A Quick Math Recap for Regression

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Vectors

- We use small **boldface** letters to denote vectors.
- ullet A column vector $oldsymbol{x} \in \mathbb{R}^d$,

$$m{x} = egin{bmatrix} x_1 \ x_2 \ dots \ x_d \end{bmatrix}$$

- ullet Alternatively, $oldsymbol{x} = egin{bmatrix} x_1, \dots, x_d \end{bmatrix}^ op$, with op denoting 'Transpose'.
- Inner product of vectors: $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^{\top} \boldsymbol{y} = \sum_{i=1}^{d} x_i y_i$
- ullet Euclidean (or L_2) norm for vectors: $\|oldsymbol{x}\|_2 = \sqrt{oldsymbol{x}^T oldsymbol{x}} = \sqrt{\sum_{i=1}^d x_i^2}$



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- ullet Euclidean (or L_2) norm for vectors: $\|m{x}\|_2 = \sqrt{m{x}^{ op}m{x}} = \sqrt{\sum_{i=1}^d x_i^2}$
- ullet $oldsymbol{x}_1,\ldots,oldsymbol{x}_n$ are linearly independent if and only if

$$a_1 \boldsymbol{x}_1 + \ldots + a_n \boldsymbol{x}_n = \boldsymbol{0} \iff a_1 = \ldots = a_n = 0$$



I.e., none of them is a linear combination of the rest.

Matrices

- We use capital **boldface** letters to denote vectors.
- A matrix $\mathbf{A} \in \mathbb{R}^{n \times d}$,

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1d} \\ A_{21} & A_{22} & \dots & A_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nd} \end{bmatrix}$$



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While $x^{\top}y$ is a scalar, xy^{\top} is a matrix.



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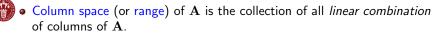
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Consider a real-valued multivariate function $f: \mathbb{R}^d \to \mathbb{R}$. The gradient of f evaluated at x, if exists:

$$\nabla f(\boldsymbol{x}) = \left[\frac{\partial f}{\partial x_1}(\boldsymbol{x}), \dots, \frac{\partial f}{\partial x_d}(\boldsymbol{x})\right]^{\top}$$



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- Note that $\nabla f : \mathbb{R}^d \to \mathbb{R}^d$.
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- ullet Example: affine function $f(oldsymbol{x}) = oldsymbol{w}^ op oldsymbol{x} + b$, with $abla f(oldsymbol{x}) = oldsymbol{w}$
- Example: quadratic function $f(x) = x^{T}Ax + b^{T}x + c$, with

$$\nabla f(\boldsymbol{x}) = (\mathbf{A} + \mathbf{A}^{\top})\boldsymbol{x} + \boldsymbol{b}$$



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$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{n \times d}.$$

A solution x exists if and only if b is in the column space of A.



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Three possibilities 'in general':

- Single solution
 - No solution (overdetermined system)
 - Infinitely many solutions (underdetermined system)



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Assume (i) n > d and (ii) \boldsymbol{b} is not in range of \boldsymbol{A} .

- An overdetermined system, hence no solution exists ...
- ullet ... yet we are interested in x s.t. $\mathbf{A}x \approx \mathbf{b}$.
- More concretely, we wish to find x s.t. $\|\mathbf{A}x \mathbf{b}\|_2$ is small:

$$\underset{x}{\operatorname{argmin}} \|\mathbf{A}x - \mathbf{b}\|_{2}$$



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An unconstrained optimization problem: To find x^\star , we solve $\nabla \|\mathbf{A}x-\mathbf{b}\|_2^2=\mathbf{0}$

$$\|\mathbf{A}\boldsymbol{x} - \mathbf{b}\|_{2}^{2} = (\mathbf{A}\boldsymbol{x} - \boldsymbol{b})^{\top}(\mathbf{A}\boldsymbol{x} - \boldsymbol{b}) = \boldsymbol{x}^{\top}\mathbf{A}^{\top}\mathbf{A}\boldsymbol{x} - 2\boldsymbol{b}^{\top}\mathbf{A}\boldsymbol{x} + \boldsymbol{b}^{\top}\boldsymbol{b}$$
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Hence, x^{\star} satisfies: $\mathbf{A}^{\top}\mathbf{A}x^{\star} = \mathbf{A}^{\top}b$

ullet If columns of ${\mathbf A}$ are linearly independent, ${\mathbf A}^{ op}{\mathbf A}$ is invertible so that

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