# The simulation of human electrocardiograph (ECG) using Fourier Series and its applications

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### *Introduction*

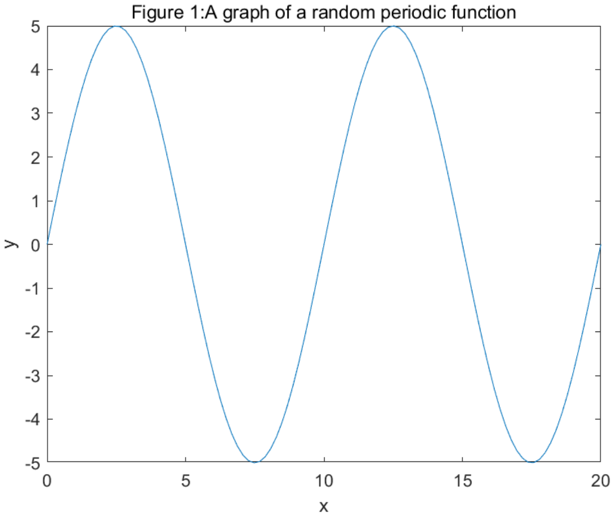
Electrocardiogram (ECG) is an innovation that utilizes an electrocardiograph to record the changes in the electrical activities of the heart during each cardiovascular cycle. The ECG is one of the most usually utilized clinical assessments. It can record the of the human body's typical heart and help analyze heart illnesses, for example, arrhythmia or myocardial ischemia. A clear understanding of the ECG diagram of patients can save millions. Using MATLAB and Fourier Series to simulate the human ECG signals with changeable clinical parameters can help doctors better understand the heart conditions of a patient simply from the outlook of the ECG diagram. This essay will focus on using MATLAB and existed algorisms to simulate the human ECG diagrams and concludes with a basic format of the ECG diagram with related heart illness. This research has a deep root with my personal life, not only because of my interests and skills in MATLAB coding and computer, but also because of its importance and significance in my family. My mom was diagnosed as mild premature beat due to genetic inheritance. My dad, as a cigar user, also causes damages on his lung and heart. With such background and environment, they intrigued me with the study of heart and the mathematical interpretation of human ECG.

### *Theoretical foundation*

**Fourier Series**

Fourier Series means that every periodic function can be expressed by an infinite series composed of a sine function and a cosine function. Why does Fourier Series use the series of functions? It is because series of a function represent the infinite subdivision to get the change of the function at each point. For example, $2345.4567 = 2000 + 300 + 40 + 5 + 0.4 + 0.05 + 0.006 + 0.0007$. By adding the changes together, they are still the same. This is very important to the accuracy of Fourier Series in which infinite series composed by trigonometry functions are the components of the mother function (the periodic function). Any periodic function (mostly trigonometry functions) can be expressed in simple harmonic motion (SHM), where $y = A\sin\left(\omega t + \varphi\right)$, $A$ is the amplitude, $\omega$ is the angular frequency, and $\varphi$ express initial phase. It is very important in Fourier Series because it is used to find the constants in Fourier Series in order to express the mother function correctly. For example, if a graph random periodic function is given:

Figure 1:A graph of a random periodic function




It can be expressed in SHM equation, where the amplitude $A$ is 5, and $\omega = \frac{2\pi}{T} = \frac{2\pi}{10} = 0.2\pi$, and the initial phase is $\varphi$. This periodic function has an equation of $y = 5\sin\left(0.2\pi t\right)$.

Back to $y = A\sin(\omega t + \varphi)$. Writing it as an infinite series: $f(x) = A_0 + \sum_{t=1}^{\infty} A_t \sin(tx + \varphi_t)$.

By applying double angle formula, the series becomes:

$A_t \sin(t\omega x + \varphi_t) = A_t \sin(\varphi_t)\cos(t\omega x) + A_t \cos(\varphi_t)\sin(t\omega x)$, further substituting

$A_0 = \frac{a_0}{2}, a_t = A_t \sin(\varphi_t), a_t = A_t \cos(\varphi_t)$. Now, this is the trigonometry series:

$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$. The trigonometry series determines whether a periodic function can be expanded into Fourier Series. If a periodic function cannot be expanded into trigonometry series, then, it cannot be expanded into Fourier Series. Now, how to expand a function into this kind of Series?

Let $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ with a $2l$ period. If a Fourier Series does exist in this function, its coefficients are $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$,

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$. The series can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right).$$

**Human electrocardiograph (ECG)**

Myocardial cell membrane shows certain semi permeable characteristics, which is closely related to the transmembrane transport of specific substances. In the resting state, a certain amount of strongly charged cations are distributed in the outer layer of the membrane, while anions with the same charge are distributed in the inner layer. When the membrane is in the polarization state, its external capacity is relatively high, and the original equilibrium state is broken. At rest, cardiomyocytes maintain a certain electrified state, and there may be no difference in the state of cardiomyocytes. In this condition, the cardiac potential bends to a horizontal level. Under the stimulation of certain factors, when the cardiomyocytes are activated, the porosity of the membrane changes to a certain extent, which leads to the influx of many cations into the cell membrane in a short time, and then leads to the potential change from negative to positive, that is, the depolarization change. Related studies have found that in this depolarization cycle, the state of cardiomyocytes will also have a certain change. The potential curve recorded by the instrument is also regarded as depolarization wave. The P wave and QRS wave of ventricle can be observed by analyzing the superficial electrocardiogram. At the end of depolarization, cardiomyocytes will enter the first polarization state to support the realization of related functions. This cyclic change, also known as repolarization, continues from the epicardium to the endocardium. It can record the change of repolarization process and the result is repolarization wave. The period of repolarization wave is lower than that of depolarization wave. The intensity of ventricular repolarization wave is small, and it is easy to be disturbed, so it is difficult to identify on the surface ECG. These kind of wave shows up as a T wave on a superficial level electrocardiogram. The polarization condition is reestablished once more, there is no expected contrast between the cardiomyocytes at different parts, and the electrocardiogram of the body surface records the isoelectric line.

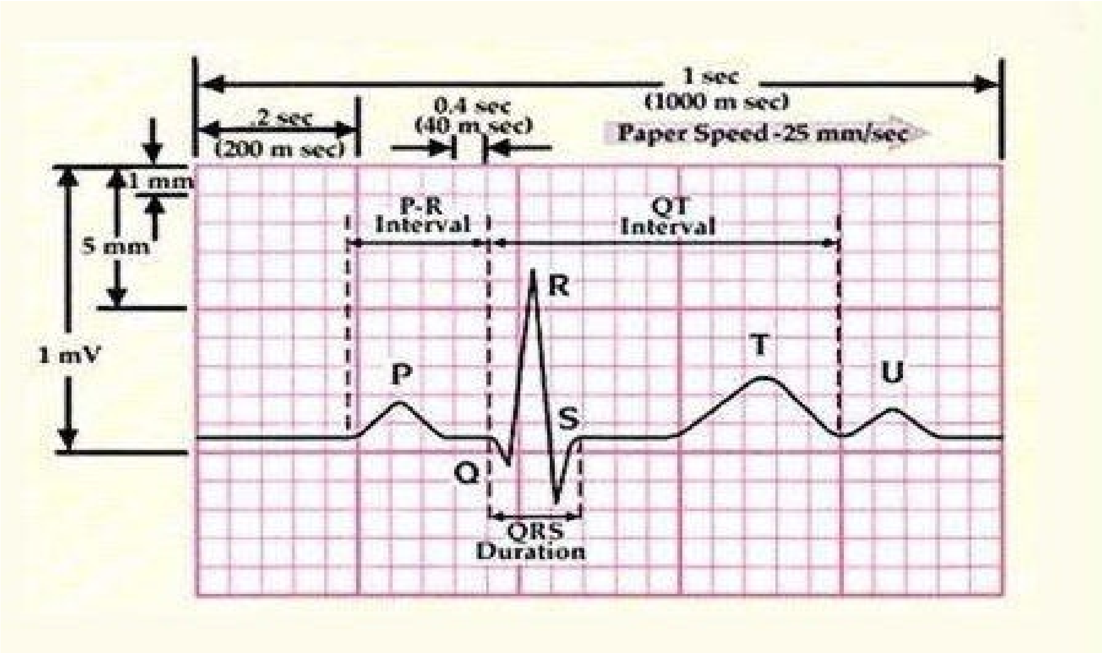



**Figure 2: the sample human ECG graph of 1 heartbeat**

| **Different sections** | **The meanings of different sections** |
|---|---|
| P wave | Depolarization |
| T wave | Ventricular repolarization |
| U wave | Possible ventricular repolarization (Very rare) |
| PR section | Atrioventricular conduction time |
| ST section | Ventricular depolarization completed |
| QT section | Time from ventricular depolarization to complete repolarization |
| QRS waveform | Ventricular depolarization |

**Anatomy of human heart**

The heart is a central empty organ, principally made out of myocardium, with four chambers: left、right atrium andventricle. The two atria are isolated by a hole and are not associated. These valves permit blood to spill out of the atria to the ventricles just, not back.

***Theoretical model of ECG simulation***

In a normal sense, the ECG signal is not periodic. Human heart is an organ, not a car engine. It works in a non-periodic way. However, for the sake of mathematics and the purpose of this essay, the ECG signal is assumed to the periodic function instead of a repetitive function. In order to use Fourier series to represent ECG signal, its structure must be analyzed using polynomials.

The p and T waves look very round with maxima, which can be approached with a trigonometric function. The QRS waveform looks like a triangle. There are also small curves with their shapes. Using basic knowledge from function, this ECG signal can be separated into different parts.

**QRS section**

The triangle shape function QRS is composed by two different linear equations with different slopes and shifts on the t-axis. But they share an elementary function $f(x)=x$. However, by closely examine the QRS waveform, the slope of the wave is clearly smaller than 1, and has discontinuity over different sections. Otherwise, the function will not have a correct frequency which differs from the required frequency as the heartbeat. Hence, the overall waveform of the QRS function should have a portion where the slope is zero, and appears as “individual triangle” on the x-axis. By applying this thought in the graph, the approximate shape of the QRS waveform should looks like the following diagram:

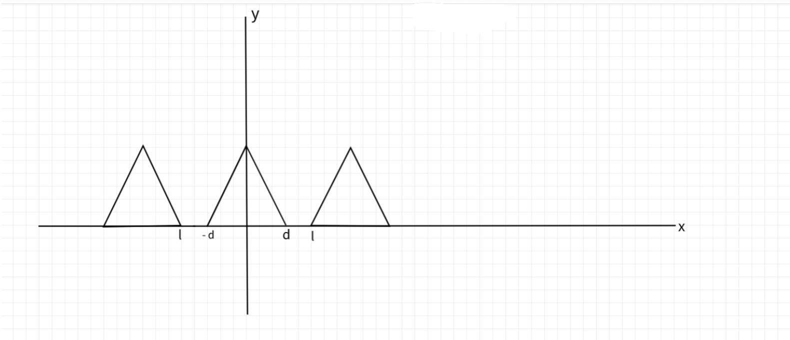



**Figure 3: the general idea of QRS waveform**

This graph presents the general idea of the QRS waveform. There are portions between triangles with zero slope, and the half period $l$. The graph has denotation of $d$ and $-d$ to show the end points of one triangle shape. In one triangle shape, let’s say that the maximum point is $a$, the slope of this wave is $\frac{a}{d}$, with a constant term of y-intercept of the maximum point $a$. Hence, this total waveform can be written as: $f(x)=\{\begin{matrix} a+\frac{a}{d}x, -d\le x\le 0 \\ a-\frac{a}{d}x, 0\le x\le d \end{matrix}$, this system of linear equations represents first, the ascending part of the QRS waveform QR, and then, the descending part, RS. Now, using the founded theoretical model of Fourier series, this function can be approximated with precision.

$$a_0=\frac{1}{l}\int_{-d}^{d} f(x)dx=\frac{1}{l}\int_{-d}^{0}\left(a+\frac{a}{d}x\right)dx+\frac{1}{l}\int_{0}^{d}\left(a-\frac{a}{d}x\right)dx=\frac{ad}{2l}+\frac{ad}{2l}=\frac{2ad}{2l}=\frac{ad}{l}$$

$$a_n = \frac{1}{l}\int_{-d}^{d} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\int_{-d}^{0}\left(a+\frac{a}{d}x\right)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$+\frac{1}{l}\int_{0}^{d}\left(a-\frac{a}{d}x\right)\cos\left(\frac{n\pi x}{l}\right)dx = -\frac{al\left[\cos\left(\frac{\pi dn}{l}\right)-1\right]}{\pi^2 dn^2} - \frac{al\left[\cos\left(\frac{\pi dn}{l}\right)-1\right]}{\pi^2 dn^2}$$

$$= -2\frac{al\left[\cos\left(\frac{\pi dn}{l}\right)-1\right]}{\pi^2 dn^2}$$

$$b_n = \frac{1}{l}\int_{-d}^{d} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\int_{-d}^{0}\left(a+\frac{a}{d}x\right)\sin\left(\frac{n\pi x}{l}\right)dx + \frac{1}{l}\int_{0}^{d}\left(a-\frac{a}{d}x\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

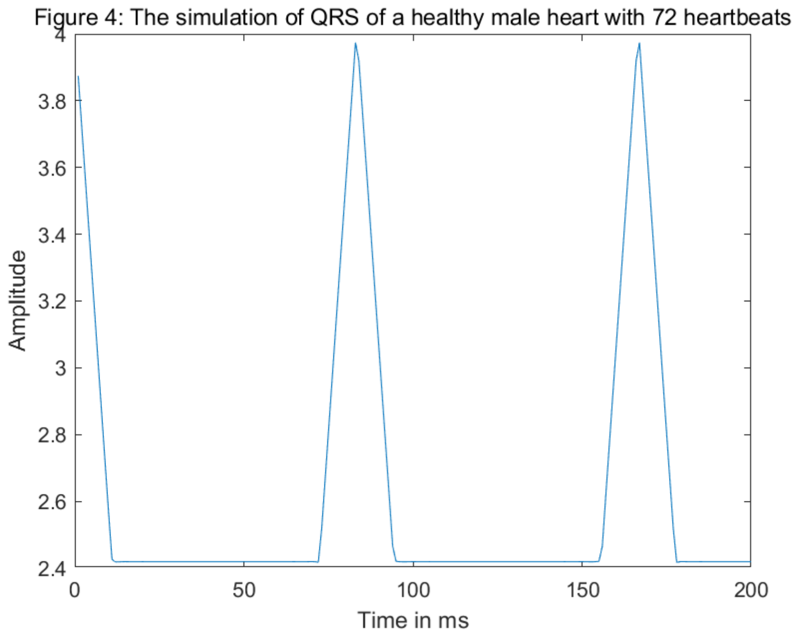
$$= \frac{a\left[l\sin\left(\frac{\pi dn}{l}\right)-\pi dn\right]}{\pi^2 dn^2} - \frac{a\left[l\sin\left(\frac{\pi dn}{l}\right)-\pi dn\right]}{\pi^2 dn^2} = 0$$

$$f(x) = \frac{\frac{ad}{l}}{2} + \sum_{n=1}^{\infty} -2\frac{al\left[\cos\left(\frac{\pi dn}{l}\right)-1\right]}{\pi^2 dn^2}\cos\left(\frac{n\pi x}{l}\right) = \frac{ad}{2l}$$

$$+\sum_{n=1}^{\infty} -2\frac{al\left[\cos\left(\frac{\pi dn}{l}\right)-1\right]}{\pi^2 dn^2}\cos\left(\frac{n\pi x}{l}\right)$$

The constant $l = \frac{30\text{seconds}}{\text{number of heartbeats}}$, which is the half period. The constant

$d = \text{the duration in seconds}$, and $a = \text{Amplitude}$. This equation not only give us the correct waveform of QRS, but also customization properties of the waveform according to the physical condition of the heart. Hence, the purpose of according simulation is fulfilled.




**PTU section**

The round and circular waveforms of P, T and U are very similar to quadratic function with its coefficient and maximum. However, waves of P, T and U cannot be simulated accurately using quadratic function. This is because quadratic function continues its function below the x-axis, which causes trouble when finalizing the overall waveform of a complete human ECG. By observing the waveform of P, T and U again, they have a very round shape, with a maximum point on the top, hence, it is very similar to basic trigonometry function such as $\sin(x)$ or $\cos(x)$. In this situation, it is very important to notice that the calculation involves integration. When the function is approaches with $\cos(x)$, its integration will not have a negative sign, which is easy to work with. Its sides are smooth and round instead of a straight line. By applying this thought in the graph, the approximate shape of the PTU waveform should look like the following diagram:

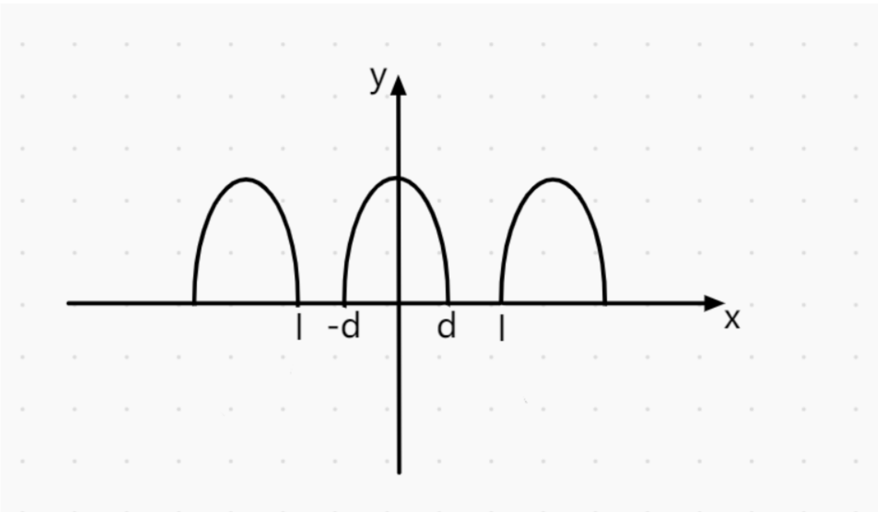



**Figure 3: the general idea of PTU waveform**

Making the function precise by determine the period of this function, which is $\frac{\pi}{2d}$. Hence, the function of this waveform is $f(x)=\cos\left(\frac{\pi}{2d}x\right)$. Now, using the founded theoretical model of Fourier series, this function can be approximated with precision.

$$a_0=\frac{1}{l}\int_{-d}^{0}\cos\left(\frac{\pi}{2d}x\right)dx+\frac{1}{l}\int_{0}^{d}\cos\left(\frac{\pi}{2d}x\right)dx=\frac{2d}{\pi l}+\frac{2d}{\pi l}=\frac{4d}{\pi l}$$

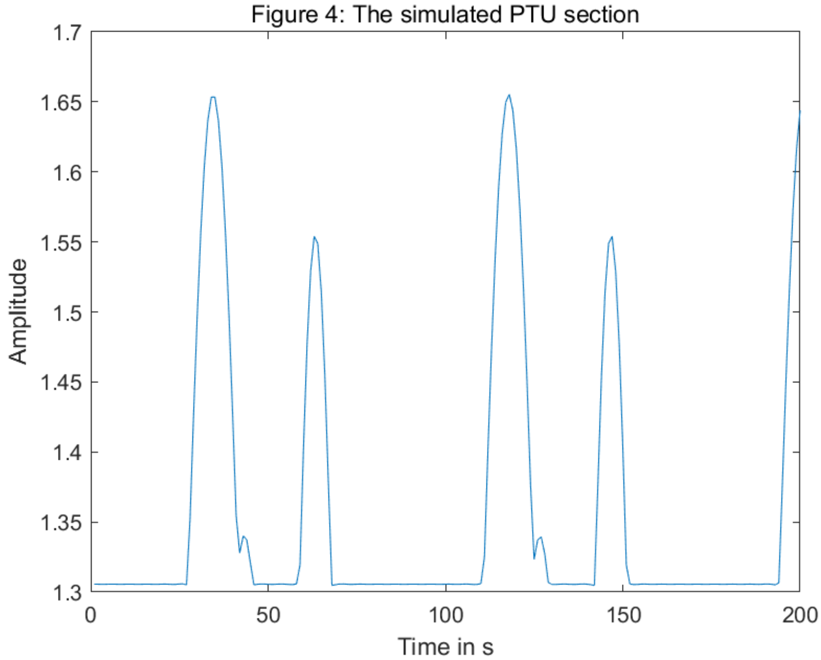
$$a_n=\frac{1}{l}\int_{-d}^{0}\cos\left(\frac{\pi}{2d}x\right)\cos\left(\frac{n\pi x}{l}\right)dx+\frac{1}{l}\int_{0}^{d}\cos\left(\frac{\pi}{2d}x\right)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$=-\frac{2dl\cos\left(\frac{\pi dn}{l}\right)}{\pi\left(4d^2n^2-l^2\right)}-\frac{2dl\cos\left(\frac{\pi dn}{l}\right)}{\pi\left(4d^2n^2-l^2\right)}=-2\frac{2dl\cos\left(\frac{\pi dn}{l}\right)}{\pi\left(4d^2n^2-l^2\right)}$$

$$b_n=\frac{1}{l}\int_{-d}^{0}\cos\left(\frac{\pi}{2d}x\right)\sin\left(\frac{n\pi x}{l}\right)dx+\frac{1}{l}\int_{0}^{d}\cos\left(\frac{\pi}{2d}x\right)\sin\left(\frac{n\pi x}{l}\right)dx=0$$

$$f(x)=\frac{\frac{4d}{\pi l}}{2}+\sum_{n=1}^{\infty}-2\frac{2dl\cos\left(\frac{\pi dn}{l}\right)}{\pi\left(4d^2n^2-l^2\right)}$$

Figure 4: The simulated PTU section




***Simulation of ECG***

Since all the sections have been studied individually, the complete ECG signal is simply the assembly of PTU and QRS section:

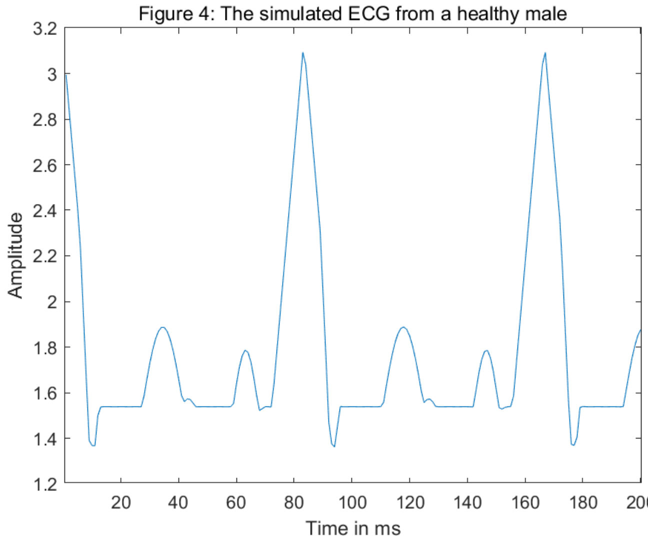
| Different sections of the ECG | Formula |
|---|---|

| | |
|---|---|
| QRS | $f(x)=\frac{\frac{ad}{l}}{2}+\sum_{n=1}^{\infty}-2\frac{al\left[\cos\left(\frac{\pi dn}{l}\right)-1\right]}{\pi^2 dn^2}\cos\left(\frac{n\pi x}{l}\right)=\frac{ad}{2l}+\sum_{n=1}^{\infty}-2\frac{al\left[\cos\left(\frac{\pi dn}{l}\right)-1\right]}{\pi^2 dn^2}\cos\left(\frac{n\pi x}{l}\right)$ |
| PT | $f(x)=\frac{\frac{4d}{\pi l}}{2}+\sum_{n=1}^{\infty}-2\frac{2dl\cos\left(\frac{\pi dn}{l}\right)}{\pi\left(4d^2n^2-l^2\right)}$ |

By deciding the amplitude $a$ and the duration $d$ of the wave, the ECG signal can be simulated which approximate to real-life ECG. Specifically, the parameters from the equation:

| Name of the parameters | Acceptable range |
|---|---|
| Amplitude $a$ of the P wave function | 0---0.4v |
| Amplitude $a$ of the QRS wave complex | 0---0.8v |
| Amplitude $a$ of the T wave function | 0---0.9v |
| Amplitude $a$ of the U wave function | 0.035v |
| Duration of the QRS wave complex | 0---111ms |
| Duration of P wave function | 0---150ms |
| Duration of the T wave function | 0---280ms |
| Duration of the U wave function | 48ms |

For a healthy male heart, the ECG signals have the following quality:




| | |
|---|---|
| $a$ of p | 0.25 |
| $a$ of T | 0.35 |
| $a$ of QRS | 0.025-1.6 |
| Time duration of T | 0.17 |
| Time duration of P | 0.16 |
| Time duration of QRS | 0.11 |
| Time duration of U | 0.0476 |

**Figure 4: The simulated ECG from a healthy male**

This graph shows the simulated results of a standard and healthy male heart ECG signals. It has a half frequency of $\frac{30}{72}$, where 72 is a standard heart rate of a young male.

### Simulation of my personal's ECG

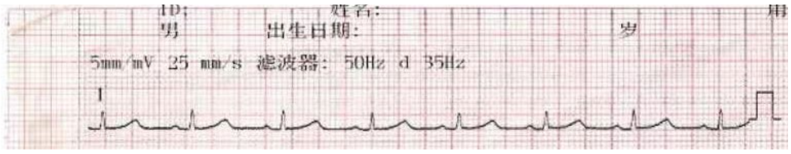



**Figure 4: The ECG graph of me during 2019**

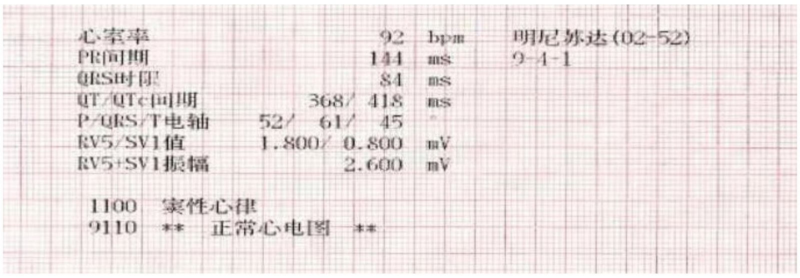
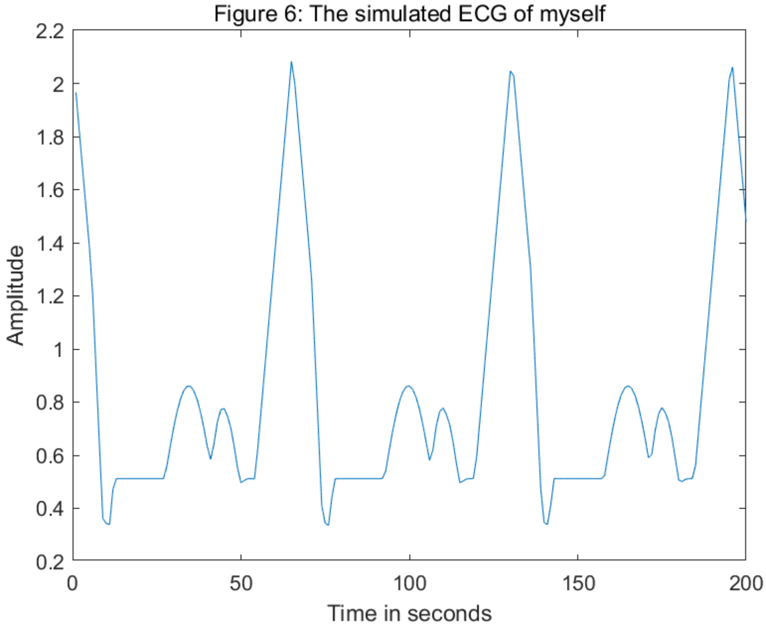



**Figure 5: The ECG graph details**

In this ECG, the P duration is about 144 ms, the QRS duration is about 84 ms, the T duration is about 368 ms. The amplitudes of P are 52mv, QRS is 61mv, and P is 45mv. Using the algorithm designed previously, the simulated ECG has the following outlook:

Figure 6: The simulated ECG of myself




This simulated ECG shows some similarities with the actual ECG. There is still difference between the P to QRS sections, where the actual ECG shows a overlap, but the simulated ECG shows distinguishable intervals. Hence, this program has limitations in simulating my ECG. It is largely because of my constant changing heartbeat during the actual measurement, while the program only allows the entry of heartbeat counts in a minute.

### Simulation of my mother' s ECG


**Figure 7: The Actual ECG and ECG details of my mother**

In this ECG, the P duration is about 166 ms, the QRS duration is about 84 ms, the T duration is about 386 ms. The amplitudes of P are 63mv, QRS is 89mv, and P is 50mv. Using the algorithm designed previously, the simulated ECG has the following outlook:

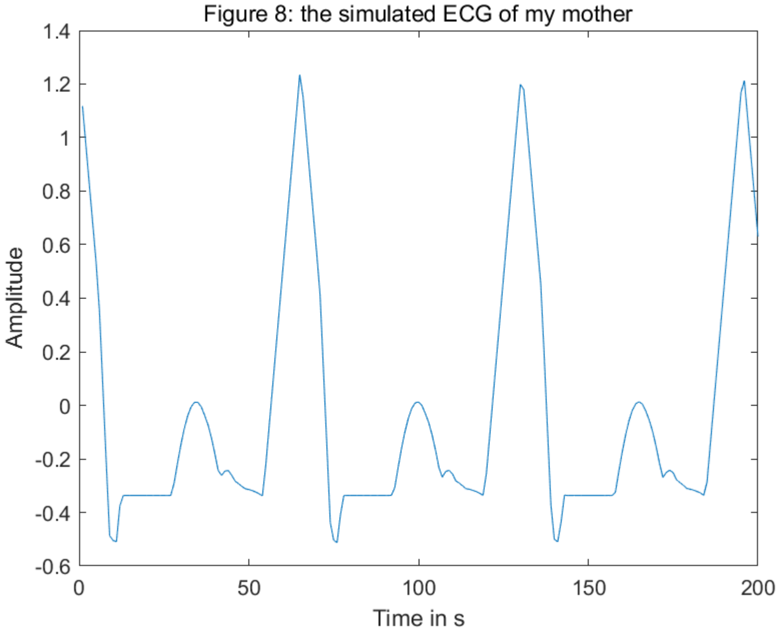



Figure 8: the simulated ECG of my mother

This simulated ECG shows some similarities with the actual ECG. There is still difference between the T to U section, where the actual ECG shows a smoother line, but the simulated ECG shows distinguishable small rises. Hence, this program has limitations in simulating my ECG. Those differences suggest a defect in the program in which physical uncertainties (instrumental errors, small heartbeat disturbances).

## *Conclusion*

In this paper, through the realization of the simulation of the various waveforms and intervals of the ECG signal, the use of triangular waves and sine waves, and the principle of Fourier series are used for simulation experiments. The ECG signal obtained by Fourier series simulation can freely adjust the length of the RR interval, and then change the heart rate value, which is used for the demonstration and discrimination of arrhythmia. Analog P, Q, R, S, T waveforms can be changed freely, large amplitude or small, conducive to the study of medically high-tip or horizontal diseases. The QRS, PR, and ST intervals provide a template for studying the mechanism of ventricular and atrial activity. Using the above simulation method, according to relevant medical knowledge, simulate the ECG signal with specified characteristics, continuously expand the ECG template library and the simulated ECG database in use, strengthen the function of the ECG database, and provide reference for the research of ECG.

## *Works cited*

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## *Appendix:*

Coding details can be viewed in this website: https://yualex1234.wixsite.com/mysite