneous beam to which all considerations seen in Sect. B.2.1 apply. The relevant geometrical definition and the reference frame used for the study are shown in Fig. B.7.

Each cross section has six degrees of freedom, three displacements, and three rotations, and the total number of degrees of freedom of the element is 12. The vector of the nodal displacements, i.e., of the generalized coordinates of the element, is

$$\mathbf{q} = [u_{x_1}, u_{y_1}, u_{z_1}, \phi_{x_1}, \phi_{y_1}, \phi_{z_1}, u_{x_2}, u_{y_2}, u_{z_2}, \phi_{x_2}, \phi_{y_2}, \phi_{z_2}]^T . \tag{B.63}$$

The beam has the properties needed to perform a complete uncoupling between axial, torsional, and flexural behavior in each of the coordinate planes; it is thus expedient to subdivide vector $\mathbf{q}$ into four smaller vectors:

$$\begin{aligned} \mathbf{q}_A &= \begin{Bmatrix} u_{z_1} \\ u_{z_2} \end{Bmatrix}, \qquad \mathbf{q}_T = \begin{Bmatrix} \phi_{z_1} \\ \phi_{z_2} \end{Bmatrix}, \\ \mathbf{q}_{F1} &= \begin{Bmatrix} u_{x_1} \\ \phi_{y_1} \\ u_{x_2} \\ \phi_{y_2} \end{Bmatrix}, \qquad \mathbf{q}_{F2} = \begin{Bmatrix} u_{y_1} \\ \phi_{x_1} \\ u_{y_2} \\ \phi_{x_2} \end{Bmatrix}. \end{aligned} \tag{B.64}$$

Remark B.16 If the rotational degree of freedom $-\phi_x$ were used instead of ϕ_x the same equations could have been used to describe the flexural behavior in both planes. This approach is, however, uncommon because it would make it more difficult to pass from the system of reference of the elements to that of the whole structure.

By reordering the various generalized coordinates with the aim of clearly showing such uncoupling, vector $\mathbf{q}$ can be written as

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_A^T & \mathbf{q}_T^T & \mathbf{q}_{F1}^T & \mathbf{q}_{F2}^T \end{bmatrix}^T . \tag{B.65}$$

The uncoupling between the various degrees of freedom makes it possible to split the matrix of the shape functions into a number of submatrices, most of which

are equal to zero. The generalized displacement of an internal point of the element whose coordinate is z can be expressed in the form of (B.54) by the equation

$$\mathbf{u}(z,t) = \begin{Bmatrix} u_z \\ \phi_z \\ \left\{ \begin{matrix} u_x \\ \phi_y \end{matrix} \right\} \\ \left\{ \begin{matrix} u_y \\ \phi_x \end{matrix} \right\} \end{Bmatrix} = \begin{bmatrix} \mathbf{N}_A & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}_{F1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{N}_{F2} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_A \\ \mathbf{q}_T \\ \mathbf{q}_{F1} \\ \mathbf{q}_{F2} \end{Bmatrix}. \tag{B.66}$$

Axial Behavior

Because each point of the element has a single degree of freedom, vector $\mathbf{u}$ has a single component u_z and matrix $\mathbf{N}_A$ has one row and two columns (the element has two degrees of freedom). u_z can be expressed as a polynomial in z, or, better, in the nondimensional axial coordinate $\zeta = z/l$:

$$u_z = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + \cdots. \tag{B.67}$$

The polynomial must yield the values of the displacements u_{z_1} and u_{z_2}, respectively at the left end (node 1, $\zeta = 0$) and at the right end (node 2, $\zeta = 1$). These two conditions allow computation of only two coefficients a_i and then the polynomial expression of the displacement must include only two terms, i.e., the constant and the linear terms. With simple computations, the matrix of the shape functions is obtained:

$$\mathbf{N}_A = [1-\zeta, \zeta]. \tag{B.68}$$

The axial strain ϵ_z can be expressed as

$$\epsilon_z = \frac{du_z}{dz}, \tag{B.69}$$

or, using vector $\boldsymbol{\epsilon}$, which in this case has only one element

$$\epsilon_z = \left[\frac{d}{dz}(1-\zeta), \frac{d}{dz}\zeta \right] \begin{Bmatrix} u_{z_1} \\ u_{z_2} \end{Bmatrix}. \tag{B.70}$$

Matrix

$$\mathbf{B} = \left[\frac{d}{dz}(1-\zeta), \frac{d}{dz}\zeta \right] = \frac{1}{l}[-1, 1] \tag{B.71}$$

has one row and two columns.

Also, vector $\boldsymbol{\sigma}$ and matrix $\mathbf{E}$ have only a single element: the axial stress σ_z and Young's modulus E, respectively. The stiffness and mass matrices can be obtained directly from (B.59) and (B.60). Remembering that $dV = A\,dz$, they reduce to

$$\mathbf{K}_A = \int_0^l A\mathbf{B}^T E\mathbf{B}\,dz = \frac{EA}{l}\int_0^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\zeta = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{B.72}$$

$$\mathbf{M}_A = \int_0^l \rho A \mathbf{N}^T \mathbf{N}\, dz = \rho A l \int_0^1 \begin{bmatrix} (1-\zeta)^2 & \zeta(1-\zeta) \\ \zeta(1-\zeta) & \zeta^2 \end{bmatrix} d\zeta$$
$$= \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{B.73}$$

If an axial force distribution $f_z(t)$ that is constant along the space coordinate z or a concentrated axial force $F_{z_k}(t)$ located in the point of coordinate z_k is acting on the bar, the nodal force vector is, respectively,

$$\mathbf{f}(t) = l \left[\int_0^l \begin{Bmatrix} (1-\zeta) \\ \zeta \end{Bmatrix} d\zeta \right] f_z(t) = f_z(t) \frac{l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \tag{B.74}$$

or

$$\mathbf{f}(t) = F_{z_k}(t) \begin{Bmatrix} 1 - \dfrac{z_k}{l} \\ \dfrac{z_k}{l} \end{Bmatrix}. \tag{B.75}$$

In this case the distributed load has been reduced to two identical forces, each equal to half of the total load acting on the bar.

An identical result would have been obtained by simply lumping the load at the nodes. This is not, however, a general rule; in other cases the consistent approach leads to a load vector that is different from that obtained using the lumped-parameters approach.

Torsional Behavior

The equations of motion governing the torsional behavior of beams are formally identical to those governing the axial behavior. Using this identity, the characterization of the beam element in torsion can be obtained from what has been seen for the axial behavior. Matrix $\mathbf{N}_T$ is identical to matrix $\mathbf{N}_A$

$$\mathbf{N}_T = [1-\zeta, \zeta]$$

and the expressions of the relevant matrices and vectors are

$$\mathbf{M}_T = \frac{\rho I_p l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{K}_T = \frac{G I_p'}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$
$$\mathbf{f}(t)_T = \frac{1}{2} l m_z(t) \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \tag{B.76}$$

Flexural Behavior in the xz-Plane

The expressions of the shape function are in this case more complex; matrix $\mathbf{N}_{F1}$ has two rows and four columns because it must yield the displacement in the x-direction

and the rotation about the y-axis of the generic cross section of the beam when it is multiplied by vector $\mathbf{q}_{F1}$, which contains four elements, namely, the displacements in the x-direction and the rotation about the y-axis of the two nodes. The simplest approach would be that of assuming polynomial expressions for the generalized displacements

$$\begin{cases} u_x = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + \cdots, \\ \phi_y = b_0 + b_1\zeta + b_2\zeta^2 + b_3\zeta^3 + \cdots. \end{cases} \tag{B.77}$$

These polynomial expressions must yield the values of the displacements u_{x1} and u_{x2} and of the rotations ϕ_{y1} and ϕ_{y2} at the left end (node 1, $\zeta = 0$) and at the right end (node 2, $\zeta = 1$), respectively. These four conditions allow the computation of only four coefficients a_i and b_i to be introduced into the polynomial expressions. Each polynomial must then include only two terms, and both rotations and displacements must vary linearly along the z-coordinate. This element formulation, although sometimes used, leads to the severe problem of locking, i.e., to the possibility of grossly overestimating the stiffness of the element.

Although locking will not be dealt with in detail here (the reader can find a detailed discussion on this matter in any good textbook on the FEM), an intuitive explanation can be seen immediately: if the beam is slender the rotation of each cross section is very close to the derivative of the displacement, as stated by the Euler–Bernoulli approach for slender beams. The polynomial shape functions, when truncated at the second term, do not allow the rotation to be equal to the derivative of the displacement, and this can be shown to lead to a severe underestimate of the displacements, i.e., of the flexibility of the beam.

A simple cure for the problem is that of resorting to the Euler–Bernoulli formulation, i.e., neglecting shear deformation and then assuming that the rotation is coincident with the derivative of the displacement. In this case, only the polynomial for u_x needs to be stated, and the aforementioned four conditions at the nodes can be used to compute the four coefficients of a cubic expression of the displacement.

To avoid locking, a Timoshenko beam element can be formulated using as shape functions the deformed shape computed using the continuous model assuming that only end forces are applied to the beam. This Timoshenko beam element reduces to the Euler–Bernoulli element as the slenderness of the beam increases and no locking occurs. The relevant shape functions are

$$\begin{aligned}
N_{11} &= \frac{1 + \Phi(1-\zeta) - 3\zeta^2 + 2\zeta^3}{1+\Phi}, & N_{12} &= l\zeta\,\frac{1 + \frac{1}{2}\Phi(1-\zeta) - 2\zeta + \zeta^2}{1+\Phi}, \\
N_{13} &= \zeta\,\frac{\Phi + 3\zeta - 2\zeta^2}{1+\Phi}, & N_{14} &= l\zeta\,\frac{-\frac{1}{2}\Phi(1-\zeta) - \zeta + \zeta^2}{1+\Phi}, \\
N_{21} &= 6\zeta\,\frac{\zeta - 1}{l(1+\Phi)}, & N_{22} &= \frac{1 + \Phi(1-\zeta) - 4\zeta + 3\zeta^2}{1+\Phi}, \\
N_{23} &= -6\zeta\,\frac{\zeta - 1}{l(1+\Phi)}, & N_{24} &= \frac{\Phi\zeta - 2\zeta + 3\zeta^2}{1+\Phi},
\end{aligned} \tag{B.78}$$

where

$$\Phi = \frac{12EI_y\chi}{GAl^2}.$$

When the slenderness of the beam increases, the value of Φ decreases, tending to zero for a Euler–Bernoulli beam.

In this case, some of the generalized coordinates are related to rotations; as a consequence, (B.59) and (B.60) cannot be used directly to express the stiffness and mass matrices. The potential energy can be computed by adding the contributions due to bending and shear deformations. By using the symbols $\mathbf{N}_1$ and $\mathbf{N}_2$ to express the first and second rows of matrix $\mathbf{N}_{F1}$, respectively, the two contributions to the potential energy of the length dz of the beam are

$$\begin{aligned} d\mathcal{U}_b &= \frac{1}{2}EI_y\left(\frac{d\phi_y}{dz}\right)^2 dz = \frac{1}{2}EI_y\{q\}^T\left[\frac{d}{dz}\mathbf{N}_2\right]^T\left[\frac{d}{dz}\mathbf{N}_2\right]\{q\}\,dz, \\ d\mathcal{U}_s &= \frac{1}{2}\frac{GA}{\chi}\left(\phi_y - \frac{du_x}{dz}\right)^2 dz \\ &= \frac{12EI_y}{2\Phi l^2}\{q\}^T\left[\mathbf{N}_2 - \frac{d}{dz}\mathbf{N}_1\right]^T\left[\mathbf{N}_2 - \frac{d}{dz}\mathbf{N}_1\right]\{q\}\,dz. \end{aligned} \tag{B.79}$$

By introducing the expressions of the shape functions into the expression of the potential energy and integrating, the bending stiffness matrix is obtained

$$\mathbf{K}_{F_1} = \frac{EI_y}{l^3(1+\Phi)}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & (4+\Phi)l^2 & -6l & (2-\Phi)l^2 \\ & & 12 & -6l \\ & \text{symm} & & (4+\Phi)l^2 \end{bmatrix}. \tag{B.80}$$

The kinetic energy of the length dz of the beam is

$$\begin{aligned} d\mathcal{T} &= \frac{1}{2}\rho A\dot{u}^2\,dz + \frac{1}{2}\rho I_y\dot{\phi}_y^{\,2}\,dz \\ &= \frac{1}{2}\rho A\dot{\mathbf{q}}^T\mathbf{N}_1^T\mathbf{N}_1\dot{\mathbf{q}}\,dz + \frac{1}{2}\rho I_y\dot{\mathbf{q}}^T\mathbf{N}_2^T\mathbf{N}_2\dot{\mathbf{q}}\,dz. \end{aligned} \tag{B.81}$$

The first term on the right-hand side is the translational kinetic energy, and the second expresses the rotational kinetic energy, which is often neglected in the case of slender beams. By introducing the expressions of the shape functions and integrating, the consistent mass matrix, which is made of two parts—one taking into account the translational inertia and the other the rotational inertia of the cross sections—is obtained

$$\mathbf{M}_{F1} = \frac{\rho Al}{420(1+\Phi)^2}\begin{bmatrix} m_1 & lm_2 & m_3 & -lm_4 \\ & l^2m_5 & lm_4 & -l^2m_6 \\ & & m_1 & -lm_2 \\ & \text{symm} & & l^2m_5 \end{bmatrix}$$

$$+\frac{\rho I_y}{30l(1+\Phi)^2}\begin{bmatrix} m_7 & lm_8 & -m_7 & lm_8 \\ & l^2m_9 & -lm_8 & -l^2m_{10} \\ & & m_7 & -lm_8 \\ & \text{symm} & & l^2m_9 \end{bmatrix}, \tag{B.82}$$

where

$$\begin{aligned} m_1 &= 156+294\Phi+140\Phi^2, & m_2 &= 22+38.5\Phi+17.5\Phi^2, \\ m_3 &= 54+126\Phi+70\Phi^2, & m_4 &= 13+31.5\Phi+17.5\Phi^2, \\ m_5 &= 4+7\Phi+3.5\Phi^2, & m_6 &= 3+7\Phi+3.5\Phi^2, \\ m_7 &= 36, & m_8 &= 3-15\Phi, \\ m_9 &= 4+5\Phi+10\Phi^2, & m_{10} &= 1+5\Phi-5\Phi^2. \end{aligned}$$

The consistent load vector due to a uniform distribution of shear force per unit length $f_x(t)$ or of bending moment $m_y(t)$ can be obtained directly from (B.61)

$$\mathbf{f}(t)_{F1} = l\left[\int_0^1 \mathbf{N}_{F1}^T\, d\zeta\right]\begin{Bmatrix} f_x(t) \\ m_y(t) \end{Bmatrix} = \frac{lf_x(t)}{12}\begin{Bmatrix} 6 \\ l \\ 6 \\ -l \end{Bmatrix} + \frac{m_y(t)}{1+\Phi}\begin{Bmatrix} -l \\ \dfrac{\Phi l}{2} \\ l \\ \dfrac{\Phi l}{2} \end{Bmatrix}. \tag{B.83}$$

Flexural Behavior in the yz-Plane

The flexural behavior in the yz-plane must be studied using equations different from those used for the xz-plane, owing to the different signs of rotation in the two planes. Matrix $\mathbf{N}_{F2}$ can, however, be obtained from matrix $\mathbf{N}_{F1}$ simply by changing the signs of elements with subscripts 12, 14, 21, and 23, and the mass and stiffness matrices related to plane yz are the same as those computed for plane xz, except for the signs of elements with subscripts 12, 14, 23, and 34 and their symmetrical ones. In the force vectors related to external forces (distributed or concentrated) or external moments, the signs of elements 2 and 4 or 1 and 3, respectively, must be changed. If the beam is not axially symmetrical and the elastic and inertial properties in the two planes are not coincident, different values of the moments of inertia and the shear factors must be introduced.

Global Behavior of the Beam

The complete expression of the mass and stiffness matrices and of the nodal force vector are, respectively,

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_A & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{F1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{F2} \end{bmatrix},$$
$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_A & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{F1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{F2} \end{bmatrix}, \qquad \mathbf{f} = \begin{Bmatrix} \mathbf{f}_A \\ \mathbf{f}_T \\ \mathbf{f}_{F1} \\ \mathbf{f}_{F2} \end{Bmatrix}. \tag{B.84}$$

B.3.3 Mass and Spring Elements

Consider a concentrated mass, or better, a rigid body located at the ith node. Let $\mathbf{q}$ be the vector of the generalized displacements of the relevant node, which may also contain rotations

$$\mathbf{q} = [u_x, u_y, u_z, \phi_x, \phi_y, \phi_z],$$

if the node is of the type of those seen in the case of beam elements.

In the latter case, if the mass also has moments of inertia, let the axes of the reference frame coincide with the principal axes of the rigid body. By writing the kinetic energy of the element, it can be shown that the mass matrix, which is diagonal because the axes of the reference frame coincide with the principal axes of the body, is

$$\mathbf{M} = \text{diag}[m, m, m, J_x, J_y, J_z]. \tag{B.85}$$

A simpler expression is obtained when only the translational degrees of freedom of the node are considered.

Remark B.17 In many computer programs, different values of the mass can be associated with the various degrees of freedom. This can account for particular physical layouts, such as the addition of a mass constrained to move with the structure in one direction and not in others.

Consider a spring element, i.e., an element that introduces a concentrated stiffness between two nodes, say node 1 and node 2, of the structure. When the nodes have a single degree of freedom, the generalized coordinates of the element are $q = [u_1, u_2]^T$, and the stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}. \tag{B.86}$$

If the nodes, like those used with beam elements, have three translational and three rotational degrees of freedom, three values of translational stiffness k_x, k_y, and k_z and three values of rotational stiffness χ_x, χ_y, and χ_z can be stated and matrices of the type shown in (B.86) can be written for each degree of freedom.

B.3.4 Assembling the Structure

The equations of motion of the element are written with reference to a local or element reference frame that has an orientation determined by the features of the element. In a beam element, for example, the z-axis can coincide with the axis of the beam, while the x- and y-axes are principal axes of inertia of the cross section. The various local reference frames of the elements have, in general, a different orientation in space. To describe the behavior of the structure as a whole, another reference frame, namely the global or structure reference frame, is defined. The orientation in space of any local frame can be expressed, with reference to the orientation of the global frame, by a suitable rotation matrix

$$\mathbf{R} = \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix}, \tag{B.87}$$

where l_i, m_i, and n_i are the direction cosines of the axes of the local frame in the global frame. The expressions $\mathbf{q}_{il}$ and $\mathbf{q}_{ig}$ of the displacement vector $\mathbf{q}_i$ of the ith node in the local and global reference frames are linked by the usual coordinate transformation

$$\mathbf{q}_{il} = \mathbf{R}\mathbf{q}_{ig}. \tag{B.88}$$

The generalized coordinates in the displacement vector of the element can be transformed from the local to the global reference frame using a similar relationship in which an expanded rotation matrix $\mathbf{R}'$ is used to deal with all the relevant generalized coordinates. It is essentially made by a number of matrices of the type of (B.87) suitably assembled together.

Remark B.18 The assumption of small displacements and rotations allows consideration of the rotations about the axes as the components of a vector, which can be rotated in the same way as displacements.

The force vectors can also be rotated using the rotation matrix $\mathbf{R}'$, and the equation of motion of the element can be written with reference to the global frame and premultiplied by the inverse of matrix $\mathbf{R}'$, obtaining

$$\mathbf{R}'^{-1}\mathbf{M}\mathbf{R}'\ddot{\mathbf{q}}_g + \mathbf{R}'^{-1}\mathbf{K}\mathbf{R}'\mathbf{q}_g = \mathbf{f}_g. \tag{B.89}$$

Because the inverse of a rotation matrix is coincident with its transpose, the expressions of the mass and stiffness matrices of the element rotated from the local to the global frame are

$$\mathbf{M}_g = \mathbf{R}'^T\mathbf{M}_l\mathbf{R}' \tag{B.90}$$

and

$$\mathbf{K}_g = \mathbf{R}'^T\mathbf{K}_l\mathbf{R}'. \tag{B.91}$$

Similarly, the nodal load vector can be rotated using the obvious relationship

$$\mathbf{f}_g = \mathbf{R}'^T \mathbf{f}_l. \tag{B.92}$$

Once the mass and stiffness matrices, referring to the global frame, of the various elements have been computed, it is possible to easily obtain the matrices of the whole structure. The n generalized coordinates of the structure can be ordered in a single vector $\mathbf{q}_g$. The matrices of the various elements can be rewritten in the form of matrices of order $n \times n$, containing all elements equal to zero except those that are in the rows and columns corresponding to the generalized coordinates of the relevant element.

Because the kinetic and potential energies of the structure can be obtained simply by adding the energies of the various elements, it follows that

$$\begin{aligned} \mathcal{T} &= \frac{1}{2} \sum_{\forall i} \dot{\mathbf{q}}_g^T \mathbf{M}_i \dot{\mathbf{q}}_g = \frac{1}{2} \dot{\mathbf{q}}_g^T \mathbf{M} \dot{\mathbf{q}}_g; \\ \mathcal{U} &= \frac{1}{2} \sum_{\forall i} \mathbf{q}_g^T \mathbf{K}_i \mathbf{q}_g = \frac{1}{2} \mathbf{q}_g^T \mathbf{K} \mathbf{q}_g. \end{aligned} \tag{B.93}$$

Matrices $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices of the whole structure and are obtained simply by adding all the mass and stiffness matrices of the elements. In practice, the various matrices of size $n \times n$ for the elements are never written: each term of the matrices of all elements is just added into the global mass and stiffness matrices in the correct place. Actually, the matrices of the structure are very easily assembled, and this is one of the easiest steps of the whole computation. If the generalized coordinates are taken into a suitable order, the assembled matrices have a band structure; many general-purpose computer codes have a suitable routine that reorders the coordinates in such a way that the bandwidth is the smallest possible.

In a similar way the nodal force vector can be easily assembled:

$$\mathbf{f} = \sum_{\forall i} \mathbf{f}_i. \tag{B.94}$$

Remark B.19 The forces that are exchanged between the elements at the nodes cancel each other in this assembling procedure, and the force vectors that must be inserted into the global equation of motion of the structure are only those related to the external forces applied to the structure.

B.3.5 Constraining the Structure

One of the advantages of the FEM is the ease with which the constraints can be defined. If the ith degree of freedom is rigidly constrained, the corresponding generalized displacement vanishes and, as a consequence, the ith column of the stiffness

and mass matrices can be neglected, because they multiply a displacement and an acceleration, respectively, that are equal to zero. Because one of the generalized displacements is known, one of the equations of motion can be neglected when solving for the deformed configuration of the system. The ith equation can thus be separated from the rest of the set of equations, which amounts to canceling the ith row of all matrices and of the force vector.

Remark B.20 The ith equation could be used, after all displacements have been computed, to obtain the value of the ith generalized nodal force that, in this case, is the unknown reaction of the constraint.

To rigidly constrain a degree of freedom it is thus sufficient to cancel the corresponding row and column in all matrices and vectors. This approach allows simplification of the formulation of the problem, which may be useful in dynamic problems, but this simplification is often marginal, since the number of constrained degrees of freedom is small, compared with the total number of degrees of freedom. To avoid restructuring the whole model and rewriting all the matrices, rigid constraints can be transformed into very stiff elastic constraints.

If the ith degree of freedom is constrained through a linear spring with stiffness k_i, the potential energy of the structure is increased by the potential energy of the spring

$$\mathcal{U} = \frac{1}{2} k_i q_i^2. \tag{B.95}$$

To take the presence of the constraint into account, it is sufficient to add the stiffness k_i to the element in the ith row and ith column of the global stiffness matrix. This procedure is simple, which explains why a very stiff elastic constraint is often added instead of canceling a degree of freedom in the case of rigid constraints. An additional advantage is that the reaction of the constraint can be obtained simply by multiplying the high generalized stiffness k_i by the corresponding small generalized displacement q_i.

B.3.6 Damping Matrices

It is possible to take into account the damping of the structure in a way that closely follows what has been said for the stiffness. If elements that can be modeled as viscous dampers are introduced into the structure between two nodes or between a node and the ground, a viscous damping matrix can be obtained using the same procedures seen for the stiffness matrix of spring elements or elastic constraints. Actually, the relevant equations are equal, once the damping coefficient is substituted for the stiffness and velocities are substituted for displacements. If the damping of some of the elements can be modeled as hysteretic damping, within the limits of validity of the complex stiffness model, an imaginary part of the element stiffness matrix can be obtained by simply multiplying the real part by the loss factor.

Viscous or structural damping matrices are then assembled following the same rules seen for mass and stiffness matrices.

Remark B.21 The real and imaginary parts of the stiffness matrices must be assembled separately, because, when the loss factor is not constant along the structure, they are not proportional to each other.

B.4 Reduction of the Number of Degrees of Freedom

It is not uncommon that the models obtained through the FEM have thousands or even millions degrees of freedom. This does not constitute a problem for modern computers when studying static problems, but the solution of an eigenproblem of that size can still be a formidable problem. Moreover, the FEM is a displacement method, i.e. first solves the displacements and then computes stresses and strains as derivatives of the displacements, and thus the precision with which displacements, and all other entities directly linked with displacements including mode shapes and natural frequencies, are obtained is far greater, for a given mesh, than that achievable for stresses and strains. Conversely, this means that the mesh needs to be much finer when solving the stress field, which is typical of static problems, than when searching natural frequencies and mode shapes.

As a consequence, there is a definite advantage in using a smaller model, in terms of the number of degrees of freedom, when performing a dynamic analysis than when doing static computations.

Remark B.22 Because it is often expedient to use the same mesh for both static and dynamic computations, or a finer mesh is required to model a complex shape, a reduction of the number of degrees of freedom for dynamic solution is useful, particularly when only a limited number of natural frequencies are required.

Two approaches can be used: reducing the size of the model or leaving the model as it is and using algorithms, such as the subspace iteration method, that search only the lowest natural frequencies. Although the two are more or less equivalent, the first leaves the choice of which degrees of freedom to retain to the user, and the second operates automatically. As a consequence, a skilled operator can use advantageously reduction techniques, which allow very good results with very few degrees of freedom. General-purpose codes for routine computations usually resort to the second approach.

Remark B.23 Before computers were available, remarkable results were obtained using models with very few (often a single) degrees of freedom, but this required great computational ability and physical insight.

B.4.1 Static Reduction

Static reduction is based on the subdivision of the generalized coordinates $\mathbf{q}$ of the model into two types: master degrees of freedom $\mathbf{q}_1$ and slave degrees of freedom $\mathbf{q}_2$. The stiffness matrix and the nodal force vector can be partitioned accordingly, and the equation expressing the static problem becomes

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{Bmatrix}. \tag{B.96}$$

Remark B.24 Matrices $\mathbf{K}_{11}$ and $\mathbf{K}_{22}$ are symmetrical, while $\mathbf{K}_{12} = \mathbf{K}_{21}^T$ are neither symmetrical nor square.

Solving the second set of (B.96) in $\mathbf{q}_2$, the following relationship linking the slave to the master coordinates is obtained:

$$\mathbf{q}_2 = -\mathbf{K}_{22}^{-1}\mathbf{K}_{21}\mathbf{q}_1 + \mathbf{K}_{22}^{-1}\mathbf{f}_2. \tag{B.97}$$

Introducing (B.97) into (B.96), the latter yields

$$\mathbf{K}_{\text{cond}}\mathbf{q}_1 = \mathbf{f}_{\text{cond}}, \tag{B.98}$$

where

$$\begin{cases} \mathbf{K}_{\text{cond}} = \mathbf{K}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{12}^T, \\ \mathbf{f}_{\text{cond}} = \mathbf{f}_1 - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{f}_2. \end{cases}$$

Equation (B.98) yields the master generalized displacements $\mathbf{q}_1$. The slave displacements can be obtained directly from (B.97) simply by multiplying some matrices.

Remark B.25 When used to solve a static problem static reduction yields exact results, i.e., the same results that would be obtained from the complete model.

The subdivision of the degrees of freedom between vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ can be based on different criteria. The master degrees of freedom can simply be those in which the user is directly interested. Another type of choice can be that of physically subdividing the structure in two parts.

The second practice, which can be generalized by subdividing the generalized coordinates into many subsets, is generally known as *solution by substructures* or *substructuring*. In particular, it may be expedient when the structure can be subdivided into many parts that are all connected to a single frame. If the generalized displacements of the connecting structure or frame are listed in vector $\mathbf{q}_0$ and those of the various substructures are included in vectors $\mathbf{q}_i$, the equation for the static

solution of the complete structure has the form

$$\begin{bmatrix} \mathbf{K}_{00} & \mathbf{K}_{01} & \mathbf{K}_{02} & \dots \\ & \mathbf{K}_{11} & \mathbf{0} & \dots \\ & & \mathbf{K}_{22} & \dots \\ & \text{symm} & & \dots \end{bmatrix} \begin{Bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \dots \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{f}_2 \\ \dots \end{Bmatrix}. \tag{B.99}$$

The equations related to the ith substructure can be solved as

$$\mathbf{q}_i = -\mathbf{K}_{ii}^{-1}\mathbf{K}_{i0}\mathbf{q}_0 + \mathbf{K}_{ii}^{-1}\mathbf{f}_i. \tag{B.100}$$

The generalized displacements of the frame can be obtained using an equation of the type of (B.98) where the condensed matrices are

$$\begin{cases} \mathbf{K}_{\text{cond}} = \mathbf{K}_{00} - \displaystyle\sum_{\forall i} \mathbf{K}_{0i}\mathbf{K}_{ii}^{-1}\mathbf{K}_{0i}^{T}, \\ \mathbf{f}_{\text{cond}} = \mathbf{f}_0 - \displaystyle\sum_{\forall i} \mathbf{K}_{0i}\mathbf{K}_{ii}^{-1}\mathbf{f}_i. \end{cases} \tag{B.101}$$

As already stated, static reduction does not introduce any further approximation into the model. A similar reduction can be used in dynamic analysis without introducing approximations only if no generalized inertia is associated with the slave degrees of freedom. In this case, static reduction is advisable because the mass matrix of the original system is singular and the condensation procedure allows removal of the singularity. When using the lumped-parameters approach with beam elements and the moments of inertia of the cross sections are neglected, no inertia is associated with half of the degrees of freedom related to bending. Static reduction allows removing all of them and then obtaining a nonsingular mass matrix. Generally speaking, however, the mass matrix is not singular and it is not possible to just neglect the inertia linked with some degrees of freedom.

B.4.2 Guyan Reduction

The so-called *Guyan reduction* is based on the assumption that the slave generalized displacements $\mathbf{q}_2$ can be computed directly from master displacements $\mathbf{q}_1$, neglecting inertia forces and external forces $\mathbf{f}_2$. In this case, (B.97), without the last term, can also be used for dynamic solution. By partitioning the mass matrix in the same way seen for the stiffness matrix, the kinetic energy of the structure can be expressed as

$$\mathcal{T} = \frac{1}{2} \begin{Bmatrix} \dot{\mathbf{q}}_1 \\ -\mathbf{K}_{22}^{-1}\mathbf{K}_{21}\dot{\mathbf{q}}_1 \end{Bmatrix}^T \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_1 \\ -\mathbf{K}_{22}^{-1}\mathbf{K}_{21}\dot{\mathbf{q}}_1 \end{Bmatrix}. \tag{B.102}$$

The kinetic energy is thus

$$\mathcal{T} = \frac{1}{2}\dot{\mathbf{q}}_1^T \mathbf{M}_{\text{cond}}\dot{\mathbf{q}}_1,$$

where the condensed mass matrix is

$$\mathbf{M}_{\text{cond}} = \mathbf{M}_{11} - \mathbf{M}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{12}^{T} - \left[\mathbf{M}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{12}^{T}\right]^{T} + \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{M}_{22}\mathbf{K}_{22}^{-1}\mathbf{K}_{12}^{T}. \tag{B.103}$$

Guyan reduction is not much more demanding from a computational viewpoint than static reduction because the only matrix inversion is that of $\mathbf{K}_{22}$, which has already been performed for the computation of the condensed stiffness matrix. If matrix $\mathbf{M}$ is diagonal, two of the terms of (B.103) vanish. Although approximate, Guyan reduction introduces errors that are usually small, at least if the choice of the slave degrees of freedom is appropriate. Inertia forces related to slave degrees of freedom are actually not neglected, but their contribution to the kinetic energy is computed from a deformed configuration obtained on the basis of the master degrees of freedom alone.

Remark B.26 If the relevant mode shapes are only slightly influenced by the presence of some of the generalized masses or if some parts of the structure are so stiff that their deflected shape can be determined by a few coordinates, the results can be good, even when few master degrees of freedom are used.

In a way similar to that seen for the mass matrix, viscous or structural damping matrices $\mathbf{C}$ and $\mathbf{K}''$ can be reduced using (B.103) in which $\mathbf{M}$ has been substituted with $\mathbf{C}$ and $\mathbf{K}''$, respectively. Also, the reduction of damping matrices introduces errors that depend on the choice of the slave degrees of freedom but are usually small when the degrees of freedom in which viscous dampers are applied or, in the case of hysteretic damping, where the loss factor of the material changes, are not eliminated. Alternatively, these degrees of freedom can be neglected when their displacement is well determined by some master displacement, as in the case of very stiff parts of the structure.

B.4.3 Component-Mode Synthesis

When substructuring is used, the degrees of freedom of each structure can be subdivided into two sets: internal degrees of freedom and boundary degrees of freedom. The latter are all degrees of freedom that the substructure has in common with other parts of the structure. They are often referred to as constraint degrees of freedom because they express how the substructure is constrained to the rest of the system. Internal degrees of freedom are those belonging only to the relevant substructure. The largest possible reduction scheme is that in which all internal degrees of freedom are considered slave coordinates and all boundary degrees of freedom are considered master coordinates. In this way, however, the approximation of all modes in which the motion of the internal points of the substructure with respect to its boundary is important, can be quite rough.

A simple way to avoid this drawback is to also consider as master coordinates, together with the boundary degrees of freedom, some of the modal coordinates of the substructure constrained at its boundary. This procedure would obviously lead to exact results if all modes were retained, but because the total number of modes is equal to the number of internal degrees of freedom, the model obtained has as many degrees of freedom as the original one. As usual with modal practices, the computational advantages grow together with the number of modes that can be neglected.

The relevant matrices are partitioned as seen for reduction techniques, with subscript 1 referring to the boundary degrees of freedom and subscript 2 to the internal degrees of freedom. The displacement vector $\mathbf{q}_2$ can be assumed to be equal to the sum of the constrained modes $\mathbf{q}_2'$, i.e., the deformation pattern due to the displacements $\mathbf{q}_1$ when no force acts on the substructure, plus the constrained normal modes $\mathbf{q}_2''$, i.e., the natural modes of free vibration of the substructure when the boundary generalized displacements $\mathbf{q}_1$ are equal to zero.

The constrained modes $\mathbf{q}_2'$ can be expressed by (B.97) once the force vector $\mathbf{f}_2$ is set equal to zero. The constrained normal modes can easily be computed by solving the eigenproblem

$$\left(-\omega^2\mathbf{M}_{22}+\mathbf{K}_{22}\right)\mathbf{q}_2''=0.$$

Once the eigenproblem has been solved, the matrix of the eigenvectors $\boldsymbol{\Phi}$ can be used to perform the modal transformation $\mathbf{q}_2''=\boldsymbol{\Phi}\boldsymbol{\eta}_2$. The generalized coordinates of the substructure can thus be expressed as

$$\begin{aligned}\begin{Bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\end{Bmatrix}&=\begin{Bmatrix}\mathbf{q}_1\\ -\mathbf{K}_{22}^{-1}\mathbf{K}_{21}\mathbf{q}_1+\boldsymbol{\Phi}\boldsymbol{\eta}_2\end{Bmatrix}\\ &=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{K}_{22}^{-1}\mathbf{K}_{21} & \boldsymbol{\Phi}\end{bmatrix}\begin{Bmatrix}\mathbf{q}_1\\ \boldsymbol{\eta}_2\end{Bmatrix}=\boldsymbol{\Psi}\begin{Bmatrix}\mathbf{q}_1\\ \boldsymbol{\eta}_2\end{Bmatrix}.\end{aligned}\tag{B.104}$$

Equation (B.104) represents a coordinate transformation, allowing to express the deformation of the internal part of the substructure in terms of constrained and normal modes. The matrix $\boldsymbol{\Psi}$ expressing this transformation can be used to compute the new mass, stiffness, and, where needed, damping matrices and the force vector

$$\begin{aligned}\mathbf{M}^*&=\boldsymbol{\Psi}^T\mathbf{M}\boldsymbol{\Psi}, & \mathbf{K}^*&=\boldsymbol{\Psi}^T\mathbf{K}\boldsymbol{\Psi},\\ \mathbf{C}^*&=\boldsymbol{\Psi}^T\mathbf{C}\boldsymbol{\Psi}, & \mathbf{f}^*&=\boldsymbol{\Psi}^T\mathbf{f}.\end{aligned}\tag{B.105}$$

If there are m constrained coordinates and n internal coordinates and if only k constrained normal modes are considered ($k<n$), then the size of the original matrices $\mathbf{M}$, $\mathbf{K}$, ... is $m+n$, while that of matrices $\mathbf{M}^*$, $\mathbf{K}^*$, ... is $m+k$.

Once the transformation of coordinates of the substructures has been performed, they can be assembled in the same way already seen for elements: The boundary coordinates are common between the substructures and are actually assembled while the modal coordinates are typical of only one substructure at a time, in the same way as the coordinates of internal nodes of elements that have nodes of this type.

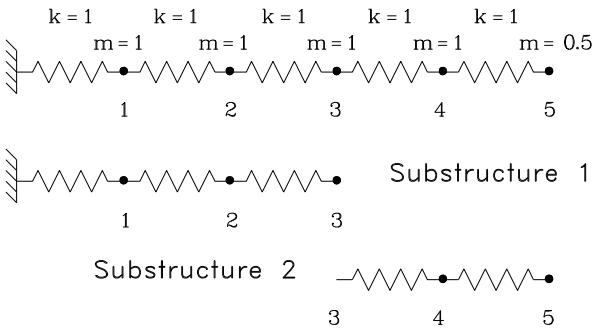

Fig. B.8 Sketch of the system and values of the relevant parameters (from G. Genta, *Vibration Dynamics and Control*, Springer, New York, 2009)

Remark B.27 Actually each substructure can be regarded as a large element, sometimes referred to as a *superelement* and the relevant procedures do not differ from those that are standard in the FEM.

The main advantage of substructuring is that of allowing the construction of the model and the analysis of the various parts of a large structure in an independent way. The results can then be assembled and the behavior of the structure can be assessed from that of its parts. If this is done, however, the connecting nodes must be defined in such a way that the same boundary degrees of freedom are considered in the analysis of the various parts. It is, however, possible to use algorithms allowing the connecting of otherwise incompatible meshes.

Remark B.28 All the methods discussed in this section, which are closely related to each other and are found in the literature in a variety of versions, are general for discrete systems and can also be used outside the FEM even if they became popular only with the use of the latter owing to the large number of degrees of freedom it yields.

An example can be useful to illustrate how the component-mode synthesis method works. Consider the discrete system sketched in Fig. B.8. Let us study its dynamic behavior and compare the results obtained using component-mode synthesis retaining different numbers of modes.

The total number of degrees of freedom of the system is five and the complete mass and stiffness matrices are

$$\mathbf{K} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

By directly solving the eigenproblem, the following matrix of the eigenvalues is obtained:

$$\left[\omega^2\right] = \text{diag}[0.0979 \quad 0.8244 \quad 2.000 \quad 3.176 \quad 3.902].$$

The structure is then subdivided into two substructures and the analysis is accordingly performed.

Substructure 1 Substructure 1 includes nodes 1, 2, and 3 with the masses located on them. The displacements at nodes 1 and 2 are internal coordinates, while the displacement at node 3 is a boundary coordinate. The mass and stiffness matrix of the substructure, partitioned with the boundary degree of freedom first and then the internal ones ordered with the displacement at node 2 before that at node 1, are

$$\mathbf{K} = \left[\begin{array}{c|cc} 1 & -1 & 0 \\ \hline -1 & 2 & -1 \\ 0 & -1 & 2 \end{array}\right], \qquad \mathbf{M} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

The matrix of the eigenvectors for the internal normal modes can be easily obtained by solving the eigenproblem related to matrices with subscript 22 and, by retaining all modes, matrices $\mathbf{K}^*$ and $\mathbf{M}^*$ of the first substructure can be computed:

$$\mathbf{\Phi} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, \qquad \mathbf{K}^* = \left[\begin{array}{c|cc} 0.3333 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 3 \end{array}\right],$$

$$\mathbf{M}^* = \left[\begin{array}{c|cc} 1.556 & 0.7071 & -0.2357 \\ \hline 0.7071 & 1 & 0 \\ -0.2357 & 0 & 1 \end{array}\right].$$

Substructure 2 The second substructure includes nodes 3, 4, and 5 with the masses located on nodes 4 and 5. The mass located on node 3 has already been taken into account in the first substructure and must not be considered again. The displacements at nodes 4 and 5 are internal coordinates, while the displacement at node 3 is a boundary coordinate. The mass and stiffness matrix of the substructure, partitioned with the boundary degree of freedom first and then the internal ones (with the displacement at node 4 and then that at node 5), are

$$\mathbf{K} = \left[\begin{array}{c|cc} 1 & -1 & 0 \\ \hline -1 & 2 & -1 \\ 0 & -1 & 1 \end{array}\right], \qquad \mathbf{M} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{array}\right].$$

Operating as seen for the first substructure, it follows that

$$\mathbf{\Phi} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{K}^* = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0.5858 & 0 \\ 0 & 0 & 3.4142 \end{array}\right],$$

$$\mathbf{M}^* = \left[\begin{array}{c|cc} 1.5 & 1.2071 & -0.2071 \\ \hline 1.2071 & 1 & 0 \\ -0.2071 & 0 & 1 \end{array}\right].$$

The substructures can be assembled in the same way as the elements. The following map can be written:

Subst. 1	d.o.f. type	1 boundary	2 modal	3 modal		
Subst. 2	d.o.f. type	1 boundary			2 modal	3 modal
Global	d.o.f.	1	2	3	4	5

yielding the following global stiffness and mass matrices:

$$\mathbf{K}^* = \left[\begin{array}{c|cccc} 0.3333 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0.5858 & 0 \\ 0 & 0 & 0 & 0 & 3.4142 \end{array}\right],$$

$$\mathbf{M}^* = \left[\begin{array}{c|cccc} 1.5 & 0.7071 & -0.2357 & 1.2071 & -0.2071 \\ \hline 0.7071 & 1 & 0 & 0 & 0 \\ -0.2357 & 0 & 1 & 0 & 0 \\ 1.2071 & 0 & 0 & 1 & 0 \\ -0.2071 & 0 & 0 & 0 & 1 \end{array}\right].$$

The matrices have been partitioned in such a way to separate the boundary displacement degree of freedom from the modal degrees of freedom. If no modal coordinate is considered, the component-mode synthesis coincides with Guyan reduction, with only one master degree of freedom. If the third and fifth rows and columns are canceled, only one internal normal mode is taken into account for each substructure. If the matrices are taken into account in complete form, all modes are considered and the result must coincide, except for computing approximations, with the exact ones. The results obtained in terms of the square of the natural frequency are

Size of matrices	5 (exact)	1 (Guyan red.)	3 (1 mode)	5 (2 modes)
Mode 1	0.0979	0.1091	0.0979	0.0979
Mode 2	0.8244	–	0.8245	0.8244
Mode 3	2.000	–	2.215	2.000
Mode 4	3.176	–	–	3.176
Mode 5	3.902	–	–	3.902

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[1] Space robotics is a strongly interdisciplinary subject and a large number of books dealing with many disciplines are relevant to the various aspects dealt with in this book. If papers published on journals are considered, the list should include thousands of titles.

The author has chosen to provide a list of books dealing with robotics in a wide sense and a further one of books dealing with terramechanics and dynamics of wheeled and legged vehicles, while leaving out books related with other types of locomotion (flying, sailing, etc) and books dealing with components used in robots (motors, gears, actuators, sensors, etc.). Even with these restrictions in mind, the author is sure that many books that should have been included were left out.

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