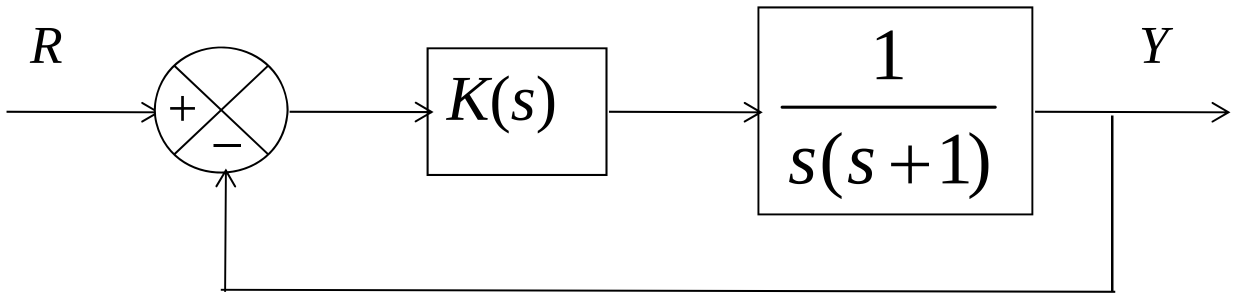
EE3011 MODEL& CONTROL
Tutorial 6 (Solutions) PID Controller Design and Simple Root Locus

1.



$$G(s) = \frac{K(s)}{s(s+1)}; \qquad \frac{Y(s)}{R(s)} = \frac{K(s)}{s(s+1)+K(s)}$$

If $K(s) = K_P$,

$$\frac{Y(s)}{R(s)} = \frac{K_P}{s^2+s+K_P} \equiv \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Standard 2nd order system!

Equating coefficients with $\zeta = 0.25$:

$$2\zeta\omega_n = 1 \quad \Rightarrow \quad \omega_n = 2$$

$$\omega_n^2 = K_P \quad \Rightarrow \quad K_P = 4$$

Overshoot: $M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.444 \text{ or } 44.4\%$

Rise time: $t_r = \dfrac{\pi - \theta}{\omega_n\sqrt{1-\zeta^2}} = 0.94 \text{ sec}$

Velocity error constant: $K_{vel} = \lim_{s\to 0} sG(s) = K_P = 4$

$$\therefore e_{ss}(\text{unit-ramp}) = \frac{1}{K_{vel}} = \frac{1}{4}$$

With $K(s) = 4 + sK_D$

$$\frac{Y(s)}{R(s)} = \frac{4 + sK_D}{s^2 + (1+K_D)s + 4} \equiv \frac{4 + sK_D}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Non-standard 2nd order system!

Equating coefficients with $\zeta = 1$:

i.e. $$s^2 + (1+K_D)s + 4 \equiv s^2 + 2\omega_n s + \omega_n^2$$

$$\Rightarrow \quad \omega_n = 2 \quad \text{and} \quad 1 + K_D = 2\times 2 \quad \Rightarrow \quad K_D = 3$$

Then $\dfrac{Y(s)}{R(s)} = \dfrac{4+3s}{(s+2)^2} = \dfrac{4}{(s+2)^2} + \dfrac{3}{4} \cdot s \cdot \dfrac{4}{(s+2)^2}$

i.e. $Y(s) = \dfrac{2}{(s+2)^2} R(s) + \dfrac{3}{4} \cdot s \cdot \dfrac{4}{(s+2)^2} R(s); \quad R(s) = \dfrac{1}{s}$

$\therefore$ time domain output, $y(t) = y_0(t) + \dfrac{3}{4}\dfrac{dy_0(t)}{dt}$

where $y_0(t) = 1 - e^{-2t}(1+2t)$

$$\frac{dy_0(t)}{dt} = 2e^{-2t}(1+2t) - e^{-2t}(2) = 4e^{-2t}t$$

$$\therefore y(t) = 1 - e^{-2t}(1+2t) + 3e^{-2t}t = 1 - e^{-2t} + e^{-2t}t$$

To find if there is overshoot, use the result similar to Q4 of Tutorial 3 to calculate the overshoot, rise time, etc.

Rise time:

$$y(t_r) = 1 = 1 - e^{-2t_r} + e^{-2t_r} t_r \quad \Rightarrow \quad t_r = 1$$

Peak time: $y(t) = 1 - e^{-2t} + e^{-2t} t$

$$\Rightarrow \quad \frac{dy}{dt} = 2e^{-2t} - 2e^{-2t} t + e^{-2t} = 0$$

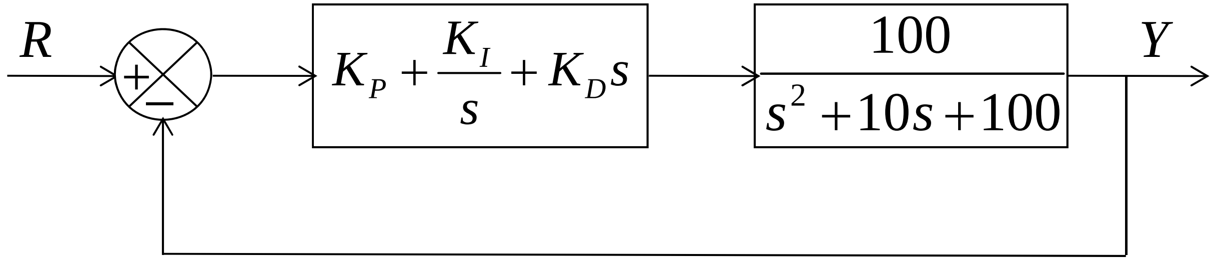
$$\Rightarrow \quad t_p = \frac{3}{2}$$

Overshoot:

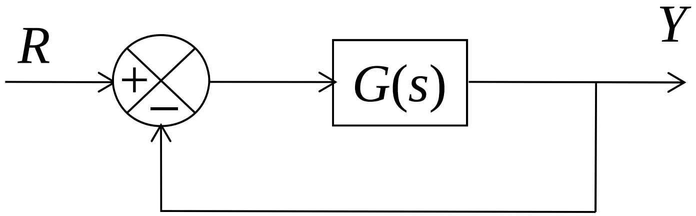
$$M_p = y(t_p) - 1 = 1 - e^{-2t_p} + e^{-2t_p} t_p - 1$$

$$= 0.5e^{-3}$$

2.



$\Rightarrow$



‘unity feedback’

where $G(s) = \dfrac{100\left(K_D s^2 + K_P s + K_I\right)}{s(s^2 + 10s + 100)}$

System is type ‘1’ provided $K_I \neq 0$

$\therefore e_{ss} = 0$ wrt to unit-step input provided $K_I \neq 0$

$$\frac{Y}{R}=\frac{G}{1+G}=\frac{100\left(K_D s^2+K_P s+K_I\right)}{s^3+\left(10+100K_D\right)s^2+\left(100+100K_P\right)s+100K_I}$$

So, C. E. $q(s)=s^3+\left(10+100K_D\right)s^2+\left(100+100K_P\right)s+100K_I$

$$\equiv\left(s^2+2\zeta\omega_n s+\omega_n^2\right)\left(s+p_3\right)$$

Besides ζ, we need spec on ω_n. We also want the pole at $s = -p_3$ to be sufficiently far away from the dominant poles so that the response is approximated by a 2nd order system.

To relate to rise time specification, we "assume" that this 3rd order system behaves like a standard 2nd order system, and use the result of standard 2nd order system.

From the result of standard 2nd order system,

$$\frac{Y}{R} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

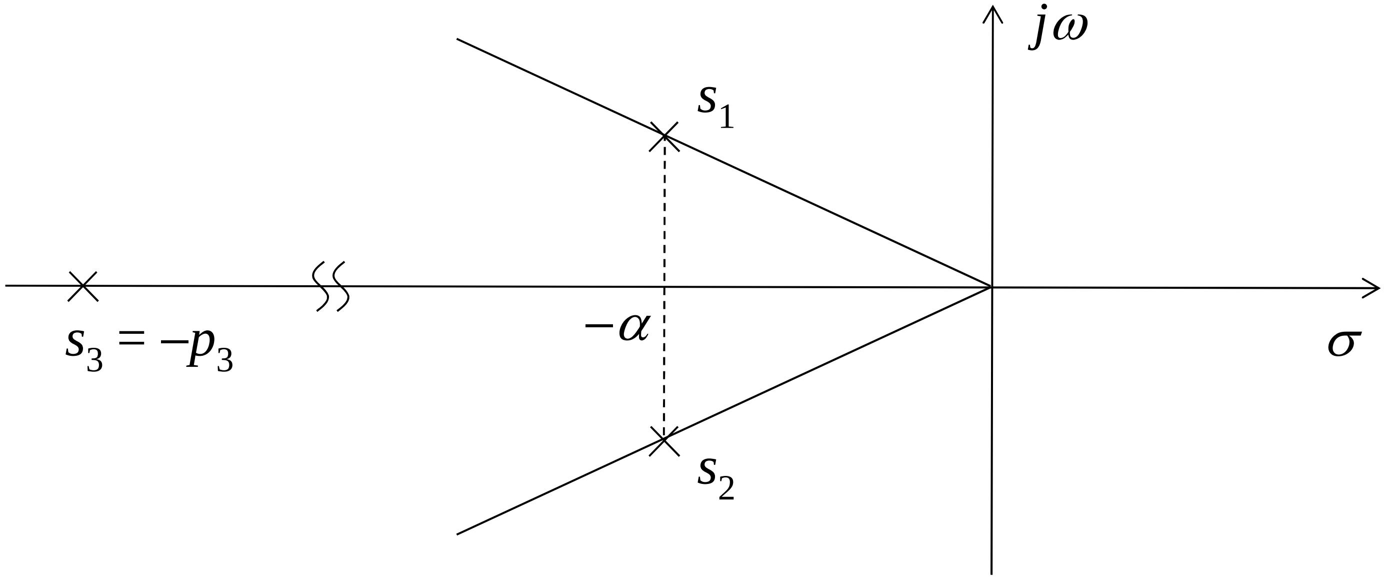
If $\zeta = 0.8$,

then $$t_r = \frac{\pi - \theta}{\omega_n \sqrt{1-\zeta^2}} = 0.5 \qquad (\theta = \cos^{-1}\zeta)$$

$$\Rightarrow \omega_n = \frac{\pi - \theta}{0.5\sqrt{1-\zeta^2}} = 8.33 \text{ rad/sec}$$

$\therefore$ the damping constant of the two dominant poles is

$$\alpha = \zeta\omega_n = 0.8 \times 8.33 = 6.66$$



Set $p_3 = 10 \text{ x} \alpha = 66.6$!

Then the 3 closed-loop poles are: s_1 and s_2 as the dominant complex poles with

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2} = -6.66 \pm j5.0$$

and $s_3 = -66.6$

Hence

$$s^3+\left(10+100K_D\right)s^2+\left(100+100K_p\right)s+100K_I$$

$$\equiv\left(s^2+2\times 6.66s+8.33^2\right)(s+66.6)$$

$$=s^3+80s^2+957s+4622$$

Equating coefficients

$$10+100K_D=80 \quad\Rightarrow\quad K_D=0.7$$

$$100+100K_P=957 \quad\Rightarrow\quad K_P=8.57$$

$$100K_I=4622 \quad\Rightarrow\quad K_I=46.22$$

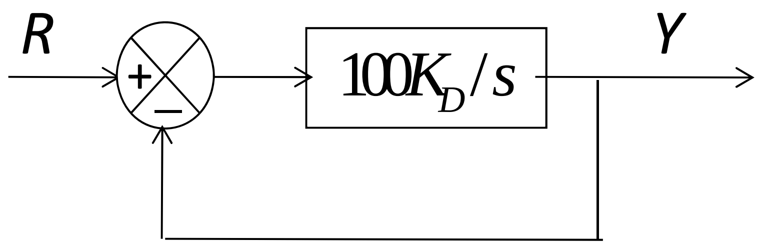
NB: The actual response is affected by the presence of the zero. With the parameters of the controller designed, the 2 zeros of the closed-loop system are $-6.12\pm j5.34$

What happen?

A simple design is to use the zeros of $K_D s^2 + K_P s + K_I$ to cancel the "nice" stable poles of $s^2 + 10s + 100$ of the given system.

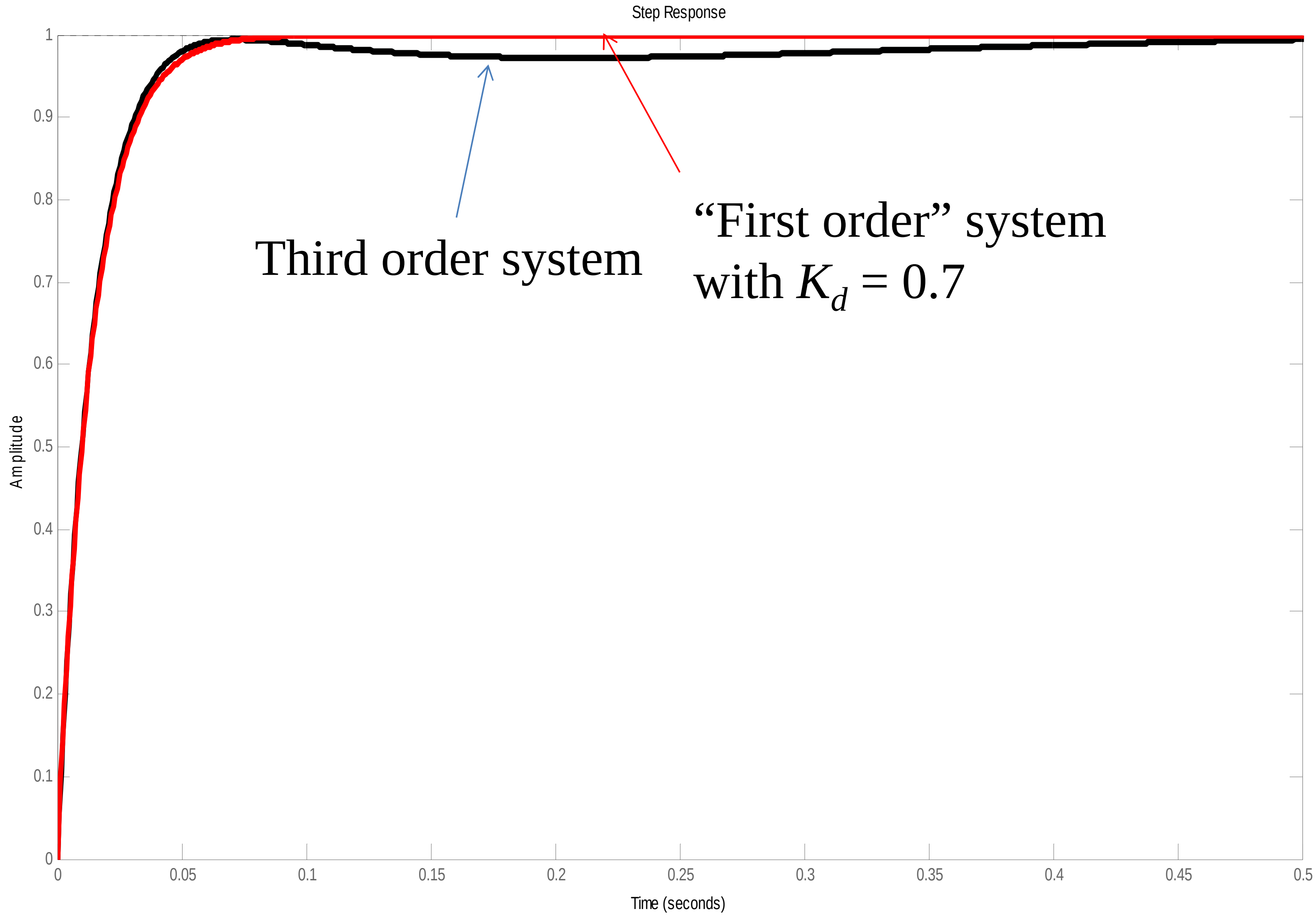
i.e. $$s^2 + \frac{K_P}{K_D} s + \frac{K_I}{K_D} \equiv s^2 + 10s + 100$$

Then the closed loop system "becomes"



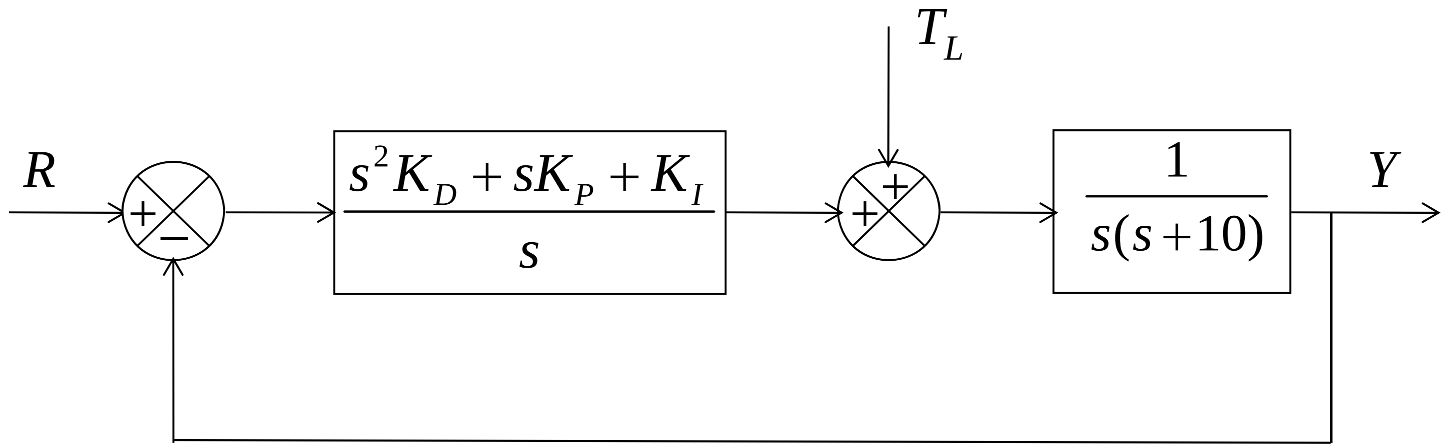
Hence $$\frac{Y}{R} = \frac{100K_D}{s + 100K_D}$$

The system behaves exactly like a 1st order system but it's in fact a 3rd order system. It's just that 2 of the modes cannot be observed from input-output measurement.



Step Response
Amplitude
Time (seconds)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Third order system
"First order" system
with $K_d = 0.7$

3(a)



$$\frac{Y_T}{T_L} = \frac{\dfrac{1}{s(s+10)}}{1 + \dfrac{s^2K_D + sK_P + K_I}{s^2(s+10)}} = \frac{s}{s^2(s+10) + s^2K_D + sK_P + K_I}$$

$$Y_T = \frac{s}{s^2(s+10) + s^2K_D + sK_P + K_I} \cdot \frac{a}{s} \quad ; \quad T_L = \frac{a}{s}$$

$$y_T(\infty) = \lim_{s \to 0} sY_T = 0 \quad \text{if} \quad K_I \neq 0$$

Now, $\dfrac{Y}{R} = \dfrac{s^2 K_D + sK_P + K_I}{s^2(s+10) + s^2 K_D + sK_P + K_I}$

$$
\begin{aligned}
& s^2(s+10) + s^2 K_D + sK_P + K_I \\
&= s^3 + (10 + K_D)s^2 + sK_P + K_I \\
&\equiv \left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)(s+p) \\
&= s^3 + (p + 2\zeta\omega_n)s^2 + \left(\omega_n^2 + 2p\zeta\omega_n\right)s + p\omega_n^2 \\
&= s^3 + (12\zeta\omega_n)s^2 + \left(\omega_n^2 + 20\zeta^2\omega_n^2\right)s + 10\zeta\omega_n^3 \quad ; \quad p = 10\zeta\omega_n
\end{aligned}
$$

Equating coefficients

$$
\begin{aligned}
10 + K_D &= 12\zeta\omega_n \\
K_P &= \omega_n^2 + 20\zeta^2\omega_n^2 \\
K_I &= 10\zeta\omega_n^3
\end{aligned}
$$

With $\zeta = 0.7$ $\Rightarrow$

$$K_D = 8.4\omega_n - 10$$
$$K_P = 10.8\omega_n^2$$
$$K_I = 7\omega_n^3$$

We have many possible solutions as ω_n is still “free”!

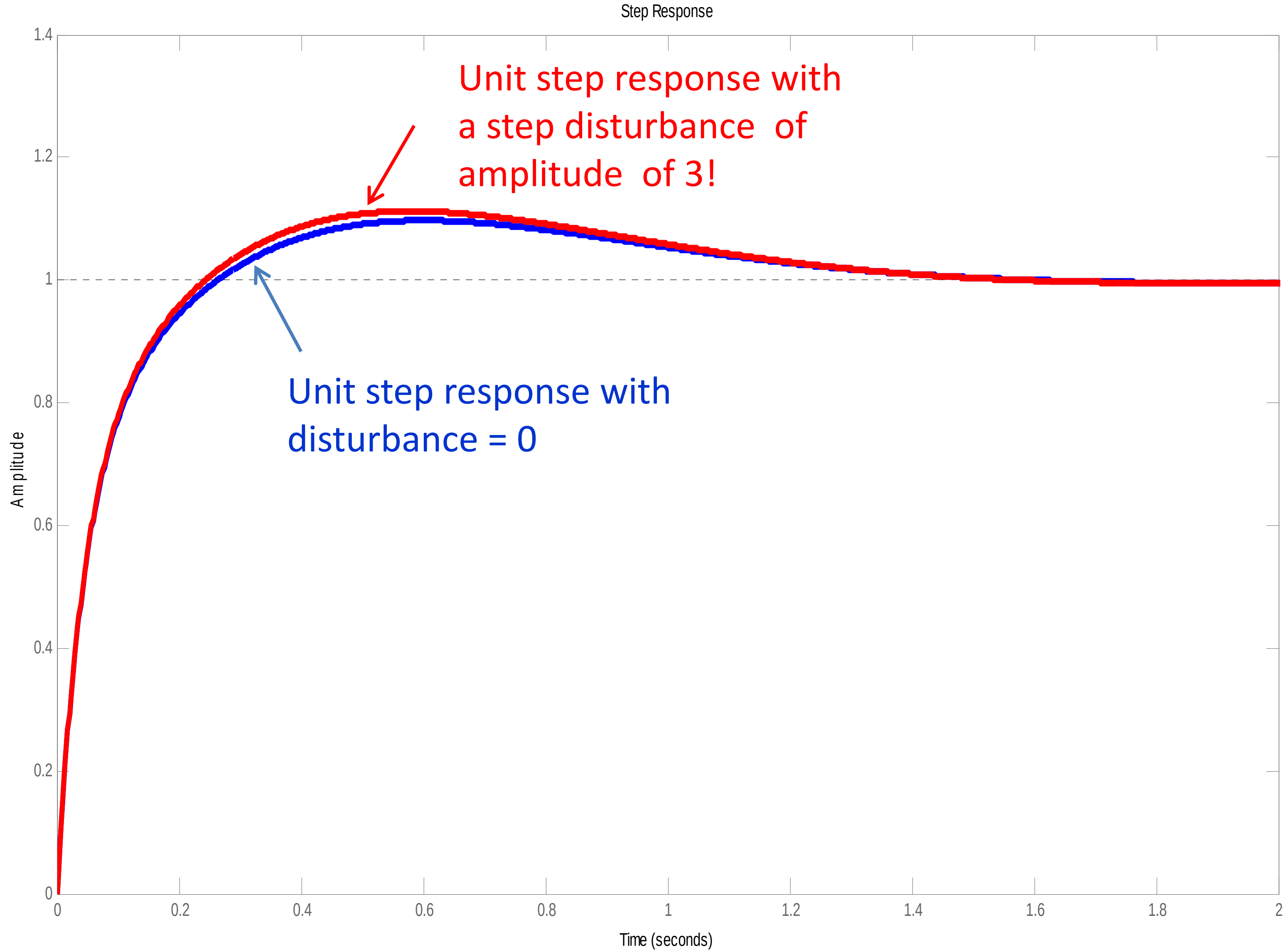
ω_n is related to t_r if we “assume” the CL system behaves like a standard 2nd-order system, and we use

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \cos^{-1}\zeta}{\omega_n\sqrt{1-\zeta^2}}$$

If we want $t_r \approx 1.0$, set $t_r = 1$ to yield $\omega_n = 3.285$. With this ω_n, we’ll get the solutions for K_P, K_I and K_D as

$$K_D = 17.59; \quad K_P = 116.55; \quad K_I = 248.14$$

With the designed PID controller, the unit step response of the 3rd order CL system with/without disturbance is given below:



3(b) From earlier derivation, even with $K_D = 0$,

$$y_T(\infty) = \lim_{s\to 0} sY_T = 0 \quad \text{if} \quad K_I \neq 0$$

To get the 2 CL poles at $-1 \pm j1$, we have

$$(s+1+j)(s+1-j) = s^2 + 2s + 2 \equiv s^2 + 2\zeta\omega_n s + \omega_n^2$$

Thus, the specifications on the 2 desired closed-loop poles is the same as imposing specifications on ζ and ω_n.

In this case,

$$\omega_n = \sqrt{2}, \quad \zeta\omega_n = 1$$

$$\Rightarrow \quad \zeta = 1/\sqrt{2}$$

With $K_D = 0$, from earlier derivation, the closed-loop CE is

$$s^2(s+10)+sK_P+K_I = s^3+10s^2+sK_P+K_I$$
$$\equiv\left(s^2+2s+2\right)(s+p); \quad \text{which yields 2 CL poles at } -1\pm j1$$
$$= s^3+(2+p)s^2+(2+2p)s+2p$$

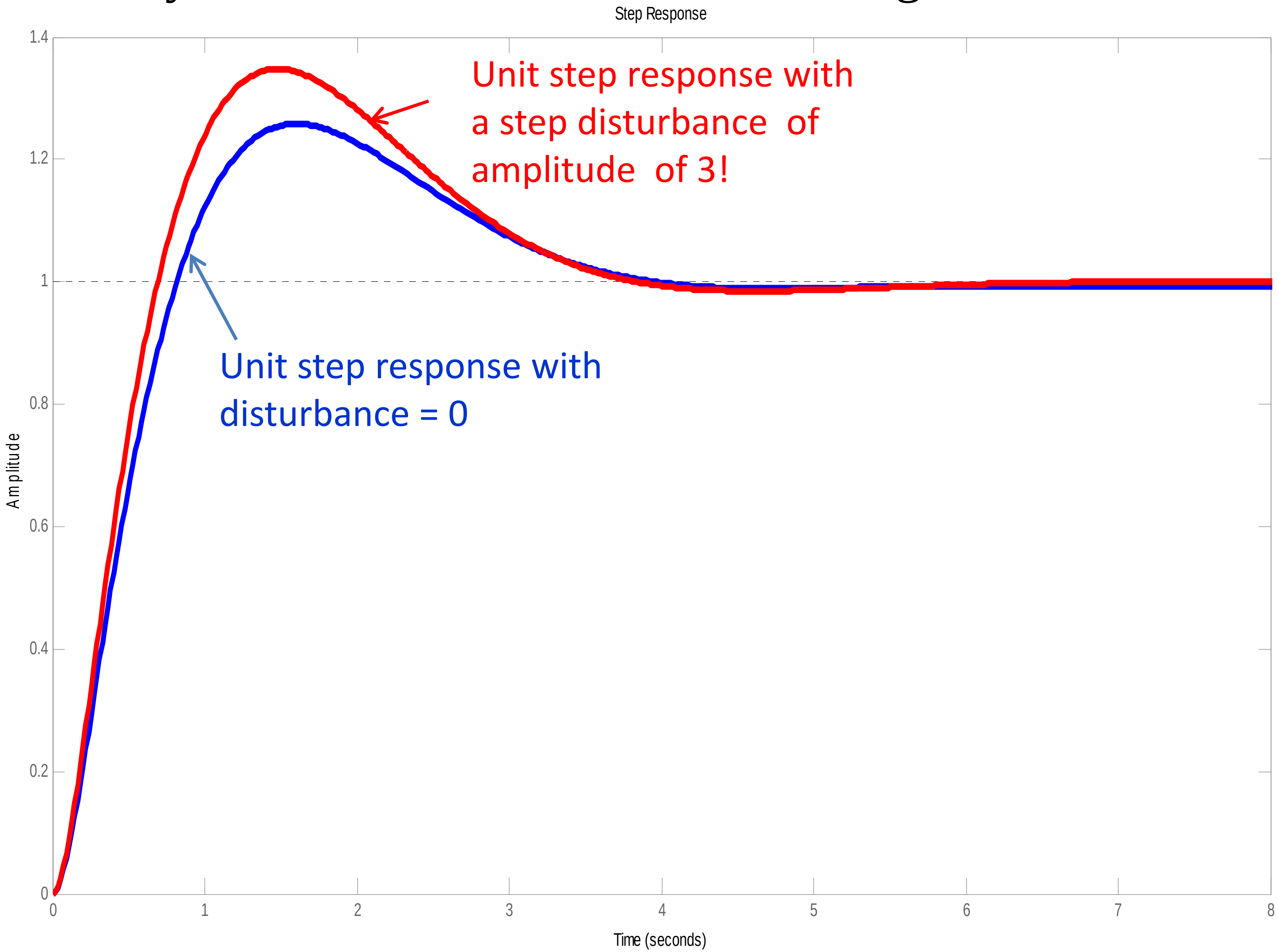
Note that we can no longer pre-fixed p in the desired CE.

Equating coefficients:

$$\begin{aligned} 10 &= 2+p \\ K_P &= 2+2p \\ K_I &= 2p \end{aligned} \quad \Rightarrow \quad \begin{aligned} p &= 8 \\ K_P &= 18 \\ K_I &= 16 \end{aligned}$$

Note that $p = 8 = 8\mathrm{x}\zeta\omega_n$ which will still yield a good dominant 2nd order response.

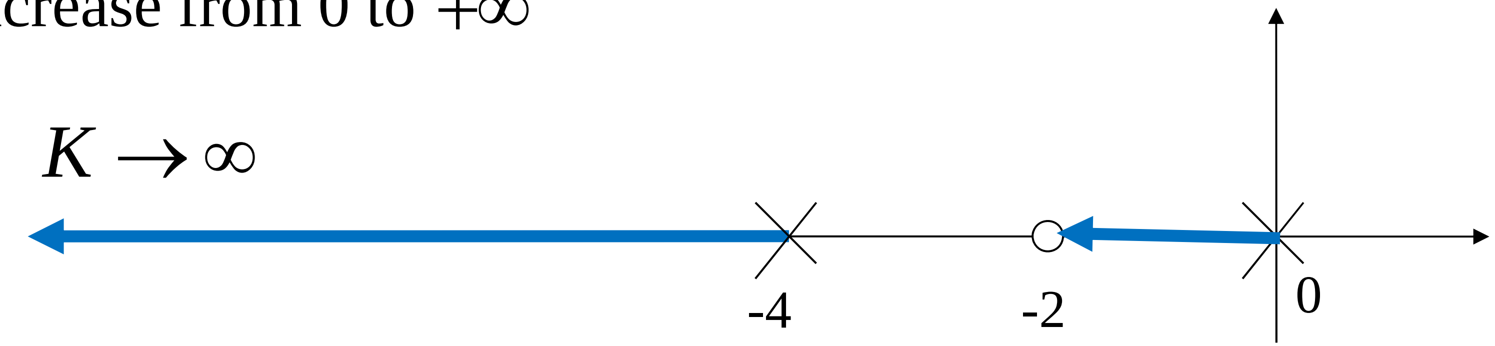
With the designed PI controller, the unit step response of the 3rd order CL system with/without disturbance is given below:



4(a) The closed loop pole is at s = -(2+K). Clearly, the pole will move from -2 to $-\infty$ as K is increase from 0 to $+\infty$

(b) The 2 closed loop poles are at $s_1, s_2 = \dfrac{-(4+K) \pm \sqrt{(4+K)^2 - 8K}}{2}$

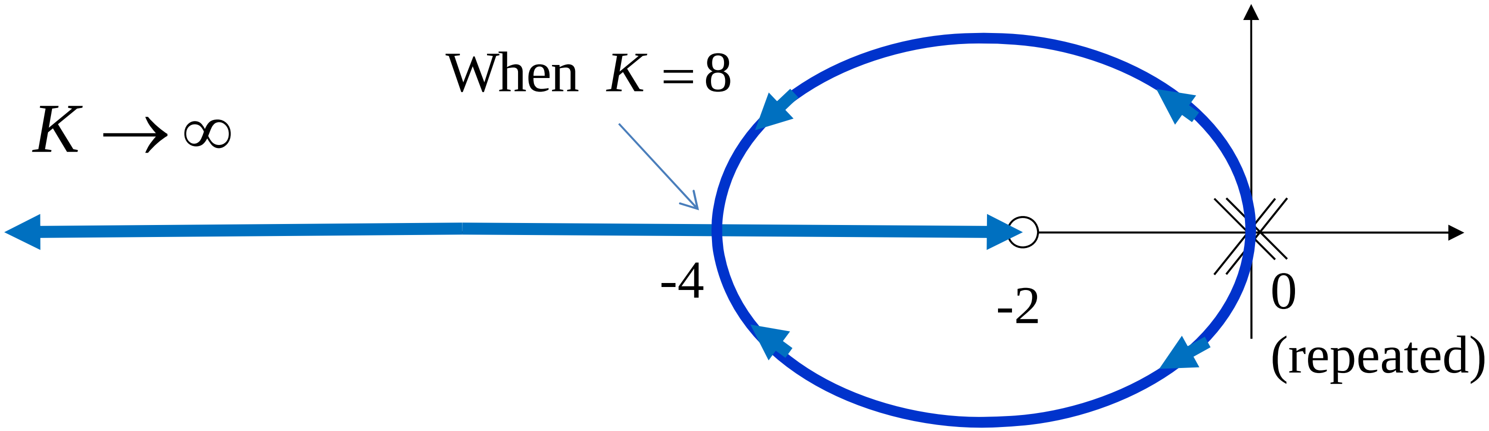
It's easy to verify that both are real and negative as K is increase from 0 to $+\infty$



One can also easily obtain this by apply rules 1, 2 and 7 of the procedure for the root locus sketch.

(c) The 2 closed loop poles are at $s_1, s_2 = \dfrac{-K \pm \sqrt{K^2 - 8K}}{2}$

It's easy to verify that the poles are complex conjugate when $0 < K < 8$ and real and negative when $K \geq 8$. When K = 8, the 2 poles are at s = -4.



From $1 + \dfrac{K(s+2)}{s^2} = 0 \Rightarrow K = -\dfrac{s^2}{(s+2)}$

Thus, $\dfrac{dK}{ds} = 0$ when $s(s+4) = 0$

So, the breakaway/in points are at s = 0 and s= -4, respectively.