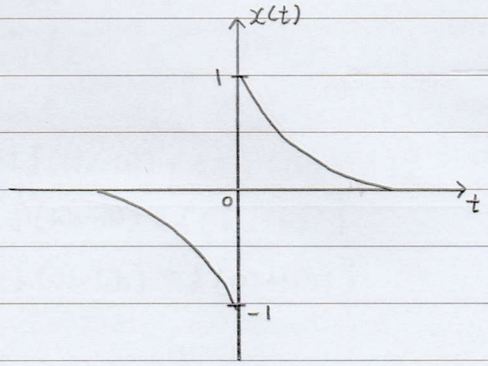
No.: Date:

Esmund Lim

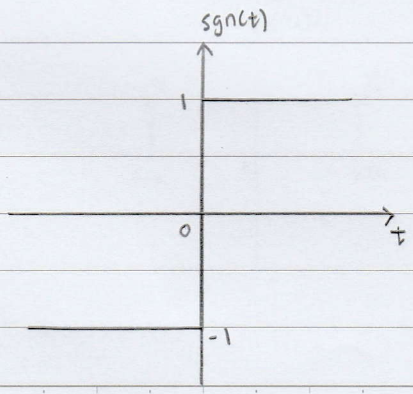
Signals Tutorial 10

1) $x(t) = \begin{cases} e^{-at} & , t > 0 \\ 0 & , t = 0 \\ -e^{at} & , t < 0 \end{cases}$



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}$$
$$= \int_{-\infty}^{0} -e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$
$$= \int_{-\infty}^{0} -e^{(a-j\omega)t} dt + \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$
$$= -\left[\frac{e^{(a-j\omega)t}}{a-j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{0}^{\infty}$$
$$= -\left[\frac{e^{(a-j\omega)(0)}}{a-j\omega} - \frac{e^{(a-j\omega)(-\infty)}}{a-j\omega}\right] + \left[\frac{e^{-(a+j\omega)(\infty)}}{-(a+j\omega)} - \frac{e^{-(a+j\omega)(0)}}{-(a+j\omega)}\right]$$
$$= -\left[\frac{1}{a-j\omega} - 0\right] + \left[0 + \frac{1}{a+j\omega}\right]$$
$$= -\frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$
$$= \frac{-(a+j\omega) + (a-j\omega)}{(a-j\omega)(a+j\omega)}$$
$$= \frac{-2j\omega}{a^2 - j^2\omega^2}$$
$$= \frac{-2j\omega}{a^2 + \omega^2}$$

b) $sgn(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t = 0 \\ -1 & , t < 0 \end{cases}$



$$sgn(t) = \lim_{a \to 0} x(t)$$
$$\mathcal{F}\{sgn(t)\} = \mathcal{F}\{\lim_{a \to 0} x(t)\}$$
$$= \lim_{a \to 0} \mathcal{F}\{x(t)\}$$
$$= \lim_{a \to 0} X(\omega)$$
$$= \lim_{a \to 0} \left(\frac{-2j\omega}{a^2 + \omega^2}\right)$$
$$= \frac{-2j\omega}{\omega^2}$$
$$= \frac{-j2}{\omega} \quad \text{or} \quad \frac{-j2 \times j}{\omega \times j}$$
$$= \frac{-j^2(2)}{j\omega}$$
$$= \frac{2}{j\omega}$$

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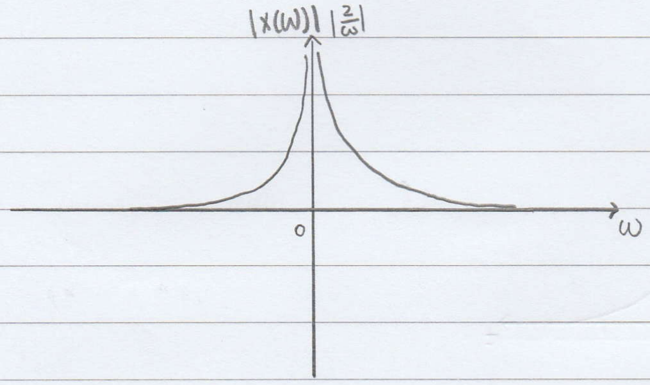
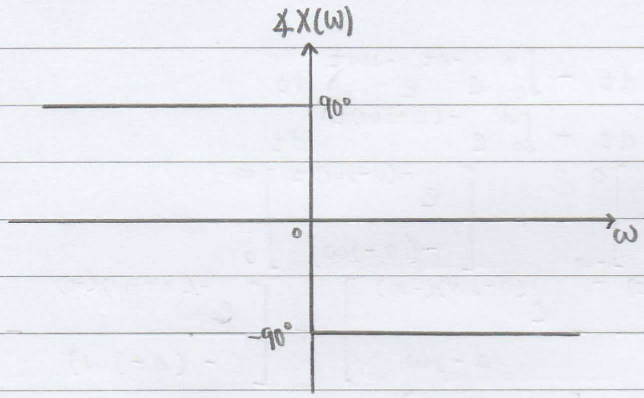
amplitude spectrum



phase spectrum

$-j\frac{2}{\omega}$

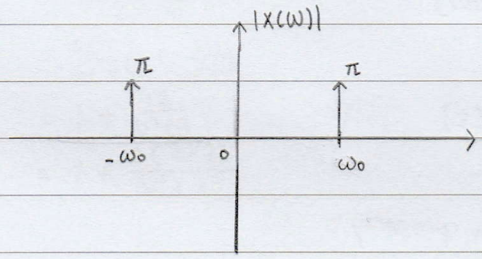


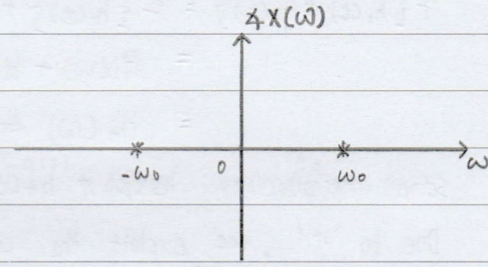
2) $x(t) = \cos \omega_0 t$

$$X(\omega) = \mathcal{F}\{\cos \omega_0 t\}$$
$$= \int_{-\infty}^{\infty} \cos \omega_0 t \, e^{-j\omega t} \, dt$$
$$= \int_{-\infty}^{\infty} \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-j\omega t} \, dt$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{j\omega_0 t} e^{-j\omega t}\right) + \left(e^{-j\omega_0 t} e^{-j\omega t}\right) dt$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{j(\omega_0-\omega)t} + e^{-j(\omega_0+\omega)t} \, dt$$

since $\delta(t)$ is an even function $\delta(-a) = \delta(a)$

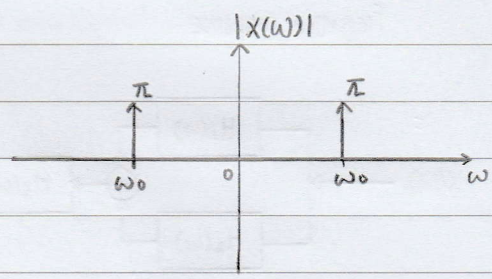
$$= \frac{1}{2}\left[2\pi\delta(\omega_0-\omega) + 2\pi\delta(\omega_0+\omega)\right]$$
$$= \pi\left[\delta(\omega_0-\omega) + \delta(\omega_0+\omega)\right]$$
$$= \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

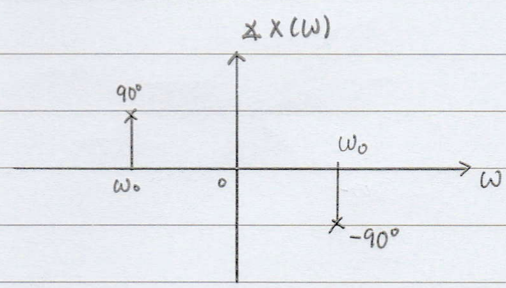




b) $x(t) = \sin \omega_0 t$

$$X(\omega) = \mathcal{F}\{\sin \omega_0 t\}$$
$$= \int_{-\infty}^{\infty} \sin \omega_0 t \, e^{-j\omega t} \, dt$$
$$= \int_{-\infty}^{\infty} \frac{1}{j2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) e^{-j\omega t} \, dt$$
$$= \frac{1}{j2}\int_{-\infty}^{\infty} \left(e^{j\omega_0 t} e^{-j\omega t}\right) - \left(e^{-j\omega_0 t} e^{-j\omega t}\right) dt$$
$$= \frac{1}{j2}\int_{-\infty}^{\infty} e^{j(\omega_0-\omega)t} - e^{-j(\omega_0+\omega)t} \, dt$$
$$= \frac{1}{j2}\left[2\pi\delta(\omega_0-\omega) - 2\pi\delta(\omega_0+\omega)\right]$$
$$= \frac{\pi}{j}\left[\delta(\omega_0-\omega) - \delta(\omega_0+\omega)\right]$$
$$= \frac{\pi \times j}{j \times j}\left[\delta(\omega_0-\omega) - \delta(\omega_0+\omega)\right]$$
$$= -j\pi\left[\delta(\omega_0-\omega) - \delta(\omega_0+\omega)\right]$$
$$= -j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$





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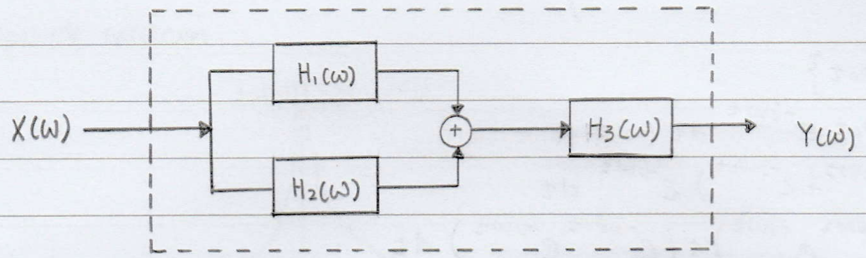
3)



$h_1(t) \leftrightarrow H_1(\omega)$, $h_2(t) \leftrightarrow H_2(\omega)$, $h_3(t) \leftrightarrow H_3(\omega)$

Parallel interconnection : $h_1(t) + h_2(t)$

Due to "+" , we exploit the linearity property :

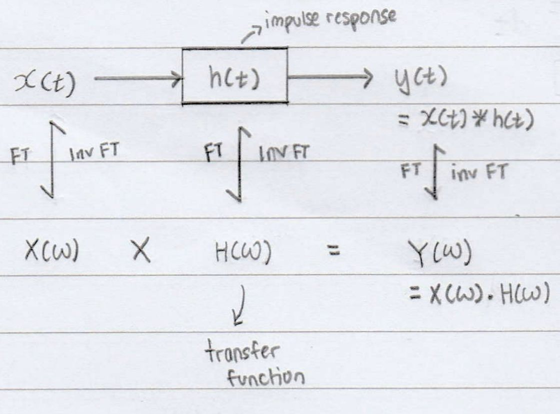
$$\mathcal{F}\{h_1(t) + h_2(t)\} = \mathcal{F}\{h_1(t)\} + \mathcal{F}\{h_2(t)\}$$
$$= H_1(\omega) + H_2(\omega)$$
$$= H_4(\omega) \leftrightarrow h_4(t)$$

Serial interconnection : $h_3(t) * h_4(t)$

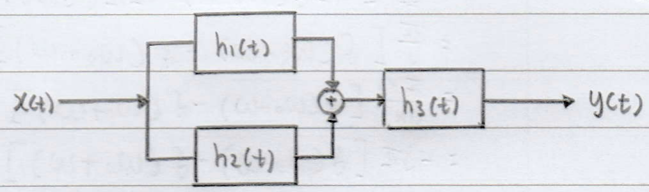
Due to '*' , we exploit the convolution property :

$$\mathcal{F}\{h_3(t) * h_4(t)\} = \mathcal{F}\{h_3(t)\} \cdot \mathcal{F}\{h_4(t)\}$$
$$= H_3(\omega) \cdot H_4(\omega)$$
$$= H_3(\omega) \cdot [H_1(\omega) + H_2(\omega)]$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$
$$= H_1(\omega) H_3(\omega) + H_2(\omega) H_3(\omega)$$

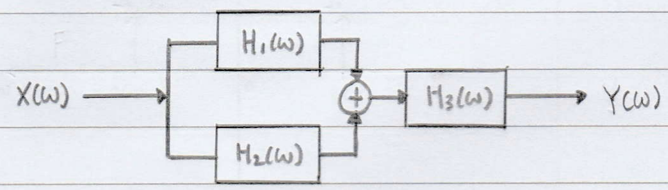


impulse response



$h(t) = [h_1(t) + h_2(t)] * h_3(t)$

Frequency response



$H(\omega) = [H_1(\omega) + H_2(\omega)] \cdot H_3(\omega)$

Multiplication in frequency domain $\leftrightarrow$ convolution in time domain

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4) $\frac{d^2 y(t)}{dt^2} - 9y(t) = \frac{d^2 x(t)}{dt^2} - 21x(t)$

$(j\omega)^2(Y(\omega)) - 9Y(\omega) = (j\omega)^2(X(\omega)) - 21X(\omega)$

$Y(\omega)[(j\omega)^2 - 9] = X(\omega)[(j\omega)^2 - 21]$

$Y(\omega)[j^2\omega^2 - 9] = X(\omega)[j^2\omega^2 - 21]$

$Y(\omega)[-\omega^2 - 9] = X(\omega)[-\omega^2 - 21]$

$Y(\omega)[\omega^2 + 9] = X(\omega)[\omega^2 + 21]$

$$
\begin{aligned}
H(\omega) &= \frac{Y(\omega)}{X(\omega)} \\
&= \frac{\omega^2 + 21}{\omega^2 + 9} \\
&= \frac{\omega^2 + 9 + 12}{\omega^2 + 9} \\
&= \frac{\omega^2 + 9}{\omega^2 + 9} + \frac{12}{\omega^2 + 9} \\
&= 1 + \frac{12}{\omega^2 + 9} \\
&= 1 + \frac{2(2 \times 3)}{\omega^2 + 3^2}
\end{aligned}
$$

$e^{-a|t|}, a > 0 \leftrightarrow \frac{2a}{a^2 + \omega^2}$

$$
\begin{aligned}
h(t) &= \mathcal{F}^{-1}\{H(\omega)\} \\
&= \mathcal{F}^{-1}\left\{1 + 2\left(\frac{2 \times 3}{\omega^2 + 3^2}\right)\right\} \\
&= \delta(t) + 2e^{-3|t|}
\end{aligned}
$$

POP URBAN