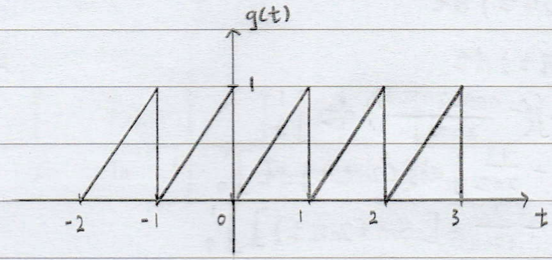
No.: Date:

Esmund Lim Signals Tutorial 8

1)



$g(t) = t \quad \text{for} \quad 0 < t < 1$

$T_0 = 1 \quad , \omega_0 = \frac{2\pi}{T_0}$
$= 2\pi$

$$DC: a_0 = \frac{1}{T_0}\int_{T_0} g(t)\,dt$$
$$= \frac{1}{1}\int_0^1 t\,dt$$
$$= 1\left[\frac{t^2}{2}\right]_0^1$$
$$= 1\left[\frac{1^2}{2} - \frac{0^2}{2}\right]$$
$$= \frac{1}{2}$$

Integration by parts
$\int u\,dv = uv - \int v\,du$
For u selection : LIATE

$$AC: a_n = \frac{2}{T_0}\int_{T_0} g(t)\cos(n\omega_0 t)\,dt$$
$$= \frac{2}{1}\int_0^1 t\cos(2n\pi t)\,dt$$

Let $u = t$ $\quad dv = \cos(2n\pi t)\,dt$

$\frac{du}{dt} = 1 \quad V = \int \cos(2n\pi t)\,dt \quad$ Let $u' = 2n\pi t$

$du = dt \quad = \int \cos(u')\left(\frac{du'}{2n\pi}\right) \quad \frac{du'}{dt} = 2n\pi$

$= \frac{1}{2n\pi}\int \cos(u')\,du' \quad dt = \frac{du'}{2n\pi}$

$= \frac{1}{2n\pi}\sin(u')$

$= \frac{1}{2n\pi}\sin(2n\pi t)$

$$= 2\left[\frac{(t)(\sin 2n\pi t)}{2n\pi} - \int \frac{1}{2n\pi}\sin(2n\pi t)\,dt\right]_0^1$$
$$= 2\left[\frac{(t)(\sin 2n\pi t)}{2n\pi} - \frac{1}{2n\pi}\int \sin(2n\pi t)\,dt\right]_0^1$$
$$= 2\left[\frac{(t)(\sin(2n\pi t))}{2n\pi} - \frac{1}{2n\pi}\int \sin(u')\frac{du'}{2n\pi}\right]_0^1$$
$$= 2\left[\frac{(t)(\sin(2n\pi t))}{2n\pi} - \frac{1}{(2n\pi)^2}\int \sin(u')\,du'\right]_0^1$$
$$= 2\left[\frac{(t)(\sin(2n\pi t))}{2n\pi} - \frac{1}{(2n\pi)^2}\left[-\cos(2n\pi t)\right]\right]_0^1$$
$$= 2\left[\frac{(t)(\sin(2n\pi t))}{2n\pi} + \frac{\cos(2n\pi t)}{(2n\pi)^2}\right]_0^1$$
$$= 2\left[\frac{(1)(\sin(2n\pi))}{2n\pi} + \frac{\cos(2n\pi)}{(2n\pi)^2} - 0 - \frac{\cos(0)}{(2n\pi)^2}\right]$$
$$= 2\left[0 + \frac{1}{(2n\pi)^2} - 0 - \frac{1}{(2n\pi)^2}\right]$$
$$= 2[0]$$
$$= 0$$

Function	value
$\cos 2n\pi$	1
$\sin 2n\pi$	0
$\cos n\pi$	$(-1)^n$
$\sin n\pi$	0
$\cos\frac{n\pi}{2}$	$\begin{cases} (-1)^{n/2} & n = \text{even} \\ 0 & n = \text{odd} \end{cases}$
$\sin\frac{n\pi}{2}$	$\begin{cases} (-1)^{(n-1)/2} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$
$e^{j2n\pi}$	1
$e^{jn\pi}$	$(-1)^n$
$e^{jn\pi/2}$	$\begin{cases} (-1)^{n/2} & n = \text{even} \\ j(-1)^{(n-1)/2} & n = \text{odd} \end{cases}$

$$b_n = \frac{2}{T_0}\int_{T_0} g(t)\sin(n\omega_0 t)\,dt$$
$$= \frac{2}{1}\int_0^1 t\sin(2n\pi t)\,dt$$
$$= 2\left[\frac{t(-\cos(2n\pi t))}{2n\pi} - \int\left(-\frac{\cos(2n\pi t)}{2n\pi}\right)dt\right]_0^1$$
$$= 2\left[\frac{t(-\cos(2n\pi t))}{2n\pi} + \frac{1}{2n\pi}\int\cos(2n\pi t)\,dt\right]_0^1$$
$$= 2\left[\frac{t(-\cos(2n\pi t))}{2n\pi} + \frac{1}{(2n\pi)^2}\left[\sin(2n\pi t)\right]\right]_0^1$$
$$= 2\left[\frac{t(-\cos(2n\pi t))}{2n\pi} + \frac{\sin(2n\pi t)}{(2n\pi)^2}\right]_0^1$$
$$= 2\left[\frac{(1)(-\cos(2n\pi))}{2n\pi} + \frac{\sin(2n\pi)}{(2n\pi)^2} - 0 - 0\right]$$
$$= 2\left[\frac{-1}{2n\pi} + 0\right]$$
$$= -\frac{2}{2n\pi}$$
$$= -\frac{1}{n\pi}$$

Let $u = t$ $\qquad dV = \sin(2n\pi t)\,dt$

$\frac{du}{dt} = 1$ $\qquad V = \int \sin(2n\pi t)\,dt$

$du = dt$ $\qquad = \frac{-\cos(2n\pi t)}{2n\pi}$

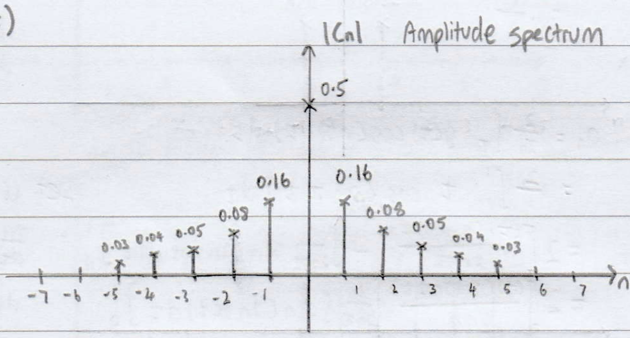
Trigonometric form

$$x(t) = a_0 + \sum_{n=1}^{\infty}\{a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\}$$

$a_0 = \frac{1}{2}$, $a_n = 0$, $b_n = -\frac{1}{n\pi}$

$$\therefore\ g(t) = \frac{1}{2} + \sum_{n=1}^{\infty} -\frac{1}{n\pi}\sin(n\omega_0 t)$$
$$= \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\omega_0 t)$$

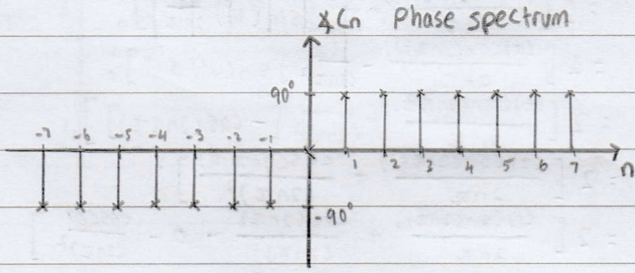
$$C_n = \frac{a_n - jb_n}{2}$$
$$= \frac{a_n}{2} - \frac{jb_n}{2}$$
$$= \frac{0}{2} - \frac{j(-\frac{1}{n\pi})}{2}$$
$$= 0 + j\frac{1}{2n\pi}$$

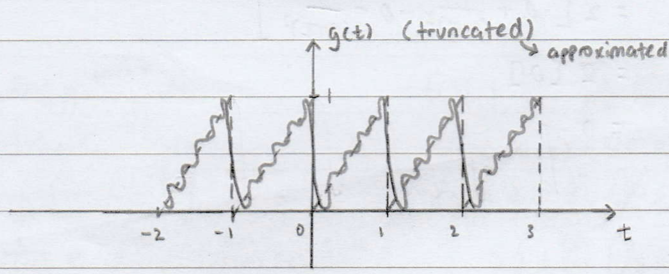


$$|C_n| = \sqrt{0^2 + \left(\frac{1}{2n\pi}\right)^2}$$
$$= \frac{1}{2n\pi}$$

$\left(0, \frac{1}{2n\pi}\right)$

$$\angle C_n = \begin{cases} 90^\circ, & n > 0 \\ -90^\circ, & n < 0 \end{cases}$$





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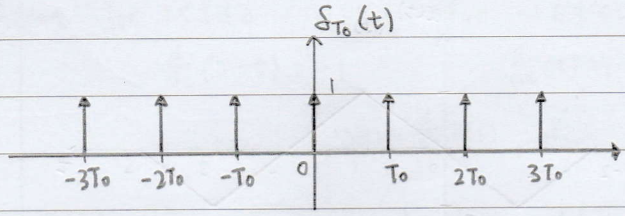

$$\delta(t) = \begin{cases} \infty, & t=0 \\ 0, & t\neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$T_0 = T_0 \qquad \omega_0 = \frac{2\pi}{T_0} \qquad \delta_{T_0}(t) = \delta(t) \qquad = \sum_{n=-\infty}^{+\infty} \delta(t-nT_0)$

$$C_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta_{T_0}(t)\,dt$$
$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\,dt$$
$$= \frac{1}{T_0}[1]$$
$$= \frac{1}{T_0}$$

$$C_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\,dt$$
$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta_{T_0}(t)e^{-jn\omega_0 t}\,dt$$
$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)e^{-jn\omega_0(0)}\,dt \qquad \text{(Prove on page 103)}$$
$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\,(1)\,dt$$
$$= \frac{1}{T_0}(1)$$
$$= \frac{1}{T_0}$$

$$\therefore\ \delta_{T_0}(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t}$$
$$= \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} e^{jn\omega_0 t}$$

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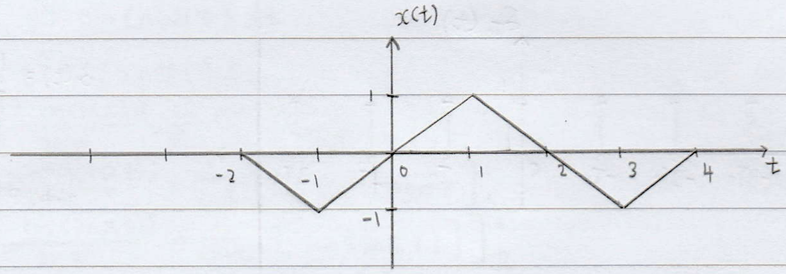


a) $T_0 = 4$, $\omega_0 = \frac{2\pi}{T_0}$
$= \frac{2\pi}{4}$
$= \frac{\pi}{2}$

$$x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \end{cases}$$

The signal $x(t)$ is an odd function. Due to odd symmetry

$a_0 = a_n = 0$

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin n\omega_0 t\,dt$$

$$= \frac{4}{4}\left[\int_0^1 (t)(\sin n\omega_0 t)\,dt + \int_1^2 (2-t)(\sin n\omega_0 t\,dt)\right]$$

$$= 1\left[\int_0^1 (t)(\sin(n\tfrac{\pi}{2}t))\,dt + \int_1^2 (2-t)(\sin(n\tfrac{\pi}{2}t))\,dt\right]$$

$$= \left[(t)\left(-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}t\right)\right) - \int -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}t\right)dt\right]_0^1$$

$$+ \int_1^2 (2)(\sin n\tfrac{\pi}{2}t) - (t)(\sin n\tfrac{\pi}{2}t)\,dt$$

$$= \left[(t)\left(-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}t\right) + \frac{2}{n\pi}\int\cos\left(\frac{n\pi}{2}t\right)dt\right]_0^1$$

$$+ \left[2\int_1^2 \sin\left(\frac{n\pi}{2}t\right)dt - \int_1^2 (t)\left(\sin\frac{n\pi}{2}t\right)dt\right]$$

$$= \left[(t)\left(-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}t\right)\right) + \frac{2}{n\pi}\left(\frac{\sin\left(\frac{n\pi}{2}t\right)}{n\pi/2}\right)\right]_0^1$$

$$+ 2\left(\frac{-\cos\left(\frac{n\pi}{2}t\right)}{n\pi/2}\right)_1^2 - \left[(t)\left(\frac{-2}{n\pi}\cos\left(\frac{n\pi}{2}t\right) + \frac{2}{n\pi}\left(\frac{\sin\left(\frac{n\pi}{2}t\right)}{n\pi/2}\right)\right]_1^2$$

$$= \left[\frac{-2t}{n\pi}\cos\left(\frac{n\pi t}{2}\right)\right]_0^1 + \left[\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi t}{2}\right)\right]_0^1 + \left[\frac{-4}{n\pi}\cos\left(\frac{n\pi t}{2}\right)\right]_1^2 + \left[\frac{2t}{n\pi}\cos\left(\frac{n\pi t}{2}\right)\right]_1^2$$

$$+ \left[\frac{-4}{n^2\pi^2}\sin\left(\frac{n\pi t}{2}\right)\right]_1^2$$

$$= \left[\frac{-2}{n\pi}\cancel{\cos\left(\frac{n\pi}{2}\right)}\right] + \left[\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right] + \left[\frac{-4}{n\pi}\cancel{\cos(n\pi)} + \frac{4}{n\pi}\cancel{\cos\left(\frac{n\pi}{2}\right)}\right] + \left[\frac{4}{n\pi}\cancel{\cos(n\pi)} - \frac{2}{n\pi}\cancel{\cos\left(\frac{n\pi}{2}\right)}\right]$$

$$+ \left[\cancel{\frac{-4}{n^2\pi^2}}\underbrace{\cancel{\sin(n\pi)}}_{0} + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore\ x(t) = \sum_{n=1}^{\infty}\left(\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)\sin(n\omega_0 t)$$

$$= \sum_{n=1}^{\infty}\left(\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi t}{2}\right)$$

Let $u = t$ $\quad dv = \sin(n\frac{\pi}{2}t)\,dt$

$\frac{du}{dt} = 1$ $\quad v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}t\right)$

$du = dt$

Even symmetry $f(t) = f(-t)$

$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos n\omega_0 t$

$a_0 = \frac{2}{T_0}\int_0^{T_0/2} x(t)\,dt$

$a_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt$

Note: $b_n = 0$, for all n

Odd symmetry $f(t) = -f(-t)$

$x(t) = \sum_{n=1}^{\infty} b_n\sin n\omega_0 t$

$b_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\sin n\omega_0 t\,dt$

Note: $a_n = 0$, for all n

Half wave $f(t) = -f(t \pm \frac{T}{2})$

$a_0 = 0$

$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt$, for odd n

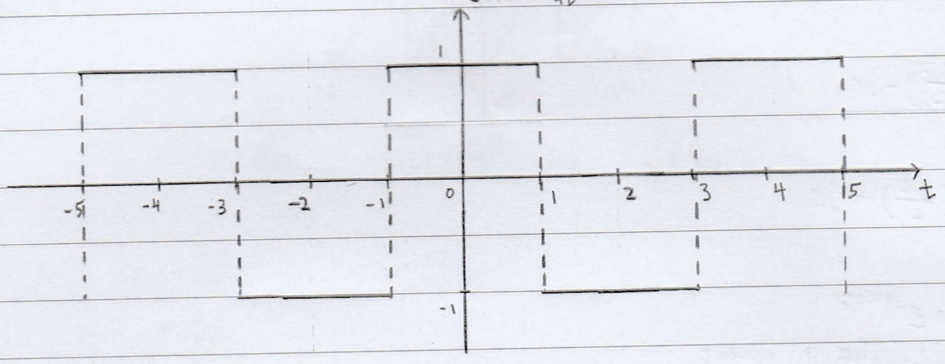
$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt$, for odd n

$a_n = b_n = 0$ for even n

b) For $0 \leq t \leq 1$ $x(t) = t$, For $1 < t \leq 2$ $x(t) = 2 - t$, For $-1 \leq t < 0$ $x(t) = t$, For $-2 \leq t < -1$ $x(t) = 2 - t$

$\frac{d}{dt}(t) = 1$ $\qquad \frac{d}{dt}(2-t) = -1$ $\qquad \frac{d}{dt}(t) = 1$ $\qquad \frac{d}{dt} = -1$

$y(t) = \frac{d}{dt} x(t)$



$$
\begin{aligned}
y(t) &= \frac{d}{dt} x(t) \\
&= \frac{d}{dt}\left[\sum_{n=1}^{\infty}\left(\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi t}{2}\right)\right] \\
&= \sum_{n=1}^{\infty}\left(\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)\frac{d}{dt}\sin\left(\frac{n\pi t}{2}\right) \\
&= \sum_{n=1}^{\infty}\left(\frac{\cancel{8}\,4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi t}{2}\right) \\
&= \sum_{n=1}^{\infty}\left(\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)\cos\Big(\underbrace{\frac{n\pi t}{2}}_{n\omega_0 t}\Big)
\end{aligned}
$$

$a_0 = 0 \qquad a_n = \frac{4}{n\pi}\sin\frac{n\pi}{2} \qquad b_n = 0$

* Note that this result is consistent with the plot of $y(t)$ as shown above, which is an even function (thus, $b_n = 0$) with zero DC bias ($a_0 = 0$)

c) $C_0 = a_0$

$= 0$

$$C_n = \frac{a_n - jb_n}{2}$$

$$= \frac{\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right) - j0}{2}$$

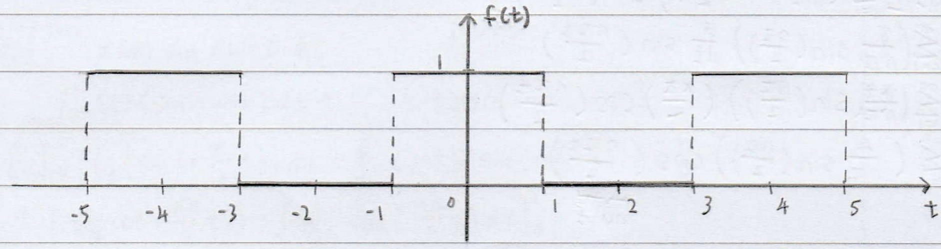
$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + j0$$

$$y(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]e^{jn\omega_0 t}$$

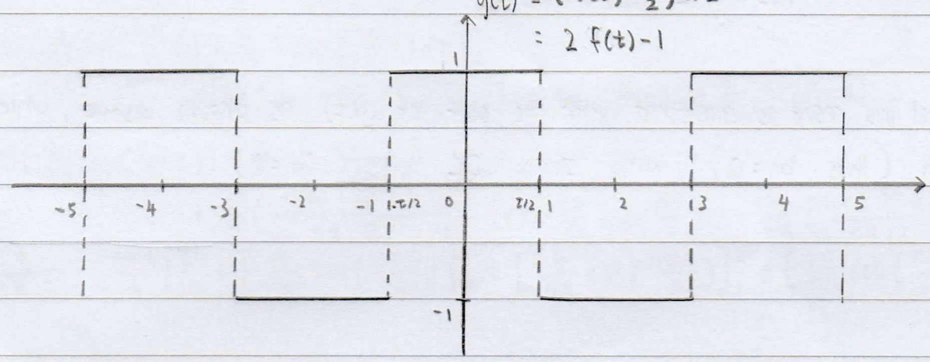
$$= \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]e^{jn\left(\frac{\pi}{2}\right)t}$$

d)



$y(t) = \left(f(t) - \frac{1}{2}\right) \times 2$

$= 2f(t) - 1$



Based on $f(t)$

$A = 1$, $T_0 = 4$ $\tau = 2$

$$C_0 = \frac{1}{T_0}\int_{T_0} f(t)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{4}\int_{-T_0/2}^{T_0/2} 1\,(1)\,dt$$

$$= \frac{1}{4}\left[t\right]_{-1}^{1}$$

$$= \frac{1}{4}\left[1-(-1)\right]$$

$$= \frac{2}{4}$$

$$= \frac{1}{2}$$

$$C_n = \frac{1}{T_0}\int_{T_0} f(t)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{4}\int_{-1}^{1}(1)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{4}\left[\frac{1}{-jn\omega_0}e^{-jn\omega_0 t}\right]_{-1}^{1}$$

$$= \frac{1}{4}\left[\frac{1}{-jn\omega_0}e^{-jn\omega_0} - \left(\frac{1}{-jn\omega_0}\right)e^{jn\omega_0}\right]$$

$$= \frac{1}{4}\left(-\frac{1}{jn\omega_0}\right)\left[e^{-jn\omega_0} - e^{jn\omega_0}\right]$$

$$= \frac{1}{4}\left(-\frac{1}{jn\omega_0}\right)\left[-j2\sin n\omega_0\right]$$

$$= \left(\frac{1}{4\cancel{2}}\right)\frac{\cancel{2}\sin n(\pi/2)}{\cancel{2}n(\pi/2)}$$

$$= \left(\frac{1}{2}\right)\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)} = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}\left(\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)e^{jn\left(\frac{\pi}{2}\right)t}$$

since $y(t) = \left(f(t) - \frac{1}{2}\right) \times 2$

$$f(t) - \frac{1}{2} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}\left(\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)e^{jn\left(\frac{\pi}{2}\right)t}$$

$$\times 2 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)e^{jn\left(\frac{\pi}{2}\right)t}$$

$= y(t)$ (Proven)