MAT244 Notes

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February 2023

Contents

1 Basic Concepts in Differential Equations

1.1 Basic Definitions

Basic Definitions in Differential Equations

Definition 1.1 (Differential Equation)**.** Any equation which contains derivatives, either ordinary derivatives or partial derivatives.
For example:
$ay'' + by' + cy = g(t)$
$\sin(y)\frac{d^2y}{dx^2} = (1-y)\frac{dy}{dx} + y^2\mathbf{e}^{-5y}$

Definition 1.2 (Order)**.** The largest derivative present in the differential equation.

Definition 1.3 (Ordinary and Partial Differential Equations)**.** A differential equation is called an ordinary differential equation if it has ordinary derivatives in it. Likewise, a differential equation is called a partial differential equation.

Definition 1.4 (Linear Differential Equation)**.** It is any differential equation that can be written as:

$$a_n(t)\,y^{(n)}(t) + a_{n-1}(t)\,y^{(n-1)}(t) + \cdots + a_1(t)\,y'(t) + a_0(t)\,y(t) = g(t)$$

Definition 1.5 (Solution to a Differential Equation)**.** A solution to a differential equation on an interval $\alpha < t < \beta$ is any function $y(t)$ which satisfies the differential equation in question on the interval $\alpha < t < \beta$.

Definition 1.6 (Initial Condition)**.** They are values of the solution and/or its derivative(s) at specific points:

$$y(t_0) = y_0 \text{ and/or } y^{(k)}(t_0) = y_k$$

Definition 1.7 (Initial Value Problem)**.** Initial value problem (IVP) is an ordinary differential equation together with an initial condition which specifies the value of the unknown function.

Definition 1.8 (Interval of Validity)**.** The interval of validity for an IVP with initial condition(s) is the largest possible interval on which the solution is valid and contains the initial condition(s).

Definition 1.9 (General Solutions)**.** The general solution to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

Definition 1.10 (Particular Solution)**.** The Particular solution to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Definition 1.11 (Explicit Solution and Implicit Solution)**.** An explicit solution is any solution that is given in the form $y = y(t)$. An implicit solution is any solution that isn't in explicit form.

1.2 Direction Field

Direction Field

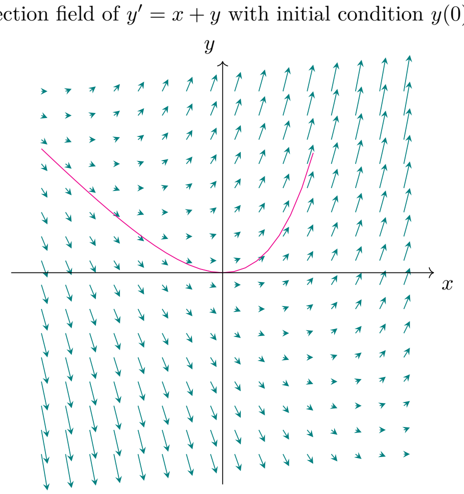
Definition 1.12. The direction field can be defined for the following type of differential equations:

$$y^{'} = f(x, y)$$

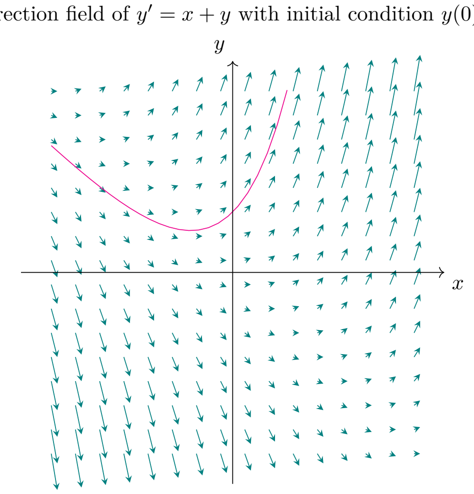
which can be interpreted geometrically as giving the slope of the tangent to the graph of the differential equation's solution (integral curve) at each point (x, y) as a function of the point coordinates.

Example 1.1.

Direction field of $y' = x + y$ with initial condition $y(0) = 0$



Direction field of $y' = x + y$ with initial condition $y(0) = 1$



A direction field for equations of the form can be constructed by evaluating f at each point of a rectangular grid. At each point of the grid, a short line segment is drawn whose slope is the value of f at that point. Thus each line segment is tangent to the graph of the solution passing through that point.

2 First-Order Differential Equation

2.1 First-Order Linear Differential Equation and Integrating Factor

First-Order Linear Differential Equation

Definition 2.1. The First-Order linear Differential Equation can be written as:

$$\frac{dy}{dt} + p(t)\, y = g(t)$$

where p and g are given continuous functions of the independent variable t.
It can also be written in the form:

$$P(t)\frac{dy}{dt} + Q(t)\, y = G(t)$$

The steps to solve this type of differential equation are:

1. Put the differential equation in the correct initial form,

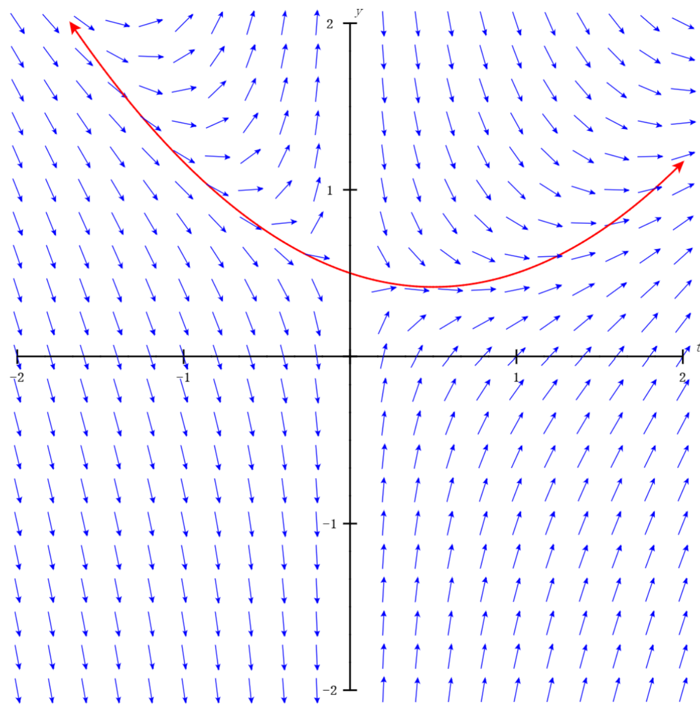
$$\frac{dy}{dt} + p(t)\, y = g(t)$$

2. Find the integrating factor, $\mu(t) = \mathbf{e}^{\int p(t)dt}$
3. Multiply everything in the differential equation by $\mu(t)$ and verify that the left side becomes the product rule $(\mu(t)\, y(t))'$ and write it as such.
4. Solve for the solution $y(t)$

Example 2.1. *Find the solution to the following IVP.*

$$ty' + 2y = t^2 - t + 1, y(1) = \frac{1}{2}$$

Sol: $y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$
$\mu(t) = \mathbf{e}^{\int \frac{2}{t}dt} = \mathbf{e}^{2\ln|t|} = t^2$
$\left(t^2 y\right)' = t^3 - t^2 + t$
$t^2 y \ = \int t^3 - t^2 + t dt \ = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + cy(t) \ = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2}$
$\frac{1}{2} = y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \Rightarrow c = \frac{1}{12}$
$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$
The Direction Field and its Solution is:


y
2
1
−1
−2
t
−2
−1
1
2

2.2 First-Order non-linear Differential Equation and separable variable

First-Order Non-Linear Differential Equation

Definition 2.2. The general first-order differential equation is $\frac{dy}{dx} = f(x, y)$, and we can rewrite this general equation into form.

$$M(x) + N(y)\frac{dy}{dx} = 0$$

This from is called as separable differential equation. This form can be written as

$$M(x)\,dx + N(y)\,dy = 0$$

or

$$N(y)\,dy = M(x)\,dx \Rightarrow \int N(y)\,dy = \int M(x)\,dx$$

Noted that Most of the solutions that we will get from separable differential equations will not be valid for all values of x, which we need to determine the interval of Validity which include the initial condition.

Example 2.2. *Solve the following IVP and find the interval of validity of the solution* $y' = \frac{xy^3}{\sqrt{1+x^2}}, y(0) = -1$

Sol:

$$y^{-3}dy = x\left(1+x^2\right)^{-\frac{1}{2}}dx$$
$$\int y^{-3}dy = \int x\left(1+x^2\right)^{-\frac{1}{2}}dx$$
$$-\frac{1}{2y^2} = \sqrt{1+x^2} + c$$

$-\frac{1}{2} = \sqrt{1} + c, c = -\frac{3}{2}$ $-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$
$$y^2 = \frac{1}{3-2\sqrt{1+x^2}}$$
$$y(x) = \pm\frac{1}{\sqrt{3-2\sqrt{1+x^2}}} \Rightarrow y(x) = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

The interval of Validity is determined as:

$$3 - 2\sqrt{1+x^2} > 0$$
$$3 > 2\sqrt{1+x^2}9 > 4\left(1+x^2\right)$$
$$\frac{9}{4} > 1 + x^2$$
$$\frac{5}{4} > x^2 \Rightarrow -\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

and the interval of validity include the initial condition.

2.3 Autonomous Equations

Autonomous Equations

Definition 2.3. A differential equation is called autonomous if it can be written as

$$\frac{dy}{dt} = f(y)$$

As this equation is separable, we can separate the equation into form.

$$-1 + \frac{1}{f(y)} \frac{dy}{dt} = 0$$

One of the simplest autonomous differential equations is the one that models exponential growth.

Exponential Growth Differential Equation

Definition 2.4.

$$\begin{cases} \frac{dy}{dt} = ry \\ y(0) = y_0 \end{cases}, y(t)$$

is the population at time t, $r > 0$ is some constant, $y_0 > 0$ is initial population.

In exponential growth, the population's growth rate increases over time, in proportion to the size of the population.

Solution to Exponential growth with IVP

We can solve this IVP: $-r + \frac{1}{y(t)} \frac{dy}{dt} = 0 \Rightarrow \frac{d}{dt}(-rt) + \frac{d}{dt}(\ln(|y|)) = 0 \Rightarrow$ $-rt + \ln(|y|) = C \Rightarrow y(t) = e^{rt}e^{c}$
$\because y(0) = y_0$
$\therefore y(t) = y_0 e^{rt}$ as the solution to IVP problem

To take account of the fact that the growth rate actually depends on the population, we have the logistic growth model.

Logistic Growth Differential Equation

Definition 2.5.

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y = f(y), K = \frac{r}{a}, (r, a) > (0, 0)$$

By considering the direction field of logistic growth, we have the following definitions:

Definitions derived from directional field of logistic growth

Definition 2.6 (Equilibrium Solution)**.** An equilibrium solution is a solution to a $y^{'} = f(y)$ whose derivative is zero everywhere. If $y(0) = y_0 = K$, $y(t) = K$ is the solution, and this solution is equilibrium solution. Same for $y(0) = y_0 = 0$

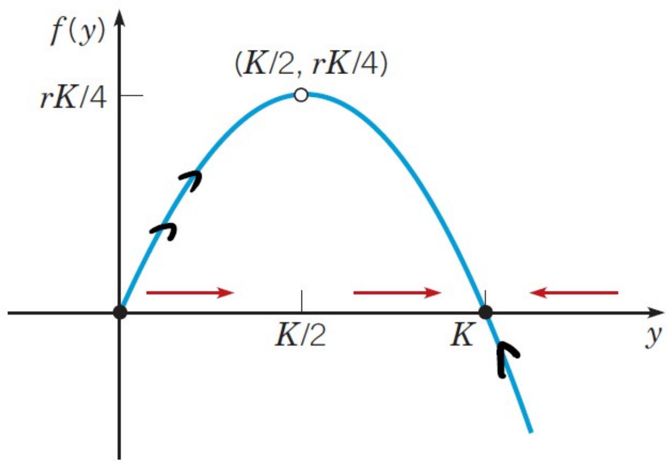
Remark. If $y_0 > K$, $y(t)$ decreases as $t \to \infty$, and $y(t)$ converges to the function $y(t) = K$
If $0 < y_0 > K$, $y(t)$ increases as $t \to \infty$, and $y(t)$ converges to the function $y(t) = K$

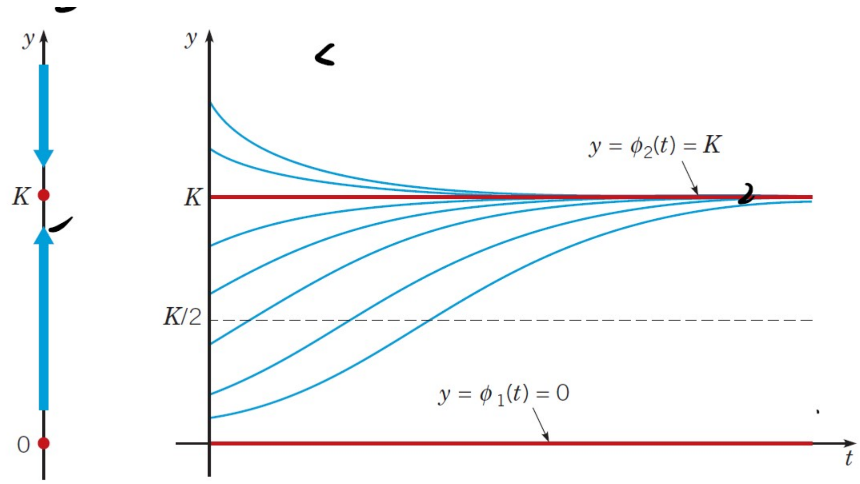
Definition 2.7 (integral curves)**.** integral curve is a parametric curve that represents a specific solution to an ordinary differential equation or system of equations.

Definition 2.8 (Asymptotically stable equilibrium solution)**.** When the nearby integral curves all converge towards an equilibrium solution as t increases, this solution is called asymptotically stable equilibrium solution.
In this case, $y(t) = K$ is an asymptotically stable equilibrium solution

Definition 2.9 (Asymptotically unstable equilibrium solution)**.** If the nearby integral curves all diverge away from an equilibrium solution as t increases, this solution is called asymptotically unstable equilibrium solution.
In this case, $y(t) = 0$ is an asymptotically unstable equilibrium solution.

Example 2.3.





2.4 Exact Equation

Exact Equation

Definition 2.10. Suppose that we have the following differential equation:

$$M\left(x,y\right)+N\left(x,y\right)\frac{dy}{dx}=0$$

or in the form:

$$M\left(x,y\right)dy+N\left(x,y\right)dx=0$$

Theorem to determine Exact Equation

Theorem 2.1. Suppose $\exists$ open set $U\subseteq\mathbb{R}^2$ such that M,N is $C^k, k\geq 1$ on U and are continuous. A differential equation

$$M\left(x,y\right)+N\left(x,y\right)\frac{dy}{dx}=0$$

is exact iff $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\Leftrightarrow\frac{\partial^2\Psi}{\partial y\partial x}=\frac{\partial^2\Psi}{\partial x\partial y}$

There are steps to solve exact equations:

First order exact differential equations of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

can be written in terms of the potential function $\Psi\left(x,y\right)$:

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx} = 0$$

, where

$$\begin{cases} \Psi_x(x,y) = M(x,y) \\ \Psi_y(x,y) = N(x,y) \end{cases}$$

Which is equivalent to:

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0 \Leftrightarrow \frac{d}{dx}\psi(x,y(x)) = 0$$

The solutions to an exact differential equation are then given by

$$\psi(x,y(x)) = c$$

and the problem reduces to finding $\Psi\left(x,y\right)$.
Take integral $\int M(x,y)dx = Q(x,y) + h(y) = \Psi\left(x,y\right)$, and $h(y)$ is unknown.
Doing the derivative with respect to y: $\Psi_y\left(x,y\right) = \frac{d}{dy}(Q(x,y) + h(y)) = N(x,y)$, we have the equality $\frac{d}{dy}(Q(x,y) + h(y)) = N(x,y)$, and we can solve $\Psi_y\left(x,y\right)$.
It can aslo be solved the other way around.
With IVP, we can solve for specific IVP and get the function $\Psi_y\left(x,y\right)$

Example 2.4. *Find the solution and interval of validity for the following IVP.*

$$2xy^2 + 4 = 2\left(3 - x^2y\right)y' \qquad y\left(-1\right) = 8$$

Sol:

$$\begin{aligned} 2xy^2 + 4 - 2\left(3 - x^2y\right)y' &= 0 \\ 2xy^2 + 4 + 2\left(x^2y - 3\right)y' &= 0 \end{aligned}$$

So, we have:

$$\begin{aligned} M &= 2xy^2 + 4 \qquad & M_y &= 4xy \\ N &= 2x^2y - 6 \qquad & N_x &= 4xy \end{aligned}$$

$\Psi(x,y) = \int 2x^2y - 6\,dy = x^2y^2 - 6y + h(x)$
$\Psi_x = 2xy^2 + h'(x) = 2xy^2 + 4 = M$
$h'(x) = 4 \qquad \Rightarrow \qquad h(x) = 4x$
Writing everything down gives us the following for $\Psi(x,y)$*:*
$\Psi(x,y) = x^2y^2 - 6y + 4x$
$x^2y^2 - 6y + 4x = c$
$64 - 48 - 4 = c \qquad c = 12$
$x^2y^2 - 6y + 4x - 12 = 0$
Using the quadratic formula gives us:

$$
\begin{aligned}
y(x) &= \frac{6 \pm \sqrt{36 - 4x^2(4x - 12)}}{2x^2} \\
&= \frac{6 \pm \sqrt{36 + 48x^2 - 16x^3}}{2x^2} \\
&= \frac{6 \pm 2\sqrt{9 + 12x^2 - 4x^3}}{2x^2} \\
&= \frac{3 \pm \sqrt{9 + 12x^2 - 4x^3}}{x^2}
\end{aligned}
$$

Reapplying the initial condition shows that $y(x) = \frac{3+\sqrt{9+12x^2-4x^3}}{x^2}$

2.5 First Order Differential Equation—Existence and Uniqueness Theorem

Existence and Uniqueness Theorem for First-Order Linear Equations

Theorem 2.2. Consider the following IVP.

$$y' + p(t)\, y = g(t), y(t_0) = y_0$$

If $p(t)$ and $g(t)$ are continuous functions on an open interval $\alpha < t < \beta$ and the interval contains t_0, then there is a unique solution to the IVP on that interval.

Existence and Uniqueness Theorem for First-Order Nonlinear Equations

Theorem 2.3. Consider the following IVP

$$y' = f(t, y), y(t_0) = y_0$$

If $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous functions in some rectangle $\alpha < t < \beta, \gamma < y < \delta$ containing the point

$$(t_0, y_0)$$

then there is a unique solution to the IVP in some interval

$$t_0 - h < t < t_0 + h$$

that is contained in

$$\alpha < t < \beta$$

.

Existence and Uniqueness of Solutions

Theorem 2.4. Consider the IVP

$$y' = f(t, y), y(0) = 0$$

If $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous functions in some rectangle $R : |t| \leq a, |y| \leq b$, then there is some interval $|t| \leq h \leq a$ in which there exists a unique solution to the IVP.

2.6 Method of successive approximations or Picard's iteration method

If we suppose temporarily that there is a differentiable function $y = \phi(t)$ that satisfies the initial value problem, then $f(t, \phi(t))$ is a continuous function of t. Hence we can integrate $y^{'} = f(t, y)$ from an initial point 0 to an arbitrary variable t and obtain

$$\phi(t) = \int_0^t f(s, \phi(s))\, ds$$

With initial condition $\phi(0) = 0$. This type of integration is called integral equation.

Now consider the simplest function:

$$\phi_0(t) = 0$$

This function satisfies the initial condition and though this function need not be a solution to our differential equation. So

$$\phi_0(t) = 0$$

approximates the solution to this IVP. For a closer approximation:

$$\begin{aligned}
\phi_1(t) &= \textstyle\int_0^t f(s, \phi_0(s))\, ds \\
\phi_2(t) &= \textstyle\int_0^t f(s, \phi_1(s))\, ds \\
&\vdots \\
\phi_n(t) &= \textstyle\int_0^t f(s, \phi_{n-1}(s))\, ds
\end{aligned}$$

In this fashion, we generate $\{\phi_n\}_{n=0}^{n=\infty} = \{\phi_0, \phi_1, \phi_2, ..., \phi_n, ...\}$

Picard–Lindelöf theorem

Theorem 2.5. If $\frac{dy}{dt} = f(t, y)$ is a first order differential equation with the initial condition

$$y(0) = 0$$

and if f and $\frac{\partial f}{\partial y}$ are both continuous on some Rectangle $R : |t| \leq a, |y| \leq b$, then $\lim\limits_{n\to\infty} \phi_n = \lim\limits_{n\to\infty} \int_0^t f(s, \phi_{n-1}(s))\, ds = \phi(t)$ where $y = \phi(t)$ is the guaranteed unique solution contained in the interval $|t| \leq h \leq a$.

Example 2.5. *Let $y(t) = \tan(t)$, the solution to the equation $y'(t) = 1 + y(t)^2$ with initial condition $y(t_0) = y_0 = 0, t_0 = 0$. Starting with $\varphi_0(t) = 0$, we iterate $\varphi_{k+1}(t) = \int_0^t (1 + (\varphi_k(s))^2)\, ds$*

so that $\varphi_n(t) \to y(t)$

$\varphi_1(t) = \int_0^t (1 + 0^2)\, ds = t$

$\varphi_2(t) = \int_0^t (1 + s^2)\, ds = t + \frac{t^3}{3}$

$\varphi_3(t) = \int_0^t \left(1 + \left(s + \frac{s^3}{3}\right)^2\right) ds = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{t^7}{63}$

and so on. Evidently, the functions are computing the Taylor series expansion of our known solution $y(t) = \tan(t)$*, and the serious converged only when* $|t| < \frac{\pi}{2}$.

3 Second-Order Differential Equations

3.1 second-order ordinary homogeneous linear differential equation with constant coefficients

Basic Definitions

Definition 3.1 (Second-order ordinary Differential Equations)**.** second-order ordinary differential equations have the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

Definition 3.2 (Second-Order ordinary Linear Differential Equation)**.** Equation is called linear if it can be written as

$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)\,y \Leftrightarrow y^{''} + p(t)\,y^{'} + q(t)\,y = g(t)$$

Sometimes equations are also written as

$$P(t)\,y^{''} + Q(t)\,y^{'} + R(t)\,y = G(t)\,, P(t) \neq 0$$

Notice $p(t) = \frac{Q(t)}{P(t)}, q(t) = \frac{R(t)}{P(t)}, g(t) = \frac{G(t)}{P(t)}$ If the equation is not in this form, then this equation is nonlinear.

Definition 3.3 (IVP in Second-Order ordinary differential equations)**.** An initial value problem consists a pair of initial conditions $y(t_0) = y_0, y^{'}(t_0) = y_0^{'}$ and equation $\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$

Definition 3.4 (Second-Order ordinary homogeneous Linear Differential Equation)**.** A second-order linear differential equation is homogeneous iff

$$y^{''} + p(t)\,y^{'} + q(t)\,y = 0$$
$$P(t)\,y^{''} + Q(t)\,y^{'} + R(t)\,y = 0$$

Otherwise, the equation is non-homogeneous.

Definition 3.5 (A second-order ordinary linear differential equation with constant coefficients—Homogeneous and Non-homogeneous)**.** A second-order linear differential equation with constant coefficients is

$$ay'' + by' + cy = G(t)$$

A second-order homogeneous linear differential equation with constant coefficients is

$$ay'' + by' + cy = 0$$

Definition 3.6 (Characteristic Equation)**.** Given the above second-order homogeneous linear differential equation with constant coefficients, the characteristic equation is

$$ar^2 + br + c = 0$$

The characteristic equation is quadratic and so will have two roots r_1, r_2. The roots will have three possible forms. These are
1. Real, distinct roots
2. Complex root
3. Same root

3.2 The Existence and Uniqueness theorem for Linear Homogeneous Differential Equations

Wronskian of Solutions

Suppose y_1, y_2 are two solutions of the differential equation

$$L[y] = y^{''} + p(t)\, y^{'} + q(t)\, y = 0$$

. The Wronskian of Solutions y_1, y_2 or Wronskian Determinant is defined as $W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix} = y_1 y_2^{'} - y_2 y_1^{'}$

Fundamental Sets of Solutions

Suppose y_1, y_2 are two solutions of the differential equation

$$L[y] = y^{''} + p(t)\, y^{'} + q(t)\, y = 0$$

If $W[y_1, y_2]$ is a nonzero solution, we say y_1, y_2 are two fundamental sets of solutions.

Existence and Uniqueness Theorem for linear homogeneous DE

Theorem 3.1. Consider the IVP

$$y^{''} + p(t)\, y^{'} + q(t)\, y = g(t)\,, y(t_0) = y_0, y^{'}(t_0) = y_0^{'}$$

, where p, q, and g are continuous on an open interval I that contains t_0. This IVP has exactly one solution $y = \phi(t)$, and the solution exists through interval I.

Principle of Superposition

Theorem 3.2. If y_1, y_2 are two solutions of the differential equation

$$L[y] = y^{''} + p(t)\, y^{'} + q(t)\, y = 0$$

Then the linear combination $c_1 y_1 + c_2 y_2$ is also a solution for any constants c_1, c_2

Abel's theorem

Theorem 3.3. If y_1, y_2 are two solutions of the differential equation.

$$L[y] = y^{''} + p(t)\, y^{'} + q(t)\, y = 0$$

where p and q are continuous on an open interval I, then the Wronskian $W[y_1, y_2](t)$ is given by $W[y_1, y_2](t) = c \exp \int -p(t)\, dt$ where c is a certain constant that depends on y_1 and y_2, but not on t. $W[y_1, y_2](t)$ is either 0 for all t in I if $c = 0$, or never 0 for all t in I and $c \neq 0$

Main Theorem—Spanning of foundamental Solutions And Wronskian

Theorem 3.4. If y_1, y_2 are two solutions of the differential equation.

$$L[y] = y^{''} + p(t)\, y^{'} + q(t)\, y = 0$$

Then the linear combination $c_1 y_1 + c_2 y_2$ is include every possible solution for any constants c_1, c_2 iff $W[y_1, y_2]$ is a nonzero solution.

Corollary 3.4.1. Consider the IVP

$$L[y] = y^{''} + p(t)\, y^{'} + q(t)\, y = 0, y(t_0) = y_0, y^{'}(t_0) = y_0^{'}$$

. Suppose y_1, y_2 are two solutions to this IVP. There are always possible constant c_1, c_2 such that $y = c_1 y_1 + c_2 y_2$ is the solution to this IVP iff Wronskian of Solutions at t_0 is not 0.

3.3 Solutions to Linear homogeneous DE with Constant Coefficients

In general, we assume the DE has two solutions in the form $y_1(t) = \mathbf{e}^{r_1 t}, y_2(t) = \mathbf{e}^{r_2 t}$

Consider two roots r_1, r_2 from the characteristic equation:

If the two roots are real and distinct $r_1 \neq r_2$, the general solution will be in the nice form

$$y(t) = c_1 y_1 + c_2 y_2 = c_1 \mathbf{e}^{r_1 t} + c_2 \mathbf{e}^{r_2 t}$$

Example 3.1. *Solve the IVP* $y'' + 11y' + 24y = 0, y(0) = 0, y'(0) = -7$

Sol: $r^2 + 11r + 24 = 0 \Leftrightarrow (r+8)(r+3) = 0 \Rightarrow r_1 = -8, r_2 = -3$

$$\begin{gathered} y(t) = c_1 \mathbf{e}^{-8t} + c_2 \mathbf{e}^{-3t} \\ y'(t) = -8c_1 \mathbf{e}^{-8t} - 3c_2 \mathbf{e}^{-3t} \end{gathered}$$

$$\begin{gathered} 0 = y(0) = c_1 + c_2 \\ -7 = y'(0) = -8c_1 - 3c_2 \\ \Rightarrow c_1 = \frac{7}{5}, c_2 = -\frac{7}{5} \end{gathered}$$

$y(t) = \frac{7}{5}\mathbf{e}^{-8t} - \frac{7}{5}\mathbf{e}^{-3t}$

If we have two complex roots from the characteristic function in the form $r_{1,2} = \lambda \pm \mu i$, the general solution is calculated as the following: Plugging our two roots into the general form of the solution gives the following solutions to the differential equation $y_1(t) = \mathbf{e}^{(\lambda+\mu i)t}, y_2(t) = \mathbf{e}^{(\lambda-\mu i)t}$ And they can be rewritten as

$$y_1(t) = \mathbf{e}^{\lambda t}\mathbf{e}^{i\mu t} = \mathbf{e}^{\lambda t}\left(\cos(\mu t) + i\sin(\mu t)\right)$$
$$y_2(t) = \mathbf{e}^{\lambda t}\mathbf{e}^{-i\mu t} = \mathbf{e}^{\lambda t}\left(\cos(\mu t) - i\sin(\mu t)\right)$$

$y_1(t) + y_2(t) = 2\mathbf{e}^{\lambda t}\cos(\mu t) \Rightarrow u(t) = \frac{1}{2}y_1(t) + \frac{1}{2}y_2(t) = \mathbf{e}^{\lambda t}\cos(\mu t)$
$y_1(t) - y_2(t) = 2i\mathbf{e}^{\lambda t}\sin(\mu t) \Rightarrow v(t) = \frac{1}{2i}y_1(t) - \frac{1}{2i}y_2(t) = \mathbf{e}^{\lambda t}\sin(\mu t)$
This gives general solution $y(t) = c_1\mathbf{e}^{\lambda t}\cos(\mu t) + c_2\mathbf{e}^{\lambda t}\sin(\mu t)$

Example 3.2. *Solve the IVP* $4y'' + 24y' + 37y = 0, y(\pi) = 1, y'(\pi) = 0$
Sol:
$4r^2 + 24r + 37 = 0 \Rightarrow r_{1,2} = -3 \pm \frac{1}{2}i$

$$y(t) = c_1\mathbf{e}^{-3t}\cos\left(\frac{t}{2}\right) + c_2\mathbf{e}^{-3t}\sin\left(\frac{t}{2}\right)$$
$$y'(t) =$$
$$-3c_1\mathbf{e}^{-3t}\cos\left(\frac{t}{2}\right) - \frac{c_1}{2}\mathbf{e}^{-3t}\sin\left(\frac{t}{2}\right) - 3c_2\mathbf{e}^{-3t}\sin\left(\frac{t}{2}\right) + \frac{c_2}{2}\mathbf{e}^{-3t}\cos\left(\frac{t}{2}\right)$$

$$\begin{cases} 1 = y(\pi) = c_1\mathbf{e}^{-3\pi}\cos\left(\frac{\pi}{2}\right) + c_2\mathbf{e}^{-3\pi}\sin\left(\frac{\pi}{2}\right) = c_2\mathbf{e}^{-3\pi} \\ 0 = y'(\pi) = -\frac{c_1}{2}\mathbf{e}^{-3\pi} - 3c_2\mathbf{e}^{-3\pi} \end{cases}$$

$c_1 = -6\mathbf{e}^{3\pi}, c_2 = \mathbf{e}^{3\pi}$
$y(t) = -6\mathbf{e}^{3\pi}\mathbf{e}^{-3t}\cos\left(\frac{t}{2}\right) + \mathbf{e}^{3\pi}\mathbf{e}^{-3t}\sin\left(\frac{t}{2}\right) \Rightarrow y(t) = -6\mathbf{e}^{-3(t-\pi)}\cos\left(\frac{t}{2}\right) + \mathbf{e}^{-3(t-\pi)}\sin\left(\frac{t}{2}\right)$

When we have repeated roots such that

$$r_1 = r_2 = r$$

This means $y_1(t) = \mathbf{e}^{r_1 t} = \mathbf{e}^{rt} = y_2(t) = \mathbf{e}^{r_2 t} = \mathbf{e}^{rt}$, and the general solution is $y(t) = c_1\mathbf{e}^{rt} + c_2 t\mathbf{e}^{rt}$

Example 3.3. *Solve the IVP* $y'' - 4y' + 4y = 0, y(0) = 12, y'(0) = -3$
Sol:
$r^2 - 4r + 4 = (r-2)^2 = 0, r_{1,2} = 2$
$y(t) \; = c_1\mathbf{e}^{2t} + c_2 t\mathbf{e}^{2t}, y'(t) \; = 2c_1\mathbf{e}^{2t} + c_2\mathbf{e}^{2t} + 2c_2 t\mathbf{e}^{2t}$
$\begin{cases} 12 = y(0) = c_1 \\ -3 = y'(0) = 2c_1 + c_2 \end{cases}$
$y(t) = 12\mathbf{e}^{2t} - 27t\mathbf{e}^{2t}$

3.4 Non-homogeneous DE and Method of Undetermined Coefficients

Solution for Non-homogeneous DE

Theorem 3.5. The general solution of the nonhomogeneous equation

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

can be written in the form $y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) = Y_c(t) + Y_p(t)$ where y_1, y_2 form a fundamental set of solutions of the corresponding homogeneous equation $y'' + p(t)\,y' + q(t)\,y = 0$, where c_1, c_2 are any arbitrary constants, and Y is any solution to

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

. Notes $c_1 y_1(t) + c_2 y_2(t) = Y_c(t)$, which is the complimentary solution, and $Y(t) = Y_p(t)$ is called the solution.

Facts

If

$$Y_{P_1}(t)$$

is a particular solution for

$$y'' + p(t)\, y' + q(t)\, y = g_1(t)$$

And if

$$Y_{P_2}(t)$$

is a particular solution for

$$y'' + p(t)\, y' + q(t)\, y = g_2(t)$$

Then

$$Y_{P_1}(t) + Y_{P_2}(t)$$

is a particular solution for

$$y'' + p(t)\, y' + q(t)\, y = g_1(t) + g_2(t)$$

In order to find the particular solution, we need to 'guess' its form, with some coefficients left as variables to be solved for, and we use the following:

Here are some possible forms of $g(t)$ and the corresponding guesses for $Y_p(t)$:

1. $g(t) = k$, where k is a constant:
   Guess: $Y_p(t) = A$
2. $g(t) = kt$, where k is a constant:
   Guess: $y_p(t) = At + B$
3. $g(t) = kt^n$, where k is a constant and n is a positive integer:
   Guess: $Y_p(t) = At^n + Bt^{n-1} + \cdots + K$
4. $g(t) = ke^{mt}$, where k and m are constants:
   Guess: $Y_p(t) = Ae^{mt}$
5. $g(t) = k\cos(mt)$ or $g(t) = k\sin(mt)$, where k and m are constants:
   Guess: $Y_p(t) = A\cos(mt) + B\sin(mt)$
6. $g(t) = ke^{mt}\cos(nt)$ or $g(t) = ke^{mt}\sin(nt)$, where k, m, and n are constants:
   Guess: $Y_p(t) = Ae^{mt}\cos(nt) + Be^{mt}\sin(nt)$

Example 3.4. *Find a particular solution for the following differential equation.*

$$y'' - 4y' - 12y = 3\mathbf{e}^{5t} + \sin(2t) + t\mathbf{e}^{4t}$$

Sol:
$Y_P(t) = -\frac{3}{7}\mathbf{e}^{5t} + \frac{1}{40}\cos(2t) - \frac{1}{20}\sin(2t) - \frac{1}{36}(3t+1)\mathbf{e}^{4t}$

3.5 Non-homogeneous DE and Variation Of Parameters

Method of undetermined coefficients for finding a particular solution to $p(t)y'' + q(t)y' + r(t)y = g(t)$ has really messy algebra. And Undetermined coefficient only works for a small class of functions.

Variation of Parameters

Theorem 3.6. Consider the differential equation, $y'' + q(t)y' + r(t)y = g(t)$,
Assume $y_1(t)$ and $y_2(t)$ are fundamental set of solutions for $y'' + q(t)y' + r(t)y = 0$
Then a particular solution to the nonhomogeneous differential equation is,

$$Y_P(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)}\,dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)}\,dt$$

Example 3.5. *Find a general solution to the following differential equation.*

$$2y'' + 18y = 6\tan(3t)$$

Sol:
$y'' + 9y = 3\tan(3t)$
Using previous section, the complementary solution is $Y_c(t) = c_1\cos(3t) + c_2\sin(3t)$ *with*

$$y_1(t) = \cos(3t) \qquad y_2(t) = \sin(3t)$$

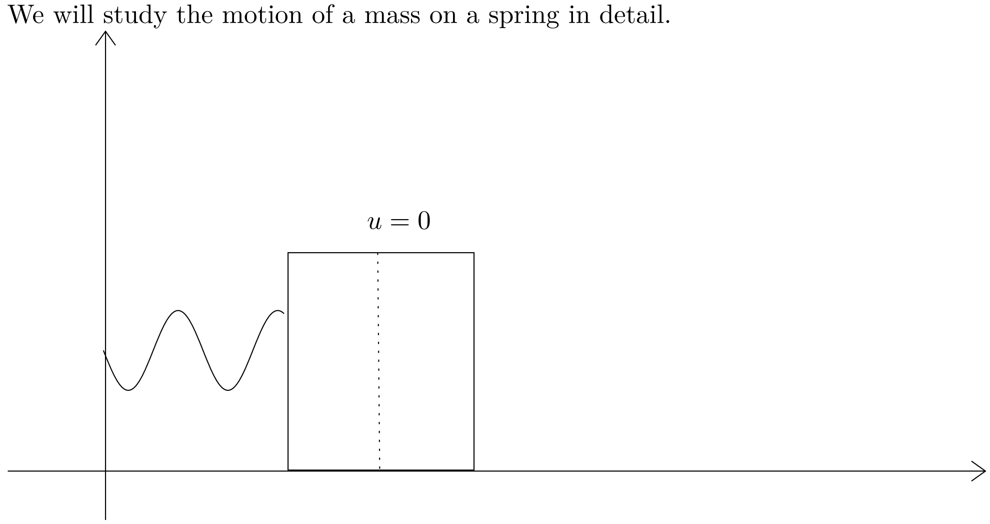
$$W = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3\cos^2(3t) + 3\sin^2(3t) = 3$$

$$\begin{aligned}
Y_P(t) &= -\cos(3t)\int \frac{3\sin(3t)\tan(3t)}{3}\,dt + \sin(3t)\int \frac{3\cos(3t)\tan(3t)}{3}\,dt \\
&= -\cos(3t)\int \frac{\sin^2(3t)}{\cos(3t)}\,dt + \sin(3t)\int \sin(3t)\,dt \\
&= -\cos(3t)\int \frac{1-\cos^2(3t)}{\cos(3t)}\,dt + \sin(3t)\int \sin(3t)\,dt \\
&= -\cos(3t)\int \sec(3t) - \cos(3t)\,dt + \sin(3t)\int \sin(3t)\,dt \\
&= -\frac{\cos(3t)}{3}\left(\ln|\sec(3t)+\tan(3t)| - \sin(3t)\right) + \frac{\sin(3t)}{3}\left(-\cos(3t)\right) \\
&= -\frac{\cos(3t)}{3}\ln|\sec(3t)+\tan(3t)|
\end{aligned}$$

The general solution is $y(t) = c_1\cos(3t) + c_2\sin(3t) - \frac{\cos(3t)}{3}\ln|\sec(3t)+\tan(3t)|$

3.6 Application of second-order DE—Spring Mass Model

One of the reasons why second-order linear differential equations with constant coefficients are worth studying is that they serve as mathematical models of many important physical processes. Two important areas of application are the fields of mechanical and electrical oscillations.
We will study the motion of a mass on a spring in detail.



In general, a list of force will act upon the object:

1. We are going to assume that Hooke's Law will govern the force that the spring exerts on the object. The fource F_s equals to $F_s = -ku$, where $k > 0$ is the spring constant, and u is the displacement from the equilibrium position.

2. There is a frictional force $F_s = -\gamma\, u' = -\gamma\, v$, where γ is a constant, and v is the velocity, which is the derivative of u.

3. There is an external force $F(t)$

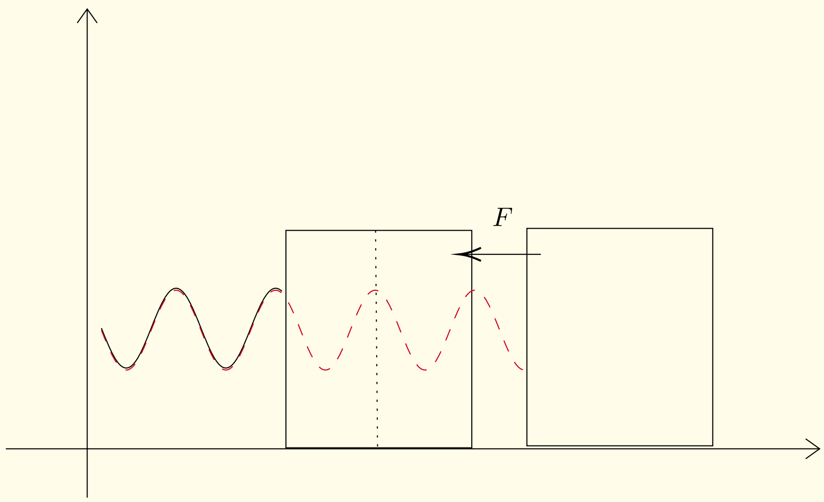
Using this in Newton's Second Law gives us the final version of the differential equation that we'll work with:

$$mu'' + \gamma u' + ku = F\left(t\right)$$

Along with this differential equation we will have the following initial conditions:

$$u\left(0\right) = u_0 \qquad \text{Initial displacement from the equilibrium position.} \qquad (1)$$
$$u'\left(0\right) = {u'}_0 \qquad \text{Initial velocity.} \qquad (2)$$

Case 1: No friction and No external force



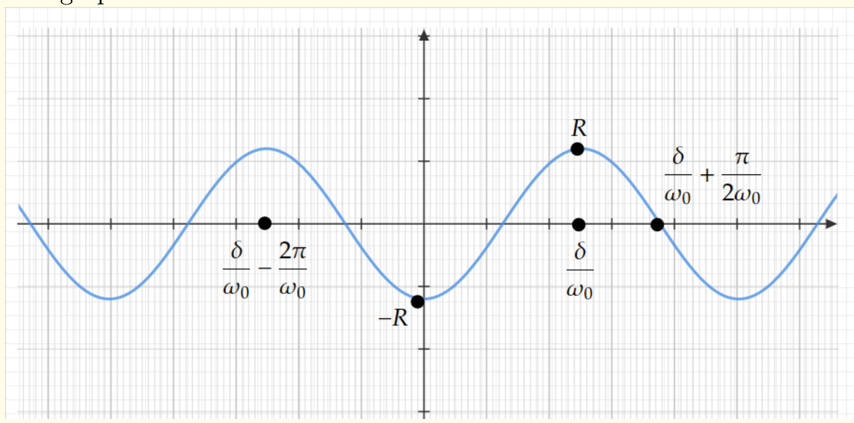
In this case the differential equation becomes $mu'' + ku = 0$

The characteristic equation has the roots $r = \pm\, i\sqrt{\frac{k}{m}} = \pm\,\omega_0\, i$, where $\omega_0 = \sqrt{\frac{k}{m}}$ and it is called natural frequency.

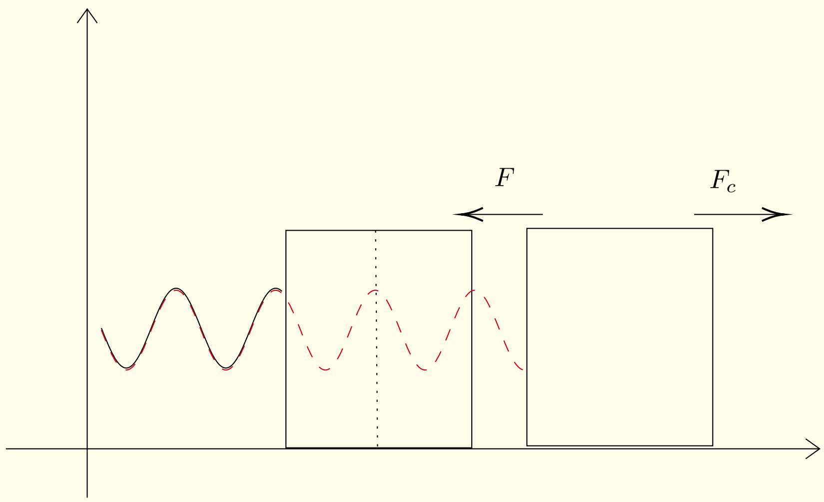
The solution in this case is then $u\,(t) = A\cos\,(\omega_0\, t) + B\sin\,(\omega_0\, t)$. And We can write the solution as: $u\,(t) = R\cos\,(\omega_0 t - \delta) = R\cos(\omega_0(t - \frac{\delta}{\omega_0}))$, where $R = \sqrt{A^2 + B^2}$ is called an amplitude, and $\delta \in [0, 2\pi)$ such that $\begin{cases} sin(\delta) = \frac{B}{R} \\ cos(\delta) = \frac{A}{R} \end{cases}$, or $\delta = \tan^{-1}(\frac{B}{A})$

The period is $T = \frac{2\pi}{\omega_0}$

The graph of the solution is:



Case 2: No external force and with friction



We are still going to assume that there will be no external forces acting on the system, with the exception of friction. In this case the differential equation will be $mu'' + \gamma u' + ku = 0$.
Upon solving for the roots of the characteristic equation we get the following

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

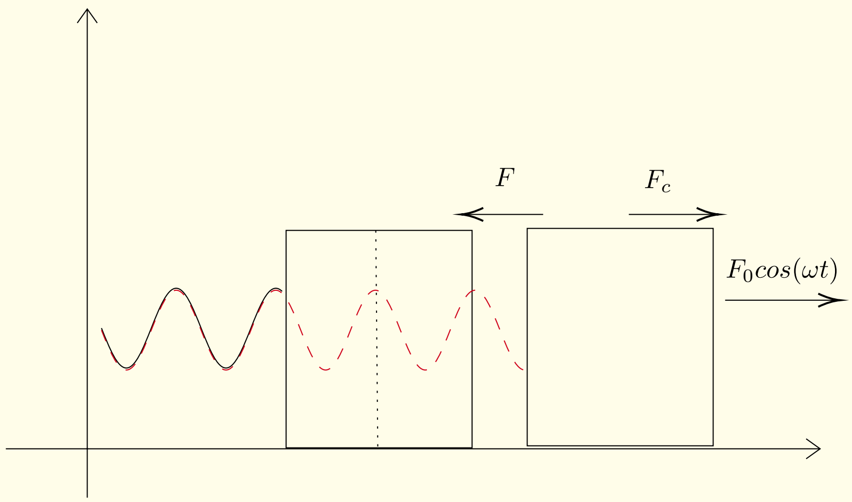
And we have three cases:

1. $\gamma^2 - 4mk = 0$, and we have double root. And the displacement at any time t will be $u(t) = A\mathbf{e}^{-\frac{\gamma t}{2m}} + Bt\mathbf{e}^{-\frac{\gamma t}{2m}}$
2. $\gamma^2 - 4mk > 0$, and $u(t) = A\mathbf{e}^{r_1 t} + B\mathbf{e}^{r_2 t}$
3. $\gamma^2 - 4mk < 0$, and we have complex root $r_{1,2} = \frac{-\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m} = \lambda \pm \mu i$
   So the displacement is

$$\begin{aligned} u(t) &= A\mathbf{e}^{\lambda t}\cos(\mu t) + B\mathbf{e}^{\lambda t}\sin(\mu t) \\ &= \mathbf{e}^{\lambda t}\left(A\cos(\mu t) + B\sin(\mu t)\right) \\ &= R\mathbf{e}^{\lambda t}\cos(\mu t - \delta) \end{aligned}$$

   The period is $T = \frac{w\pi}{\mu}$, and amplitude is decaying in the rate of $e^{\frac{-\delta}{2\mu}t}$

Case 3: With friction and External forces



This is the full blown case where we consider every last possible force that can act upon the system. The differential equation for this case is $mu'' + \gamma u' + ku = F(t)$. The displacement function this time will be $u(t) = U_c(t) + U_P(t) = e^{-\frac{\delta}{2m}t}Rcos(\omega t - \delta) + Acos(\omega t) + Bsin(\omega t)$
As $t \to \infty$, $u(t)$ approaches $U_0(t) = Rcos(\omega t - \delta)$

4 Higher-Order Linear Differential Equations

4.1 General ideas of nth Order Linear Differential Equations

nth Order Linear Differential Equations

Definition 4.1. it is in the form:

$$P_n(t)\, y^{(n)} + P_{n-1}(t)\, y^{(n-1)} + \cdots + P_1(t)\, y' + P_0(t)\, y = G(t)$$

Functions $P_0, P_1, ..., P_n, G$ are continuous real-valued function on interval I, and P_0 is non-zero on the entire domain.
By Dividing P_0, we have another form:

$$y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = g(t)$$

An IVP for an nth order differential equation will require the following n initial conditions: $y(t_0) = \overline{y}_0, \quad y'(t_0) = \overline{y}_1, \quad \cdots, \quad y^{(n-1)}(t_0) = \overline{y}_{n-1}$

Existence and Uniquness theorem for nth linear DE IVP

Theorem 4.1. Suppose the functions $p_0, p_1, \ldots, p_{n-1}$ and $g(t)$ are all continuous in some open interval I containing t_0 then there is a unique solution to the IVP given by:

$$y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = g(t)$$
$$y(t_0) = \overline{y}_0, \quad y'(t_0) = \overline{y}_1, \quad \cdots, \quad y^{(n-1)}(t_0) = \overline{y}_{n-1}$$

and the unique solution will exist for all $t \in I$

4.2 Solutions to a nth linear homogeneous DE

Linear Combination of solutions is a solution to nth linear homogeneous DE

Theorem 4.2. Consider the following linear Operators:

- $C((a,b))=$ the set of continuous functions on (a,b)
- $C^n((a,b))=$ the set of n times differential functions on (a,b)

Consider a linear operator $L : C^n((a,b)) \to C((a,b))$ with $\forall \phi \in C^n((a,b))$, $L(\phi) = y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y$
L is a linear operator such that $\forall \phi_1,\ \phi_2$ and for all constant $C_1,\ C_2$, we have $L(C_1\phi_1 + C_2\phi_2) = C_1 L(\phi_1) + C_2 L(\phi_2)$.
Using linear operators, we can prove if $y_1, ..., y_k$ are solutions to the DE $y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = 0$, then any linear combinations $C_1 y_1 + ... + C_n y_n$ is also a solution.

Wronskian of n functions

Definition 4.2. Given $y_1, ..., y_n$ functions on (a,b), the wronskain is defined as $W[y_1, ..., y_n](t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$

Fundamental Set of Solutions

Definition 4.3. A set of functions $y_1, y_2, ..., y_n$ form a fundamental set of solutions to $y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = 0$ if the linear combinations $C_1 y_1 + ... + C_n y_n$ include every solution to $y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = 0$

Wronskain and Span of Solutions

Theorem 4.3. Suppose the functions $p_0, p_1, \ldots, p_{n-1}$ are all continuous functions on (a,b). Let $y_1, y_2, ..., y_n$ be solutions to $y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = 0$. The linear combination $C_1 y_1 + ... + C_n y_n$ include every solutions to this DE iff Wronskain $W[y_1, ..., y_n]$ is a nonzero function. And $y_1, y_2, ..., y_n$ is called the Foundamental Set of solutions.

Linear Independence and Wronskain

Theorem 4.4. Suppose the functions $p_0, p_1, \ldots, p_{n-1}$ are all continuous functions on (a, b). Let $y_1, y_2, ..., y_n$ be solutions to $y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = 0$. $y_1, y_2, ..., y_n$ are linear independent iff Wronskain $W[y_1, ..., y_n]$ is a nonzero function.

Linear independence and Span of Solutions

Theorem 4.5. Suppose the functions $p_0, p_1, \ldots, p_{n-1}$ are all continuous functions on (a, b). Let $y_1, y_2, ..., y_n$ be solutions to $y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = 0$. $y_1, y_2, ..., y_n$ are linear independent iff $y_1, y_2, ..., y_n$ is the Foundamental Set of solutions.

4.3 Solve nth linear homogeneous DE with constant coefficients

linear homogeneous differential equation of the nth order with constant coefficients

Definition 4.4. The linear homogeneous differential equation of the nth order with constant coefficients can be written as $y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = 0$
where $a_1, a_2, .., a_n$ can be real or complex

Characteristic equation

Definition 4.5. $Z(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$
This n degree polynomial has n roots $r_1,\ r_2,\ , ..., r_n$ by Fundamental Theorem of Algebra

There are different cases when solving this type of DE:

Case 1: All Roots of the Characteristic Equation are Real and Distinct

Assume that the characteristic equation $Z(\lambda)$ has n real roots $r_1,\ r_2,\ , ..., r_n$. In this case the general solution of the differential equation is written in a simple form:

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x},$$

Case 2: The Roots of the Characteristic Equation are Real and Multiple

Let the characteristic equation has m roots $r_1,\ r_2,\ ,...,r_m$, the multiplicity of which, respectively, is equal to $k_1,\ k_2\ ,...,\ k_m$, and $k_1+...+k_m=n$. Then the general solution of the homogeneous differential equations with constant coefficients has the form

$$y\left(x\right)=C_1e^{\lambda_1 x}+C_2xe^{\lambda_1 x}+\cdots+C_{k_1}x^{k_1-1}e^{\lambda_1 x}+\cdots+C_{n-k_m+1}e^{\lambda_m x}+ C_{n-k_m+2}xe^{\lambda_m x}+\cdots+C_nx^{k_m-1}e^{\lambda_m x}$$

Case 3: The Roots of the Characteristic Equation are Complex and Distinct

If the coefficients of the differential equation are real numbers, the complex roots of the characteristic equation will be presented in the form of conjugate pairs of complex numbers: $\lambda_{1,2}=\alpha\pm i\beta,\ \ \lambda_{3,4}=\gamma\pm i\delta,\ \ldots$ In this case the general solution is written as:

$$y\left(x\right)=e^{\alpha x}\left(C_1\cos\beta x+C_2\sin\beta x\right)+e^{\gamma x}\left(C_3\cos\delta x+C_4\sin\delta x\right)+\cdots$$

Case 4: The Roots of the Characteristic Equation are Complex and Multiple

Each pair of complex conjugate roots $\alpha\pm i\beta$ of multiplicity k produces $2k$ particular solutions:
$e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x,\ e^{\alpha x}x\cos\beta x,\ e^{\alpha x}x\sin\beta x,\ldots,$
$e^{\alpha x}x^{k-1}\cos\beta x,\ e^{\alpha x}x^{k-1}\sin\beta x.$
Then the part of the general solution of the differential equation corresponding to a given pair of complex conjugate roots is constructed as follows:

$$y\left(x\right)=e^{\alpha x}\left(C_1\cos\beta x+C_2\sin\beta x\right)+xe^{\alpha x}\left(C_3\cos\beta x+C_4\sin\beta x\right)+ \cdots+x^{k-1}e^{\alpha x}\left(C_{2k-1}\cos\beta x+C_{2k}\sin\beta x\right).$$

Example 4.1. *Solve the equation* $y^{(5)}+18y'''+81y'=0$.
Sol: The characteristic equation can be written as $\lambda^5+18\lambda^3+81\lambda=0$.
The equation has the following roots: $\lambda_1=0,\ \ \lambda_{2,3}=\pm 3i$, *and imaginary roots have multiplicity 2*
We write the general solution in the form:

$$y\left(x\right)=C_1+\left(C_2+C_3x\right)\cos 3x+\left(C_4+C_5x\right)\sin 3x,$$

4.4 nth Linear Non-homogeneous Differential Equations with Constant Coefficients

nth Linear Non-homogeneous Differential Equations with Constant Coefficients

Definition 4.6. This type of DE is defined as $y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = g(x)\,,$
where $a_1, a_2, .., a_n$ can be real or complex constant, and $g(x)$ is a continuous function on an interval (a, b)

The general solution $y(x)$ of the non-homogeneous equation is the sum of the complimentary solution $Y_C(x)$ of the corresponding homogeneous equation and a particular solution $Y_P(x)$ of the non-homogeneous equation:

$$y(x) = Y_C(x) + Y_P(x)\,.$$

The complement solutions can be found as the method introduced in the previous section. The particular solution of this type of DE can be found using **method of Variation of parameters** or **Method of undetermined coefficients**.

Method Of Variation of parameters

Assume that we found a fundamental set of solutions, $y_1(x), y_2(x), \ldots, y_n(x)$ for the associated homogeneous differential equation. A particular solution $Y_P(x)$ to the DE is therefore:

$$u_1 = \int \frac{g(x)W_1(x)}{W(x)}\,dx, \quad u_2 = \int \frac{g(x)W_2(x)}{W(x)}\,dx, \quad \cdots, \quad u_n = \int \frac{g(x)W_n(x)}{W(x)}\,dx$$

$$Y_P(x) = y_1(x)\int \frac{g(x)W_1(x)}{W(x)}\,dx + y_2(x)\int \frac{g(x)W_2(x)}{W(x)}\,dx + \cdots + y_n(x)\int \frac{g(x)W_n(x)}{W(x)}\,dx$$

Example 4.2. *Solve the following differential equation* $y^{(3)} - 2y'' - 21y' - 18y = 3 + 4\mathbf{e}^{-t}$ *Sol: The characteristic equation is* $r^3 - 2r^2 - 21r - 18 = (r+3)(r+1)(r-6) = 0 \quad \Rightarrow \quad r_1 = -3,\ r_2 = -1,\ r_3 = 6$

So, we have three real distinct roots here and so the complimentary solution is $Y_c(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t} + c_3\mathbf{e}^{6t}$

$$W = \begin{vmatrix} \mathbf{e}^{-3t} & \mathbf{e}^{-t} & \mathbf{e}^{6t} \\ -3\mathbf{e}^{-3t} & -\mathbf{e}^{-t} & 6\mathbf{e}^{6t} \\ 9\mathbf{e}^{-3t} & \mathbf{e}^{-t} & 36\mathbf{e}^{6t} \end{vmatrix} = 126\mathbf{e}^{2t} \qquad W_1 = \begin{vmatrix} 0 & \mathbf{e}^{-t} & \mathbf{e}^{6t} \\ 0 & -\mathbf{e}^{-t} & 6\mathbf{e}^{6t} \\ 1 & \mathbf{e}^{-t} & 36\mathbf{e}^{6t} \end{vmatrix} = 7\mathbf{e}^{5t}$$

$$W_2 = \begin{vmatrix} \mathbf{e}^{-3t} & 0 & \mathbf{e}^{6t} \\ -3\mathbf{e}^{-3t} & 0 & 6\mathbf{e}^{6t} \\ 9\mathbf{e}^{-3t} & 1 & 36\mathbf{e}^{6t} \end{vmatrix} = -9\mathbf{e}^{3t} \qquad W_3 = \begin{vmatrix} \mathbf{e}^{-3t} & \mathbf{e}^{-t} & 0 \\ -3\mathbf{e}^{-3t} & -\mathbf{e}^{-t} & 0 \\ 9\mathbf{e}^{-3t} & \mathbf{e}^{-t} & 1 \end{vmatrix} = 2\mathbf{e}^{-4t}$$

Given that $g(t) = 3 + 4\mathbf{e}^{-t}$*, we have:*

$$u_1 = \int \frac{\left(3+4\mathbf{e}^{-t}\right)\left(7\mathbf{e}^{5t}\right)}{126\mathbf{e}^{2t}}\,dt = \frac{1}{18}\int 3\mathbf{e}^{3t} + 4\mathbf{e}^{2t}\,dt = \frac{1}{18}\left(\mathbf{e}^{3t} + 2\mathbf{e}^{2t}\right)$$
$$u_2 = \int \frac{\left(3+4\mathbf{e}^{-t}\right)\left(-9\mathbf{e}^{3t}\right)}{126\mathbf{e}^{2t}}\,dt = -\frac{1}{14}\int 3\mathbf{e}^{t} + 4\,dt = -\frac{1}{14}\left(3\mathbf{e}^{t} + 4t\right)$$
$$u_3 = \int \frac{\left(3+4\mathbf{e}^{-t}\right)\left(2\mathbf{e}^{-4t}\right)}{126\mathbf{e}^{2t}}\,dt = \frac{1}{63}\int 3\mathbf{e}^{-6t} + 4\mathbf{e}^{-7t}\,dt = \frac{1}{63}\left(-\frac{1}{2}\mathbf{e}^{-6t} - \frac{4}{7}\mathbf{e}^{-7t}\right)$$

A particular solution for this differential equation is then:

$$\begin{aligned} Y_P &= u_1 y_1 + u_2 y_2 + u_3 y_3 \\ &= \frac{1}{18}\left(\mathbf{e}^{3t} + 2\mathbf{e}^{2t}\right)\mathbf{e}^{-3t} - \frac{1}{14}\left(3\mathbf{e}^{t} + 4t\right)\mathbf{e}^{-t} + \frac{1}{63}\left(-\frac{1}{2}\mathbf{e}^{-6t} - \frac{4}{7}\mathbf{e}^{-7t}\right)\mathbf{e}^{6t} \\ &= -\frac{1}{6} + \frac{5}{49}\mathbf{e}^{-t} - \frac{2}{7}t\mathbf{e}^{-t} \end{aligned}$$

The general solution is then:

$$y(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t} + c_3\mathbf{e}^{6t} - \frac{1}{6} + \frac{5}{49}\mathbf{e}^{-t} - \frac{2}{7}t\mathbf{e}^{-t}$$

Undetermined Coefficients

This method is the same as the undetermined coefficients as introduced before.

Example 4.3. *Solve the following differential equation* $y^{(3)} - 12y'' + 48y' - 64y = 12 - 32\mathbf{e}^{-8t} + 2\mathbf{e}^{4t}$

Sol: The Characteristic Equation is $r^3 - 12r^2 + 48r - 64 = (r-4)^3 = 0 \quad \Rightarrow$ $r = 4$ *(multiplicity 3)*

$Y_c(t) = c_1\mathbf{e}^{4t} + c_2 t\mathbf{e}^{4t} + c_3 t^2\mathbf{e}^{4t}$ *by the method introduced before.*

The guess of the particular solution is: $Y_P = A + B\mathbf{e}^{-8t} + Ct^3\mathbf{e}^{4t}$

By plugging in the DE, we have equality $-64A - 1728B\mathbf{e}^{-8t} + 6C\mathbf{e}^{4t} = 12 - 32\mathbf{e}^{-8t} + 2\mathbf{e}^{4t}$

$$\begin{array}{rl} t^0: & -64A = 12 \\ \mathbf{e}^{-8t}: & -1728B = -32 \\ \mathbf{e}^{4t}: & 6C = 2 \end{array} \quad \Rightarrow \quad \begin{array}{l} A = -\frac{3}{16} \\ B = \frac{1}{54} \\ C = \frac{1}{3} \end{array}$$

A particular solution is then:

$$Y_P = -\frac{3}{16} + \frac{1}{54}\mathbf{e}^{-8t} + \frac{1}{3}t^3\mathbf{e}^{4t}$$

The general solution is then:

$$y(t) = c_1\mathbf{e}^{4t} + c_2 t\mathbf{e}^{4t} + c_3 t^2\mathbf{e}^{4t} - \frac{3}{16} + \frac{1}{54}\mathbf{e}^{-8t} + \frac{1}{3}t^3\mathbf{e}^{4t}$$

5 System of Differential Equations

5.1 Review of System of equations

Argumented matrix and system of equations

Definition 5.1. Given a system of n equations with n unknowns $x_1,\ x_2,\ ....,\ x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

The argument matrix of this system of equations is the matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{pmatrix}$$

Triangular Form

Definition 5.2. Considering the argument matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{pmatrix}$$

The matrix will be in triangular form if all the entries below the main diagonal (the diagonal containing $a_{11}, a_{22}, ..., a_{nn}$ are zeros).

Row operations

- Interchange two rows. This is exactly what it says. We will interchange row i and row j. The notation that we'll use to denote this operation is $R_i \leftrightarrow R_j$ and does not change the system of equations
- Multiple any row R_i with any constant c does not change the system of equations
- Add the a multiple of row cR_i to row R_j does not change the system of equations

Fact about the solution of the system of equations

Given the system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

we will have one of the three possibilities for the number of solutions:

- No solution
- Exactly one Solution
- Infinitely many solutions

Fact of solution of system of homogeneous equations

Consider a system of homogeneous equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0 \end{aligned}$$

- Exactly one solution, the trivial solution
- Infinitely many non-zero solutions in addition to the trivial solution.

Singular Matrix

Given a square matrix A

- If A is nonsingular then A^{-1} does exists
- If A is singular then A^-1 does not exists.

Fact about singular matrix and solutions

Given the system of equations $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \Rightarrow A\overrightarrow{x} = b$, we have:

- If A is nonsingular then there will be exactly one solution to the system.
- If A is singular then there will either be no solution or infinitely many solutions to the system.

Fact about singular matrix and homogeneous solutions

Given the homogeneous system $A\vec{x} = \vec{0}$, we have:

- If A is nonsingular then its solution is only trivial solution
- If A is singular then there will be infinitely many solutions to the system.

Linear Independence and singular matrix

Consider matrix $X = \begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{pmatrix}$, which each x_i is a column vector.

- If x_i are linear independent, the matrix is singular, and determinant is non zero.
- If x_i are not linear independent, the matrix is nonsingular, and the determinant is zero.

Calculus on Matrix

$$A\left(t\right)=\begin{pmatrix} a_{11}\left(t\right) & a_{12}\left(t\right) & \cdots & a_{1n}\left(t\right) \\ a_{21}\left(t\right) & a_{22}\left(t\right) & \cdots & a_{2n}\left(t\right) \\ \vdots & \vdots & & \vdots \\ a_{m1}\left(t\right) & a_{m2}\left(t\right) & \cdots & a_{mn}\left(t\right) \end{pmatrix}$$

$$A'\left(t\right)=\begin{pmatrix} a'_{11}\left(t\right) & a'_{12}\left(t\right) & \cdots & a'_{1n}\left(t\right) \\ a'_{21}\left(t\right) & a'_{22}\left(t\right) & \cdots & a'_{2n}\left(t\right) \\ \vdots & \vdots & & \vdots \\ a'_{m1}\left(t\right) & a'_{m2}\left(t\right) & \cdots & a'_{mn}\left(t\right) \end{pmatrix}$$

$$\int A\left(t\right)\,dt=\begin{pmatrix} \int a_{11}\left(t\right)\,dt & \int a_{12}\left(t\right)\,dt & \cdots & \int a_{1n}\left(t\right)\,dt \\ \int a_{21}\left(t\right)\,dt & \int a_{22}\left(t\right)\,dt & \cdots & \int a_{2n}\left(t\right)\,dt \\ \vdots & \vdots & & \vdots \\ \int a_{m1}\left(t\right)\,dt & \int a_{m2}\left(t\right)\,dt & \cdots & \int a_{mn}\left(t\right)\,dt \end{pmatrix}$$

Matrix Exponential

The matrix exponential is denoted as $\exp(A)$ for a square matrix A. It is defined as:

$$\exp(A)=\sum_{k=0}^{\infty}\frac{1}{k!}A^k$$

where A^k denotes the matrix product of k copies of A.
Some important properties of the matrix exponential include:
1: $\exp(A+B)=\exp(A)\exp(B)$ for any two matrices A and B that commute with each other, i.e., $AB=BA$.
2: If A is diagonalizable with eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ and corresponding eigenvectors $v_1,v_2,\ldots,v_n$, then $\exp(A)$ is also diagonalizable with eigenvalues $\exp(\lambda_1),\exp(\lambda_2),\ldots,\exp(\lambda_n)$ and corresponding eigenvectors $v_1,v_2,\ldots,v_n$.
3: $\frac{d}{dt}\left(e^{tA}\right)=Ae^{tA}$.
4: If A is invertible, then $\exp(A^{-1})=(\exp(A))^{-1}$.
5: For any scalar c, $\exp(cA)=(\exp(A))^c$.

5.2 Review on Eigenvalues and Eigenvectors

Given a matrix A, an eigenvector is a vector $\mathbf{x}$ that satisfies $A\mathbf{x}=\lambda\mathbf{x}$ for some λ. We call λ the associated eigenvalue. In some sense, these vectors are not modified by the matrix, and are just scaled up by the matrix. We will look at the properties of eigenvectors and eigenvalues, and see their importance in diagonalizing matrices.

Foundamental theorem of Algebra

Theorem 5.1. Let $p(z)$ be a polynomial of degree $m \geq 1$, i.e. $p(z) = \sum_{j=0}^{m} c_j z^j$, where $c_j \in \mathbb{C}$ and $c_m \neq 0$. Then $p(z) = 0$ has precisely m (not necessarily distinct) roots in the complex plane, accounting for multiplicity.

Note that we have the disclaimer "accounting for multiplicity". For example, $x^2 - 2x + 1 = 0$ has only one distinct root, 1, but we say that this root has multiplicity 2, and is thus counted twice. Formally, multiplicity is defined as follows:

Multiplicity and Roots

Definition 5.3 (Multiplicity of root)**.** The root $z = \omega$ has *multiplicity* k if $(z - \omega)^k$ is a factor of $p(z)$ but $(z - \omega)^{k+1}$ is not.

Definition 5.4 (Algebraic multiplicity)**.** The algebraic multiplicity of λ is its multiplicity as a root of the characteristic polynomial of a $n \times n$ dimensional matrix A.

Definition 5.5 (Geometric multiplicity)**.** It is the dimension of the eigenspace corresponding to λ.

Example 5.1. *Let* $p(z) = z^3 - z^2 - z + 1 = (z - 1)^2(z + 1)$*. So* $p(z) = 0$ *has roots* $1, 1, -1$*, where* $z = 1$ *has multiplicity* 2*.*

Definitions regarding Eigenvectors and Eigenvalues

Definition 5.6 (Eigenvectors and Eigenvalues)**.** Let $\alpha : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map with associated matrix A. Then $\mathbf{x} \neq \mathbf{0}$ is an *eigenvector* of A if

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some λ. λ is the associated *eigenvalue*. This means that the direction of the eigenvector is preserved by the mapping, but is scaled up by λ.

Definition 5.7 (Characteristic Polynomial of a equation)**.** The *characteristic equation* of A is

$$\det(A - \lambda I) = 0.$$

Definition 5.8 (Characteristic Polynomial of a Matrix)**.** The *characteristic polynomial* of A is

$$p_A(\lambda) = \det(A - \lambda I).$$

Process to find Eigenvectors and Eigenvalues

First use $\det(A - \lambda I) = 0$ to find the eigenvalues

$$A\vec{\eta} - \lambda \mathbf{x} = \vec{0}$$
$$A\mathbf{x} - \lambda I_n \mathbf{x} = \vec{0}$$
$$(A - \lambda I_n)\,\mathbf{x} = \vec{0}$$

Then we solve for $\mathbf{x}$, and it is the eigenvector.

Important results of Eigenvectors and Eigenvalues

Theorem 5.2 (Distinct eigenvalues and linear independent)*.* Suppose $n \times n$ matrix A has distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Then the corresponding eigenvectors $x_1, \dots, x_n$ are linearly independent

Theorem 5.3 (Diagonal matrix and Eigenvalues)*.* If $n \times n$ matrix A has n distinct eigenvalues, and hence has n linearly independent eigenvectors, and with respect to this eigen basis, A is diagonal. Such that:

$$\text{For } v_i = e_i, Av_i = \lambda_i v_i, \text{ and } A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

Generalized eigenvetor of rank m and Jordan Chain

Definition 5.9 (Generalized eigenvector of rank m)**.** A vector x_m is a generalized eigenvector of rank m of a matrix A and corresponding to eigenvalue λ if $(A - \lambda I)^m \mathbf{x}_m = \mathbf{0}$

A generalized eigenvector of rank 1 is a ordinary eigenvector.
The set of all generalized eigenvector of λ span the generalized eigenspace of λ

Definition 5.10 (Jordan Chains)**.** Let x_m be a generalized eigenvector of rank m corresponding to a matrix A and eigenvalue λ. The chain generated by x_m if a set of vectors $\{\mathbf{x}_m, \mathbf{x}_{m-1}, \ldots, \mathbf{x}_1\}$ given by:

$$\begin{aligned}
x_{m-1} &= (A-\lambda I)x_m \\
x_{m-2} &= (A-\lambda I)^2 x_m = (A-\lambda I)x_{m-1}, \\
x_{m-3} &= (A-\lambda I)^3 x_m = (A-\lambda I)x_{m-2}, \\
&\vdots \\
x_1 &= (A-\lambda I)^{m-1} x_m = (A-\lambda I)x_2.
\end{aligned}$$

In general:

$$x_j = (A-\lambda I)^{m-j} x_m = (A-\lambda I)x_{j+1} \qquad (j = 1, 2, \ldots, m-1)$$

5.3 General knowledge of Systems Of Differential Equations

System of Differential Equations

Definition 5.11. A system of differential equations is a finite set of differential equations. Such a system can be either linear or non-linear. Also, such a system can be either a system of ordinary differential equations or a system of partial differential equations.

The motivation is we can turn a nth order linear differential equation into a system of first order linear differential equation.

Example 5.2. *Write the following 2nd order differential equation as a system of first order, linear differential equations.*

$$2y'' - 5y' + y = 0 \qquad y(3) = 6 \qquad y'(3) = -1$$

Sol: Define

$$\begin{aligned}
x_1(t) &= y(t) \\
x_2(t) &= y'(t)
\end{aligned}$$

Notice:

$$\begin{aligned}
{x'}_1 &= y' = x_2 \\
{x'}_2 &= y'' = -\frac{1}{2}y + \frac{5}{2}y' = -\frac{1}{2}x_1 + \frac{5}{2}x_2
\end{aligned}$$

$$\begin{aligned}
x_1(3) &= y(3) = 6 \\
x_2(3) &= y'(3) = -1
\end{aligned}$$

Put together we have:

$$x'_1 = x_2 \qquad\qquad x_1(3) = 6$$
$$x'_2 = -\frac{1}{2}x_1 + \frac{5}{2}x_2 \qquad\qquad x_2(3) = -1$$

Converting into form of a matrix:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The system becomes:

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix}\vec{x} \qquad \vec{x}(3) = \begin{pmatrix} x_1(3) \\ x_2(3) \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

Example 5.3. *Write the following 4th order differential equation as a system of first order, linear differential equations.*

$$y^{(4)} + 3y'' - \sin(t)\,y' + 8y = t^2 \qquad y(0) = 1 \quad y'(0) = 2 \quad y''(0) = 3 \quad y'''(0) = 4$$

Sol: We have the following functions:

$$x_1 = y \quad \Rightarrow \quad x'_1 = y' = x_2$$
$$x_2 = y' \quad \Rightarrow \quad x'_2 = y'' = x_3$$
$$x_3 = y'' \quad \Rightarrow \quad x'_3 = y''' = x_4$$
$$x_4 = y''' \quad \Rightarrow \quad x'_4 = y^{(4)} = -8y + \sin(t)\,y' - 3y'' + t^2 = -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2$$

The system along with the initial conditions is then:

$$x'_1 = x_2 \qquad\qquad x_1(0) = 1$$
$$x'_2 = x_3 \qquad\qquad x_2(0) = 2$$
$$x'_3 = x_4 \qquad\qquad x_3(0) = 3$$
$$x'_4 = -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 \qquad\qquad x_4(0) = 4$$

Write it in matrix form, we have:

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -8x_1 + \sin(t)\,x_2 - 3x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix}$$

where, $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}$

We have the system $\vec{x}' = A\vec{x} + \vec{g}(t)$

Remark. For a system $\vec{x}' = P\vec{x} + \vec{g}(t)$, the system is homogeneous if $\vec{g}(t) = \vec{0}$ and we say the system is nonhomogeneous if $\vec{g}(t) \neq \vec{0}$

5.4 Theories about Solutions To Systems

Consider the following matrices:

$$P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{bmatrix}, g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

Existence and Uniqueness theorem

Theorem 5.4. Consider this system of equations and its initial conditions $x_1(t_0) = x_1^0, x_2(t_0) = x_2^0, ..., x_n(t_0) = x_n^0$:

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned}$$

If the functions $p_{11}, p_{12}, ..., p_{nn}, g_1, g_2, ..., g_n$ are continuous on an open interval (α, β), then there exists a unique solution $x_1 = \phi_1(t), ..., x_n = \phi_n(t)$ such that is satisfies the system and the initial conditions

Principle of Superposition

Theorem 5.5. If vector functions $x^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, x^{(2)}(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{pmatrix}$ are solutions to the homogeneous system $\vec{x}' = P(t)x$, then any linear combination of them $c_1x^{(1)} + c_2x^{(2)}$ is also a solution to this homogeneous system.

Wronskian for solutions of a system of linear equations

Definition 5.12. Suppose vector functions $x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)$ are solutions to the homogeneous system $\vec{x}^{'} = P(t)x$ on interval (α, β). the wronskian of $x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)$ is deinfed as $W\left[x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)\right](t) = \det\left(x\left(t\right)\right) = \begin{vmatrix} \vdots & \vdots & \vdots \\ x^{(1)}(t) & \cdots & x^{(n)}(t) \\ \vdots & \vdots & \vdots \end{vmatrix} = \begin{vmatrix} x_{11}\left(t\right) & \cdots & \cdots & x_{1n}\left(t\right) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}\left(t\right) & \cdots & \cdots & x_{nn}\left(t\right) \end{vmatrix}$

Foundamental Set of Solutions

Definition 5.13. Suppose vector functions $x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)$ are solutions to the homogeneous system $\vec{x}^{'} = P(t)x$ on interval (α, β). And for every point on the open interval (α, β), if the solutions $x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)$ are all linear independent (Wronskian is never 0 for all possible t), then set $\left\{x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)\right\}$ is called the foundamental set of solutions

Fundamental matrix

Definition 5.14. Suppose vector functions $x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)$ are fundamental solutions to the homogeneous system $\vec{x}^{'} = P(t)x$ on interval (α, β). The matrix $\Phi(x) = (x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t))$ is said to be a fundamental matrix.

Linear Independence and span of Solutions

Theorem 5.6. Suppose vector functions $x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)$ are solutions to the homogeneous system $\vec{x}^{'} = P(t)x$ on interval (α, β). And for every point on the open interval (α, β), if the solutions $x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)$ are all linear independent, then each possible solution $x = x(t)$ to this system of equation can be expressed as the linear combination of $x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)$ in exactly one way. So we have $x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) + ... + c_n x^{(n)}(t)$

Able's Theorem

Theorem 5.7. Suppose vector functions $x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)$ are solutions to the homogeneous system $\vec{x}^{'} = P(t)x$ on interval (α, β). The wrongskian $W[x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}](t)$ is either zero on the entire interval or never zero on the entire interval

System always has at least one set of fundamental solutions

Let

$$e^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e^{(2)} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, e^{(n)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Let $x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)$ be the solutions of the system $\vec{x}^{'} = P(t)x$ that satisfy the initial conditions

$$x^{(1)}(t_0) = e^{(1)}, x^{(2)}(t_0) = e^{(2)}, ..., x^{(n)}(t_0) = e^{(n)}$$

Where t_0 is any point between $\alpha < t < \beta$. Then $x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)$ form a fundamental set of solutions to $\vec{x}^{'} = P(t)x$

5.5 Solve system of equations using Eigenvectors and Eigenvalues

Consider a linear homogeneous system of n differential equations with constant coefficients, which can be written in matrix form as $\mathbf{X}'(t) = A\mathbf{X}(t)$

Where the following notation is used: $\mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $\mathbf{X}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$.

We look for non-trivial solutions of the homogeneous system in the form of $\mathbf{X}(t) = e^{\lambda t}\mathbf{V}$,, where $V \neq 0$

Substituting the above expression for $\mathbf{X}(t)$ into the system of equations, we obtain $\lambda e^{\lambda t}\mathbf{V} = Ae^{\lambda t}\mathbf{V}, \ \Rightarrow A\mathbf{V} = \lambda\mathbf{V}$.

Thus, we conclude that in order the vector function $\mathbf{X}(t) = e^{\lambda t}\mathbf{V}$ be a solution of the homogeneous linear system, it is necessary and sufficient that the number λ be an eigenvalue of the matrix A, and vector V be the corresponding eigenvector.

We have three different cases: Real Eigenvalues, Complex eigenvalues, Repeated eigenvalues.

Real and distinct Eigenvalues

Consider system $\vec{x}' = A\vec{x}$

Suppose A has n distinct real eigenvalues λ_i, the general solution is in the form $\vec{x}\,(t) = c_1\mathbf{e}^{\lambda_1 t}\vec{v}_1 + ... + c_n\mathbf{e}^{\lambda_n t}\vec{v}_n$

Example 5.4. *Find the solution to the following system*

$$\begin{aligned} {x'}_1 &= x_1 + 2x_2 \qquad x_1(0) = 0 \\ {x'}_2 &= 3x_1 + 2x_2 \qquad x_2(0) = -4 \end{aligned}$$

Sol: $\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x}, \qquad \vec{x}\,(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda+1)(\lambda-4) \qquad \Rightarrow \qquad \lambda_1 = -1,\ \lambda_2 = 4 \end{aligned}$$

For $\lambda_1 = -1$, *we have* $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 4$, *we have* $v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ *Then general solution is then:* $\vec{x}\,(t) = c_1\mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

we simply need to apply the initial conditions: $\begin{pmatrix} 0 \\ -4 \end{pmatrix} = \vec{x}\,(0) = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

We solve the constant as: $\left.\begin{matrix} -c_1 + 2c_2 = 0 \\ c_1 + 3c_2 = -4 \end{matrix}\right\} \qquad \Rightarrow \qquad c_1 = -\frac{8}{5},\ c_2 = -\frac{4}{5}$

So the solution is: $\vec{x}\,(t) = -\frac{8}{5}\mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{4}{5}\mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Repeated Eigenvalues—When the algebraic multiplicity equals to the geometric multiplicity, and bigger than 1

Theorem 5.8. Suppose $n \times n$ matrix P has n real eigenvalues (not necessarily distinct), $\lambda_1, \lambda_2, \ldots, \lambda_n$ and there are n linear independent corresponding eigenvectors $\vec{v}_1$,$\vec{v}_2, \ldots \vec{v}_n$. Then the general solution to $\vec{x}' = P\vec{x}$

can be written as:

$$\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \cdots + c_n\vec{v}_n e^{\lambda_n t}$$

Repeated Eigenvalues—Defective eigenvalues

If an $n \times n$ matrix has less than n linear independent eigenvectors, it is said to be deficient. Then there is at least one eigenvalue with an algebraic multiplicity that is higher than its geometric multiplicity. Suppose that matrix A has an eigenvalue λ of multiplicity m. We find vectors such that

$$(A-\lambda I)^k \vec{v} = \vec{0}, \qquad \text{but} \qquad (A-\lambda I)^{k-1}\vec{v} \neq \vec{0}.$$

Such vectors are called generalized eigenvectors. For the eigenvector $\vec{v}_1$ there is a chain of generalized eigenvectors $\vec{v}_2$ through $\vec{v}_k$ such that:

$$\begin{aligned}
(A-\lambda I)\vec{v}_1 &= \vec{0},\\
(A-\lambda I)\vec{v}_2 &= \vec{v}_1,\\
&\vdots\\
(A-\lambda I)\vec{v}_k &= \vec{v}_{k-1}.
\end{aligned}$$

We form linearly independent solutions

$$\begin{aligned}
\vec{x}_1 &= \vec{v}_1 e^{\lambda t},\\
\vec{x}_2 &= (\vec{v}_2 + \vec{v}_1 t)\, e^{\lambda t},\\
&\vdots\\
\vec{x}_k &= \left(\vec{v}_k + \vec{v}_{k-1} t + \vec{v}_{k-2}\frac{t^2}{2} + \cdots + \vec{v}_2 \frac{t^{k-2}}{(k-2)!} + \vec{v}_1 \frac{t^{k-1}}{(k-1)!}\right) e^{\lambda t}.
\end{aligned}$$

We go until we form m linearly independent solutions where m is the algebraic multiplicity. And the general solution is the linear combination of the independent solutions.

Example 5.5. *Solve system*

$$\vec{x}' = \begin{pmatrix} -1 & \frac{3}{2} \\ -\frac{1}{6} & -2 \end{pmatrix} \vec{x} \qquad \vec{x}\,(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Sol:

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & \frac{3}{2} \\ -\frac{1}{6} & -2-\lambda \end{vmatrix}$$
$$= \lambda^2 + 3\lambda + \frac{9}{4}$$
$$= \left(\lambda + \frac{3}{2}\right)^2 \qquad \Rightarrow \qquad \lambda_{1,2} = -\frac{3}{2}$$

For the regular eigenvector:

$$\vec{\eta} = \begin{pmatrix} -3\eta_2 \\ \eta_2 \end{pmatrix} \qquad\qquad \eta_2 \neq 0$$
$$\vec{\eta}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad\qquad \eta_2 = 1$$

The algebraic multiplicity is 2, we need one more generalized eigenvector:
$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \frac{1}{2}\rho_1 + \frac{3}{2}\rho_2 = -3 \quad \rho_1 = -6 - 3\rho_2$
The general solution for the system is then:

$$\vec{x}(t) = c_1 \mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2\left(t\mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -6 \\ 0 \end{pmatrix}\right)$$

Applying the initial condition, we have:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{x}(2) = c_1 \mathbf{e}^{-3} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2\left(2\mathbf{e}^{-3} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mathbf{e}^{-3} \begin{pmatrix} -6 \\ 0 \end{pmatrix}\right)$$

$$\left.\begin{aligned} -3\mathbf{e}^{-3}c_1 - 12\mathbf{e}^{-3}c_2 = 1 \\ \mathbf{e}^{-3}c_1 + 2\mathbf{e}^{-3}c_2 = 0 \end{aligned}\right\} \quad \Rightarrow \quad c_1 = \frac{\mathbf{e}^3}{3}, \;\; c_2 = -\frac{\mathbf{e}^3}{6}$$

$$\vec{x}(t) = \frac{\mathbf{e}^3}{3}\mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} - \frac{\mathbf{e}^3}{6}\left(t\mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -6 \\ 0 \end{pmatrix}\right)$$
$$= \mathbf{e}^{-\frac{3t}{2}+3} \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} + t\mathbf{e}^{-\frac{3t}{2}+3} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{6} \end{pmatrix}$$

Complex Eigenvalues

Theorem 5.9. Let P be a constant real-valued matrix. If P has a complex eigenvalue $a+bi$ and a corresponding eigenvector v, then P also has a complex eigenvalue $a-bi$ with a corresponding eigenvector v. Furthermore, $\vec{x}' = P\vec{x}$ has two linearly independent real-valued solutions:

$$\vec{x}_1 = \mathrm{Re}\,\vec{v}e^{(a+ib)t}, \qquad \text{and} \qquad \vec{x}_2 = \mathrm{Im}\,\vec{v}e^{(a+ib)t}.$$

And the general solution is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1\left(\mathrm{Re}\,\vec{v}e^{(a+ib)t}\right) + c_2\left(\mathrm{Im}\,\vec{v}e^{(a+ib)t}\right).$$

Example 5.6. *Solve the equation:*

$$\vec{x}' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \vec{x}$$

Sol:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{vmatrix} \\ &= \lambda^2 - 4\lambda + 68 \qquad \lambda_{1,2} = 2 \pm 8\,i \end{aligned}$$

For $\lambda_1 = 2 + 8i$, the solution corresponding to this eigenvalue and eigenvector is:

$$\begin{aligned} \vec{x}_1(t) &= \mathbf{e}^{(2+8i)t} \begin{pmatrix} 1+8\,i \\ 5 \end{pmatrix} \\ &= \mathbf{e}^{2t}\mathbf{e}^{8i\,t} \begin{pmatrix} 1+8\,i \\ 5 \end{pmatrix} \\ &= \mathbf{e}^{2t}\left(\cos(8t) + i\sin(8t)\right) \begin{pmatrix} 1+8\,i \\ 5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{x}_1(t) &= \mathbf{e}^{2t} \begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + i\mathbf{e}^{2t} \begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix} \\ &= \vec{u}(t) + i\,\vec{v}(t) \end{aligned}$$

The general solution is therefore:

$$\vec{x}(t) = c_1\mathbf{e}^{2t} \begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + c_2\mathbf{e}^{2t} \begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix}$$

5.6 Method of Matrix exponential

Matrix Exponential and solution to the system

Theorem 5.10. Let P be an $n \times n$ matrix, then the general solution to $\vec{x}' = Px$ is

$$\vec{x} = e^{tP}\vec{c}$$

, where $\vec{c}$ is a constant vector, and $x(0) = c$

In general: Assume P is diagonalizable, and let λ_1,λ_2,..., λ_n be the eigenvalues of P with corresponding $v_1, ..., v_n$ be eigenvectors. Take

$$E = [\,\vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_n\,], \text{ and } D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$e^{tP} = Ee^{tD}E^{-1} = E \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} E^{-1}$$

Example 5.7. *Compute a fundamental matrix solution using the matrix exponential for the system:*

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then compute the particular solution for the initial conditions $x(0) = 4, y(0) = 2$
Sol:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}}_{E^{-1}}.$$

We write:

$$\begin{aligned} e^{tA} = Ee^{tD}E^{-1} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{-1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} -e^{3t} - e^{-t} & -e^{3t} + e^{-t} \\ -e^{3t} + e^{-t} & -e^{3t} - e^{-t} \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{3t}-e^{-t}}{2} \\ \frac{e^{3t}-e^{-t}}{2} & \frac{e^{3t}+e^{-t}}{2} \end{bmatrix}. \end{aligned}$$

Applying the initial conditions, we have:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{3t}-e^{-t}}{2} \\ \frac{e^{3t}-e^{-t}}{2} & \frac{e^{3t}+e^{-t}}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2e^{3t} + 2e^{-t} + e^{3t} - e^{-t} \\ 2e^{3t} - 2e^{-t} + e^{3t} + e^{-t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} + e^{-t} \\ 3e^{3t} - e^{-t} \end{bmatrix}.$$

5.7 Nonhomogeneous Systems—Undetermined Coefficients

The method of Undetermined Coefficients for systems is pretty much identical to the second order differential equation case. The only difference is that the coefficients will need to be vectors now.

Example 5.8. *Solve the system of DE:*

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x} + t \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Sol: The complementary solution is to set this into homogeneous system:

$$\vec{x}_c(t) = c_1 \mathbf{e}^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

We have a linear polynomial and so our guess will need to be a linear polynomial. The only difference is that the "coefficients" will need to be vectors instead of constants. The particular solution will have the form:

$$\vec{x}_P = t\vec{a} + \vec{b} = t \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

So, we need to differentiate the guess::

$$\vec{x}_P' = \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

We'll rewrite the system as:

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x} + t \begin{pmatrix} 2 \\ -4 \end{pmatrix} = A\vec{x} + t\vec{g}$$

Now, let's plug things into the system:

$$\begin{aligned} \vec{a} &= A\left(t\vec{a} + \vec{b}\right) + t\vec{g} \\ \vec{a} &= tA\vec{a} + A\vec{b} + t\vec{g} \\ \vec{0} &= t\left(A\vec{a} + \vec{g}\right) + \left(A\vec{b} - \vec{a}\right) \end{aligned}$$

Now we need to set the coefficients equal:

$$\begin{array}{lcc} t^1: & A\vec{a} + \vec{g} = \vec{0} & A\vec{a} = -\vec{g} \\ t^0: & A\vec{b} - \vec{a} = \vec{0} & A\vec{b} = \vec{a} \end{array}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\begin{pmatrix} 2 \\ -4 \end{pmatrix} \quad \Rightarrow \quad \vec{a} = \begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix} \quad \Rightarrow \quad \vec{b} = \begin{pmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{pmatrix}$$

So the particular solution is therefore:

$$\vec{x}_P = t \begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix} + \begin{pmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{pmatrix}$$

And we have the general solution given by:

$$\vec{x}(t) = c_1 \mathbf{e}^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix} + \begin{pmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{pmatrix}$$

5.8 Nonhomogeneous Systems—Variation of Parameters

Consider the system of differential equation:

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

Let $X(t)$ be the matrix whose *ith* column is the *ith* linearly independent solution to the system:

$$\vec{x}' = A\vec{x}$$

The particular solution is given by

$$\vec{X}_P = X \int X^{-1}\vec{g}\, dt$$

And the general solution to this system is:

$$X + X_P$$

Example 5.9. *Find the general solution to the following system:*

$$\vec{x}' = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} \vec{x} + \mathbf{e}^{2t} \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

We found the complementary solution to this system using the real eigenvalue. It is:

$$\vec{x}_c(t) = c_1\mathbf{e}^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2\mathbf{e}^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The matrix X is:

$$X = \begin{pmatrix} \mathbf{e}^{-t} & -\mathbf{e}^{-6t} \\ 4\mathbf{e}^{-t} & \mathbf{e}^{-6t} \end{pmatrix}, X^{-1} = \begin{pmatrix} \frac{1}{5}\mathbf{e}^{t} & \frac{1}{5}\mathbf{e}^{t} \\ -\frac{4}{5}\mathbf{e}^{6t} & \frac{1}{5}\mathbf{e}^{6t} \end{pmatrix}$$

Now do the multiplication in the integral:

$$X^{-1}\vec{g} = \begin{pmatrix} \frac{1}{5}\mathbf{e}^{t} & \frac{1}{5}\mathbf{e}^{t} \\ -\frac{4}{5}\mathbf{e}^{6t} & \frac{1}{5}\mathbf{e}^{6t} \end{pmatrix} \begin{pmatrix} 6\mathbf{e}^{2t} \\ -\mathbf{e}^{2t} \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{3t} \\ -5\mathbf{e}^{8t} \end{pmatrix}$$

Now do the integral:

$$\int X^{-1}\vec{g}\, dt = \int \begin{pmatrix} \mathbf{e}^{3t} \\ -5\mathbf{e}^{8t} \end{pmatrix} dt = \begin{pmatrix} \int \mathbf{e}^{3t}\, dt \\ \int -5\mathbf{e}^{8t}\, dt \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\mathbf{e}^{3t} \\ -\frac{5}{8}\mathbf{e}^{8t} \end{pmatrix}$$

So the particular solution is:

$$\begin{aligned} \vec{x}_P &= X \int X^{-1}\vec{g}\, dt \\ &= \begin{pmatrix} \mathbf{e}^{-t} & -\mathbf{e}^{-6t} \\ 4\mathbf{e}^{-t} & \mathbf{e}^{-6t} \end{pmatrix} \begin{pmatrix} \frac{1}{3}\mathbf{e}^{3t} \\ -\frac{5}{8}\mathbf{e}^{8t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{23}{24}\mathbf{e}^{2t} \\ \frac{17}{24}\mathbf{e}^{2t} \end{pmatrix} \\ &= \mathbf{e}^{2t}\begin{pmatrix} \frac{23}{24} \\ \frac{17}{24} \end{pmatrix} \end{aligned}$$

The general solution is:

$$\vec{x}\,(t) = c_1 \mathbf{e}^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 \mathbf{e}^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mathbf{e}^{2t} \begin{pmatrix} \frac{23}{24} \\ \frac{17}{24} \end{pmatrix}$$

6 Nonlinear system

6.1 Phase Line of the system of DE

Consider the following system:

$$\vec{x}^{'} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x} = A\vec{x}$$

Solutions to this system will be of the form:

$$\vec{x} = \begin{pmatrix} x_1\,(t) \\ x_2\,(t) \end{pmatrix}$$

Definition 6.1 (Trajectory)**.** A sketch of a particular solution in the phase plane is called the trajectory of the solution.

Definition 6.2 (Critical points)**.** Critical points are points (x, y) such that

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \vec{0}.$$

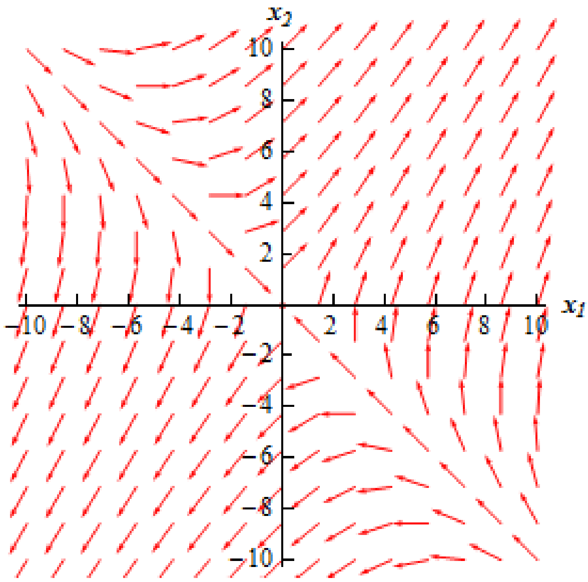
Example 6.1. *Graph the phase line of the system of DE:*

$$\vec{x}^{'} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \vec{x}$$

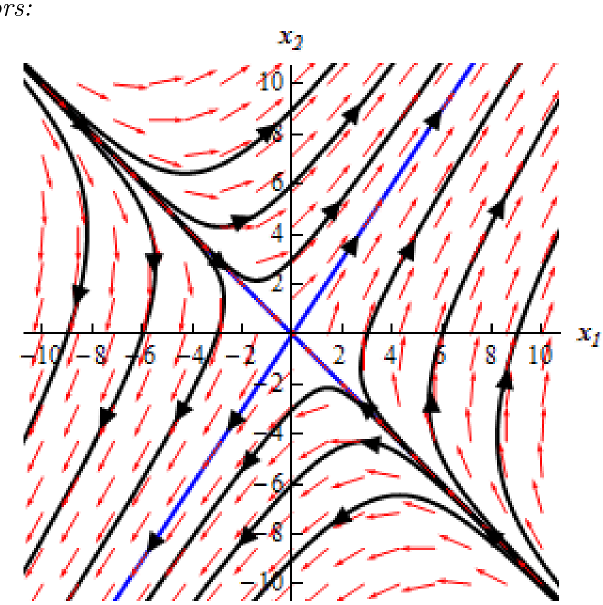
Sol:

$$\begin{aligned}
\vec{x} &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} & \Rightarrow \quad \vec{x}' &= \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\vec{x} &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \Rightarrow \quad \vec{x}' &= \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \\
\vec{x} &= \begin{pmatrix} -3 \\ -2 \end{pmatrix} & \Rightarrow \quad \vec{x}' &= \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -7 \\ -13 \end{pmatrix}
\end{aligned}$$

At the point $(-1, 1)$, we draw vector $< 1, -1 >$, etc. By drawing infinitely many points,we have the phase plane:



Phase portrait only include the trajectories of the solutions and not any vectors:



There can be classifications of critical points using the eigenvalues:

Table 1: Stability Properties of Linear Systems $x^{'} = Ax$ with $det(A - rI) = 0$ and $det(A) \neq 0$

Eigenvalues	Types of critical points	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asymptotically stable
$r_2 < 0 < r_1$	Saddle point	Unstable
$r_1 = r_2 > 0$	Proper or improper node	Unstable
$r_1 = r_2 < 0$	Proper or improper node	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$		
$\lambda > 0$	Spiral point	Unstable
$\lambda < 0$	Spiral point	Asymptotically stable
$\lambda = 0$	Center	Stable

Critical point x

As we define $det(A) \neq 0$, the only critical point is $x = 0$.
So, the critical point $x = 0$ to the system $x^{'} = Ax$ is the only critical point.
The critical point is:

- The critical point is asymptotically stable if the eigenvalues r_1, r_2 are real and negative or have negative real part
- is stable, but not asymptotically stable, if r_1 and r_2 are pure imaginary
- is unstable if r_1 and r_2 are real and either is positive, or if they have real part.

6.2 Autonomous systems and Stability

We restrict our attention to a two-dimensional autonomous system:

$$x' = f(x, y), \qquad y' = g(x, y),$$

where $f(x, y)$ and $g(x, y)$ are functions of two variables, and the derivatives are taken with respect to t. Solutions are functions $x(t)$ and $y(t)$ such that

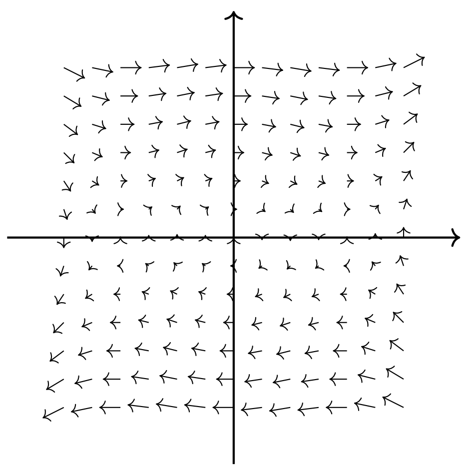
$$x'(t) = f\big(x(t), y(t)\big), \qquad y'(t) = g\big(x(t), y(t)\big).$$

AS introduced, we can draw the phase plane as introduced:

Example 6.2. *Consider the second order equation $x'' = -x + x^2$. Write this equation as a first order nonlinear system*

$$x' = y, \qquad y' = -x + x^2.$$

The corresponding phase plane is:



Stability and Instability

Definition 6.3 (Stable)**.** A critical point x^0 in an autonomous system $x^{'} = f(x)$ is called **stable** if for any $\epsilon > 0$, there exists a $\delta > 0$ such that for every solution of the autonomous system $x = x(t)$ which at $t = 0$ satisfies $||x(0) - x^0|| < 0$ both exist for all positive t, and satisfies $||x(t) - x^0|| < \epsilon$ for all $t \leq 0$

Definition 6.4 (Asymptotically stable)**.** Critical point x^0 is asymptotically stable when it is stable and $\exists \delta_0 > 0$ such that if a solution $x(t)$ to the system satisfies

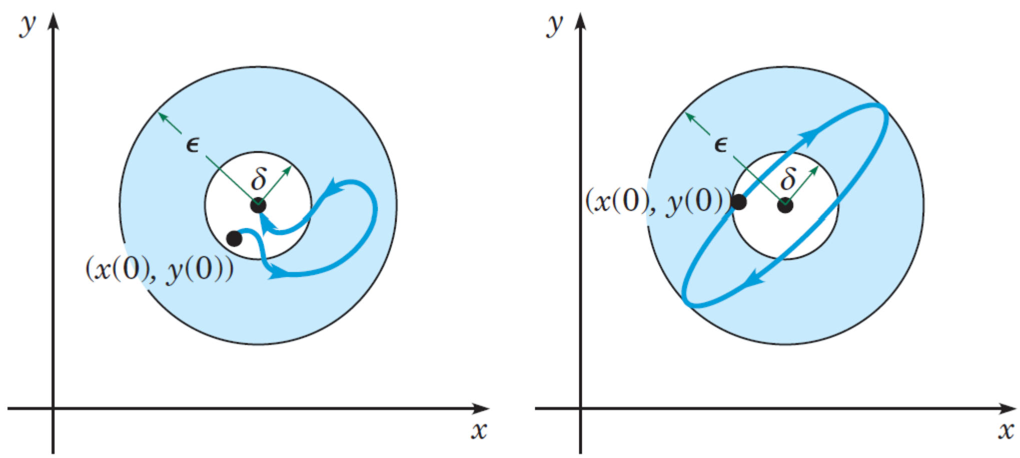
$$||x(0) - x^0|| < \delta_0$$

, then

$$\lim_{t\to\infty} x(t) = x^0$$

Definition 6.5 (Unstable)**.** A critical point is unstable when it is not stable.

The graphic representation of asymptotically stable and stable is:



6.3 Locally Linear Systems

Locally linear system

Consider non-linear system:

$$x' = F(x, y), \quad y' = G(x, y)$$

that is, $\mathbf{x} = (x, y)^T$ and $\mathbf{f}(\mathbf{x}) = (F(x, y), G(x, y))^T$. The above system is locally linear in the neighborhood of a critical point (x_0, y_0) whenever the function F and G have continuous partial derivatives up to order 2

Use Taylor expansions about the point (x_0, y_0) to write $F(x, y)$ and $G(x, y)$ in the form

$$\begin{aligned}
F(x, y) &= F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) + \eta_1(x, y), \\
G(x, y) &= G(x_0, y_0) + G_x(x_0, y_0)(x - x_0) + G_y(x_0, y_0)(y - y_0) + \eta_2(x, y),
\end{aligned}$$

where $\eta_1(x, y) / \left((x - x_0)^2 + (y - y_0)^2\right)^{1/2} \to 0$ as $(x, y) \to (x_0, y_0)$, and similarly for η_2. Note that $F(x_0, y_0) = G(x_0, y_0) = 0$, and that $dx/dt = d(x - x_0)/dt$

and $dy/dt = d\left(y - y_0\right)/dt$. Then the system $x' = F(x, y), \quad y' = G(x, y)$ reduces to

$$\frac{d}{dt}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} F_x\left(x_0, y_0\right) & F_y\left(x_0, y_0\right) \\ G_x\left(x_0, y_0\right) & G_y\left(x_0, y_0\right) \end{pmatrix}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} \eta_1(x, y) \\ \eta_2(x, y) \end{pmatrix},$$

or, in vector notation,

$$\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{f}}{d\mathbf{x}}\left(\mathbf{x}^0\right)\mathbf{u} + \eta(\mathbf{x})$$

where $\mathbf{u} = \left(x - x_0, y - y_0\right)^T$ and $\eta = \left(\eta_1, \eta_2\right)^T$. the linear system that approximates the nonlinear system near $\left(x_0, y_0\right)$ is given by the linear part of equations $\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{f}}{d\mathbf{x}}\left(\mathbf{x}^0\right)\mathbf{u} + \eta(\mathbf{x})$:

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} F_x\left(x_0, y_0\right) & F_y\left(x_0, y_0\right) \\ G_x\left(x_0, y_0\right) & G_y\left(x_0, y_0\right) \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where $u_1 = x - x_0$ and $u_2 = y - y_0$. Equation above provides a simple and general method for finding the linear system corresponding to a locally linear system near a given critical point. The matrix

$$\mathbf{J} = \mathbf{J}[F, G](x, y) = \begin{pmatrix} F_x(x, y) & F_y(x, y) \\ G_x(x, y) & G_y(x, y) \end{pmatrix}$$

Stability and Instability properties of linear and locally linear systems

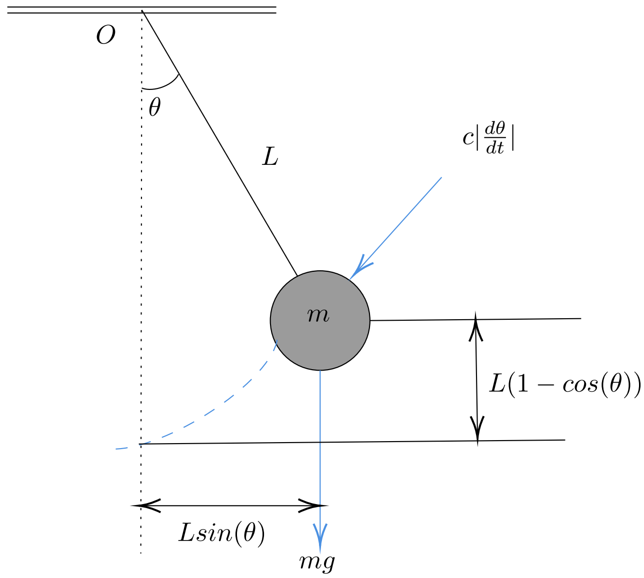
Let r_1 and r_2 be the eigenvalues of the linear system (1), $\mathbf{x}' = \mathbf{A}\mathbf{x}$, corresponding to the locally linear system (4), $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$. Then the type and stability of the critical point of the linear system (1) and the locally linear system (4) are as shown in the table below:

N: Node, IN: Improper Node, PN: Proper Node, SP: Saddle point, SpP: Spiral point, C: Center

Eigenvalues	Linear System		Locally Linear System	
	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	PN or W	Unstable	N or SpP	Unstable
$r_1 = r_2 < 0$	PN or IN	Asymptotically stable	N or SpP	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$\lambda = 0$	C	Stable	C or SpP	Indeterminate

6.4 The Oscillating Pendulum

Consider the following oscillating pendulum:



The governing equation is:

$$mL^2 \frac{d^2\theta}{dt^2} = -cL\frac{d\theta}{dt} - mgL\sin\theta$$

By straightforward algebraic operations, rewrite the equation above in the standard form

$$\frac{d^2\theta}{dt^2} + \frac{c}{mL}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0,$$

or

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2\sin\theta = 0$$

where $\gamma = \frac{c}{mL}$ and $\omega^2 = \frac{g}{L}$. To convert equation to a system of two first-order equations, let $x = \theta$ and $y = d\theta/dt$; then

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\omega^2 \sin x - \gamma y$$

The critical points from this system are:

$$(x, y) = (n\pi, 0) \ and \ (-n\pi, 0) \ and \ (0, 0)$$

These points correspond to two physical equilibrium positions, one with the mass directly below the point of support and the other with the mass directly above the point of support.

Using the theorem above, the system is C^2, so it is locally linear in the neighborhood of the critical points.
We can find the linear system using the equations above near the origin; near the critical point $(\pi, 0)$. In this case we have,

$$F(x, y) = y, \quad G(x, y) = -\omega^2 \sin x - \gamma y$$

since these functions are differentiable as many times as necessary, the system is locally linear near each critical point. The first partial derivatives of F and G are

$$F_x = 0, \quad F_y = 1, \quad G_x = -\omega^2 \cos x, \quad G_y = -\gamma.$$

Thus, at the origin the corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

Similarly, evaluating the partial derivatives in equation (16) at $(\pi, 0)$, we obtain

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

where $u = x - \pi, v = y$. This is the linear system near the point $(\pi, 0)$ Near the origin, the non-linear system $F(x, y) = y, \quad G(x, y) = -\omega^2 \sin x - \gamma y$ is approximated by linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

The eigenvalues of this matrix is

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}$$

1. If $\gamma^2 - 4\omega^2 > 0$, the eigenvalues are real, unequal and negative. The critical point $(0, 0)$ is an asymptotically stable node of the linear system and of the locally linear system

2. If $\gamma^2 - 4\omega^2 = 0$, the eigenvalues are real, equal and negative. The critical point $(0, 0)$ is an asymptotically stable (proper or improper) node of the linear system. It may be either an asymptotically stable node or spiral point of the locally linear system.

3. If $\gamma^2 - 4\omega^2 < 0$, then the eigenvalues are complex with negative real part The critical point $(0, 0)$ is an asymptotically stable spiral point of the linear system and of the locally linear system.

6.5 Liapunov's second method

Liapunov's stability theorem

Consider the autonomous system

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y),$$

Suppose that the autonomous system has an critical point (x_0, y_0).
let $V : \mathcal{O} \to \mathbb{R}$ be a differentiable function defined on an open set $\mathcal{O}$ containing (x_0, y_0). Suppose further that:

1. $V(x_0, y_0) = 0$ and $V(x, y) > 0$ if $(x, y) \neq (x_0, y_0)$
2. $\dot{V}(x, y) = V_x(x, y)F(x, y) + V_y(x, y)G(x, y) \leq 0$ in $\mathcal{O} - \{(x_0, y_0)\}$

Then (x_0, y_0) is stable. Furthermore, if V also satisfies: 3: $\dot{V}(x, y) < 0$ in $\mathcal{O} - \{(x_0, y_0)\}$, then (x_0, y_0) is asymptotic ally stable.

Liapunov's function and strict Liapunov's function

Definition 6.6 (Liapunov's function)**.** Function V is called Liapunov's function if it satisfies criteria 1 and 2

Definition 6.7 (Strict Liapunov's function)**.** Function V is called a strict Liapunov's function if it satisfies 1 and 2 and 3

Consider the following oscillating pendulum:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\omega^2 \sin x - \gamma y$$

$(0, 0)$ is a critical point of the system. Show that $V(x, y) = mgL(1 - \cos x) + \frac{1}{2}mL^2y^2$. is a Liapunov's function for $(0, 0)$.
Actually, $V(x, y)$ is the total energy of the pendulum:

- $mgL(1 - \cos x)$: potential energy which measures how much work it takes to lift the mass
- $\frac{1}{2}mL^2y^2$ = kinetic energy.

Sol:

$$V(0, 0) = mgL(1 - \cos 0) + \frac{1}{2}mL^2 0^2 = 0$$

Set $\mathcal{O} = \left\{(x, y) : -\frac{\pi}{2} < x < \frac{1}{2}, -1 < y < 1\right\}$

$mgL(1-\cos x): -\dfrac{\pi}{2} < -x < \dfrac{\pi}{2}, \quad mgL(1-\cos x) \geqslant 0.$
and $mgL(1-\cos x) = 0 \Leftrightarrow x = 0.$

$\frac{1}{2}mL^2y^2: \quad -ky < 1, \quad \frac{1}{2}mL^2y^2 \geqslant 0$ and $\quad \frac{1}{2}mL^2y^2 \neq\Leftrightarrow y_0.$
So $V(x,y) > 0$ on $\theta - \{(0,0)\}$.

$$\begin{aligned}
\dot{V}(x,y) &= V_x(x,y)F(x,y) + V_y(x,y)G(x,y) \\
&= V_x(x,y)y + V_y(x,y)\left(-\gamma y - \omega^2 \sin x\right) \\
&= mgL\sin x \cdot y + mL^2y\left(-\gamma y - \omega^2 \sin x\right) \\
&= -mL^2\gamma y^2 \leqslant 0 \text{ on } \theta - \{(0,0)\}.
\end{aligned}$$

By Liapunov' stability theonem, $(0,0)$ is stable.

Positive definite and negative definition

Definition 6.8. Positive definite function:

1. $V(\mathbf{x}) > 0 \quad \forall \mathbf{x} \neq \mathbf{0}$ (strictly positive for all non-zero inputs)
2. $V(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$ (equal to zero only at the origin)

Definition 6.9. Negative definite function:

1. $W(\mathbf{x}) < 0 \quad \forall \mathbf{x} \neq \mathbf{0}$ (strictly negative for all non-zero inputs)
2. $W(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$ (equal to zero only at the origin)

Theorem about positive definite and negative definite

Theorem 6.1. The function

$$V(x,y) = ax^2 + bxy + cy^2$$

is positive definite if, and only if,

$$a > 0 \text{ and } 4ac - b^2 > 0$$

and is negative definite if, and only if,

$$a < 0 \text{ and } 4ac - b^2 > 0.$$

Example 6.3. *Show that the critical point $(x,y) = (0,0)$ of the autonomous system*

$$\frac{dx}{dt} = -x - xy^2, \quad \frac{dy}{dt} = -y - x^2y$$

is asymptotically stable. Solution: The only equilibrium solution of system is $(x,y) = (0,0)$. We try to construct a Liapunov function of the form $V(x,y) =$

$ax^2+bxy+cy^2$. Then $V_x(x,y)=2ax+by, V_y(x,y)=bx+2cy$, so

$$\begin{aligned}\dot{V}(x,y)&=(2ax+by)\left(-x-xy^2\right)+(bx+2cy)\left(-y-x^2y\right)\\&=-\left(2a\left(x^2+x^2y^2\right)+b\left(2xy+xy^3+x^3y\right)+2c\left(y^2+x^2y^2\right)\right).\end{aligned}$$

If we choose $b=0$, and a and c to be any positive numbers, then $\dot{V}$ is negative definite and V is positive definite by Theorem. Thus, by Theorem, the origin is an asymptotically stable critical point.

6.6 Competing species

Suppose that in some closed environment there are two similar species competing for a limited food supply. Let x and y be the populations of the two species at time t. As discussed in previous section, we assume that the population of each of the species, in the absence of the other, is governed by a logistic equation. Thus

$$\begin{aligned}\frac{dx}{dt}&=x\left(\epsilon_1-\sigma_1x\right)\\\frac{dy}{dt}&=y\left(\epsilon_2-\sigma_2y\right)\end{aligned}$$

respectively, where ϵ_1 and ϵ_2 are the growth rates of the two populations, and ϵ_1/σ_1 and ϵ_2/σ_2 are their saturation levels. However, when both species are present, each will tend to diminish the available food supply for the other. In effect, they reduce each other's growth rates and saturation populations. The simplest expression for reducing the growth rate of species x due to the presence of species y is to replace the growth rate factor $\epsilon_1-\sigma_1x$ in the first of equations by $\epsilon_1-\sigma_1x-\alpha_1y$, where α_1 is a measure of the degree to which species y interferes with species x. Similarly, in the second of equations we replace $\epsilon_2-\sigma_2y$ by $\epsilon_2-\sigma_2y-\alpha_2x$. Thus we have the system of equations

$$\begin{aligned}\frac{dx}{dt}&=x\left(\epsilon_1-\sigma_1x-\alpha_1y\right),\\\frac{dy}{dt}&=y\left(\epsilon_2-\sigma_2y-\alpha_2x\right).\end{aligned}$$

The values of the positive constants $\epsilon_1,\sigma_1,\alpha_1,\epsilon_2,\sigma_2$, and α_2 depend on the particular species under consideration and, in general, must be determined from observations. We are interested in solutions of the above equations for which x and y are nonnegative.

Coexistence occurs when equilibrium solution with both (x,y) are bigger than 0. So all species exist.

To find the x- and y-nullclines, set the derivatives dx/dt and dy/dt to zero, and solve for x and y. The nullclines represent the points in the phase plane where the rate of change of one of the variables is zero, and they can help you understand the dynamics and equilibrium points of the system.

Given the system:

$$\frac{dx}{dt} = x(\epsilon_1 - \sigma_1 x - \alpha_1 y) \qquad \frac{dy}{dt} = y(\epsilon_2 - \sigma_2 y - \alpha_2 x)$$

To find the x-nullcline, set dx/dt = 0:

$$x(\epsilon_1 - \sigma_1 x - \alpha_1 y) = 0$$

This equation is satisfied when:
1: x = 0
2: $\epsilon_1 - \sigma_1 x - \alpha_1 y = 0$
To find the y-nullcline, set dy/dt = 0:

$$y(\epsilon_2 - \sigma_2 y - \alpha_2 x) = 0$$

This equation is satisfied when:
1: y = 0
2: $\epsilon_2 - \sigma_2 y - \alpha_2 x = 0$
In a competing species model, x and y typically represent the populations of two different species. The x-nullcline represents the points in the phase plane where the population growth rate of species x is zero, i.e., the population of species x is neither increasing nor decreasing. Similarly, the y-nullcline represents the points where the population growth rate of species y is zero.

The intersections of the x- and y-nullclines are the equilibrium points of the system. At these points, the population growth rates of both species are zero, and the populations of both species remain constant. By analyzing the stability of these equilibrium points, you can determine the long-term behavior of the competing species populations.

When $\sigma_1\sigma_2 > \alpha_1\alpha_2$, interaction (competition) is "weak" and the species can coexist; when $\sigma_1\sigma_2 < \alpha_1\alpha_2$, interaction (competition) is "strong" and the species cannot coexist-one must die out.

6.7 Predator-Prey Equations

We will denote by x and y the populations of the prey and predator, respectively, at time t. In constructing a model of the interaction of the two species, we make the following assumptions:

$$\frac{dx}{dt} = ax - \alpha xy = x(a - \alpha y)$$
$$\frac{dy}{dt} = -cy + \gamma xy = y(-c + \gamma x).$$

The constants a, c, a, and γ are all positive; a and c are the growth rate of the prey and the death rate of the predator, respectively, and α and γ are measures of the effect of the interaction between the two species.

The general system can be analyzed in exactly the same way as in the example. The critical points of the system are the solutions of

$$x(a - \alpha y) = 0, \quad y(-c + \gamma x) = 0,$$

that is, the points $(0,0)$ and $(c/\gamma, a/\alpha)$.

In the neighborhood of the origin, the corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues and eigenvectors are

$$r_1 = a, \quad \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = -c, \quad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{at} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ct}$$

The Jacobian matrix is

$$\mathbf{J} = \begin{pmatrix} a - \alpha y & -\alpha x \\ \gamma y & -c + \gamma x \end{pmatrix}.$$

Thus the origin is a saddle point and hence unstable. Entrance to the saddle point is along the (positive) y-axis; all other trajectories depart from the neighborhood of the critical point.