Tutorial - ~~5~~ 4
(Suggested solutions)

(1)

$$\nabla f = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(3x^2y - y^3z^2)$$

$$= \vec{i}\,\frac{\partial}{\partial x}(3x^2y - y^3z^2)$$

$$+ \vec{j}\,\frac{\partial}{\partial y}(3x^2y - y^3z^2)$$

$$+ \vec{k}\,\frac{\partial}{\partial z}(3x^2y - y^2z^2)$$

$$= 6xy\,\vec{i} + (3x^2 - 3y^2z^2)\vec{j} - 2y^3z\,\vec{k}$$

$$\nabla f(1,-2,-1) = 6(1)(-2)\vec{i} + (3(1)^2 - 3(-2)^2(-1)^2)\vec{j} - 2(-2)^3(-1)\vec{k}$$

$$= -12\vec{i} - 9\vec{j} - 16\vec{k}$$

∇f is the vector that gives the direction of the maximum rate of change of f at a point. Its norm $\| \nabla f \|$ give the maximum rate.

(5) let $f = x^2y + 2xz = 4.$

A normal of the surface can be given by

$$\vec{N} = \nabla f = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(x^2y + 2xz)$$

$$= (2xy + 2z)\vec{i} + x^2\vec{j} + 2x\vec{k}$$

At point $(2,-2,3)$

$$\vec{N} = \left(2(2)(-2) + 2(3)\right)\vec{i} + 2^2\vec{j} + 2(2)\vec{k}$$

$$= -2\vec{i} + 4\vec{j} + 4\vec{k}$$

a unit normal at $(2,-2,3)$ is

$$\vec{n} = \frac{1}{|\vec{N}|}\vec{N}$$

$$= \frac{1}{\sqrt{2^2+4^2+4^2}}(-2\vec{i} + 4\vec{j} + 4\vec{k})$$

$$= -\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$$

Another unit normal is in the opposite direction

i.e $\frac{1}{3}\vec{i} - \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}$

3. $\nabla f = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot(x^2 e^y)$

$$= 2xe^y\,\vec{i} + x^2 e^y\,\vec{j} + 0\,\vec{k}$$

Directional derivative along direction $-\mathbf{j}$ is found by taking the dot product of ∇f with the unit vector of the direction $-\mathbf{j}$

$$= \nabla f \bullet (0\mathbf{i} - \mathbf{j} + 0\mathbf{k}) = -x^2 e^y$$

At point (-2, 0, 0), the directional derivative in direction of $-\mathbf{j}$ $= -(-2)^2 e^0 = -4$

At point (-2, 0, 0), the maximum directional derivative is in the direction of ∇f and has value $= \|\nabla f\| = \|2(-2)e^0\,\mathbf{i} + (-2)^2 e^0\,\mathbf{j} + 0\,\mathbf{k}\| = \|-4\,\mathbf{i} + 4\,\mathbf{j}\| = 5.6569$

(4) $r = |\vec{r}| = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{\frac{1}{2}}$

$r^n = (x^2+y^2+z^2)^{\frac{n}{2}}$

$\nabla r^n = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(x^2+y^2+z^2)^{\frac{n}{2}}$

$$\frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{n}{2}} = \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2x)$$

$$= n\,(x^2+y^2+z^2)^{\frac{1}{2}(n-2)}\,x$$

$$= n\,r^{n-2}\,x$$

Similarly,

$$\frac{\partial}{\partial y}(x^2+y^2+z^2)^{\frac{n}{2}} = n\,r^{n-2}\,y$$

$$\frac{\partial}{\partial z}(x^2+y^2+z^2)^{\frac{n}{2}} = n\,r^{n-2}\,z$$

$$\therefore \nabla r^n = nr^{n-2}x\,\vec{i} + nr^{n-2}y\,\vec{j} + nr^{n-2}z\,\vec{k}$$

$$= nr^{n-2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= n\,r^{n-2}\,\vec{r}$$

5. let $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\nabla \times \vec{v} = \nabla \times (\vec{w} \times \vec{r})$$

$$= \nabla \times \begin{vmatrix} i & j & k \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[(w_2 z - w_3 y)\vec{i} + (w_3 x - w_1 z)\vec{j} + (w_1 y - w_2 x)\vec{k} \right]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (w_2 z - w_3 y) & (w_3 x - w_1 z) & (w_1 y - w_2 x) \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(w_1 y - w_2 x) - \frac{\partial}{\partial z}(w_3 x - w_1 z)\right]$$

$$+ \vec{j}\left[\frac{\partial}{\partial z}(w_2 z - w_3 y) - \frac{\partial}{\partial x}(w_1 y - w_2 x)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(w_3 x - w_1 z) - \frac{\partial}{\partial y}(w_2 z - w_3 y)\right]$$

$$= 2w_1\vec{i} + 2w_2\vec{j} + 2w_3\vec{k}$$

$$= 2(w_1\vec{i} + w_2\vec{j} + w_3\vec{k}) = 2\vec{w}$$

i.e $\vec{w} = \frac{1}{2}\nabla \times \vec{v}$

(b)

Curl $(xy^2z\,\underset{\sim}{i} + 2x^3y\,\underset{\sim}{j} + 4x^2y^2\,\underset{\sim}{k})$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z & 2x^3y & 4x^2y^2 \end{vmatrix}$$

$$= \underset{\sim}{i}(8x^2y - 0) - \underset{\sim}{j}(8xy^2 - xy^2) + \underset{\sim}{k}(6x^2y - 2xyz)$$

At Point $(1, 1, -1)$,

Curl $(\cdot) = 8\underset{\sim}{i} - 7\underset{\sim}{j} + 8\underset{\sim}{k}$ #

And, curl $(yz^3\,\underset{\sim}{i} + xz\,\underset{\sim}{j} + 2x\,\underset{\sim}{k})$

$$= \underset{\sim}{i}(0 - x) - \underset{\sim}{j}(2 - 3yz^2) + \underset{\sim}{k}(z - z^3)$$

At Point $(1, 1, -1)$,

Curl $(\cdot) = -\underset{\sim}{i} + \underset{\sim}{j} + 0\underset{\sim}{k}$ #