Valuation of common stock

Ideas of the valuation

The fundamental principle in finance is that value of any asset is the present value of its expected future cash flows. For stocks, they comes from

— Dividends : made by corporation to the share holders.

— Future sale proceeds : profits or loss made from selling the stock.

Pure dividends and future sales

If the stock is held forever :

$$\boxed{\text{Stock Price} = P_0 = PV \text{ of } Div_1 + PV \text{ of } Div_2 + \cdots = \sum_{t=1}^{\infty} PV_0 \text{ of } Div_t}$$

This means the infinite sum of the present value of the dividends.

If the stock is held until time k: The stock price is the present value of dividends until time k plus the present value of the stock price at time k.

$$\text{Stock Price} = P_0 = PV_0 \text{ of } Div_1 + \cdots + PV_0 \text{ of } Div_k + PV_0 \text{ of } P_k$$

$$P_k = PV_k \text{ of } Div_{k+1} + \cdots + PV_k \text{ of } Div_{k+2} + \cdots$$

$$PV_0 \text{ of } P_k = PV_0 \text{ of } Div_{k+1} + PV_0 \text{ of } Div_{k+2} + \cdots$$

$$\Rightarrow P_0 = \sum_{t=1}^{\infty} PV_0 \text{ of } Div_t$$

Some important terminology in stock valuation

Definition (Earning): The profit the company made after all expenses, taxes, and costs subtracted from total revenue. This is also known as the net income.

Definition (Earning per shares): The portion of a company's profit allocated to each outstanding share of common stock.

$$EPS = \frac{\text{Earnings}}{\text{Total common stock}}$$

Definition (Dividend payout ratio): $\text{Payout ratio} = \dfrac{\text{Total dividends}}{\text{Earnings}} = \dfrac{\text{Div. per share}}{EPS}$

Definition (Retained earning): The portion of net income that is not distributed as dividends but is retained in the business.

Definition (Retention ratio): $\text{Retention ratio} = \dfrac{\text{Retained earning this year}}{\text{earning this year}} = 1 - \text{payout ratio}$

Definition (Return on equity): $RoE = \dfrac{\text{Net income}}{\text{Total equity}} = \dfrac{\text{Net income}}{\text{Total assets} - \text{total liabilities}}$

Definition (Dividend yield): $\text{Dividend yield} = \dfrac{\text{Dividend per share}}{\text{Share price}}$

The dividend discount model

The true value of a stock arises from the future dividend cash flows it will generate. This is shown above.

Even if the stock is currently not paying dividends, they will be paid eventually.

The DDM model evaluates the value of common stocks based on the future dividends.

we have three cases:

- zero growth
- Constant growth
- Differential growth

zero growth

Here, we assume dividends remain the same forever: $Div_1 = Div_2 = \cdots = Div$

$$P_0 = \text{stock price} = \frac{Div}{r}, \quad r = \text{relevant discount rate per paying period} = \text{the required return on stock.}$$

Constant growth

Here, we assume dividends grow at a constant rate g forever: $Div_t = Div_1 (1+g)^{t-1}$

$$\text{Stock price} = P_0 = \frac{Div_1}{r-g}$$

Differentiate growth

Here, we assume dividends grow at rate g_1 for N years and at g_2 afterwards forever.

$$Div_2 = Div_1 (1+g_1)$$

$$\vdots$$

$$Div_N = Div_{N-1} (1+g_1) = Div_1 \cdot (1+g_1)^{N-1}$$

$$Div_{N+1} = Div_N \cdot (1+g_2) = Div_1 \cdot (1+g_1)^{N-1} \cdot (1+g_2)$$

$$\vdots$$

$$P_0 = \text{Stock Price} = P_{0A} + P_{0B} = \underbrace{\frac{Div_1}{r-g_1} \cdot \left[1 + \left(\frac{1+g_1}{1+r}\right)^N\right]}_{\substack{g_1 \text{ Period} \\ \text{Annuity growing}}} + \underbrace{\frac{\dfrac{\overbrace{Div_1 \cdot (1+g_1)^{N-1} \cdot (1+g_2)}^{= Div_{N+1}}}{r-g_2}}{(1+r)^N}}_{\substack{g_2 \text{ Period} \\ \text{Perpetuity growing}}}$$

Estimates of g in DDM

The growth rate g reflects how fast a company's earnings and dividends are expected to grow.

We can derive g from retention ratio and its ROE.

we have:

Earning next year = Earning this year + Retained earning this year · return on retained earning

$$\Rightarrow \frac{\text{Earning next year}}{\text{Earning this year}} = \frac{\text{Earning this year}}{\text{Earning this year}} + \frac{\text{Retained earning this year}}{\text{Earning this year}} \cdot \text{return on retained earning}$$

$$\Rightarrow 1+g = 1 + \text{retention ratio} \cdot \text{return on retained earning}$$

$$\Rightarrow \boxed{g = \text{retention ratio} \cdot \text{return on retained earning}}$$

Estimates of r in DDM

The discount rate r is the rate of return that investors require to hold the stock.

For the constant growth case, we have $P_0 = \frac{Div_1}{r-g} \Rightarrow \frac{Div_1}{P_0} = r-g$

$$\Rightarrow r = \frac{Div_1}{P_0} + g = \text{Dividend yield} + g$$

The NPVGO model

The NPVGO (Net present value of growth opportunities) model helps in valuing stocks that do not pay dividends by considering the value of growth opportunities.

The value of the firm is the sum of:

- The value of a firm that pays out all its earnings as dividends.
- The net present value of growth opportunities (NPVGO).

The total value of the firm is the sum of the value derived from its current earnings and the value from its growth opportunities. And this market valuation of the firm is equal to the stock price of the firm:

$$P = \frac{EPS}{r} + \text{NPVGO per share}$$

For example: Zybrid has EPS of \$1.5 with 1 million shares outstanding, and currently pays out all earnings as dividends.

The company plans to invest \$1.5 million in a project at date 1, expected to increase earnings by \$250000 annually, yielding 16.7% return on the project.

The firm's discount rate is 12%.

NPV at date 1 $= -1500000 + \frac{250000}{0.12} = \$583333$

Value at date 0 $= \frac{583333}{1.12} = \$520833$

NPVGO per share $= \frac{520833}{1000000} = \$0.52$

$\Rightarrow P = \frac{1.5}{0.12} + 0.52 = \$13.02$

The DDM v.s. NPVGO model

For example: A firm has EPS of $5 at the end of the first year, a dividend payoff ratio of 30%, a discount rate of 16%, and a return on retained earnings of 20%.

Dividend $= 5 \cdot 0.3 = \$1.5$ per share

Retention ratio $= 1 - 0.3 = 0.7$

Growth rate $= 0.7 \cdot 0.2 = 0.14$

Using DDM, $P_0 = \frac{1.5}{0.16 - 0.14} = \$75$

$\frac{EPS}{r} = \frac{5}{0.16} = \$31.25$

$NPVGO = \frac{-3.5 + \frac{3.5 \cdot 0.2}{0.16}}{0.16 - 0.14} = \$43.75$

$\Rightarrow$ Using NPVGO, $P_0 = 31.25 + 43.75 = \$75$

Since both model yield the same price, it means the intrinsic value calculated from both perspective are equivalent. They are just different perspective.

Price-Earing ratio

The PE ratio or PE multiple is a key metric to relate a company's EPS to its stock prices.

$$PE \text{ multiple} = \frac{P}{EPS} = \frac{1}{r} + \frac{NPVGO}{EPS}$$

— PE multiple is negatively related to r.

— PE multiple is positively related to g because increasing g drives up stock price and vice versa.

Definition (Growth stocks): Shares in company that are expected to grow above average compare with other companies.

Definition (value stock): shares in the company that are considered undervalued.

Growth stocks tend to have high PE multiple because investors expect substantial earning growth. this leads to a higher stock price.

Value stocks tend to have low PE multiple because they are preceived to be undervalued.

Investment rules

The net present value (NPV) rule

The net present value (NPV) is the difference between the present value of cash inflows and the present value of cash inflows over a period of time.

$$NPV = PV \text{ of cash inflows} - PV \text{ of cash outflows}$$

or

$$NPV = PV \text{ of future net cash flows} - \text{initial investment}$$

In order to estimate NPV:

— Determine the amounts and future cash inflows and outflows.

— Determine the discount rate, which reflects the riskness of cash flows and return on alternative investment.

— Determine the initial cost of investments.

We follow the decision criteria:

— We accept the project if $NPV > 0$.

– We always choose the project with the highest NPV.

Reinvestment assumption: Assume that all cash flows can be reinvested at the discount rate. This means the discount rate is constant for all cash flows no matter what.

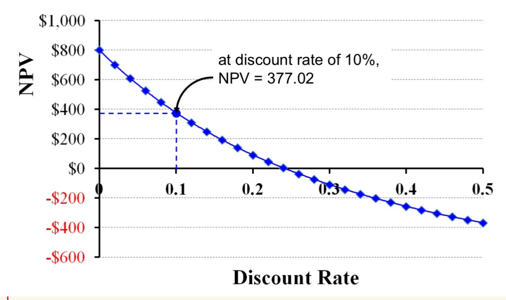
For example: Consider project X with initial outlay (investment cost) of \$1100, and a discount rate of 10%, and we have the following cash flows:

Year	Revenues	Expenses	Net cash flows
1	\$1000	\$500	\$500
2	\$2000	\$1300	\$700
3	\$2200	\$2700	- \$500
4	\$2600	\$1400	\$1200

$$NPV = -1100 + \frac{500}{1.1} + \frac{700}{1.1^2} + \frac{-500}{1.1^3} + \frac{1200}{1.1^4} = \$375.01 > 0$$

We accept this project by the criteria.

NPV - discount rate profile (curve)



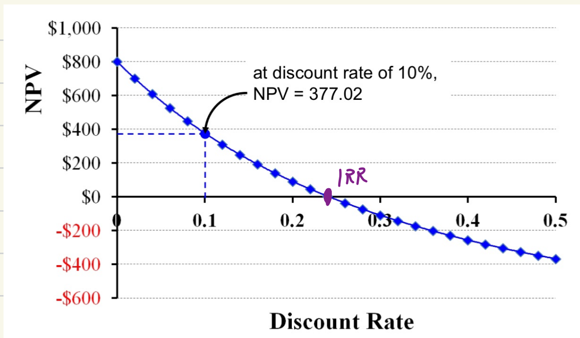
A graphic representation between discount rate and NPV of a project.

It generally slopes downwards.

However, a upward sloping NPV-discount rate curve is possible when they have a significantly negative cash flows in the distant future. This means the increasing discount rate will lead to increasing in the present value of the negative cash flows. Thus, increasing the NPV.

The internal rate of return (IRR) rule

Definition (Internal rate of return): The discount rate that makes NPV of a project 0.



Decision criteria:

- we accept investment project if IRR > discount rate. This is because the project's NPV would be > 0.
- we choose the project with the highest IRR.

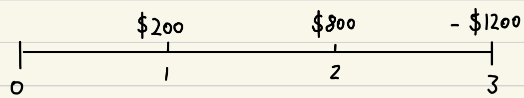
Reinvestment assumption: All future cash flows assumed reinvested at IRR.

The main advantage of using IRR is because it is easy to understand since it compares percentages only. And we understand the project with only one rate.

However, there are some issues about IRR rule when there are unconventional cash flows or mutually exclusive projects:

When NPV-discout curve is upward sloping

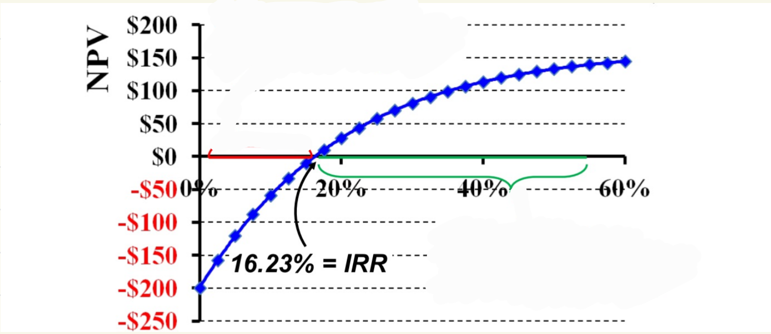
Consider the following cash flows with 10% discount rate:



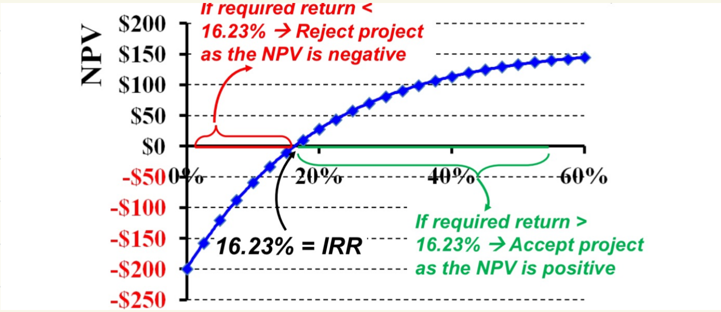
Take $NPV = 0 = \frac{200}{1+IRR} + \frac{800}{(1+IRR)^2} - \frac{1200}{(1+IRR)^3} \Rightarrow IRR = 16.23\% > 10\%$

By conventional criteria, we should accept this project.

However, this project has a upwards NPV curve:



At 10%, the NPV<0. Thus, we need to reverse the criteria: reject when IRR > discount rate.

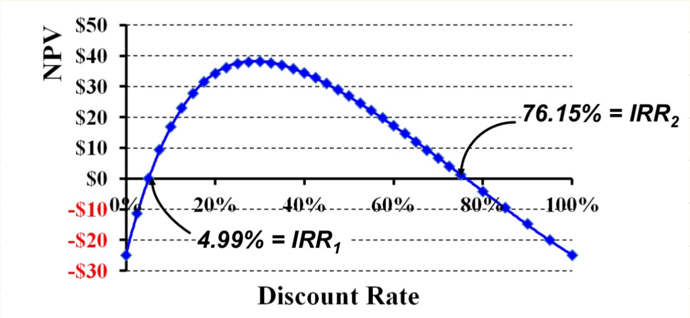


Multiple IRR

Projects with non-conventional cash flows (cash flows that changes signs more than once) can have multiple IRRs, making it unclear to which IRR should be used.

For example: Consider the following project with the following cash flows

0	1	2	3
-$225	$200	$800	-$800



To counter this problem, we introduce Modified IRR (MIRR) to give us a specific rate.

– Calculate the present value of all cash outflows using the borrowing rate.
- Calculate the future value of all cash inflows using the investment rate (required rate).
– Find the rate equating these values.

For example: with the same cash flow, assume required rate = borrowing rate = 4%.

$FV_{in} = 200 \cdot (1.04)^2 + 800 \cdot 1.04 = \$1048.32$, $PV_{out} = 255 + \frac{800}{1.04^3} = \$936.19$

$PV_{out} \cdot (1+MIRR)^3 = FV_{in} \Rightarrow MIRR = 3.84\% < 4\%$ ⇒ we reject the project following the conventional criteria.

Mutually exclusive projects

Definition (Mutually exclusive projects): only one potential project can be chosen.

Using NPV rule, we choose the project with the highest NPV.

Using IRR rule, our result may not align with the NPV rule due to scale problem and timing problem.

Definition (scale problem): This problem arises when comparing projects of different sizes. IRR might favor a smaller project with a higher rate over a larger project with a lower rate, even if the larger projects adds more absolute value to the firm. So, IRR can be misleading in this case.

For example: consider two investments A, B

initial investments:	\$20,	\$1000
Final value:	\$40,	\$1500
return:	100%,	50%

For investment A: $0 = -20 + \frac{40}{1+IRR_A}$

$\Rightarrow IRR_A = 100\%$

For investment B: $0 = -1000 + \frac{1500}{1+IRR_B}$

$\Rightarrow IRR_B = 50\%$

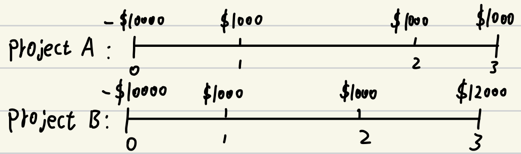
Using IRR rule, investment A is prefered, however, investment B brings more value to the company.

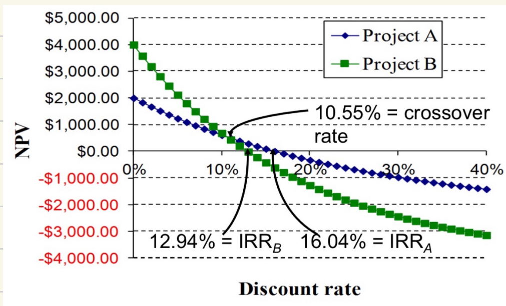
This shows IRR leads to a sub-optimal condition.

The NPV rule handles better in this case because it measures the absolute value added.

Definition (Timing Problem): The timing problem arises when comparing projects with different cash flows timing. the IRR favors the project with early returns over one with later returns.

For example: consider two projects





After calculation, $IRR_A = 16.04\%$, and $IRR_B = 12.94\%$.

This means under the IRR rule, Project A is prefered.

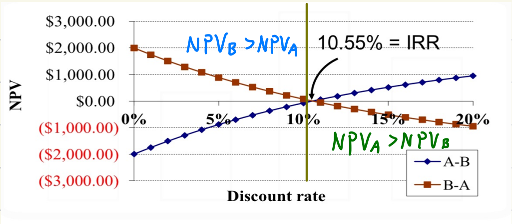
However, at a discount rate of 10%, Project B might have more NPV than Project A.

Thus, the actual preferred project depends on the discount rate rather than IRR.

Definition (Cross over rate): It is a discount rate at which the NPV of two projects are equal.

To calculate, we simply take NPV of two projects, equate them, then solve for r.

By calculating the cross over rate, we have 10.55%



Thus, we conclude:

- If the discount rate < 10.55%, we choose project B.
- If the discount rate > 10.55% and < 16.04%, we pick project A.
- If the discount rate > 16.44%, we pick none since $NPV_B < NPV_A < 0$.

Thus, the preferred project in the timing problem depends on the discount rate rather than the IRR.

Independent Projects

Definition (Independent projects): Projects who accepting or rejecting does not impact the decision of other projects.

Using NPV rule, we should accept all projects with a positive NPV.

Using IRR rule, we should accept all projects with IRR > discount rate.

Risk and return

Holding period return (HPR)

The holding period return (HPR) is a measure of the total return received from holding an asset ove a period of time.

— For bonds: $HPR = \frac{\text{coupons} + \text{change in bond price}}{\text{purchase price}}$

— For stocks: $HPR = \frac{\text{dividends} + \text{change in market value}}{\text{begining market price}}$

For example: Suppose you bought 100 shares of BCE two years ago at \$25 per share. Over the last year, you received \$0.2 per share in dividends. The stocks now sell for \$30 per share.

$$HPR = \frac{0.2 \cdot 100 + (30-25) \cdot 100}{25 \cdot 100} = 20.8\% \quad , (1+EAR)^2 = 1+20.8\% \Rightarrow EAR = (1+20.8\%)^{\frac{1}{2}} - 1 = 9.91\%$$

Average return: Arithmetic vs. Geometric.

We take r_i as return at period i.

We have:

— Arithmetic average return (mean return) $= \bar{r}_A = \frac{1}{n} \cdot \sum_{i=1}^{n} r_i$

— Geometric average return $= \bar{r}_G = [(1+r_1) \cdots (1+r_n)]^{\frac{1}{n}} - 1 \Rightarrow (1+\bar{r}_G)^n = (1+r_1) \cdot (1+r_2) \cdots (1+r_n)$

Remark: We refer to average return as $\bar{r}_A$

Remark: $\bar{r}_A$ is also the expected return over multiple periods. or it is return earned in average over multiple periods.

Remark: $\bar{r}_G$ is the average compounded return over multiple periods.

Return statistics for history of financial market

The history of capital market returns have some statistics:

— The mear return:

$$\bar{r} = \frac{r_1 + r_2 + \cdots + r_N}{N}$$

— The variance of returns:

$$var = \frac{(r_1 - \bar{r})^2 + (r_2 - \bar{r})^2 + \cdots + (r_N - \bar{r})^2}{N-1}$$

— The standard deviation of returns:

$$sd = \sqrt{var}$$

Risk Premium

Definition (Risk premium): The risk premium is the additional return over the risk-free rate for taking on additional risk.

- The risk-free rate is usually the T-bill.
- The risk premium for stocks and bonds is calculated as the difference between their average return and their risk free rate.

For example: Take the risk-free rate = 5%, average excess return (risk premium) = 4.45%. The expected return would be 5% + 4.45% = 9.45%.

Risk statistics

The statistical measure of risk we mainly discuess are based on returns:

- The variance of returns:

$$var = \frac{(r_1-\bar{r})^2+(r_2-\bar{r})^2+\cdots+(r_N-\bar{r})^2}{N-1}$$

The higher the variance, the higher the fluctuations in returns, thus higher risk.

- The standard deviation of returns:

$$sd = \sqrt{var}$$

This is also referred as the volatility of returns. It represents how the return moves around the average return as indicator of risk.

Risk and returns for investors

Investors are concerned with both risks and returns when choosing a financial assets.

For risk, it is usually measured by the standard deviation of returns. It indicates the uncertainty in the returns.

For return, it is uncertained and needed to be estimated. The actual profit or loss is certain only when the asset is sold.

- If we know the probability distribution of return, we have:

$$E(r) = \sum_{i=1}^{N} pr_i \cdot r_i \text{ , } pr_i = \text{probability of } r_i \text{ occurs.}$$

$$var = \sigma^2 = E\left((r_i - E(r))^2\right) = \sum_{i=1}^{N} pr_i \cdot (r_i - E(r))^2$$

$$sd = \sigma = \sqrt{var}$$

— If the probability distribution of return is unknown:

$$E(r) = \bar{r} = \frac{1}{N} \cdot (r_1 + r_2 + \cdots + r_N)$$

$$var = \frac{(r_1 - \bar{r})^2 + (r_2 - \bar{r})^2 + \cdots + (r_N - \bar{r})^2}{N-1} \quad , \quad \sigma^2 = \frac{(r_1 - \bar{r})^2 + (r_2 - \bar{r})^2 + \cdots + (r_N - \bar{r})^2}{N}$$

= sample variance   = population variance.

$$sd = \sigma = \sqrt{var}$$

For example: consider a hypothetical with two risky assets, stock and bond. The economy can be in one of the three stages: Recession, Normal, or Boom. Each with $\frac{1}{3}$ probability.

Scenario		Probability	Rate of Return: Stock	Rate of Return: Bond	
Recession	Pr_1	33.3%	-7%	17%	r_1
Normal	Pr_2	33.3%	12%	7%	r_2
Boom	Pr_3	33.3%	28%	-3%	r_3

$$E(r_s) = \frac{1}{3} \cdot -0.07 + \frac{1}{3} \cdot 0.12 + \frac{1}{3} \cdot 0.28 = 11\%$$

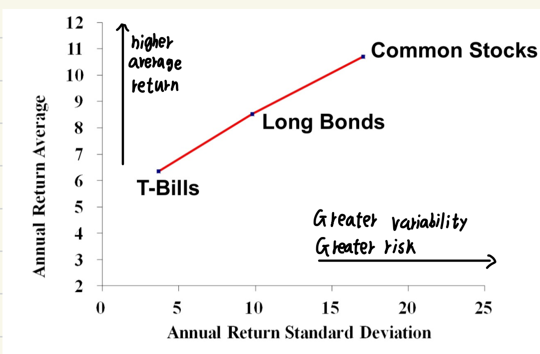
$$E(r_B) = \frac{1}{3} \cdot 0.17 + \frac{1}{3} \cdot 0.07 + \frac{1}{3} \cdot -0.03 = 7\%$$

$$var_s = \frac{(-7\% - 11\%)^2 + (12\% - 11\%)^2 + (28\% - 11\%)^2}{3} = 0.0205 \quad , \quad \sigma_s = \sqrt{0.0205}$$

$$var_B = \frac{(17\% - 7\%)^2 + (7\% - 7\%)^2 + (-3\% - 7\%)^2}{3} = 0.0607 \quad , \quad \sigma_B = \sqrt{0.0607}$$

The risk-return trade off example

The graph below illustrates the relationship between the annual return average and annual return standard deviation of T-bills, long bonds, and common stocks.



From the graph, we can see that:

- T-bill: minimal risk, minimal return
- Long bonds: moderate risk, moderate return
- Common stock: highest risk, highest return.

Portfolio risk and return

Definition (Portfolio): A portfolio is a combination of different assets or securities.

The expected return of a portfolio depends on:

- The expected return of the individual asset.
- The valued weighted propotion of each asset in the portfolio is measured by the dollar amount invested in an asset relative to the total dollar amount invested in the whole portfolio.

The standard deviation (risk) of a portfolio depends on:

- The standard deviation of individual assets.
- The weight of each asset.
- The correlation of returns among different assets.

For a portfolio with two assets, where w_1 is the weight of asset 1 and w_2 is the weight of asset 2, the expected return of the portfolio $E(r_p)$ is calculated as:

$$E(r_p) = w_1 \cdot E(r_1) + w_2 \cdot E(r_2)$$

The variance of the portfolio return σ_p^2 is given by:

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \operatorname{cov}(r_1, r_2)$$

- $\operatorname{cov}(r_1, r_2) = \rho_{1,2}\, \sigma_1 \sigma_2$ is the covariance between asset 1 and 2.
- $\rho_{1,2}$ is the correlation coefficient between the returns of asset 1 and 2.
- σ_1, σ_2 are standard deviation of returns for asset 1 and 2.

The standard deviation of the portfolio return σ_p is $\sigma_p = \sqrt{\sigma_p^2}$

Suppose the probability distribution is known: $\operatorname{cov}(r_1, r_2) = E\big((r_1 - E(r_1)) \cdot (r_2 - E(r_2))\big)$

Suppose the probability distribution is unknown:

- $$\operatorname{cov}(r_1, r_2) = \frac{(r_{11} - \bar{r}_1)^2 (r_{21} - \bar{r}_2) + \cdots + (r_{1N} - \bar{r}_1) \cdot (r_{2N} - \bar{r}_2)}{N-1} = \text{sample covariance}$$

- $$\operatorname{cov}(r_1, r_2) = \frac{(r_{11} - \bar{r}_1) \cdot (r_{21} - \bar{r}_2) + \cdots + (r_{1N} - \bar{r}_1) \cdot (r_{2N} - \bar{r}_2)}{N} = \text{population covariance}$$

where $\bar{r}_1 = \dfrac{\sum_{i=1}^{N} r_{1i}}{N}$, $r_2 = \dfrac{\sum_{i=1}^{N} r_{2i}}{N}$

For example: Consider the same example. You have $1000 and invest $500 in stock and $500 in bond.

Scenario	Stock Return	(r-E(r)) Deviation	Sqr. Dev.	Bond Return	(r-E(r)) Deviation	Sqr. Dev.
Recession	-7%	-18%	0.0324	17%	10%	0.0100
Normal	12%	1%	0.0001	7%	0%	0.0000
Boom	28%	17%	0.0289	-3%	-10%	0.0100
Expected Return	11.00%			7.00%		
Variance	0.0205			0.0067		
Standard Dev.	14.3%			8.2%		

$$W_S = \frac{500}{1000} = W_B = 0.5.$$

$$E(r_P) = W_S \cdot E(r_S) + W_B \cdot E(r_B) = 0.5 \cdot 11\% + 0.5 \cdot 7\% = 9\%$$

$$Cov(S,B) = \frac{-18\% \cdot 10\% + 1\% \cdot 0\% + 17\% \cdot -10\%}{3} = -0.01167$$

$$\rho_{S,B} = \frac{Cov(S,B)}{\sigma_S \cdot \sigma_B} = \frac{-0.01167}{14.3\% \cdot 8.2\%} = -0.995$$

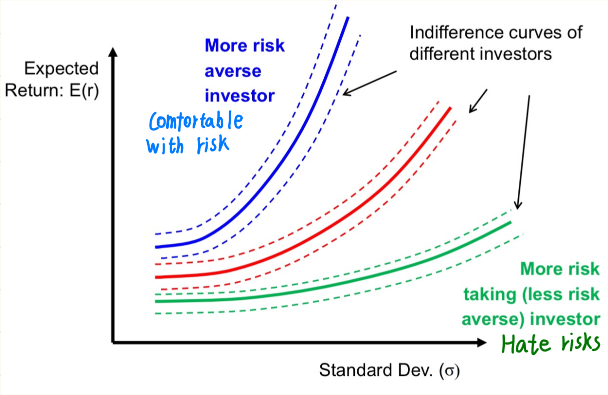
$$var(P) = (W_S\, \sigma_S)^2 + (W_B\, \sigma_B)^2 + 2 W_S W_B\, Cov(S,B) = (0.5\ 0.175)^2 + (0.5 \cdot 0.1)^2 +$$
$$2 \cdot 0.5 \cdot 0.5 \cdot -0.01167 = 0.00432125$$
$$\sigma_P = \sqrt{var(P)} \approx 6.75\%$$

Risk - return preferences

Investors have preference regarding the tradeoff between risk and return. This relationship can be illustrated using indifference curve.

They always have the following preference:

- Given a certain level of expected return, investors prefer a lower sd (less risk).
- Given a certain level of sd (risk), investors want more expected return to compensate the additional risk.



The indifference curve is upward sloping because as sd (risk) increase, return must also increase to maintain the same level of satisfaction.

Portfolio choice

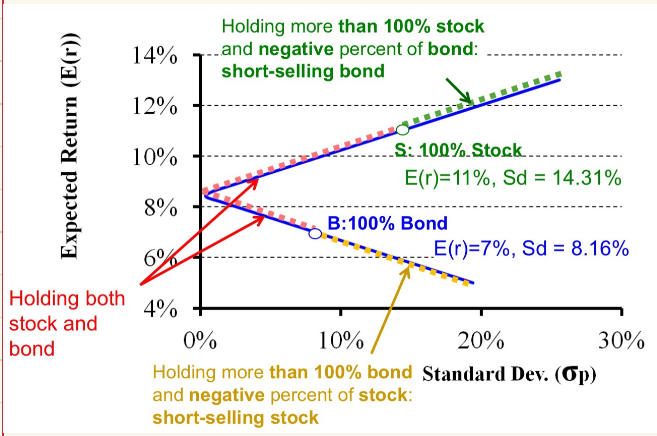
Investment opportunity set: two assets

Definition (Investment opportunity set): It represents all possible combinations of expected return and sd (risk) that can be achieved by varying the weights of assets in the portfolio. In this case, we have two assets.

By changing the propotion (weight) of each asset in the portfolio, investors can create different portfolios with varying levels of risks and returns.

Consider the same example:

Scenario	Stock Return	(r-E(r)) Deviation	Sqr. Dev.	Bond Return	(r-E(r)) Deviation	Sqr. Dev.
Recession	-7%	-18%	0.0324	17%	10%	0.0100
Normal	12%	1%	0.0001	7%	0%	0.0000
Boom	28%	17%	0.0289	-3%	-10%	0.0100
Expected Return	11.00%			7.00%		
Variance	0.0205	S		0.0067	B	
Standard Dev.	14.3%			8.2%		



At S: $E(P) = 11\% = W_S \cdot E(r_S) + W_B \cdot E(r_B) = 1 \cdot 11\% + 0 \cdot 7\%$

So, at S, no bond is bought.

At B: $E(P) = E(r) = 7\%$, no stock is bought since $W_S = 0$.

For the green section beyond points:

- You have already invested 100% in stock, so you need to borrow money to buy more stocks.
- To finance the borrowing, you sell bonds you do not own, and wish to buy them back later at a lower price. This is call short sell.

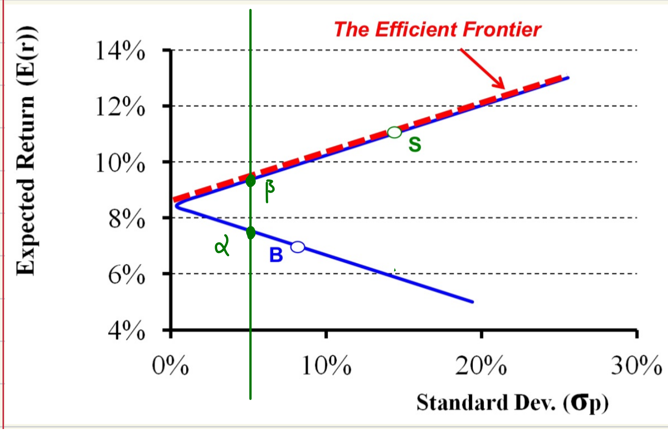
For the yellow section beyond points:

- You have already invested 100% in bond, so you need to borrow money to buy more bonds.
- To finance the borrowing, you sell stocks you do not own, and wish to buy them back later at a lower price.

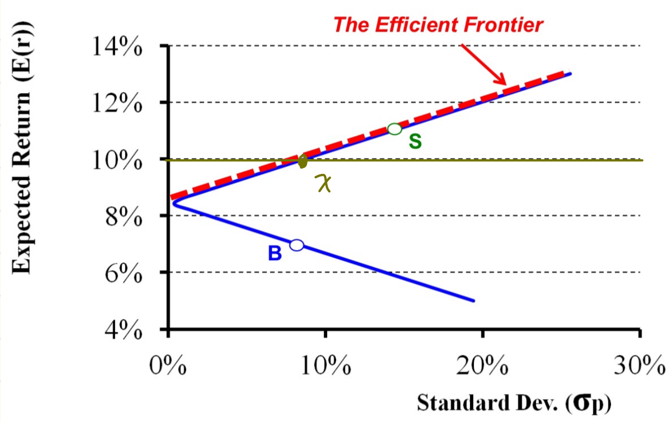
For the red section:

- None of W_S or W_B <0 or $=0$, thus, we are buying both stocks and bonds.

Definition (Efficient frontier): It is a subset of the investment opportunity set. It represents the set of portfolios that offer the highest expected return for a given level of risk, or, equivalently, the lowest risk for a given level of expected return.

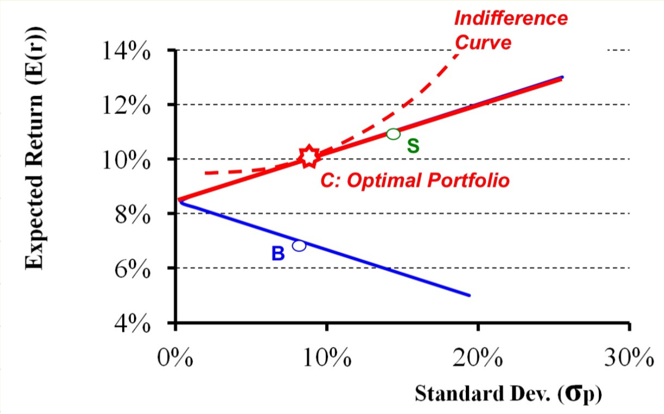


At any given level of risk, point α and β have the same level of risk. But point β on the efficient frontier offers more expected return.



At any given level of return, the point x has the lowest one achievable risk.
Further decrease risk will drop the return.
Further increase risk will rise the return.

Definition (optimal portfolio): The point where investor's highest indifference curve is tangent to the efficient frontier. Given their risk tolerance, this point is the best possible combination of risk and return.



Consider the correlation coefficient ρ measures the degree to which two assets move in relation to each other, ranging -1 to 1.

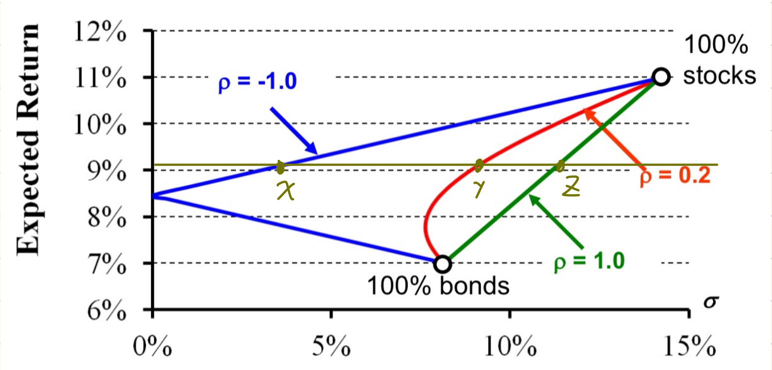
The shape of the efficient frontier depends on the correlation coefficient (ρ) between the returns of the two assets.

Consider varience equation $\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho_{1,2} \sigma_1 \sigma_2$:

— when $\rho = -1$, one asset return increases, the other asset will have return decrease by the proportional amount. And one asset will immediately cancels out the other asset in terms of risk. So, the risk is minimized at the level of expected return. And complete risk reduction is possible.

— when $\rho = +1$, no risk reduction is possible, so the combinations of them will not minimize risk.

— when $-1 < \rho < 1$, partial risk reduction is possible, the smaller the correlation the better.



on the same level of expected return, x has minimum risk, and then it is y, and lastly z, which no risk reduction is possible.

Thus, curvature is a indicator for the risk of the assets.

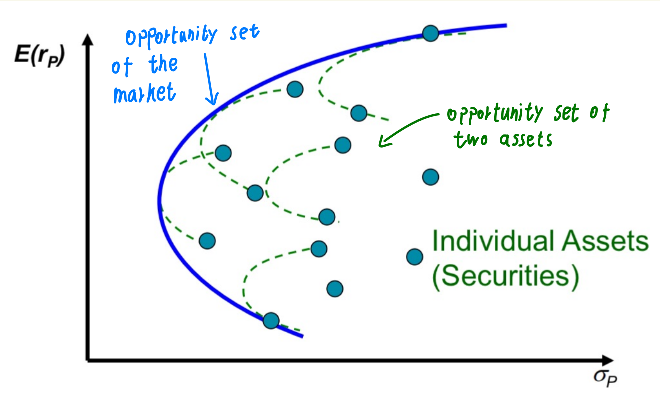
we have two cases:

— For positive correlation ($\rho > 0$): the shape is less curved and closer to a straight line, indicates limited diversification benefits.

— For negative correlation ($\rho < 0$): the shape bows more torward the y-axis, showing significant risk reduction.

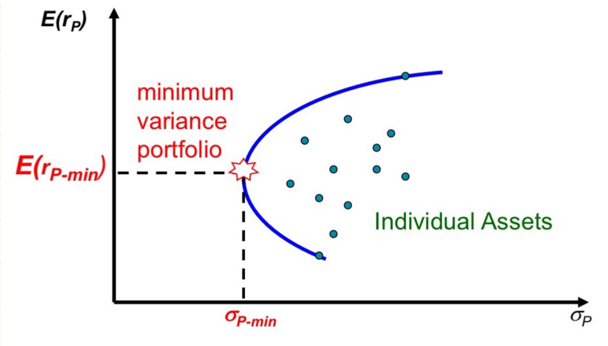
General investment opportunity set and minimum variance portfolio

Consider a world with many risky assets (more than 2), the concepts of efficient frontier and the opportunity set still apply. We can move from the opportunity set of only two elements to the opportunity set of the whole market, and investors can still identify optimal risk-return combinations.



Definition (minimum variance portfolio): A specific portfolio with the lowest possible risk (sd) for a given level of return.

The section of the opportunity set above the minimum variance portfolio forms effective frontier.



We have two cases:

— when we have only two assets in the portfolio, we have constraint $w_1 + w_2 = 1$

$$\sigma_P^2 = (w_1\sigma_1)^2 + (w_2\sigma_2)^2 + 2w_1w_2\sigma_1\sigma_2\rho$$

We substitute $w_2 = 1 - w_1 \Rightarrow \sigma_P^2 = (w_1\sigma_1)^2 + ((1-w_1)\cdot\sigma_2))^2 + 2\cdot w_1\cdot(1-w_1)\cdot\sigma_1\cdot\sigma_2\cdot\rho$

$$\Rightarrow \frac{\partial \sigma_P^2}{\partial w_1} = 2\sigma_1^2 w_1 - 2\sigma_2^2(1-w_1) + 2\sigma_1\sigma_2\rho(1-2w_1) = 0$$

$$\Rightarrow w_1^* = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}, \quad w_2^* = 1 - w_1^*$$

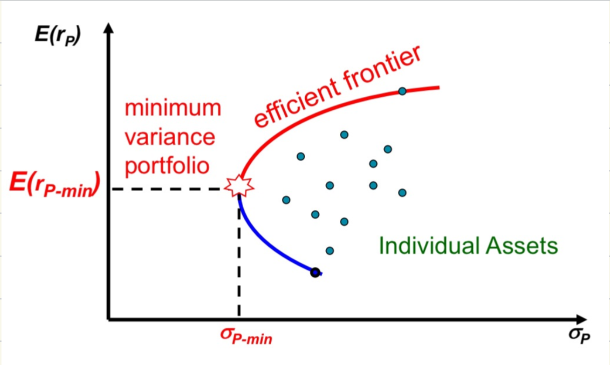
This means $\sigma_{P\text{-}min}^2 = (w_1^*\sigma_1)^2 + (w_2^*\sigma_2)^2 + 2w_1^*w_2^*\sigma_1\sigma_2\rho$

$$E(r_{P\text{-}min}) = w_1^*\cdot E(r_1) + w_2^*\cdot E(r_2)$$

— when we have multiple assets, we subjected to constraint $\sum_{i=1}^{N} w_i = 1$

$$\sigma_P^2 = \sum_{i=1}^{N} (w_i \sigma_i)^2 + 2 \cdot \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \rho_{i,j}$$

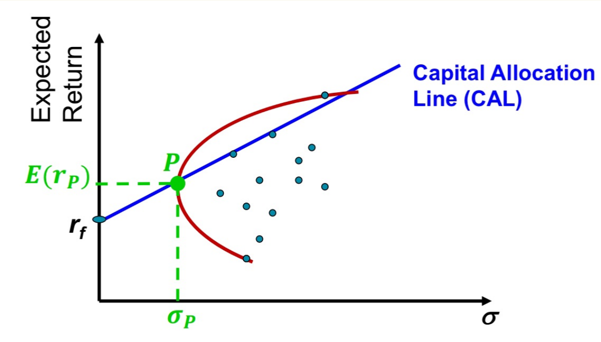
And we minimize σ_P^2 for $w_1^*, w_2^*, \cdots, w_N^*$. And it is the same as the two asset case.



Risky and risk-free asset allocation

In addition to risky assets like stocks and bonds, the investment would also includes risk-free assets like T-bills. The introduction of risk-free securities allows the creation of the capital market line.

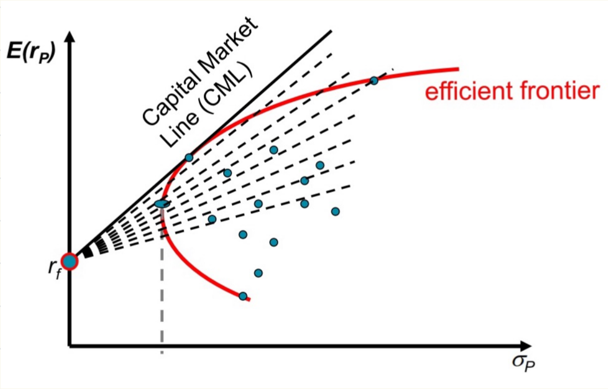
Definition (capital allocation line CAL): Visual representation shows all feasible risk-return combinations of a riskey and risk-free asset. It is a straight line on the plot starting at the risk-free rate (r_f) and passing through combinations of risky and risk-free assets.



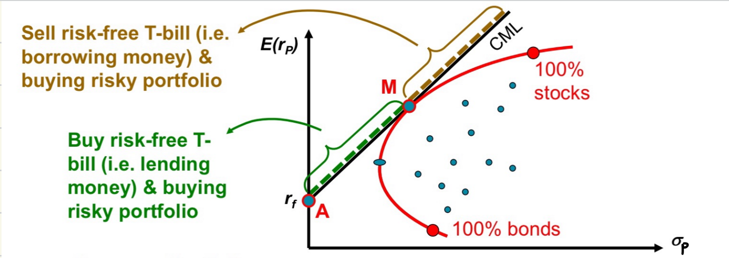
Definition (sharpe ratio): The slope of the CAL

$$\text{sharpe ratio} = \frac{E(r_P) - r_f}{\sigma_P}$$

Definition (capital market line CML): The CAL with the steepest slope ⇒ highest sharpe ratio. And all investors will choose the CML because it maximize the returns for a given risk.



Specifically:



For point A: $E(r_P) = r_f = W_f \cdot E(r_f) + W_R \cdot E(r_R) = 1 \cdot E(r_f) + 0 \cdot E(r_R)$

Thus, point A represents the risk-free set with expected return r_f.

For point M: point M is a risky portfolio that contains only risky asset.

Definition (Tangency Portfolio): Point M, which is the tangency point between CML and efficient frontier. It is called the optimal risky portfolio.

For the green section: As you move from A to M, your portfolio involves more % of your money to buy the M portfolio and less % of your money to buy risk-free assets. Thus, you are still buying T-bill (borrow money) and buying M portfolio.

For the yellow section: passing point M, which holds 100% risky asset. You are short-selling T-bills to buy more risky portfolio

The CML is given by equation:

$$E(r_P) = r_f + \frac{\overbrace{E(r_M) - r_f}^{\text{Market risk Premium}}}{\sigma_M} \cdot \sigma_P$$

The separation property under assumptions

Definition (Homogeneous expectation): It is the assumption that all investors have the same expectation about the future of all securities. This ensures all investors will have the same efficient frontier.

In addition, we assume investors are rational optimizers with single period investment period.

We also assume the market is perfectly competitive.

Due to homogeneous expectation, all investors will have the same efficient frontier, and since they want to maximize sharpe ratio, this means they will also share the same CML. As a result, we have this property.

The separation property states that:

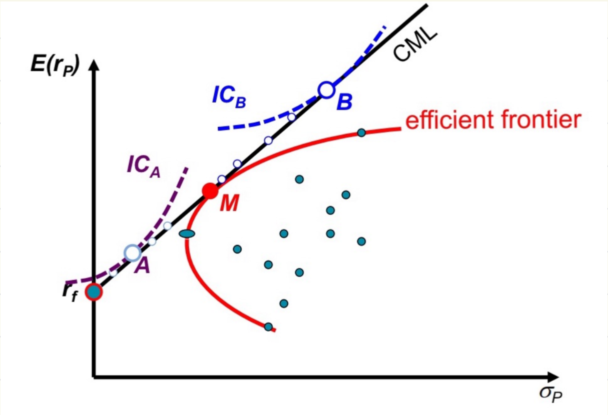
> All investors, regardless of their individual risk tolerence, will agree on the same M tangent portfolio.

This means investors will have the same $E(r_m)$ and σ_m, meaning they will have the same sharpe ratio $= \frac{E(r_m) - r_f}{\sigma_m}$. And the CML will the same for all investors given r_F.

This implies investors then choose their desired mix by selecting a point along CML.

— Risk-averse investor will choose a point closer to A.

— Risk-tolerant investor will choose a point beyond M.



Market securities

Every investor holds the same set of securities in point M, though the amount of money invested varies. Every investor buys M and likes M because it has optimal return.

The proportion of each securities in M is consistent across all investors.

The total market capitalization is the aggregate amount of money invested by all investors.

So, market share of each security = weight of each security in the investor's portfolio.

Definition (Market portfolio): A portfolio that includes all securities traded in the market with each security weighted by its market share. In this case, M is the market portfolio.

Having market portfolio can reduce risk, and under homogeneous expectation, everyone holds market portfolio.

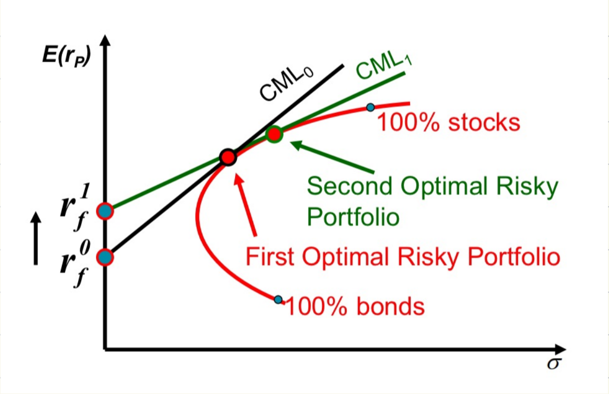
For example: Consider the following market

Securities	Price	Units		Market Cap.	Market Share
Stock A	$ 5	1000	$	5,000	27.40%
Stock B	$ 3	2000	$	6,000	32.88%
Corp Bond A	$ 150	30	$	4,500	24.66%
Corp Bond B	$ 90	20	$	1,800	9.86%
Municipal Bond C	$ 95	10	$	950	5.21%
Total Market Cap.			**$**	**18,250**	

The weight in the market portfolio = its market share

- Stock A: $\frac{5000}{18250} \approx 27.4\%$
- Stock B: $\frac{6000}{18250} \approx 32.88\%$
- Corp bond A: $\frac{4500}{18250} \approx 24.66\%$
- Corp bond B: $\frac{1800}{18250} \approx 9.86\%$
- Municipal bond C: $\frac{950}{18250} \approx 5.21\%$

Changes in M



The optimal risky portfolio depends on the risk-free rate.

The capital asset pricing model CAPM

The CAPM describes a linear relation between the expected return of a individual security and the expected return of the market portfolio.

$$E(R_i) = R_f + \beta_i \cdot (E(R_m) - R_f)$$

- $E(R_i)$ = Expected return on security i
- R_f = risk-free rate
- β_i = Beta of security i, a measure of its sensitivity to market movement
- $E(R_m)$ = Expected return of the market portfolio

Remark: $E(R_m) - R_f$ = Market risk premium

Definition (β_i): It measures the co-movement of the return of individual security i with that of the market portfolio.

$$\beta_i = \frac{cov(r_i, r_m)}{\sigma_m^2}$$

- $cov(r_i, r_m)$ = covariance of the return of security i and the return of the market portfolio.
- σ_m^2 = variance of the market

There are some implications about β_i:

- if $\beta_i = 0$, $E(R_i) = R_f$. This means the return is impacted by market movement.
- if $\beta_i = 1$, $E(R_i) = R_f + E(R_m) - R_f = E(R_m)$. This means the security moves in perfect correlation with the market.
- $\beta_i \cdot (E(R_m) - R_f)$ = risk premium of asset i. So, risk premium is proportional to market risk premium.

Assumptions of CAPM

① Homogeneous expectations.

② competitive market

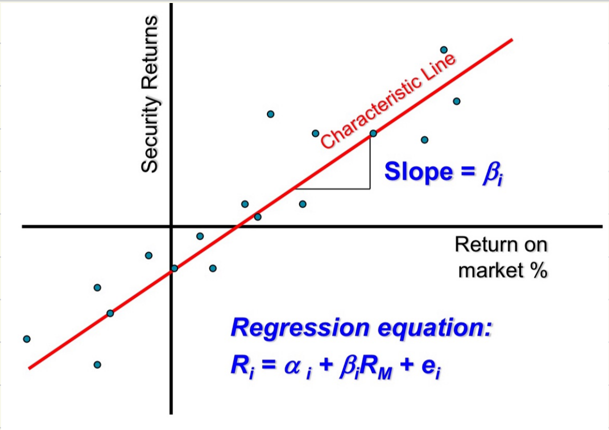
③ investors are rational mean-variance optimizers.

④ No taxes and transaction costs.

⑤ Free information

⑥ Investors are limited in trading financial assets.

Estimate β with regression



- R_i = return on the security i
- α_i = Intercept of the regression line
- e_i = error term.
- β_i = slope of the regression line
- R_M = return on market portfolio

We use this formula to estimate beta:

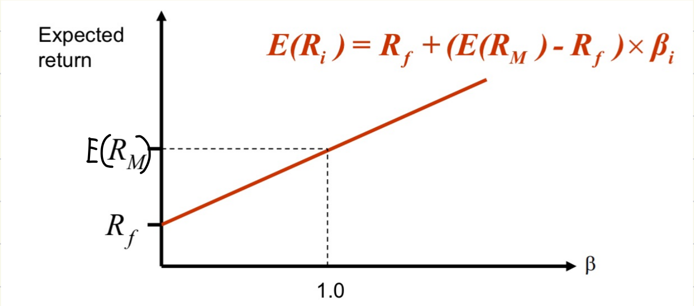
$$\hat{\beta}_i = \frac{cov(r_i, r_m)}{\sigma^2(r_m)}$$

- $\sigma^2(r_m)$ is the variance of the market return

We often use proxy to derive the $\sigma^2(r_m)$ in our calculation.

The security market line

Definition (The security market line): Graphical representation for $E(R_i)$ and β_i using CAPM.



Risk: Systematic and unsystematic

Definition (Diversification): Holding more varieties of assets in the portfolios.

Diversification can substantially reduce risk without equivalent reduction in expected returns. This is because the risk effect from one asset is offset by expected return from others.

Definition (Systematic risk): Type of risk that cannot be eliminated using diversification. This type of risk often affects a large number of assets.

For example: inflation rates, interest rate changes, global economic condition

Definition (Unsystematic risk): Type of risk that can be eliminated through diversification. They usually affect individual or group of assets. We also call it idiosyncratic risk.

For example: regulatory changes, launch of a new product.

Decomposition of risk

The actual return on a asset (r_i) can be expressed as:

$$r_i = E(r_i) + u_i$$

- $E(r_i)$ = expected return on asset i
- u_i = unexpected return or risk

And we have $u_i = m + \varepsilon_i$, m = systematic risk, ε_i = unsystematic risk

Thus, $r_i = E(r_i) + m + \varepsilon_i$

Remark: For two different assets i, j, we have $Corr(\varepsilon_i, \varepsilon_j) = 0$.

suppose we hold a portfolio composed of N similar assets: