MAT334 Introduction to complex variables

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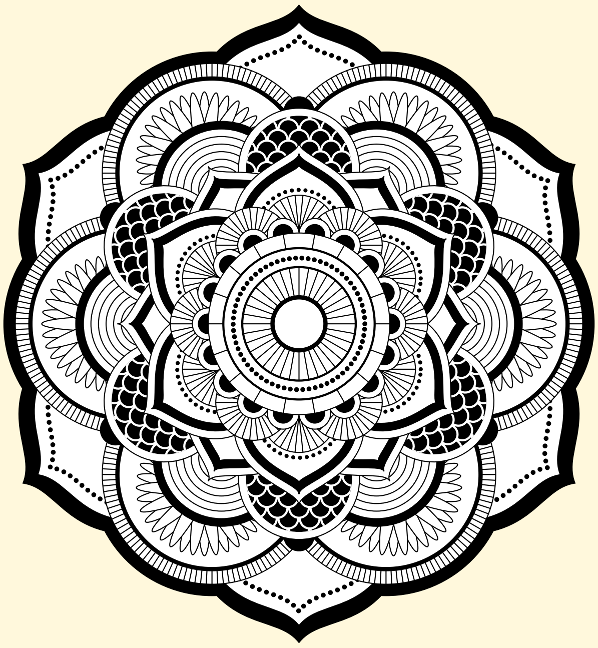

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1 Complex plane

1.1 Introduction to complex number and complex plane

Definition(Complex number): Complex numbers are pairs of real numbers (x, y) written as $z := x + iy$. When x, y run a real line $\mathbb{R}$ and i satisfies the rule

$$(i)^2 = (i)(i) = -1$$

Definition(Real part and imaginary part): The number x is called the real part of z and is written

$$x = \operatorname{Re} z$$

The number y, despite the fact that it is also a real number, is called the imaginary part of z and is written

$$y = \operatorname{Im} z$$

We identify $x + i0$ with a real number x and $0 + iy$ with an imaginary number $0 + iy$. In particular, we write $0 + i0$ simply as 0.

Definition(Set of complex number C): The set of all complex numbers is denoted by $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

Definition(Modulus): The modulus, or absolute value, of z is defined by

$$|z| := \sqrt{x^2 + y^2}, \quad z = x + iy.$$

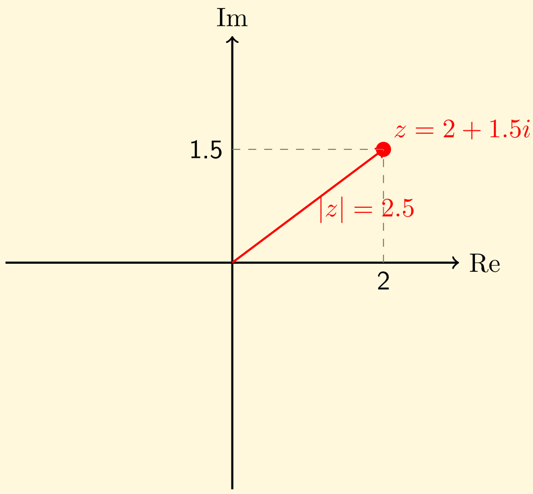
Definition(Complex conjugate): The complex conjugate of $z = x + iy$ is given by

$$\bar{z} := x - iy.$$

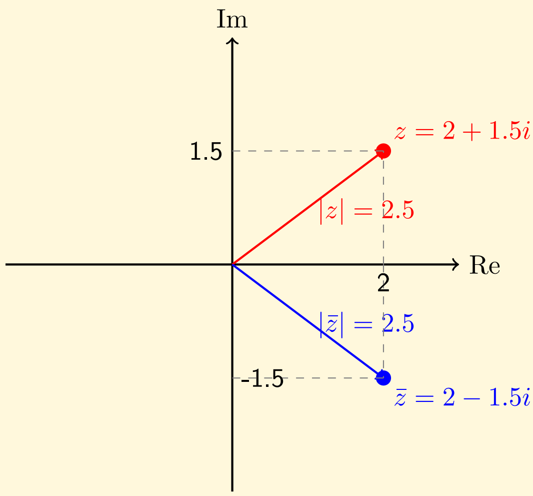
Remark: $|z| = \sqrt{x^2 + y^2} = |\bar{z}|$
Remark: $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$

Definition(Complex plane): The complex plane (also called the Argand plane or Gauss plane) is a way to represent complex numbers geometrically. It is basically a modified Cartesian plane, with the real part of a complex number represented by a displacement along the x-axis, and the imaginary part by a displacement along the y-axis. $|x| \leqslant |z|, |y| \leqslant |z|$, and $|z| \leqslant |x| + |y|$.

Visually, the complex plane looks like:



The conjugate of a complex number can be visualized as follows:



Addition, subtraction, multiplication, and division of complex numbers follow the ordinary rules of arithmetic. (Keep in mind that $i^2 = -1$, and, as usual, division by zero is not allowed.) Specifically, if

$$z = x + iy \quad \text{and} \quad w = s + it$$

then

$$\begin{aligned}
z + w &= (x + s) + i(y + t) \\
z - w &= (x - s) + i(y - t) \\
zw &= (xs - yt) + i(xt + ys) \\
\frac{z}{w} &= \frac{\bar{w}z}{\bar{w}w} = \frac{(xs + yt) + i(ys - xt)}{s^2 + t^2}, \quad w \neq 0 \\
z\bar{z} &= (x + iy)(x - iy) = x^2 + y^2 = |z|^2 \\
|zw|^2 &= (xs - yt)^2 + (xt + ys)^2 \\
&= x^2s^2 + y^2t^2 + x^2t^2 + y^2s^2 \\
&= (x^2 + y^2)(s^2 + t^2) \\
&= |z|^2|w|^2 \\
|zw| &= |z||w| \\
\overline{zw} &= (xs - yt) - i(xt + ys) \\
&= (x - iy)(s - it) \\
&= \bar{z}\bar{w}
\end{aligned}$$

1.2 Polar representation of complex number and arguments

The polar coordinate system gives

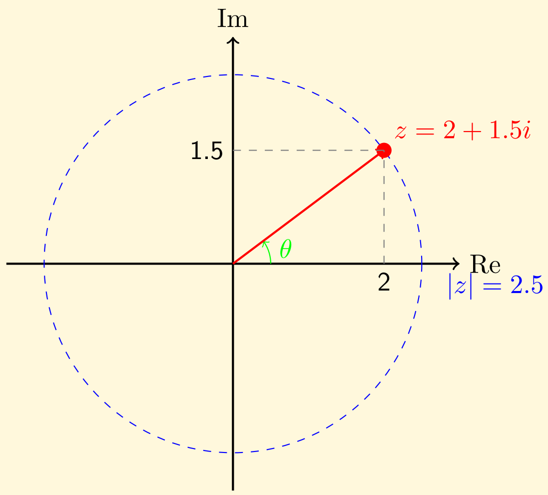
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta,$$

where $r = \sqrt{x^2 + y^2}$ and θ is the angle measured from the positive x-axis to the line segment from the origin to point. We immediately see that $r = |z|$, so

$$z = |z|(\cos\theta + i\sin\theta)$$

This is called the polar representation of z

For example:



Notice that θ could equally well be replaced in the formulas by $\theta + 2\pi$, by $\theta - 4\pi$, or, indeed, by $\theta + 2\pi n$, where n is any integer.

Definition(Argument): We define an argument of the nonzero complex number z to be ANY angle θ for which

$$z = |z|(\cos\theta + i\sin\theta)$$

we write $\theta = \arg z$.

However, there is always one angle $\theta \in [-\pi, \pi)$ that works by properties of trignometry.

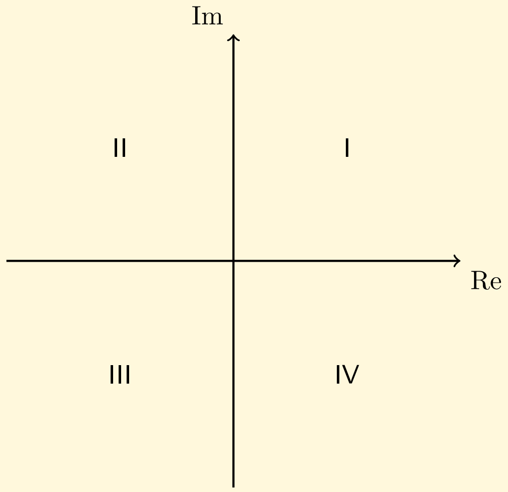
Definition(Principle argument): We will call the unique argument of z That lies in $[-\pi, \pi)$ the principal argument, denoted by $\mathrm{Arg}(z)$.

We have an algorithm to calculate θ:

For finite non-zero values of y/x, the principal value of the arctangent function lies inside the interval $0 < \mathrm{Arctan}(y/x) < \frac{1}{2}\pi$ if $y/x > 0$ and $-\frac{1}{2}\pi < \mathrm{Arctan}(y/x) < 0$ if $y/x < 0$. For completeness, we note that

$$\mathrm{Arctan}(y/x) = \begin{cases} 0, & \text{if } y = 0 \text{ and } x \neq 0, \\ \frac{1}{2}\pi, & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{1}{2}\pi, & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined}, & \text{if } x = y = 0. \end{cases}$$

Since $-\pi \leq \operatorname{Arg} z < \pi$, it follows that $\operatorname{Arg} z$ cannot be identified with $\arctan(y/x)$ in all regions of the complex plane. The correct relation between these two quantities is easily ascertained by considering the four quadrants of the complex plane separately. The quadrants of the complex plane (called regions I, II, III and IV) are illustrated in the figure below:



The relation is summarize in the table below:

Quadrant	Sign of x and y	$\operatorname{Arg} z$
I	$x > 0, y > 0$	$\mathrm{Arctan}(y/x)$
II	$x < 0, y > 0$	$\pi + \mathrm{Arctan}(y/x)$
III	$x < 0, y < 0$	$-\pi + \mathrm{Arctan}(y/x)$
IV	$x > 0, y < 0$	$-\mathrm{Arctan}(y/x)$

Formulae for the argument of a complex number $z = x + iy$ when z is real or pure imaginary.

Quadrant border	type of complex number z	Conditions on x and y	$\operatorname{Arg} z$
IV/I	real and positive	$x > 0, y = 0$	0
I/II	pure imaginary with Im $z > 0$	$x = 0, y > 0$	$\frac{1}{2}\pi$
II/III	real and negative	$x < 0, y = 0$	π
III/IV	pure imaginary with Im $z < 0$	$x = 0, y < 0$	$-\frac{1}{2}\pi$
origin	zero	$x = y = 0$	undefined

One of the important theorem is:

Theorem (De Moivre's Theorem)**:**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for any positive integer n and any angle θ.

Proof. We aim to prove:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

for all positive integers n and any angle θ.

Base Case: Let $n = 1$

$$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$$

which is clearly true.

Inductive Step: Assume the statement holds true for some positive integer k, i.e.,

$$(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$$

We need to prove it for $n = k + 1$.

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)(\cos(k\theta) + i\sin(k\theta)) \\ &= \cos\theta\cos(k\theta) - \sin\theta\sin(k\theta) + i(\cos\theta\sin(k\theta) + \sin\theta\cos(k\theta)) \\ &= \cos(\theta + k\theta) + i\sin(\theta + k\theta) \\ &= \cos((k+1)\theta) + i\sin((k+1)\theta)\end{aligned}$$

Thus, by the principle of mathematical induction, the formula holds true for all positive integers n and any angle θ. ■

Suppose that$z = |z|(\cos\theta + i\sin\theta)$ and $w = |w|(\cos\psi + i\sin\psi)$. We have the following computation properties:

$$\begin{aligned}zw &= |z||w|\{(\cos\theta\cos\psi - \sin\theta\sin\psi) + i(\cos\theta\sin\psi + \cos\psi\sin\theta)\} \\ &= |zw|\{\cos(\theta + \psi) + i\sin(\theta + \psi)\}\end{aligned}$$

And:

$$\begin{aligned}\frac{z}{w} &= \frac{|z|(\cos\theta + i\sin\theta)}{|w|(\cos\psi + i\sin\psi)} \\ &= \left(\frac{|z|}{|w|}\right)\{\cos\theta\cos\psi + \sin\theta\sin\psi + i(\cos\psi\sin\theta - \cos\theta\sin\psi)\} \\ &= \left(\frac{|z|}{|w|}\right)\{\cos(\theta - \psi) + i\sin(\theta - \psi)\}\end{aligned}$$

For argument $\arg$, suppose that $z = |z|(\cos\theta + i\sin\theta)$ and $w = |w|(\cos\psi + i\sin\psi)$. We have the following properties on arguments:

- $\arg(zw) = \arg z + \arg w$
- $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$
- $\arg\left(\frac{1}{z}\right) = \arg\bar{z} = -\arg z$

Argument and principle argument are also related:

$$\arg z \equiv \operatorname{Arg} z + 2\pi n = \theta + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

- $\arg z + \arg w = \operatorname{Arg} z + \operatorname{Arg} w + 2\pi(n_1 + n_2)$
- $\arg(zw) = \operatorname{Arg} z + \operatorname{Arg} z + 2\pi n_{12}, \quad n_{12} = n_1 + n_2$

For principle arguments, we have the following properties:

1. For the product $z_1 z_2$: When you multiply two complex numbers, the arguments add up. However, the result might not necessarily fall within our new range $[-\pi, \pi)$. To adjust for this, we introduce a factor of $2\pi N_+$such that the resulting argument is shifted into the correct interval.

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2\pi N_+$$

2. For the quotient z_1/z_2 : Similarly, when you divide two complex numbers, you subtract their arguments. But, again, the result might not necessarily be within $[-\pi, \pi)$. We introduce $2\pi N_-$to correct for this:

$$\operatorname{Arg}(z_1/z_2) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 + 2\pi N_-$$

where the integers $N_\pm$are determined as follows:

$$N_\pm = \begin{cases} -1, \text{ if } \operatorname{Arg} z_1 \pm \operatorname{Arg} z_2 > \pi \\ \ \ 0, \text{ if } -\pi \le \operatorname{Arg} z_1 \pm \operatorname{Arg} z_2 < \pi \\ \ \ 1, \text{ if } \operatorname{Arg} z_1 \pm \operatorname{Arg} z_2 \le -\pi \end{cases}$$

$$\operatorname{Arg}(1/z) = \operatorname{Arg}\bar{z} = \begin{cases} \ \ \operatorname{Arg} z, \text{ if } \operatorname{Im} z = 0 \text{ and } z \ne 0 \\ -\operatorname{Arg} z, \text{ if } \operatorname{Im} z \ne 0 \end{cases}$$

1.3 Roots of complex number

Definition(nth root of complex number)**:** Let w be a nonzero complex number. A complex number z satisfying $z^n = w$ is called an $\mathbf{n}$ th root of w.

Suppose:

$$w = |w|(\cos\psi + i\sin\psi),$$
$$z = |z|(\cos\theta + i\sin\theta).$$

There are n distinct solutions to the equation $z^n = w$, and they are

$$|w|^{\frac{1}{n}}\left(\cos\left(\frac{\psi}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\psi}{n} + \frac{2k\pi}{n}\right)\right)$$

with $k = 0, 1, \ldots, n-1$.

1.4 Complex number as fields

The complex numbers, denoted by $\mathbb{C}$, form a field. To define the complex numbers as a field, we need to provide the set, the two operations, and verify that all field axioms are satisfied.

The set of complex numbers is given by:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

where a and b are real numbers and i is the imaginary unit satisfying $i^2 = -1$.

Two operations are:

- Addition: The sum of two complex numbers $(a + bi)$ and $(c + di)$ is defined as:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

- Multiplication: The product of two complex numbers $(a + bi)$ and $(c + di)$ is defined as:

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$

For the complex numbers to be a field, they must satisfy the following axioms:

1. Additive Axioms:
   - Closure: For all $z, w \in \mathbb{C}, z + w \in \mathbb{C}$.
   - Associativity: For all $z, w, u \in \mathbb{C}, z + (w + u) = (z + w) + u$.
   - Commutativity: For all $z, w \in \mathbb{C}, z + w = w + z$.
   - Additive Identity: There exists a number $0 \in \mathbb{C}$ such that for all $z \in \mathbb{C}, z + 0 = z$.
   - Additive Inverse: For every $z \in \mathbb{C}$, there exists a $-z \in \mathbb{C}$ such that $z + (-z) = 0$.
2. Multiplicative Axioms:
   - Closure: For all $z, w \in \mathbb{C}, z \cdot w \in \mathbb{C}$.
   - Associativity: For all $z, w, u \in \mathbb{C}, z \cdot (w \cdot u) = (z \cdot w) \cdot u$.
   - Commutativity: For all $z, w \in \mathbb{C}, z \cdot w = w \cdot z$.
   - Multiplicative Identity: There exists a number $1 \in \mathbb{C}$ such that for all $z \in \mathbb{C}, z \cdot 1 = z$.
   - Multiplicative Inverse: For every $z \neq 0 \in \mathbb{C}$, there exists a $z^{-1} \in \mathbb{C}$ such that $z.\ z^{-1} = 1$.
3. Distributive Property: For all $z, w, u \in \mathbb{C}, z \cdot (w + u) = z \cdot w + z \cdot u$.

Proof. Let $z = a + bi$ and $w = c + di$ be two complex numbers:

1. **Additive Axioms:**
   - *Closure:* $z + w = (a + c) + (b + d)i$ is a complex number.
   - *Associativity:* Let $u = e + fi$, then $z + (w + u) = z + ((c + e) + (d + f)i) = (a + c + e) + (b + d + f)i = (a + c) + (b + d)i + e + fi = (z + w) + u$.
   - *Commutativity:* $z + w = (a + c) + (b + d)i = (c + a) + (d + b)i = w + z$.
   - *Additive Identity:* If $z = 0$, $z + 0 = a + bi = z$.
   - *Additive Inverse:* The inverse is $-z = -a - bi$ since $z + (-z) = (a - a) + (b - b)i = 0$.
2. **Multiplicative Axioms:**

- *Closure:* $z \cdot w = (ac - bd) + (ad + bc)i$ is a complex number.
- *Associativity:* Let $u = e + fi$, then $z(wu) = z((ce - df) + (cf + de)i) = a(ce - df) - b(cf + de) + (a(cf + de) + b(ce - df))i = (z \cdot w) \cdot u$.
- *Commutativity:* $z \cdot w = (ac - bd) + (ad + bc)i = (ca - db) + (da + cb)i = w \cdot z$.
- *Multiplicative Identity:* If $w = 1$, $z \cdot 1 = a + bi = z$.
- *Multiplicative Inverse:* If $z \neq 0$, the inverse is $\frac{1}{z} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$ since $z \cdot \frac{1}{z} = 1$.

3. **Distributive Property:** $z(w + u) = z(c + e + (d + f)i) = (a(c + e) - b(d + f)) + (a(d + f) + b(c + e))i = ac - bd + ad + ae - bf + be + af + bc = (ac - bd) + ad + bc + (ae - bf) = z \cdot w + z \cdot u$.

Thus, all the field axioms hold for the set of complex numbers, proving that the complex numbers form a field. ∎

1.5 Geometry of plane

Theorem (Triangle inequality)**:** Suppose $z = x + iy$ and $w = s + it$ are two complex numbers. Then

$$|z + w| \leqslant |z| + |w|$$

Proof.

$$\begin{aligned} |z + w|^2 &= (x + s)^2 + (y + t)^2 \\ &= x^2 + s^2 + y^2 + t^2 + 2(xs + yt) \\ &= |z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w}) \\ &\leqslant |z|^2 + |w|^2 + 2|z\bar{w}| \\ &= |z|^2 + |w|^2 + 2|z||w| \\ &= (|z| + |w|)^2. \end{aligned}$$

Taking the square root of both sides yields the inequality

$$|z + w| \leqslant |z| + |w|.$$

∎

Remark: If ζ and ξ are two (other) complex numbers, then by putting $z = \zeta - \xi$ and $w = \xi$ we get $|\zeta| \leqslant |\zeta - \xi| + |\xi|$ or

$$|\zeta| - |\xi| \leqslant |\zeta - \xi|.$$

Likewise,

$$|\xi| - |\zeta| \leqslant |\zeta - \xi|,$$

which together yield a variation of the triangle inequality,

$$||\zeta| - |\xi|| \leqslant |\zeta - \xi|$$

Definition(Line in complex plane)**:** All lines intro complex plane can be described algebraically by the equation: $\operatorname{Re}(az+b) = 0$ for some $a, b \in \mathbb{C}$ and $a \neq 0$

$$\text{line } = \{z \in \mathbb{C} \mid \operatorname{Re}(az + b) = 0\}\}$$

More generally, if $a = A + iB$ is a nonzero complex number and b is any complex number (not just a real number), then

$$0 = \operatorname{Re}(az + b)$$

is exactly the straight line $Ax - By + \operatorname{Re}(b) = 0$; this formulation also includes the vertical lines, $x = \operatorname{Re} z =$ constant.

Example: Let's find the line in the complex plane represented by a and b where:

$$a = 2 + 3i$$
$$b = 1 - 4i$$

Given a complex number $z = x + yi$, the equation for our line is:

$$\operatorname{Re}(az + b) = 0$$

Substituting the values of a, b, and z, we get:

$$\operatorname{Re}((2 + 3i)(x + yi) + (1 - 4i)) = 0$$

Expanding this:

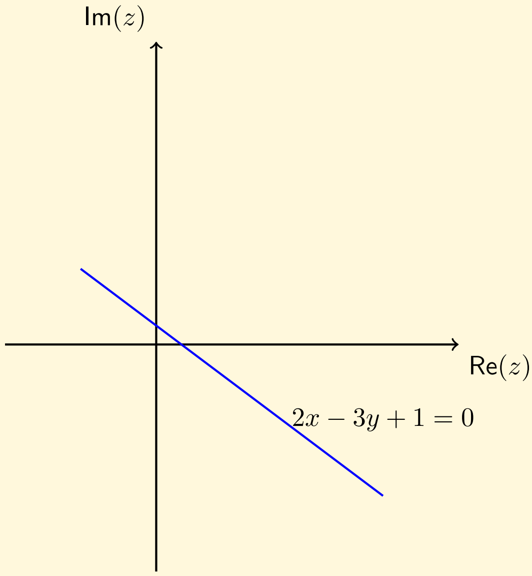
$$\operatorname{Re}(2x - 3y + (3x + 2y)i + 1 - 4i) = 0$$

Which becomes:

$$2x - 3y + 1 = 0$$

Thus, the equation of the line in the complex plane described by the given complex numbers a and b is:

$$2x - 3y + 1 = 0$$



Example: Sketch the locus of all complex numbers z satisfying

$$\mathrm{Re}((1+i)z + 2 + 3i) = 0.$$

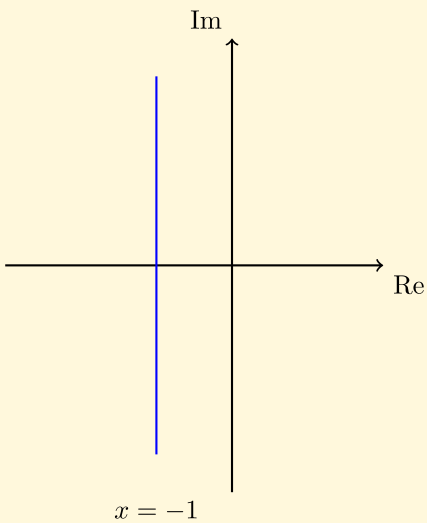
Solution: Given the equation:

$$\mathrm{Re}((1+i)z + 2 + 3i) = 0$$

To find the locus of complex numbers $z = x + yi$:

1. Expand the multiplication:
$$(1+i)z = x - y + (x+y)i$$
2. Add the complex number $2 + 3i$:
$$x - y + x + y + 2 + (x + y + 3)i = 2x + 2 + (2x + 2y + 3)i$$
3. Taking the real part:
$$\mathrm{Re}(2x + 2 + (2x + 2y + 3)i) = 2x + 2$$
4. Setting this to zero gives:
$$2x + 2 = 0 \implies x = -1$$

This corresponds to a vertical line in the complex plane:



1.5.1 Circles in complex plane

Definition: Let C be the set of all complex numbers z that form a circle centered at z_0 with radius r in the complex plane. This can be defined as:

$$C = \{z \in \mathbb{C} | |z - z_0 | = r\}$$

Here:

- z is a complex number.
- z_0 is the center of the circle in the complex plane.
- r is the radius of the circle.

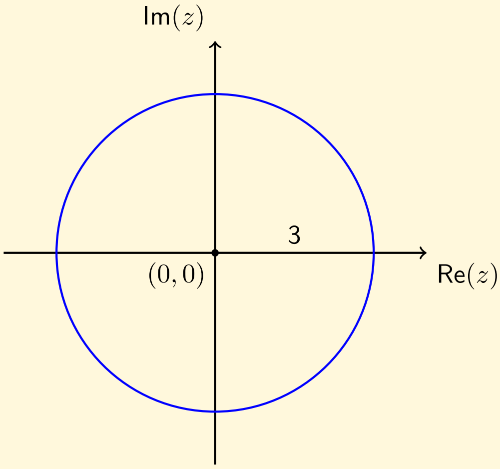
Example: Sketch the locus of all the complex numbers z satisfying

(a) $|z| = 3$

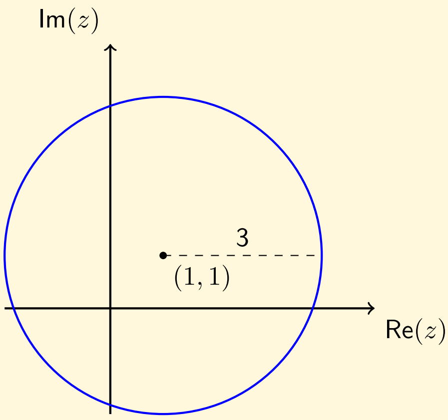
(b) $|z - (1 + i)| = 3$

(c) $|z - (1 + i)| < 3$

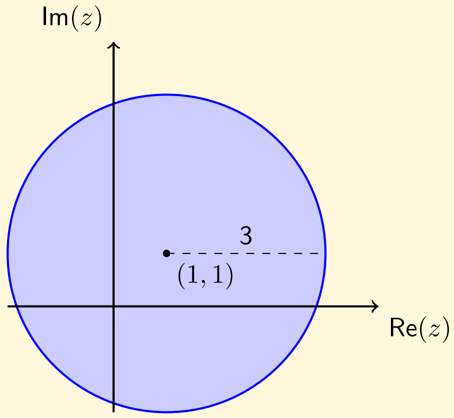
Solution: $|z| = 3$ represents all complex numbers at a distance of 3 units from the origin. This is a circle centered at the origin with a radius of 3.



Solution: $|z - (1 + i)| = 3$ represents all complex numbers at a distance of 3 units from the point $1 + i$. This is a circle centered at the point $1 + i$ with a radius of 3.



Solution: $|z - (1 + i)| < 3$ represents all complex numbers within a distance of 3 units from the point $1 + i$. This is the interior of a circle centered at the point $1 + i$ with a radius of 3.



There are, however, other ways to use complex numbers to describe circles. The new way of describing the circle is:

$$|z-p|=\rho|z-q|$$

where:

- z is any complex number on the circle.
- p and q are given distinct complex numbers.
- ρ is a positive real number that is not equal to 1.

Through the derivation presented, the circle described by this equation has:

- Center: $z_0=\frac{1}{1-\rho^2}p-\frac{\rho^2}{1-\rho^2}q$
- Radius: $R=|c|\frac{\rho}{1-\rho^2}$ where $c=p-q$.

The proof of this new formula is:

Proof. Given the equation:

$$|z-p|=\rho|z-q|$$

where $0<\rho<1$. Let's shift the origin to point q by setting $w=z-q$ and $c=p-q$, which gives:

$$|w-c|=\rho|w|$$

Upon squaring and transposing terms, this can be written as

$$|w|^2\left(1-\rho^2\right)-2\operatorname{Re}w\vec{c}+|c|^2=0.$$

We complete the square of the left side and find that

$$\left(1-\rho^2\right)|w|^2-2\operatorname{Re}w\bar{c}+\frac{|c|^2}{1-\rho^2}=\frac{|c|^2\rho^2}{1-\rho^2}.$$

Equivalently,

$$\left|w-\frac{c}{1-\rho^2}\right|=|c|\frac{\rho}{1-\rho^2}.$$

Thus, w lies on the circle of radius $R=|c|\rho/\left(1-\rho^2\right)$ centered at the point $c/\left(1-\rho^2\right)$, and so z lies on the circle of the same radius R centered at the point

$$z_0=\frac{p-\rho^2q}{1-\rho^2}=\frac{1}{1-\rho^2}p-\frac{\rho^2}{1-\rho^2}q$$

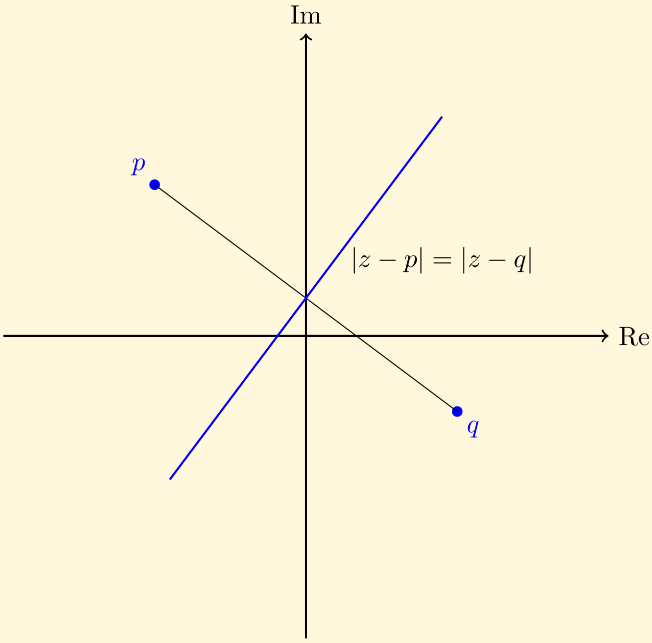
∎

It is worth noticing that the equation $|z-p|=|z-q|$ itself is a straight line that is the perpendicular bisector of the line segment joining p and q. This is because the equation stipulates that any point z on this line is equidistant from p and q.

However, when you generalize this equation to $|z-p|=\rho|z-q|$ where ρ is a positive real number not equal to 1, this describes a circle (under specific conditions). The reason for this is due to the relationship between the distances from z to p and q.

For instance, if $\rho<1$, then the set of points z that satisfy the equation will lie closer to p than to q by a factor of ρ. This will form a circle with a center that lies on the line segment between p and q but shifted more towards p.

Conversely, if $\rho>1$, the set of points z will lie closer to q than to p by a factor of ρ. This will also form a circle but shifted more towards q.



Example:

$$|z-i| = \frac{1}{2}|z-1|$$

After multiplying both sides by 2 and squaring,

$$4\left\{|z|^2 - 2\,\mathrm{Re}(z\bar{i}) + |i|^2\right\} = |z|^2 - 2\,\mathrm{Re}\,z + |1|^2,$$

or after simplifying,

$$3|z|^2 - 8y + 2x = -3.$$

More algebra yields

$$3x^2 + 2x + \frac{1}{3} + 3y^2 - 8y + \frac{16}{3} = -3 + \frac{17}{3} = \frac{8}{3}.$$

Thus, the locus is

$$\left(x + \frac{1}{3}\right)^2 + \left(y - \frac{4}{3}\right)^2 = \frac{8}{9}$$

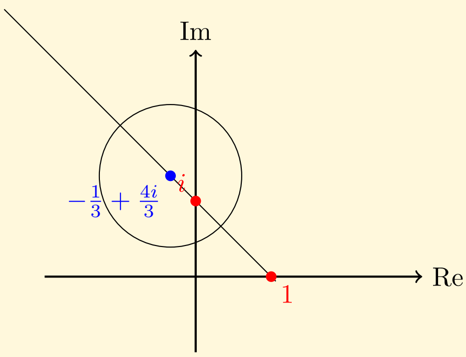
This is a circle of radius $2(\sqrt{2}/3)$ centered at $-1/3 + 4i/3$. Now, in the notation that preceded the example, $p = i, q = 1$, and $\rho = \frac{1}{2}$. The radius should be

$$R = \frac{|p-q|\rho}{1-\rho^2} = \frac{\sqrt{2}\left(\frac{1}{2}\right)}{\frac{3}{4}} = \frac{2\sqrt{2}}{3},$$

and the center at

$$z_0 = \frac{p - \rho^2 q}{1-\rho^2} = \frac{-\frac{1}{4} + i}{\frac{3}{4}} = -\frac{1}{3} + \frac{4}{3}i$$

Visually, it looks like:



1.6 Topology of complex plane

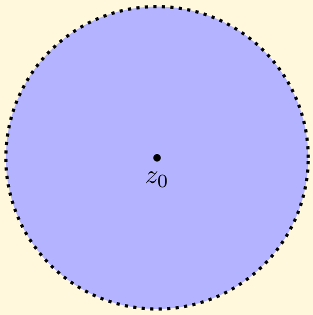
1.6.1 Open sets

Definition(Open disc): The set consisting of all points z satisfying $|z - z_0| < R$ is called the open disc of radius R centered at z_0.

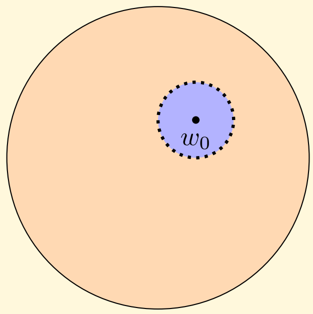
Definition(Interior points): A point w_0 in a set D in the complex plane is called an interior point of D if there is some open disc centered at w_0 that lies entirely within D. A set D is called open if all of its points are interior points.

Definition(Open sets): A set D is called open if all of its points are interior points.

Open disc of radius R centered at z_0

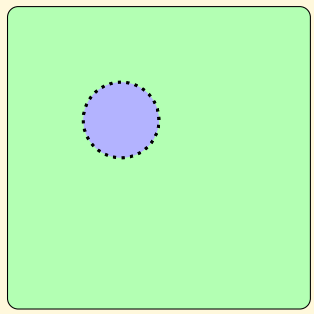


Set D



Interior point w_0 of set D

Open set D



Remark: Let $z_0 \in \mathbb{C}$ and $r > 0$. The open disc $|z - z_0| < r$ is an open set.

Proof. Let w be any point in the disc $|z - z_0| < r$. This means $|w - z_0| < r$.

Let's define $\delta = r - |w - z_0|$. Clearly, $0 < \delta < r$.

Consider the open disc centered at w with radius δ, that is, the set of all points z such that $|z - w| < \delta$. We will show this disc lies entirely within the original disc $|z - z_0| < r$.

For any point z in the open disc centered at w, by the triangle inequality, we have:

$$\begin{aligned} |z - z_0| &\leq |z - w| + |w - z_0| \\ &< \delta + |w - z_0| \\ &= (r - |w - z_0|) + |w - z_0| \\ &= r. \end{aligned}$$

Thus, every point z in the open disc centered at w also lies in the open disc $|z - z_0| < r$.

Therefore, for every point w in the disc $|z - z_0| < r$, there exists an open disc centered at w (with radius less than r) that lies entirely within the given disc. Hence, by definition, the open disc $|z - z_0| < r$ is an open set. ∎

Remark: The set $\{z : \operatorname{Im} z > 1\}$ is an open set.

Proof. To prove that the set $S = \{z : \operatorname{Im} z > 1\}$ is open, we will show that for any point w in S, there exists an open disc centered at w that is entirely contained in S.

Let w be any point in S, so $\operatorname{Im} w > 1$.

Let $\delta = \operatorname{Im} w - 1$. Clearly, $\delta > 0$.

Consider an open disc D centered at w with radius δ.

For any point z in D:

$$|z - w| < \delta$$

This implies:

$$|\operatorname{Im} z - \operatorname{Im} w| < \delta$$

$$\Rightarrow -\delta < \operatorname{Im} z - \operatorname{Im} w < \delta$$

$$\Rightarrow \operatorname{Im} w - \delta < \operatorname{Im} z < \operatorname{Im} w + \delta$$

Given that $\delta = \operatorname{Im} w - 1$, the lower bound yields:

$$1 < \operatorname{Im} z$$

So, D is entirely contained in S.

As every point w in S has an open disc centered at w that lies entirely within S, S is an open set. ∎

Remark: The set $\{z : \operatorname{Im} z \geq 1\}$ is not an open set.

Proof. To determine if S is an open set, let's consider a point z such that $\operatorname{Im} z = 1$ (i.e., a point on the boundary). For any open disc centered at z, irrespective of the radius, the disc will contain points with imaginary parts less than 1, as it would straddle the line $\operatorname{Im} z = 1$.

Thus, there does not exist an open disc centered at z that is completely contained within S for these boundary points. Hence, S is *not* an open set. ■

1.6.2 Closed set

Definition(Boundary point): Let D be a subset in $\mathbb{C}$. A point w_0 (w_0 can be in D or it can be outside of D) is called a boundary point of D is every open disc centered at w_0 contains both points in D and points not in D.

Definition(Boundary set): The set of all boundary points of D is called the boundary of D.

Definition(Closed set): A set D in $\mathbb{C}$ is called closed if it contains its boundary.

Remark: A set D in $\mathbb{C}$ is open if and only if it contains no points of its boundary.

Proof. **(i) Forward Direction:** Assume D is open. For every point z in D, there exists some $\epsilon > 0$ such that the open disk of radius ϵ centered at z is entirely contained in D. If z were a boundary point of D, then every open disk centered at z would contain points both in D and not in D. This is a contradiction. Hence, D contains no points of its boundary.
(ii) Reverse Direction: Assume D contains no points of its boundary. For any point z in D, there exists an open disk centered at z that is either entirely contained in D or entirely outside D. Since z is in D, the disk must be entirely contained in D. Thus, D is open. ■

Remark: $\mathbb{C}$ is both open and closed.

Proof. Take any point z in $\mathbb{C}$. For any $\epsilon > 0$, the open disk of radius ϵ centered at z is entirely contained in $\mathbb{C}$ because $\mathbb{C}$ contains all complex numbers. Thus, by definition, $\mathbb{C}$ is open.
For a set to be closed, it must contain all its boundary points. Since $\mathbb{C}$ contains all complex numbers, it contains all its boundary points (even though $\mathbb{C}$ technically has no boundary points). Thus, $\mathbb{C}$ is closed. ■

Example: Consider the set $\{z \in \mathbb{C} : \operatorname{Im} z \geq 1, \operatorname{Re} z > 2\}$. Is it an open set? Is it a closed set?

Proof. This set consists of all complex numbers whose imaginary part is greater than or equal to 1 and whose real part is strictly greater than 2.
Is it an open set?
Consider the condition $\operatorname{Re} z > 2$. This means that the real part of z can get arbitrarily close to 2, but never actually be 2. For any point z in S that has a real part very close to 2, we cannot find a sufficiently small open disk around z that is entirely contained in S because some part of that disk will spill over to the region where $\operatorname{Re} z \leq 2$. Therefore, S is not open.
Is it a closed set?
Consider the condition $\operatorname{Im} z \geq 1$. This means that the imaginary part of z can be exactly 1. For any point z on the line $\operatorname{Im} z = 1$ (with $\operatorname{Re} z > 2$), every open disk around z will contain points from both inside and outside of S. This means that S does not contain all its boundary points, and therefore, S is not closed.
In conclusion, the set S is neither open nor closed in $\mathbb{C}$. ■

1.6.3 Closed set and its complement

Theorem: A set C in $\mathbb{C}$ is closed if and only if its complement $D = \{z \in \mathbb{C} : z \notin C\}$ is open.

Proof. **(i) Forward Direction:** Assume C is closed. By previous remark, the complement of C is open. Hence, D is open.
(ii) Reverse Direction: Assume D is open. Then, by previous remark of a closed set, C is closed. ■

1.6.4 Connected set

Definition(Polygonal): A polygonal curve in $\mathbb{C}$ is the union of a finite number of directed line segments

$$P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$$

Definition(Connected set): An open set D in $\mathbb{C}$ is connected if each pair p, q of points in D can be joined by a polygonal curve lying completely in D.

Example: The set $\{z \in \mathbb{C} : \operatorname{Im} z > 1\}$ is connected.

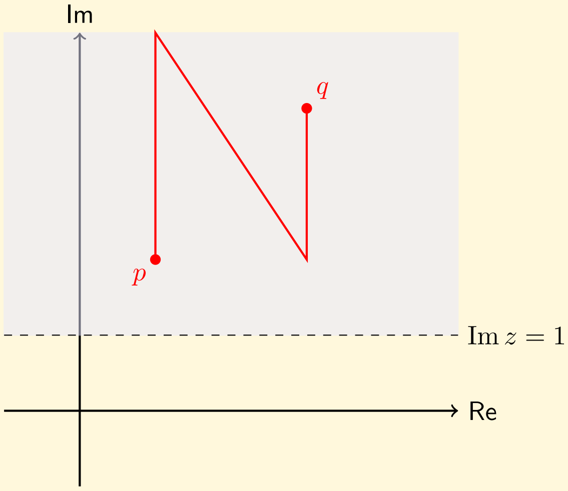
Proof. Consider any two points p, q in S. Without loss of generality, let $p = a + bi$ and $q = c + di$ where $a, c \in \mathbb{R}$ and $b, d > 1$.
We want to show that p and q can be joined by a polygonal curve lying completely in S.
Case 1: If $a = c$, then the straight line segment joining p and q lies entirely in S.
Case 2: If $a \neq c$, construct a polygonal curve as follows:

1. Start at point p.

2. Move vertically to the point $(a, \max(b, d) + 1)$. This point is in S since its imaginary part is greater than 1.

3. Move horizontally to the point $(c, \max(b, d) + 1)$. This segment lies in S since the imaginary part is greater than 1.

4. Move vertically to reach point q.



The entire polygonal curve lies in S and connects p and q.
Since p and q were arbitrary points in S, any two points in S can be joined by a polygonal curve lying completely in S. Thus, S is connected. ■

Example: The set $\{z \in \mathbb{C} : |\operatorname{Im} z| > 1\}$ is not connected.

Proof. Consider the set S. It consists of two disjoint regions: one where $\operatorname{Im} z > 1$ and one where $\operatorname{Im} z < -1$. Take two points:

1. p such that $\operatorname{Im} p = 2$ (in the upper region).

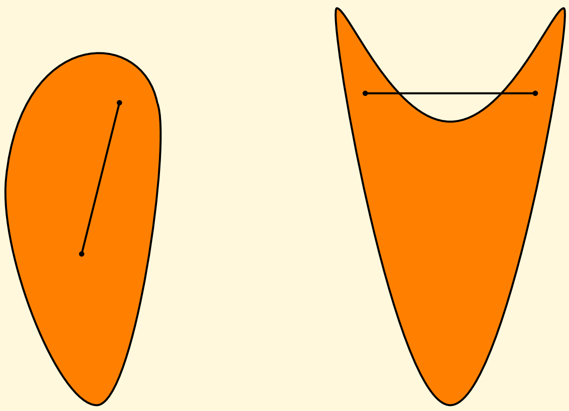
2. q such that $\operatorname{Im} q = -2$ (in the lower region).

Any polygonal curve in $\mathbb{C}$ joining p and q must cross either the line $\operatorname{Im} z = 1$ or the line $\operatorname{Im} z = -1$. Neither of these lines is in S. Thus, there's no polygonal curve entirely in S joining p and q.
Since we've found a pair p, q in S that cannot be joined by a polygonal curve in S, S is not connected. ■

Definition(Domain): An open connected set is called a domain.

Definition(Convex set): A set C in $\mathbb{C}$ is convex if the line segment pq joining each pair of points p, q in C also lies in C.

Example: Visually, a convex set and non-convex set is easy to tell:



Example: The open set $\left\{z = x + iy \in \mathbb{C} : x^2 > y\right\}$ is connected but not convex.

Proof. **1. S is connected:**
Consider any two points p, q in S. Without loss of generality, let $p = a + bi$ and $q = c + di$ where $a^2 > b$ and $c^2 > d$. We can join p and q by a polygonal curve lying completely in S as follows:

1. Start at point p.

2. Move vertically to the point on the curve $x^2 = y$ with the same x-coordinate as p. This point is in S since $x^2 > y$.

3. Move horizontally to the point on the curve $x^2 = y$ with the same x-coordinate as q. This segment lies in S since it is above the curve $x^2 = y$.

4. Move vertically to reach point q.

Since p and q were arbitrary points in S, any two points in S can be joined by a polygonal curve lying completely in S. Thus, S is connected.

2. S is not convex:

To show that S is not convex, consider the points $p = 2 + i(3)$ and $q = -2 + i(3)$. Both points are in S since $2^2 > 3$ and $(-2)^2 > 3$. However, the midpoint of the segment joining p and q is $0 + i(3)$, which is not in S since 0^2 is not greater than 3.

Thus, the line segment joining p and q does not lie entirely in S, and S is not convex.

In conclusion, the set S is connected but not convex. ∎

1.7 Functions and limits

Definition(Functions, domain and range): Let D be a subset in $\mathbb{C}$. A function f defined on D is a rule that assigns a complex number to each z in D. D is called the domain of the function f. The collection of all possible values of the function f is called the range of the function.

Example: Show that the range of the function $T(z) = \frac{1+z}{1-z}$ on the disc $|z| < 1$ is the set $\{w \in \mathbb{C} : \operatorname{Re} w > 0\}$.

Proof. Solution Compute the real part of w :

$$\operatorname{Re} w = \operatorname{Re}\left(\frac{1+z}{1-z}\right) = \operatorname{Re}\frac{(1+z)(1-\bar{z})}{|1-z|^2} = \frac{1-|z|^2}{|1-z|^2}$$

This last quantity is positive when $|z| < 1$. This shows that the range of T is a subset of those w with $\operatorname{Re} w > 0$. Now let w' be any point with $\operatorname{Re} w' > 0$; we shall show that $z' = (w'-1)/(w'+1)$ satisfies $|z'| < 1$. Indeed, $1 > |(w'-1)/(w'+1)|$ exactly when $|w'+1|^2 > |w'-1|^2$. We expand both $|w'+1|^2$ and $|w'-1|^2$ and obtain

$$|w'|^2 + 2\operatorname{Re} w' + 1 > |w'|^2 - 2\operatorname{Re} w' + 1.$$

This is a correct inequality because $\operatorname{Re} w' > 0$. Thus, $z' = (w'-1)/(w'+1)$ lies in the disc $|z| < 1$, and

$$Tz' = \frac{1+z'}{1-z'} = \frac{1+\frac{w'-1}{w'+1}}{1-\frac{w'-1}{w'+1}} = \frac{2w'}{2} = w'.$$

∎

1.8 Limits

1.8.1 Limits of a sequence

Definition(Limit of a sequence of complex numbers): Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. We say that $\{z_n\}_{n=1}^{\infty}$ has the complex number A as a limit, or that $\{z_n\}_{n=1}^{\infty}$ converges to A, and we write

$$\lim_{n\to\infty} z_n = A \text{ or } z_n \to A,$$

if given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$|z_n - A| < \epsilon \text{ for all } n \geq N.$$

Definition(Divergence): A sequence $\{z_n\}_{n=1}^{\infty}$ that does not converge is called divergent.

Theorem: If $z_n = x_n + iy_n$ and $A = s + it$, then $z_n \to A$ if and only if $x_n \to s$ and $y_n \to t$.

Proof. Suppose $z_n \to A$ for some $A = s + it$ in the complex plane. Then, for any $\epsilon > 0$, there exists an N such that for all $n \geq N$, $|z_n - A| < \epsilon$. Using the given inequalities, we have:

$$|x_n - s| \leq |z_n - A| < \epsilon$$

and

$$|y_n - t| \leq |z_n - A| < \epsilon$$

which implies $x_n \to s$ and $y_n \to t$.
Conversely, suppose $x_n \to s$ and $y_n \to t$. Then, for any $\epsilon > 0$, there exist natural numbers N_1 and N_2 such that for all $n \geq N_1$, $|x_n - s| < \frac{\epsilon}{2}$ and for all $n \geq N_2$, $|y_n - t| < \frac{\epsilon}{2}$. Let $N = \max(N_1, N_2)$. For $n \geq N$, using the third given inequality:

$$|z_n - A| \leq |x_n - s| + |y_n - t| < \epsilon$$

Thus, $z_n \to A$.
Hence, $z_n \to A$ if and only if $x_n \to s$ and $y_n \to t$. ∎

Example: Find the limit of the sequence $z_n = \left(-\frac{1}{2}\right)^n + i\left(1 - \frac{1}{2n}\right)$.

Solution: Given the sequence

$$z_n = \left(-\frac{1}{2}\right)^n + i\left(1 - \frac{1}{2n}\right)$$

we can analyze its real and imaginary parts to determine the limit as $n \to \infty$.

1. For the real part:

$$\text{Re}(z_n) = \left(-\frac{1}{2}\right)^n$$

   As n increases, this approaches 0

2. For the imaginary part:

$$\text{Im}(z_n) = 1 - \frac{1}{2n}$$

   As $n \to \infty$, $\text{Im}(z_n)$ approaches 1.

In conclusion, it converges to $1 + i$
Example: Find the limit of the sequence $z_n = \frac{1}{n}\left(\cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right)\right)$.

Solution: Given the sequence

$$z_n = \frac{1}{n}\left(\cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right)\right)$$

we can analyze its real and imaginary parts to determine the limit as $n \to \infty$.

1. For the real part:

$$\text{Re}(z_n) = \frac{1}{n}\cos\left(\frac{n\pi}{4}\right)$$

   The real part tends to 0 as $n \to \infty$.

2. For the imaginary part:

$$\text{Im}(z_n) = \frac{1}{n}\sin\left(\frac{n\pi}{4}\right)$$

   The imaginary part also tends to 0 as $n \to \infty$.

In conclusion, the sequence z_n converges to 0 as $n \to \infty$.

Corollary: If $z_n \to A$, then $|z_n| \to |A|$.

Proof. Given that $z_n \to A$, by the previous theorem, the real and imaginary parts of z_n converge to the real and imaginary parts of A, respectively. Let $z_n = x_n + iy_n$ and $A = s + it$. Then, $x_n \to s$ and $y_n \to t$. Now, consider the modulus of z_n:

$$|z_n| = \sqrt{x_n^2 + y_n^2}$$

As $n \to \infty$, $x_n \to s$ and $y_n \to t$. By the properties of limits:

$$\lim_{n\to\infty} x_n^2 = s^2$$

$$\lim_{n\to\infty} y_n^2 = t^2$$

Using the sum rule for limits:

$$\lim_{n\to\infty} (x_n^2 + y_n^2) = s^2 + t^2$$

Now, using the continuity of the square root function:

$$\lim_{n\to\infty} \sqrt{x_n^2 + y_n^2} = \sqrt{s^2 + t^2}$$

Thus:

$$\lim_{n\to\infty} |z_n| = |A|$$

This proves that if $z_n \to A$, then $|z_n| \to |A|$. ∎

Remark: The converse of this assertion is generally false; for example, $|(-1)^n + i/n| \to 1$, but the sequence $\{(-1)^n + i/n\}$ itself has no limit.

Theorem: Let $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be two convergent sequences of complex numbers with limits A and B, respectively. Let λ be a complex number. We have

1. the sequence $\{z_n + \lambda w_n\}_{n=1}^{\infty}$ converges to $A + \lambda B$;
2. the sequence $\{z_n w_n\}_{n=1}^{\infty}$ converges to AB;
3. if $B \neq 0$, then the sequence $\left\{\frac{z_n}{w_n}\right\}_{n=1}^{\infty}$ converges to $\frac{A}{B}$.

Proof.

1. Given that $z_n \to A$ and $w_n \to B$, for any $\epsilon > 0$, there exists N_1 such that for all $n \geq N_1$, $|z_n - A| < \frac{\epsilon}{2}$, and there exists N_2 such that for all $n \geq N_2, |w_n - B| < \frac{\epsilon}{2|\lambda|}$ (assuming $\lambda \neq 0$; if $\lambda = 0$, the sequence λw_n is just 0 for all n and the result is trivial).
   Let $N = \max(N_1, N_2)$. For $n \geq N$:
$$\begin{aligned} &|(z_n + \lambda w_n) - (A + \lambda B)| \\ &= |(z_n - A) + \lambda (w_n - B)| \\ &\leq |z_n - A| + |\lambda| \, |w_n - B| \\ &< \frac{\epsilon}{2} + |\lambda| \frac{\epsilon}{2|\lambda|} \\ &= \epsilon \end{aligned}$$
   Thus, $z_n + \lambda w_n \to A + \lambda B$.

2. Given $z_n \to A$ and $w_n \to B$, we want to show that $z_n w_n \to AB$.
   Using the properties of complex multiplication and the distributive property, we have:
$$\begin{aligned} z_n w_n - AB &= z_n w_n - z_n B + z_n B - AB \\ &= z_n(w_n - B) + B(z_n - A). \end{aligned}$$
   Now, using the triangle inequality:
$$\begin{aligned} |z_n w_n - AB| &\leq |z_n(w_n - B)| + |B(z_n - A)| \\ &\leq |z_n||w_n - B| + |B||z_n - A|. \end{aligned}$$
   Given that $z_n \to A$, the sequence $|z_n|$ is bounded. Let M be an upper bound for $|z_n|$. Then:
$$|z_n w_n - AB| \leq M|w_n - B| + |B||z_n - A|$$
   For any $\epsilon > 0$, there exists N_1 such that for all $n \geq N_1$, $|z_n - A| < \frac{\epsilon}{2|B|}$, and there exists N_2 such that for all $n \geq N_2$, $|w_n - B| < \frac{\epsilon}{2M}$.
   Let $N = \max(N_1, N_2)$. For $n \geq N$:
$$|z_n w_n - AB| < \epsilon$$
   Thus, $z_n w_n \to AB$.

3. Given $w_n \to B$ and $B \neq 0$, there exists a positive distance δ such that $|w_n - B| > \delta$ for all n since the limit of w_n is B and $B \neq 0$.
   Now, consider the difference:
$$\left|\frac{z_n}{w_n} - \frac{A}{B}\right| = \left|\frac{z_n B - A w_n}{w_n B}\right|$$
   Using the triangle inequality and properties of modulus:
$$\left|\frac{z_n}{w_n} - \frac{A}{B}\right| \leq \frac{|z_n||B - w_n| + |A||w_n|}{|w_n||B|}$$
   Given that $z_n \to A$, the sequence $|z_n|$ is bounded. Let M be an upper bound for $|z_n|$. Then:
$$\left|\frac{z_n}{w_n} - \frac{A}{B}\right| \leq \frac{M|B - w_n| + |A||w_n|}{\delta |B|}$$
   For any $\epsilon > 0$, there exists N_1 such that for all $n \geq N_1$, $|z_n - A| < \frac{\epsilon\delta|B|}{2M}$, and there exists N_2 such that for all $n \geq N_2$, $|w_n - B| < \frac{\epsilon\delta|B|}{2|A|}$. Let $N = \max(N_1, N_2)$. For $n \geq N$:
$$\left|\frac{z_n}{w_n} - \frac{A}{B}\right| < \epsilon$$
   Thus, $\frac{z_n}{w_n} \to \frac{A}{B}$.

∎

Example: Let $z_n = \left(-\frac{1}{2}\right)^n + i\left(1 - \frac{1}{2n}\right)$. Find the limit of the sequence z_n^2.

Solution: We know sequence z_n converges to $1+i$, z_n^2 converges to $(1+i)^2$ by the theorem above.
Example: Suppose $\{z_n\}_{n=1}^{\infty}$ is a sequence with $z_n \to A$. Let a_0, a_1, a_2, a_3 be any complex numbers. Prove the sequence $\left\{a_0 + a_1 z_n + a_2 z_n^2 + a_3 z_n^3\right\}_{n=1}^{\infty}$ converges to $a_0 + a_1 A + a_2 A^2 + a_3 A^3$.

Proof. Using the properties of limits, we can break down the limit of the sum into the sum of the limits:

$$\lim_{n\to\infty}(a_0 + a_1 z_n + a_2 z_n^2 + a_3 z_n^3) = \lim_{n\to\infty} a_0 + \lim_{n\to\infty} a_1 z_n + \lim_{n\to\infty} a_2 z_n^2 + \lim_{n\to\infty} a_3 z_n^3$$

Now, let's evaluate each term separately:

1. $\lim_{n\to\infty} a_0 = a_0$ since a_0 is a constant.
2. $\lim_{n\to\infty} a_1 z_n = a_1 \lim_{n\to\infty} z_n = a_1 A$ using the fact that $z_n \to A$.
3. $\lim_{n\to\infty} a_2 z_n^2 = a_2 \lim_{n\to\infty} z_n^2 = a_2 A^2$ using the fact that if $z_n \to A$, then $z_n^2 \to A^2$.
4. $\lim_{n\to\infty} a_3 z_n^3 = a_3 \lim_{n\to\infty} z_n^3 = a_3 A^3$ using the fact that if $z_n \to A$, then $z_n^3 \to A^3$.

Combining the results of the four terms, we get:

$$\lim_{n\to\infty}(a_0 + a_1 z_n + a_2 z_n^2 + a_3 z_n^3) = a_0 + a_1 A + a_2 A^2 + a_3 A^3$$

Thus, the sequence $\left\{a_0 + a_1 z_n + a_2 z_n^2 + a_3 z_n^3\right\}_{n=1}^{\infty}$ converges to $a_0 + a_1 A + a_2 A^2 + a_3 A^3$. ∎

1.8.2 Limits of functions

Definition(Limit of a function of complex variable): Let D be a subset in $\mathbb{C}$ and let f be a function defined on D. Let z_0 be a point either in D or in the boundary of D. We say that f has limit L at the point z_0, and we write

$$\lim_{z\to z_0} f(z) = L \text{ or } f(z) \to L \text{ as } z \to z_0,$$

if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - L| < \epsilon \text{ for any } z \in D\setminus\{z_0\} \text{ and } |z - z_0| < \delta.$$

Definition(Limit at infinity): Let f be a function defined on $\mathbb{C}$. We say that f has a limit L at ∞, and we write

$$\lim_{z\to\infty} f(z) = L \text{ or } f(z) \to L \text{ as } z \to \infty,$$

if for any $\epsilon > 0$, there exists $M > 0$ such that

$$|f(z) - L| < \epsilon \text{ for any } |z| > M.$$

Example: Show that the function $f(z) = |z|$ has limit 2 at the point $z_0 = 2i$.

Proof. To show that the function $f(z) = |z|$ has a limit of 2 at the point $z_0 = 2i$, we'll use the given definition of a limit. Given:

$$f(z) = |z|$$

$$z_0 = 2i$$

We want to prove:

$$\lim_{z\to 2i} |z| = 2$$

According to the definition of the limit, for any $\epsilon > 0$, we need to find a $\delta > 0$ such that:

$$|f(z) - 2| < \epsilon$$

whenever

$$0 < |z - 2i| < \delta$$

Given $f(z) = |z|$, we have:

$$|f(z) - 2| = ||z| - 2|$$

Now, consider the point $z = x + yi$. The distance from z to $2i$ is:

$$|z - 2i| = |x + yi - 2i| = |x + (y-2)i| = \sqrt{x^2 + (y-2)^2}$$

For $z = 2i$, $f(z) = |2i| = 2$.
Using the triangle inequality, which states that for any complex numbers a and b:

$$|a + b| \leq |a| + |b|$$

We can deduce:

$$||z| - 2| \leq |z - 2i|$$

Thus, for $|z - 2i| < \delta$, we have:

$$||z| - 2| < \delta$$

If we choose $\delta = \epsilon$, then for any z such that $0 < |z - 2i| < \epsilon$, we have:

$$||z| - 2| < \epsilon$$

This implies that:

$$\lim_{z \to 2i} |z| = 2$$

This completes the proof. ∎

Example: Consider the function $f(z) = \frac{z}{\bar{z}}, z \neq 0$. Decide whether it has a limit at $z_0 = 0$.

Proof. Consider the function:

$$f(z) = \frac{z}{\bar{z}}, \quad z \neq 0$$

To determine if $f(z)$ has a limit at $z_0 = 0$, we'll approach z_0 from different paths.

1. Approach z_0 along the real axis ($y = 0$):
   Let $z = x$ (where x is real). Then $\bar{z} = x$ and:
   $$f(z) = \frac{x}{x} = 1$$
2. Approach z_0 along the imaginary axis ($x = 0$):
   Let $z = yi$ (where y is real). Then $\bar{z} = -yi$ and:
   $$f(z) = \frac{yi}{-yi} = -1$$

From the two different paths, we observe that the function approaches two distinct values (1 and -1) as z approaches $z_0 = 0$. Therefore, the function $f(z) = \frac{z}{\bar{z}}$ does not have a limit at $z_0 = 0$. ∎

Example: Consider the function $f(z) = \frac{1}{z^{100}}, z \neq 0$. Find $\lim_{z \to \infty} f(z)$.

Proof. Consider the function:

$$f(z) = \frac{1}{z^{100}}, \quad z \neq 0$$

We want to determine:

$$\lim_{z \to \infty} f(z)$$

According to the given definition of the limit at infinity, for any $\epsilon > 0$, there exists $M > 0$ such that:

$$|f(z) - L| < \epsilon$$

for any $|z| > M$.
For the function $f(z) = \frac{1}{z^{100}}$, as $|z|$ grows larger, the value of $f(z)$ approaches 0. Thus, we hypothesize that:

$$L = 0$$

To prove this, for any $\epsilon > 0$, we need to find an $M > 0$ such that:

$$\left| \frac{1}{z^{100}} \right| < \epsilon$$

whenever $|z| > M$.
Given that $|z^{100}| = |z|^{100}$, we can rewrite the inequality as:

$$|z|^{100} > \frac{1}{\epsilon}$$

Taking the 100th root of both sides, we get:

$$|z| > \left(\frac{1}{\epsilon}\right)^{\frac{1}{100}}$$

Let's choose $M = \left(\frac{1}{\epsilon}\right)^{\frac{1}{100}}$. Then, for any $|z| > M$, we have:

$$\left|\frac{1}{z^{100}}\right| < \epsilon$$

Thus, by the definition of the limit at infinity:

$$\lim_{z\to\infty} f(z) = 0$$

This completes the proof. ∎

Theorem: Let f and g be two functions of one complex variable. Suppose that

$$\lim_{z\to z_0} f(z) = L \text{ and } \lim_{z\to z_0} g(z) = M.$$

Let λ be any complex number. Then we have

1. $\lim_{z\to z_0}(f + \lambda g)(z) = L + \lambda M$;
2. $\lim_{z\to z_0}(fg)(z) = LM$;
3. if $M \neq 0$, then $\lim_{z\to z_0}\left(\frac{f}{g}\right)(z) = \frac{L}{M}$.

Proof. Let f and g be two functions of one complex variable. Given:

$$\lim_{z\to z_0} f(z) = L$$
$$\lim_{z\to z_0} g(z) = M$$

and let λ be any complex number.

1. **Sum of Functions:** For any $\epsilon > 0$, there exist δ_1 and δ_2 such that:

   $$|f(z) - L| < \frac{\epsilon}{2}$$

   whenever $|z - z_0| < \delta_1$ and

   $$|g(z) - M| < \frac{\epsilon}{2|\lambda|}$$

   whenever $|z - z_0| < \delta_2$. Choose $\delta = \min(\delta_1, \delta_2)$. Then, for $|z - z_0| < \delta$:

   $$\begin{aligned}|(f + \lambda g)(z) - (L + \lambda M)| &\leq |f(z) - L| + |\lambda||g(z) - M| \\ &< \frac{\epsilon}{2} + |\lambda|\frac{\epsilon}{2|\lambda|} \\ &= \epsilon\end{aligned}$$

   Hence, $\lim_{z\to z_0}(f + \lambda g)(z) = L + \lambda M$.

2. **Product of Functions:** Given any $\epsilon > 0$, we want to find $\delta > 0$ such that:

   $$|f(z)g(z) - LM| < \epsilon$$

   whenever $|z - z_0| < \delta$. Using the fact that:

   $$\begin{aligned}&|f(z)g(z) - LM| = |f(z)g(z) - Lg(z) + Lg(z) - LM| \\ &\leq |f(z) - L||g(z)| + |L||g(z) - M|\end{aligned}$$

   Since $\lim_{z\to z_0} g(z) = M$, for a sufficiently small neighborhood around z_0, the value of $|g(z)|$ can be bounded by some positive number G, i.e., $|g(z)| \leq G$. Choosing $\epsilon_1 = \frac{\epsilon}{2G}$ and $\epsilon_2 = \frac{\epsilon}{2|L|}$, we can find δ_1 and δ_2 such that:

   $$|f(z) - L| < \epsilon_1$$

   and

   $$|g(z) - M| < \epsilon_2$$

   whenever $|z - z_0| < \delta_1$ and $|z - z_0| < \delta_2$ respectively. Choosing $\delta = \min(\delta_1, \delta_2)$, we ensure that:

   $$|f(z)g(z) - LM| < \epsilon$$

   Thus, $\lim_{z\to z_0} f(z)g(z) = LM$.

3. **Quotient of Functions:** Given any $\epsilon > 0$, we want to find $\delta > 0$ such that:

$$\left|\frac{f(z)}{g(z)} - \frac{L}{M}\right| < \epsilon$$

whenever $|z - z_0| < \delta$. Using the fact that:

$$\begin{aligned} \left|\frac{f(z)}{g(z)} - \frac{L}{M}\right| &= \left|\frac{Mf(z) - Lg(z)}{Mg(z)}\right| \\ &\leq \frac{|f(z) - L|}{|M|} + \frac{|L||g(z) - M|}{|M^2|} \end{aligned}$$

Since $M \neq 0$, we can choose $\epsilon_1 = |M|\epsilon/2$ and $\epsilon_2 = \left|M^2\right|\epsilon/2|L|$ to ensure:

$$|f(z) - L| < \epsilon_1$$

and

$$|g(z) - M| < \epsilon_2$$

whenever $|z - z_0| < \delta_1$ and $|z - z_0| < \delta_2$ respectively. Choosing $\delta = \min(\delta_1, \delta_2)$, we ensure that:

$$\left|\frac{f(z)}{g(z)} - \frac{L}{M}\right| < \epsilon$$

Thus, $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$.

This completes the proof. ∎

1.9 Continuity

Definition(Continuity): Let D be a subset in $\mathbb{C}$ and let f be a function defined on D. Let z_0 be a point in D. We say that f is continuous at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0).$$

Example: Consider the function

$$f(z) = \begin{cases} \frac{z^4-1}{z-1}, & z \neq 1 \\ 4, & z = 1. \end{cases}$$

Prove f is continuous at 1 .

Proof. To prove the continuity of f at $z_0 = 1$, we need to demonstrate that:

$$\lim_{z\to 1} f(z) = f(1)$$

Given the function:

$$f(z) = \begin{cases} \frac{z^4-1}{z-1} & \text{if } z \neq 1 \\ 4 & \text{if } z = 1 \end{cases}$$

For $z \neq 1$, we can factorize $z^4 - 1$ using the difference of squares:

$$z^4 - 1 = (z^2 + 1)(z^2 - 1) = (z^2 + 1)(z + 1)(z - 1)$$

This simplifies the expression for $f(z)$ to:

$$f(z) = z^2 + z + 2$$

Evaluating the limit as z approaches 1:

$$\lim_{z\to 1} f(z) = 1^2 + 1 + 2 = 4$$

Given that $f(1) = 4$, it follows that:

$$\lim_{z\to 1} f(z) = f(1)$$

By the given definition of continuity, it is evident that f is continuous at $z_0 = 1$. ∎

Theorem: Let f and g be two functions of one complex variable. Suppose that both f and g are continuous at z_0. Let λ be any complex number. Then:

1. $f + \lambda g$ is continuous at z_0.

2. $f \cdot g$ is continuous at z_0.

3. If $g(z_0) \neq 0$, then $\frac{f}{g}$ is continuous at z_0.
4. (Composition Function) If h is a function continuous at each point of some disc centered at $f(z_0)$, then $h(f(z))$ is continuous at z_0.

Proof. Given that f and g are continuous at z_0 and λ is any complex number:

1. Using properties of limits, we have:

$$\lim_{z \to z_0} (f(z) + \lambda g(z)) = f(z_0) + \lambda g(z_0)$$

   Thus, $f + \lambda g$ is continuous at z_0.

2. Using properties of limits, we have:

$$\lim_{z \to z_0} f(z)g(z) = f(z_0)g(z_0)$$

   Thus, $f \cdot g$ is continuous at z_0.

3. Given $g(z_0) \neq 0$, using properties of limits, we have:

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g(z_0)}$$

   Thus, if $g(z_0) \neq 0$, then $\frac{f}{g}$ is continuous at z_0.

4. Given h is continuous at each point of some disc centered at $f(z_0)$, by definition:

$$\lim_{z \to z_0} h(f(z)) = h(f(z_0))$$

   Thus, $h(f(z))$ is continuous at z_0.

This completes the proof of the theorem. ■

1.10 Infinite series

Definition: Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. We define infinite series as:

$$z_1 + z_2 + z_3 + \ldots = \sum_{n=1}^{\infty} z_n$$

Definition(nth partial sum): nth partial sum for a sequence $\{z_n\}_{n=1}^{\infty} s_n = z_1 + \ldots + z_n = \sum_{j=1}^{n} z_j$.

Definition(Converges): We say that the infinite series $\sum_{j=1}^{\infty} z_j$ converges if the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ converges.

Definition(Diverges): We say that the infinite series $\sum_{j=1}^{\infty} z_j$ diverges if the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ diverges.

The convergence (or divergence) of the infinite series $\sum_{j=1}^{\infty} z_j$ of complex numbers can be formulated in terms of the convergence (or divergence) of two infinite series of real numbers. This follows directly by writing $z_j = x_j + iy_j$, so

$$s_n = \sum_{j=1}^{n} z_j = \sum_{j=1}^{n} x_j + i \sum_{j=1}^{n} y_j = \sigma_n + i\tau_n$$

The sequence $\{s_n\}$ converges if and only if both of the sequences $\{\sigma_n\}$ and $\{\tau_n\}$ of real numbers converge, and, this being the case,

$$s = \lim s_n = \lim \sigma_n + i \lim \tau_n$$

Thus, we have the following result:

Theorem: Let $z_j = x_j + iy_j; j = 1, 2, 3, \ldots$ The infinite series $\sum_{j=1}^{\infty} z_j$ converges, $z_j = x_j + iy_j$, if and only if both

$$\sum_{j=1}^{\infty} x_j \quad \text{and} \quad \sum_{j=1}^{\infty} y_j$$

converge. Furthermore, if $\sum z_j$ converges, then

$$\sum_{j=1}^{\infty} z_j = \sum_{j=1}^{\infty} x_j + i \sum_{j=1}^{\infty} y_j$$

Corollary: If a series of complex numbers converges, the nth term converges to zero as n tends to infinity.

Lemma: If $\sum_{j=1}^{\infty} |z_j|$ converges, then $\sum_{j=1}^{\infty} z_j$ converges.

1.11 Exponential, Logarithm, and Trigonometric Functions

1.11.1 Exponential functions

Definition(Exponential function): For any complex number $z = x + iy$, we define

$$e^z = e^x(\cos y + i\sin y)$$

to be the exponential function with domain of $e^z : \mathbb{C}$ and range of $e^z : \mathbb{C} - \{0\}$.

Property: $e^{z+w} = e^z e^w$ for any $z, w \in \mathbb{C}$.

Proof. Let $z = x + iy$ and $w = s + it$. Then, making use of two basic trigonometric identities for the sine and cosine of the sum of two numbers,

$$\begin{aligned} e^{z+w} &= e^{x+s}[\cos(y+t) + i\sin(y+t)] \\ &= e^x e^s[(\cos y\cos t - \sin y\sin t) + i(\sin y\cos t + \sin t\cos y)] \\ &= e^x(\cos y + i\sin y)e^s(\cos t + i\sin t) \\ &= e^z e^w \end{aligned}$$

∎

Property: $|e^z| = e^{\operatorname{Re} z}$

Proof. $$\begin{aligned} |e^z| &= \left((e^x\cos y)^2 + (e^x\sin y)^2\right)^{1/2} \\ &= e^x = e^{\operatorname{Re} z} \end{aligned}$$ ∎

Property: $e^{2\pi i m} = 1$ for $m = 0, \pm1, \ldots$
Property: $e^{x+iy} = e^{\ln r}e^{i(\psi+2\pi m)} = r[\cos\psi + i\sin\psi] = w$. If $e^z = w$, then

$$r = |w| = |e^z| = e^x,$$

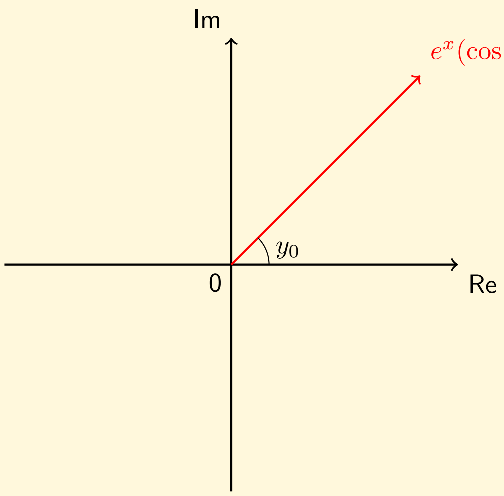
Example: For any $y_0 \in \mathbb{R}$, consider the horizontal line $\{x + iy_0 : x \in \mathbb{R}\}$. What is its image under $e^z = e^x(\cos y + i\sin y)$.
Solution: Given the function $f(z) = e^z$ where $z = x + yi$, the expression for $f(z)$ in terms of x and y is:

$$f(z) = e^x(\cos y + i\sin y)$$

Now, for a horizontal line $y = y_0$ in the complex plane, the image under f is:

$$f(x + iy_0) = e^x(\cos y_0 + i\sin y_0)$$

This represents a curve in the complex plane. The real part is $e^x\cos y_0$ and the imaginary part is $e^x\sin y_0$.
Given that x ranges over all real numbers, e^x ranges from 0 to ∞. Thus, the image of the horizontal line $y = y_0$ under the function f is a half-line (or ray) emanating from the origin in the direction specified by the angle y_0 and extending infinitely in that direction.



Example: What is the image of the strip $\{x + iy : y_0 \le y < y_0 + 2\pi, x \in \mathbb{R}\}$?

Solution: Given the function $f(z) = e^z$ where $z = x + yi$, the expression for $f(z)$ in terms of x and y is:

$$f(z) = e^x(\cos y + i\sin y)$$

For the strip $y_0 \leq y < y_0 + 2\pi$, the image under f is:

$$f(x+iy) = e^x(\cos y + i \sin y)$$

Given that x ranges over all real numbers, e^x ranges from 0 to ∞. The term $\cos y + i \sin y$ represents a point on the unit circle in the complex plane, and as y ranges from y_0 to $y_0+\ 2\pi$, this term traces out a full circle. Thus, the image of the strip under the function f is the entire complex plane without the origin. This is because for every point in the complex plane (except the origin), there exists an x and y in the given strip such that the point is the image of $x+iy$ under f. And function is bijective on this domain.

Example: Consider the vertical line $\{it : t \in \mathbb{R}\}$. What is its image under e^z?

Solution: Given the function definition:

$$e^z = e^x(\cos y + i \sin y)$$

where $z = x + iy$, we want to determine the image of the vertical line $\{it : t \in \mathbb{R}\}$ under e^z.

For the vertical line $\{it : t \in \mathbb{R}\}, x = 0$ and $y = t$. Plugging these values into our function, we get:

$$e^{it} = e^0(\cos t + i \sin t)$$
$$e^{it} = \cos t + i \sin t$$

This is the representation of a complex number in its polar form. The magnitude (or modulus) of this number is 1 , and the argument (or angle) is t. This represents a point on the unit circle in the complex plane, with the angle t radians from the positive real axis.

As t varies over all real numbers, the image will trace out the entire unit circle in the complex plane. Therefore, the image of the vertical line $\{it : t \in \mathbb{R}\}$ under e^z is the unit circle in the complex plane, centered at the origin.

Example: In general, for any $x_0 \in \mathbb{R}$, consider the vertical line $\{x_0 + it : t \in \mathbb{R}\}$. What is its image?

Solution: Given the function definition:

$$e^z = e^x(\cos y + i \sin y)$$

where $z = x + iy$, we want to determine the image of the vertical line $\{x_0 + it : t \in \mathbb{R}\}$ under e^z.

For the vertical line $\{x_0 + it : t \in \mathbb{R}\}, x = x_0$ and $y = t$. Plugging these values into our function, we get:

$$e^{x_0+it} = e^{x_0}(\cos t + i \sin t)$$

This is still a representation of a complex number in its polar form. However, the magnitude (or modulus) of this number is now e^{x_0} instead of 1 , and the argument (or angle) is t. This represents a point on the circle in the complex plane with radius e^{x_0} and angle t radians from the positive real axis.

As t varies over all real numbers, the image will trace out the entire circle in the complex plane with radius e^{x_0}, centered at the origin.

Therefore, the image of the vertical line $\{x_0 + it : t \in \mathbb{R}\}$ under e^z is the circle in the complex plane with radius e^{x_0}, centered at the origin.

1.11.2 Logarithm function

Consider z any nonzero complex number. We would like to solve for w, the equation

$$e^w = z$$

If $\Theta = \operatorname{Arg}(z)$ with $-\pi < \Theta \leq \pi$, then z and w can be written as follows

$$z = re^{i\Theta} \text{ and } \quad w = u + iv.$$

Then the $e^w = z$ becomes

$$e^u e^{iv} = re^{i\Theta}$$

Thus, we have

$$e^u = r \quad \text{and} \quad v = \Theta + 2n\pi$$

where $n \in \mathbb{Z}$. Since $e^u = r$ is the same as $u = \ln r$, it follows that $e^w = z$ is satisfied if and only if w has one of the values

$$w = \ln r + i(\Theta + 2n\pi) \quad (n \in \mathbb{Z}).$$

Therefore, the (multiple-valued) logarithmic function of a nonzero complex variable $z = re^{i\Theta}$ is defined by the formula

$$\log z = \ln r + i(\Theta + 2n\pi) \quad (n \in \mathbb{Z})$$

Definition(Logarithmic function): The (multiple-valued) logarithmic function of a nonzero complex variable $z = re^{i\Theta}$ is defined by the formula

$$\log z = \ln r + i(\Theta + 2n\pi) \quad (n \in \mathbb{Z})$$

Definition(Principal branch): The principal value of $\log z$ is the value obtained from equation $\log z = \ln r + i(\Theta + 2n\pi)$ when $n = 0$ and is denoted by $\log z$. Thus

$$\log z = \ln r + i\Theta$$

Remark: The function $\log z$ is well defined and single-valued when $z \neq 0$ and that

$$\log z = \log z + 2n\pi i \quad (n \in \mathbb{Z})$$

The logrithm function has some basic properties:

1. $\log(z_1 z_2) = \log z_1 + \log z_2$

   Proof. Let $z_1 = r_1 e^{i\Theta_1}$ and $z_2 = r_2 e^{i\Theta_2}$. Using properties of exponential:

   $$z_1 z_2 = r_1 e^{i\Theta_1} r_2 e^{i\Theta_2} = r_1 r_2 e^{i(\Theta_1 + \Theta_2)}$$

   Now, using the definition of the principal branch of the logarithm:

   $$\begin{aligned} \log(z_1 z_2) &= \ln(r_1 r_2) + i(\Theta_1 + \Theta_2) \\ \log z_1 &= \ln r_1 + i\Theta_1 \\ \log z_2 &= \ln r_2 + i\Theta_2 \end{aligned}$$

   Adding $\log z_1$ and $\log z_2$:

   $$\log z_1 + \log z_2 = \ln r_1 + \ln r_2 + i(\Theta_1 + \Theta_2) = \ln(r_1 r_2) + i(\Theta_1 + \Theta_2)$$

   This is the same as $\log(z_1 z_2)$, proving the first property for the principal branch. ■

2. $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$

   Proof. Using properties of exponential:

   $$\frac{z_1}{z_2} = \frac{r_1 e^{i\Theta_1}}{r_2 e^{i\Theta_2}} = \frac{r_1}{r_2} e^{i(\Theta_1 - \Theta_2)}$$

   Using the definition of the principal branch of the logarithm:

   $$\log\left(\frac{z_1}{z_2}\right) = \ln\left(\frac{r_1}{r_2}\right) + i(\Theta_1 - \Theta_2)$$

   Subtracting $\log z_2$ from $\log z_1$:

   $$\log z_1 - \log z_2 = (\ln r_1 - \ln r_2) + i(\Theta_1 - \Theta_2) = \ln\left(\frac{r_1}{r_2}\right) + i(\Theta_1 - \Theta_2)$$

   This is the same as $\log\left(\frac{z_1}{z_2}\right)$, proving the second property for the principal branch. ■

3. There may hold $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$

   Proof. The principal value Log is defined for $-\pi < \Theta \leq \pi$. Consider $z_1 = e^{i\pi/2}$ and $z_2 = e^{i\pi/2}$. $\operatorname{Log} z_1 = i\pi/2$ $\operatorname{Log} z_2 = i\pi/2$

   $$\begin{aligned} z_1 z_2 &= e^{i\pi} \\ \operatorname{Log}(z_1 z_2) &= i\pi \end{aligned}$$

   However,

   $$\operatorname{Log} z_1 + \operatorname{Log} z_2 = i\pi + i\pi/2 = 3i\pi/2$$

   This is not in the range $-\pi < \Theta \leq \pi$, so it's not the principal value. Thus, $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$. This completes the proofs for the properties. ■

From definition $\log z = \ln r + i(\Theta + 2n\pi)$ let $\theta = \Theta + 2n\pi (n \in \mathbb{Z})$, so we can write

$$\log z = \ln r + i\theta$$

Now, let α be any real number. If we restrict the value of θ so that $\alpha < \theta < \alpha + 2n\pi$, then the function

$$\log z = \ln r + i\theta \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

with components

$$u(r,\theta) = \ln r, \quad v(r,\theta) = \theta$$

is a single-value and continuous function in the stated domain.

Definition(Branch)**:** A branch of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of f. The requirement of analyticity, of course, prevents F from taking on a random selection of the values of f.

Observe that for each fixed α, the single-valued function $\log z = \ln r + i\theta, (r > 0, \alpha < \theta < \alpha + 2\pi)$ is a branch of the multiplevalued function $\log z = \ln r + i\theta$, where the function is bijective on this demain. For example, the function

$$\log z = \ln r + i\Theta \quad (r > 0, -\pi < \theta < \pi),$$

is called the principal branch.

Definition(General exponential function)**:** For any nonzero complex number a and any complex number z, define

$$a^z = e^{z \log a}.$$

We can use this general exponential function and the logrithmetic relation above to do some new things.
Example: Find the values of $(-1)^i$.
Solution: To find the values of $(-1)^i$, we'll use the above relation with $a = -1$ and $z = i$.

$$(-1)^i = e^{i \log(-1)}$$

Now, we need to determine $\log(-1)$. Recall the definition of the logarithm for complex numbers:

$$\log z = \ln r + i\Theta$$

where r is the magnitude of z and Θ is the argument of z. For $z = -1 : r = |-1| = 1$ The argument Θ can be π (180 degrees) for the principal value, but remember that the logarithm is multi-valued in the complex plane, so we can also add any integer multiple of 2π to the argument. Thus, the logarithm of -1 can be:

$$\log(-1) = i(\pi + 2n\pi)$$

where n is an integer. Now, plugging this into our expression for $(-1)^i$:

$$(-1)^i = e^{i^2(\pi+2n\pi)}$$
$$(-1)^i = e^{-\pi(1+2n)}$$

For the principal value (where $n = 0$):

$$(-1)^i = e^{-\pi}$$

Example: Show that for any $z \in \mathbb{C}$, we have

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z.$$

Proof. To prove the given limit, we'll make use of the exponential and logarithmic relations provided earlier. Starting with the expression:

$$\left(1 + \frac{z}{n}\right)^n \tag{1}$$

Take the natural logarithm of both sides:

$$\ln\left(\left(1 + \frac{z}{n}\right)^n\right) = n \ln\left(1 + \frac{z}{n}\right) \tag{2}$$

Now, let's consider the limit of the right-hand side as $n \to \infty$. Using the property of logarithms that $\ln(1 + x) \approx x$ for small x:

$$\begin{aligned}\lim_{n\to\infty} n \ln\left(1 + \frac{z}{n}\right) &= \lim_{n\to\infty} n\left(\frac{z}{n}\right) \\ &= z\end{aligned}$$

Using the relation between the exponential and logarithm:

$$e^{\ln f(x)} = f(x)$$

Applying this to our expression:

$$\lim_{n\to\infty} e^{n \ln\left(1+\frac{z}{n}\right)} = e^z \tag{3}$$

Given that:

$$\left(1 + \frac{z}{n}\right)^n = e^{n \ln\left(1+\frac{z}{n}\right)}$$

We can conclude that:

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z$$

This completes the proof. ■

1.11.3 Trigonometric functions

Definition(Trigonometric functions): For any complex number z, define

$$\cos z=\frac{1}{2}\left(e^{iz}+e^{-iz}\right),$$
$$\sin z=\frac{1}{2i}\left(e^{iz}-e^{-iz}\right).$$

The other four trigonometric functions are now defined in terms of $\sin z$ and $\cos z$:

$$\tan z=\frac{\sin z}{\cos z},\quad \cot z=\frac{\cos z}{\sin z},\quad \sec z=\frac{1}{\cos z},\quad \csc z=\frac{1}{\sin z}.$$

Theorem:

$$\cos(z+2\pi k)=\cos z$$
$$\sin(z+2\pi k)=\sin z,$$

for all z and any integer k.

Theorem:

$$\cos(x+iy)=\cos x\cosh y-i\sin x\sinh y$$
$$\sin(x+iy)=\sin x\cosh y+i\cos x\sinh y$$

where

$$\begin{array}{lll}\cosh u & =\frac{1}{2}\left(e^{u}+e^{-u}\right), & u \text{ real}\\ \sinh u & =\frac{1}{2}\left(e^{u}-e^{-u}\right), & u \text{ real.}\end{array}$$

1.12 Line integral

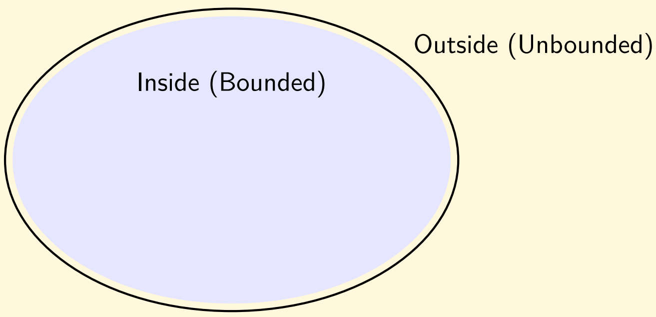
Definition(Curve): A curve γ in $\mathbb{C}$ is a continuous function $\gamma:[a,b]\to\mathbb{C}$, where $[a,b]$ is interval in $\mathbb{R}$.

1. γ is simple if $\gamma(t_1)\neq\gamma(t_2)$ for any $a\leq t_1<t_2<b$
2. γ is closed if $\gamma(a)=\gamma(b)$

Theorem (Jordan Curve Theorem)**:** Let γ be a simple closed curve in $\mathbb{C}$. Then the complement of the range of γ consists of two disjoint open connected sets, one bounded and the other unbounded.

The theorem states that if you have a loop (our simple closed curve) drawn on a sheet of paper, and you exclude the points on the loop itself, then the remaining points on the paper can be grouped into two separate regions that don't touch each other. Each of these regions is a continuous chunk of space without any gaps.

Of these two regions, one is finite in extent (bounded) and is surrounded by the loop, while the other stretches out infinitely in all directions outside the loop (unbounded).



Definition(Differentiable): Definition. Let $\gamma:[a,b]\to\mathbb{C}$ be a curve. Write $\gamma(t)=x(t)+iy(t)$. For any $t_0\in[a,b],\gamma$ is differentiable at t_0 if $x(t)$ and $y(t)$ are differentiable at t_0. Set

$$\gamma'(t_0)=x'(t_0)+iy'(t_0)$$

- γ is smooth if γ is differentiable at every $t\in[a,b]$ and $\gamma'(t)$ is continuous on $[a,b]$;
- γ is piecewise smooth if in $[a,b]$, there are points $t_0=a<t_1<\cdots<t_{n-1}<t_n=b$ such that for each interval $[t_j,t_{j+1}],\gamma$ is differentiable at every $t\in[t_j,t_{j+1}]$ and $\gamma'(t)$ is continuous on $[t_j,t_{j+1}]$.

Example: Parametrize each of the following curves.

1. Fix any $z_0,z_1\in\mathbb{C}$. Consider the line segment between z_0 and z_1.
2. Fix a point $p\in\mathbb{C}$ and a positive number $R>0$. Consider the set $\{z\in\mathbb{C}:|z-p|=R\}$.

3. Fix $0 < \epsilon < R$.

Solution:

1. For any two points in the complex plane, the line segment between them can be parametrized using a parameter t that varies between 0 and 1 . The parametrization is:

$$\gamma(t) = (1-t)z_0 + tz_1$$

where $t \in [0,1]$. When $t = 0, \gamma(t) = z_0$, and when $t = 1, \gamma(t) = z_1$.

2. The set $\{z \in \mathbb{C} : |z-p| = R\}$ represents a circle with center p and radius R. A standard parametrization for a circle in the complex plane uses trigonometric functions: $\gamma(\theta) = p + Re^{i\theta}$ where θ varies from 0 to 2π. This traces out the circle as θ goes through a full rotation.

3. The description seems to be incomplete, but if we're considering an annulus centered at the origin with inner radius ϵ and outer radius R, then the set is:

$$\{z \in \mathbb{C} : \epsilon \leq |z| \leq R\}$$

To parametrize this, we can use polar coordinates. However, an annulus is a 2D region, so a single parametric curve won't capture the entire set. Instead, you'd typically use two parameters: one for the angle and one for the radial distance. A possible parametrization is: $\gamma(r,\theta) = re^{i\theta}$
where θ varies from 0 to 2π and r varies from ϵ to R. This will trace out the entire annulus as r and θ vary over their respective ranges.

Definition: Let $g : [a,b] \to \mathbb{C}$ be a continuous function where $[a,b]$ is an interval in $\mathbb{R}$. Write $g(t) = x(t) + iy(t)$. We define the integral of g over $[a,b]$ by

$$\int_a^b g(t)dt = \int_a^b x(t)dt + i\int_a^b y(t)dt$$

Definition(Line integral): Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth curve. Let $u : \mathbb{C} \to \mathbb{C}$ be a continuous function. We define the integral of u along γ by

$$\int_\gamma u(z)dz = \int_a^b u(\gamma(t))\gamma'(t)dt.$$

Definition(Piecewise line integral): Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve. Let $u : \mathbb{C} \to \mathbb{C}$ be a continuous function. We define the integral of u along γ by

$$\int_\gamma u(z)dz = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} u(\gamma(t))\gamma'(t)dt.$$

Here the points $t_0 = a < t_1 < \cdots t_{n-1} < t_n = b$ come from the definition of "piecewise smooth".

Example: Compute $\int_\gamma zdz$, where γ is the line segment from 2 to $3+i$.
Solution: Let's parametrize the line segment using a parameter t that varies between 0 and 1 :

$$\gamma(t) = (1-t)(2) + t(3+i) = 2 + t + ti$$

where $t \in [0,1]$. Now, differentiate $\gamma(t)$ with respect to t to get $\gamma'(t)$:

$$\gamma'(t) = 1 + i$$

Now, we can express the integral in terms of t :

$$\int_\gamma zdz = \int_0^1 \gamma(t)\gamma'(t)dt$$

Substitute in our expressions for $\gamma(t)$ and $\gamma'(t)$:

$$\begin{aligned} &\int_0^1 (2+t+ti)(1+i)dt \\ &= \int_0^1 (2+2i+t+ti+ti-t)dt \\ &= \int_0^1 (2+3ti)dt \end{aligned}$$

Now, integrate with respect to t :

$$\int_0^1 2dt + \int_0^1 3tidt$$
$$= 2t|_0^1 + \frac{3}{2}t^2 i\bigg|_0^1$$
$$= 2 + \frac{3}{2}i - 0 = 2 + \frac{3}{2}i$$

Thus, $\int_\gamma zdz = 2 + \frac{3}{2}i$.

Example: Compute $\int_\gamma zdz$, where γ consists of the line segment from -1 to 1 and the semicircle from 1 to -1 passing through i.

Solution:

1. Line Segment from -1 to 1:
   Let's call this path γ_1. Parametrization:
   $$\gamma_1(t) = -1 + 2t$$
   where $t \in [0,1]$. Derivative:
   $$\gamma_1'(t) = 2$$
   Integral over γ_1 :
   $$\int_{\gamma_1} zdz = \int_0^1 (-1+2t)(2)dt$$
   $$= \int_0^1 (-2+4t)dt$$
   $$= -2t + 2t^2\Big|_0^1 = 0$$

2. Semicircle from 1 to -1 passing through i :
   Let's call this path γ_2. The radius of the semicircle is 1 , and it's centered at the origin. Parametrization:
   $$\gamma_2(\theta) = e^{i\theta}$$
   where θ varies from 0 to π (since it's the upper half of the unit circle). Derivative:
   $$\gamma_2'(\theta) = ie^{i\theta}$$
   Integral over γ_2 :
   $$\int_{\gamma_2} zdz = \int_0^\pi e^{i\theta}\left(ie^{i\theta}\right)d\theta$$
   $$= \int_0^\pi ie^{2i\theta}d\theta$$
   $$= \frac{1}{2}e^{2i\theta}\bigg|_0^\pi = \frac{1}{2}(-1-1) = -1$$

Now, summing the integrals over the two paths:

$$\int_\gamma zdz = \int_{\gamma_1} zdz + \int_{\gamma_2} zdz = 0 - 1 = -1$$

Thus, $\int_\gamma zdz = -1$.

Properties of line integral

- For any smooth curve $\gamma : [a, b] \to \mathbb{C}$, any continuous functions $u : \mathbb{C} \to \mathbb{C}$ and $v : \mathbb{C} \to \mathbb{C}$, and any complex numbers A, B, we have
$$\int_\gamma (Au + Bv)(z)dz = A\int_\gamma u(z)dz + B\int_\gamma v(z)dz.$$
- For any smooth curve $\gamma : [a, b] \to \mathbb{C}$, set
$$\begin{aligned} -\gamma : [a, b] &\to \mathbb{C}, \\ t &\mapsto \gamma(a + b - t). \end{aligned}$$
For any continuous function $u : \mathbb{C} \to \mathbb{C}$, we have
$$\int_{-\gamma} u(z)dz = -\int_\gamma u(z)dz$$
- Let $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ be two smooth curves with $\gamma_1(b_1) = \gamma_2(a_2)$. Then the sum of γ_1 and γ_2 is the curve
$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t), & a_1 \le t \le b_1, \\ \gamma_2(t + a_2 - b_1), & b_1 \le t \le b_1 + b_2 - a_2. \end{cases}$$
For any continuous function $u : \mathbb{C} \to \mathbb{C}$, we have
$$\int_{\gamma_1+\gamma_2} u(z)dz = \int_{\gamma_1} u(z)dz + \int_{\gamma_2} u(z)dz.$$
- Let $g : [a, b] \to \mathbb{C}$ be a continuous function. We have
$$\left| \int_a^b g(t)dt \right| \le \int_a^b |g(t)|dt.$$
- For any smooth curve $\gamma : [a, b] \to \mathbb{C}$. The length of γ is defined by
$$\text{length }(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$
For any continuous function $u : \mathbb{C} \to \mathbb{C}$, we have
$$\left| \int_\gamma u(z)dz \right| \le \max_{z\in\gamma} |u(z)| \cdot \text{ length }(\gamma).$$

Example: Give an upper bound to
$$\left| \int_\gamma e^{-z}dz \right|$$
where γ is the vertical segment from $-i + 1$ to $i + 1$.
Solution: If $|f(z)| \le M$ for all z on a contour γ of length L, then
$$\left| \int_\gamma f(z)dz \right| \le M \cdot L$$

1. Determine M:
   Given $f(z) = e^{-z}$, on the vertical segment γ from $-i + 1$ to $i + 1$, the real part of z is constant and equal to 1 . Thus, the magnitude of $f(z)$ is maximized when the exponential term is maximized. Since the exponential of a negative real number is always between 0 and 1, the maximum magnitude of e^{-z} on γ is e^{-1}. Therefore, $M = e^{-1}$.
2. Determine L:
   The length of the vertical segment γ from $-i + 1$ to $i + 1$ is 2 units (from $-i$ to i). Using the ML-inequality:
$$\left| \int_\gamma e^{-z}dz \right| \le e^{-1} \cdot 2 = 2e^{-1}$$

Thus, an upper bound for the integral is $2e^{-1}$.

1.13 Green theorem

Let Ω be a domain in $\mathbb{C}$ whose boundary Γ consists of a finite number of disjoint, piecewise smooth closed curves $\gamma_1, \ldots, \gamma_n$.

Orient Γ positively with respect to Ω : parametrize Γ such that Ω remains on the left as we walk along Γ.

Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function. We write $f(z) = p(z) + iq(z)$. Assume f has continuous partial derivatives with respect to x and y. More precisely, the partial derivatives of f are defined by

$$\frac{\partial f}{\partial x} = \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial p}{\partial y} + i\frac{\partial q}{\partial y}.$$

We assume that $\frac{\partial p}{\partial x}, \frac{\partial q}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial q}{\partial y}$ exist and continuous on $\mathbb{R}^2$.

Theorem (Green theorem)**:**

$$\int_\Gamma f(z)dz = i\iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dxdy$$

2 Introduction to analytic function

2.1 Analytic and harmonic functions, and the Cauchy-Riemann equations

2.1.1 Analytic function

Definition(Analytic on a domain)**:** Let f be a function defined on a domain D in $\mathbb{C}$. For $z_0 \in D$, f is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exits; the limit, if it exists, is denoted by $f'(z_0)$. If f is differentiable at each point in D, then f is called analytic in D.

Definition(Entire)**:** A function analytic on the whole complex plane is called entire.

Properties of analytic functions

1. Suppose f is differentiable at a point z_0 in a domain D. Then f is continuous at z_0.
2. Let f and g be two analytic functions defined on a domain D. Then
   - $f + g$ is analytic on D and
   $$(f+g)' = f' + g'$$
   - fg is analytic on D and
   $$(fg)' = f'g + fg'$$
   - for any $z_0 \in D$ such that $g(z_0) \neq 0$, $\frac{f}{g}$ is differentiable at z_0 and
   $$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)\,g(z_0) - f(z_0)\,g'(z_0)}{g^2(z_0)}$$
   - Let h be an entire function. Then $h \circ f$ is analytic on D and
   $$(h \circ f)'(z) = h'(f(z))f'(z) \text{ for any } z \in D.$$

We have some common analytic functions:

- Any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is an entire function.
- A rational function $r = p/q$, where p and q are polynomials, is analytic on any domain containing no zero of q.
- The exponential function $f(z) = e^z$ is an entire function.

Complex numbers have both real and imaginary parts. When we talk about functions of a complex variable, say $f(z)$, where $z = x + iy$, it's natural to wonder how such functions behave and how to differentiate them. The differentiation becomes a bit nuanced compared to real functions because of the presence of the imaginary unit.

2.1.2 Cauchy–Riemann equations

Let's assume $f(z)=u(x,y)+iv(x,y)$, where u and v are real-valued functions. For f to be differentiable, changes in f (due to a small change in z) must be consistent regardless of the direction of the change in the complex plane.

The main motivation behind the Cauchy-Riemann equations is to find a condition that ensures a complex function is differentiable (or analytic). The equations serve as a bridge between the complex function's behavior and the behavior of its real and imaginary parts. If the Cauchy-Riemann equations hold at a point and if certain continuity conditions are satisfied, then the function is differentiable at that point.

Definition(Cauchy–Riemann equations): Suppose that $f=u+iv$, the Cauchy–Riemann equations are

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Theorem (Cauchy–Riemann equations theorem)**:** Suppose that $f=u+iv$ is analytic on a domain D. Then throughout D,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Proof. ***Supposition:*** Let's assume that the complex derivative of $f(z)$ at a point z exists, which means:

$$f'(z)=\lim_{\zeta\to 0}\frac{f(z+\zeta)-f(z)}{\zeta}$$

This existence implies that for every $\epsilon>0$, there's a $\rho>0$ such that for all $|\zeta|<\rho$:

$$\left|f'(z)-\frac{f(z+\zeta)-f(z)}{\zeta}\right|<\epsilon$$

Using Definitions: Write f as $f=u+iv$ and z as $z=x+iy$.
Considering Real Changes: When ζ is purely real, this means $\zeta=\xi$, where ξ is some real number. Now, remember our definition of the complex function and its derivative:

$$f(z)=u(x,y)+iv(x,y)$$
$$f'(z)=\lim_{\zeta\to 0}\frac{f(z+\zeta)-f(z)}{\zeta}$$

Plug in the purely real change:

$$f'(z)=\lim_{\xi\to 0}\frac{f(x+\xi+iy)-f(x+iy)}{\xi}$$

Since f is $u+iv$, we can differentiate each part separately with respect to x (because the change is purely in the x direction). This results in:

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

Considering Imaginary Changes: If ζ is purely imaginary, then $\zeta=i\eta$, where η is real. Using our definition again:

$$f'(z)=\lim_{\eta\to 0}\frac{f(x+i(y+\eta))-f(x+iy)}{i\eta}$$

Notice the denominator being $i\eta$. We can simplify the fraction by multiplying both the numerator and the denominator by $-i$. This will change the denominator to $-\eta$. Thus, the rate of change is in the purely imaginary direction, and when we differentiate u and v in this direction (with respect to y), we get:

$$f'(z)=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

Combining Both: Now, if f is differentiable at z, then its derivative should be the same, irrespective of the direction of the approach. Thus, these two results for the derivative must be equal:

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

Equating the real parts gives:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

Equating the imaginary parts gives:

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

■

Remark: The converse is not necessarily true.

Theorem (Converse of Cauthy-Riemann theorem)**:** Let f be a function defined on a domain D. Write $f = u + iv$. For $z_0 \in D$, suppose

1. $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial x}{\partial x}, \frac{\partial v}{\partial y}$ are continuous in a disc centered at z_0
2. u and v satisfy the Cauchy-Riemann equations at z_0.

Then f is differentiable at z_0. In this case, the complex derivative of $f(z)$ is equal to any of the following expressions:

$$f'(z) = u_x + iv_x = v_y - iu_y$$

Definition(Harmonic and Laplace equation)**:** Let $u(x, y)$ be a continuous function defined on a domain D in $\mathbb{R}^2$. Suppose that u has continuous first and second partial derivatives. Then u is called harmonic on D if it satisfies the Laplace's equation on D :

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Here are some common differentiations:

- $(\sin z)' = \cos z$
- $(\cos z)' = -\sin z$
- $(\sinh z)' = \cosh z$
- $(\cosh z)' = \sinh z$
- $(\tan z)' = \sec^2 z$
- $(\arcsin z)' = \left(1 - z^2\right)^{-1/2}$
- $(\arctan z)' = \left(1 + z^2\right)^{-1}$
- $(e^z)' = e^z$

2.2 Power series

Definition(Power series)**:** A power series in z is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n,$$

where $a_0, a_1, \ldots$ are complex numbers, called the coefficients of the series; z_0 is a complex number, called the center of the series.

Theorem (Absolutely convergent)**:** Suppose there is some $z_1 \neq z_0$ such that $\sum a_n \left(z_1 - z_0\right)^n$ converges. Then for each z with $|z - z_0| < |z_1 - z_0|$, the series $\sum a_n \left(z - z_0\right)^n$ is absolutely convergent.

For any power series $\sum_0^{\infty} a_n \left(z - z_0\right)^n$, there are always three mutually exclusive possibilities:

1. The series $\sum a_n \left(z - z_0\right)^n$ converges only for $z = z_0$.
2. The series $\sum a_n \left(z - z_0\right)^n$ converges for all z.
3. The series $\sum a_n \left(z - z_0\right)^n$ converges for some $z \neq z_0$, but not for all z.

Definition(Radius of convergence)**:** The convergence radius R of the series $\sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n$ is the smallest number $R \in [0, \infty]$ such that

$$\left\{z \in \mathbb{C} : \sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n \text{ converges}\right\} \subset \left\{z \in \mathbb{C} : |z - z_0| \leq R\right\}.$$

We have three different possibilities for convergence radius, which are:

- if the power series converges only for $z = z_0$, then $R = 0$.
- if the power series converges for all z, then $R = \infty$.
- if the power series converges for some z, then R can be a positive number or infinity.

Theorem: Consider the power series $\sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n$. Let R be its radius of convergence. We have the followings.

1. If $\lim_{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists, then
$$\frac{1}{R}=\lim_{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_n}\right|$$
2. If $\lim_{n \rightarrow \infty} \sqrt[n]{\left|a_n\right|}$ exists, then
$$\frac{1}{R}=\lim_{n \rightarrow \infty} \sqrt[n]{\left|a_n\right|}$$

Example: Consider the power series $\sum_{n=0}^{\infty} z^n$. Find its radius of convergence.
Solution: This is a geometric series with $a_n=1$ for all n and $z_0=0$. To find the radius of convergence R, we can use the formula provided:
$$\frac{1}{R}=\lim_{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_n}\right|$$
Here, since a_n is 1 for all n, the ratio $\frac{a_{n+1}}{a_n}$ is:
$$\frac{a_{n+1}}{a_n}=\frac{1}{1}=1$$
The limit of this ratio as n approaches infinity is simply 1 :
$$\lim_{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_n}\right|=1$$
So, we have:
$$\frac{1}{R}=1$$
Therefore, the radius of convergence R is:
$$R=\frac{1}{1}=1$$
The power series $\sum_{n=0}^{\infty} z^n$ converges when $|z|<1$ and diverges when $|z|>1$. The radius of convergence is 1 .
Example: Consider the power series $\sum_{n=0}^{\infty} \frac{i^n z^{3 n}}{2^n}$. Find its radius of convergence.
Solution: we can use the formula provided:
$$\frac{1}{R}=\lim_{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_n}\right|$$
Here, the general term of the series is $a_n=\frac{i^n}{2^n}$, so the ratio $\frac{a_{n+1}}{a_n}$ for consecutive terms is:
$$\begin{aligned}
\frac{a_{n+1}}{a_n} & =\frac{\frac{i^{n+1}}{2^{n+1}}}{\frac{i}{2}^n} \\
& =\frac{i^{n+1} \cdot 2^{2^n}}{i^n \cdot 2^{n+1}} \\
& =\frac{i}{2}
\end{aligned}$$
The absolute value of this ratio is:
$$\left|\frac{i}{2}\right|=\frac{1}{2}$$
Since the modulus of i is 1 . The limit does not depend on n and is already known, so we can state that:
$$\frac{1}{R}=\frac{1}{2}$$
Therefore, the radius of convergence R is:
$$R=2$$
The power series $\sum_{n=0}^{\infty} \frac{i^n z^{3 n}}{2^n}$ converges when $\left|z^3\right|<2$, which is the same as $|z|<\sqrt[3]{2}$.

Theorem (Analytic function and power series)**:** Consider a power series $f(z)=\sum_{n=0}^{\infty} a_n\left(z-z_0\right)^n$. Assume its radius of convergence $R \in(0, \infty]$. Then f is analytic on the open disc $\left\{z \in \mathbb{C}:\left|z-z_0\right|<R\right\}$, and
$$f^{\prime}(z)=\sum_{n=1}^{\infty} n a_n\left(z-z_0\right)^{n-1} .$$
Proof. To prove the theorem, let's follow the provided steps:

1. We first show that $g(z)$ converges on the open disc $\{z \in \mathbb{C} : |z - z_0| < R\}$. For $|z - z_0| < R$, choose any r such that $|z - z_0| < r < R$. The power series $f(z)$ converges absolutely for $|z - z_0| \leq r$, hence by the comparison test, so does any series with smaller terms. Consider the series $g(z)$ whose general term is $na_n (z - z_0)^{n-1}$. We compare it with $n |a_n| r^{n-1}$, a term-by-term absolute comparison with the series whose n-th term is $n |a_n| r^n$ which is the derivative of the series $\sum n |a_n| r^n$ that converges absolutely since it is the derivative of a convergent power series within its radius of convergence. Thus $g(z)$ converges absolutely for $|z - z_0| < r$ and, by extension, for $|z - z_0| < R$.

2. Show that $g(z)$ is the derivative of $f(z)$.

   (a) Compute $f(z_1 + h) - f(z_1)$. Using the binomial formula, we have:

   $$(z_1 + h - z_0)^n = \sum_{j=0}^{n} \binom{n}{j} (z_1 - z_0)^{n-j} h^j$$

   Substituting this into the expression for $f(z_1 + h)$, we get:

   $$\begin{aligned} f(z_1 + h) &= \sum_{n=0}^{\infty} a_n (z_1 + h - z_0)^n \\ &= \sum_{n=0}^{\infty} a_n \left(\sum_{j=0}^{n} \binom{n}{j} (z_1 - z_0)^{n-j} h^j \right) \end{aligned}$$

   Now, $f(z_1)$ is:

   $$f(z_1) = \sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$$

   Subtracting $f(z_1)$ from $f(z_1 + h)$, we get:

   $$f(z_1 + h) - f(z_1) = \sum_{n=0}^{\infty} a_n \left(\sum_{j=1}^{n} \binom{n}{j} (z_1 - z_0)^{n-j} h^j \right)$$

   Note that the $j = 0$ term cancels out in the subtraction.

   (b) Compute $\frac{f(z_1+h)-f(z_1)}{h} - g(z_1)$. We can express the difference quotient as:

   $$\frac{f(z_1 + h) - f(z_1)}{h} = \sum_{n=1}^{\infty} a_n \left(\sum_{j=1}^{n} \binom{n}{j} (z_1 - z_0)^{n-j} h^{j-1} \right)$$

   Now, subtracting $g(z_1)$ from this expression:

   $$g(z_1) = \sum_{n=1}^{\infty} na_n (z_1 - z_0)^{n-1}$$

   We get:

   $$\frac{f(z_1 + h) - f(z_1)}{h} - g(z_1) = \sum_{n=1}^{\infty} a_n \left(\sum_{j=2}^{n} \binom{n}{j} (z_1 - z_0)^{n-j} h^{j-1} \right)$$

   This expression includes only terms of h to the power of at least 1 , so it vanishes as $h \to 0$.

   (c) Estimate $\left| \frac{f(z_1+h)-f(z_1)}{h} - g(z_1) \right|$. As $h \to 0$, the terms h^{j-1} for $j \geq 2$ become arbitrarily small. Since the series for $f(z)$ converges absolutely on the disc $|z - z_0| < R$, the sums involved in the expression above are uniformly convergent on the disc. Therefore, as $h \to 0$, the magnitude of the expression $\left| \frac{f(z_1+h)-f(z_1)}{h} - g(z_1) \right|$ tends to zero

   And thus:

   $$f'(z_1) = g(z_1)$$

   This holds for every z_1 in the disc $|z - z_0| < R$, hence f is analytic on the disc and $f'(z) = g(z)$. This completes the proof.

■

Example: Consider the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

1. Show that its radius of convergence is ∞.
2. Show that $e^z = f(z)$ for any $z \in \mathbb{C}$.

Solution:

1. The radius of convergence R of a power series can be found using the ratio test. For the given series
$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
we consider the limit:
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}} \right|$$
Simplifying the expression gives us:
$$\lim_{n\to\infty} \left| \frac{z^{n+1} \cdot n!}{z^n \cdot (n+1)!} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right|$$
As n approaches infinity, the limit of $\frac{z}{n+1}$ approaches 0 for any finite z. Therefore, the radius of convergence R is infinite.

2. The exponential function e^z is defined by its power series:
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
This is precisely the form of the given function $f(z)$. Hence, by the definition of the exponential function, we have $f(z) = e^z$ for any $z \in \mathbb{C}$. This is consistent with the entire function property of the exponential function, which is analytic everywhere in the complex plane. Therefore, $f(z)$, as defined by the power series, is equal to e^z and has an infinite radius of convergence, meaning it converges for all z in the complex plane.

Theorem: Consider a power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. Assume its radius of convergence $R \in (0, \infty]$. Then on the open disc $\{z \in \mathbb{C} : |z - z_0| < R\}$, f is infinitely differentiable, and
$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n (z - z_0)^{n-k}, \quad k = 1, 2, \ldots$$

2.3 Cauchy's theorem and Cauchy's formula

Theorem (Cauchy's theorem)**:** Suppose f is analytic on a domain $D \subset \mathbb{C}$. Let γ be a piecewise smooth simple closed curve in D whose inside Ω is also in D. Then we have
$$\int_\gamma f(z)dz = 0.$$

Example: Suppose f is analytic on a domain $D \subset \mathbb{C}$. Let γ be a piecewise smooth simple closed curve in D whose inside Ω is also in D. Then we have
$$\int_\gamma f(z)dz = 0.$$

Solution: According to Cauchy's theorem, for a function f that is analytic on a domain D in $\mathbb{C}$, and a piecewise smooth simple closed curve γ in D whose inside Ω is also in D, the integral $\int_\gamma f(z)dz = 0$.
In our case, the function $f(z) = \frac{z}{(z-2)^2}$ is analytic everywhere except at $z = 2$. Since the contour $|z| = 1$ is a circle of radius 1 centered at the origin, it does not enclose the point $z = 2$ where the function has a singularity. Therefore, the inside of this contour, along with the contour itself, is within the domain of analyticity of $f(z)$.
Applying Cauchy's theorem, we can conclude that the integral of $f(z)$ along this contour is zero:
$$\int_{|z|=1} \frac{z}{(z-2)^2} dz = 0.$$
This result holds because the singular point $z = 2$ is outside the contour $|z| = 1$, and thus the conditions of Cauchy's theorem are satisfied.

Definition(Simple connected domain)**:** A domain D is simply-connected if, whenever γ is a simple closed curve in D, the inside of γ is also a subset of D.

Example:

- The disc $\{z : |z - z_0| < R\}$ is simply-connected, as is the horizontal strip $\{z : a < \operatorname{Im} z < b\}$.
- Any convex domain Ω is simply-connected.

Theorem: If f is analytic in a simply-connected domain D, then there is an anatic function F on D with $F' = f$ throughout D.

Corollary: Let f be analytic on a simply-connected domain D, and let γ be a piecewise smooth closed curve in D. Then

$$\int_\gamma f(z)dz = 0.$$

- **Scope of Applicability:** Cauchy's Theorem is more specific in its requirements regarding the curve γ and the region it encloses. The corollary, on the other hand, applies more broadly to any closed curve in a simply-connected domain, without the need to consider the interior of the curve specifically.
- **Analyticity Requirements:** Both statements require the function f to be analytic, but Cauchy's Theorem also requires that f be analytic inside the region enclosed by γ. The corollary, by focusing on simply-connected domains, inherently ensures that f is analytic on all points within any closed curve in the domain.
- **Type of Domain:** The corollary is explicitly concerned with simply-connected domains, while Cauchy's Theorem can apply to more complex domains as long as the conditions regarding the curve γ and its interior are met.

Corollary: Let f be an analytic function on a simply-connected domain D. And let γ be a piecewise smooth curve in D. Then

$$\int_\gamma f(z)dz = F(\text{ end pt of } \gamma) - F(\text{ initial pt of } \gamma)$$

where F is an antiderivative of f.

Example: Compute the following integral

$$\int_\gamma e^z dz,$$

where γ is the semicircle from -1 to 1 passing through i.
Solution: To evaluate the integral $\int_\gamma e^z dz$, where γ is the semicircle from -1 to 1 passing through i, we follow these steps:

1. **Analyticity of e^z:** The function e^z is analytic everywhere in the complex plane. It has no singularities and its derivative exists at every point.
2. **Domain and Curve:** The complex plane is a simply-connected domain. The curve γ, a semicircle from -1 to 1 passing through i, is a piecewise smooth curve in this domain.
3. **Antiderivative of e^z:** The antiderivative of e^z is e^z itself, since $\frac{d}{dz}e^z = e^z$.
4. **Application of the Corollary:** The corollary states that $\int_\gamma f(z)dz = F(\text{end point of } \gamma) - F(\text{initial point of } \gamma)$, where F is an antiderivative of f. For our case, the initial point of γ is -1 and the end point is 1.
5. **Computing the Integral:**
$$\int_\gamma e^z dz = F(1) - F(-1) = e^1 - e^{-1} = e - \frac{1}{e}.$$

Thus, the integral $\int_\gamma e^z dz$ along the semicircle from -1 to 1 passing through i is $e - \frac{1}{e}$.
Example: Compute the following integral

$$\int_\gamma \frac{z}{z+1} dz$$

where γ is any piecewise smooth curve in the domain $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, which joins $-1 + 2i$ to $1 + 2i$.
Solution: To evaluate the integral $\int_\gamma \frac{z}{z+1} dz$, where γ is a curve joining $-1+2i$ to $1+2i$ in the domain $\{z \in \mathbb{C} : \text{Im} z > 0\}$, we follow these steps:

1. **Analyticity of $f(z) = \frac{z}{z+1}$:** The function is analytic in the given domain since it has no singularities where Im$z > 0$.
2. **Antiderivative of $f(z)$:** An antiderivative $F(z)$ of $f(z)$ can be found as $F(z) = z\log(z+1) - \int \log(z+1)dz$. For simplicity, we consider $F(z) = z\log(z+1)$, where $\log$ represents the complex logarithm.
3. **Applying the Corollary:** According to the corollary, $\int_\gamma f(z)dz = F(\text{end point of } \gamma) - F(\text{initial point of } \gamma)$. For our case, the initial point of γ is $-1 + 2i$ and the end point is $1 + 2i$.

4. **Computing the Integral:**

$$\int_\gamma \frac{z}{z+1} dz = F(1+2i) - F(-1+2i) = (1+2i)\log(2+2i) - (-1+2i)\log(1+2i).$$

Thus, the value of the integral $\int_\gamma \frac{z}{z+1} dz$ is $(1+2i)\log(2+2i) - (-1+2i)\log(1+2i)$, where the logarithm is the complex logarithm.

Theorem: Theorem 2. If f is analytic in a simply-connected domain D, then there is analytic function F on D with $F' = f$.

Theorem (Cauchy's formula)**:** Suppose that f is analytic on a domain D and that γ is a piecewise smooth, positively oriented simple closed curve in D whose inside Ω also lies in D. Then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \quad \text{for all } z \in \Omega$$

Example: Compute the line integral

$$\int_\gamma \frac{\sin z}{z} dz,$$

where $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ and it is negatively oriented.
Solution: To evaluate the integral $\int_\gamma \frac{\sin z}{z} dz$, where γ is the circle $|z| = 1$ in the complex plane, negatively oriented, we use the Cauchy Integral Formula. This formula states that for a function f analytic on a domain D, and a piecewise smooth, positively oriented simple closed curve γ in D, with its inside Ω also in D, then for all $z \in \Omega$:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

In our case, $f(z) = \sin z$, which is analytic everywhere in the complex plane, and γ is the unit circle $|z| = 1$. We choose $z = 0$ since it is inside the unit circle and simplifies the integral due to the singularity of $\frac{\sin z}{z}$ at $z = 0$. The Cauchy Integral Formula gives us:

$$\sin(0) = \frac{1}{2\pi i} \int_\gamma \frac{\sin \zeta}{\zeta} d\zeta.$$

Since $\sin(0) = 0$, the integral $\int_\gamma \frac{\sin \zeta}{\zeta} d\zeta = 0$ when γ is positively oriented. However, since our curve γ is negatively oriented, we have:

$$\int_\gamma \frac{\sin z}{z} dz = -0 = 0.$$

Therefore, the value of the integral $\int_\gamma \frac{\sin z}{z} dz$ over the negatively oriented circle $|z| = 1$ is 0.
Example: Compute the definite integral

$$\int_0^{2\pi} \frac{1}{2 + \cos\theta} d\theta.$$

Solution: To find the value of the integral $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$, we use a complex analysis approach.
Step 1: Transform to Complex Integral

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad z = e^{i\theta}$$
$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right), \quad z = e^{i\theta}$$
$$d\theta = \frac{1}{i}\frac{dz}{z}$$

This transforms the integral to:

$$\frac{d\theta}{2+\sin\theta} = \frac{2dz}{iz\left(4 - iz + \frac{i}{z}\right)} = \frac{2dz}{z^2 + 4iz - 1}$$

Step 2: Factorize the Denominator

$$z^2 + 4iz - 1 = [z - i(\sqrt{3} - 2)][z + i(\sqrt{3} + 2)]$$

Set $p = i(\sqrt{3} - 2)$ and $q = -i(\sqrt{3} + 2)$. p lies within the circle $|z| = 1$, while q lies outside.
Step 3: Apply Cauchy's Formula Since $(z - q)^{-1}$ is analytic in the disc $|z| < \sqrt{3} + 2$, Cauchy's Formula gives:

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{(z-q)(z-p)} = \frac{1}{p-q} = \frac{1}{2\sqrt{3}i}$$

Step 4: Relate Back to Original Integral

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{dz}{(z-q)(z-p)}=\frac{1}{2\pi i}\int_{0}^{2\pi}\frac{ie^{i\theta}d\theta}{2ie^{i\theta}(2+\sin\theta)}=\frac{1}{4\pi i}\int_{0}^{2\pi}\frac{d\theta}{2+\sin\theta}$$

This yields

$$\int_{0}^{2\pi}\frac{d\theta}{2+\sin\theta}=\frac{2\pi}{\sqrt{3}}$$

2.4 Consequences of Cauchy's formula

Theorem: Suppose f is analytic on a domain D. Then for any $z_0\in D$ and any closed disc $\{z\in\mathbb{C}:|z-z_0|\leq R\}\subset D, f$ has a power series expansion

$$f(z)=\sum_{k=0}^{\infty}a_k\left(z-z_0\right)^k$$

on the open disc $\{z\in\mathbb{C}:|z-z_0|<R\}$. The coefficients are given by

$$a_k=\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{\left(\zeta-z_0\right)^{k+1}}d\zeta,$$

where γ is the positively oriented circle $\{z\in\mathbb{C}:|z-z_0|=R\}$.

Corollary: Corollary. Suppose f is analytic on a domain D. Then

- f' is also analytic on D;
- f has derivatives of all orders, and each derivative is analytic on D;
- for any $z_0\in D$ and any closed disc $\{z\in\mathbb{C}:|z-z_0|\leq R\}\subset D, f$ has the power series expansion

  $$f(z)=\sum_{k=0}^{\infty}a_k\left(z-z_0\right)^k$$

  on the open disc $\{z\in\mathbb{C}:|z-z_0|<R\}$. The coefficients are given by

  $$a_k=\frac{f^{(k)}\left(z_0\right)}{k!}.$$

- if at some point $z_0\in D, f^{(k)}\left(z_0\right)=0$ for $k=0,1,2,\ldots$, then $f(z)=0$ for any $z\in D$.

Example: Find the power series expansion of $\log(1-z)$ about 0 . Find the largest disc in which the series is valid.
Solution: To find the power series expansion of $\log(1-z)$ about $z=0$, we use the fact that $\log(1-z)$ is analytic in the domain where $1-z$ is not zero or a negative real number. The largest such domain containing the origin is the open unit disc $|z|<1$.
Derivatives and Coefficients:

- The function $\log(1-z)$ has derivatives of all orders in this domain.
- The k-th coefficient a_k of the power series expansion is given by $a_k=\frac{f^{(k)}(0)}{k!}$, where $f^{(k)}(0)$ is the k-th derivative of $\log(1-z)$ evaluated at $z=0$.

The first few derivatives yield:

$$\begin{aligned}
a_0&=0,\\
a_1&=-1,\\
a_2&=-\frac{1}{2},\\
a_3&=-\frac{1}{3},\\
a_4&=-\frac{1}{4}.
\end{aligned}$$

Power Series Expansion:

$$\log(1-z)=\sum_{k=1}^{\infty}-\frac{1}{k}z^k.$$

Radius of Convergence:

- The radius of convergence of this series is determined by the distance from the expansion point to the nearest singularity of $\log(1-z)$.
- Since the nearest singularity is at $z=1$, the radius of convergence is 1.
- Therefore, the series converges in the open unit disc $|z|<1$.

Definition: Let $m \in \mathbb{N}$ be the smallest natural number such that

$$f^{(m)}\left(z_0\right) \neq 0.$$

We say that f has an order of m at z_0.

Lemma: Suppose f is analytic and not identically zero on a domain D. Let $z_0 \in D$ be such that $f\left(z_0\right)=0$. If f has a zero of order m at z_0, then

$$f(z)=\left(z-z_0\right)^m g(z),$$

where g is analytic on D and $g\left(z_0\right) \neq 0$.

Example: Give the order of each of zeros of the function

$$\log(1-z), \quad |z|<1.$$

Solution: We are interested in determining the order of the zero of the function $\log(1-z)$ within the domain $|z|<1$.
Identifying the Zero:

- The function $\log(1-z)$ has a zero at $z=0$ because $\log(1-0)=\log(1)=0$.

Determining the Order of the Zero:

- The order of a zero at a point z_0 is the smallest positive integer n such that the n-th derivative of the function at z_0 is non-zero.
- For $\log(1-z)$, at $z=0$, the function itself is zero.
- The first derivative of $\log(1-z)$ is $-\frac{1}{1-z}$. Evaluating this at $z=0$ gives -1, which is non-zero.

Therefore, since the first derivative of $\log(1-z)$ at $z=0$ is non-zero, the zero at $z=0$ is a simple zero, meaning its order is 1.

Theorem (Morera's Theorem)**:** If f is a continuous function on a domain D and if

$$\int_\gamma f(z) dz=0$$

for every triangle γ that lies, together with its interior, in D, then f is analytic on D.

Theorem (Liouville's theorem)**:** Suppose F is entire and there is a constant $M>0$ such that $|F(z)| \leq M$ for all $z \in \mathbb{C}$. Then F is identically constant.

2.5 Isolated singularities

Definition(Isolated singuarities): An analytic function f has an isolated singularity at a point z_0 if f is analytic on the punctured disc $\{z: 0<|z-z_0|<r\}$ for some $r>0$.

Example: Given the functions, we identify their isolated singularities:
1. $f(z)=\frac{z^2-i^2}{z-i}$**:**

- Simplify the function by factoring: $f(z)=\frac{(z-i)(z+i)}{z-i}$.
- The apparent singularity at $z=i$ is removable, as the factor $z-i$ cancels out.
- The simplified function $f(z)=z+i$ is analytic everywhere; hence, no isolated singularities.

2. $f(z)=\frac{1}{(z+2)^3}$**:**

- The function has a singularity at $z=-2$ (division by zero).
- Since $f(z)$ is analytic elsewhere, $z=-2$ is an isolated singularity.

3. $f(z)=e^{\frac{1}{z+i}}$**:**

- The exponent $\frac{1}{z+i}$ becomes undefined at $z=-i$.

- The function is analytic elsewhere in the complex plane, so $z=-i$ is an isolated singularity.

Proposition: Suppose f is an analytic on a punctured disc $\Omega=\{z: 0<|z-z_0|<r\}$. Assume $\lim_{z \rightarrow z_0} f(z)$ exits. Define

$$g(z)=\left\{\begin{array}{ll} f(z), & z \neq z_0, \\ \lim_{z \rightarrow z_0} f(z), & z=z_0 .\end{array}\right.$$

We have that g is analytic on $\{z:|z-z_0|<r\}$.
Remark: In this case, z_0 is called a removable singularity. We can replace the assumption " $\lim_{z \rightarrow z_0} f(z)$ exits" by $\lim_{z \rightarrow z_0}|f(z_0)|$ bounded, and the same conclusion holds.
Example: Consider

$$f(z)=\frac{z^2-i^2}{z-i} .$$

Find an entire function $g(z)$ such that $g(z)=f(z)$ for all $z \in \mathbb{C} \backslash\{i\}$.
Solution: Given the function $f(z)=\frac{z^2-i^2}{z-i}$, we are to find an entire function $g(z)$ such that $g(z)=f(z)$ for all $z \in \mathbb{C} \backslash\{i\}$.
Simplification of $f(z)$:

- Simplify $f(z)$ by factoring z^2-i^2 (difference of squares):

$$f(z)=\frac{(z-i)(z+i)}{z-i} .$$

- This simplifies to:

$$f(z)=z+i \quad \text{for} \quad z \neq i .$$

Defining $g(z)$:

- Based on the proposition, if the singularity at $z=i$ is removable, then we define $g(z)$ as:

$$g(z)= \begin{cases}f(z), & \text{for} \quad z \neq i, \\ \lim_{z \rightarrow i} f(z), & \text{for} \quad z=i .\end{cases}$$

- Since $f(z)$ simplifies to $z+i$, and the limit as z approaches i equals $2i$.

Entire Function $g(z)$:

- The entire function $g(z)$ is defined for all $z \in \mathbb{C}$ as:

$$g(z)=z+i .$$

- This function is entire and agrees with $f(z)$ for all $z \neq i$.

Therefore, the entire function corresponding to $f(z)$ for all $z \in \mathbb{C} \backslash\{i\}$ is $g(z)=z+i$.
Proposition: Suppose f is analytic on a punctured disc $\Omega=\{z: 0<|z-z_0|<r\}$. If $\lim_{z \rightarrow z_0}|f(z)|=\infty$, then on a smaller punctured disc $\{z: 0<|z-z_0|<r_1\}$ with $r_1<r$, we can write f in the form:

$$f(z)=\frac{H(z)}{(z-z_0)^m}$$

where $m \in \mathbb{N}$ and H is an analytic function on $\{z:|z-z_0|<r_1\}$ with $H(z_0) \neq 0$.
Remark: In this case, z_0 is called a pole of f and m is the order of the pole.
Given a fraction $\frac{f(z)}{g(z)}$, a general strategy to find the poles of $\frac{f(z)}{g(z)}$ is as follows:

1. identify all the zeros in $g(z)$, along with their orders;
2. identify any zeros in $f(z)$ that are also zeros of $g(z)$, along with their orders;
3. for each zero z_0 of $g(z)$, if

$$m:= \text{ order of } g \text{ at } z_0 \text{ - order of } f \text{ at } z_0>0,$$

then z_0 is a pole of $\frac{f}{g}$ of order m. Otherwise, z_0 is not a pole.

Example: Consider the function

$$f(z)=\frac{z}{\sin z} .$$

Find its poles and their order.
Solution: Given the function $f(z)=\frac{z}{\sin z}$, we aim to find its poles and their orders.
Step 1: Identify Zeros of $g(z)$, where $g(z)=\sin z$

- The zeros of $\sin z$ occur at $z = n\pi$ for $n \in \mathbb{Z}$.
- These are simple zeros (order 1).

Step 2: Identify Common Zeros in $f(z)$ and $g(z)$

- The numerator z in $f(z)$ has a simple zero (order 1) at $z = 0$.
- The common zero with $g(z) = \sin z$ is at $z = 0$.

Step 3: Determine the Order of Poles

- For each zero z_0 of $g(z)$, compute:

$$m = \text{order of } g \text{ at } z_0 - \text{order of } f \text{ at } z_0.$$

- At $z = 0$, both $f(z)$ and $g(z)$ have a zero of order 1, so $m = 1 - 1 = 0$. Thus, $z = 0$ is not a pole.
- At $z = n\pi$ for $n \neq 0$, $g(z)$ has a zero of order 1, but $f(z)$ does not have a zero. Hence, $m = 1$. Therefore, each $z = n\pi$ (for $n \neq 0$) is a pole of order 1.

Conclusion: The function $f(z) = \frac{z}{\sin z}$ has poles of order 1 at each $z = n\pi$, where n is any non-zero integer.

2.6 The residue

Definition(Residue): Suppose f is analytic on a punctured disc $\{z : 0 < |z - z_0| < r\}$. Define the residue of f at z_0 by

$$\operatorname{Res}(f; z_0) = \frac{1}{2\pi i} \int_{\gamma_s} f(\zeta) d\zeta$$

where s is any number in $(0, r)$ and $\gamma_s = \{z : |z - z_0| = s\}$ positively oriented.

1. If f is analytic on $0 < |z - z_0| < \gamma$, and f has a pole of order m at $z = z_0$, then:

$$\operatorname{Res}(f; z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}|_{z=z_0} \left[(z - z_0)^m f(z)\right] = \frac{1}{(m-1)!} \lim_{z \to z_0} \left((z - z_0)^m f(z)\right)^{(m-1)}$$

2. If z_0 is a removable singularity of f, then $R(f, z_0) = 0$.

Example: Find the residue of $f(z) = \left(z^2 + 3z - 1\right)/(z + 2)$ at its pole.
Solution: The pole of f is at $z_0 = -2$. Expand the numerator in powers of $z + 2$:

$$z^2 + 3z - 1 = (z + 2)^2 - (z + 2) - 3$$

Therefore $\operatorname{Res}(f; -2) = -3$
Example: Find the residue of $g(z) = e^z/(z - 1)^3$ at $z_0 = 1$,
Solution: Expand e^z in powers of $z - 1$:

$$e^z = ee^{z-1} = e\left(1 + (z - 1) + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \cdots\right)$$

Thus,

$$g(z) = \frac{e}{(z-1)^3} + \frac{e}{(z-1)^2} + \frac{e}{2(z-1)} + \frac{e}{6} + \frac{e}{24}(z - 1) + \cdots$$

so

$$\operatorname{Res}(g; 1) = \frac{e}{2}$$

2.7 Laurent series

The Laurent series of a complex function $f(z)$ is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied.

Theorem: Suppose that a function f is analytic throughout an annular domain $R_1 < |z - c| < R_2$, centred at c, and let γ denote any positively oriented simple closed contour around c and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - c)^n$$

And

$$a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - c)^{n+1}} dz$$

Definition(Principal part of the LS): The principal part of a Laurent series is the series of terms with negative degree, that is

$$\sum_{k=-\infty}^{-1} a_k(z-c)^k$$

Definition(Essential singularity): The point a is called an essential singularity of the function f if the singularity is neither a pole nor a removable singularity.

1. c is a pole of order m if and only if $a_k = 0 \quad \forall k < -m$ and $a_{-m} \neq 0$
2. c is a removable singularity if and only if $a_k = 0 \quad \forall k < 0$
3. c is an essential singularity if and only if $a_k \neq 0$ for infinitely many $k < 0$
4. $\mathrm{Res}(f; c) = a_{-1}$

Theorem: Suppose that f is analytic on some domain D except for a pole of order m at a point z_0 of D. Then f may be written as

$$f(z) = P\left(\frac{1}{z-z_0}\right) + g(z),$$

where P is a polynomial of degree m with zero constant term and g is analytic on D and also at $z_0 . P\left(1/\left(z-z_0\right)\right)$ is the principal part of f at z_0 and is analytic on the whole plane except at z_0; it is also analytic at ∞.

Definition(Principal value): Given the above theorem, $P\left(1/\left(z-z_0\right)\right)$ is the principal value of f at the pole z_0

2.8 Residue theorem and its application in evaluating integrals

Theorem (Residue theorem)**:** Suppose that:

1. f is analytic on a simply connected domain D except for a finite number of isolated singularities at the points $z_1, \ldots, z_n$ of D.
2. γ is a piecewise smooth, positively oriented, simple closed curve in D such that it does not pass through any of the points $z_1, \ldots, z_n$.

Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{z_k \text{ inside } \gamma} \mathrm{Res}\left(f; z_k\right).$$

2.8.1 Integrals of rational functions

Proposition: Suppose P and Q are polynomials that are real-valued on the real axis and for which the degree of Q exceeds the degree of P by 2 or more. If $Q(x) \neq 0$ for all real x, then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_U \mathrm{Res}\left(\frac{P}{Q}; z_j\right),$$

where the sum is taken over all poles of P/Q that lie in the upper half-plane $U = \{z : \mathrm{Im}\, z > 0\}$
Example: Compute

$$\int_{-\infty}^{\infty} \frac{x^2}{\left(1+x^2\right)\left(4+x^2\right)} dx$$

Solution: Solution The polynomials are $F(z) = z^2$ and $Q(z) = \left(1+z^2\right)\left(4+z^2\right)$, respectively, and Q has zeros at $z_1 = i$ and $z_2 = 2i$ in U, the upper half-plane. Now

$$\frac{P(z)}{Q(z)} = \frac{z^2}{(z-i)(z+i)(z-2i)(z+2i)}.$$

We have

$$\mathrm{Res}\left(\frac{P}{Q}; i\right) = \frac{-1}{(2i)3} = \frac{-1}{6i}$$

and

$$\mathrm{Res}\left(\frac{P}{Q}; 2i\right) = \frac{-4}{(-3)(4i)} = \frac{1}{3i}$$

Hence,

$$\int_{-\infty}^{\infty} \frac{x^2}{\left(1+x^2\right)\left(4+x^2\right)} dx = 2\pi i\left[\frac{-1}{6i} + \frac{1}{3i}\right] = \frac{\pi}{3}$$

2.8.2 Integrals over the real axis involving trigonometric functions

Lemma: The inequality

$$\left| \int_{\substack{|z|=R \\ \text{lm}\, z \geqslant 0}} e^{iz} dz \right| < \pi \quad \forall R$$

$$\int_0^\pi Re^{-R\sin t} dt < \pi \quad \forall R$$

This is called Jordan lemma.

Example: Compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+\alpha^2} dx, \quad \alpha > 0$$

Solution: Set

$$f(z) = \frac{e^{iz}}{z^2+\alpha^2}$$

f has a pole at $i\alpha$ in U with residue

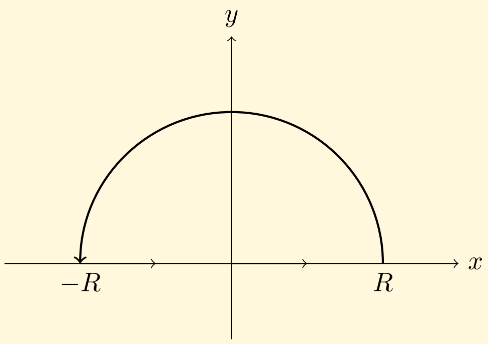
$$\text{Res}(f; i\alpha) = \frac{e^{-\alpha}}{2i\alpha}$$

So by Residue theorem:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+\alpha^2} dx = \pi \frac{e^{-\alpha}}{\alpha}$$

Now we are going to prove this. Let B_R be the contour below, with the circle being γ_R so

$$\int_{B_R} \frac{e^{iz}}{z^2+\alpha^2} = \int_{-R}^{R} \frac{e^{ix}}{x^2+\alpha^2} dx + \int_{\gamma_R} \frac{e^{iz}}{z^2+\alpha^2} dz$$



$$\left| \int_{\gamma_R} \frac{e^{iz}}{z^2+\alpha^2} dz \right| = \left| \int_0^\pi \frac{e^{iRe^{it}}}{R^{2it}+\alpha^2} \cdot Ri^{ie^{it}} dt \right| \leq \int_0^\pi \frac{e^{-R\sin t}}{|R^2e^{2it}+\alpha^2|} Rdt \leq \frac{R}{R^2-\alpha^2} \int_0^\pi e^{-R\sin t} dt \leq \frac{R}{R^2-\alpha^2} \cdot \pi$$

As R approaches ∞, $\frac{R}{R^2-\alpha^2} \cdot \pi$ approaches 0. Thus, we take R approaches ∞, we have:

$$\int_{B_R} \frac{e^{iz}}{z^2+\alpha^2} = \int_{-R}^{R} \frac{e^{ix}}{x^2+\alpha^2} dx = \pi \frac{e^{-\alpha}}{\alpha}$$

Thus:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+\alpha^2} dx = \pi \frac{e^{-\alpha}}{\alpha}$$

Example: In the foregoing result, replace x with βx and set $\delta = \alpha/\beta$. This results in the formula

$$H(\delta, \beta) = \int_{-\infty}^{\infty} \frac{\cos \beta x}{x^2+\delta^2} dx = \left(\frac{\pi}{\delta}\right) e^{-\delta\beta}, \quad \beta, \delta > 0$$

And we have:

$$\frac{\partial H}{\partial \beta} = -\int_{-\infty}^{\infty} \frac{x \sin \beta x}{x^2+\delta^2} dx = -\pi e^{-\delta\beta}, \quad \beta, \delta > 0$$

Thus:

$$\int_{-\infty}^{\infty} \frac{x \sin \beta x}{x^2+\delta^2} dx = \pi e^{-\delta\beta}, \quad \beta, \delta > 0$$

Take β approaches 1 and δ approaches 0:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

Example: Find the value of

$$\int_{0}^{\infty} \frac{\sin^2 x}{x^2} dx$$

Solution: We use the identity $2\sin^2 x = 1 - \cos 2x = \operatorname{Re}\left(1 - e^{2ix}\right)$. Set:

$$f(z) = \frac{1 - e^{2iz}}{z^2}$$

and take γ to be the contour below because we have a singurity at $z = 0$:

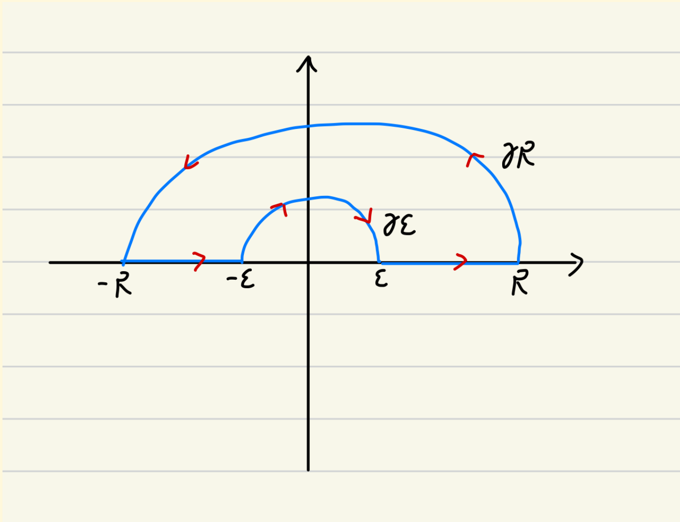


Figure 1: Desired contour

$$\begin{aligned} \int_{B_R} \frac{1 - e^{2iz}}{z^2} dz &= \int_{-R}^{-\varepsilon} \frac{1 - e^{2ix}}{x^2} dx + \int_{\varepsilon}^{R} \frac{1 - e^{2ix}}{x^2} dx \\ &+ \int_{\gamma_\varepsilon} \frac{1 - e^{2iz}}{z^2} dz + \int_{\gamma R}^{1-e^{2iz}} \frac{1 - e^{2iz}}{z^2} dz \end{aligned}$$

$$\left| \int_{\gamma R} \frac{1 - e^{2iz}}{z^2} dz \right| = \left| \int_0^{\pi} \frac{1 - e^{2iRe^{it}}}{R^2 e^{2it}} Rie^{it} dt \right| \leq \int_0^{\pi} \frac{1 + e^{-2R\sin t}}{R} dt \leq \frac{2\pi}{R}$$

Note as R approaches ∞, $\left| \int_{\gamma R} \frac{1-e^{2iz}}{z^2} dz \right|$ approaches 0.

$$\begin{aligned} \int_{\gamma_\varepsilon} \frac{1 - e^{2iz}}{z^2} dz &= -\int_0^{\pi} f\left(\varepsilon e^{it}\right) i\varepsilon e^{it} dt \\ &= -\int_0^{\pi} \frac{1 - e^{2i\gamma(t)}}{\varepsilon^2 e^{2it}} \varepsilon i e^{it} dt \\ &= \frac{-1}{\varepsilon} \int_0^{\pi} e^{-it} \left[-2i\gamma(t) - \frac{(2i\gamma(t))^2}{2} - \cdots\cdots \right] idt \\ &= -\int_0^{\pi} 2dt + O(\varepsilon) \\ &= -2\pi + O(\varepsilon) \end{aligned}$$

Let ε approaches 0, then $O(\varepsilon)$ approaches 0. And by Cauthy theorem we know that $\int_{B_R} \frac{1-e^{2iz}}{z^2} dz = 0$. Thus,

$$0 = \int_{\varepsilon}^{R} \frac{1 - e^{2ix}}{x^2} dx + \int_{-R}^{-\varepsilon} \frac{1 - e^{2ix}}{x^2} dx - 2\pi + O(\varepsilon)$$

Now we take $R \to \infty$ and $\varepsilon \to 0$, we have:

$$0 = \int_{B_R} \frac{1-e^{2iz}}{z^2} dz = \int_{-\infty}^{\infty} \frac{1-e^{2ix}}{x^2} dx + 2\pi$$

$$\theta = \int_{-\infty}^{\infty} \left[\frac{\cos 2x}{x^2} - i \frac{\sin 2x}{x^2} \right] dx - 2\pi$$

Thus:

$$\int_{-\infty}^{\infty} \frac{2\sin^2 x}{x^2} = 2\pi$$

And we have:

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} = \frac{\pi}{2}$$

2.8.3 Integrals involving $\log x$ or fractional powers of x

Example: Compute $\int_0^\infty \frac{\ln x}{(1+x^2)^2} dx$
Solution: Let

$$f(z) = \frac{g(z)}{(1+z^2)^2}$$

on the domain D obtained by deleting the negative imaginary axis; $\ln z$ is determined on D by requiring the imaginary part to have values in the interval $(-\pi/2, 3\pi/2)$ because we want it to be.
We still have the same contour:

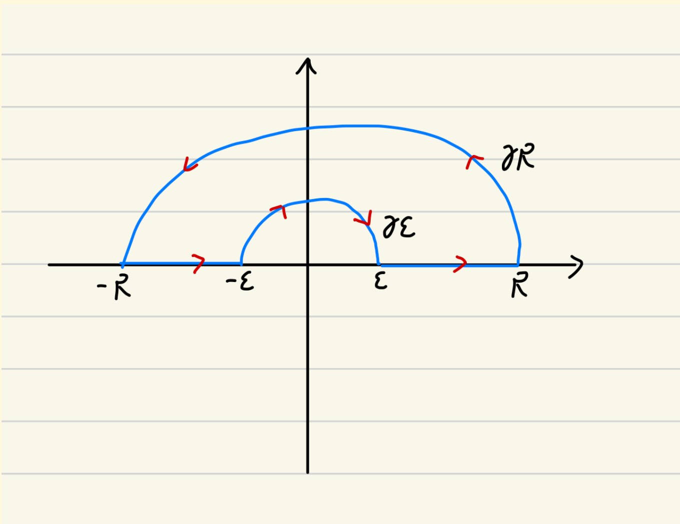


Figure 2: The contour

$z = i$ is The only singularity in the upper half plane.

$$\begin{aligned}
&\operatorname{Res}\left(\frac{g(z)}{(1+z^2)^2}, i\right) \\
&= \frac{1}{11} \frac{d}{dz}\bigg|_{z=i} (z-i^2)^2 \cdot \frac{g(z)}{(z+i)^2(z-i)^2} \\
&= \frac{d}{dz}\bigg|_{z=i} \frac{g(z)}{(z+i)^2} \\
&= \left[\frac{1}{z} \cdot \frac{1}{(z+i)^2} + g(z) \cdot \frac{-1}{(z+i)^3}\right]\bigg|_{z=i} \\
&= \frac{\pi + 2i}{8}
\end{aligned}$$

And by Residue theorem:

$$\int_{B_R} \frac{g(z)}{(1+z^2)^2} dz$$
$$= 2\pi i \operatorname{Res}\left(\frac{g(z)}{(-z^2)^2}, i\right)$$
$$= \frac{\pi^2 i}{4} - \frac{\pi}{2}$$

As before, the integral is written as:

$$\int_{B_R} \frac{g(z)}{(1+z^2)^2} dz$$
$$= \int_{-R}^{-\varepsilon} \frac{g(z)}{(1+z^2)^2} dz + \int_{\varepsilon}^{R} \frac{g(z)}{(1+z^2)^2} dz$$
$$+ \int_{\gamma_\varepsilon} \frac{g(z)}{(1+z^2)^2} dz + \int_{\gamma_R}^{1-e^{2iz}} \frac{g(z)}{(1+z^2)^2} dz$$

We have:

$$\left| \int_{\gamma R} \frac{g(t)}{(1+z^2)^2} dz \right| \leq \max_{\substack{|z|=R \\ Im(z)\geq 0}} \frac{|g(z)|}{|1+z^2|^2} \cdot \pi R$$
$$\leq \frac{\ln R + 3\pi/2}{(R^2-1)^2} \cdot \pi R$$

The above equality approaches 0 as R approaches ∞.

$$\left| \int_{\gamma_\varepsilon} \frac{g(z)}{(1+z^2)^2} dz \right|$$
$$\leq \max_{\substack{|z|=\varepsilon \\ Im(z)>0}} \frac{|g(z)|}{|1+z^2|^2} \cdot \pi\varepsilon$$
$$\leq \frac{\ln \varepsilon + 3\pi/2}{(1-\varepsilon^2)^2} \cdot \pi\varepsilon$$

The above equality approaches 0 as ε approaches 0.
Take the limit as $R \to \infty$ and $\varepsilon \to 0$:

$$\frac{-\pi}{2} + \frac{\pi i}{4} = \int_{-\infty}^{0} \frac{\ln(-x) + i\pi}{(1+x^2)^2} dx + \int_{0}^{\infty} \frac{\ln x}{(1+x^2)^2} dx$$
$$= \int_{0}^{\infty} \frac{\ln x + i\pi}{(1+x^2)^2} dx + \int_{0}^{\infty} \frac{\ln x}{(1+x^2)^2} dx$$
$$= 2\int_{0}^{\infty} \frac{\ln x}{(1+x^2)^2} dx + i\pi \int_{0}^{\infty} \frac{1}{(1+x^2)^2}$$

Therefore:

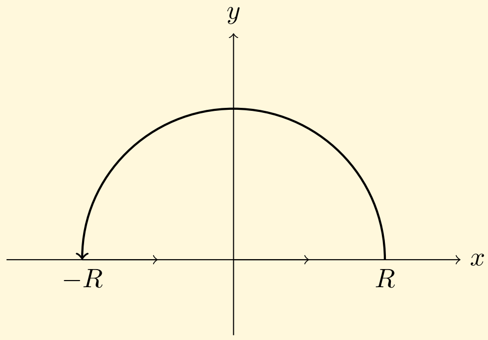
$$\int_{0}^{\infty} \frac{\ln x}{(1+x^2)^2} = \frac{-\pi}{4}$$

Example: Compute:

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{x^4+16} dx$$

Solution: Consider the following contour and $f(z)$:

$$f(z) = \frac{e^{iz} z^3}{z^4+16}$$



We have two poles of order 1:

$$z_1 = \sqrt{2}(-1+i)$$
$$z_2 = \sqrt{2}(1+i)$$

$$\begin{aligned}
\operatorname{Res}\left(\frac{e^{iz}z^3}{z^4+16}, z_1\right) &= \lim_{z\to z_1} \frac{e^{iz}z^3}{z^4+16}\cdot(z-z_1) \\
&= \lim_{z\to z_1} \frac{e^{iz}z^3}{(z-z_2)(z-z_3)(z-z_4)} \\
&= \frac{1}{4}e^{\sqrt{2}(-1+i^i)}
\end{aligned}$$

$$\operatorname{Res}\left(\frac{e^{iz}z^3}{z^4+16}, z_2\right) = \frac{1}{4}e^{\sqrt{2}(-1+i)}$$

Then by residue theorem, $\int_{B_R} \frac{e^{iz}z^3}{z^4+10}dz = 2\pi i\left[\operatorname{Res}\left(\frac{e^{iz}z^3}{z^4+16}, z_1\right) + \operatorname{Res}\left(\frac{e^{iz}z^3}{z^4+16}, z_2\right)\right] = \pi i e^{\sqrt{2}(-1+i)}$ We also know:

$$\int_{B_R} \frac{e^{iz}z^3}{z^4+10}dz = \int_{-R}^{R} \frac{e^{iz}z^3}{z^4+10}dz + \int_{\gamma_R} \frac{e^{iz}}{z^2+\alpha^2}dz$$

And we have:

$$\begin{aligned}
\left|\int \gamma_R \frac{e^{iz}z^3}{z^4+16}dz\right| &= \int_0^\pi \frac{e^{iRe^t}R^3e^{3it}}{R^4e^{40t}+16} iRe^{it}dt \,| \\
&\le \int_0^\pi \frac{e^{-R\sin t}R^3}{|R^4e^{4it}+16|}Rdt \\
&\le \frac{R^3}{R^4-16}\int_0^\pi e^{-R\sin t}Rdt \\
&\le \frac{R^3}{R^4-16}\pi
\end{aligned}$$

The inequality approaches 0 as R approaches ∞. Thus take R approaches ∞:

$$\pi i e^{\sqrt{2}(-1+i)} = \int_{-\infty}^{\infty} \frac{e^{ix}x^3}{x^4+16}dx$$

3 Analytic function as mapping

3.1 The zeros of analytic function

Proposition: Suppose that f is analytic in a domain D. If

1. there are distinct points $z_1, z_2 \ldots$ in D with $f(z_n) = 0, n = 1, 2, \ldots$;
2. the sequence $\{z_n\}$ converges to a point z_0 in D.

Then

$$f(z) = 0 \text{ for all } z \in D.$$

And note that the zeros of a non-constant analytic function are isolated.

Suppose:

- γ piece wise smooth, simple closed curve, positively oriented;
- f analytic on and inside γ, except for (possibly) some isolated poles inside (not on) γ and some zeros inside (not on) γ.

- Let $p_1, \ldots, p_m$ be the poles of f inside γ.
- Let $z_1, \ldots, z_n$ be the zeros of f inside γ.

Write $\operatorname{mult}(z_k)=$ the order of the zero at z_k. Write $\operatorname{mult}(p_k)=$ the order of the pole at p_k. Then:

Theorem:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz=\sum \operatorname{mult}(z_k)-\sum \operatorname{mult}(p_k).$$

Definition(Winding number): Let γ be a piecewise smooth closed curve, and z_0 be a point not on γ. Then the winding number (or index) of γ about z_0 is defined as

$$\operatorname{Ind}(\gamma, z_0)=\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}dz.$$

Roughly speaking: $\operatorname{Ind}(\gamma, z_0)=\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}dz=$ number of times γ winding around z_0 anticlockwise $-$ number of times γ winding around z_0 clockwise.

Example: Let γ be the unit circle, taking the anticlockwise orientation. Understand the geometric meaning of

$$\operatorname{Ind}(\gamma, 0)=\frac{1}{2\pi i}\int_\gamma \frac{1}{z}dz$$

Solution: The integral $\frac{1}{2\pi i}\int_\gamma \frac{1}{z}dz$ effectively measures this winding. As γ is a circle of radius $1, z$ can be parameterized as e^{it} for t ranging from 0 to 2π. The integral thus counts how many times γ winds around the origin. Given the anticlockwise orientation, this will be a positive count.

Therefore, the geometric meaning of $\operatorname{Ind}(\gamma, 0)$ in this context is that the unit circle winds around the origin exactly once in the anticlockwise direction. Hence, the value of this integral will be 1 , indicating a single anticlockwise winding around the point $z_0=0$.

Example: Let γ be the unit circle, taking the anticlockwise orientation. Let $f(z)=z^2$. Describe the curve $f\circ\gamma$.

Solution:

1. Doubling the Angle: The transformation $z\rightarrow z^2$ in the complex plane doubles the angle of each point on the unit circle. If z has an angle t, then z^2 will have an angle of $2t$.
2. Radius Remains Unchanged: Since the unit circle has a radius of 1 , and squaring a complex number of magnitude 1 still results in a magnitude of 1 , the radius of the curve $f\circ\gamma$ remains 1 .
3. Traversing the Circle Twice: Due to the doubling of the angle, as t goes from 0 to 2π, the angle $2t$ actually goes from 0 to 4π. This means that the curve $f\circ\gamma$ will complete two full circles around the origin as we traverse the unit circle once.
4. Anticlockwise Orientation: The original curve γ is oriented anticlockwise, and since the squaring operation doesn't change the direction of traversal, $f\circ\gamma$ will also be traversed anticlockwise.

The curve $f\circ\gamma$ for $f(z)=z^2$ and γ being the unit circle will be another unit circle, but it will be traversed twice as t goes from 0 to 2π. The direction of traversal remains anticlockwise.

Suppose:

- γ piecewise smooth, simple closed curve, positively oriented.
- f analytic on and inside γ, except for (possibly) some isolated poles inside (not on) γ and some zeros inside (not on) γ.

And

1. Let $p_1, \ldots, p_m$ be the poles of f inside γ.
2. Let $z_1, \ldots, z_n$ be the zeros of f inside γ.

Theorem (Argument principal)**:** Write $\operatorname{mult}(z_k)=$ the order of the zero at z_k. Write $\operatorname{mult}(p_k)=$ the order of the pole at p_k.

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz=\sum \operatorname{mult}(z_k)-\sum \operatorname{mult}(p_k)=\operatorname{Ind}(f\circ\gamma, 0)$$

Theorem (Rouche's theorem)**:** Let γ be a piecewise smooth simple closed curve. Suppose that

1. two function $f(z)$ and $g(z)$ analytic inside and on γ.
2. $|f(z)|>|g(z)|$ at each $z\in\gamma$.

Then f and $f+g$ have the same number of zeros (counting multiplicities) inside γ.

Example: Determine the number of zeros (counting multiplicities) of the equation

$$z^7 - 4z^3 + z - 1 = 0.$$

inside the circle $|z| = 1$.
Solution: To determine the number of zeros inside the unit circle for the equation $z^7 - 4z^3 + z - 1 = 0$, we apply Rouche's theorem. We split the polynomial into two parts, $f(z)$ and $g(z)$, and check if $|f(z)| > |g(z)|$ on the circle $|z| = 1$. We choose $f(z) = z^7 - 4z^3$ and $g(z) = z - 1$.
First, we compute the magnitudes of $f(z)$ and $g(z)$ on $|z| = 1$. On the unit circle $z = e^{it}$, the magnitudes simplify to:

$$|f(z)| = \sqrt{-64\sin^4(t) + 64\sin^2(t) + 9},$$
$$|g(z)| = \sqrt{2 - 2\cos(t)}.$$

The maximum of $|g(z)|$ is 2, occurring when $t = \pi$. To apply Rouche's theorem, we need $|f(z)| > |g(z)|$ for all t in $[0, 2\pi]$. We find the minimum value of $|f(z)|$ on the unit circle by solving $\frac{d}{dt}|f(z)| = 0$ and evaluating $|f(z)|$ at these points. The minimum value of $|f(z)|$ turns out to be 3, which is greater than the maximum of $|g(z)|$.
Since $|f(z)| > |g(z)|$ on $|z| = 1$, by Rouche's theorem, $f(z)$ and $f(z) + g(z)$ (the original polynomial) have the same number of zeros inside the unit circle. Since $f(z) = z^7 - 4z^3$ has seven zeros inside the unit circle (all at the origin, counting multiplicities), the polynomial $z^7 - 4z^3 + z - 1$ also has seven zeros inside the unit circle, counting multiplicities.
Example: Determine the number of zeros (counting multiplicities) of the equation $2z^5 - 6z^2 + z + 1 = 0$ in the annulus $1 \le |z| < 2$.
Solution: To determine the number of zeros of the equation $2z^5 - 6z^2 + z + 1 = 0$ in the annulus $1 \le |z| < 2$, we apply Rouche's theorem in two cases: for $|z| = 1$ and for $|z| = 2$.
Case 1: $|z| = 1$
Let $f(z) = 2z^5$ and $g(z) = -6z^2 + z + 1$. For $|z| = 1$, $|f(z)| = |2z^5| = 2$. The magnitude of $g(z)$ can be estimated as $|g(z)| \le |-6z^2| + |z| + |1| = 6 + 1 + 1 = 8$. Thus, $|f(z)|$ may not be greater than $|g(z)|$ on $|z| = 1$, and we need to reconsider our choice of functions for Rouche's theorem.
Case 2: $|z| = 2$
Again, let $f(z) = 2z^5$ and $g(z) = -6z^2 + z + 1$, but now for $|z| = 2$. Here, $|f(z)| = |2z^5| = 64$, and $|g(z)| \le |-6z^2| + |z| + |1| = 25$. Therefore, $|f(z)| > |g(z)|$ on $|z| = 2$.
By Rouche's theorem, $f(z)$ and $f(z) + g(z)$ (the original polynomial) have the same number of zeros inside the circle $|z| = 2$. Since $f(z) = 2z^5$ has five zeros inside this circle, the original polynomial also has five zeros within $|z| = 2$.
Conclusion:
The number of zeros in the annulus $1 \le |z| < 2$ would be the number of zeros inside $|z| = 2$ minus the number inside $|z| = 1$. From the analysis, we conclude that there are five zeros inside $|z| = 2$. However, we could not apply Rouche's theorem successfully for $|z| = 1$, so a different approach or choice of functions is needed to determine the number of zeros inside $|z| = 1$. Without this information, the exact number of zeros in the specified annulus cannot be precisely determined.
Example: Consider a degree n polynomial as foundamental theorem of algebra:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + z_0,$$

where $a_0, \ldots a_n \in \mathbb{C}$ and $a_n \neq 0$. Show that it has n zeros (counting multiplicities).
Solution:

1. *Setup:* Define $f(z) = a_n z^n$ and $g(z) = P(z) - f(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

2. *Applying Rouche's Theorem:* We need to show that for a sufficiently large circle $|z| = R$ centered at the origin, $|f(z)| > |g(z)|$. This will imply that $f(z)$ and $P(z)$ have the same number of zeros inside this circle.

3. *Choosing the Radius R:* Select R large enough such that $|a_n z^n| > |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|$ for all z on the circle $|z| = R$.

4. *Verifying the Condition of Rouche's Theorem:* On $|z| = R$, $|f(z)| = |a_n| R^n$. We need to ensure $|g(z)| < |f(z)|$. By the triangle inequality,
$$|g(z)| \le |a_{n-1}| R^{n-1} + \cdots + |a_1| R + |a_0|.$$
For sufficiently large R, the term $|a_n| R^n$ dominates, ensuring $|f(z)| > |g(z)|$.

5. *Conclusion:* Since $|f(z)| > |g(z)|$ on the circle $|z| = R$, by Rouche's theorem, $f(z)$ and $P(z)$ have the same number of zeros inside this circle. As $f(z) = a_n z^n$ has n zeros, $P(z)$ also has n zeros within the circle. As R can be made arbitrarily large, $P(z)$ has exactly n zeros in the complex plane, proving the theorem.

3.2 Maximum modulus and mean value

Definition(Open function): Let D be a domain in $\mathbb{C}$. A function $f : D \to \mathbb{C}$ is called open if it maps open sets to open sets.

Theorem (Open mapping theorem)**:** Suppose f is a non-constant analytic function on a domain D. Then f is open.

Theorem (Maximum modulus principle)**:** If f is a non-constant analytic function on a domain D, then $|f|$ cannot attain a maximum in D.

Example: Let $f(z) = \frac{z^2}{z+2}$. Find the maximum value of $|f(z)|$ as z varies over the disc $|z| \leq 1$.
Solution: To find the maximum value of $|f(z)|$ for $f(z) = \frac{z^2}{z+2}$ as z varies over the disc $|z| \leq 1$, we use the Maximum Modulus Principle. This principle asserts that if f is a non-constant analytic function on a domain D, then $|f|$ cannot attain a maximum in D. Therefore, the maximum of $|f(z)|$ must occur on the boundary of the disc, i.e., the circle $|z| = 1$. We parameterize z on the circle $|z| = 1$ as $z = e^{i\theta}$, where θ ranges from 0 to 2π. The function becomes:

$$f(e^{i\theta}) = \frac{e^{2i\theta}}{e^{i\theta} + 2}.$$

The absolute value of $f(z)$ simplifies to:

$$|f(z)| = \frac{1}{\sqrt{4\cos(\theta) + 5}}.$$

To find the maximum value of $|f(z)|$, we need to maximize this expression, which is achieved by minimizing the denominator $\sqrt{4\cos(\theta) + 5}$. The denominator is minimized when $\cos(\theta)$ is minimized, which occurs at $\theta = \pi$. Substituting this into the expression, we find the maximum value of $|f(z)|$ on the unit circle.
After calculation, the maximum value of $|f(z)|$ over the disc $|z| \leq 1$ is found to be 1.
Example: Suppose that p is a polynomial of degree n and that $|p(z)| \leq M$ for $|z| = 1$. Show that $|p(z)| \leq M|z|^n$ for $|z| \geq 1$.
Solution: Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n. Consider the polynomial $q(z) = \frac{p(z)}{z^n}$. For $|z| = 1$, $q(z) = p(z)$ as $|z|^n = 1$, so $|q(z)| = |p(z)| \leq M$.
The function $q(z)$ is analytic inside and on the unit circle $|z| = 1$, except at $z = 0$. By the Maximum Modulus Principle, $|q(z)|$ cannot attain its maximum inside this circle, so the maximum of $|q(z)|$ on the closed disc $|z| \leq 1$ occurs on its boundary, $|z| = 1$.
For $|z| > 1$, $|q(z)| = \frac{|p(z)|}{|z|^n}$. Since $|p(z)|$ is bounded by $M|z|^n$, $|q(z)|$ must be less than or equal to M for $|z| > 1$.
Therefore, for $|z| \geq 1$, $|p(z)| \leq M|z|^n$. This conclusion follows since $|p(z)| = |q(z)| \cdot |z|^n$ and $|q(z)| \leq M$ for $|z| \geq 1$.

Theorem (Schwarz's theorem)**:** Let f be an analytic function in the disc $|z| < 1$ such that

- $f(0) = 0$
- $|f(z)| \leq 1$ for any $|z| < 1$

Then

$$|f(z)| \leq |z| \text{ for any } |z| < 1.$$