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A Survey of Maneuvering Target Tracking: Dynamic Models

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Abstract

This is the first part of a series of papers that provide a comprehensive and up-to-date survey of the problems and techniques of tracking maneuvering targets in the absence of the so-called measurement-origin uncertainty. It surveys the various mathematical models of target dynamics proposed for maneuvering target tracking, including 2D and 3D maneuver models as well as coordinate-uncoupled generic models for target dynamics. This survey emphasizes the underlying ideas and assumptions of the models. Interrelationships among the models surveyed and insight to the pros and cons of the models are provided. Some material presented here has not appeared elsewhere.

Key Words: Target Tracking, Maneuvering Target, Dynamic Model, Survey

1 Introduction

The key to successful target tracking lies in the optimal extraction of useful information about the target's state from the observations. A good model of the target will certainly facilitate this information extraction to a great extent. In general, one can say without exaggeration that a good model is worth a thousand pieces of data. This saying has an even stronger positive connotation in target tracking where observation data are rather limited. Most tracking algorithms are model based because a good model-based tracking algorithm will greatly outperform any model-free tracking algorithm if the underlying model turns out to be a good one. As such, it is hard to overstate the importance of the role of a good model here.

Various mathematical models of target motion have been developed over the past three decades. They are, however, scattered in the literature. Many of them have never appeared in any periodical in the public domain. As a result, few people have a good knowledge of these models. This is partly due to the lack of a comprehensive and updated survey. The importance of such a survey for both practitioners and researchers in the tracking community is evident. The single best source so far is, in the authors' opinion, the recent book by Blackman and Popoli [1], which is still far from complete.

This paper is the first part of a comprehensive and updated survey of the techniques for maneuvering target tracking. It is well known that the two major challenges for target tracking are the so-called measurement-origin uncertainty and target motion uncertainty. To limit the scope of the work, this survey deals only with the second uncertainty, leaving the techniques unique for the data-association problems untouched.

Target detection, tracking, classification and identification (recognition) are closely interrelated areas, with significant overlaps. It is not easy to draw a clear line to separate them. To be relatively more focused, this survey covers only dynamic models of a "point target." In other words, the models surveyed include only those of the dynamic (temporal) behaviors, rather than spatial characteristics, of a target. While many of these models are also useful for target detection and recognition, this survey is only concerned with their values for target tracking. This of course does not prevent us from developing or applying a model that describes both the temporal evolution and spatial characteristics of a target.

Needless to say, target dynamic models and tracking algorithms have intimate ties. The applicability of a target dynamic model for a practical problem can hardly be evaluated without referring to the corresponding tracking algorithms used. In other words, some target models and tracking algorithms work well as a team. To be more focused and concise, however, this interdependence is largely ignored here.

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This survey emphasizes the underlying ideas and assumptions of the models. This should help the reader understand not only how these models work but also their pros and cons. It is hoped that a distinctive feature of this survey is that it reveals well the interrelationships among various models. However, the reader should keep in mind that much of such discussion is based on the authors' personal views and preferences, not always accurate or unbiased, although a great deal of effort has been made toward this goal. In addition to such discussions, certain material included in this survey has not appeared elsewhere.

Regrettably, many important issues associated with the target dynamic models, particularly those of implementation, cannot be discussed (at least to the desired degree) due to space limitation as well as the authors' background and experience. Nevertheless, the authors would appreciate it very much receiving comments and any missing material that should be covered in this survey.

The remaining part of the paper is organized as follows. Section gives briefly the state-space representation of target dynamics and the observation system. Section describes the nonmaneuver model. The presentation of the dynamic models of target maneuvers breaks down as follows: while Section deals with those that are uncoupled along different spatial coordinates, two- and three-dimensional coupled models are reviewed in Sections 4.12 and 5.4, respectively. The final section provides concluding remarks.

2 Mathematical Models for Maneuvering Target Tracking

The primary objective of target tracking is to estimate the state trajectories of a target — a moving object. Although a target is almost never really a point in the space and the information about its orientation is valuable for tracking, a target is usually treated as a point object without a shape in tracking, especially in target dynamic models. A target dynamic model or motion model describes the evolution of the target state x with respect to time.

Almost all maneuvering target tracking methods are model-based. They assume that the target motion and its observations can be represented by some known mathematical models accurately. The most commonly used such models are those known as state-space models, in the following form[1]

$$
\begin{aligned}
x_{k+1} &= f_k(x_k, u_k, w_k) && (1)\\
z_k &= h_k(x_k) + v_k && (2)
\end{aligned}
$$

where x_k, z_k, and u_k are the target state, observation, and control input vectors, respectively, at the discrete time t_k, which are indexed by k; w_k and v_k are process and measurement noise sequences, respectively; and f_k and h_k are some vector-valued time-varying functions. Such a *discrete-time* model can be (deemed to have been) obtained by discretizing the following *continuous-time* model [2]

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t), w(t), t), \qquad x(t_0) = x_0 && (3)\\
z(t) &= h(x(t), t) + v(t) && (4)
\end{aligned}
$$

where $x_k = x(t_k)$, $z_k = z(t_k)$, $v_k = v(t_k)$, $u_k = u(t_k)$, $h_k(x_k) = h(x(t_k), t_k)$. Note that

$$
w_k \neq w(t_k), \qquad f_k(x_k, u_k, w_k) \neq f(x(t_k), u(t_k), w(t_k), t_k)
$$

In fact, it is more appropriate to use the following *mixed-time* models for most tracking problems

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t), t) + w(t), \qquad x(t_0) = x_0 && (5)\\
z_k &= h_k(x_k) + v_k && (6)
\end{aligned}
$$

because while observations are usually available only at discrete time instants, the target motion is more accurately modeled in continuous time. For example, target motion behaviors should not depend on how and when samples are taken, which is the case, however, for a discrete-time model.

The continuous-, discrete-, and mixed-time linear counterparts of the above models are the corresponding pairs of the following equations

$$
x_{k+1} = F_k x_k + G_k^u u_k + G_k w_k \qquad (7)
$$

[1]Note that the symbol x is used to denote both the vector-valued state vector and the position along the x axis whenever no ambiguity arises from the context. Likewise for z.

$$\dot{x}(t) = A(t)x(t) + B^u(t)u(t) + B(t)w_c(t), \qquad x(t_0) = x_0 \tag{8}$$
$$z_k = H_k x_k + v_k \tag{9}$$
$$z(t) = C(t)x(t) + v(t) \tag{10}$$

The above state-space model consists of two parts: the target dynamics/motion model and its observation model, respectively. They are described below. In target tracking, the control input u is usually not known.

One of the two major challenges for target tracking arises from the target motion uncertainty. This uncertainty refers to the fact that an accurate dynamic model of the target being tracked is not available to the tracker. Specifically, although the general form of the model (5) is usually adequate, a track lacks knowledge/information about the actual control input u of the target, and possibly the actual form of f, its parameters, or statistical properties of the noise w for the particular target being tracked. Target motion modeling is thus one of the first tasks for maneuvering target tracking. It aims at developing a model that well accounts for the effect of target motion and is easily tractable.

In this paper, we describe the efforts and results in modeling the target motion for tracking a maneuvering target without knowing its true dynamic behaviors. These efforts have been made along two lines: (a) approximate the actually nonrandom control input u as a random process of certain properties, and (b) describe typical target trajectories by some representative motion models with properly designed parameters.

Target motions are normally classified into two classes of modes: maneuver and nonmaneuver. A *nonmaneuvering motion* is the *straight and level motion* at a constant velocity[2], sometimes also referred to as the *uniform motion*. Loosely speaking, all the other motions belong to the maneuvering mode.

3 Nonmaneuver Target Dynamic Models

It is well-known that a point moving in the three-dimensional physical world can be described by its three-dimensional position and velocity vectors. For instance, $x = [x, \dot{x}, y, \dot{y}, z, \dot{z}]'$ can be used as a state vector of such a point in the Cartesian coordinate system, where (x, y, z) are the position coordinates along x, y, and z axes, respectively, and $[\dot{x}, \dot{y}, \dot{z}]'$ are the velocity vector. When a target is treated as a point object, the nonmaneuvering motion is thus described by the vector-valued equation $\dot{x}(t) = 0$, where $x = [\dot{x}, \dot{y}, z]'$. In practice, this ideal equation is usually modified as $\dot{x}(t) = w(t) \approx 0$, where $w(t)$ is a white noise process, with a "small" effect on x, that accounts for unpredictable modeling errors due to turbulence, etc. The corresponding state-space model is given by, with state vector $x = [x, \dot{x}, y, \dot{y}, z]'$,

$$\dot{x}(t) = \text{diag}\,[A_{cv}, 0]\, x(t) + \text{diag}\,[B_{cv}, 1]\, w(t) \tag{11}$$

where $w(t) = [w_x(t), w_y(t), w_z(t)]'$ is a continuous-time vector-valued white noise process, and

$$A_{cv} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{cv} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} w(t) \tag{12}$$

The discrete-time counterpart of the above continuous-time model is[3]

$$x_{k+1} = \text{diag}\,[F_{cv}, 1]\, x_k + \text{diag}\,[G_{cv}, T]\, w_k = \text{diag}\,[F_2, F_2, 1]\, x_k + \text{diag}\,[G_2, G_2, T]\, w_k \tag{15}$$

where

$$F_{cv} = \text{diag}\,[F_2, F_2], \qquad G_{cv} = \text{diag}\,[G_2, G_2], \qquad F_2 = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \tag{16}$$

[2]Note that *velocity* is a vector and *speed* is its magnitude.

[3]It is directly from a discrete-time framework, rather than from discretizing the above continuous-time model, which would lead to [2]

$$x_{k+1} = \text{diag}\,[F_2, F_2, 1]\, x_k + w_k \tag{13}$$

where, with $E[w(t+\tau)w(t)'] = \text{diag}[q_x, q_y, q_z]\delta(\tau)$,

$$\text{cov}(w_k) = \text{diag}\left[\frac{q_x}{T}\tilde{Q}_2, \frac{q_y}{T}\tilde{Q}_2, \frac{q_z}{T}\right], \qquad \tilde{Q}_2 = \begin{bmatrix} T^4/3 & T^3/2 \\ T^3/2 & T^2 \end{bmatrix} \tag{14}$$

$w_k = [w_x, w_y, w_z]'_k$ is a discrete-time white noise sequence and T is the sampling interval. Note that w_x and w_y correspond to noisy "accelerations" along x and y axes, respectively, while w_z corresponds to noisy "velocity" along z axis. If w is uncoupled across its components, then the nonmaneuvering motion modeled by the above models is uncoupled across x, y, and z directions. In this case, the covariance of the noise term in (15) is given by

$$\text{cov}(Gw_k) = \text{diag}\left[\text{var}(w_x)Q_2, \text{var}(w_y)Q_2, \text{var}(w_z)\right], \quad Q_2 = \begin{bmatrix} T^4/4 & T^3/2 \\ T^3/2 & T^2 \end{bmatrix}$$

In a two-dimensional scenario where the altitude z is not considered, the above model takes the more popular form:

$$x_{k+1} = F_{cv} x_k + G_{cv} w_k \tag{17}$$

The above models (11) and (17) are known as the continuous- and discrete-time *constant-velocity (CV) models*, or more precisely "nearly-constant-velocity models," respectively. Note that the control input $u = 0$ in the nonmaneuvering models, although the actual thrust of the target has to be present to maintain the motion. Also, inclusion of any unnecessary component (e.g., acceleration) in the state vector would lead to a tracking performance deterioration.

4 Coordinate-Uncoupled Target Maneuver Models

Most target maneuvering motions are coupled across different coordinates. For simplicity, however, many maneuver models developed assume that this coordinate coupling is weak and can be neglected. This is particularly the case for those that model the actually nonrandom control input u as a random process. As a consequence, we need to consider only a generic coordinate direction.

The control input u for a target can usually be assumed to be an unknown *acceleration* in tracking. As u is unknown, a natural way is to assume that it can be modeled by a random process. These models proposed in the literature can be classified into three groups:

- White Noise Models: The control input is modeled as a white noise process. This includes CV, CA, and polynomial models.
- Markov Process Models: The control input is modeled as a Markov process. This includes the well-known Singer model, its various extensions, and some other models.
- Semi-Markov Jump Process Models: The control input is modeled as a semi-Markov jump process.

Let x, $\dot{x}$, and $\ddot{x}$ be the position, velocity, and acceleration along a generic direction, respectively. Specifically,

$$\ddot{x}(t) = a(t) \tag{18}$$

The models discussed in this subsection differ in how the function $a(t)$ is defined.

In this subsection, the state vector is always taken to be $x = [\text{position}, \text{velocity}, \text{acceleration}]'$, unless stated otherwise explicitly.

4.1 White-Noise Acceleration Model

The simplest model for a target maneuver is the so-called *white-noise acceleration model* [2]. It assumes that the target acceleration $\ddot{x}(t)$ is an independent (i.e., "white noise") process. It differs from the nonmaneuver model of Subsection only in the noise level: the white noise process w used to model the effect of the control input u has a much higher intensity than the one used in a nonmaneuvering model. A maneuver by its very nature aims at accomplishing certain task and thus is rarely independent with respect to time. The only attractive feature of this model is its simplicity. It is used when the maneuver is quite small or random. It is also used in the noise-level adjustment approach, discussed later.

4.2 Wiener-Process Acceleration Model

The second simplest model is the so-called *Wiener-process acceleration model* [2]. It assumes that the acceleration is a Wiener process, or more generally and precisely, the acceleration is a process with independent increments, which is not necessarily a Wiener process. It is also referred to simply as the *constant-acceleration (CA) model* or more precisely "nearly-constant-acceleration model."

This model has two commonly used versions. The first one, referred to as the *white-noise jerk model*, assumes that the *acceleration derivative* (i.e., the "jerk") $\dot{a}(t)$ is an independent (white noise) process $w(t)$: $\dot{a}(t) = w(t)$. The corresponding state-space representation is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t) \tag{19}$$

Its discrete-time equivalent is

$$x_{k+1} = F_3 x_k + w_k, \qquad F_3 = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

where, with $E[w(t+\tau)w(t)] = S_w \delta(\tau)$,

$$Q = \text{cov}(w_k) = S_w Q_3, \qquad Q_3 = \begin{bmatrix} T^5/20 & T^4/8 & T^3/6 \\ T^4/8 & T^3/3 & T^2/2 \\ T^3/6 & T^2/2 & T \end{bmatrix} \tag{21}$$

Note that S_w is the power spectral density, not the variance, of the continuous-time white noise.

The second version can be called *Wiener-sequence acceleration model*. It assumes that the *acceleration increment* is an independent (white noise) process. Note that an acceleration increment over a time period is the integral of the jerk over the period. This model is most conveniently expressed in discrete time, given by

$$x_{k+1} = F_3 x_k + G_3 w_k, \qquad G_3 = \begin{bmatrix} T^2/2 \\ T \\ 1 \end{bmatrix} \tag{22}$$

Note that its noise term has a covariance different from that of the white-noise jerk model:

$$Q = \text{cov}(G_3 w_k) = \text{var}(w_k) \begin{bmatrix} T^4/4 & T^3/2 & T^2/2 \\ T^3/2 & T^2/2 & T \\ T^2/2 & T & 1 \end{bmatrix} \tag{23}$$

The above models are simple but crude. Actual maneuvers seldom have (nearly) constant accelerations that are uncoupled across coordinate directions.

As explained before, a continuous-time model is more accurate than its discrete-time counterpart for most practical situations since a target moves continuously over time. The assumption of the discrete-time CA model (i.e., the second version above) that the acceleration increment $\Delta a_k = a_{k+1} - a_k = a(t_{k+1}) - a(t_k)$ is independent across different sampling intervals is hardly justifiable, except for its mathematical simplicity and tractability. Were this assumption true for a sampling period T, it would not be true in general for any other sampling period T' unless one is a multiple of the other: $T' = nT$ or $T = nT'$. Even if such periods exist, we would not be so lucky that one of them is used by chance.

4.3 General Polynomial Models

It is well-known that any continuous target trajectory can be approximated by a polynomial of a certain order to an arbitrary accuracy. As such, it is possible to model target motion by an nth-order polynomial [2] in the Cartesian coordinates:

$$x(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \\ c_0 & c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} + \begin{bmatrix} w_x(t) \\ w_y(t) \\ w_z(t) \end{bmatrix} \tag{24}$$

with different choices of the coefficients a_i, b_i, c_i, where (x, y, z) are the position coordinates and (w_x, w_y, w_z) are the corresponding noise terms. Such an nth-order polynomial model amounts to assuming the nth time derivative of the position is (nearly) constant (i.e., it derivative is equal to the noise w). The CV and CA models described in the previous sections are special cases (for $n = 1, 2$, respectively) of this general nth-order model with white noise $w(t)$. Note that an nth-order polynomial has $(n+1)$ parameters per coordinate and thus the dimension of the corresponding state vector is $n+1$. That is why an nth-order polynomial model is often call an $(n+1)$th-order model.

This model in its general setting is not very attractive for tracking for several reasons. Such models are usually good for fitting to a set of data, that is, for smoothing problem; however, the purpose of tracking is filtering and prediction, rather than fitting/smoothing. It is difficult to develop an uncomplicated and efficient method to determine the coefficients a_i, b_i, c_i in a general setting. Nevertheless, many special polynomial models have been developed for target tracking. In fact, most of the models discussed in this section can be viewed as special cases of this general polynomial model with different models for the noise $w(t)$.

4.4 Singer Acceleration Model — Zero-Mean First-Order Markov Model

In stochastic modeling, a random variable is used to represent an unknown time-invariant quantity, while an unknown time-varying quantity is modeled by a random process. As far as temporal properties are concerned, white noise constitutes the simplest class of random processes. The second simplest class is either the processes with independent increments, represented by the Wiener processes, or the so-called Markov processes, which include the Wiener processes and white noise as special cases.

A white noise process is "isolated" in time since its value at one time is uncoupled of any other time, while a Markov process is "local" in time because its value at one time depends on values at other time instants only through its nearest neighbors. Consequently, it is natural to consider the use of a Markov process model whenever white noise models are not good enough.

The *Singer model* [3] assumes that the target acceleration $a(t)$ is a zero-mean stationary first-order Markov process with autocorrelation $R_a(\tau) = E[a(t+\tau)a(t)] = \sigma^2 e^{-\alpha|\tau|}$, or equivalently, power spectrum $S_a(\omega) = \frac{2\alpha\sigma^2}{\omega^2+\alpha^2}$. Such a process $a(t)$ is the state process of a linear time-invariant system[4]

$$\dot{a}(t) = -\alpha a(t) + w(t) \tag{25}$$

where $w(t)$ is zero-mean white noise with constant power spectral density $S_w = 2\alpha\sigma^2$. Its discrete-time equivalent is

$$a_{k+1} = \beta a_k + w_k^a \tag{26}$$

where w_k^a is a zero-mean white noise sequence with variance $\sigma^2(1-\beta^2)$. It is related to the continuous-time model by $\beta = e^{-\alpha T}$ if it is obtained by the usual discretization (sample and hold). The state-space representation of the continuous-time Singer model is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t) \tag{27}$$

The corresponding discrete-time version by the usual discretization is

$$x_{k+1} = F_\alpha x_k + w_k = \begin{bmatrix} 1 & T & (\alpha T - 1 + e^{-\alpha T})/\alpha^2 \\ 0 & 1 & (1 - e^{-\alpha T})/\alpha \\ 0 & 0 & e^{-\alpha T} \end{bmatrix} x_k + w_k \tag{28}$$

The exact covariance of w_k is a function of α and T and can be found in [3, 2, 1].

The success of the Singer model relies on an accurate determination of the parameters α and σ^2 [4]. The parameter $\alpha = 1/\tau_m$ is the reciprocal of the maneuver time constant τ_m and thus depends on how many seconds the maneuver lasts. For example for an aircraft, $\tau_m \approx 60$ for a lazy turn and $\tau_m \approx 10-20$ for an evasive maneuver, as suggested by Singer [3]. Nothing has been proposed for the determination of α. The parameter σ^2 is the instantaneous variance of the acceleration, treated as a random variable. Singer [3] proposed to model the distribution of the acceleration by the following *ternary-uniform mixture*: The target may move without acceleration with probability P_0; accelerate or decelerate

[4]A Markov process with a rational power spectrum (as is the case here) is equivalent to the state of a linear time-invariant system excited by white noise: Every such process can be represented as the state of such a system and the state of such a system is a Markov process with a rational power spectrum.

at a maximum rate $\pm a_{\max}$ with equal probability $P_{\max}$; or accelerate or decelerate at a rate uniformly distributed over $(-a_{\max}, a_{\max})$. It turns out that

$$\sigma^2 = \frac{a_{\max}^2}{3}(1 + 4P_{\max} - P_0)$$

where $P_{\max}$, P_0, and $a_{\max}$ are design parameters. Note that this ternary-uniform mixture distribution of acceleration can obviously be used for other maneuver models.

It is clear from (27)–(28) that in the limit:

- As the maneuver time constant τ_m increases (i.e., αT decreases), the Singer model reduces to the nearly-constant-acceleration (CA) model [more precisely, the white-noise jerk model since cov(w_k) reduces to Q_3 of (21) instead of Q of (23)]. If the following direct discrete-time state-space model is set up based on (26)

$$x_{k+1} = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & \beta \end{bmatrix} x_k + G_3 w_k \tag{29}$$

  then its limit as τ_m increases would be the Wiener-sequence acceleration model. This relationship between the Singer and CA models make sense since the deterministic part of the acceleration in the Singer model becomes constant in the limit as τ_m increases.

- On the other hand, as the maneuver time constant τ_m decreases (i.e., αT increases), the Singer model reduces to the nearly-constant-velocity (CV) model. In this case, the acceleration becomes noise.

Consequently, for a choice of $0 < \alpha T < \infty$, the Singer model corresponds to a motion in between of constant velocity and constant acceleration. It should thus be clear that the Singer model has much wider coverage than the CV and CA models.

The Singer acceleration model is a standard model for target maneuvers. It was the first model that characterizes the unknown target acceleration as a time-correlated (i.e., colored) stochastic process, and has served as a basis for the further development of effective target maneuver models.

Many other models have been proposed (see, e.g., [5, 6, 7]), which are equivalent to or are simple variants of the Singer model.

The Singer model is in essence an a priori model since it does not use on-line information about the target maneuver, although it can be made adaptive through an adaptation of its design parameters α, T, $P_{\max}$, P_0, and $a_{\max}$. We cannot reasonably expect any a priori model to have a remarkable effectiveness for the diverse acceleration situations of actual target maneuvers. As a consequence of its a priori nature, the Singer model is also symmetric in that the assumed ternary-uniform mixture distribution of the acceleration is symmetric. One of the main shortcomings of the Singer model stems from this symmetry; that is, the target acceleration has zero mean at any moment. Indeed, this is almost the best one can do a priori without on-line information about the target maneuver. Otherwise towards where the mean should be? However, nothing really prevents us from using on-line information if we can withstand a little more sophistication. Several more sophisticated acceleration models have been proposed to remedy this shortcoming.

4.5 Mean-Adaptive Acceleration Model

An acceleration model, called the *"current" model*, proposed in [8], is in essence a Singer model with an adaptive, hence nonzero, mean; that is, a Singer model modified to have a nonzero mean of the acceleration: $a(t) = \tilde{a}(t) + \overline{a}(t)$, where $\tilde{a}(t)$ is the zero-mean Singer acceleration process, defined by (25), and $\overline{a}(t)$ is the mean of the acceleration, assumed constant over each sampling interval. Such a nonzero-mean acceleration satisfies

$$\dot{a}(t) = -\alpha\tilde{a}(t) + w(t) \qquad \text{or} \qquad \dot{a}(t) = -\alpha a(t) + \alpha\overline{a}(t) + w(t) \tag{30}$$

since $\dot{a}(t) = \dot{\tilde{a}}(t)$. The estimate $\hat{a}_k$ of a_k from all available on-line information (i.e., the sequence z^k of observations through time k) is taken to be the "current" value of the mean $\overline{a}_{k+1}$, hence, the name. It has the potential of being more effective than the Singer model.

The state-space representation of the "current" model is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} \overline{a}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t) \tag{31}$$

excluding the time instants at which samples are taken. The corresponding discrete-time version by the usual discretization is

$$x_{k+1} = F_\alpha x_k + \left(\begin{bmatrix} T^2/2 \\ T \\ 1 \end{bmatrix} - \begin{bmatrix} (\alpha T - 1 + e^{-\alpha T})/\alpha^2 \\ (1 - e^{-\alpha T})/\alpha \\ e^{-\alpha T} \end{bmatrix} \right) \overline{a}_k + w_k \tag{32}$$

where F_α was given in (28). These two equations differ from the Singer model only in the additional terms associated with $\overline{a}(t)$ and $\overline{a}_k$, respectively. For example, the noise $w(t)$ and w_k are identical to those in the Singer model. Note also that the "current" model corresponds to a Singer model with noise w of a nonzero mean.

A key assumption of the "current" model, as proposed in [8], is that $\overline{a}_{k+1} = \hat{a}_k$, or more specifically,

$$\overline{a}_{k+1} \triangleq E[a_{k+1}|z^k] = E[a_k|z^k] \triangleq \hat{a}_k$$

where z^k stands for all measurements through time t_k. This is questionable and can actually be avoided. Since the last equation of (32) reads

$$a_{k+1} = e^{-\alpha T} a_k + (1 - e^{-\alpha T})\overline{a}_k + w_k^a$$

the authors of this survey propose to improve the "current" model by replacing $\overline{a}_{k+1} = \hat{a}_k$with the following recursion

$$\overline{a}_{k+1} = E[a_{k+1}|z^k] = e^{-\alpha T} E[a_k|z^k] + (1 - e^{-\alpha T})\overline{a}_k = e^{-\alpha T}\hat{a}_k + (1 - e^{-\alpha T})\overline{a}_k \tag{33}$$

Such a relationship makes much better sense because $\overline{a}_{k+1}$ depends on the current information $\hat{a}_k$ as well as past information $\overline{a}_k$. However, what is resulting is no longer a purely *current* model.

In the "current" model, the a priori (unconditional) probability density $p(a_{k+1})$ of the acceleration a_{k+1} at time $k+1$ in the Singer model is replaced by a conditional density $p(a_{k+1}|\hat{a}_k)$. Clearly, this conditional density carries more accurate information than the a priori density and is better to be used than the a priori density, as explained before. The following *conditional* Rayleigh density was proposed in [8] for a_{k+1}:

$$p(a|\hat{a}_k) = \begin{cases} c_k^{-2}(a_{\max} - a)\exp\left[-(a_{\max} - a)^2/(2c_k^2)\right] \mathbb{1}(a_{\max} - a) & \hat{a}_k > 0 \\ c_k^{-2}(a - a_{-\max})\exp\left[-(a - a_{-\max})^2/(2c_k^2)\right] \mathbb{1}(a - a_{-\max}) & \hat{a}_k < 0 \end{cases}$$

where $\mathbb{1}(\cdot)$ is the unit step function; $a_{-\max}$ is the negative acceleration limit, not necessarily equal to $-a_{\max}$; and c_k is a $\hat{a}_k$-dependent parameter. Note that the ternary-uniform distribution assumption in the Singer model was made only out of the need to calculate the error variance of the acceleration. Similarly, this conditional Rayleigh assumption was made for the sole purpose of obtaining the variance of the acceleration prediction, which turns out to be

$$\sigma_k^2 \triangleq E[(a_{k+1} - \overline{a}_{k+1})^2|\hat{a}_k] = \begin{cases} \frac{4-\pi}{\pi}(a_{\max} - \hat{a}_k)^2 & \hat{a}_k > 0 \\ \frac{4-\pi}{\pi}(a_{-\max} + \hat{a}_k)^2 & \hat{a}_k < 0 \end{cases}$$

This result is valid only under the simplifying assumption that $E[a_{k+1}|z^k] = E[a_{k+1}|\hat{a}_k]$, or $\hat{a}_k$ is a sufficient statistics of z^k for a_{k+1}.

4.6 Second-Order Markov Acceleration Model

The Singer model is not very suitable for practical maneuvers in which target acceleration is oscillatory. The autocorrelation of such acceleration is better described by

$$R_a(\tau) = \sigma^2 e^{-\alpha|\tau|}\cos(\omega\tau) \tag{34}$$

where ω is the oscillation frequency of the maneuver. Such an acceleration process is the output of the following second-order system driven by white noise $w(t)$ with power spectral density $2\alpha\sigma^2$ and its time derivative $\dot{w}(t)$.

$$\begin{bmatrix} \dot{a} \\ \ddot{a} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\alpha^2 + \omega^2) & -2\alpha \end{bmatrix} \begin{bmatrix} a \\ \dot{a} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \sqrt{\alpha^2 + \omega^2} & 1 \end{bmatrix} \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix} \tag{35}$$

and the augmented state-space model is, for $x = [\text{position}, \text{velocity}, \text{acceleration}, \text{jerk}]'$,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(\alpha^2 + \omega^2) & -2\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sqrt{\alpha^2 + \omega^2} & 1 \end{bmatrix} \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix} \tag{36}$$

4.7 Markov Acceleration Model for Coordinated Turns

A typical target maneuver, such as a turn, usually has an approximately constant forward speed and turn rate. The original Singer model is not very good for such motion, known as the coordinated turn in aviation and target tracking. Let the acceleration a along the x and y directions of the Cartesian coordinates be a_x and a_y, respectively: $a = [a_x, a_y]'$. Denote by V the constant forward speed, $\phi(t)$ the heading angle, and $\omega = \dot{\phi}$ the constant turn rate. Under the assumption that both ϕ and ω as random variables have symmetric distributions and are mutually independent, it can be easily shown that the zero-mean processes $a_x(t)$ and $a_y(t)$ are uncoupled (uncorrelated) [9]. Note, however, that while the symmetry assumption is reasonable in practice for a priori models, the independence assumption is questionable since the turn rate ω and the heading ϕ are related by $\omega = \dot{\phi}$.

Only the model for $a_x(t)$ is considered below since the model for $a_y(t)$ can be similarly obtained. It can be easily shown from $a_x = \ddot{x} = -\omega\dot{y} = -\omega V \sin\phi$ and $\phi(t+\tau) = \phi(t) + \omega\tau$ that the autocorrelation of $a_x(t)$ is given by

$$R_{a_x}(t+\tau, t) = \frac{1}{2}V^2 E\{\omega^2[\cos(\omega\tau)(1-\cos 2\phi(t)) + \sin(\omega\tau)\sin 2\phi(t)]\}e^{-\alpha|\tau|} \tag{37}$$

assuming that the *magnitude* of the total acceleration $a(t)$ obeys the Singer model. Clearly, $a_x(t)$ is nonstationary because the second statistics $R_{a_x}(t+\tau, t)$ is time varying (since it depends on t as well as τ). This is the major difference between this model and the Singer model. Simply put, this model is a modified Singer model for a typical coordinated turn (i.e., with constant forward speed and turn rate). Its nonstationarity is a consequence of the coordinated-turn constraint. This model is clearly more accurate than the Singer model for a typical coordinated turn. The price paid is that the precise state-space form of this model is complicated. To have a time-invariant model of $a_x(t)$, it is necessary to consider only those distributions of ϕ for which $R_{a_x}(t+\tau, t) = R_{a_x}(\tau)$. The uniform distribution of ϕ over $[-\pi, \pi]$ is one of such distributions. To obtain a state-space model of $a_x(t)$, its power spectrum is derived, which is, however, not of a rational form and is dependent on the distribution of ω. In [9], it was assumed that the constant turn rate is uniformly distributed over a given interval $[-\omega_{\max}, \omega_{\max}]$ and the heading range is uniformly distributed over $[-\pi, \pi]$, and they are independent[5]. It was proposed in [9] to approximate the irrational power spectrum of $a_x(t)$ by an nth-order rational spectrum to obtain a simplified state-space model. This yields an *nth-order Markov model*. The rational approximation may be obtained numerically, as in [9]. For instance, if the power spectrum of $a_x(t)$ has the following second-order rational approximation

$$S(s) = H(s)H(-s) \qquad \text{with} \qquad H(s) = \frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2} \tag{38}$$

then the prewhitening system for $a_x(t) = [\beta_2, \beta_1]\,[a_x, \dot{a}_x]'$ is given by the causal and stable $H(s)$, or equivalently, in state-space form:

$$\begin{bmatrix} \dot{a}_x \\ \ddot{a}_x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha_2 & -\alpha_1 \end{bmatrix} \begin{bmatrix} a_x \\ \dot{a}_x \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) \tag{39}$$

where $w(t)$ is a white noise process with unity power spectral density. The corresponding augmented state-space model is, for $x = [p_x, v_x, a_x, \dot{a}_x]'$,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\alpha_2 & -\alpha_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w(t) \tag{40}$$

Note that the above uniform distribution assumption of the turn rate is better replaced by those of [10], the Single's ternary-uniform mixture or its variants (e.g., a binary-uniform mixture or a single-point and uniform mixture) in many practical situations with additional information of the turn rate. Note also the similarity and difference between the one in Subsection **4.6** and this second-order Markov model: The acceleration of this model is an autoregressive-moving average (ARMA) process of order $(2, 1)$ while the former is an ARMA$(2, 2)$ process.

4.8 Asymmetrically Distributed Normal Acceleration Model

Target acceleration can be decomposed along two directions: lift (normal to the target velocity and wing directions) and thrust or drag (along the velocity direction). Each component can be modeled by a time-correlated random process. The

[5] The assumptions made for this model are not consistent with each other. For example, $\phi(t+\tau) = \phi(t) + \omega\tau$ indicates that if $\phi(t) \sim U[-\pi, \pi], \omega \sim U[-\omega_{\max}, \omega_{\max}]$ and they are independent, then $\phi(t+\tau)$ has a trapezoidal distribution rather than a uniform distribution.

nonnormal component can be modeled well by the Singer model. However, this is often not the case for lift, which is usually the dominant one, especially during maneuvers. Its direction is determined by the target aspect angle and its magnitude can be modeled as a colored random process with an asymmetrical distribution. It was proposed in [11] that the normal acceleration $a_n(t)$ be modeled as an asymmetric and deterministic function of a zero-mean first-order Gauss-Markov process $b(t)$:

$$a_n(t) = \alpha + \beta e^{\gamma b(t)} \tag{41}$$

where α, β, γ are design parameters, depending on the particular type of target; $b(t)$ satisfies the equation of the Single-model type $\dot{b}(t) = -\frac{1}{\tau}b(t) + w(t)$, where τ is the correlation time constant of $b(t)$, which in general different from that of $a_n(t)$. Three typical choices of α, β, γ and the corresponding (highly asymmetrical) probability density functions were given in [11], including one that is considered typical of modern piloted aircraft in evasive maneuvers.

Both pros and cons of this model stem from the fact that the parameters α, β, γ are target-type specific. It is more accurate than the Singer model at the cost of designing these parameters, which requires knowledge of the target type, obtained either a priori or a posteriori. The shortcoming of the Singer model with a symmetric pdf is overcome by the use of additional information of target type in this model.

Note that the temporal correlation of the acceleration in this model does not have a rational power spectrum and is much more complicated than in the Singer model. It would be interesting to compare this model with a Singer model of the normal acceleration a_n having the initial asymmetric pdf $a_n(t_0) = \alpha + \beta e^{\gamma b(t_0)}$ with the same parameters α, β, γ as above.

4.9 Semi-Markov Jump Process Models

The Singer model approximates the target acceleration as a *continuous-time zero-mean* Markov process. In practice, many target maneuvers involve an acceleration of a *nonzero* mean that may be reasonably assumed piecewise constant. The difficulty lies in that neither the time intervals over which the acceleration mean is piecewise constant nor the corresponding constant levels of the nonzero mean are known to a tracker.

One of the simplest models for such a piecewise-constant random process, which belongs to the class of the so-called jump processes[6], is the so-called semi-Markov jump process. It differs from a Markov jump process only in that the time it stays in a mode (called *sojourn time*) is random (i.e., it has jumps at *random times*) [12], while a Markov jump process has jumps only at some deterministic time instants.

Several Markovian jump-mean acceleration models have been proposed. The first was the one given in [13, 14, 15, 16]. In this method, the unknown input $u(t)$ (assumed to be equal to the nonzero mean of the acceleration a) is modeled as a finite-mode semi-Markov jump process. Specifically, it was assumed that the possible *mean values* of the acceleration are quantized into n known levels $\overline{a}_1, \ldots, \overline{a}_n$, and the sequence $\langle u(t_k) \rangle$ of the input among these levels is a semi-Markov process (to be specific, a sojourn-time dependent Markov chain) with known transition probability $P\{u(t_k) = \overline{a}_j | u(t_{k-1}) = \overline{a}_i\}$, $i, j = 1, \ldots, n$, and sojourn-time probability distribution function $P_{ij}(\tau) = P\{\tau_{ij} \leq \tau\}$, where $\tau_{ij} = t_k - t_{k-1}$ is the sojourn time in mode $\overline{a}_i$ before it jumps to mode $\overline{a}_j$. Although such a semi-Markov jump process formulism was introduced by Moose et al., only nonrandom or exponentially distributed sojourn time was considered by them**.** Note that if the sojourn time is nonrandom or has an exponential distribution, the input sequence $\langle u(t_k) \rangle$ is in fact a Markov chain — the semi-Markov formulation is not needed. Moreover, they proposed the following model of the acceleration $a(t)$ as a combination of the above jump-mean model and the Singer model:

$$a(t) = -\beta v(t) + u(t) + \tilde{a}(t) \tag{42}$$

where $\tilde{a}(t)$ is the Singer acceleration of (25), v is the velocity, and β is a drag coefficient. The corresponding continuous-time state-space representation is, for $x =$ [position,velocity,acceleration]$'$,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\beta & 1 \\ 0 & 0 & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t) \tag{43}$$

In other words, the target acceleration is modeled as the Singer acceleration with *nonzero mean* that is a (semi-)Markov process. This model can be referred to as a *Markovian jump-mean acceleration model*. The unknown acceleration mean is estimated by a weighted sum of the quantization levels: $\hat{u}(t) = \sum_{i=1}^{n} \overline{a}_i P\{u(t) = \overline{a}_i | z(s), s \leq t\}$, where as in

[6] A jump process is one that has a staircase-type state trajectory.

a multiple-model formulation, the weight is the posterior probability of each level being the correct one, using all online measurement information $z(s), s \leq t$, as well as the initial mode probabilities, mode transition probabilities, and the sojourn time distribution. Note that a semi-Markov jump process model for the acceleration is more effective than a white noise model with randomly jumping mean.

Three important issues associated with this model are the design of the input quantization levels (i.e., the modes), the transition probabilities, and the sojourn time. This is similar to model-set design in the multiple-model method, discussed later. This should not be a surprise because this jump-mean model actually amounts to multiple models of the input (as quantized or partitioned) in a degenerated form.

The model proposed by Moose et al. for the unknown input u is a discrete-time, finite-mode (semi-)Markov process. A continuous-time counterpart was proposed in [17]. A sojourn-time dependent Markov chain model for target motion is proposed in [18] in the context of the multiple-model approach.

For simplicity, the sojourn time is oft modeled as having an exponential distribution[7] in the above semi-Markov formulism. The point (or counting) process associated with such a sojourn time is a Poisson process. Such a sojourn time τ has a monotonically decreasing probability density. This implies that very small τ would occur most frequently of all τ within a time interval of the same length. This is not consistent with the distribution of the durations of practical uniform motions as well as maneuvers, in particular for highly agile targets. To correct this deficiency, it was proposed in [19] that the sojourn time between any two consecutive mode transitions be assumed to be independent of any other sojourn time, and the sojourn time have a gamma distribution, which includes exponential as a special case. Such a model for the point process is known as a gamma-*renewal process*. The word "renewal" was coined with regard to the failure times of certain equipment (corresponding to the sojourn times in our case), and thus the value of the counting process models the number of renewals that must be made to keep the equipment in working order.

Although intuitively quite appealing, the *renewal model* is rather complicated since the maneuver process so modeled cannot be represented precisely as the state of a system driven by white noise due to its non-Markovian nature. Its predictable part can be approximated by a Markov process, as the output of a prewhitening system driven by white noise. The transfer function of this (usually first- or second-order) prewhitening system is obtained by spectral factorization of the power spectra of the Markov process matched to the equilibrium power spectrum of the semi-Markov maneuver process. This leads to a Markovian approximation of the renewal model. Probably more importantly, simulation results of an application of the renewal model to an agile target that executes coordinated turns indicates that the performance improvement is not commensurate with the model and algorithm complexity [19], but is significant for an image-enhanced tracking system based on the fusion of, e.g., a microwave radar and an infrared imaging sensor [20].

The renewal model was proposed in [19] for the turn rate per se. It can clearly be applied for target acceleration, its mean value, or something else. In fact, following the idea of Moose et al. for acceleration process, the turn rate can be modeled as the sum of its mean that obeys the above renewal model and a zero-mean first-order Markov process that obeys Singer model. It would be interesting to compare its applicability with that of the original renewal model.

4.10 Acceleration Models Versus Jerk Models

Jerk is the derivative of acceleration. In most coordinate-uncoupled models, it is the target acceleration that is chosen to be the descriptor of a target maneuver and the acceleration is modeled as a random process. This is most natural from mechanics, kinematics, and vehicle dynamics since acceleration is directly related to the force acting on the target. However, for some targets, particularly agile targets, it is more convenient to use a random jerk process to model the target maneuvers. Although an acceleration model can always be obtained from a jerk model by integration, a jerk model differs from an acceleration model usually in simplicity, depending on whether the target motion is better described (usually in terms of simplicity) by a random process model of the jerk or the acceleration.

4.11 First-Order Markov Jerk Model

The first-order Markov model as proposed by Singer is for target acceleration. However, the same modeling method can be applied to other target functions, e.g., target jerk. This method was indeed applied in [21, 22] to the jerk process as the maneuver forcing function; that is, the jerk is modeled as a zero-mean first-order Markov process, exactly the same as the Singer model for acceleration. The derivation is straightforward in a completely analogous manner as that of the Singer model. The resulting state-space model has four dimensions (position, velocity, acceleration, and jerk) per coordinate and is

[7] A semi-Markov process with exponentially distributed sojourn time in each mode is actually a Markov process. This is a consequence of the unique memoryless property of the exponential distribution. Similarly, a Markov sequence is semi-Markov only if the sojourn time has a geometric distribution.

supposed to be potentially capable of describing agile target maneuvers more accurately than the Singer acceleration model. A tracker using this model has a higher dimension and would be more responsive than using the Singer acceleration model, possibly at the cost that the resulting tracks are more noisy.

4.12 Nonzero-Mean Jerk Model

Note first that although the Singer model is usually regarded as a model for the target acceleration, it can also be interpreted as a zero-mean jerk model in which the target jerk $\dot{\tilde{a}}(t)$ is defined by $\dot{\tilde{a}}(t) = -\alpha\tilde{a}(t) + w(t)$ with $\tilde{a}(0) = 0$, which is a zero-mean process. Berg proposed in [23] a nonzero-mean jerk model by introducing an additional term in the Singer-model equation

$$\dot{a}(t) = -\alpha a(t) + w(t) + \overline{\dot{a}} \tag{44}$$

where $\overline{\dot{a}}$ is a nonzero expected jerk. It turns out that this is not equivalent to assuming $\dot{a}(t) = \dot{\tilde{a}}(t) + \overline{\dot{a}}$ directly, where $\dot{\tilde{a}}(t)$ is the zero-mean Singer jerk process.

This model and the "current" model are the same in spirit — they aim at improving the Singer model by adding a nonzero-mean term to equations that describe the acceleration. As is clear from a comparison of (31) and (44), they would be equivalent if $\overline{\dot{a}} = \alpha\overline{a}(t)$, where $\overline{a}(t)$ is the assumed piecewise constant mean of the acceleration. Consequently, their difference lies mainly in how mean jerk $\overline{\dot{a}}$ and mean acceleration $\overline{a}$ are obtained.

Berg proposed [23] to determine the mean jerk $\overline{\dot{a}}$ adaptively from the most recent estimates of the target velocity and acceleration under the assumption of a *coordinated-turn* maneuver. Such a maneuver has a zero-mean target lift rate, thrust rate, and roll rate. This will be discussed in detail later. The noise level of w was also proposed to be determined adaptively from the most recent estimate of target orientation and expected maneuver level.

The above model was slightly refined in [24] for the coordinated-turn maneuvers by removing the acceleration mean as

$$\dot{a}(t) = -\alpha[a(t) - \overline{a}(t)] + w(t) + \overline{\dot{a}} \tag{45}$$

On the other hand, since mean jerk is equal to the derivative of the expected acceleration, this equation also follows from taking derivative on both sides of $a(t) = \tilde{a}(t) + \overline{a}(t)$, where $\tilde{a}(t)$ is the Singer acceleration. Thus, this model differs from the "current" model only in that the expected acceleration $\overline{a}(t)$ here is not assumed constant over each sampling interval.

5 2D Horizontal Motion Models

Most two- and three-dimensional target maneuver models are naturally turn motion models, in particular, the so-called coordinated turn models. These models are usually established relying on target *kinematics*, in contrast to those of the previous section that are based on random processes. This difference is easily understandable since random processes are more natural for modeling time correlation than spatial coupling where kinematics is a more appropriate tool.

Two-dimensional horizontal motion models are described in this section generally in an order from the simplest to the most sophisticated. Three-dimensional motion models are described in the next section.

Coordinate-coupled target models are highly dependent on the choice of the state components. To describe a planar motion, the state vector is at least four-dimensional. The choice of the state components (and implicitly the respective kinematic model) is a not trivial problem [25] in which consideration needs to be given to target dynamics (usually modeled as a trajectory with random perturbation), the accuracy of unavoidable approximations, sensor coordinate system, among others.

Various (noiseless) kinematic models proposed for tracking of a target moving in the horizontal plane can be comprised from the following *standard curvilinear-motion model* from kinematics:

$$\dot{x}(t) = V(t)\cos\phi(t) \tag{46}$$
$$\dot{y}(t) = V(t)\sin\phi(t) \tag{47}$$
$$\dot{V}(t) = a_t(t) \tag{48}$$
$$\dot{\phi}(t) = \frac{a_n(t)}{V(t)} \tag{49}$$

where $(x, y), V, \phi$ are the target position in Cartesian coordinates, speed, and heading angle, and a_t, a_n denote the target tangential (along-track) and normal (cross-track) accelerations in the horizontal plane, respectively. This is generic — it accounts for along- and cross-track input accelerations, and trivially reduces to the following special cases:

1. $a_n = 0, a_t = 0$ — rectilinear, constant velocity motion;
2. $a_n = 0, a_t \neq 0$ — rectilinear, accelerated motion (CA motion if $a_t =$ constant);
3. $a_n \neq 0, a_t = 0$ — circular, constant speed motion (CT motion if $a_n =$ constant).

The last case above is known as a (standard) *coordinated turn* (CT), which has a constant speed and constant turn rate, and in this case the target motion is preferably specified in terms of the turn rate $\omega = \dot{\phi}$.

5.1 CT Model with Known Turn Rate

This model presumes that the target moves with (nearly) *constant speed* V and (nearly) *constant angular (turn) rate* ω. Assuming ω is **known** leads to (only four-dimensional) state vector, e.g., $x = (x, \dot{x}, y, \dot{y})'$, in the Cartesian coordinates. It follows immediately from (46)–(49) that such a circular motion is described by equation (5) with $f(x) = (\dot{x}, -\omega\dot{y}, \dot{y}, \omega\dot{x})'$; that is,

$$\dot{x}(t) = (\dot{x}, -\omega\dot{y}, \dot{y}, \omega\dot{x})'(t) + w(t) = A(\omega)x + w(t), \quad A(\omega) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & 0 & 1 \\ 0 & \omega & 0 & 0 \end{bmatrix} \tag{50}$$

This CT model is linear since ω is known. By the usual discretization, its discrete-time version is shown to be (see [2], p.187 or [1], p.206):

$$x_{k+1} = F_{ct}(\omega)x_k + w_k = \begin{bmatrix} 1 & \frac{\sin\omega T}{\omega} & 0 & -\frac{1-\cos\omega T}{\omega} \\ 0 & \cos\omega T & 0 & -\sin\omega T \\ 0 & \frac{1-\cos\omega T}{\omega} & 1 & \frac{\sin\omega T}{\omega} \\ 0 & \sin\omega T & 0 & \cos\omega T \end{bmatrix} x_k + w_k \tag{51}$$

It appears that this model was first introduced to the tracking community in [26, 27]. The zero-mean (Gaussian) white noise w in the above is used to model the perturbation of the trajectory from the ideal CT motion.

An approximation[8] of the above model (51) is [28, 29]:

$$F_{ct}(\omega)x_k \approx \begin{bmatrix} 1 & T & 0 & -\omega T^2/2 \\ 0 & 1-(\omega T)^2/2 & 0 & -\omega T \\ 0 & \omega T^2/2 & 1 & T \\ 0 & \omega T & 0 & 1-(\omega T)^2/2 \end{bmatrix} x_k = \begin{bmatrix} x + T\dot{x} - (1/2)T^2\dot{y}\omega \\ \dot{x} - T\dot{y}\omega - (1/2)T^2\dot{x}\omega^2 \\ y + T\dot{y} + (1/2)T^2\dot{x}\omega \\ \dot{y} + T\dot{x}\omega - (1/2)T^2\dot{y}\omega^2 \end{bmatrix} \tag{52}$$

which is a second-order polynomial in ω. It provides a simple but less accurate alternative to the exact CT model. It is of certain value when a nonlinear tracker (e.g. EKF) is designed with state vector that includes the (unknown) turn rate. However, it is valid only if $\omega T \approx 0$, which may be violated in many cases with large T.

In the rare cases where the constant turn rate is (approximately) known a priori, the above CT model gives good tracking performance. The necessity of an exact knowledge about the value of the turn rate makes the direct use of this model unrealistic for most practical applications. A natural idea is to replace the above ω by its ad hoc estimate, based on e.g., the latest velocity estimates, as used in [30] and [31, 32]. However, this may inject unacceptably large nonlinearities into the system. Additional efforts are obviously requisite to model the motion with an unknown turn-rate within this framework.

Multiple turn-rate models Another natural solution is based on the use of multiple models with different, fixed turn rates. This approach alleviates the effect of the uncertainty in the turn rate and takes advantage of the simple and linear form of the dynamic model (63) given the turn rate. In this approach, the sequence of turn rates $\{\omega_k\}$ is modeled as a Markov chain (or a semi-Markov sequence) taking values in the set $\{\omega_i, i = 1, \ldots, n\}$, governed by the transitional probabilities $P\{\omega_k = \omega_i | \omega_{k-1} = \omega_j\}, i, j = 1, \ldots, n$, as well as initial probabilities. This approach has been well established (see, e.g., [31], [32], and [28] or [29]), mostly for civil ATC tracking application. Therefore, a main application of this CT model with known turn rate is serving as one or more (with different ω) submodels in a multiple-model architecture.

[8] By expanding sin and cos up to the second-order terms: $\sin\xi \approx \xi$ and $\cos\xi \approx 1 - \xi^2/2$ for $\xi \approx 0$.

5.2 CT Model with Unknown Turn Rate

This model differs from the above CT model only in that the turn rate is included as a state component, to be estimated. As such, this model is described by (50) in continuous-time, or (51) in discrete-time, plus an additional equation for ω. The two most popular models for ω are the *Wiener process model*

$$\begin{aligned} \dot{\omega}(t) &= w_\omega(t), \quad \text{in continuous-time} & (53)\\ \omega_{k+1} &= \omega_k + w_{\omega,k}, \quad \text{in discrete-time} & (54)\end{aligned}$$

and the *first-order Markov process model*

$$\begin{aligned} \dot{\omega}(t) &= -\frac{1}{\tau_\omega}\omega + w_\omega(t), \quad \text{in continuous-time} & (55)\\ \omega_{k+1} &= e^{-T/\tau_\omega}\omega_k + w_{\omega,k}, \quad \text{in discrete-time} & (56)\end{aligned}$$

where τ_ω is the correlation time constant for the turn rate, and w is zero-mean white noise of a suitable noise level, which can be determined exactly as for the corresponding models for acceleration described in the previous section. Some models, as described in the previous section (for acceleration), can also be used for the turn rate. For example, the renewal model of [19] is one of them.

Discretization of a continuous-time model has a unique issue: which discrete-time ω should be used in $F_{ct}(\omega)$ of (51)? The standard way is to use ω_k here. Alternatively, it was proposed in [33] to use instead ω_{k+1} in $F_{ct}(\omega)$. Their difference is similar in spirit to that of approximating a derivative by the forward difference and backward difference in the finite difference method. With this analogy in mind, the authors of this survey believe that replacing ω in (51) by $\bar{\omega} = \frac{1}{2}(\omega_k + \omega_{k+1})$ would be superior to the above two schemes since the center point is usually a better approximation than either of the end points, just like the central difference is superior to the forward difference and backward difference at the cost of more computation. For the current problem, however, if the procedure of solving the linearized equation as proposed in [33] is used, no extra computation is required by the new scheme. These three schemes lead to the following three different linearized models by the first-order Taylor series expansions at $[x,\omega]' = [\hat{x}_{k|k}, \hat{\omega}_{k|k}]'$:

$$F_{ct}(\omega_k)x_k = F(\hat{\omega}_{k|k})x_k + F_\omega(\hat{\omega}_{k|k})\hat{\eta}_{k|k}(\omega_k - \hat{\omega}_{k|k}) \tag{57}$$

$$F_{ct}(\omega_{k+1})x_k = F(\hat{\omega}_{k|k})x_k + F_\omega(\hat{\omega}_{k|k})\hat{x}_{k|k}(\omega_{k+1} - \hat{\omega}_{k|k}) \tag{58}$$

$$F(\bar{\omega})x_k = F(\hat{\omega}_{k|k})x_k + F_\omega(\hat{\omega}_{k|k})\hat{x}_{k|k}(\bar{\omega} - \hat{\omega}_{k|k}) \tag{59}$$

where $[x',\omega]' = [x,\dot{x},y,\dot{y},\omega]'$, $F_\omega(\hat{\omega}_{k|k}) = \frac{\partial}{\partial\omega}F_{ct}(\omega)|_{\omega=\hat{\omega}_{k|k}}$. If (54) is used, however, these models have an identical state prediction because

$$E[x_{k+1}|z^k] = E[F_{ct}(\omega)x_k|z^k] = F(\hat{\omega}_{k|k})\hat{x}_{k|k} \tag{60}$$

where z^k stands for all measurements through time t_k. This follows from $E[\bar{\omega}|z^k] = E[\omega_{k+1}|z^k] = E[\omega_k|z^k] = \hat{\omega}_{k|k}$, a consequence of (54). Nevertheless, the corresponding covariances differ because different models are used. That is why these models result in very similar tracking performance, as shown in [34] for two IMM-EKF algorithms using the first two schemes of the above, respectively, with real ATC data. Obviously, the state prediction would also be different if model (56) is used.

An alternative suggested here is to obtain a discrete-time model for $[x',\omega]' = [x,\dot{x},y,\dot{y},\omega]'$ by discretizing the continuous-time equation for ω above and the nonlinear equation (50) jointly:

$$\begin{aligned} \dot{x}(t) &= (\dot{x}, -\omega\dot{y}, \dot{y}, \omega\dot{x})'(t) + w(t) & (61)\\ \dot{\omega}(t) &= -\alpha\omega + w_\omega(t) & (62)\end{aligned}$$

where $\alpha = 0$ or $1/\tau_\omega$. It is reasonable to expect an improvement in performance with this alternative.

As stated before, the choice of the state vector is not trivial for the turn models. Essentially two classes have been proposed. They differ in the representation of the velocity vector: in Cartesian and polar coordinates, respectively.

5.2.1 CT Models with Cartesian Velocity

In this model, the state vector is chosen to be $x = [x,\dot{x},y,\dot{y},\omega]'$; that is, the velocity vector $(\dot{x},\dot{y})'$ is represented in the Cartesian coordinates. Consequently, the discrete-time version of the model is given by (see [2], [31]):

$$x_{k+1} = \begin{bmatrix} F_{ct}(\omega^*) & 0 \\ 0 & \beta \end{bmatrix} x_k + \text{diag}[G_2, G_2, 1]w_k \tag{63}$$

where $\beta = 1$ or e^{-T/τ_ω}, $\omega^* = \omega_k, \omega_{k+1}, \bar{\omega}$, or something similar, G_2 was given in (16), $w_k = [w_x, w_y, w_\omega]'_k$ is zero-mean Gaussian white noise with suitable statistics — they are noise terms for acceleration in x and y directions and for turn rate. In [30], [1] the random turn-rate noise w_ω is coupled with the acceleration noise w_x, w_y through a common process noise transition matrix.

This model is known to be used successfully as a submodel in numerous multiple-model configurations (see, e.g., [35],[31], [2], [36], [32]).

A continuous-time model is more accurate than its discrete-time counterparts, but the latter is needed for most applications. In the case where the continuous-time model is nonlinear, such as the CT models with unknown turn rate, there are in general two approaches to acquiring its discrete-time *linear* approximate models. Such models are needed in the application of a linear filter (e.g., Kalman filter) based nonlinear filtering technique. The first is to linearize the nonlinear equation for the state first and then discretize the linearized differential equation (by integration). This approach is more commonly used because it is easy. An alternative approach is to discretize the nonlinear state-space equation first and then linearize the discrete-time model. These two approaches are referred to as *discretized linearization* and *linearized discretization*, respectively, in [25]. The second approach seems more accurate in general than the first since linearization usually lose more information than discretization and thus should be done later. However, the second approach does not work in general because discretization requires solving (by integration) the nonlinear differential equation, which is often a great challenge. Fortunately, for the CT motion with an unknown turn rate, the corresponding differential equations for the state are simple and the solution can be readily obtained. The equations of the nearly CT model, obtained by the first approach, and some performance comparison results are also given in [25].

5.2.2 CT Models with Polar Velocity

Obviously, the velocity vector can also be represented in the polar coordinates as $(V,\ \phi)'$ through speed $V = \sqrt{\dot{x}^2 + \dot{y}^2}$ and heading angle $\phi = \tan^{-1}(y/x)$. The state vector is thus $x = (x,\ y,\ V,\ \phi,\ \omega)'$. The corresponding differential equation is given by (5) with $f(x) = (V\cos\phi,\ V\sin\phi,\ 0,\ \omega,\ 0)'$; that is:

$$\dot{x}(t) = [V\cos\phi, V\sin\phi, 0, \omega, 0]' + w(t) \tag{64}$$

which follows immediately from (46)–(49) by setting ω = constant, V = constant. By linearization first and then discretization, its discrete-time linearized model is given in [37] (see also [1]):

$$x_{k+1} = \begin{bmatrix} x + \frac{2}{\omega}V\sin\frac{\omega T}{2}\cos(\phi + \frac{\omega T}{2}) \\ y + \frac{2}{\omega}V\sin\frac{\omega T}{2}\sin(\phi + \frac{\omega T}{2}) \\ V \\ \phi + \omega T \\ \omega \end{bmatrix}_k + w_k \tag{65}$$

where w_k is a white Gaussian noise sequence with the covariance Q:

$$Q = \text{diag}\left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, T^2\sigma_{\dot{V}}^2, \begin{bmatrix} T^3\sigma_{\dot{\omega}}^2/3 & T^2\sigma_{\dot{\omega}}^2/2 \\ T^2\sigma_{\dot{\omega}}^2/2 & T^2\sigma_{\dot{\omega}}^2 \end{bmatrix}\right] \tag{66}$$

This model was successfully used as a building block of a multiple-model algorithm for aircraft tracking application in an air defense system [38], [39].

The above model uses (53) for the turn rate per se. Instead, it may be better to use (53). The corresponding discrete-time model need be modified accordingly.

The discretized linearization (i.e. linearize first and then discretization) alternative of this model can be found in [25], along with a comparison of its performance with that of the linearized discretization based on a theoretical error analysis[9] and Monte Carlo simulations. It concluded that whenever possible linearized discretization is preferable to discretized linearization. For the two CT models with polar velocity, the former slightly outperforms the latter. It is also claimed that the CT model with polar velocity outperforms the CT model with Cartesian velocity.

The use of the following variant of the above CT model with polar velocity was reported in [40]. Let the state vector be $x = (x,\ y,\ V,\ \phi,\ a_n)'$; that is, replace the turn rate ω in the above model with the normal (transversal) acceleration a_n. Then, it follows immediately from (46)–(49) that the continuous-time state-space model for the CT motion is given by

[9]Based on estimation of the Frobenius norm of the neglected terms in the approximations.

(5) with $f(x) = (V \cos\phi,\ V \sin\phi,\ 0,\ a_n/V,\ 0)'$. A discrete-time approximation of this model can be obtained based on a refined Euler-Cauchy scheme as:

$$x_{k+1} = x_k + Tf\left(x_k + \frac{1}{2}Tf(x_k)\right) + w_k \tag{67}$$

where w_k is white noise, obtained by a backward difference of (the derivative of) the continuous-time white noise. Note that $x_{k+\frac{1}{2}} \approx \frac{1}{2}Tf(x_k)$. This model, followed by the standard EKF linearization, was included in an MM tracker for air traffic control to account for possible target horizontal turns [40].

5.3 Maneuver-Centered Circular Motion Models

For a circular motion of a target, if its center were known, the simplest model would be to represent the circle in the polar coordinates and place the origin at the circle center. In this coordinate system, the target state vector is $x = (\rho,\ \theta,\ \dot{\theta})'$ and the target dynamic model is linear

$$x_{k+1} = \text{diag}[1, F_2]x_k + \text{diag}[1, G_2/T]w_k \tag{68}$$

where F_2 and G_2 were given by (16), and w_k is Gaussian white noise. The corresponding measurement equation is a pseudo-linear one because the noise covariance is actually state dependent. As a result, the Kalman filter is the optimal state estimator and can be implemented straightforwardly. This *maneuver-centered* CT model was first introduced in [41]. While the idea underlying this model is intuitively appealing, the inherent nonlinearity of the problem is not avoided. It obviously relies on an accurate determination of the center of the turn in terms of the sensor coordinate system, which is inherently a nonlinear estimation problem. The following simple geometrically oriented procedure of estimating the center was proposed in [41]: Assume that each target position measurements are points on the circle; replace the chord between any two consecutive measurement points with the straight line segment connecting them; the center can then be determined from the (average) intersection of the perpendicular bisectors of two or more such straight line segments. An essentially the same procedure was used in [42] for estimating the center. Note that using the center estimates injects additional *unmodeled nonlinearities* into to the system, which are not accounted for in the above linear model.

5.4 Curvilinear Motion Model

This model, proposed in [43] recently, is more general. It accounts for possibly nonzero normal (cross-track) and tangential (along-track) target maneuver accelerations simultaneously. For the Cartesian state vector $x = (x, \dot{x}, y, \dot{y})'$, it follows from the standard equations of curvilinear motion (46)–(49)[10] that this model in continuous-time is given by

$$\dot{x}(t) = A_{cv}x(t) + B(x(t))a(t) + w(t) \tag{69}$$

where $a = (a_t, a_n)'$ is the maneuver induced acceleration, A_{cv} was given by (16), and

$$B(x(t)) = \begin{bmatrix} 0 & 0 \\ \frac{\dot{x}(t)}{\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}} & -\frac{\dot{y}(t)}{\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}} \\ 0 & 0 \\ \frac{\dot{y}(t)}{\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}} & \frac{\dot{x}(t)}{\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}} \end{bmatrix} \tag{70}$$

Its usual discrete-time equivalent model in the form

$$x_{k+1} = F_{cv}x_k + G_k(x)a_k + w_k \tag{71}$$

is highly nonlinear because of its dependence on the target state via matrix B:

$$G_k(x) = \int_0^T e^{A(T-\tau)}B(x(kT+\tau))d\tau \tag{72}$$

[10] The heading angle ϕ defined in [43] differs from the traditional one, which is adopted in this paper.

and the integration involved in $G_k(x)$ is hard to evaluate exactly. It was shown there that $G_k(x)$ can be found approximately to be

$$G_k(x) \approx G_a(\phi_k,\omega_k) = \begin{bmatrix} G_{a_t}(\phi_k,\omega_k), \begin{bmatrix} \frac{1}{\omega_k^2}\sin\varphi_{k+1} - \frac{1}{\omega_k^2}\sin\phi_k - \frac{1}{\omega_k}T\cos\phi_k \\ \frac{1}{\omega_k}\cos\varphi_{k+1} - \frac{1}{\omega_k}\cos\phi_k \\ -\frac{1}{\omega_k^2}\cos\varphi_{k+1} + \frac{1}{\omega_k^2}\cos\phi_k - \frac{1}{\omega_k}T\sin\phi_k \\ \frac{1}{\omega_k}\sin\varphi_{k+1} - \frac{1}{\omega_k}\sin\phi_k \end{bmatrix} \end{bmatrix} \tag{73}$$

$$G_{a_t}(\phi_k,\omega_k) = \begin{bmatrix} -\frac{1}{\omega_k^2}\cos\varphi_{k+1} + \frac{1}{\omega_k^2}\cos\phi_k - \frac{1}{\omega_k}T\sin\phi_k \\ \frac{1}{\omega_k}\sin\varphi_{k+1} - \frac{1}{\omega_k}\sin\phi_k \\ -\frac{1}{\omega_k^2}\sin\varphi_{k+1} + \frac{1}{\omega_k^2}\sin\phi_k + \frac{1}{\omega_k}T\cos\phi_k \\ -\frac{1}{\omega_k}\cos\varphi_{k+1} + \frac{1}{\omega_k}\cos\phi_k \end{bmatrix} \tag{74}$$

where $\varphi_{k+1} \triangleq \phi_k + \omega_k T$, under the following simplifying assumptions: (a) the acceleration a is piecewise constant over each sampling interval $[kT,\, kT+T)$; and (b) the speed change over a sampling interval is much smaller than the speed itself: $a_{t_k}T \ll V_k$. An analytically equivalent form[11] of (71) is, with a_t being the only explicit forcing term,

$$x_{k+1} = F_{ct}(\omega_k)x_k + G_{a_t}(\phi_k,\omega_k)a_{t_k} + w_k \tag{75}$$

Note that while in (71) the maneuver acceleration term $G_a(\phi_k,\omega_k)a_k$ is "added" to CV motion, in (75) the effect of the tangential acceleration a_t (i.e., the term $G_{a_t}(\phi_k,\omega_k)a_{t_k}$) is "added" to the CT (constant-speed constant turn-rate) motion. (75) makes it clear the capability of this model to account for "relatively small" tangential accelerations as well as the normal accelerations, while the popular CT model accounts only for the latter. Both forms are applicable to tracking targets performing maneuvers with concurrent nonzero along- and cross-track accelerations, but attention should be paid to the approximating assumptions of the models.

This model, combined with a suitable model for turn rate, is one of the most sophisticated target maneuver models for 2D horizontal motions.

6 3D Motion Models

Many of the 2D horizontal models reviewed above have been considered for application to 3D tracking of civilian aircraft in the ATC systems. Such targets maneuver mostly in a horizontal plane, with nearly constant speed and turn rate and the above 2D models can give satisfactory tracking performances. On the other hand, the vertical maneuvers of civilian aircraft are limited, usually performed not at the time of a horizontal turn. Thus, the altitude changes are most often modeled independently by a CV model along z direction corrupted by white noise, with an acceptable practical accuracy. However, when the task is to track agile military aircraft, capable of performing "high-g" turns (e.g. for tracking in air defense systems), decoupled models may be completely inadequate. Many efforts have been devoted to solving this challenging problem, and fairly satisfactory results have been achieved. The most important of these are 3D turn models, surveyed next.

6.1 Constant Turn-Rate Models

6.1.1 Fixed-Center Constant Turn-Rate Models

Denote by $\mathbf{p}$, $\mathbf{v}$, and $\mathbf{a}$ the target position, velocity, and acceleration vectors, respectively, in an inertial coordinate system. A general *rotational* motion of a particle around a *fixed* center $\mathbf{p}_0$ is described by the following vector equation of a rigid body motion [44]:

$$\mathbf{v} = \mathbf{\Omega} \times (\mathbf{p} - \mathbf{p}_0) \tag{76}$$

where $\mathbf{\Omega}$ denotes the angular velocity vector and $\times$ denotes the vector (cross) product operation. Assuming the angular velocity is constant: $\dot{\mathbf{\Omega}} = \mathbf{0}$, and differentiating[12] (76) yields the basic kinematic equation for the 3D fixed-center constant angular-velocity (CAV) model:

$$\mathbf{a} = \mathbf{\Omega} \times \mathbf{v} \tag{77}$$

[11] It can be obtained by using $\dot{x}_k = \frac{a_n}{\omega_k}\cos\phi_k$ and $\dot{y}_k = \frac{a_n}{\omega_k}\cos\phi_k$.

[12] $\mathbf{a} = \dot{\mathbf{v}} = \dot{\mathbf{\Omega}} \times (\mathbf{p} - \mathbf{p}_0) + \mathbf{\Omega} \times \dot{\mathbf{p}} - \mathbf{\Omega} \times \dot{\mathbf{p}}_0 = \mathbf{\Omega} \times \mathbf{v}$ since $\dot{\mathbf{\Omega}} = \mathbf{0}$ and $\dot{\mathbf{p}}_0 = \mathbf{0}$.

From (77), the angular velocity can be expressed in terms of the velocity and acceleration vector[13]:

$$\mathbf{\Omega} = \frac{\mathbf{v} \times \mathbf{a}}{\mathbf{v}^2} \tag{78}$$

where $\mathbf{v}^2 \triangleq \mathbf{v} \cdot \mathbf{v}$. (78) can be referred to as the basic equation for 3D turn motion because it is the basis of most 3D turn models. It is more general than (77), although it is derived here from (77).

The CAV model (77) can be alternatively represented[14] in terms of the turn rate ω as: $\dot{\mathbf{a}} = -\omega^2 \mathbf{v}$, where

$$\omega \triangleq \|\mathbf{\Omega}\| = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\mathbf{v}^2} \tag{79}$$

Thus, the 3D fixed-center *constant turn-rate* (CTR) maneuver model takes a form of a second-order Markov process

$$\dot{\mathbf{a}} = -\omega^2 \mathbf{v} + \mathbf{w} \tag{80}$$

with constant turn rate ω given by (79), where $\mathbf{w}$ is white noise with power spectral density $\sigma_w^2 I$.

This well-established model can be traced back to [26, 45]. The vector model (77) is discussed in [46], along with an analysis and comparison with some other models. Several implementations referred to as coordinated turn models are surveyed in [1].

The state-space form of this model for each Cartesian coordinate with state $x = [\text{position, velocity, acceleration}]'$ is clearly given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & 0 \end{bmatrix} x(t) + w(t) \tag{81}$$

Its usual discretization leads to the following discrete-time model [30] for each Cartesian coordinate with state $x = (x,\ \dot{x},\ \ddot{x})'$

$$x_{k+1} = \begin{bmatrix} 1 & \frac{\sin \omega T}{\omega} & \frac{1-\cos \omega T}{\omega^2} \\ 0 & \cos \omega T & \frac{\sin \omega T}{\omega} \\ 0 & -\omega \sin \omega T & \cos \omega T \end{bmatrix} x_k + \begin{bmatrix} \frac{\omega T - \sin \omega T}{\omega^3} \\ \frac{1-\cos \omega T}{\omega^2} \\ \frac{\sin \omega T}{\omega} \end{bmatrix} w_k \tag{82}$$

where the $\text{cov}(w_k) = \sigma_w^2$.

The motions in $x,\ y,\ z$-coordinate directions in this model are coupled only through the common ω. This model specifies a circular motion at a constant turn rate in the so-called maneuver plane[15], defined by the velocity and acceleration vectors.

For a target with a constant speed, the velocity and acceleration vectors are orthogonal (i.e. $\mathbf{a} \cdot \mathbf{v} = 0$) and then the equation (79) for the magnitude of the turn rate simplifies to

$$\omega = \frac{\|\mathbf{a}\|}{\|\mathbf{v}\|} \tag{83}$$

The 3D fixed-center *constant-speed constant turn-rate model* for a motion of a fixed center takes the same form as above. The only difference is that the turn rate is governed now by (83), rather than (79), which is also valid for turn motion with a variable speed.

In order to use this constant turn-rate model for state estimation, an estimate of ω is needed. It can be obtained from the latest velocity and acceleration estimates in general by (79) or (83). This will, however, injects additional nonlinearities into the model and may lead to accuracy degradation, especially when the orthogonality property $\mathbf{a} \cdot \mathbf{v} = 0$ is clearly violated for constant-speed motion. A possible rescue is to impose $\mathbf{a} \cdot \mathbf{v} = 0$ as a *kinematic constraint* [47], [30], [48]. This constraint can be incorporated into the above target dynamic model, which, however, turns the model into a highly nonlinear one. Mainly to avoid this nonlinearity, an alternative approach was chosen in [47], [30], [48] where the constraint was incorporated into the pseudo-measurement model. This is addressed later.

[13] $\mathbf{v} \times \mathbf{a} = \mathbf{v} \times (\mathbf{\Omega} \times \mathbf{v}) = (\mathbf{v} \cdot \mathbf{v})\,\mathbf{\Omega} - (\mathbf{v} \cdot \mathbf{\Omega})\,\mathbf{v} = \mathbf{v}^2 \mathbf{\Omega}$, since $\mathbf{\Omega}$ and $\mathbf{v}$ are orthogonal.

[14] $\dot{\mathbf{a}} = \dot{\mathbf{\Omega}} \times \mathbf{v} + \mathbf{\Omega} \times \dot{\mathbf{v}} = \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{v}) = (\mathbf{\Omega} \cdot \mathbf{v})\,\mathbf{\Omega} - (\mathbf{\Omega} \cdot \mathbf{\Omega})\,\mathbf{v} = -\boldsymbol{\omega}^2 \mathbf{v}$.

[15] In the special case where the maneuver plane is horizontal, this model simplifies greatly and it is more commonly used than (the more general) model (46)–(49) for derivation of some of the horizontal CT models [1], considered in the previous section. In this case (i.e. $a_t = 0$ and $\omega = a_n/V$), however, both models are essentially equivalent.

An alternative form of the CAV model (77) is the following. Let $\mathbf{x} = (\mathbf{p}', \mathbf{v}', \mathbf{\Omega}')' = (x, y,\ z,\ \dot{x}\ , \dot{y},\ \dot{z},\ \Omega_x, \Omega_y,\ \Omega_z\ )'$. Then (77) becomes[16]

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \times \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \Omega_y \dot{z} - \Omega_z \dot{y} \\ \Omega_z \dot{x} - \Omega_x \dot{z} \\ \Omega_x \dot{y} - \Omega_y \dot{x} \end{bmatrix} \tag{84}$$

The continuous-time model is then given by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & I & 0 \\ 0 & A_\Omega & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + w(t) \quad \text{with} \quad A_\Omega = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \tag{85}$$

In this form, the angular velocity vector is part of the state vector and will be estimated directly, while in the form (81) the acceleration vector is part of the state vector and will be estimated directly.

For a horizontal constant turn-rate motion, (84) reduces to the 2D coordinated-turn model (50) since $\omega = \Omega_z, \Omega_x = \Omega_y = 0$. The same reduction should be true for (81), which is, however, not clearly seen.

6.1.2 Nearly Constant Turn-Rate Models

The above models based on the rotational motion equation (76) are restrictive in describing the variety of possible maneuvers. In particular, they implicitly presume that the center of the circular motion is fixed. A more sophisticated and expectably more accurate model was proposed in [49]. It assumes that the target can have a *general* motion of a rigid body in space, only subject to the constraint that the angular velocity vector $\mathbf{\Omega}$ satisfies (78), which essentially corresponds to a constant turn-rate motion. This results in modeling the target acceleration as a second-order Gauss-Markov process with state dependent coefficients[17], in the sensor inertial coordinate system,

$$\dot{\mathbf{a}} = -2\alpha\mathbf{a} - \left(\alpha^2 + \omega^2\right)\mathbf{v} + \mathbf{w} \tag{86}$$

where ω is the turn rate and[18]

$$\alpha = -\frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{v}^2}, \qquad \mathbf{w} = \ddot{\mathbf{v}}^{[b]} + \dot{\mathbf{\Omega}}^{[b]} \times \mathbf{v} \tag{87}$$

The term $\mathbf{w}$ in (86) reflects the effect of the forces and moments applied to the target. It can be modeled as zero-mean (Gaussian) white noise with covariance $\sigma_w^2 I$ that is to be designed. The damping coefficient α is a normalized target drag (i.e., the ratio of negative tangential acceleration to target speed). Both α and ω can be interpreted as unknown model parameters. This model, taking a slightly different form, in rotating LOS coordinates is also given in [49]. Both forms were utilized therein in designing two highly accurate trackers.

This is the most general and perhaps most accurate (nearly) constant turn-rate model developed so far. It reduces to the model (80) when the center of the target circular motion is fixed. For a constant speed motion, both models take the form of (80) with ω given by (83) since $\mathbf{a}\cdot\mathbf{v} = 0$. When the speed is not constant, the existence of the damping coefficient α allows the model to automatically adapt itself to target maneuvers. Both α and ω vanish for an unaccelerated motion.

It is straightforward from (86) to obtain the following continuous-time state-space form of this model for each Cartesian coordinate with state $x = (x,\ \dot{x}\ ,\ \ddot{x})'$:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\left(\alpha^2 + \omega^2\right) & -2\alpha \end{bmatrix} x\,(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t) \tag{88}$$

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[16] By using the vector product coordinate representation $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \mathbf{e}_x + \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix} \mathbf{e}_y + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \mathbf{e}_z$ for vectors $\mathbf{a} = (a_x\, a_y\, a_z)'$, $\mathbf{b} = (b_x\, b_y\, b_z)'$ where $\mathbf{e}_x = (1\,0\,0)'$, $\mathbf{e}_y = (0\,1\,0)'$, $\mathbf{e}_z = (0\,0\,1)'$ are the basis vectors

[17] The formulation given here follows from (5a) and (5b) of [49] in view of the identity

$$\frac{\mathbf{a}^2}{\mathbf{v}^2} = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{v}^2}\right)^2 + \left(\frac{\|\mathbf{v}\times\mathbf{a}\|}{\mathbf{v}^2}\right)^2 = \alpha^2 + \omega^2$$

[18] A superscript $[b]$ refers to the target body-frame coordinate system. The differentiation (dot sign) without such a superscript is with reference to an inertial (e.g. sensor) coordinate system.

The corresponding discrete-time model is given here as follows:

$$x_{k+1} = \begin{bmatrix} 1 & \frac{2\alpha\omega - e^{-\alpha T}\left(2\alpha\omega\cos\omega T + \left(\alpha^2-\omega^2\right)\sin\omega T\right)}{\omega(\alpha^2+\omega^2)} & \frac{\omega - e^{-\alpha T}(\omega\cos\omega T + \alpha\sin\omega T)}{\omega(\alpha^2+\omega^2)} \\ 0 & \frac{e^{-\alpha T}(\omega\cos\omega T + \alpha\sin\omega T)}{\omega} & \frac{e^{-\alpha T}\sin\omega T}{\omega} \\ 0 & -\frac{\left(\alpha^2+\omega^2\right)e^{-\alpha T}\sin\omega T}{\omega} & \frac{e^{-\alpha T}(\omega\cos\omega T - \alpha\sin\omega T)}{\omega} \end{bmatrix} x_k + w_k \tag{89}$$

where cov$(w_k) = \sigma_w^2 G(\alpha,\omega)G'(\alpha,\omega)$ with

$$G(\alpha,\omega) = \begin{bmatrix} \frac{\omega\left(-2\alpha + \left(\alpha^2+\omega^2\right)T\right) + e^{-\alpha T}\left(2\alpha\omega\cos\omega T + \left(\alpha^2-\omega^2\right)\sin\omega T\right)}{\omega(\alpha^2+\omega^2)^2} \\ \frac{\omega - e^{-\alpha T}(\omega\cos\omega T + \alpha\sin\omega T)}{\omega(\alpha^2+\omega^2)} \\ \frac{e^{-\alpha T}\sin\omega T}{\omega} \end{bmatrix}$$

Like the CTR model above, different coordinates in this model are coupled only through the common α and ω.

It can be seen by comparing (88) with (35) that if the white noise term of this model is replaced by that of (35), the velocity component along each Cartesian coordinate is implicitly modeled in this model as a random process having a oscillatory exponentially decaying autocorrelation $R_v(\tau) = \sigma_w^2 e^{-\alpha|\tau|}\cos(\omega\tau)$. This makes perfect sense in view of the physical meanings of α and ω.

6.2 3D Coordinated-Turn Models

6.2.1 Berg's 3D Coordinated-Turn Model

The coordinated-turn (CT) motion in the 3D space is defined (for aircraft) by the following conditions [23]: The *thrust* T (acceleration along the aircraft velocity direction), *lift* L (acceleration normal to the aircraft wing plane), and *roll angle* γ (angle around the longitudinal axis) are all constant. These conditions confine the average target motion in a plane, known as the *maneuver plane*. This definition is consistent with the bank-to-turn characteristics of fixed-wing aircraft. Note that such an (average) motion does not necessarily have a constant turn rate or constant speed. In reality, the conditions for the CT motion can be relaxed to that T, L, and γ all have constant expected values.

The following 3D CT model was proposed in [23]:

$$\dot{x}_m = V\cos\phi_m \tag{90}$$
$$\dot{y}_m = V\sin\phi_m \tag{91}$$
$$\dot{V} = a_t = T + g_t \tag{92}$$
$$V\dot{\phi} = a_n = L\cos\gamma + g_n \tag{93}$$

where (x_m, y_m), ϕ_m,and (g_t, g_n) are the target position, the bearing of the velocity vector, and gravity components along target velocity and normal directions, respectively, all in the *maneuver plane coordinates*, and V is the target speed. This set of equations is a special case of the curvilinear motion equations (46)–(49) *in the maneuver plane* for acceleration (a_t, a_n) determined by the thrust, lift, and gravity. Note that $L\cos\gamma$ is the projection of the lift along the normal direction, and $g_t = g_{x_m}\cos\phi_m + g_{y_m}\sin\phi_m$ and $g_n = -g_{x_m}\sin\phi_m + g_{y_m}\cos\phi_m$, where g_{x_m} and g_{y_m} are gravity components along x and y directions, respectively, in the maneuver plane.

Clearly, the approximate curvilinear model of Subsection 5.4 in the state-space form can be used here for the 3D CT motion in the maneuver plane. However, the small speed-change assumption as well as the piecewise-constant acceleration assumption on which this approximate model is based should be kept in mind. These assumptions are not valid for a large thrust and sampling interval.

This approach of modeling the target accelerations in the maneuver plane reduces the difficulties arising when these highly coupled accelerations are treated in the sensor inertial coordinates. It, however, requires estimating thrust and lift. Further, when applying this model, the fact that it is in a coordinate system different than the sensor inertial system induces difficulties. In essence, all ways around these difficulties involve explicitly or implicitly a conversion from one coordinate system to the other, which can be done accurately in general only if accurate knowledge of the target position and model parameters is available. In addition, while this model is intuitive appealing and provides an accurate target position prediction, its implementation is rather complicated mostly due to the representation of the turn via thrust, lift and roll parameters.

This model was proposed in [23] for the purpose of predicting target position in an anti-aircraft fire control application.

6.2.2 Nonconstant-Speed Coordinated-Turn Model

With an additional assumption that the angle of attack and the sideslip[19] are zero, the CT motion defined above becomes a kinematics problem where the velocity vector coincides with the heading direction and the maneuver plane coordinates differ from the target body coordinates only by the nonzero roll angle. This makes it simple to derive a coordinated-turn model directly in the inertial coordinate system, resulting in what is sometimes called a *nonconstant-speed coordinated-turn (NCS-CT)* model, proposed originally and implemented in [50], and later analyzed and validated in [46] using processed real data. This model generalizes a number of constant-speed constant turn-rate models proposed before and appears to be more realistic and accurate.

This model takes the following form [51],[46]

$$\dot{\mathbf{a}} = \mathbf{\Omega} \times (\mathbf{a} - \mathbf{g}) + \mathbf{T}(\mathbf{v}, \mathbf{a})\,\mathbf{u} \tag{94}$$

where $\mathbf{p}, \mathbf{v}, \mathbf{a}, \mathbf{g}$ are the position, velocity, acceleration, and gravity vectors, respectively, in an inertial coordinate system; $\mathbf{\Omega}$ was given by (78); $\mathbf{T}(\mathbf{v}, \mathbf{a}) = [\mathbf{t}\ \mathbf{b}\ \mathbf{n}]$ is the coordinate transformation matrix from the target body frame to the inertial frame, where $\mathbf{t} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$, $\mathbf{b} = \frac{(\mathbf{a}-\mathbf{g})\times\mathbf{v}}{\|(\mathbf{a}-\mathbf{g})\times\mathbf{v}\|}$, $\mathbf{n} = \mathbf{b} \times \mathbf{t}$ are the unit tangential, normal, and binormal vectors, respectively; and $\mathbf{u}$ is a vector-valued Singer (zero-mean first-order Gauss-Markov) process that models the random perturbations in acceleration *with respect to the body frame*.

In this model, the (unperturbed) motions in the three spatial directions are coupled and confined to the maneuver plane, but not necessarily at a constant speed. The time-correlated perturbations $\mathbf{u}$ accounts for the (mostly out-of-plane) unmodeled motions and is modeled in the body frame to be more precise. This, however, necessitates a coordinate transformation. This in turn introduces a dependency of the model on velocity and acceleration estimates, which may lead to a dynamic model error and thus model degradation. This NCS-CT model does not estimate the turn rate explicitly. It is highly nonlinear and requires nonlinear filtering techniques for implementation.

The state-space form of this NCS-CT model in Cartesian coordinates can be obtained straightforwardly, although somewhat tedious. For the state vector $\mathbf{x} = (\mathbf{p}', \mathbf{v}', \mathbf{a}', \mathbf{u}')'$, it is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{v} \\ \mathbf{a} \\ \frac{\mathbf{v}\times\mathbf{a}}{\mathbf{v}^2} \times (\mathbf{a} - \mathbf{g}) + \mathbf{T}(\mathbf{v}, \mathbf{a})\,\mathbf{u} \\ -\alpha\mathbf{u} \end{bmatrix} + \begin{bmatrix} 0_{3\times3} \\ 0_{3\times3} \\ 0_{3\times3} \\ I_3 \end{bmatrix} \mathbf{w}_b \tag{95}$$

6.3 Discussions

While the constant turn-rate (CTR) models of Subsection 6.1 and the coordinated-turn (CT) models of 6.2 are similar, they are based on two quite diffferent ideas. The CTR models are of kinematic type, which aim at fitting typical target trajectories during a maneuver, while the development of the CT models relies on flight dynamics explicitly,

The last model above can also be derived by simply assuming that (78) holds and that the thrust and lift accelerations are constant. It is more convenient than the Berg's model (e.g., without the need to determine the thrust and lift accelerations). However, the nearly CTR model (86) essentially relies only on (78) and is even more convenient for most applications. The underlying assumptions for the fixed-center CTR or CAV model are that (78) holds and that the turn rate or the angular velocity vector is constant. Note that differential equation for (77) is $\dot{\mathbf{a}} = \mathbf{\Omega} \times \mathbf{a}$. Comparing it with the NCS-CT model, it can be seen that the essential difference is the $\mathbf{\Omega}\times\ \mathbf{g}$ term, which is constant if $\mathbf{\Omega}$ is constant. In particular, the two models coincide when the maneuver plane is horizontal. Overall, the general CTR model (86) seems a top choice.

Some other 3D target maneuver models cannot be grouped into the above classes. For example, the following maneuver model: $\ddot{\theta} = 0, \ddot{\phi} = 0$, was proposed in [52].with a state $x = [x, y, z, V, \theta, \dot{\theta}, \phi, \dot{\phi}]'$, where (x, y, z), V are the position and speed of the target, and θ and ϕ are the bearing and elevation of the velocity vector, respectively.

7 Concluding Remarks

The models developed for target dynamics can be classified into 1D, 2D and 3D categories, according to the coupling along different coordinates.

[19]The angle of attack and the sideslip are the angles between the heading and the velocity vector around wing axis and lift axis, respectively.

The 1D models have no (or weak) coupling among coordinates. Most of them are based on modeling the driving force (usually acceleration or jerk) of the target maneuver as a random process without recourse to the actual target dynamics or kinematics directly. These models form three families, corresponding to three classes of random processes: white noise, Markov processes, and semi-Markov jump processes. The simplest white-noise family models the target position derivative (or difference) of a certain order as white noise. It includes the classical CV and CA models. In the simple and widely used family of Markov processes, the target acceleration or jerk is modeled as a Markov process of various degrees of complexity. The most well-known representative of this family is the Singer model. The semi-Markov jump process models are the most sophisticated family of the three. It has quite good potential, but the intricacy involved in these models makes it hard for the practitioners to apply them.

The 2D and 3D models differ from the 1D models by not only their correlation across coordinates, but also their explicit dependence on the target dynamics and kinematics. These models try to capture the important characteristics of the target's behavior during maneuvers, which typically involve various turning motions. Many versions of (nearly) constant turn-rate (or coordinated-turn) models have been developed. Few models are available that are valid for other maneuver motions. Apart from the popular coordinated-turn models, it seems that the 2D approximate curvilinear motion model and the 3D nearly constant turn-rate model of (86) deserve more attention than they have received due to their attractive simplicity, applicability and flexibility.

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