Tutorial - 6

(Suggested solutions)

① (a) points on the path can be represented by

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= t\vec{i} + t^2\vec{j} + t^3\vec{k}, \quad 0 \le t \le 1$$

$$d\vec{r} = \left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right)dt$$

$$= (\vec{i} + 2t\vec{j} + 3t^2\vec{k})\,dt$$

$$\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$$

$$= (3t^2 + 6t^2)\vec{i} - 14t^2t^3\vec{j} + 20t\,t^6\vec{k}$$

$$= 9t^2\vec{i} - 14t^5\vec{j} + 20t^7\vec{k}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (9t^2\vec{i} - 14t^5\vec{j} + 20t^7\vec{k})\cdot(\vec{i} + 2t\vec{j} + 3t^2\vec{k})\,dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= \left[\frac{9}{3}t^3 - \frac{28}{7}t^7 + \frac{60}{10}t^{10}\right]_0^1$$

$$= 3 - 4 + 6$$

$$= 5.$$

(b) let path from $(0,0,0)$ to $(1,0,0)$ be C_1

" " $(1,0,0)$ to $(1,1,0)$ be C_2

" " $(1,1,0)$ to $(1,1,1)$ be C_3.

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{x=0}^{x=1} \left[(3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}\right]\cdot(dx\vec{i} + \underset{0}{\underset{\parallel}{dy}}\vec{j} + \underset{0}{\underset{\parallel}{dz}}\vec{k})$$

$$= \int_0^1 (3x^2 + \underset{0}{\underset{\parallel}{6y}})dx$$

$$= \frac{3}{3}x^3\Big|_0^1 = 1$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{y=0}^{y=1} \vec{F}\cdot dy\,\vec{j} \qquad \left(\overset{x=1,\ z=0}{dx = dz = 0}\right)$$

$$= \int_0^1 -14y\underset{0}{\underset{\parallel}{z}}\,dy$$

$$= 0$$

$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_{z=0}^{z=1} \vec{F}\cdot dz\,\vec{k} \quad \left(\begin{matrix} x=1,\ y=1 \\ dx=dy=0 \end{matrix}\right)$$

$$= \int_0^1 20xz^2\,dz$$

$$= \frac{20}{3}z^3\Big|_0^1 = \frac{20}{3}$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$

(c) the straight line joining $(0,0,0)$ and $(1,1,1)$ can be given in parametric form by

$$x=t,\quad y=t,\quad z=t\ ,\quad 0\leq t\leq 1$$

$$\vec{r} = t\vec{i} + t\vec{j} + t\vec{k}$$

$$d\vec{r} = (\vec{i} + \vec{j} + \vec{k})\,dt$$

$$\vec{F} = (3t^2+6t)\vec{i} - 14(t)(t)\vec{j} + 20(t)(t)^2\vec{k}$$

$$= (3t^2+6t)\vec{i} - 14t^2\vec{j} + 20t^3\vec{k}$$

$$\vec{F}\cdot d\vec{r} = (3t^2+6t-14t^2+20t^3)\,dt$$

$$= (6t - 11t^2 + 20t^3)\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (6t - 11t^2 + 20t^3)\,dt$$

$$= \left[\frac{6}{2}t^2 - \frac{11}{3}t^3 + \frac{20}{4}t^4\right]_0^1 = \frac{13}{3}$$

(2) $x = t^2+1,\ y = 2t^2,\ z = t^3,\ 1 \le t \le 2$

$$d\vec{r} = \left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \right) dt$$

$$= \left(2t\,\vec{i} + 4t\,\vec{j} + 3t^2\vec{k} \right) dt$$

$$\vec{F} = 3xy\,\vec{i} - 5z\,\vec{j} + 10x\,\vec{k}$$

$$= 3(t^2+1)(2t^2)\,\vec{i} - 5(t^3)\,\vec{j} + 10(t^2+1)\,\vec{k}$$

$$= 6(t^4+t^2)\,\vec{i} - 5t^3\,\vec{j} + 10(t^2+1)\,\vec{k}$$

$$\vec{F}\cdot d\vec{r} = \left[12(t^5+t^3) - 20t^4 + 30(t^4+t^2) \right] dt$$

$$= \left(12t^5 + 10t^4 + 12t^3 + 30t^2 \right) dt$$

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\, dt$$

$$= \left[\frac{12}{6}t^6 + \frac{10}{5}t^5 + \frac{12}{4}t^4 + \frac{30}{3}t^3 \right]_1^2$$

$$= 2(2^6-1) + 2(2^5-1) + 3(2^4-1) + 10(2^3-1)$$

$$= 303$$

③ points on the x-y plane can be represented by

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C F(x,y)\cdot(dx\,\vec{i} + dy\,\vec{j})$$

$$= \int_C F(t)\cdot\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}\right)dt$$

Method ①,

$$\int_C F(x,y)\cdot(dx\,\vec{i} + dy\,\vec{j})$$

$$= \int_C (3xy\,\vec{i} - y^2\vec{j})\cdot(dx\,\vec{i} + dy\,\vec{j})$$

$$= \int_C 3xy\,dx - y^2\,dy$$

(since path is $y = 2x^2$
replace y by $2x^2$
replace dy by $4x\,dx$
$0 \le x \le 1$)

$$= \int_0^1 3x(2x^2)dx - (2x^2)^2 4x\,dx$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx$$

$$= \left[\frac{6}{4}x^4 - \frac{16}{6}x^6\right]_0^1$$

$$= -\frac{7}{6}$$

method ②

let $x = t$, then $y = 2t^2$, $0 \le t \le 1$

$$\int_C F(t) \cdot \left(\frac{dx}{dt} \vec{i} + \frac{dy}{dt} \vec{j} \right) dt$$

$$= \int_0^1 \left[3(t)(2t^2) \vec{i} - (2t^2)^2 \vec{j} \right] \cdot \left(\vec{i} + 4t \vec{j} \right) dt$$

$$= \int_0^1 \left(6t^3 - 16t^5 \right) dt = -\frac{7}{6}$$

(4) (a) If $\nabla \times \vec{F} = 0$, $\vec{F}$ is conservative

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2)\right]\vec{i} + \left[\frac{\partial}{\partial z}(2xy + z^3) - \frac{\partial}{\partial x}(3xz^2)\right]\vec{j}$$

$$+ \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^3)\right]\vec{k}$$

$$= 0\,\vec{i} + (3z^2 - 3z^2)\,\vec{j} + (2x - 2x)\,\vec{k} = 0$$

Thus $\vec{F}$ is conservative.

(b) let ϕ be scalar potential of $\vec{F}$ i.e

$$\vec{F} = \nabla\phi$$

$$= \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$= (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$$

$$\frac{\partial\phi}{\partial x} = 2xy + z^3 \Rightarrow \phi = x^2y + xz^3 + f(y,z) + c$$

$$\frac{\partial\phi}{\partial y} = x^2 \Rightarrow \phi = x^2y + g(x,z) + c$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 \Rightarrow \phi = xz^3 + h(x,y) + c$$

constant term

all these agree if

$$\phi = x^2y + xz^3 + c$$

(c) for conservative field, work done is is given by the scalar potential at start and end points

$$\text{Work} = \phi(3,1,4) - \phi(1,-2,1)$$

$$= \left(3^2(1) + 3(4^3) + c\right) - \left(1(-2) + 1(1)^3 + c\right)$$

$$= 201 - (-1) = 202$$

Alternatively, calculate line integral along any path from $(1,-2,1)$ to $(3,1,4)$.

The straight line from $(1,-2,1)$ to $(3,1,4)$ can be parameterize by

$$\begin{cases} x = 1 + \frac{2}{3}t \\ y = -2 + t \\ z = 1 + t \end{cases} \qquad 0 \le t \le 3$$

$$\vec{F} = \left[2\left(1+\tfrac{2}{3}t\right)(-2+t) + (1+t)^3\right]\vec{i} + \left(1+\tfrac{2}{3}t\right)^2\vec{j} + 3\left(1+\tfrac{2}{3}t\right)(1+t)^2\vec{k}$$

$$= \left(t^3 + \tfrac{13}{3}t^2 + \tfrac{7}{3}t - 3\right)\vec{i} + \left(1 + \tfrac{4}{9}t^2 + \tfrac{4}{3}t\right)\vec{j} + \left(2t^3 + 7t^2 + 8t + 3\right)\vec{k}$$

$$\vec{F}\cdot d\vec{r} = \vec{F}\cdot\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right)dt$$

$$= \vec{F}\cdot\left(\tfrac{2}{3}\vec{i} + \vec{j} + \vec{k}\right)dt$$

$$= \left(\tfrac{2}{3}t^3 + \tfrac{26}{9}t^2 + \tfrac{14}{9}t - 2 + \tfrac{4}{9}t^2 + \tfrac{4}{3}t + 1 + 2t^3 + 7t^2 + 8t + 3\right)dt$$

$$= \left(\tfrac{8}{3}t^3 + \tfrac{93}{9}t^2 + \tfrac{98}{9}t + 2\right)dt$$

$$W = \int_0^3 \left(\tfrac{8}{3}t^3 + \tfrac{93}{9}t^2 + \tfrac{98}{9}t + 2\right)dt$$

$$= \left[\frac{8}{3\times4}t^4 + \frac{93}{9\times3}t^3 + \frac{98}{9\times2}t^2 + 2t\right]_0^3$$

$$= 54 + 93 + 49 + 6$$

$$= 202$$

5. To calculate flux $= \oiint_S \vec{F} \cdot \vec{n}\, dA$

parameterize surface using u, v such that

$$\begin{cases} x = a \cos u \sin v \\ y = a \sin u \sin v \\ z = a \cos v \end{cases} \qquad \begin{matrix} 0 \le u \le 2\pi \\ 0 \le v \le \pi \end{matrix}$$

(u, v are actually θ, ϕ of spherical coordinates)

then any point on the surface of the sphere can be represented by

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= a\cos u \sin v\,\vec{i} + a \sin u \sin v\,\vec{j} + a\cos v\,\vec{k}$$

A surface normal $\vec{N}$ can be given by

$$\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \quad \text{i.e. } r_u \times r_v$$

where $\frac{\partial r}{\partial u} = -a \sin u \sin v\,\vec{i} + a\cos u \sin v\,\vec{j}$

$$\frac{\partial r}{\partial v} = a\cos u \cos v\,\vec{i} + a \sin u \cos v\,\vec{j} - a \sin v\,\vec{k}$$

$$\vec{N} = -a^2 \sin^2 u \sin v \cos v\,\vec{k} + a^2 \sin u \sin^2 v\,(-\vec{j})$$
$$+ a^2 \cos^2 u \sin v \cos v\,(-\vec{k}) - a^2 \cos u \sin^2 v\,\vec{i}$$

$$\bar{N} = -\left(a^2 \cos u \sin^2 v\,\vec{i} + a^2 \sin u \sin^2 v\,\vec{j} + a^2 \sin v \cos v\,\vec{k}\right)$$

$$N = -a \sin v\,(xi + yj + zk)$$

An outward normal is

$$N = a\sin v\left(a\cos u\sin v\,\vec{i} + a\sin u\sin v\,\vec{j} + a\cos v\,\vec{k}\right)$$

$$= a\sin v\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_S \vec{F}\cdot N\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\pi} (z\vec{k})\cdot a\sin v\,(x\vec{i} + y\vec{j} + z\vec{k})\,dv\,du$$

$$= \int_0^{2\pi}\int_0^{\pi} a\sin v\, z^2\,dv\,du$$

$$= \int_0^{2\pi}\int_0^{\pi} a\sin v\, a^2\cos^2 v\,dv\,du$$

$$= a^3\int_0^{2\pi}\left[-\frac{\cos^3 v}{3}\right]_0^{\pi} du$$

$$= a^3\left(\frac{2}{3}\right)2\pi$$

$$= \frac{4}{3}\pi a^3$$