EE3011 MODEL& CONTROL

Tutorial 4 (Solutions) Stability Analysis

1. $q(s) = s^3 + a_2 s^2 + a_1 s + a_0\ ; \quad a_0, a_1, a_2 > 0$

s^3	1	a_1
s^2	a_2	a_0
s^1	$\dfrac{a_2 a_1 - a_0}{a_2}$	
s^0	a_0	

Since $a_0, a_1, a_2 > 0$, for stability we require that

$$a_2 a_1 - a_0 > 0$$

2(a). $q(s) = s^4 + 2s^3 + 10s^2 + 20s + 5$

$$
\begin{array}{c|ccc}
s^4 & 1 & 10 & 5 \\
s^3 & 2 & 20 & \\
\hline
s^2 & \dfrac{2\times 10 - 1\times 20}{2} = 0 \to \varepsilon & 5 & \\
s^1 & \dfrac{\varepsilon\times 20 - 2\times 5}{\varepsilon} & & \\
s^0 & 5 & &
\end{array}
$$

As $\varepsilon \to 0^+$, $\dfrac{20\varepsilon - 10}{\varepsilon} < 0$ (i.e. $-ve$)

Hence we get 2 sign changes.

$\Rightarrow$ 2 roots in the RHP (+2 in the LHP)

2(b). $q(s) = s^6 + s^5 + 4s^4 + 4s^3 + 5s^2 + 4s + 2$

$$
\begin{array}{c|cccc}
s^6 & 1 & 4 & 5 & 2 \\
s^5 & 1 & 4 & 4 & \\
\hline
s^4 & \frac{4-4}{1} = 0 \to \varepsilon & 1 & &
\end{array}
$$

If we proceed, we will get a messy expression in ε. *Try it!*

Consider

$$q''(z) = z^6 q(z^{-1}) = 2z^6 + 4z^5 + 5z^4 + 4z^3 + 4z^2 + z + 1$$

z^6	2	5	4	1	
z^5	4	4	1		
z^4	$\frac{12}{4}$	$\frac{14}{4}$	1		
z^3	$-\frac{2}{3}$	$-\frac{1}{3}$			
z^2	2	1			$A(z)$
z^1	$0 \rightarrow 4$		'zero row'		$A'(z)$
z^0	1				

Clearly, we get 2 sign changes $\Rightarrow$ 2 roots in the RHP.

One auxiliary equation: $A(z) = 2z^2 + 1 \;\Rightarrow\; A'(z) = 4z$

$\Rightarrow$ 2 roots of $q''(z)$ on the $j\omega$-axis at $z = \pm j1/\sqrt{2}$.

(+ another 2 roots in the LHP!).

The "same" conclusion applies to the roots of $q(s)$! (But where exactly are the 2 roots of $q(s)$ that are on the $j\omega$-axis?)

Auxiliary equation in z-variable: $A(z) = 2z^2 + 1 = 0$

$\Rightarrow$

Auxiliary equation in s-variable: $A''(s) = s^2\,(2s^{-2} + 1) = s^2 + 2 = 0$

Factorize $\quad q(s) = s^6 + s^5 + 4s^4 + 4s^3 + 5s^2 + 4s + 2$

$$= (s^2 + 2)(s^4 + s^3 + 2s^2 + 2s + 1) = 0$$

Clearly, $A''(s)$ is a factor of $q(s)$.

3(a). $q(s) = s^4 + 10s^3 + 25s^2 + 50s + K$

Necessary condition for stability is that $K > 0$.

s^4	1	25	K
s^3	10	50	
s^2	$\frac{250-50}{10} = 20$	K	
s^1	$\frac{1000-10K}{20}$		
s^0	K		

Thus, an additional condition for stability is that $1000 - 10K > 0 \quad \Rightarrow \quad K < 100$

So, for stability, $0 < K < 100$.

When $K = 100$, s^1-row contains all zeros, $\therefore A(s) = 20s^2 + K$; $K = 100$

$$\Rightarrow \; s^2 + 5 = 0 \qquad \Rightarrow \qquad s = \pm j\sqrt{5}$$

i.e. system has undamped oscillation when $K = 100$, and $\omega_n = \sqrt{5}$ rad/s

To check if all roots lie to the left of $s = -1.0$, we set $s := z - 1$ in $q(s)$, then

$$q'(z) = (z-1)^4 + 10(z-1)^3 + 25(z-1)^2 + 50(z-1) + K$$

$$= \left(z^4 - 4z^3 + 6z^2 - 4z + 1\right)$$

$$+ 10\left(z^3 - 3z^2 + 3z - 1\right) + 25\left(z^2 - 2z + 1\right) + 50(z-1) + K$$

$$= z^4 + 6z^3 + z^2 + 26z + K - 34$$

Necessary condition is that $K > 34$

z^4	1	1	K-34
z^3	6	26	
z^2	$\frac{6-26}{6} = -\frac{10}{3}$	K-34	
z^1	$\frac{352-18K}{-10}$		
z^0	K-34		

There is at least one change of sign in the 1st column of the array, independent of K, so the system is not stable w.r.t. z-plane for all values of K.

Thus, for all values of K, the characteristic equation CANNOT have all its roots lying to the left of s = -1.0.

3(b). $q(s) = s^3 + (K+3)s^2 + 3Ks + 2K$

Necessary condition for stability is $K > 0$

$$\begin{array}{c|cc} s^3 & 1 & 3K \\ s^2 & K+3 & 2K \\ \hline s^1 & \dfrac{3K(K+3)-2K}{K+3} & \\ s^0 & 2K & \end{array}$$

Thus, additional condition for stability is that

$$3K(K+3) - 2K > 0 \quad \Rightarrow \quad K(3K+7) > 0 \quad \Rightarrow \quad K > 0$$

So, for stability, $K > 0$.

When $K = 0$, $q(s) = s^3 + 3s^2$, so there are 2 roots at the origin and the system will be unstable. There is no other value of K that will yield an auxiliary equation. Thus, in this case, the system will not have undamped oscillation for all values of K.

To check if all roots lie to the left of $s = -1.0$, we set $s := z - 1$ in $q(s)$, then

$$
\begin{aligned}
q'(z) &= (z-1)^3 + (K+3)(z-1)^2 + 3K(z-1) + 2K \\
&= \left(z^3 - 3z^2 + 3z - 1\right) + (K+3)\left(z^2 - 2z + 1\right) + 3K(z-1) + 2K \\
&= z^3 + Kz^2 + (K-3)z + 2
\end{aligned}
$$

Thus, necessary condition for 'stability w.r.t. z-plane' is $K > 3$ and

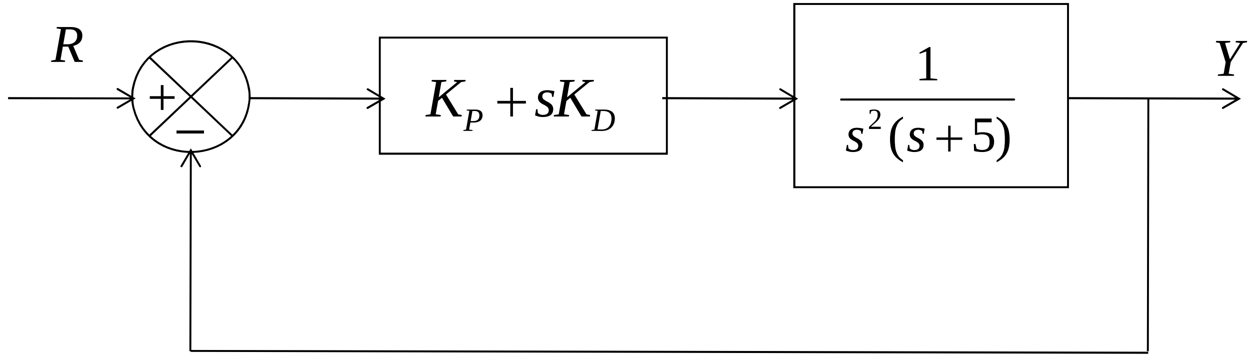
$$
\begin{array}{c|cc}
z^3 & 1 & K-3 \\
z^2 & K & 2 \\
\hline
z^1 & \dfrac{K(K-3)-2}{K} & \\
z^0 & 2 &
\end{array}
$$

For stability w.r.t. z-plane, the additional condition is

$$K^2 - 3K - 2 > 0 \qquad \Rightarrow \qquad K > \frac{3+\sqrt{17}}{2}$$

Thus, all roots of $q(s)$ will lie to the left of $s = -1.0$ if $\quad K > \dfrac{3+\sqrt{17}}{2}$

4.

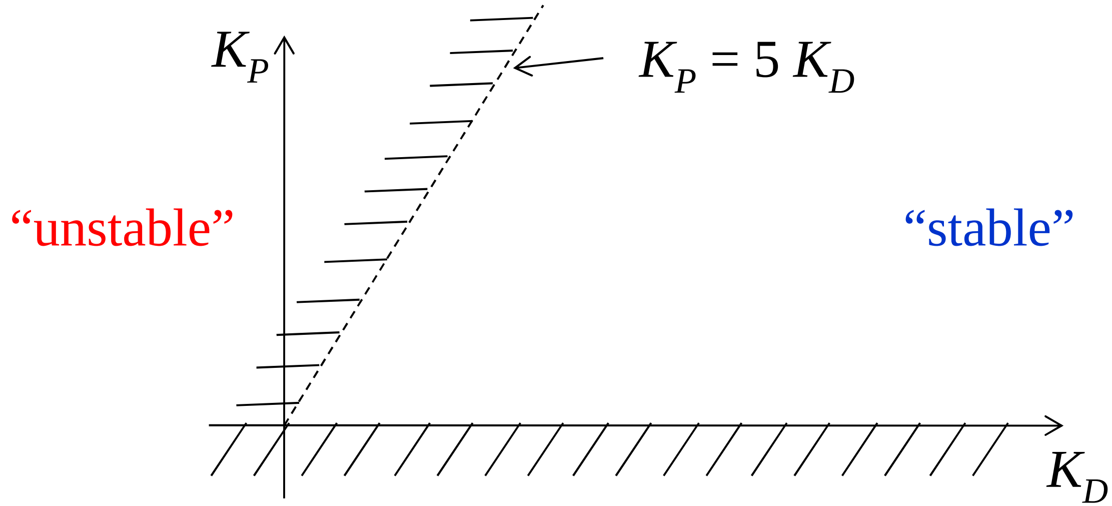


CLTF: $$\frac{Y}{R} = \frac{K_P + sK_D}{s^2(s+5) + sK_D + K_P}$$

C.E.: $$q(s) = s^3 + 5s^2 + K_D s + K_P$$

s^3	1	K_D
s^2	5	K_P
s^1	$\frac{5K_D - K_P}{5}$	0
s^0	K_P	

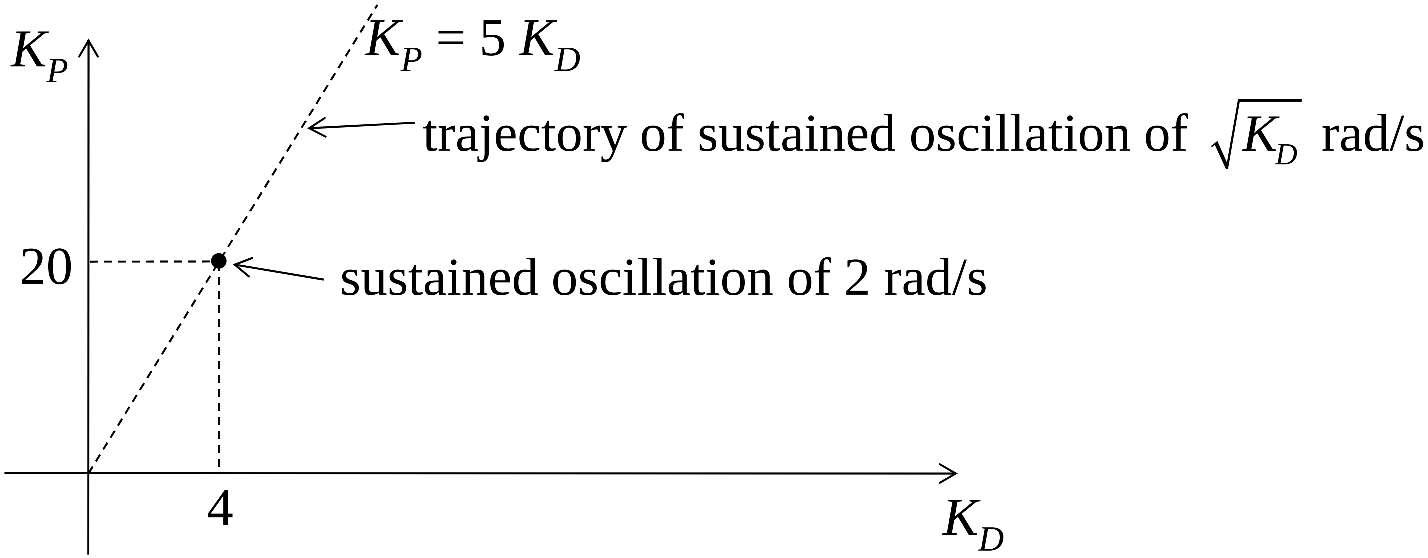
For stability, we need $K_P > 0$ and $5K_D - K_P > 0 \Rightarrow K_P < 5\,K_D$



When $K_P = 5\,K_D$, we get an auxiliary equation

$$A(s) = 5s^2 + K_P$$

$$= 5s^2 + 5K_D \Rightarrow s = \pm j\sqrt{K_D}$$



$$\sqrt{K_D} = 2 \qquad \Rightarrow \qquad K_D = 4 \ \ (\text{and } K_P = 20)$$

To check if all roots lie to the left of s = –1.0, we set s := z – 1 in $q(s)$, so

$$\begin{aligned} q'(z) &= (z-1)^3 + 5(z-1)^2 + K_D(z-1) + K_P \\ &= z^3 - 3z^2 + 3z - 1 + 5\left(z^2 - 2z + 1\right) + K_D(z-1) + K_P \\ &= z^3 + 2z^2 + \left(K_D - 7\right)z + K_P - K_D + 4 \end{aligned}$$

For stability w.r.t. *z*-plane, we use the result of Q1 and we need

$$K_D - 7 > 0 \quad \Rightarrow \quad K_D > 7$$

$$4 + K_P - K_D > 0 \quad \Rightarrow \quad K_P > K_D - 4$$

$$2(K_D - 7) > 4 + K_P - K_D \quad \Rightarrow \quad K_P < 3K_D - 18$$

We find the intersection of the above 3 regions to give: