Basics of financial markets

Introduction to Financial market

Definition (Real assets): Real assets are physical or intangible assets that have intrinsic value due to their substance and properties.

For example: land, building, precious metal, human capital, IPs, etc.

Definition (Financial asset): A non-physical asset whose value is derived from a contractual claim. This means it is a claim on real assets, or a claim on the cash flow generated from real assets.

For example: bank deposit, bonds, company stock, etc.

Remark: Financial assets $\cup$ Real assets = All forms of assets.

Definition (Financial instrument): A monetary contract of financial assets between parties. This means this contract give rise to both a financial asset for one party and a financial liability or equity for another party. They can be created, traded, modified, and settled.

For example: corporate bond, corporate share, etc.

Definition (Financial market): A market where financial instruments are traded. There are some fundamental roles of financial market:

- **Reduce transaction costs:** A efficient financial market streamline the process of making transactions, and reduce the costs of transaction of financial market. This encourages more trading and investments since it makes financial activities more appealing.
- **Promote efficient allocation of resources:** we can match right borrowers and savers in financial market, meaning investors and lenders supply the right capital to the right people who need it. This efficient allocation boosts economic growth and lead to optimal outcomes.
- **Determine the prices of financial instruments:** The prices of stocks, bonds, derivatives, etc. are determined based on supply and demand dynamics.
- **Provide information:** This market reveals information about economic environments.

Example of efficient resource allocation in financial market:

Corp. A has invented a new product but lack the funds to produce:

- Equipment fixed cost: $100 million.
- Labor expense: $10 million per year.
- Potential revenue: $20 million per year for 20 years.

Corp. A issues shares (a financial instrument) worth of $110 million on financial market. And investors purchase the share at $110 million.

With the money, Corp. A start purchase equipment, employ workers, start production, and sells products. The corp. A collect revenue of $20 million per year, and receive $10 million profit per year. Corp. A pay a dividend of $8 million, and this continuous 20 years.

- For investors: end up with $50 million of profit, 45% on investment.
- For workers: they receive wages.
- For Corp. A: It earns profit.
- For consumers: they consume new product, and thus increase their utility.

Thus, financial market leads to pareto efficiency

Types of financial market

Financial market can be classified into different types. Money vs. capital and primary vs. secondary.

Definition (Securities): They are specific types of financial intruments that represent ownership or a creditor relationship with an entity. They can be classified as:

- Definition (Equity securities): It represent ownership interest held by the share holders in a company.
- Definition (Debt securities): It represent a loan made by an investor to a borrower (typically corporate or government.

Definition (issue securities): The process of entities create and sell new financial instruments to raise capital.

Money vs. capital market

Definition (Money market): The money market is a sector of the financial market for debt securities that pay off in short term (less than one year).

Remark: Money market is organized into a dealer market. This means dealers such as chartered banks or investment dealers hold inventories of money market instruments and quote prices at which they are willing to buy and sell these instruments. And investors will contact the dealer directly looking to buy or sell money market instruments. Thus, the market is two sided.

Remark: This type of market is also called over the counter (OTC) market.

For example: Treasury bills (short term debt obligations of a government), commercial paper (short term debt instrument issued by corporations to finance their short term liabilities and operation needs), etc. are traded in money market. So. a part of bond market is money market.

Definition (capital market): The capital market is a segment of the financial market where long-term debt and equity securities are traded. Meaning the maturities longer than one year.

Remark: capital market is a broker/auction market. It means transactions are conducted on an exchange and prices are determined through auction.

- Transaction take place on a centralized platform.
- Investment brokers facilitate transaction by matching buy and sell orders from their clients. They do not hold an inventory of securities but act on behalf of buyers and sellers.
- Prices are determined through an auction mechanism where buyers and sellers submit bids and offers. If there is a match, the transaction is executed. The maximum price a buyer WTP is matched with the lowest price a seller is WTS. So, the market is efficient.

For example: Stock market is a strictly capital market. And a part of bond market strictly for long-term bond is a capital market.

Remark: Stock exchange usually organized as an auction market or a hybrid of dealer and auction market.

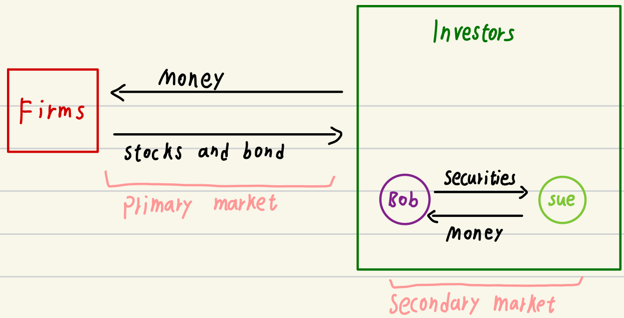
primary vs. secondary market

The financial market consists of two main segments: primary market and secondary market.

Definition (primary market): It is market where new securities are issued and sold for the first time. So funds flow from investors to the firm to issue new securities.

For example: An initial public offering (IPO) is when a corporation issues securities to the general public for the first time. Usually a underwriter is involved. An underwritter is financial institution who advise and determine the price, then purchase the entire issuance at a discounted price, then sell to the general public. And this whole process is taken place in primary market.

Definition (Secondary market): It is the place where previously issued securities are bought and sold among investors. This market provides liquidity and the ability to trade securities after they have been issued.



Participants of financial market

Here are the participants of financial market:

• House holds: Net supplies of funds

Households typically act as net suppliers of funds. They will save money surplus, and the money is used for investments.

• Firms: Net demanders of funds.

Firms or businesses are usually net demanders of funds. They need money to finance their operations.

• Government: Can be both

Government can either supply or demand funds. Government issues bonds when there is a budget deficit, and invest the surplus budget.

• Financial intermediaries

They are important in connecting suppliers and demanders of capital.
They include commercial banks. investment banks, insurance companies, and etc.
Here are its key functions:

- Match those with excess funds to those who needs funds.
- They often act as dealers in financial markets. They often hold inventories of Securities, and they buy and sell financial instruments.
- They pool funds from many small investors, and invest in large investment port folios.
- They benefit from specilzations and economies of scale, reducing their cost and increase their efficiency. For example, bank specialize in assessing credit risk and manage loans, which reduce costs and spreads risk across many borrowers.

Here are different kinds of financial intermediaries:

- **Borrowing and lending institutions:** They focus on facilitating borrowing and lending activities. They include commercial bank and credit union.
  - Commercial bank uses deposits from individuals and business to lend the money to other entities.
    They match lenders (depositors) to borrowers (loan recipients).

They earn the spread between the interest rate paid on deposits and interest rate charged on loans.
They screen potential borrowers to access creditworthliness and monitor loans to manage risks.

- Credit union is member-owned financial cooperatives that provide similar service and work similar to commercial banks.
  They offer loans and accept deposits from members, often at more favorable terms.
  They earn from spread between deposit rated and loan rates.

- Investment companies: They manage and invest pooled funds from individual investors to achieve invest goals.
They include pension fund, mutual fund, hedge fund, insurance company
  - Pension funds manage retirement savings of individual. They invest these contributions.
    They earn management fees based on the assets under management.
  - Mutual funds pool money from many investors to invest in diversified portfolio of stocks, bonds, or other securities.
    They earn management and service fees, often a percentage of assets under management.
  - Hedge funds use pooled funds to engage in a wide range of investments, often very complex and high risk.
    They aim to achieve high returns.
    They earn management fees and performance fees.
  - Insurance companies collect premiums from policyholders and invest them to pay future claims.
    They earn premiums and investment services.

- Investment banks provide a range of financial services to entities.
  They are typically highly capitalized to undertake significant financial transactions and risks.
  They offer advisory services for strategic business decisions, such as mergers, acquisitions, and restructurings. They earn service fees.
  They perform underwrite for issuing company. They earn service fees and the spread between the purchase price of securities from the issuer and the sale price to investors.
  They are able to reduce costs through specialized process, and they take risks of underwriting by guaranteeing the sale of securities.

Financial instruments

There are 3 types of financial instruments: Debt, equity, and derivatives.

- **Debt instruments:** Debt instrument represent agreements where borrowers receive funds from lenders and commit to repay the principal amount and often interest at a specific date.

Definition (maturity date): The date on which the borrower must repay the entire amount.

Definition (principal payment): The lump sum that borrower must pay back to the lender at the maturity date.

Definition (coupon payment): Periodic interest payment made to the debt holder.

Definition (Time deposit): A bank deposit that has a fixed term of period during which the funds cannot be withdraw without incurring a penalty.

Remark. Coupon payment is common in long-term debt.

Definition (Denomination): face value or the stated value of a financial instrument.

Money market instruments

Treasury bill (T-bill): issued by government, short maturities (30, 60, 90 days, 6 months, 1 year), highly liquid, backed by fiscal power of government. Usually purchased by investment dealers, chartered banks, central bank.

Certificate of deposit: issued by chartered banks. It is a time deposit. This time deposit is also known as Guaranteed investment certificate in canada. It is generally not transferable and not marketable in canada.

Bearer deposit notes (BDN): It is issued by chartered banks. It is similar to a CD except for it is transferable, large denominations \$100000+, not registered under a specific individual, and marketable. This means BDN is more liquid.

Commercial papers: It is issued by large corporation with large denomination ≥ \$50000. It is often short term 1-2 months. Issuing

Capital market instrument

Government bond: They are long-term debt securities issued by the federal government. They can mature up to 40 years with monthly, semi-annual, annual coupon interest. They are backed by the government fiscal power.

Provincial and municipal bonds: They are bonds issued usually by provincial and municipal government. They are used to finance local projects. They are backed by tax revenue or project revenue.

Corporate bond: They are issued by the government with semi-annual coupon interest. They are highly liquid, actively traded, and subject to credit risk.

commercial paper is usually cheaper than bank loan because the interest is lower. The investors are usually money market funds seeking higher yields than bank deposits. The credit quality of the commercial paper is rated by companies.

Repurchase agreements (Repo): It is the mechanism which financial intermediaries or central bank sell government securities and promise to buy them back the next day at a predetermined price. The difference between sell price and repurchase price is the interest rate of the loan. Thus, financial institutions can meet short liquidity needs, and investors can earn interest.

Reverse repo: It is the mechanism which the buyers (investors) of the securities agreeing to sell them back to the original seller at a predetermined price. The buyer provides cash to the seller and earn a interest since it sells back the securities at a higher price.

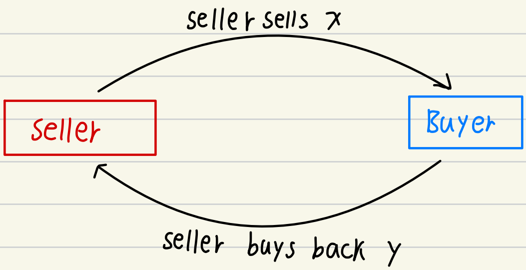
Euro dollars: They are U.S. dollar held in time deposit in foreign banks or foreign branches of American banks. They typically mature less than 6 months. And they are not regulated by Federal reserve. They are less liquid, risker, but has higher yields than domestic deposit.

Remark: Certificate of deposit and GIC in canada are insured by canada deposit insurance corporation up to $100000.

Special examples of repo and reverse repo:

- Repo: A bank needs short-term liquidity and sells government securities worth $1 mill to another bank, agreeing to repurchase them the next day for $1.002 mill. The $2000 difference is the interest cost for borrowing.
- Reverse Repo: A investment firm with excess cash buys the same securities from the bank and agree to sell them back for $1.002 mill the next day, earing $2000 interest.

Graphic illustration of repo and reverse repo:



$y - x$ = Repo interest
Buyer = investor / lender
seller = borrower.
Repo interest is earned by buyers.

$y - x$ = Reverse repo interest
Buyer = investor / lender
Seller = borrower.
Reverse repo interest is earned by buyers.

Remark: Repo and reverse repo are low risk because:

- Backed by high quality securities implies high quality collateral. Because this is a safeguard if seller in repo and buyer in reverse repo default.
- Short duration minimize credit risk. Meaning probability of adverse event is low.
- Institutions participate in repo / reverse repo are highly credit standing.

- **Equity:** Equity represents ownership in a corporation and consists of two main types: Common stocks and preferred stock.

Equity securities are created and distributed by the corporation to raise capital. When investor buy equity, it equals purchasing a portion of ownership in the company.

Definition (common stock): This type of shares gives stock holder the right to share profits, vote on corporate decisions, and claim on remaining corporate assets after clearing all debts and obligation in case of a liquidation.

Definition (preferred stock): This type of shares do not have voting rights. they have share on profits, and have a higher claim on assets and earning than common stock holders.
They receive dividends befor common stockholders and have a priority claim on assets in the event of bankruptcy.

Remark: share holders enjoy limited liabilities.

Remark: In most firms, management $\neq$ ownership, they are separated.

Remark: In claiming dividends and assets, we have debt holders > preferred stock holder > common stockholder.

- **Derivatives:** Derivatives are contractual agreements between two or more parties where the value of the contract is dependent on the value of a underlying assets (financial assets, real assets, commodities).

Definition (Futures): A future contract obligates the buyer to purchase, and the seller to sell, an asset at a predetermined future date and price.

- Long position: The party that agrees to buy the asset in the future is said to have a long position. The party benefits if the price of the asset increases.
- Short position: The party that agrees to sell the asset in the future holds a short position. This party benefits if the price of the asset decreases.

Definition (options): Options give the buyer the right, but not the obligation, to buy or sell an asset at a specified price before or on a certain date.

- Call option: Gives the holder the right to buy an asset at a specified price within a specific period. The buyer expects the asset price to rise.
- Put option: Gives the holder the right to sell an asset at a specified price within a specific period. The buyer expects the asset price to fall.

Remark: The buyer pays a fee calls premium for this right.

Remark: The buyer can choose not to exercise the option.

Definition (warrents): Warrents are similar to call options but are typically issued by the company itself and have longer durations.

- Stock warrents give the holder the right to buy the company's stock at a specified price before the expiry date. They are often used to raise capital.

stock market indicies

Stock market indicies are measures that track the performance of a specific group of stocks.

Definition (Market capitalization): share price × share outstanding (Total # of shares).

Definition (Market value weighted index): Index value $= \frac{\sum_i (P_i \cdot \# shares_i)}{D}$

- D = a fixed number (divisor). subjected to adjustment.
- P_i = Share price for company i
- $\# shares_i$ = number of shares of company i.

For example: TSX composite index, standard & Poor 500.

Definition (Price weighted index): index value $= \frac{\sum_i P_i}{D_P}$

- P_i = share price for company i
- D_P = a fixed number (divisor). subjected to adjustment.

For example: Dow Jones industrial average.

Definition (Stock split): A increase in the number of shares by issuing more shares to the current shareholders. For example: 2-for-1 stock split means each shareholders receives an additional share per owned share.

Remark: D_P must be adjusted for stock splits, so a new divisor will be set until a new stock split. So, a stock split does not affect stock market performance, so the price-weighted index has to be adjusted s.t. the index stays the same after the split.

Remark: value weighted index tends to bias the company with large market capitalization, so it may distort the purpose of index.

Inter-temporal consumption choice model and investment decision

The financial market economy and financial intermediation

Definition (intertemporal consumption): An economic theory that explains individual preferences for consuming now or saving for later.

Individuals and institutions have different income patterns and different inter-temporal consumption preferences.

Because of this, a market for money arises. This market helps match those who want to borrow money to consume or invest now with those who want to lend money and defer their consumption for future returns.

The interest rate becomes the price of borrowing money:

- it is a compensation for lenders.
- it is a cost for borrowers.

Definition (Financial intermediation): Process through which financial intermediaries facilitate transaction between savers and borrowers.

Motivating example:

- The dentist earns \$200000 per year and choose to consume \$80000, leaving \$120000 to invest.
- Instead of lending \$30000 to each of four college seniors individually, the dentist can lend the entire \$120000 to a bank (a financial intermediation).
- The bank then lends \$30000 to each of the student.

⇓

The bank promoted:

- Reduce transaction cost: save time and money for dentist and efficiency in payment.
- Shift and control credit risk

⇓

Financial intermediation facilitates efficient resource allocation in the market of money.

Three forms of financial intermediations:

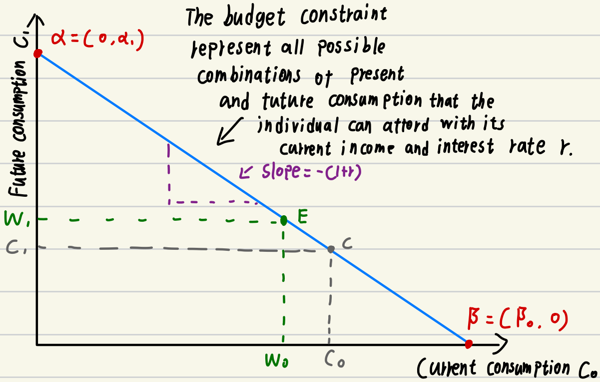
- Size intermediation: They pool resources for many savers and allocate them efficiently.
- Term intermediation: They turn short-term deposit to long-term loans.
- Risk intermediation: They offer different products tailored to varying risk tolerances.

inter-temporal consumption opportunity set

The inter-temporal consumption opportunity set is a model used in economics to understand how individual make decisions about consumption and saving over time.

Individuals can alter their consumption patterns by either borrowing (to consume more now) or saving (to consume more future).

We have this model:



The endowment point E represent individual's initial income distribution across two periods (today and future). It is denoted as (W_0, W_1), W_0 is the initial endowment to denote income today, and W_1 is the future endowment as future income.

The extreme points (α and β) show maximum possible consumption in one period if individual allocate all resources to that period, either save everthing or consume everthing.

Definition (Gross return on savings): The total amount you receive on savings including the principal per dollar. It is $1+r$, where r is the interest rate.

The slope of the budget line is $-(1+r)$, it means how much future consumption you must give up to consume more today.

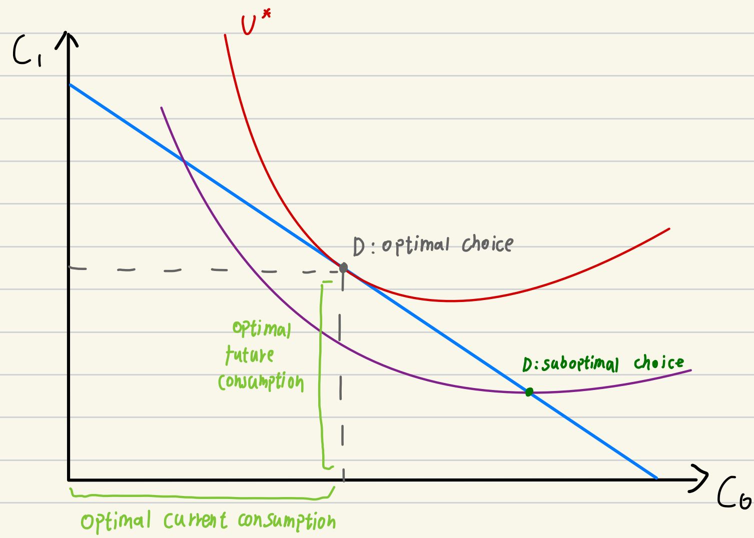
Remark: A downward sloping straight line reflecting the trade off between C_0 and C_1.

we have the following formula:

- For $c=(c_0, c_1)$, $\underset{\text{Future Consumption}}{\boxed{c_1}} = \underset{\text{Future endowment}}{\boxed{w_1}} + \overbrace{(1+r)\cdot\underbrace{(w_0-c_0)}_{\text{Current Saving}}}^{\text{Gross return on saving.}} = -(1+r)\,c_0 + (1+r)\cdot w_0 + w_1$
- For $\alpha=(0,\alpha_1)$, $\alpha_1 = w_1 + (1+r)\cdot(w_0-0) = w_1 + (1+r)\cdot w_0$
- For $\beta=(\beta_0, 0)$, $\beta_0 = w_0 + \frac{w_1}{1+r}$

This means given $E=(w_0, w_1)$ and the market interest rate, individual can reach any point on the budget line by setting saving (lending) and borrowing activities.

The indifference curve in this model represent the individuals combinations of consumption that gives the individual the same utility.
The optimal choice is the tangency point between budget line and the indifference curve.



Remark: From the model we can see that individual derive more utilities from balanced consumption patterns with stability. This is because the optimal choice is always close to the center of the indifference curve.

Market clearings

we can derive the definitions of net savers and net borrowers:

Definition (Net savers): A individual is a net saver when $C_0 < W_0$, and net saving = $W_0 - C_0$. Net Saver supplies capital.

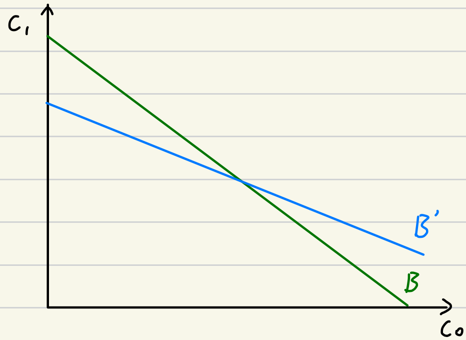
Definition (Net borrower): A individual is a net borrower when $C_0 > W_0$. The net borrowing = $C_0 - W_0$. Net borrower demand capital.

In the loan market where price of money (funds) = interest rate:

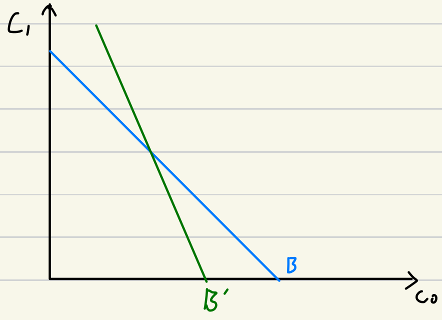
- Net saver supplies funds
- Net borrower demand funds
- when $NS > NB \Rightarrow$ Supply $> D \Rightarrow$ Money is not cleared and too expensive.

  The interest rate must drop $\Rightarrow$ This means $-(1+r)$ becomes bigger $\Rightarrow$ Budget constraint becomes flatter and C_0 is cheaper.

  The interest rate drop until $NS = NB \Rightarrow$ Then will achieve equilibrium interest rate or market clearing rate.



- when $NB > NS \Rightarrow$ Demand $>$ supply $\Rightarrow$ Money is too cheap $\Rightarrow$ interest rate will rise $\Rightarrow$ Slope $-(1+r)$ of the budget constraint will increase $\Rightarrow$ budget curve becomes more skewed and C_0 becomes more expensive $\Rightarrow NB \downarrow$, $NS \uparrow \Rightarrow$ until $NB = NS$ to achieve equilibrium r.



Remark: Example above assume the market is competitive.

In a competitive market:
- There are many traders or investors, no individual can move market price.
- Trading is costless.
- Both side has perfect information.

There can be only one equilibrium interest rate in a competitive market, otherwise a arbitrage opportunity would rise.

Definition (Arbitrage): The process of making risk-free profit through exploiting price difference.

Example of arbitrage as a result of having two interest rate:

Consider two bank A, B with 3% interest and 5% interest respectively.
- Investor will borrow from bank A with $r = 0.03$
- Then they save the borrowed money in bank B with $r = 0.05$.
- The profit will be $5\% - 3\% = 2\%$ interest.
- The profit is risk-free as long as interest rate does not change.

$\Downarrow$

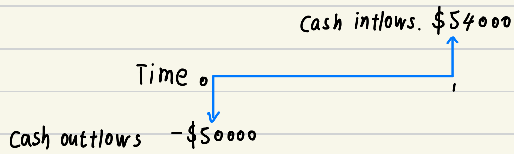
- This would mean bank A will have increasing borrowing demand
  Bank B will have increasing supply of money.
- This means bank A will increase its interest rate to deal with the increasing demand.
  Bank B will drop interest rate to deal with increasing supply.
- Eventually, bank A and B will reach equilibrium interest rate $\Rightarrow$ no more arbitrage.

The basic principle of investment

An investment must be at least as desirable as the opportunities avaliable in the financial market

Example 1:

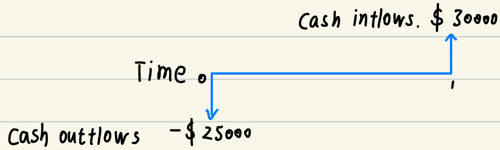
Consider an investment opportunity that costs \$50000 this year and provides a certain cash flow of \$54000 next year.



The rate of return on this investment project is $50000 \cdot (1+r) = 54000 \Rightarrow r = 8\%$.
If the financial market cannot offer a rate higher or equal to 8% risk free, the project is at least worth investment by the principle.

Example 2:

Consider an investor who has a initial endowment $w_0 = \$40000$ and $w_1 = \$55000$.
Suppose she faces 10% interest rate on saving and borrowing. And following investment is offered:



The rate of return on this new investment is $r = 20\%$.
Now, we consider the inter-temporal consumption model:

We have point A since we first consume $40000 - 25000 = 15000$ today, save 25000 and receive 10% rate. The next year we will consume $C_1 = 55000 + (1+0.1) \cdot 25000 = 82500$. We are apply the formula here.
Now, we consider the alternative plan of doing the investment. She will still consume

15000, invest 25000 to the project, and consume 55000 + 25000 · 1.2 = 85000 next year:

we are clearly better off since $U_1 < U_2$. we achieve better utility.

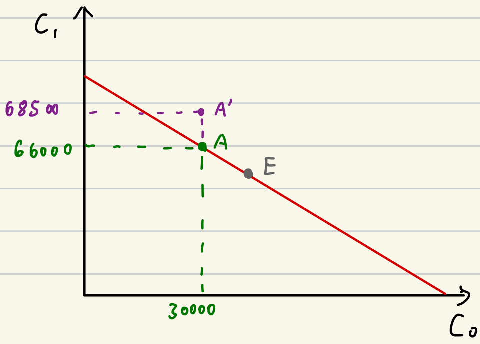
Example 3:

with the same set up:

what if $C_0 = 30000$ instead of 15000? she can still take 10000 and borrow 15000 from the bank at 10% rate. Then, she invest in project.

She will receive 30000 from the project, and pay $1.1 \cdot 15000 = 16500$ to the bank. And she still has 13500 left. This means she can still consume 55000 + 13500 = 68500.

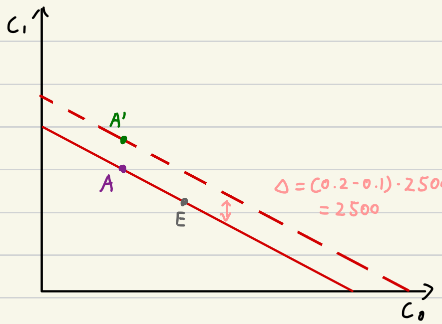
So, she is still better off.



The outside investment opportunity shifts the consumption opportunity upwards to dotted line.

With proper arrangement, individual can now consume at any point on the dotted line.

The amount of shift is the difference between rate of return on outside investment and the rate offered in the financial market.



In conclusion:

- Any outside investment options that offers a rate of return higher than the rate of saving and borrowing on financial market effectively shifts the consumption budget upwards.
- Individual will become necessarily better off.
- The optimal consumption will shift upwards.

Additional formula summary:

Take r' = rate of return on investment, r = market interest rate / loan rate.

I_0 = investment cost, I_1 = investment profit

$I_1 = I_0 \cdot (1+r')$, $C_1 = W_1 + I_1 + (1+r)(W_0 - C_0 - I_0)$

Δ shift of budget curve $= (r'-r)\cdot I_0$

The fisher separation theorem

> The best investment decisions is separated from individual's consumption decisions regardless of their personal preferences.

The firms should choose investments that maximize the present value of future cash flow, which is considered the best investment decision.

All shareholders will agree on the firms investment decision if they are made to maxmize their wealth.

optimal investment decisions by the firm increases the value of the firm, thereby expanding the potential opportunities for shareholders. shareholders can use financial market to adjust their personal consumption patterns to suit their preferences.

The theorem assumes on competitive capital market:

- No transaction cost
- Perfect information
- All investors are price takers.

And if the assumptions are violated, different individual will arrive at different investment decisions.

For example: consider the case when there is a transaction cost.

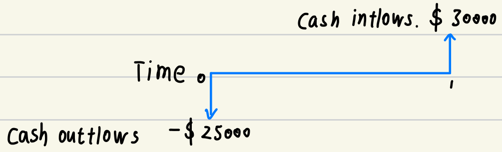
A transaction cost can be:

- cost of adinistrative and operation
- cost of credit analysis
- Regulatory compliance cost
- Cost of default

Therefore, in order to cover the cost, saving rate is set to be smaller than loan rate to cover the expenses.

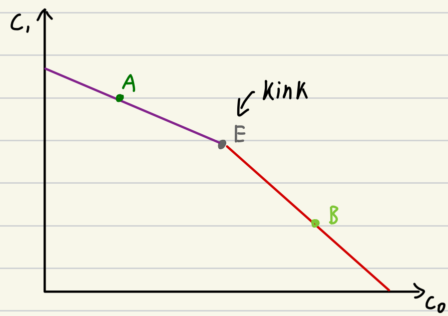
Consider an investor who has a initial endowment $W_0 = \$40000$ and $W_1 = \$55000$.

Suppose she faces 10% saving rate and 50% loan rate. And following investment is offered:



The rate of return on this new investment is $r = 20\%$.

- To the left of the endowment, the person is a net saver, and earn 10% saving rate.
- To the right of the endowment, the person is a net borrower, and pay 50% loan rate.
- This means to the left E, rate is smaller, so the curve is flatter.
- This means to the right E, rate is higher, so curve is steeper.



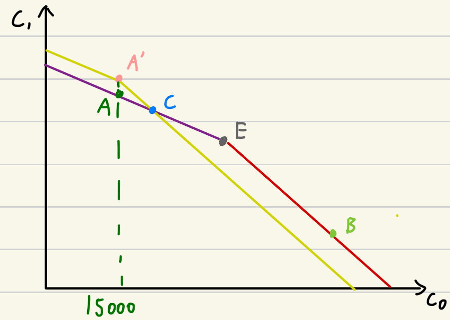
Net saver would like to pick A, Net borrower would like to pick B.

For person A, she initially prefers to spend $15000 = C_0$ and choose A.

Now, she invest the rest of the money $40000 - 15000 = 25000$ and get 30000 as profit.

Now, she has $C_1 = 55000 + 30000 = 85000$ and reaches A'.

And it is not just person A, any one with $C_0 < 15000$ will have shift up budget curve.



For those who $C_1 > 15000$, they have to borrow at high loan rate. So as demand rise, the line will be steeper to the right of A'.

we notice that to the left of C, the budget line shifts up, so they are better off to invest. Thus, they will take the investment.

we notice that to the right of C, the budget line shifts down, so they are worse off to invest. Thus, they will not invest.

This example show that personal preferences impacted investment decision, so the fisher information does not work. This is also because the market is not competitive.

Time value of money

Introduction

Definition (Future value / compound value): Value of a sum after investing it over one or more periods.

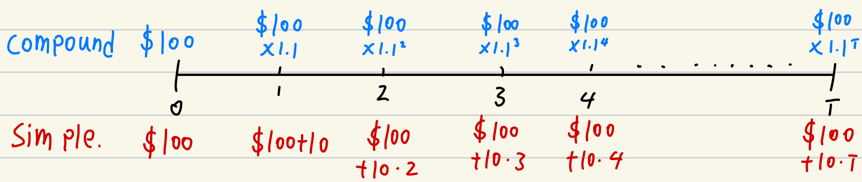
Definition (Gross return): The gross return on an investment or saving includes the initial principle and interest earned.

Definition (Present value): The current value of a sum of money that will be received or paid in the future, discounted at a rate.

Simple vs. compounding

Definition (simple interest): It is a kind of interest that is not re-invested and interest does not earn future interest.

Definition (compound interest): It is a kind of interest that is immediately re-invested and earns interest on interest.



Future value - one period case

In one period case: $FV = C_0 \cdot (1+r)$, where C_0 is the cash flow at 0 and r is the appropriate interest rate.

Future value - Multi-period case with constant interest rate

Assumptions of the general formula case:

- Investment yields payoff at the end of each period
- The principle and the payoff is immediately reinvested at the same rate of interest per period.

$$FV = C_0 \cdot (1+r)^T$$

, C_0 is the initial cash flow or PV

T is the numbers of period

r is the constant interest rate

Future vale - multiple period with changing interest rate

we have $FV = C_0 \cdot (1+r_1) \cdot (1+r_2) \cdots (1+r_n)$

- n = number of periods
- $r_1, r_2, \ldots, r_n$ = interest rate per period
- C_0 = initial cash flow or PV.

Rule of 72

The time to double the initial investment is about $\frac{72}{r}$, where r is constant growth rate per period.

present value - one period

In the one-period case: $PV = \frac{C_1}{1+r} = C_1 \cdot (1+r)^{-1}$, C_1 is the cash flow at date 1

r is the interest rate (discount rate).

present value - multiple period with constant r

$PV = C_T \cdot (1+r)^{-T}$, C_T is the lump-sum cash flow at period T.

r is the appropriate interest rate, called discount rate.

T is the number of periods.

Present value - Multiple period with changing r

$$PV = \sum_{t=1}^{n} \frac{C_t}{(1+r_t)^t},$$

C_t = cash flow at period t

r_t = interest rate for period t

t = time period

n = total number of period

present value - Multiple period with changing cash flow at constant r

$$PV = \sum_{t=1}^{n} \frac{C_t}{(1+r)^t},$$

C_t = cash flow at period t

r = interest rate

t = time period

n = total number of period

Basic type of cash flow model

perpetuity

Definition (Perpetuity): constant stream of identical cash flows that continuous indefinitely.

$$PV = \frac{C}{r}, \quad C = \text{cash flow per period}, \ r = \text{discount rate or interest rate.}$$

Growing perpetuity

Definition (Growing perpetuity): cash flows that increase at a constant rate g indefinitely.

$$PV = \frac{C}{r-g},$$

C = initial cash flow
r = discount rate
g = growth rate
$g < r$.

For $g \geq r$, this means present value is infinitely large, which is unrealistic.

Annuity

Definition (Annuity): series of equal cash flows made at regular intervals for a fixed number of periods.

$$PV = C \cdot \frac{1-(1+r)^{-n}}{r},$$

C is the first payment
r is the discount rate
n is the number of payments (period)

- The present value of a perpetuity that starts making payments of C at the end of period 1 is $PV_1 = \frac{C}{r}$.
- The present value of a perpetuity that starts making payment of C at the end of period $T+1$ is $PV_2 = \frac{C}{r} \cdot \frac{1}{(1+r)^T}$

we notice that $PV_1 - PV_2 = \frac{C}{r}\left(1 - \frac{1}{(1+r)^T}\right)$, this means by subtracting the second perpetuity from the first, we remove the payments that would occur from $T+1$ period onward. This leaves us with value of an annuity that only lasts for T periods.

Annuity with delayed payments

Annuity with delayed payments means the payments begin at future date rather than immediately. During the delay period, no payments are made.

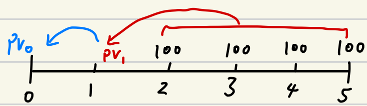
The present value of a delayed annuity is calculated by first determine the value of the annuity at the start of the payment period, then discounting the value back to today:

$$PV_{annuity} = \frac{c}{r} \cdot \left(1 - (1+r)^{-t}\right)$$

$$PV_0 = \frac{PV_{annuity}}{(1+r)^n} \quad , n = \text{number of payments before the start payments.}$$

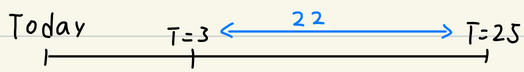
For example: what is the present value of a four year annuity of \$100 per year that makes its first payment two years from today if the discount rate is 9% ?

$$PV_1 = \frac{100}{0.09}\left(1 - \frac{1}{(1+0.09)^4}\right) = 323.97, \quad PV_0 = \frac{PV_1}{1+r} = 297.22$$



For example:

5.28 What is the PV of an annuity of \$6,500 per year, with the first cash flow received three years from today and the last one received 25 years from today? Use a discount rate of 7 percent.



$$PV_3 = \frac{6300}{0.07}\left(1 - \frac{1}{1.07^{22}}\right) \qquad PV_0 = \frac{PV_3}{(1+0.07)^3}$$

Growing annuity

Definition(Growing annuity): cash flows that grow at a constant rate g for a fixed number of periods.

$$PV = \frac{1 - \left(\frac{1+g}{1+r}\right)^n}{r-g} \cdot C \quad , \quad \begin{array}{l} r = \text{discount rate} \\ g = \text{growth rate} \\ n = \text{total periods.} \end{array}$$

For example:

Growing Annuity

5.33 Winnipeg Publishing Company is trying to decide whether to revise its popular textbook, *Financial Psychoanalysis Made Simple*. The company has estimated that the revision will cost \$75,000. Cash flows from increased sales will be \$21,000 the first year. These cash flows will increase by 4 percent per year. The book will go out of print five years from now. Assume that the initial cost is paid now and revenues are received at the end of each year. If the company requires a 10 percent return for such an investment, should it undertake the revision?

$$PV = C \cdot \frac{1-\left(\frac{1+g}{1+r}\right)^n}{r-g} = 21000 \cdot \frac{1-\left(\frac{1+0.04}{1+0.1}\right)^5}{0.1-0.04} = \$88459.07 > \$75000 = \text{investment cost}$$

The company should undertake the revision as its cost is smaller than its PV.

For example:

Calculating Growing Annuities

5.55 You have 30 years left until retirement and want to retire with $2 million. Your salary is paid annually, and you will receive $70,000 at the end of the current year. Your salary will increase at 3 percent per year, and you can earn a 9 percent return on the money you invest. If you save a constant percentage of your salary, what percentage of your salary must you save each year?

$$PV = \frac{FV}{(1+r)^n} = \frac{2000000}{1.09^{30}} = \$150808.97$$

$$C = \frac{PV}{\left(\frac{1-\left(\frac{1+g}{1+r}\right)^n}{r-g}\right)} = \frac{150808.97}{12.44065} = \$12119.66$$

$\frac{12119.66}{70000} \cdot 100 = 17.31\%$, you need to save 17.31% each year.

For example:

Growing Annuities

5.53 Farhan Qureshi has received a job offer from a large investment bank as a clerk to an associate banker. His base salary will be $55,000. He will receive his first annual salary payment one year from the day he begins to work. In addition, he will get an immediate $10,000 bonus for joining the company. His salary will grow at 3.5 percent each year. Each year he will receive a bonus equal to 10 percent of his salary. Farhan is expected to work for 25 years. What is the PV of the offer if the discount rate is 9 percent?

$$PV_S = S_0 \cdot \left(\frac{1-\left(\frac{1+g}{1+r}\right)^n}{r-g}\right) = 55000 \cdot \left(\frac{1-\left(\frac{1+0.035}{1+0.09}\right)^{25}}{0.09-0.035}\right) = \$756604.2$$

$$PV_B = 0.1 \cdot 55000 \cdot \left(\frac{1-\left(\frac{1+0.035}{1+0.09}\right)^{25}}{0.09-0.035}\right) = \$75660.42$$

$PV_{B_0} = 10000$ $\quad PV_{total} = PV_S + PV_B + PV_{B_0} = \$842264$

Growing annuity with delayed payments

Growing annuity with delayed payments is defined similar to annuity with delayed payments but the cash flow is growing and the first payment of the annuity is delayed.

For example:

Now suppose the rent is $5000 for the first year and $8500 in the second year. And is expected to increase 7% each year after year 2. Rent payments are collected at the end of each year.

What is the payment value of the estimated income stream over the first 5 years if the discount rate is 12% ?

- Rent for the 1st year = \$5000
- Rent for the 2nd year = \$8500
- Rent increase by 7% each year after the second year.
- Discount rate = 12%

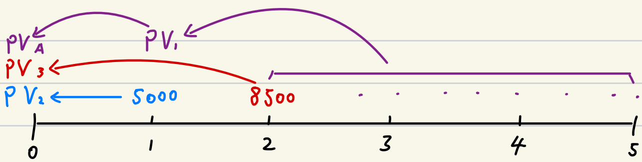
$$PV_1 = \frac{8500}{0.12-0.07} \cdot \left(1-\left(\frac{1+0.07}{1+0.12}\right)^4\right) = \$28384.12$$

$$PV_A = \frac{PV_1}{1.12} = \$25342.96$$

$$PV_2 = \frac{5000}{1.12} = \$4464.29$$

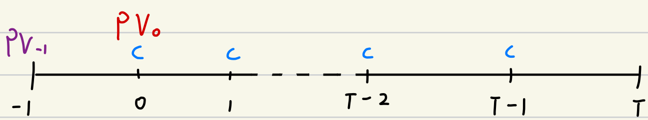
$$PV_3 = \frac{8500}{(1.12)^2} = \$6777.41$$

$$PV_T = PV_A + PV_2 + PV_3 = \$29807.25$$



Annuity due

Definition (Annuity due): A series of equal payments made at the begining of each period for a fixed number of periods.



$$PV_{-1} = \frac{c}{r} \cdot \left(1-(1+r)^{-T}\right), \quad PV_0 = (1+r) \cdot PV_1$$

More example on annuity

Calculating Annuity Payments

5.65 Your friend is celebrating her 30th birthday today and wants to start saving for her anticipated retirement at age 65. She wants to be able to withdraw $100,000 from her savings account on each birthday for 25 years following her retirement, with the first withdrawal on her 66th birthday. Your friend intends to invest her money in the local credit union, which offers 8 percent interest per year. She wants to make equal annual payments on each birthday into the account established at the credit union for her retirement fund.

a. If she starts making these deposits on her 30th birthday and continues to make deposits until she is 65 (the last deposit will be on her 65th birthday), what amount must she deposit annually to be able to make the desired withdrawals at retirement?

b. Suppose your friend has just inherited a large sum of money. Rather than making equal annual payments, she has decided to make one lump-sum payment on her 30th birthday to cover her retirement needs. What amount does she have to deposit?

c. Suppose your friend's employer will contribute $1,300 to the account every year as part of the company's profit-sharing plan. In addition, your friend expects a $45,000 distribution from a family trust fund on her 55th birthday, which she will also put into the retirement account. What amount must she deposit annually now to be able to make the desired withdrawals at retirement?

a: $PV = 100000 \cdot \frac{1-\left(\frac{1}{1.08}\right)^{25}}{0.08} = \$1067477.62$

$FV = 1067477.62 = C \cdot \frac{1.08^{36}-1}{0.08} \Rightarrow C = \$5705.32$

b: $FV = 1067477.42 = PV \cdot (1.08)^{36} \Rightarrow PV = \$66850.33$

C: From the trust deposit: $FV = 45000 \cdot 1.08^{10} = \$97151.62$

we have $1067477.62 - 97151.62 = \$970326$

$970326 = C \cdot \frac{1.08^{26}-1}{0.08} \Rightarrow C = 5186.08$

The friend must contribute $5186.08 - 1300 = \$3886.08$

Compound frequency and compound within a year

Definition (Compound frequency): The number of times interest is added to the principle within one year.

Definition (Annual percentage rate): The nominal annual interest rate for the whole year.

Definition (Effective rate per period): The real return applied in each compound period within a year.

Definition (Effective annual rate): It is the interest rate that accounts for the compounding frequency. It gives us the same amount of saving / investment at the end of the period.

Remark: we denote annual percentage rate as APR, and effective annual rate as EAR.

For compounding within a year, we have the following formula:

$FV = C_0 \cdot (1+\frac{r}{m})^{m \cdot T}$	C_0 = initial savings r = Annual percentage rate m = compounding frequency $i = \frac{r}{m}$ = effective rate per period $n = m \cdot T$ = Total number of payment periods T = number of years
$PV = C_T \cdot (1+\frac{r}{m})^{-m \cdot T}$	T = number of years C_T = value at year T r = Annual percentage rate m = compounding frequency $i = \frac{r}{m}$ = effective rate per period $n = m \cdot T$ = Total number of payment periods
Effective annual rate = $(1+\frac{r}{m})^{m} - 1$	r = Annual percentage rate m = compounding frequency $i = \frac{r}{m}$ = effective rate per period

Remark: The discount rate reflects the idea that money today is worth more than the same amount in the future due to its potential earning capacity. Thus, discount rate is viewed as the opportunity cost. And this represents the minimum rate of return needed for investors since it represents the minimum willingness to pay for investors. So, discount rate can also be viewed as the required yield, which can be measured with APR. Thus, we can take APR = discount rate for the purpose of calculation.

Continuous Compounding

Given APR, as compounding frequency increases, EAR $= (1+\frac{r}{m})^{m} - 1$ will increase.

If the compounding frequency is infinite (continuous compounding), then

$$FV = C_0 \cdot e^{rT}, \quad r = APR$$

e = the constant e

T = number of years.

Canadian mortage as a application

In canada, mortage have unique characteristics and quoting conventions.

Canadian banks quote the annual interest rate (APR) compounded semi-annually for mortage. This means the nominal interest rate stated on the mortage is applied twice a year.

While the APR is quoted with semi-annual compounding, mortage payments are typically made monthly. To align with this monthly payment, interest is effectively compounded monthly. To calculate the effective rate per month and monthly payment, we will use EAR.

Definition (terms of the mortage): The length of time over which the conditions of the mortage including interest rate, payment schedule, etc. are agreed upon with the lender.

Definition (Amortization period): The total time over which the mortage loan is scheduled to be paid off. Usually monthly payments are scheduled.

The terms of a mortage can often be renegotiated periodically even if the amortization period is much longer. For example, the amortization period can be 25 years, and the term can be 5 years. And at the end of each term, you can renegotiate for a new term.

There are two kinds of mortage rate:

- Fixed rate mortage: The interest rate remains constant for the term of the agreement.
- Variable rate mortage: The interest rate fluctuates based on a benchmark interest rate in Canada.

Remark: Saving at APR of r per year compounded m times a year is equivalent to saving at EAR per year, compounded annually.

For example: You have negotiated a 25-year, \$300000 fixed rate mortgage at 2.9% per year compounded semi-annually with TD bank. What is your monthly payment?

EAR_{quote} = the EAR under APR $= (1+\frac{0.029}{2})^2 - 1 = 0.0292$.

$EAR_{payment}$ = EAR under monthly payments $= (1+i)^{12} - 1$

We solve for i by take $EAR_{quote} = EAR_{payment}$. This is because the effective interest rate per month provides the same annual yield as semi-annual compounded. This makes the mortage consistent over the year.

Here, $i \approx 0.002402$. Now, we calculate monthly payment using annuity formula.

$$PV = \frac{C}{i}\cdot\left(1 - \frac{1}{(1+i)^n}\right) \Rightarrow 300000 = \frac{C}{0.002402}\left(1 - \frac{1}{1.002402^{300}}\right) \Rightarrow C \approx \$1432.25$$

Cash flow at different frequencies – APR as discount rate

To determine which investment project is worth more to the investor, we need to calculate the present value of the cash flow from projects.

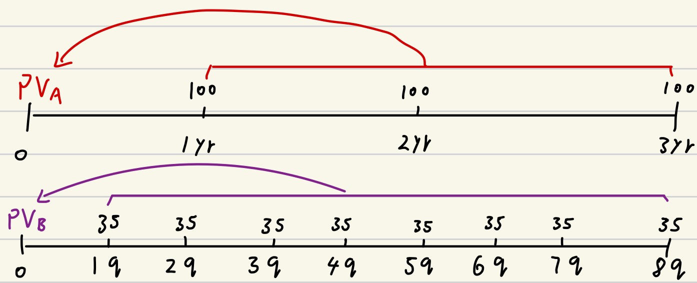
For example: Suppose there are two investment projects: Project A pays \$100 at the end of each year for 3 years; Project B pays \$35 at the end of each quarter for 2 years. The investor can earn 10% per year on the investment elsewhere. which project is worth more to the investor?

The 10% rate as the opportunity cost represents the discount rate. we take the APR as 10%, and we have $(1+r_Q)^4 = 1+0.1 \Rightarrow r_Q \approx 0.024114$.

This is because we ensure that the future cash flows from the projects are discounted at the rate that represents the investor's expected return.

So, $PV_A = \frac{100}{0.1}(1-1.1^{-3}) = \$248.69$

$$PV_B = \frac{35}{0.024114}(1-1.024114^{-8}) = \$251.91 > \$248.69$$



So, project B is worth more to the investor.

Net present value (NPV)

Definition (NPV): It is the difference as

$$NPV = PV \text{ of cash inflows} - PV \text{ of cash outflow}$$

or

$$NPV = PV \text{ of investment income} - PV \text{ of cost of investment.}$$

To calculate the PV of future cash flows, we use a discount rate, which is typically:

- The interest rate avaliable in the financial market.
- The rate of return on similar investment.

The fundamental rule for using the NPV in investment decision is:

Accept investment if $NPV > 0$

This rule is derived from the basic principle of investment.

For example: we will illustrate the NPV investment principle now.

Suppose we have \$C to invest for one year. The financial market offers an interest rate of r_F per year. An investment opportunity offers a payment of $\$C\cdot(1+r)$ in one year, but requires an initial investment of \$C.

We have $NPV = -C + \frac{C\cdot(1+r_I)}{1+r_F} = C\cdot\left(\frac{1+r_I}{1+r_F} - 1\right) \geq 0 \Rightarrow r_I > r_F$

Therefore, the $r_I > r_F$ indicates investment is acceptable.

Valuation of financial securities - A idea

The value of a financial security is determined by the PV of the expected future cash flows that the security will generate.

The estimate future cash flows is determined by size (the amount of cash flow) and timing (when these cash flows will occur).

We will use appropriate discount rate to convert future cash flows into their present value.

The discount rate is higher when the risk of the security is higher. This is to compensate investors for taking more risk.

Valuation of bonds

Definition of bonds

Definition (Bond): A bond is a legally binding agreement between a borrower (bond issuer) and a lender (bond holder).

- The principle amount of the loan is the lump-sum payment due at the end of the borrowing term.
- Bonds usually involve coupon interest payment. And the paying of principle is mandatory.
- There are usually two types of interest rate:
  - Fixed rate bond: The coupon interest rate is fixed.
  - Variable rate bond: The coupon interest rate is variable.

Terminology	Explain
Par (face) value	The par value or face value of the bond is the principle. The quoted price of a bond represents the current market price at which bond can be bought or sold. And this price is typically expressed as a percentage of their par value. We usually quote bond per \$100. This means the quoted price is the price you would pay for each \$100 of the par value. Or equivalently, it is equals to a percentage (higher than 100%) of the par value. For example: Suppose a bond has \$1000000 par value. If this bond is quoted at \$107.05, and quote is per \$100 of per value, this means the ratio is given by $\frac{1\,000\,000}{100} = 10000$. Since the quote price is \$107.05 per \$100 in par value. The actual price for purchasing this bond is $10000 \cdot 107.05 = \$1070500$.
Maturity date	This is when the principle amount of the debt must be repaid. Time to maturity is the remaining time until the maturity date, usually measured in years.
Yield to maturity (YTM)	The annual rate of return on a bond if held until maturity, expressed as an Annual percentage rate. This means it is the rate of return an investor will earn if bond is held until maturity, assuming all payments are made as scheduled.
Coupon rate	The interest rate used to calculate coupon payment, expressed as a percentage of the par (face) value.
Premium	If the bond price is greater than the par value, the difference price - par is called premium. Such bond is called premium bond and called "sell at premium"

Discount	If the bond price is less than the par value, the difference price-par is called the "discount". These bonds are called discount bonds and sell at a discount.
At par	If the bond price equals the par value, the bond sells at par.
Holding period return (HPR)	The rate of return earned during the period you hold the bond, not necessarily until maturity. $$HPR = \frac{\text{Coupon payment} + P_{n+1} - P_n}{P_n}$$ P_{n+1} = Price at future P_n = Current Price
Current yield	The ratio of annual coupon payment(s) to the bond price. $$\text{Current yield} = \frac{\text{Total annual coupons}}{\text{Bond Price}}$$ Total annual coupons $= \sum$ all coupons over a year

Remark: In reality, bond quoted price changes all the time like stocks.
Remark: Bond price is another word for bond selling price.
Remark: Bond quoted price = percentage relative to the par value s.t. Bond Price = quoted price · Bond par value.

Bond value determination

The bond value is determined by the present value of the coupon (if any) and the face value payments.

This means the value of a bond is the sum of the present values of its future cash flows. These cash flows include periodic coupon payments (if any) and the repayment of the value value at maturity.

We have to identify the size and timing of cash flows:

- The size of the coupon payments.
- The timing of these payments.
- The face value (principal) repayment at maturity.

If we have the price of a bond and the size and the timing of cash flows, then the discount rate is the yield to maturity (YTM).

Discount rates are inversely related to present (i.e. bond) values. This means when the discount rate (YTM) increases, the present value (price) of the bond decreases, and vice versa.

Based on the condition of the bond, we have two major types: pure discount (zero-coupon) bond and level-coupon bonds.

Pure discount (zero-coupon) bonds

Definition (pure discount bonds): The bonds with no coupon payments. The only cash flow is the repayment of the face value at maturity.

Remark: Since there is no periodic interest payment, coupon rate = 0%.

The yield to maturity arises from the difference between the purchase price and the par value. It is usually sold for less than their face value because the only return to the investor is the appreciation to face value at maturity.

In order to value pure discount bond, we need:

- Time to maturity (T): The number of periods from now until the bond matures
- Face value (F): The amount paid to the bondholder at maturity
- Discount rate (i): The YTM per period

$$\boxed{\text{The price (PV) of a pure discount bond at time } 0 = PV = \frac{F}{(1+i)^T}}$$

For example: Find the value of a 30 year zero-coupon bond with a face value of \$1000 and a YTM of 6%:

$$PV = \frac{1000}{(1+0.06)^{30}} = 174.11$$

The quoted price on the bond market per \$100 of face value:

$$\text{Quoted price} = \frac{174.11}{\frac{1000}{100}} = \$17.41$$

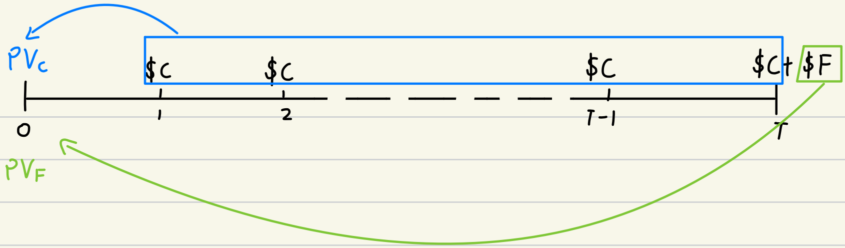
level-coupon bond

Definition (level-coupon bond): It is bonds that make periodic payments in addition to face value payment. The coupon payments are fixed and typically made semi-annually.

In order to value this bond, you need:

- Coupon payments schedule and maturity (T): The number of periods until maturity
- Coupon payment per period (c) from the coupon rate and face value (F)
- Discount rate (i) per coupon payment period $= \frac{YTM}{T}$ = expected rate per period

$$\boxed{\begin{aligned}\text{The price at time } 0 &= PV = PV \text{ of coupon payment} + PV \text{ of face value} \\ &= \frac{c}{i}\left[1 - \frac{1}{(1+i)^T}\right] + \frac{F}{(1+i)^T}\end{aligned}}$$



For example: Find the present value as of January 1, 2011, of a $6\text{-}\frac{3}{8}\%$ coupon bond with semi-annual payments, and a maturity date of December 31, 2016. The YTM is 5%. The face value is \$1000.

The effective rate semi-annually is $\frac{6.375\%}{2} = 3.1875\%$

The required annual rate (discount rate) is 5%, and the semi-annual required rate is $\frac{5\%}{2} = 2.5\%$

This means $C = 1000 \cdot 3.1875\% = \$31.875$

So $PV = \frac{31.875}{0.025} \cdot \left[1 - \frac{1}{1.025^{12}}\right] + \frac{1000}{1.025^{12}} = \$1070.52$

Bond Prices between coupon dates

Coupon dates are the specific dates on which these interest payments are made to bond holders. When a bond is traded (bought or sold) on any day that is not one of these coupon dates, it is said to be trading between coupon dates.

Definition (Accrued interest): This is the interest that has accumulated since the last coupon payment up to the date of sale.

When a bond is sold between coupon dates, the seller is entitled to the interest earned during their holding period up to the sale date.

Definition (clean or quoted price): It is the price of the bond that is quoted in the bond market. It reflects the bond's value based solely on the present value of its future cash flows without include any accrued interest since the last coupon payment.

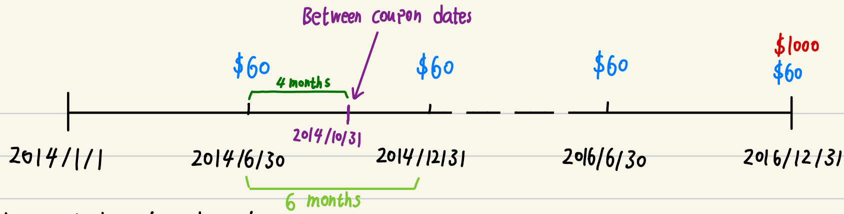
For example: Imagine a bond that pays coupons semi-annually on June 30 and December 31. Suppose today is October 31, the next coupon payment date is 2 months away. The last coupon payment was on June 30.

The clean price is the value of the bond assuming it is still June 30, the last coupon payment. It does not include the interest that has accrued from July 1 to october 31.

Definition (Dirty price): It is also known as full or invoice price. It is the actual price the buyer pays, which includes the clean price plus the accrued interest.

$$\text{Dirty price} = \text{clean price} + \text{accrued interest}$$

Complete example: Suppose a bond pays a \$60 coupon payment every six months. If a bond is sold on october 31, which is between coupon dates.



Let the quoted price be \$1080. So, the clean price = \$1080.

The fraction of period accrued = $\frac{4 \text{ mon}}{6 \text{ mon}} = \frac{2}{3}$. So, the accrued interest $= \frac{2}{3} \cdot 60 = \$40$

Thus, the dirty price = 1080 + 40 = \$1120

Calculating yield to maturity YTM

YTM is the annual percentage rate of return on a bond if it is purchased and held until maturity.

As we introduced, take m = number of coupon payments per year:

$$\text{Bond Price} = \frac{C}{\frac{YTM}{m}} \cdot \left[1 - \frac{1}{(1+\frac{YTM}{m})^{m \cdot T}}\right] + \frac{F}{(1+\frac{YTM}{m})^{m \cdot T}}$$

Given the current price of a bond, the YTM becomes the discount rate in solving the equation above.

The YTM is also the minimum rate of return that investors expect to earn on the bond until maturity.

For example: A bond pays a semi-annual coupon at coupon rate of 10%. Maturity date is 2.5 years. And the current quoted price is \$102.05.

This means the coupon payment $C = \$5 = \frac{100 \cdot 10\%}{2}$, and $n = 2 \cdot 2.5 = 5$.

Since the coupon pays semi-annually, the effective rate per period $= \frac{YTM}{2}$

The price is \$102.05, and face value = \$100.

We have:

$$102.05 = \frac{5}{\frac{YTM}{2}} \cdot \left[1 - \frac{1}{(1+\frac{YTM}{2})^5}\right] + \frac{100}{(1+\frac{YTM}{2})^5}$$

$$\Rightarrow i = \frac{YTM}{2} = 4.5326\% \Rightarrow YTM = 2 \cdot 4.5326\% = 9.0652\%$$

For example: A zero-coupon bond is selling for \$95, it has 2.5 years until maturity.

We have

$$95 = \frac{100}{(1+YTM)^{2.5}} \Rightarrow YTM = 2.0729\%.$$

Bond price and YTM relationship

We define the market interest rate as the current rate of return that investors require for investing in bonds of a specific type and maturity.

We have the following relationship:

— When market interest rate rise, new bonds are issued with higher coupon rates to attract investors. Existing bonds with lower coupon rates becomes less attractive because investors can get a better return with new bonds.

— To make yield of existing bond competitive with the new higher market rate, the price of the existing bonds must decrease. This decrease in price increases the YTM of existing bonds.

— Conversely, decreasing YTM align with lower market rate.

$\Downarrow$

$$YTM = f(r_m),\quad \frac{d(YTM)}{d(r_m)} > 1,\quad r_m = \text{market interest rate}$$

— When market interest rate fall, existing bonds with higher coupon rates becomes more attractive. Investors are then willing to pay more for these bonds. So, price will increase.

— Vice versa, when market interest rate increase, price will decrease.

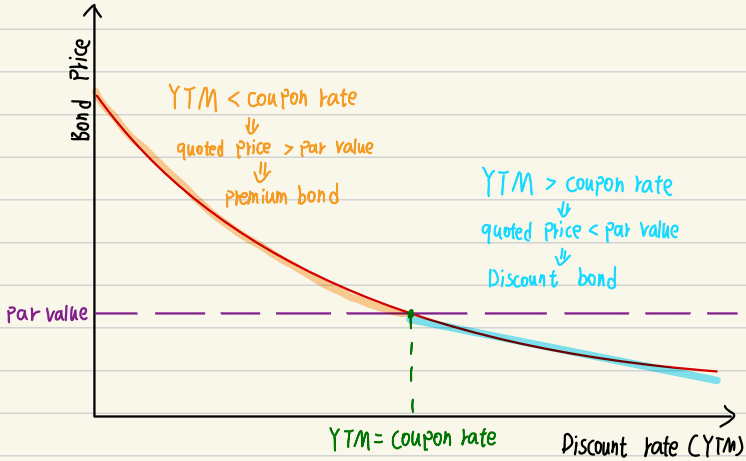
$\Downarrow$

$$P = f(r_m),\quad \frac{dP}{dr_m} < 1,\quad r_m = \text{market interest rate},\ P = \text{bond price}$$

Thus, we conclude:

—As $r_m \uparrow$, we have $P\downarrow$ and $YTM\uparrow$
—As $r_m \downarrow$, we have $P\uparrow$ and $YTM\downarrow$
$\Rightarrow$ bond price and YTM are inversely propotional.

And we have the following graphic representation:



From the graph, we have:

— When the coupon rate = YTM, the bond price = its par value. This is because when the coupon rate equals the YTM, the bond's interest payments are exactly what the market demands for that bond's level of risk and maturity. Because the bond is offering exactly the return that investor required.

there is no need for a price adjustment. The bond does not need to be priced higher (premium) or lower (discount) than its par value.

— when coupon rate > YTM, the bond offers more interest than the market requires. Thus, it is more attractive to investors, leading to a higher price than the par value. This is known as premium bond.

— when coupon rate < YTM, the bond offers less interest than the market requires. Thus, it is less attractive to investors, leading to a lower price than the par value. This is known as discount bond.

Bond Price Volatility and maturity

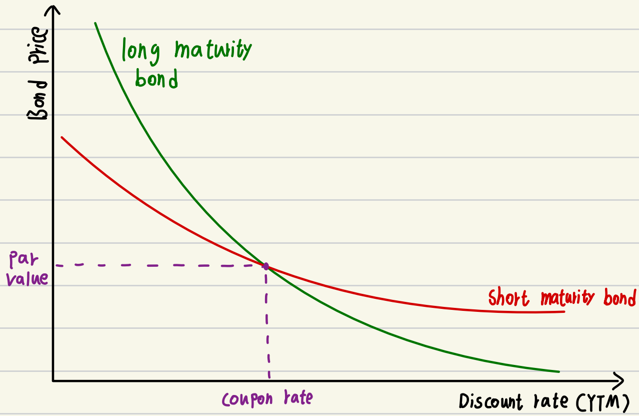
Consider bond with long maturity:

— Bonds with longer maturities have more future coupon payments and the final face value payment. They need to be discounted back using YTM. More cash flows being discounted means more sensitivity to changes in the discount rate.

— The farther into the future a cash flow is, the more it gets discounted. This means the present value of these distant cash flows will be much lower than nearer-term cash flows. Changes in YTM have a larger impact on these deeply discounted cash flows, increasing the bond's price volatility.

⇓

Thus, bonds with long maturity are more volatile in response to changes in YTM compare with bonds with short maturity.

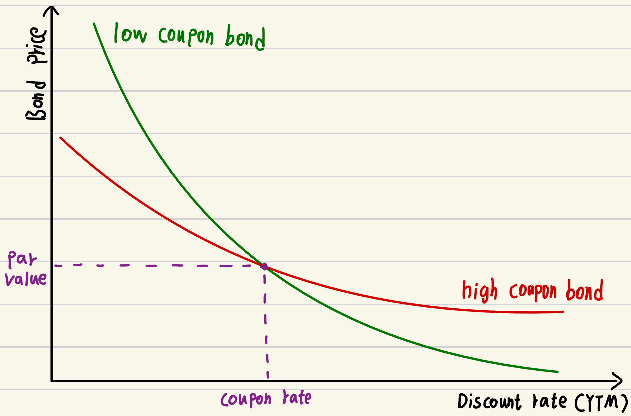


We have bond price = PV of coupon + PV of face value

— For bonds with lower coupon rates, a larger proposition of the bond's total value comes from the face value payment, which occurs far in the future and is therefore more heavily discounted. This makes the bond more sensitive to changes in YTM because the distant face value payment's present value changes significantly with YTM fluctuations.

⇓

Therefore, bonds with lower coupon rates exhibits greater price volatility than high coupon rate bonds.



Holding period return HPR

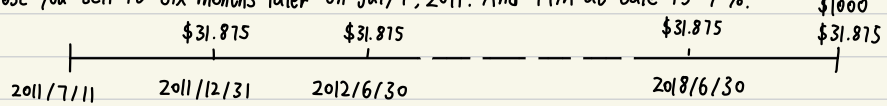
The gains or losses from holding a bond for resale can be:

- Increase or decrease in bond prices.
- coupon payments received.

The holding period return measures the total return earned from holding a bond over a specific period.

$$r_{Hold} = \frac{P_{sell} - P_{purchase} + \sum \text{coupon payments.}}{P_{purchase}}$$

For example: suppose on January 1, 2011, you purchase a 6.375% coupon government bond with semi-annual payments. It matures at December 31, 2018. The YTM at purchase is 5%. The purchase price = $1089.75. Suppose you sell it six months later on July 1, 2011. And YTM at sale is 4%.



Given that YTM at sale is 4%, the price of the bond is calculated using the present value formula:

$$PV = \frac{31.875}{\frac{0.04}{2}} \left[1 - \frac{1}{(1.02)^{15}} \right] + \frac{1000}{1.02^{15}} = \$1152.59$$

So:

$$r_{Hold} = \frac{1152.59 - 1089.75 + 31.875}{1089.75} = 8.69\%$$

And this leads to a effective annual rate $= (1 + r_{Hold})^2 - 1 = 18.14\%$

Things that affects the YTM

Definition (Default risk): This is the risk that the bond issuer will not be able to make the required interest payments or repay the principle.

— when there is a higher risk that the issuer might default, investors demand a higher yield to compensate for this increased risk. Thus, the price will be lower, which in turn increases the YTM. And vice versa.

— The market interest rate influence the YTM of bonds as introduced above.

— Investors' preference for liquidity also affects YTM. More liquid bonds generally have lower YTM, while less liquid bonds have higher YTM to compensate for the liquidity risk.

The yield curve, spot and forward rate

The YTM of bonds can vary depending on how long it is until the bond matures. This occurs even when the bonds have the same issuers.

Definition (Yield curve): It is a graphic representation of the relationship between the YTM and time to maturity for a series of bonds issued by the same entity.

The yield curve can have different shapes:

— The shape of the yield curve can indicate market expectations about future interest rates.

— Investor's preferences for bonds of different maturities can also influence the yield curve shape.

Spot and forward rate

Definition (spot rate): It is essentially the yield of a bond if you make the investment today and matures at a specific date in the future (YTM). Usually, we use zero-coupon bonds of various maturities to derive and calculate spot rates. Spot rates vary based on the bond's maturity, each spot rate corresponds to a specific term.

Definition (forward rate): They are the interest rates agreed upon today that will occur in the future. They can be interpreted as the market's expectation of future short-term interest rates.

They are derived from the current spot rates.

In an ideal world with no risk or liquidity preference, forward rate = future short-term interest rate. However, this is not possible in the real world.

The relationship between spot and forward rate has the general formula:

$$f_n = \frac{(1+S_n)^n}{(1+S_{n-1})^{n-1}} - 1,$$

f_n = forward rate from period $n-1$ to n

S_n = n-year spot rate

As we discussed, in a perfect world f_n = future short-term interest rates. otherwise, discrepancies between forward rates and actual future short-term rates r_n can lead to arbitrage opportunities.

For example: Consider two zero-coupon with short-term and a long-term.

The price for long-term (n year) zero-coupon bond is $P_0^L = \frac{FV}{(1+S_n)^n}$

The price for short-term (n-1 year) zero-coupon bond is $P_0^S = \frac{FV}{(1+S_{n-1})^{n-1}}$

suppose $f_n > S_n$, we have the following arbitrage strategy:

— At time 0:

we will borrow and sell the shorter-term bond in the market. We receive cash:

$$+P_0^S = \frac{1000}{(1+S_{n-1})^{n-1}}$$

This is called "short".

we will use the received cash to by $\frac{P_0^S}{P_0^L}$ units of long-term bond. We have outflow:

$$-\frac{P_0^S}{P_0^L} \cdot P_0^L = -P_0^S$$

— At time n-1:

we will buy back the shorter-term bond and return to the lender. The cash outflow is

$$-\$1000$$

And we borrow \$1000 to close the short position.

— At time n:

we receive the proceeds from longer-term bond adjusted for the initial purchase:

$$+\frac{P_0^S}{P_0^L} \cdot FV = +\frac{1000 \cdot (1+S_n)^n}{(1+S_{n-1})^{n-1}}$$

we repay the borrowed \$1000 at time n-1: $-1000 \cdot (1+r_n)$

Thus, we have total net cash flow $\frac{1000 \cdot (1+S_n)^n}{(1+S_{n-1})^{n-1}} - 1000 \cdot (1+r_n) = 1000 \cdot (f_n - r_n) > 0$.

Remark: The following arbitrage strategy does not work if $f_n < r_n$.

Remark: We assume zero transaction cost, perfect information, and arbitrage is risk free.

Remark: The above arbitrage is not possible because r_n can deviate from f_n.

Expectation hypothesis

The expectation hypothesis of the yield curve is:

The yield curve reflects the market consensus on future short-term interest rate i.e. $f_n = E(r_n)$.

This expectation has the assumption that short-term and long-term bonds are perfect substitutes for each other. This means they should have the same expected return.

Under this hypothesis:

- if investor expect future short-term rates to rise, the yield curve will slope upwards.
- if investor expect future short-term rates to fall, the yield curve will slope downwards.
- if investor expect future short-term rates to remain unchanged, the curve will be flat.

Liquidity preference hypothesis

This hypothesis suggest that investors prefer bonds for different maturities based on their liquidity needs.

— Investors prefer short-term bonds because they provide liquidity. And they typically offer lower yields. This means $f_n > E(r_n) \Rightarrow$ They want more yields.

— Investors prefer long-term bonds when they do not require immediate liquidity. To attract investors to hold longer-term bonds, issuers offer liquidity premium to compensate investors for the reduced liquidity and higher risk $\Rightarrow f_n = E(r_n) + LP$.

— long-term investors do not require liquidity, and they want stability. This means they do not require liquidity premium, and they accepts lower yields. This means $f_n < E(r_n) \Rightarrow f_n = E(r_n) - LP$.

This leads to the hypothesis:

> Investors demand a premium for holding long-term bonds due to lower liquidity.
>
> $$f_n = E(r_n) + \text{liquidity premium}$$

Remark: liquidity premium can be positive or negative

Assuming constant liquidity premium, we have $E(r_n) = f_n - LP$, where f_n can be calculated from the spot rates provided on the yield curve. This means we can predict future interest rate movements and economic conditions.

However, measuring exact liquidity premium is difficult and they are not constant over time.

Shapes of the yield curve

The yield curve has different shapes:

— Steeply rising yield curve:

This means the market expects the short-term interest rate to rise, and thus shifts up the interest rate of all maturities because the investors demand higher yields on the long term bond.

This often indicates economy is expanding.

— Modestly rising yield curve:

This means the market expects the short-term interest rate to rise somewhat.

we cannot predict the future economy with this curve because it lacks strong signals.

— Declining yield curve:

This means the market expects the short term interest rate to fall and economy is worsen.