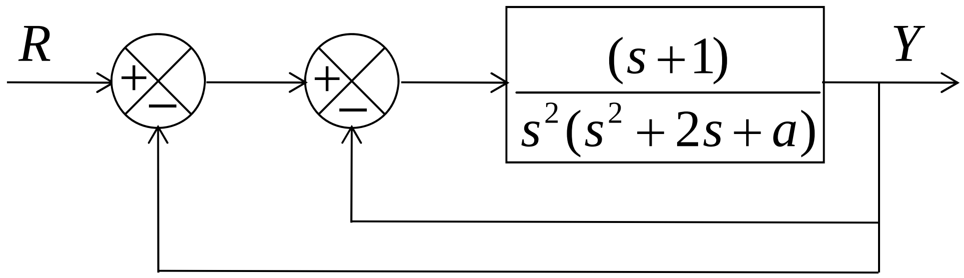
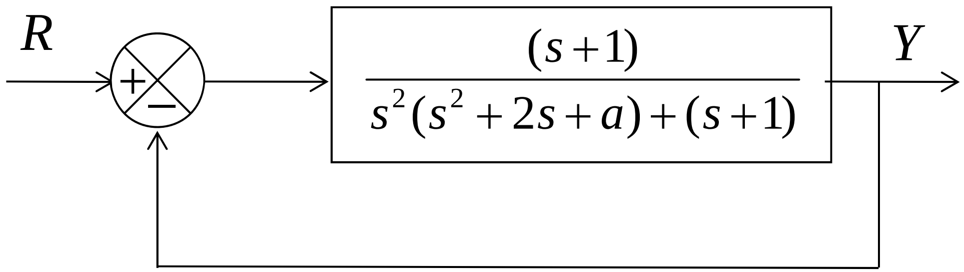
EE3011 MODEL& CONTROL
Tutorial 5 (Solutions) System Performance

1.



$\Rightarrow$



$$G(s)=\frac{(s+1)}{s^2\left(s^2+2s+a\right)+(s+1)}$$

The system is type '0' (with respect to $R(s)$)!

$$\frac{Y}{R}=\frac{G(s)}{1+G(s)}=\frac{s+1}{s^2\left(s^2+2s+a\right)+2(s+1)}$$

When $a = 5$, C.E.: $q(s) = s^4 + 2s^3 + 5s^2 + 2s + 2 = 0$

The Routh array is

$$
\begin{array}{c|ccc}
s^4 & 1 & 5 & 2 \\
s^3 & 2 & 2 & \\
s^2 & 4 & 2 & \\
s^1 & 1 & & \\
s^0 & 2 & &
\end{array}
$$

The system is stable and hence

$$K_{pos} = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{(s+1)}{s^2\left(s^2 + 2s + 5\right) + (s+1)} = 1$$

K_{vel} and K_{acc} are both zero because the system is type '0'.

The steady-state value of $y(t)$ w.r.t unit step input is

$$y_{ss}(\text{unit} - \text{step}) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s \frac{(s+1)}{s^2\left(s^2 + 2s + 5\right) + 2(s+1)} \times \frac{1}{s} = \frac{1}{2}$$

When $a = 2$, C.E.: $q(s) = s^4 + 2s^3 + 2s^2 + 2s + 2 = 0$

The Routh array is

s^4	1	2	2
s^3	2	2	
s^2	1	2	
s^1	-2		
s^0	2		

The system is unstable and hence the notions of error constants are not useful here. If one simply compute steady-state value of $y(t)$ w.r.t unit step input using final value theorem,

$$y_{ss}(\text{unit}-\text{step}) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s \frac{(s+1)}{s^2\left(s^2+2s+2\right)+2(s+1)} \times \frac{1}{s} = \frac{1}{2}$$

one would get a wrong answer. Here, the system is unstable and hence the steady state error w.r.t. unit-step input is infinity!

- 2a. Let's focus on the output due to disturbance input $W(s)=\dfrac{a}{s}$

$$\frac{Y_W}{W}=\frac{\dfrac{1}{s(s+10)}}{1+\dfrac{sK_P+K_I}{s^2(s+10)}}$$

$$=\frac{s}{s^3+10s^2+sK_P+K_I} \qquad \text{(notice the } s \text{ term in the numerator)}$$

$$Y_W=\frac{s}{s^3+10s^2+sK_P+K_I}\times\frac{a}{s}$$

$$y_{ssw}=\lim_{s\to 0} sY_W(s)=\begin{cases}0 & \text{if } K_I\neq 0 \text{ and } 10K_P>K_I>0\\ \dfrac{a}{K_P} & \text{if } K_I=0 \text{ and } K_P>0\end{cases}$$

· 2b. With disturbance input enters at the output:

$$\frac{Y_W}{W} = \frac{1}{1 + \dfrac{sK_P + K_I}{s^2(s+10)}}$$

$$= \frac{s^2(s+10)}{s^3 + 10s^2 + sK_P + K_I}$$ (notice the s^2 term in the numerator)

$$Y_W = \frac{s^2(s+10)}{s^3 + 10s^2 + sK_P + K_I} \times \frac{a}{s}$$

$$y_{ssw} = \lim_{s\to 0} sY_W(s) = \begin{cases} 0 & \text{if } K_I \neq 0 \text{ and } 10K_P > K_I > 0 \\ 0 & \text{if } K_I = 0 \text{ and } K_P > 0 \end{cases}$$

3. $$G(s)=\frac{K}{s(s+a)(s+b)}=\frac{K}{s^3+(a+b)s^2+abs}$$

$$T(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{K}{s(s+a)(s+b)+K}$$

$$S_K^T=\frac{dT(s)}{dK}\bullet\frac{K}{T(s)}=\frac{(s(s+a)(s+b)+K)-K}{(s(s+a)(s+b)+K)^2}\bullet\frac{s(s+a)(s+b)+K}{1}$$

$$=\frac{s(s+a)(s+b)}{s(s+a)(s+b)+K}=\frac{1}{1+\frac{K}{s(s+a)(s+b)}}$$

Or

$$S_K^T=\frac{dT(s)}{dG(s)}\bullet\frac{dG(s)}{dK}\bullet\frac{K}{T(s)}=\frac{1}{(1+G)^2}\bullet\frac{1}{s(s+a)(s+b)}\bullet\frac{s(s+a)(s+b)+K}{1}$$

$$=\frac{1}{1+\frac{K}{s(s+a)(s+b)}}$$

$$S_a^T = \frac{dT(s)}{dG(s)} \bullet \frac{dG(s)}{da} \bullet \frac{a}{T(s)}$$

$$= \frac{1}{(1+G)^2} \bullet \frac{-K(s^2+bs)}{(s(s+a)(s+b))^2} \bullet \frac{a(s(s+a)(s+b)+K)}{K}$$

$$= \frac{-a(s^2+bs)}{s(s+a)(s+b)+K} = \frac{-1}{1+\frac{s^3+bs^2+K}{a(s^2+bs)}}$$

$$S_b^T = \frac{dT(s)}{dG(s)} \bullet \frac{dG(s)}{db} \bullet \frac{b}{T(s)}$$

$$= \frac{1}{(1+G)^2} \bullet \frac{-K(s^2+as)}{(s(s+a)(s+b))^2} \bullet \frac{b(s(s+a)(s+b)+K)}{K}$$

$$= \frac{-b(s^2+as)}{s(s+a)(s+b)+K} = \frac{-1}{1+\frac{s^3+as^2+K}{b(s^2+as)}}$$

4. For the open-loop system:

$$T_1(s) = \frac{Y(s)}{R(s)} = \frac{K}{s+\alpha} = \frac{K}{s+10} \quad \text{when} \quad \alpha = 10$$

$$S_\alpha^{T_1} = \frac{dT_1(s)}{d\alpha} \bullet \frac{\alpha}{T_1(s)} = \frac{-K}{(s+\alpha)^2} \bullet \frac{\alpha(s+\alpha)}{K} = \frac{-\alpha}{s+\alpha} = \frac{-10}{s+10} \quad \text{when } \alpha = 10$$

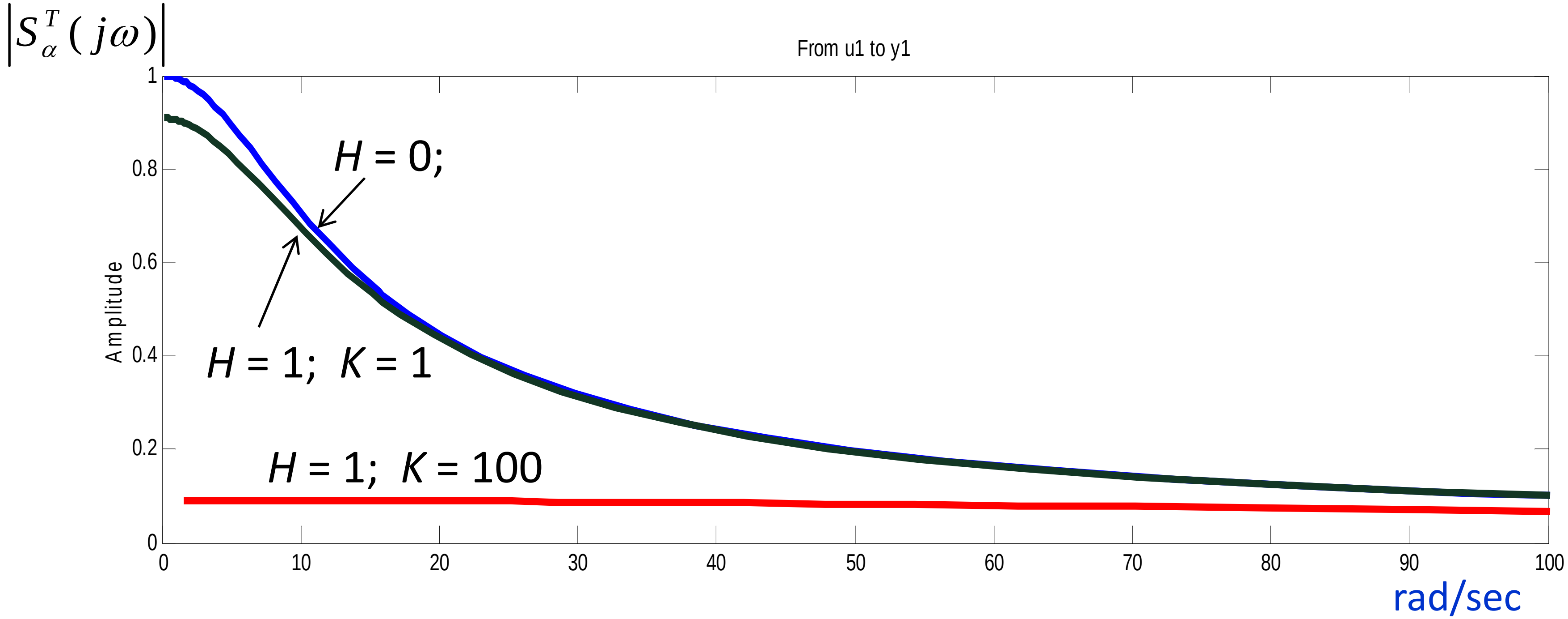
The sensitivity function is independent of *K*!

For the closed-loop system:

$$T_2(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s+\alpha+K} = \frac{K}{s+10+K} \quad \text{when} \quad \alpha = 10$$

$$S_\alpha^{T_2} = \frac{dT_2(s)}{d\alpha} \bullet \frac{\alpha}{T_2(s)} = \frac{-K}{(s+\alpha+K)^2} \bullet \frac{\alpha(s+\alpha+K)}{K} = \frac{-\alpha}{s+\alpha+K}$$

$$= \frac{-10}{s+10+K} \quad \text{when} \quad \alpha = 10$$



The sensitivity of both the open-loop and closed-loop systems decreases with increasing frequency, with the closed-loop system having lower sensitivity (provided $K > 0$)

The sensitivity for the closed-loop system decreases with increasing K.

5. For the open-loop system:

$$T_1(s) = \frac{Y(s)}{R(s)} = \frac{K}{s^2 + 4s + 5} = \frac{2}{s^2 + 4s + 5} \quad \text{when } \; K = 2$$

For the closed-loop system:

$$T_2(s) = \frac{Y(s)}{R(s)} = \frac{K}{(s+1)(s+3) + K} = \frac{2}{s^2 + 4s + 5} \quad \text{when } \; K = 2$$

The sensitivity of the open-loop system:

$$S_K^{T_1} = \frac{dT_1(s)}{dK} \bullet \frac{K}{T_1(s)} = \frac{1}{s^2 + 4s + 5} \bullet \frac{s^2 + 4s + 5}{1} = 1$$

The sensitivity function is 1 for all *K*, and all frequencies.

The sensitivity of the closed-loop system:

$$S_K^{T_2} = \frac{dT_2(s)}{dK} \bullet \frac{K}{T_2(s)} = \frac{(s+1)(s+3)+K-K}{((s+1)(s+3)+K)^2} \bullet \frac{(s+1)(s+3)+K}{1}$$

$$= \frac{1}{1+\dfrac{K}{(s+1)(s+3)}} = \frac{1}{1+\dfrac{2}{(s+1)(s+3)}} \quad \text{when } K=2$$

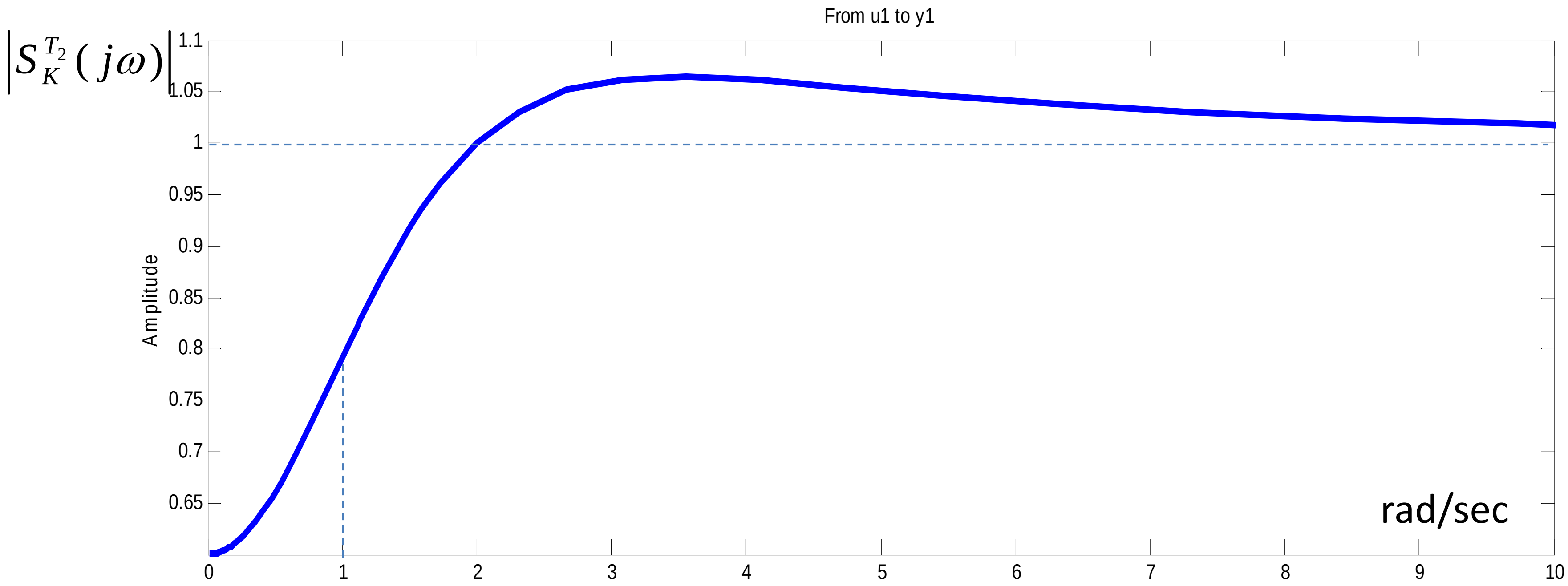
The sensitivity function is dependent on *K*, and the complex variable *s*, i.e. it also depends on the frequencies.

Thus, the sensitivity function can be adjusted by varying *K* or restricting the frequencies of the input function to within an appropriate range.

With $s = j\omega$: $$S_K^{T_2}(j\omega) = \frac{1}{1+\dfrac{2}{(3-\omega^2+j4\omega)}} \quad \text{when} \quad K=2$$

When $\omega = 0$, $$S_K^{T_2}(j0) = \frac{1}{1+\frac{2}{3}} = 0.6$$

When $\omega = 1$ rad/sec, $$\left|S_K^{T_2}(j1)\right| = \left|\frac{1}{1+\frac{2}{2+j4}}\right| = 0.79$$



Thus, the closed-loop system is 40% less sensitive than open-loop system when at steady-state, and 21% less sensitive than the open-loop system for low frequencies (< 1 rad/s). For high frequencies, the sensitivity of the closed-loop system approaches 1, the same as that of the open-loop system.

It’s interesting to note that for this problem, the sensitivity of the closed-loop system is higher than that of the open-loop system for certain range of frequencies (> 2 rad/s)!