EECE 574 - Adaptive Control

Model-Reference Adaptive Control - An Overview

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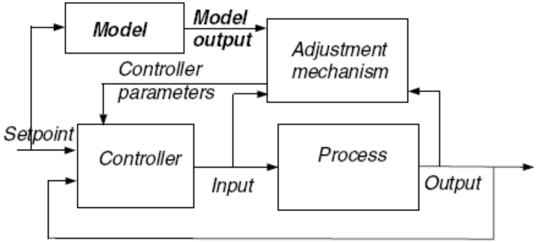
The MRAC or MRAS is an important adaptive control methodology [1]



[1] see Chapter 5 of the Åström and Wittenmark textbook, or H. Butler, "Model-Reference Adaptive Control-From Theory to Practice", Prentice-Hall, 1992

Model-Reference Adaptive Systems

- The MIT rule
- Lyapunov stability theory
- Design of MRAS based on Lyapunov stability theory
- Hyperstability and passivity theory
- The error model
- Augmented error
- A model-following MRAS

The MIT Rule

- Original approach to MRAC developed around 1960 at MIT for aerospace applications
- With $e = y - y_m$, adjust the parameters θ to minimize

$$J(\theta) = \frac{1}{2}e^2$$

- It is reasonable to adjust the parameters in the direction of the negative gradient of J:

$$\frac{d\theta}{dt} = -\gamma\frac{\partial J}{\partial \theta} = -\gamma e \frac{\partial e}{\partial \theta}$$

- $\partial e/\partial \theta$ is called the sensitivity derivative of the system and is evaluated under the assumption that θ varies slowly

The MIT Rule

- The derivative of J is then described by

$$\frac{dJ}{dt} = e\frac{\partial e}{\partial t} = -\gamma e^2 \left(\frac{\partial e}{\partial \theta}\right)^2$$

- Alternatively, one may consider $J(e) = |e|$ in which case

$$\frac{d\theta}{dt} = -\gamma\frac{\partial J}{\partial \theta} = -\gamma\frac{\partial e}{\partial \theta}\mathrm{sign(e)}$$

- The sign-sign algorithm used in telecommunications where simple implementation and fast computations are required, is

$$\frac{d\theta}{dt} = -\gamma\frac{\partial J}{\partial \theta} = -\gamma \mathrm{sign}\left(\frac{\partial \mathrm{e}}{\partial \theta}\right)\mathrm{sign(e)}$$

MIT Rule: Example 1

- Process: $y = kG(s)$ where $G(s)$ is known but k is unknown
- The desired response is $y_m = k_0 G(s) u_c$
- Controller is $u = \theta u_c$
- Then $e = y - y_m = kG(p)\theta u_c - k_0 G(p) u_c$
- Sensitivity derivative

$$\frac{\partial e}{\partial \theta} = kG(p)u_c = \frac{k}{k_0} y_m$$

- MIT rule

$$\frac{d\theta}{dt} = \gamma' \frac{k}{k_0} y_m e = -\gamma y_m e$$

MIT Rule: Example 1

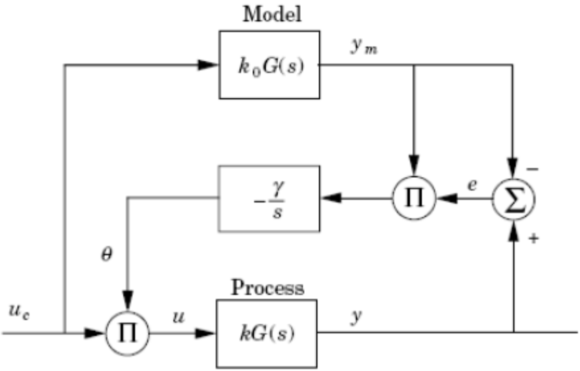


Figure: MIT rule for adjustment of feedforward gain (from textbook).

MIT Rule: Example 1

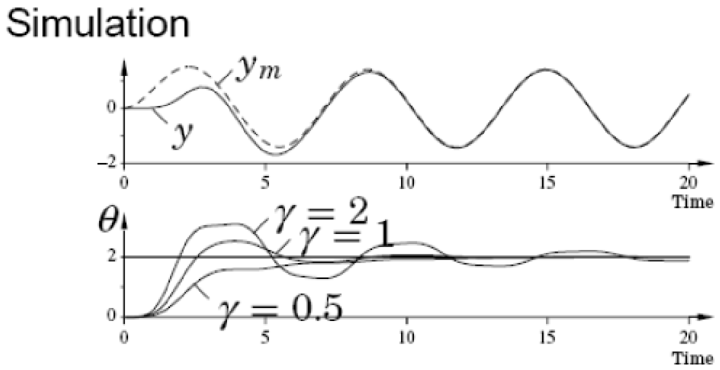


Figure: MIT rule for adjustment of feedforward gain: Simulation results (from textbook).

MIT Rule: Example 2

Consider the first-order system

$$\frac{dy}{dt} = -ay + bu$$

The desired closed-loop system is

$$\frac{dy_m}{dt} = -a_m y_m + b_m u_c$$

Applying model-following design (see lecture notes on pole placement)

$$\deg A_0 \geq 0 \qquad \deg S = \deg R = \deg T = 0$$

MIT Rule: Example 2

The controller is then

$$u(t) = t_0 u_c(t) - s_0 y(t)$$

For perfect model-following

$$\begin{aligned}\frac{dy}{dt} &= -ay(t) + b[t_0 u_c(t) - s_0 y(t)] \\ &= -(a + bs_0)y(t) + bt_0 u_c \\ &= -a_m y_m(t) + b_m u_c\end{aligned}$$

This implies

$$\begin{aligned}s_0 &= \frac{a_m - a}{b} \\ t_0 &= \frac{b_m}{b}\end{aligned}$$

MIT Rule: Example 2

With the controller we can write[2]

$$y = \frac{bt_0}{p + a + bs_0} u_c$$

With $e = y - y_m$, the sensitivity derivatives are

$$\begin{aligned} \frac{\partial e}{\partial t_0} &= \frac{b}{p + a + bs_0} u_c \\ \frac{\partial e}{\partial s_0} &= \frac{b^2 t_0}{(p + a + bs_0)^2} u_c = \frac{b}{p + a + bs_0} y \end{aligned}$$

[2] p is the differential operator $d(.)/dt$

MIT Rule: Example 2

However, a and b are unknown[3]. For the nominal parameters, we know that

$$a + bs_0 = a_m$$

$$p + a + bs_0 \approx p + a_m$$

Then

$$\begin{aligned}
\frac{dt_0}{dt} &= -\gamma' b \left(\frac{1}{p + a_m} u_c \right) e = -\gamma \left(\frac{a_m}{p + a_m} u_c \right) e \\
\frac{ds_0}{dt} &= \gamma' b \left(\frac{1}{p + a_m} y \right) e = \gamma \left(\frac{a_m}{p + a_m} y \right) e
\end{aligned}$$

with $\gamma = \gamma' b / a_m$, i.e. b is absorbed in γ and the filter is normalized with static gain of one.

[3] after all, we want to design an adaptive controller!

MIT Rule: Example 2

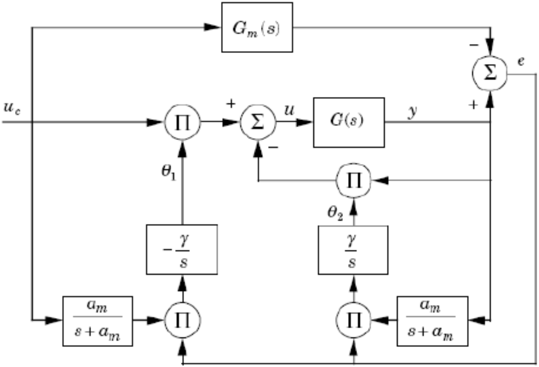


Figure: MIT rule for first-order (from textbook)

MIT Rule: Example 2

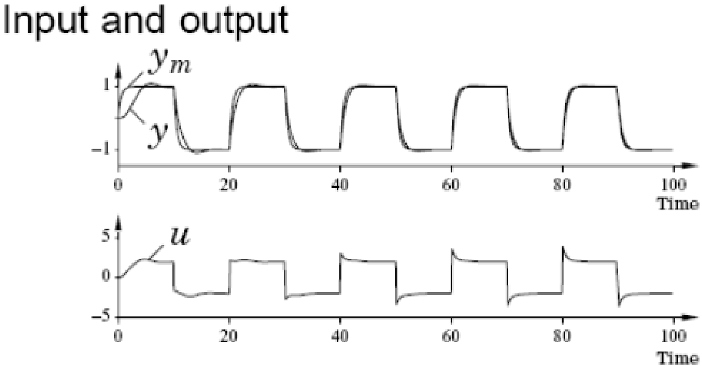


Figure: Simulation of MIT rule for first-order with $a = 1$, $b = 0.5$, $a_m = b_m = 2$ and $\gamma = 1$(from textbook)

MIT Rule: Example 2

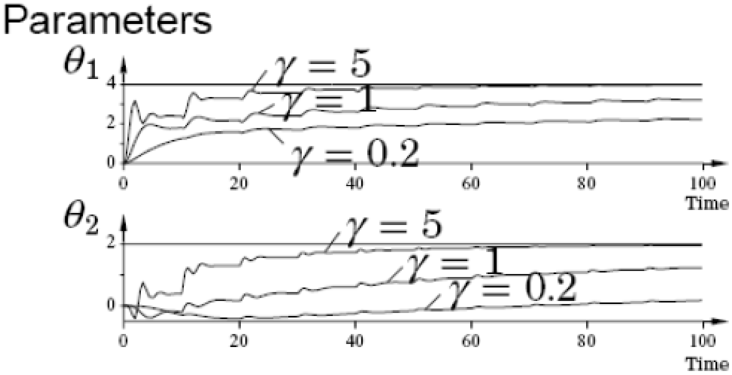


Figure: Simulation of MIT rule for first-order with $a = 1$, $b = 0.5$, $a_m = b_m = 2$. Controller parameters for $\gamma = 0.2$, 1 and 5(from textbook)

MIT Rule: Example 2

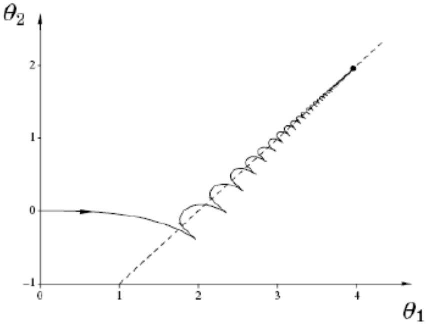


Figure: Simulation of MIT rule for first-order with $a = 1$, $b = 0.5$, $a_m = b_m = 2$. Relation between controller parameters θ_1 and θ_2. The dashed line is $\theta_2 = \theta_1 - a/b$. (from textbook)

Stability of the Adaptive Loop

Consider again the adaptation of a feedforward gain

$$G = \frac{1}{s^2 + a_1 s + a_2}$$

$$e = (\theta - \theta_0) G u_c$$

$$\begin{aligned} \frac{\partial e}{\partial \theta} &= G u_c = \frac{y_m}{\theta_0} \\ \frac{d\theta}{dt} &= -\gamma' e \frac{\partial e}{\partial \theta} = -\gamma' e \frac{y_m}{\theta_0} = -\gamma e y_m \end{aligned}$$

Stability of the Adaptive Loop

Thus the system can be described as

$$\begin{aligned}
\frac{d^2 y_m}{dt^2} + a_1 \frac{dy_m}{dt} + a_2 y_m &= \theta_0 u_c \\
\frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y &= \theta u_c \\
\frac{d\theta}{dt} &= -\gamma (y - y_m) y_m
\end{aligned}$$

This is difficult to solve analytically

Stability of the Adaptive Loop

$$\frac{d^3y}{dt^3} + a_1\frac{d^2y}{dt^2} + a_2\frac{dy}{dt} + \gamma u_c y_m y = \theta\frac{du_c}{dt} + \gamma u_c y_m^2$$

Assume u_c^o and y_m^o constant, equilibrium is

$$y(t) = y_m^o = \frac{\theta_0 u_c^o}{a_2}$$

which is stable if

$$a_1 a_2 > \gamma u_c^o y_m^o = \frac{\gamma}{a_2}(u_c^o)^2$$

Thus, if γ or u_c is sufficiently large, the system will be unstable

Stability of the Adaptive Loop

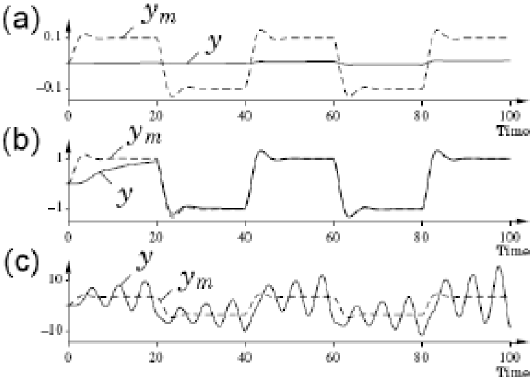


Figure: Influence on convergence and stability of signal amplitude for MIT rule (from textbook)

Modified MIT Rule

- Normalization can be used to protect against dependence on the signal amplitudes
- With $\varphi = \partial e/\partial\theta$, the MIT rule can be written as

$$\frac{d\theta}{dt} = -\gamma\varphi e$$

- The **normalized** MIT rule is then

$$\frac{d\theta}{dt} = \frac{-\gamma\varphi e}{\alpha^2 + \varphi^T\varphi}$$

- For the previous example,

$$\frac{d\theta}{dt} = \frac{-\gamma e y_m/\theta_0}{\alpha^2 + y_m^2/\theta_0^2}$$

Modified MIT Rule

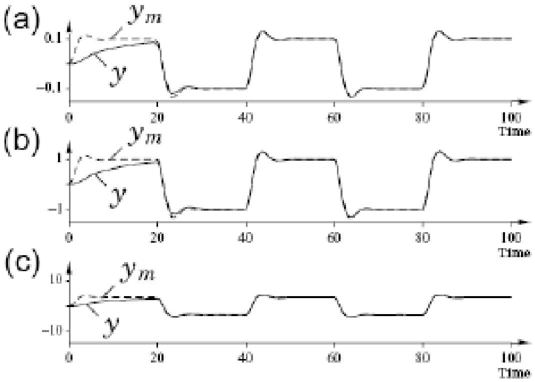


Figure: Influence on convergence and stability of signal amplitude for modified MIT rule (from textbook)

Lyapunov Stability Theory

- Aleksandr M Lyapunov (1857-1918, Russia)
- Classmate of Markov, taught by Chebyshev
- Doctoral thesis *The general problem of the stability of motion* became a fundamental contribution to the study of dynamic systems stability
- He shot himself three days after his wife died of tuberculosis

Lyapunov Stability Theory

- Consider the nonlinear time-varying system

$$\dot{x} = f(x,t) \qquad \text{with } f(0,t) = 0$$

- If at time $t = t_0$ $\|x(0)\| = \delta$, i.e. if the initial state is not the equlibrium state $x^* = 0$, what will happen?
- There are four possibilities

Lyapunov Stability Theory

- The system is stable. For a sufficiently small δ, x stays within ε of x^*. If δ can be chosen independently of t_0, then the system is said to be uniformly stable.
- The system is unstable. It is possible to find ε which does not allow any δ.
- The system is asymptotically stable. For $\delta < R$ and an arbitrary ε, there exists t^* such that for all $t > t^*$, $\|x - x^*\| < \varepsilon$. It implies that $\|x - x^*\| \to 0$ as $t \to \infty$.
- If asymptotic stability is guaranteed for any δ, the system is globally asymptotically stable.

Lyapunov Stability Theory

- A scalar time-varying function is locally positive definite if $V(o,t) = 0$ and there exists a time-invariant positive definite function $V_0(x)$ such that $\forall t \geq t_0$, $V(x,t) \geq V_0(x)$. In other words, it dominates a time-invariant positive definite function.
- $V(x,t)$ is radially unbounded if $0 < \alpha\|x\| \leq V(x,t)$, $\alpha > 0$.
- A scalar time-varying function is said to be decrescent if $V(o,t) = 0$ and there exists a time-invariant positive definite function $V_1(x)$ such that $\forall t \geq t_0$, $V(x,t) \leq V_1(x)$.
- In other words, it is dominated by a time-invariant positive definite function.

Lyapunov Stability Theory

Theorem (Lyapunov Theorem)

- ***Stability:*** *if in a ball B_R around the equilibrium point 0, there exists a scalar function $V(x,t)$ with continuous partial derivatives such that*
  1. *V is positive definite*
  2. *$\dot{V}$ is negative semi-definite*

  then the equilibrium point is ***stable****.*
- ***Uniform stability and uniform asymptotic stability:*** *If furthermore,*
  3. *V is decrescent*

  then the origin is ***uniformly stable****. If condition 2 is strengthened by requiring that $\dot{V}$ be negative definite, then the equilibrium is* ***uniformly asymptotically stable****.*
- ***Global uniform asymptotic stability:*** *If B_R is replaced by the whole state space, and condition 1, the strengthened condition 2, condition 3 and the condition*
  4. *$V(x,t)$ is radially unbounded*

  are all satisfied, then the equilibrium point at 0 is ***globally uniformly asymptotically stable****.*

Lyapunov Stability Theory

- The Lyapunov function has similarities with the **energy content** of the system and must be **decreasing** with time.
- This result can be used in the following way to design a stable a stable adaptive controller
  1. First, the **error equation**, i.e. a differential equation describing either the output error $y - y_m$ or the state error $x - x_m$ is derived.
  2. Second, a Lyapunov function, function of both the signal error $e = x - x_m$ and the parameter error $\phi = \theta - \theta_m$ is chosen. A typical choice is

     $$V = e^T P e + \phi^T \Gamma^{-1} \phi$$

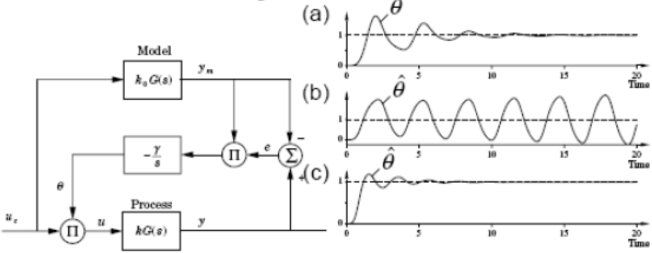
     where both P and Γ^{-1} are positive definite matrices.
  3. The time derivative of V is calculated. Typically, it will have the form

     $$\dot{V} = -e^T Q e + \text{ some terms including } \phi$$

     Putting the extra terms to zero will ensure negative definiteness for $\dot{V}$ if Q is positive definite, and will provide the adaptive law.

Stability of Adaptive Loop

Consider the MRAS designed with MIT rule



$u_c = sin\omega t$ $\omega =$ 1(a), 2(b) and 3(c)

Lyapunov Design of MRAC

- Determine controller structure
- Derive the error equation
- Find a Lyapunov equation
- Determine adaptation law that satisfies Lyapunov theorem

Adaptation of Feedforward Gain

- Process model

$$\frac{dy}{dt} = -ay + ku$$

- Desired response

$$\frac{dy_m}{dt} = -ay_m + k_0 u_c$$

- Controller

$$u = \theta u_c$$

- Introduce error $e = y - y_m$
- The error equation is

$$\frac{de}{dt} = -ae + (k\theta - k_0)u_c$$

Adaptation of Feedforward Gain

- System model

$$\begin{aligned}\frac{dy}{dt} &= -ay + ku \\ \frac{d\theta}{dt} &= ??\end{aligned}$$

- Desired equilibrium

$$\begin{aligned}e &= 0 \\ \theta &= \theta_0 = \frac{k_0}{k}\end{aligned}$$

Adaptation of Feedforward Gain

- Consider the Lyapunov function

$$
\begin{aligned}
V(e,\theta) &= \frac{\gamma}{2}e^2 + \frac{k}{2}(\theta-\theta_0)^2 \\
\frac{dV}{dt} &= -\gamma a e^2 + (k\theta - k_0)(\frac{d\theta}{dt} + \gamma u_c e)
\end{aligned}
$$

- Choosing the adjustment rule

$$\frac{d\theta}{dt} = -\gamma u_c e$$

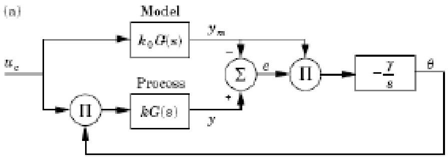
gives

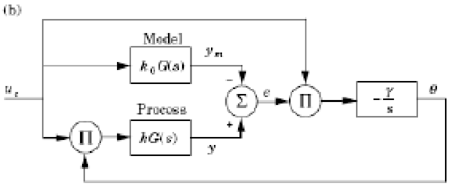
$$\frac{dV}{dt} = -\gamma a e^2$$

Adaptation of Feedforward Gain

Lyapunov rule: $\frac{d\theta}{dt} = -\gamma u_c e$

MIT rule: $\frac{d\theta}{dt} = -\gamma y_m e$

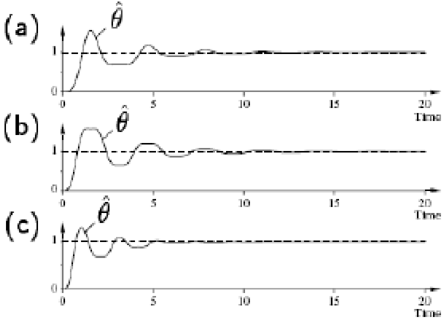




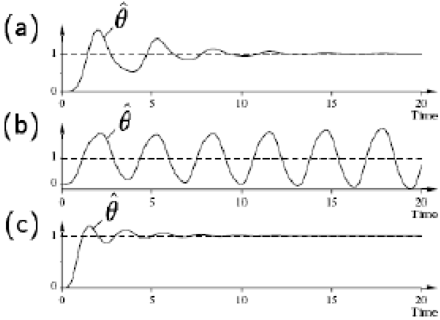
 Lyapunov Design of MRAC

Adaptation of Feedforward Gain

Lyapunov rule:



MIT rule:



First-order System

- Process model

$$\frac{dy}{dt} = -ay + bu$$

- Desired response

$$\frac{dy_m}{dt} = -a_m y_m + b_m u_c$$

- Controller

$$u = \theta_1 u_c - \theta_2 y$$

- Introduce error $e = y - y_m$
- The error equation is

$$\frac{de}{dt} = -a_m e - (b\theta_2 + a - a_m)y + (b\theta_1 - b_m)u_c$$

First-order System

- Candidate Lyapunov function

$$V(t,\theta_1,\theta_2) = \frac{1}{2}\left(e^2 + \frac{1}{b\gamma}(b\theta_2 + a - a_m)^2 + \frac{1}{b\gamma}(b\theta_1 - b_m)^2\right)$$

- Derivative

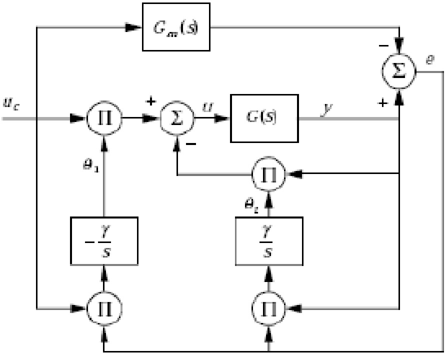
$$\begin{aligned}\frac{dV}{dt} &= \frac{de}{dt} + \frac{1}{\gamma}(b\theta_2 + a - a_m)\frac{d\theta_2}{dt} + \frac{1}{\gamma}(b\theta_1 - b_m)\frac{d\theta_1}{dt} \\ &= -a_m e^2 + \frac{1}{\gamma}(b\theta_2 + a - a_m)(\frac{d\theta_2}{dt} - \gamma y e) + \frac{1}{\gamma}(b\theta_1 - b_m)(\frac{d\theta_1}{dt} + \gamma u_c e)\end{aligned}$$

- This suggests the adaptation law

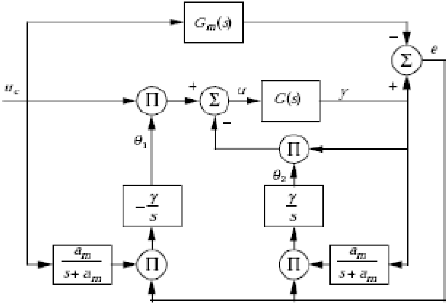
$$\begin{aligned}\frac{d\theta_1}{dt} &= -\gamma u_c e \\ \frac{d\theta_2}{dt} &= \gamma y e\end{aligned}$$

First-order System

Lyapunov Rule

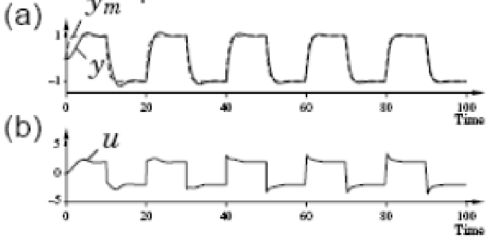


MIT Rule

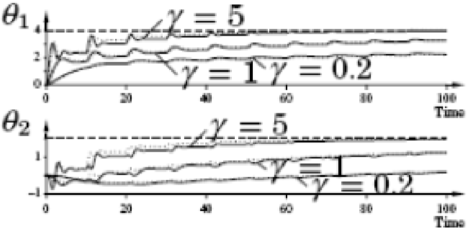


First-order System

Process inputs and outputs



Parameters



Lyapunov-based MRAC Design

- The main advantage of Lyapunov design is that it guarantees a closed-loop system.
- For a linear, asymptotically stable governed by a matrix A, a positive symmetric matrix Q yields a positive symmetric matrix P by the equation

$$A^T P + PA = -Q$$

This equation is known as Lyapunov's equation.

Lyapunov-based MRAC Design

- The main drawback of Lyapunov design is that there is no systematic way of finding a suitable Lyapunov function V leading to a specific adaptive law.
- For example, if one wants to add a proportional term to the adaptive law, it is not trivial to find the corresponding Lyapunov function.
- The hyperstability approach is more flexible than the Lyapunov approach.

The Lyapunov Equation

- Let the linear system
$$\dot{x} = AX$$
be stable
- Let Q be an arbitrary positive definite matrix
- Then the Lyapunov equation

$$A^T P + PA = -Q$$

always has a unique solution where P is positive definite.

- The function
$$V(x) = x^T P x$$
is then a Lyapunov function

State Feedback

- Process: $\dot{x} = Ax + Bu$
- Desired response: $\dot{x}_m = A_m x_m + B_m u_c$
- Control law: $u = Mu_c - Lx$
- Closed-loop system

$$\dot{x} = Ax + Bu = (A - BL)x + BMu_c = A_c(\theta)x + B_c(\theta)u_c$$

- The parametrization is $A_c(\theta) = A_m$ and $B_c(\theta) = B_m$
- For compatibility we need $A - A_m = BL$ and $B_m = BM$

Error Equation

- The error $e = x - x_m$ satisfies

$$\dot{e} = \dot{x} - \dot{x}_m = Ax + Bu - Ax_m - B_m u_c$$

- Hence

$$\begin{aligned} \dot{e} &= A_m e + (A - A_m - BL)x + (BM - B_m)u_c \\ &= A_m e + (A_c(\theta) - A_m)x + (B_c(\theta) - B_m)u_c \\ &= A_m e + \Psi(\theta - \theta_0) \end{aligned}$$

The Lyapunov Function

- Try

$$V(e,\theta) = \frac{1}{2}\left(\gamma e^T P e + (\theta - \theta_0)^T (\theta - \theta_0)\right)$$

- Differentiating V

$$\dot{V} = \frac{\gamma}{2}(\dot{e}^T P e + e^T P \dot{e}) + (\theta - \theta_0)^T \dot{\theta}$$

- Then with $\dot{e} = A_m e + \Psi(\theta - \theta_0)$ we can write

$$\begin{aligned} \dot{V} &= \frac{\gamma}{2}\left[\left(e^T A_m^T + \Psi(\theta - \theta_0)\right) P e + e^T \left(A_m e + \Psi(\theta - \theta_0)\right]\right. \\ &\quad + (\theta - \theta_0)^T \dot{\theta} \end{aligned}$$

The Lyapunov Function

- Now, consider the Lyapunov equation $A_m^T P + PA_m = -Q$ (possible because reference model always stable)

$$\begin{aligned} \dot{V} &= -\frac{\gamma}{2} e^T Q e + \gamma(\theta - \theta_0)^T \Psi^T P e + (\theta - \theta_0)^T \dot{\theta} \\ &= -\frac{\gamma}{2} e^T Q e + (\theta - \theta_0)^T (\dot{\theta} + \gamma \Psi^T P e) \end{aligned}$$

- If the parameter adjustment law is chosen as

$$\frac{d\theta}{dt} = -\gamma \Psi^T P e$$

- We obtain

$$\frac{dV}{dt} = -\frac{\gamma}{2} e^T Q e$$

  which is negative semi-definite

- Note that this assumes knowledge of the state.

Adaptation of Feedforward Gain

- Consider the error

$$e = (kG(p)\theta - k_0 G(p))u_c = kG(p)(\theta - \theta_0)u_c$$

- With

$$\begin{aligned} \dot{x} &= Ax + B(\theta - \theta_0)u_c \\ e &= Cx \end{aligned}$$

- A candidate Lyapunov function is

$$V = \frac{1}{2}(\gamma x^T P x + (\theta - \theta_0)^2)$$

- Let

$$A^T P + PA = -Q$$

Adaptation of Feedforward Gain

- Differentiating V

$$\dot{V} = \frac{\gamma}{2}\left(\dot{x}^T P x + x^T P \dot{x}\right) + (\theta - \theta_0)\dot{\theta}$$

- Which after using the model for $\dot{x}$ and the Lyapunov equation gives

$$\dot{V} = -\frac{\gamma}{2} x^T Q x + (\theta - \theta_0)\left(\dot{\theta} + \gamma u_c B^T P x\right)$$

- Using the adaptation law

$$\frac{d\theta}{dt} = -\gamma u_c B^T P x$$

gives

$$\frac{dV}{dt} = -\frac{\gamma}{2} x^T Q x$$

Output Feedback

- The adaptation law

$$\frac{d\theta}{dt} = -\gamma u_c B^T P x$$

assumes knowledge of the state x

- If we can find P such that

$$B^T P = C$$

then the adaptation law becomes

$$\frac{d\theta}{dt} = -\gamma u_c e$$

i.e. this is now output feedback and the state x is not required

- When is it possible to do so?

Kalman-Yakubovich Lemma

Definition: Positive real transfer function

A rational transfer function G with real coefficients is positive real (PR) if

$$\text{Re}\, G(s) \geq 0 \qquad \text{for} \qquad \text{Re}\, s \geq 0$$

A rational transfer function G with real coefficients is strictly positive real (SPR) if $G(s-\varepsilon)$ is PR for some $\varepsilon > 0$

- $G(s) = 1/(s+1)$ is SPR
- $G(s) = 1/s$ is PR bur not SPR

Kalman-Yakubovich Lemma

Theorem (Kalman-Yakubovich Lemma)

Let the linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

be completely controllable and completely observable. The transfer function

$$G(s) = C(sI - A)^{-1}B$$

is strictly positive real if and only if there exist positive definite matrices P and Q such that

$$\begin{aligned} A^T P + PA &= -Q \\ B^T P &= C \end{aligned}$$

Operator View of Dynamical Systems

L_2 norm

$$\|u\| = \left(\int_{-\infty}^{\infty} u^2(t)dt \right)^{\frac{1}{2}}$$

L_∞ norm

$$\|u\| = \sup_{0 \le t < \infty} |u(t)|$$

Truncation operator:

$$x_T(t) = \begin{cases} x(t) & 0 \le t \le T \\ 0 & t > T \end{cases}$$

$$L_{2e} = \left\{ x | \; \|x_T\|^2 = \langle x_T | x_T \rangle < \infty \right\}$$

The Notion of Gain

Gain of a nonlinear system

Let the signal space be X_e. The gain $\gamma(S)$ of a system S is defined as

$$\gamma(S) = \sup_{u \in X_e} \frac{\|Su\|}{\|u\|}$$

where u is the input signal to the system.
The gain is thus the smallest value γ such that

$$\|Su\| \leq \gamma(S)\|u\| \quad \text{for all } u \in X_e$$

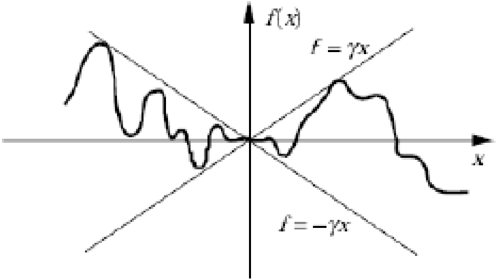
The Notion of Gain

- For a linear system G with signals in L_{2e},

$$\gamma(G) = \max_{\omega} |G(i\omega)|$$

- Static nonlinear system $y(t) = f(u(t))$, the gain of the system is

$$\gamma = \max_{u} \frac{|f(u)|}{|u|}$$



BIBO Stability

Definition

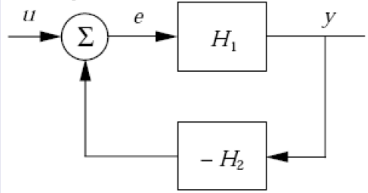
A system is bounded-input, bounded-output (BIBO) stable if it has bounded gain

- A BIBO stable system is a system for which the outputs will remain bounded for all time, for any finite initial condition and input.
- A continuous-time linear time-invariant system is BIBO stable if and only if all the poles of the system have real parts less than 0.

The Small Gain Theorem

Theorem (The Small Gain Theorem)

Consider the system



Let γ_1 *and* γ_2 *be the gains of the systems* H_1 *and* H_2*. The closed-loop system is then BIBO* ***stable*** *if*

$$\gamma_1\gamma_2 < 1 \qquad \textit{and its gain is less than} \qquad \gamma = \frac{\gamma_1}{1-\gamma_1\gamma_2}$$

Positivity and Passivity

- Positivity and passivity are nearly equivalent concepts.
- Positivity is a property of linear systems, while passivity is a more general concept, applicable to both linear and nonlinear systems.
- The scalar, controllable system

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= c^T x \end{aligned}$$

with a transfer function

$$H(s) = c^T (sI - A)^{-1} b$$

is said to be positive real (PR) if $\mathrm{Re[H(s)]} \geq 0$ for all $\mathrm{Re(s)} \geq 0$.

Positivity and Passivity

- If $s = j\omega$, it means that the Nyquist diagram of $H(j\omega)$ must lies in the **right half plane**, including the imaginary axis.
- Thus, the phase shift must be between $-90\,\mathrm{deg}$ and $+90\,\mathrm{deg}$.
- The **Kalman-Yakubovich** lemma states that the system $H(s)$ is strictly positive real if there exist positive definite matrices P and Q such that:

$$\begin{aligned} A^T P + PA &= -Q \\ Pb &= c \end{aligned}$$

Positivity and Passivity

- Passivity is a general form of positivity
- A **passive** linear system must be **positive real** and vice-versa.
- Consider a system with input u and output y. The energy of the system is such that

$$\frac{d}{dt}\text{[Stored energy]=[external power input]+[internal power generation]}$$

- If the internal power generation is negative, then the system is **dissipative** or **strictly passive**. If the internal power generation is less than or equal to zero the system is **passive**.

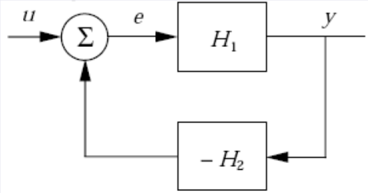
Positivity and Passivity

- If the input u and output y are such that the external power input is expressed as $u^T y$, then
  - The system is **passive** if $\langle y|u\rangle \geq 0$
  - The system is **input strictly passive** (ISP) if there exists $\varepsilon > 0$ such that $\langle y|u\rangle \geq \|u\|^2$
  - The system is **output strictly passive** (OSP) if there exists $\varepsilon > 0$ such that $\langle y|u\rangle \geq \|y\|^2$
- Examples include electrical networks (power proportional to product of current and voltage), mass-spring systems (mechanical power is force $\times$ velocity).

The Passivity Theorem

Theorem (Passivity Theorem)

Consider the system



*Let H_1 be **strictly output passive** and H_2 be **passive**. The closed-loop system is then **BIBO stable***

Positive Real Transfer Functions

A rational transfer function $G(s)$ with real coefficients is **PR** if and only if the following conditions hold

1. The function has no poles in the right half-plane
2. If the function has poles on the imaginary axis or at infinity, they are simple poles with positive residues
3. The real part of G is nonnegative along the $i\omega$ axis

$$\mathrm{Re}(G(i\omega)) \geq 0$$

G is **SPR** if conditions 1 and 3 hold and if $G(s)$ has no poles or zeros on the imaginary axis

Positivity and Passivity

- Positive real (PR): $\text{Re } G(j\omega) \geq 0$
- Input strictly passive (ISP): $\text{Re } G(j\omega) \geq \varepsilon > 0$
- Input strictly passive (ISP): $\text{Re } G(j\omega) \geq \varepsilon |G(j\omega)|^2$
- Examples
  - $G(s) = s+1$ is SPR and ISP but not OSP
  - $G(s) = 1/(s+1)$ is SPR and OSP but not ISP
  - $G(s) = (s^2+1)/(s+1)^2$ is OSP and ISP but not SPR
  - $G(s) = 1/s$ is PR, but not SPR, OSP or ISP

Positive Real Transfer Functions

Theorem (Lemma)

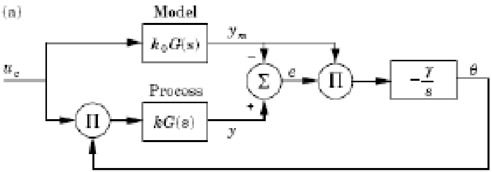
Let r be a bounded square integrable function, and let $G(s)$ be a positive real transfer function. The system whose input-output relation is given by

$$y = r(G(p)ru)$$

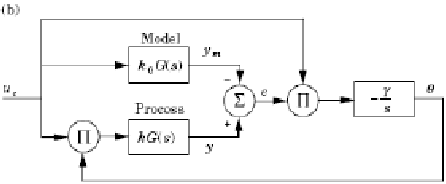
is then passive.

Adaptation of Feedforward Gain

MIT rule:

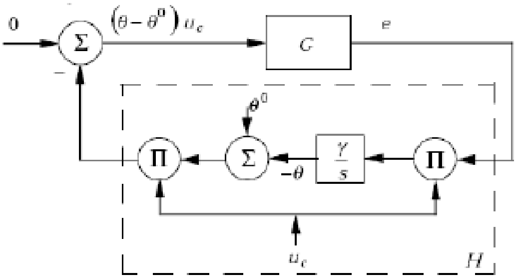


Lyapunov rule:



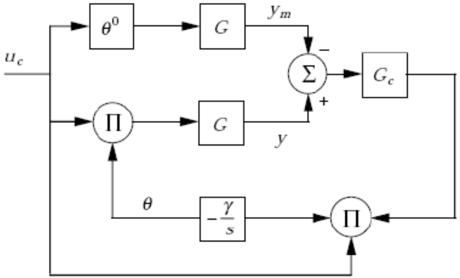
Analysis

Redraw (b) as:



- Because γ/s is PR, H is **passive**
- Thus, if G is SPR, the passivity theorem implies that the system is L_2 (BIBO) stable
- If non-zero initial conditions, $e(t)$ is still in L_2 and thus goes to zero as $t \to \infty$
- This is stable for all $\gamma > 0$, so the adaptation can be made arbitrarily fast as long as G is SPR
- If G is not SPR, then may be possible to introduce a compensator G_c such that GG_c is SPR

Analysis



- The Kalman-Yakubovich lemma can be used to find G_c such that GG_c is PR
- However, when relative degree of G is greater than 1, G_c will contain derivatives
- Augmented error

Augmented Error

- Factor G as

$$G = G_1 G_2$$

where G_1 is SPR

$$\begin{aligned} e &= G(\theta - \theta_0)u_c = (G_1 G_2)(\theta - \theta_0)u_c \\ &= G_1 \left(G_2(\theta - \theta_0)u_c(\theta - \theta_0)G_2 u_c - (\theta - \theta_0)G_2 u_c \right) \end{aligned}$$

- Introduce the augmented error

$$\varepsilon = e + \eta$$

where η is the error augmentation defined by

$$\eta = G_1(\theta - \theta_0)G2u_c - G(\theta - \theta_0)u_c = G_1(\theta G_2 uc) - G\theta u_c$$

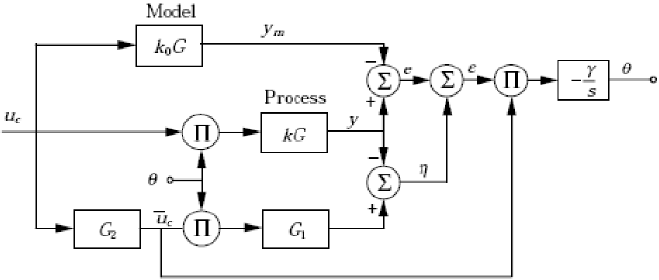
- The correction term η vanishes when θ is constant

Augmented Error

- The adaptation law

$$\dot{\theta} = -\gamma \varepsilon G_2 u_c$$

gives a closed-loop system where $e(t)$ goes to zero as t goes to infinity



Augmented Error

- Error augmentation is a fundamental idea in MRAC
- Details can sometimes be messy...
- See example in Section 5.8 for an output feedback design
    - Error signal
    $$e = \frac{b_0}{A_0 A_m}(Ru + Sy - Tu_c)$$
    - Key is introduction of a filter Q such that $b_0 Q / A_0 A_m$ is SPR and $e_f = Qe/P$
    - Then the closed-loop system is BIBO stable

Active Suspension

IEEE TRANSACTIONS ON INDUSTRIAL ELECTRONICS, VOL. 38, NO. 3, JUNE 1991 217

Model Reference Adaptive Control for Vehicle Active Suspension Systems

Myounghe Sunwoo, Ka C. Cheok, *Member, IEEE*, and N. J. Huang, *Student Member, IEEE*

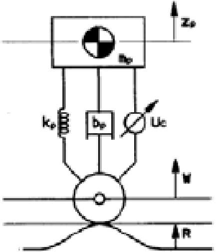


Fig. 3. Active suspension model.

Active Suspension

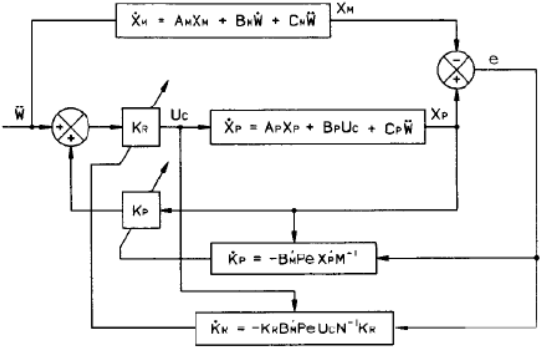


Fig. 1. Direct MRAC scheme via Lyapunov stability criteria.

Active Suspension

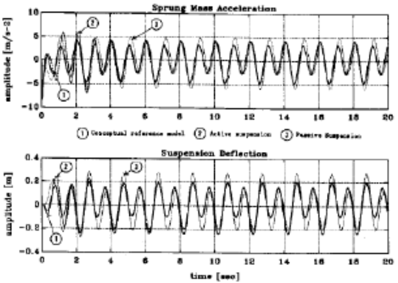


Fig. 6. Response of sprung mass acceleration and suspension deflection for the case (30% sprung mass increment and 30% decrement of sprint and damper coefficients).

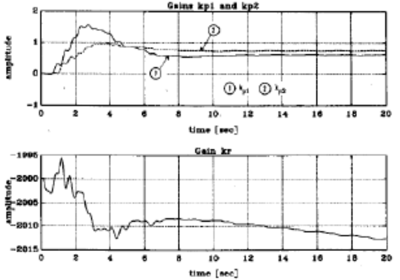


Fig. 8. Convergence of controller gains (K_{p1}, K_{p2}, and K_r) for the case of Fig. 6.