Tutorial 1 (Solutions)

1a) Let $y = \ln(i^{1/2})$

$$e^{y} = i^{1/2}$$

$$= \left[e^{i(\frac{\pi}{2} + 2n\pi)}\right]^{1/2} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$= e^{i(\frac{\pi}{4} + n\pi)}$$

$$\therefore \underline{\underline{y = i\left(\frac{\pi}{4} + n\pi\right) \qquad n = 0, \pm 1, \pm 2, \ldots}}$$

b) Let $y = i^{i}$

$$\ln y = i \ln i$$

$$= i \ln e^{i(\frac{\pi}{2} + 2n\pi)} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$= i\left[i\left(\frac{\pi}{2} + 2n\pi\right)\right]$$

$$= -\left(\frac{\pi}{2} + 2n\pi\right)$$

$$\therefore \underline{\underline{y = e^{-(\frac{\pi}{2} + 2n\pi)} \qquad n = 0, \pm 1, \pm 2, \ldots}}$$

<u>which is real-valued.</u>

(b) (Cont'd).

Let $y = z^{i}$

$\ln y = i \ln z$

$= i \ln re^{i(\theta + 2n\pi)} \qquad n = 0, \pm 1, \pm 2, \ldots$

$= i \ln r - (\theta + 2n\pi)$

$y = z^{i} = e^{i \ln r} \cdot e^{-(\theta + 2n\pi)}$

$e^{i \ln r} = \cos(\ln r) + i \sin(\ln r)$.

For $y = z^{i}$ to be real,

$\sin(\ln r) = 0$.

$\ln r = \pm k\pi \qquad k = 0, 1, 2, \ldots$

$\underline{r = e^{\pm k\pi}}$.

$\therefore z^{i} = (re^{i\theta})^{i}$

$= (e^{\pm k\pi} \cdot e^{i\theta})^{i}$ is real.

The values of z for real z^{i} are

$$\underline{\underline{z = e^{\pm k\pi} \cdot e^{i\theta} \qquad k = 0, 1, 2, \ldots}}$$

2 (~~1~~ ~~1(a)~~)

2 a)

$$f(z) = \frac{x^2 y}{x^3 + y^3} + i x y$$

For the limit to exist, $\lim_{z \to z_0} f(z)$ need to be unique and independent of the directions in which z approaches z_0.

Let the direction be given by $y = kx$, k is a constant.

$$\lim_{z \to 0} f(z) = \lim_{x \to 0,\, y = kx} f(z)$$

$$= \lim_{x \to 0} \frac{k x^3}{x^3 + k^3 x^3} + i k x^2$$

$$= \frac{k}{1 + k^3} \quad \text{which depends on } k$$

(the direction in which x, y approach zero)

$\Rightarrow$ the limit does not exist

3 ~~2-2~~

2b) let $z = re^{i\theta}$

$$\lim_{z \to 0} f(z) = \lim_{r \to 0} f(z)$$

$$= \lim_{r \to 0} \left[\frac{re^{-i\theta}}{re^{i\theta}} - \frac{re^{i\theta}}{re^{-i\theta}} - \frac{r^2 e^{i2\theta}}{r^2 e^{-i2\theta}} \right].$$

$$= e^{-i2\theta} - e^{i2\theta} - e^{i4\theta}$$

$\Rightarrow$ the limit does not exist.

3 a). A function $f(z)$ is continuous at $z = z_0$

if (a) $f(z_0)$ is defined, and

(b) $\lim_{z \to z_0} f(z) = f(z_0)$.

$$f(z) = \begin{cases} \operatorname{Re}\left[\frac{z}{|z|}\right] & z \neq 0 \\ 0 & z = 0 \end{cases}.$$

$$\lim_{z \to 0} f(z) = \lim_{r \to 0} \operatorname{Re}\left[\frac{re^{i\theta}}{|re^{i\theta}|}\right]$$

$$= \lim_{r \to 0} \operatorname{Re}\left[\cos\theta + i\sin\theta\right]$$

$$= \cos\theta.$$

$\Rightarrow$ limit does not exist

$\Rightarrow$ $f(z)$ is <u>not continuous at $z = 0$</u>.

b) $$\lim_{z \to 0} f(z) = \lim_{r \to 0} \operatorname{Im}\left[\frac{re^{i\theta}}{1 + |re^{i\theta}|}\right]$$

$$= \lim_{r \to 0} \operatorname{Im}\left[\frac{re^{i\theta}}{1+r}\right]$$

$$= \lim_{r \to 0} \frac{r}{1+r} \sin\theta$$

$$= 0.$$

At $z = 0$, $f(z) = 0$

$\Rightarrow$ $f(z)$ is <u>continuous at $z = 0$</u>.

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4) $f(z) = \begin{cases} \operatorname{Im}\left[\frac{z}{|z|}\right] & z \neq 0 \\ 0 & z = 0 \end{cases}$.

$f(z)$ is continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

<u>For $z_0 = 0$</u>.

$$\lim_{z \to 0} f(z) = \lim_{r \to 0} \operatorname{Im}\left[\frac{\cancel{r}e^{i\theta}}{|\cancel{r}e^{i\theta}|}\right]$$

$$= \sin\theta$$

$\Rightarrow$ limit does not exist

$\Rightarrow$ $f(z)$ is <u>not</u> continuous at <u>$z = 0$</u>.

<u>For $z_0 = 5$</u>

$$\lim_{z \to z_0} f(z) = \lim_{r \to 0} \operatorname{Im}\left[\frac{z_0 + re^{i\theta}}{|z_0 + re^{i\theta}|}\right], \quad z_0 = 5.$$

$$= \lim_{r \to 0} \frac{r \sin\theta}{5}$$

$$= 0$$

$$f(5) = \operatorname{Im}\left[\frac{5}{|5|}\right] = 0.$$

$\Rightarrow$ $\lim_{z \to 5} f(z) = f(5)$

$\Rightarrow$ <u>$f(z)$ at $z=5$ is continuous</u>.

4) (Cont'd).

For $z_0 = 5+i$

$$\lim_{z \to z_0} f(z) = \lim_{r \to 0} \text{Im}\left[\frac{z_0 + re^{i\theta}}{|z_0 + re^{i\theta}|}\right] \qquad z_0 = 5+i$$

$$= \lim_{r \to 0} \text{Im} \frac{5+i + re^{i\theta}}{|5+i+\underbrace{re^{i\theta}}_{\to 0}|}.$$

$$= \lim_{r \to 0}\left[\frac{1 + r\sin\theta}{|5+i|}\right]$$

$$= \frac{1}{|5+i|} = \frac{1}{\sqrt{26}}.$$

$$f(5+i) = \text{Im}\left[\frac{5+i}{|5+i|}\right]$$

$$= \frac{1}{\sqrt{26}}.$$

$$\therefore \lim_{z \to z_0} f(z) = f(z_0), \quad z_0 = 5+i.$$

$\Rightarrow$ function is continuous at $z = 5+i$.

Tutorial 2 (Solutions)

1 a). $f(z) = (2x - y) + i(ax + by)$.

Using the C-R equations.

$u_x = 2 \qquad v_x = a$.

$u_y = -1 \qquad v_y = b$

To satisfy the C-R equations.

$u_x = v_y \quad , \quad u_y = -v_x$.

$\therefore \quad \underline{a = 1}$

$\underline{b = 2}$

The function is differentiable for all z.

$f'(z) = u_x + i v_x$

$= 2 + i$

b) (i) $f(z) = Re[z^2]$

$= Re[(x + iy)^2]$

$= Re[x^2 - y^2 + i2xy]$

$= x^2 - y^2$

$u_x = 2x \qquad v_x = 0$

$u_y = -2y \qquad v_y = 0$.

The C-R equations are only satisfied at $z = 0$

$\Rightarrow$ $f(z)$ is nowhere analytic

1 b) (ii) $f(z) = \frac{1}{z^4}$

$= \frac{1}{r^4} e^{i\left(\frac{\pi}{2} - 4\theta\right)}$

$= \frac{1}{r^4}\left[\sin 4\theta + i \cos 4\theta \right]$

$u_r = -\frac{4}{r^5} \sin 4\theta$ $v_r = -\frac{4}{r^5} \cos 4\theta$

$u_\theta = \frac{4}{r^4} \cos 4\theta$ $v_\theta = -\frac{4}{r^4} \sin 4\theta$

$u_r = \frac{1}{r} v_\theta$ and $v_r = -\frac{1}{r} u_\theta$

the C-R equations are satisfied everywhere except at $z = 0$ (where the functions u and v are not continuous)

(iii) $f(z) = z - \bar{z} = (x + iy) - (x - iy)$

$= i 2y$

$u_x = 0$ $v_x = 0$

$u_y = 0$ $v_y = 2$

C.R equations are not satisfied except at $z = 0$

$\Rightarrow$ Not analytic

1 b) (iv) $f(z) = e^x(\sin y - i\cos y)$

$u_x = e^x \sin y$. $\quad v_x = -e^x \cos y$.

$u_y = e^x \cos y$ $\quad v_y = e^x \sin y$.

$u_x = v_y$ and $v_x = -u_y$

$f(z)$ is analytic everywhere in the complex plane

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2 a) $f(z) = 2xy - i x^2$

$u_x = 2y \quad v_x = -2x$

$u_y = 2x \quad v_y = 0$.

For the C-R equations to be satisfied,

$u_x = v_y \Rightarrow y = 0$ (x-axis)

$v_x = -u_y$

$\Rightarrow$ C-R equations are satisfied only at x-axis

$\Rightarrow$ $f'(z)$ exists only on x-axis

$f'(z) = u_x + i v_x$

$= 2y - i2x$

$= -i2x$.

2 b)

$$f(z) = z^2 - 2z + 3$$

$$= (x + iy)^2 - 2(x + iy) + 3$$

$$= (x^2 - y^2 - 2x + 3) + i(+2xy - 2y)$$

$u_x = 2x - 2$ $\qquad$ $v_x = +2y$

$u_y = -2y$ $\qquad$ $v_y = 2x - 2$

$u_x = v_y$ and $v_x = -u_y$

CR equations are satisfied for all z

$\Rightarrow$ $f(z)$ is analytic for all z.

$$f'(z) = u_x + i v_x$$

$$= 2x - 2 + i\,2y = 2(x+iy) - 2$$

$$= 2z - 2$$

Note - Polynomials in z are analytic in the entire z plane and the usual differentiation applies

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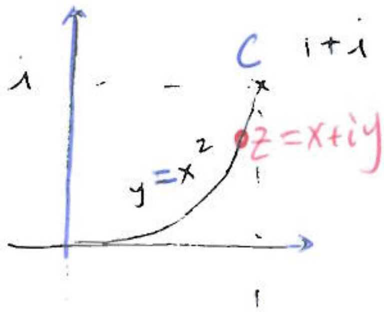
3 a)

$f(z) = Re[z]$

$C: y = x^2$. Let $x = t$. Then $y = t^2$

$z(t) = t + i t^2 \quad 0 \le t \le 1$



$dz = (1 + i2t) \cdot dt$

$$\therefore \int_C f(z)dz = \int_0^1 t \cdot (1+i2t)dt$$

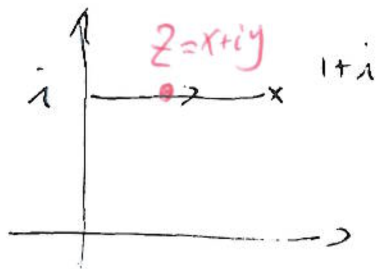
$$= \int_0^1 (t + i2t^2) dt$$

$$= \left[\frac{t^2}{2} + i\frac{2t^3}{3}\right]_0^1 = \frac{1}{2} + i\frac{2}{3}$$

b) $f(z) = 4z - 3$

$C: y=1$. Let $x=t$, Then

$z(t) = t + i, \quad 0 \le t \le 1$



$dz = dt$

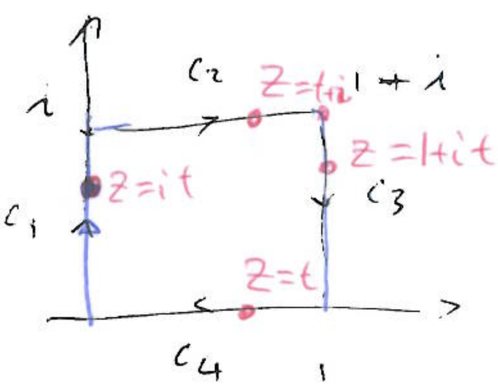
$$\int_c f(z)dz = \int_0^1 [4(t+i) - 3]\, dt$$

$$= \left[2t^2 + (4i-3)t\right]_0^1$$

$$= 2 + 4i - 3 = -1 + 4i$$

3 (b) $f(z) = e^{z}$

$z = x + iy$



C_1: $z(t) = it \quad 0 \le t \le 1$

$dz = i\,dt$

C_2: $z(t) = t + i \quad 0 \le t \le 1$

$dz = dt$

C_3: $z(t) = 1 + it \quad 0 \le t \le 1$.

$dz = i\,dt$

C_4: $z(t) = t \quad 0 \le t \le 1$

$dz = dt$

$$\int_C f(z)dz = \int_0^1 e^{it} \cdot i\,dt + \int_0^1 e^{t+i} \cdot dt$$

$$+ \int_1^0 e^{1+it} \cdot i\,dt + \int_1^0 e^{t}\,dt.$$

$$= \left[i \cdot \frac{e^{it}}{i} \right]_0^1 + \left[\frac{e^{t+i}}{1} \right]_0^1$$

$$+ \left[i \cdot \frac{e^{1+it}}{i} \right]_1^0 + \left[e^{t} \right]_1^0$$

$$= (e^{i} - e^{0}) + (e^{1+i} - e^{i}) + (e^{1} - e^{1+i}) + (e^{0} - e^{1})$$

$$= 0$$

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3d). $f(z) = \text{Im}\left[z^2\right]$

Referencing Q 3c solution,

C_1: $f(z) = \text{Im}\left[z^2\right] = \text{Im}\left[t^2\right] = 0$.

C_2: $f(z) = \text{Im}\left[(t+i)(t+i)\right]$

$= \text{Im}\left[t^2 - 1 + i2t\right] = 2t$.

C_3. $f(z) = \text{Im}\left[(1+it)(1+it)\right]$

$= \text{Im}\left[1 - t^2 + i2t\right] = 2t$.

C_4: $f(z) = \text{Im}\left[t^2\right] = 0$

$\int_c f(z)\,dz = \int_0^1 0 \cdot i\,dt + \int_0^1 2t\,dt$

$+ \int_1^0 2t \cdot i\,dt + \int_1^0 0\,dt$

$= \left[t^2\right]_0^1 + i\left[t^2\right]_1^0$

$= 1 - i$ //.

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