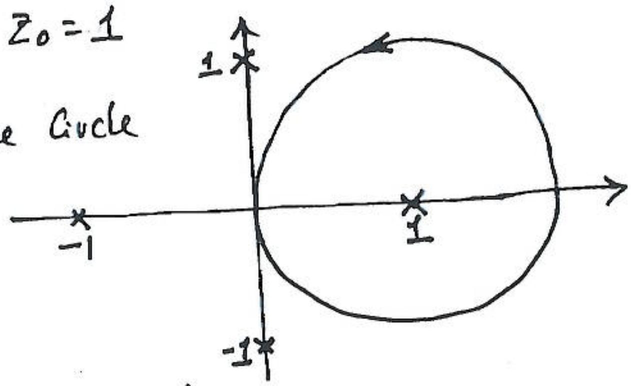
Tutorial - 3 (Solutions)

1 (a) The function $f(z) = \frac{1}{z^4 - 1}$ is not analytic at $z^4 - 1 = 0$, or $z = \pm 1, \pm i$.

The path $|z-1| = 1$ is a circle of radius 1 and centred at $z = 1$.

The singular point $z_0 = 1$ is enclosed inside the circle



Factor $z^4 - 1$ as

$$z^4 - 1 = (z-1)(z^3 + z^2 + z + 1)$$

and make use of Cauchy's Integral Formula

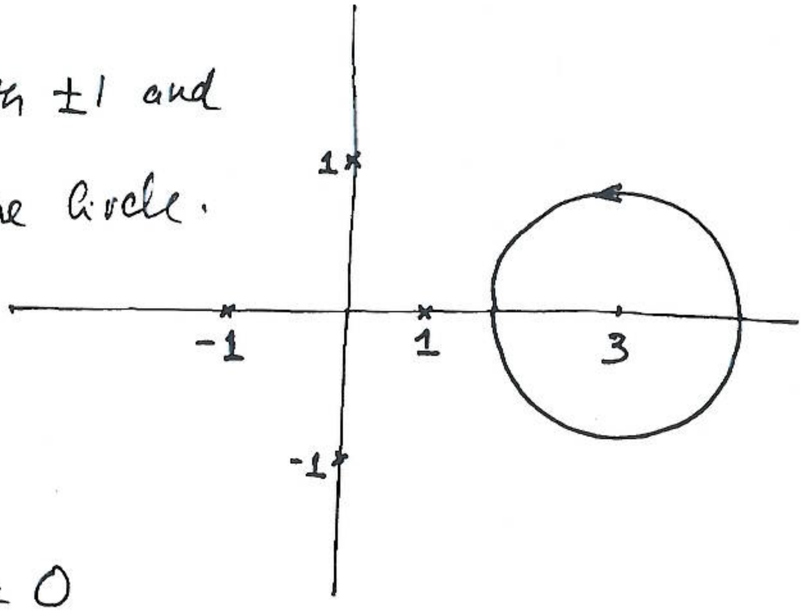
$$\int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0)$$

$$\int_C \frac{(z^3 + z^2 + z + 1)^{-1}}{(z-1)}\, dz = 2\pi i\, [1+1+1+1]^{-1} = \underline{\underline{\frac{\pi i}{2}}}.$$

Note: $\left.\begin{array}{l} f(z) = (z^3 + z^2 + z + 1)^{-1} \\ f(z_0) = (1+1+1+1)^{-1} \end{array}\right\}$

1(b)

In this case both ± 1 and $\pm i$ are outside the circle.



Hence

$$\int_C \frac{1}{(z^4-1)}\,dz = \underline{\underline{0}}.$$

2 (a) $\displaystyle\int_C \frac{5z}{z^2+4}\,dz$; $z^2+4=0$ gives simple poles

at $z = \pm 2i$

Hence $$\int_C \frac{5z}{z^2+4}\,dz = \int_{C_1} \frac{\frac{5z}{(z+2i)}}{(z-2i)}\,dz + \int_{C_2} \frac{\frac{5z}{(z-2i)}}{(z+2i)}\,dz$$

$$= 2\pi i\left[\frac{5z}{z+2i}\right]_{z=2i} + 2\pi i\left[\frac{5z}{z-2i}\right]_{z=-2i}$$

$$= \underline{\underline{10\pi i}}.$$

2(b) $\displaystyle\int_C \frac{z+e^z}{z^3-z}\,dz$; $z^3-z=0$ gives simple poles at $z=0$ and $z=\pm 1$. Hence

$$\int_C \frac{z+e^z}{z(z+1)(z-1)} = \int_{C_1} \frac{\left[\frac{z+e^z}{(z+1)(z-1)}\right]}{z}\,dz + \int_{C_2} \frac{\left[\frac{z+e^z}{z(z+1)}\right]}{(z-1)}\,dz$$

$$+ \int_{C_3} \frac{\left[\frac{z+e^z}{z(z-1)}\right]}{(z+1)}\,dz$$

$$= 2\pi i \left[\left.\frac{z+e^z}{(z+1)(z-1)}\right|_{z=0} + \left.\frac{z+e^z}{z(z+1)}\right|_{z=1} + \left.\frac{z+e^z}{z(z-1)}\right|_{z=-1} \right.$$

$$= 2\pi i \left[-1 + \frac{1+e}{2} + \frac{-1+e^{-1}}{2} \right]$$

$$= 2\pi i \left[-1 + \frac{e+e^{-1}}{2} \right]$$

3 (a) Let $z = e^{i\theta}$; $\sin\theta = \frac{1}{2i}\left[z - z^{-1}\right]$

$dz = iz\,d\theta$

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\sin\theta} = \int_C \frac{1}{5 - 3 \cdot \frac{1}{2i}\left[z - z^{-1}\right]} \frac{dz}{iz}$$

$$= \int_C \frac{-2 \quad dz}{(3z^2 - 10iz - 3)}$$

$$= \int_C \frac{-2 \quad dz}{(3z - i)(z - 3i)} = \int_C \frac{-2 \quad dz}{3(z - \frac{i}{3})(z - 3i)}$$

$$= \int_C \frac{\left[\frac{-2}{3(z-3i)}\right]}{(z - i/3)} dz$$

$$= 2\pi i \left[\left. \frac{-2}{3(z-3i)} \right|_{z=\frac{i}{3}} \right] = \underline{\underline{\frac{\pi}{2}}}$$

[Since only the simple pole $z = \frac{i}{3}$ is inside C, the unit circle.]

3(b) Let $z = e^{i\theta}$, Then

$$\cos\theta = \frac{1}{2}\left[z + z^{-1}\right]; \quad \cos 2\theta = \frac{1}{2}\left[z^2 + z^{-2}\right]$$

$$dz = iz\, d\theta$$

$$\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta}\, d\theta = \int_C \frac{\frac{1}{2}(z + z^{-1})}{13 - 6(z^2 + z^{-2})}\, \frac{dz}{iz}$$

$$= \int_C \frac{(z^2+1)\quad dz}{-2i(6z^4 - 13z^2 + 6)} = \int_C \frac{(z^2+1)\quad dz}{-2i(3z^2-2)(2z^2-3)}$$

$$= \int_C \frac{(z^2+1)\quad dz}{-2i\,3\left(z^2 - \frac{2}{3}\right)(2z^2-3)} = \int_C \frac{\left[\frac{(z^2+1)}{-6i(2z^2-3)}\right]}{\left(z^2 - 2/3\right)}\, dz$$

$$= \int_{C_1} \frac{\frac{(z^2+1)\quad dz}{-6i(2z^2-3)\left(z+\sqrt{\frac{2}{3}}\right)}}{z - \sqrt{2/3}} + \int_{C_2} \frac{\frac{(z^2+1)\quad dz}{-6i(2z^2-3)\left(z-\sqrt{2/3}\right)}}{z + \sqrt{\frac{2}{3}}}$$

$$= \underline{\underline{0}}$$ [Note :- only $z = \pm\sqrt{\frac{2}{3}}$ are inside the unit circle]

4 a) $$\int_{-\infty}^{+\infty} \frac{x \, dx}{(x^2 - 2x + 2)^2} = \int_{UHP} \frac{z \, dz}{(z^2 - 2z + 2)^2}$$

$$\frac{z}{(z^2 - 2z + 2)^2} = \frac{z}{[(z-1+i)(z-1-i)]^2}$$

This has second order poles at $z = 1 \pm i$. Only $z = 1+i$ is inside the upper half plane (UHP)

Hence

$$\int_{UHP} \frac{z \, dz}{[(z-(1-i))(z-(1+i))]^2} = 2\pi i \left.\frac{d}{dz}\left(\frac{z}{[z-(1-i)]^2}\right)\right|_{z=1+i}$$

$$= 2\pi i \left(-\frac{i}{4}\right) = \underline{\underline{\frac{\pi}{2}}}$$

4 (b) $$\int_{-\infty}^{\infty} \frac{1\,dx}{(4+x^2)^2} = \int_{UHP} \frac{1\,dz}{(4+z^2)^2}$$

$$\frac{1}{(4+z^2)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$$

This has a second order pole at $z = 2i$ inside the UHP.

$$\int_{UHP} \frac{dz}{(z+2i)^2(z-2i)^2} = 2\pi i \left. \frac{d}{dz} \frac{1}{(z+2i)^2} \right|_{z=2i}$$

$$= 2\pi i \frac{2}{4^3 i} = \frac{\pi}{16}$$