EE3011 MODEL& CONTROL

Tutorial 3 (Solutions) Time Domain Analysis

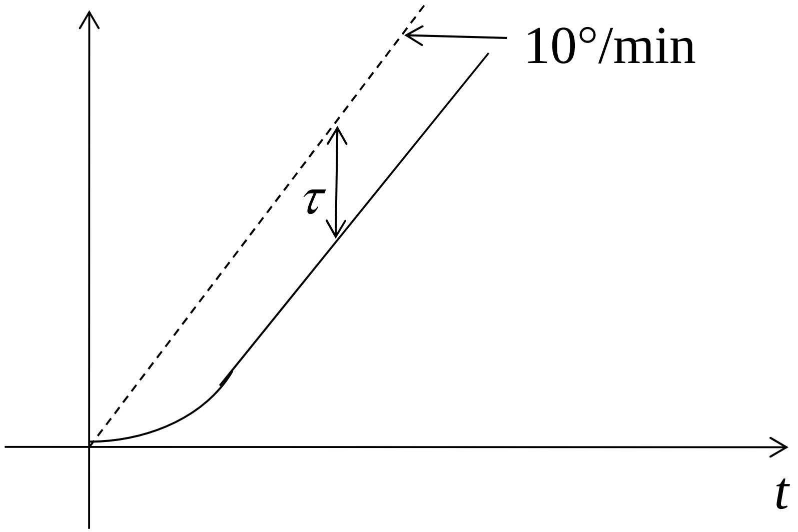
1. For 1st order system, the response to a unit-step input is

$$y(t) = 1 - e^{-\frac{t}{\tau}}$$

$$\therefore \quad 0.98 = 1 - e^{-\frac{1}{\tau}} \quad \Rightarrow \quad e^{-\frac{1}{\tau}} = 0.02 \qquad (\tau \text{ in min})$$

$$\Rightarrow \tau = 0.256 \text{ min}$$

Response to a unit-ramp input:



Steady-state error $\equiv \tau = 0.256$ min

$= 0.256 \text{ min} \times 10°/\text{min} = 2.56°$

Alternatively, treat the input as a ramp input with an amplitude of $a = 10°/\text{min}$. Then

$$e_{ss} = a\tau = 10°/\text{min} \times \tau = 2.56°$$

2. The key to this problem is to realize that the input can be decomposed into 2 step inputs with one a shifted step input. With a standard step input, the output is given as follow:

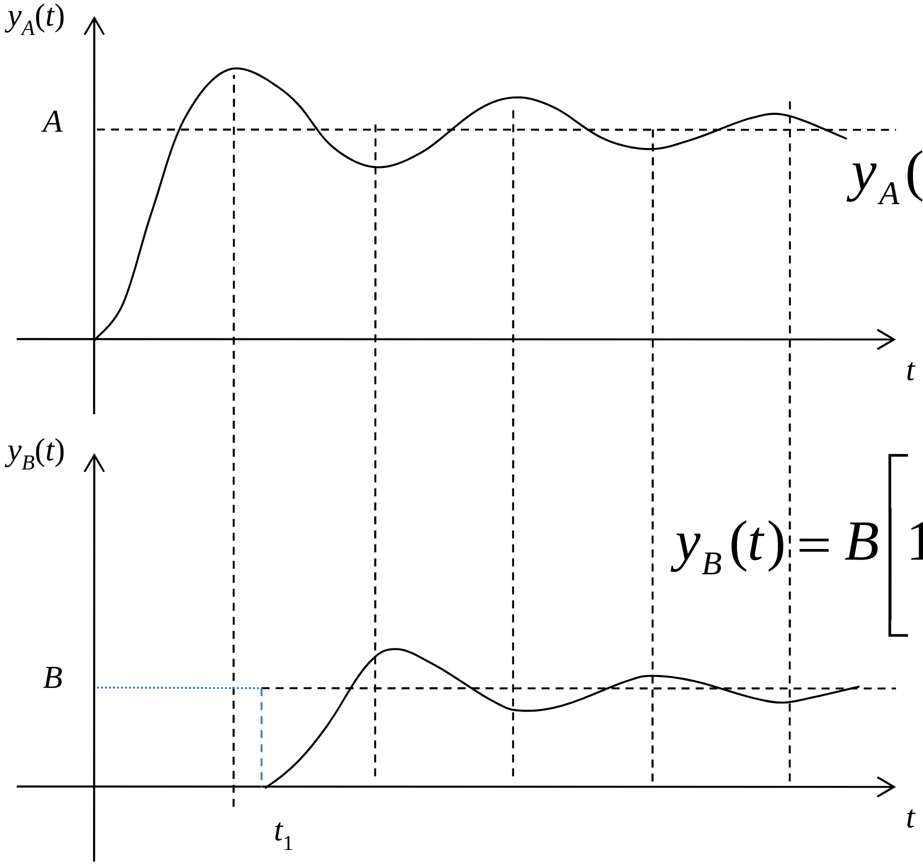
$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}; \quad R(s) = \frac{a}{s} \qquad \therefore \quad Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \bullet \frac{a}{s}$$

i.e. $$y(t) = a\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \theta\right)\right)$$

where $$\omega_d = \omega_n \sqrt{1-\zeta^2}; \theta = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$$

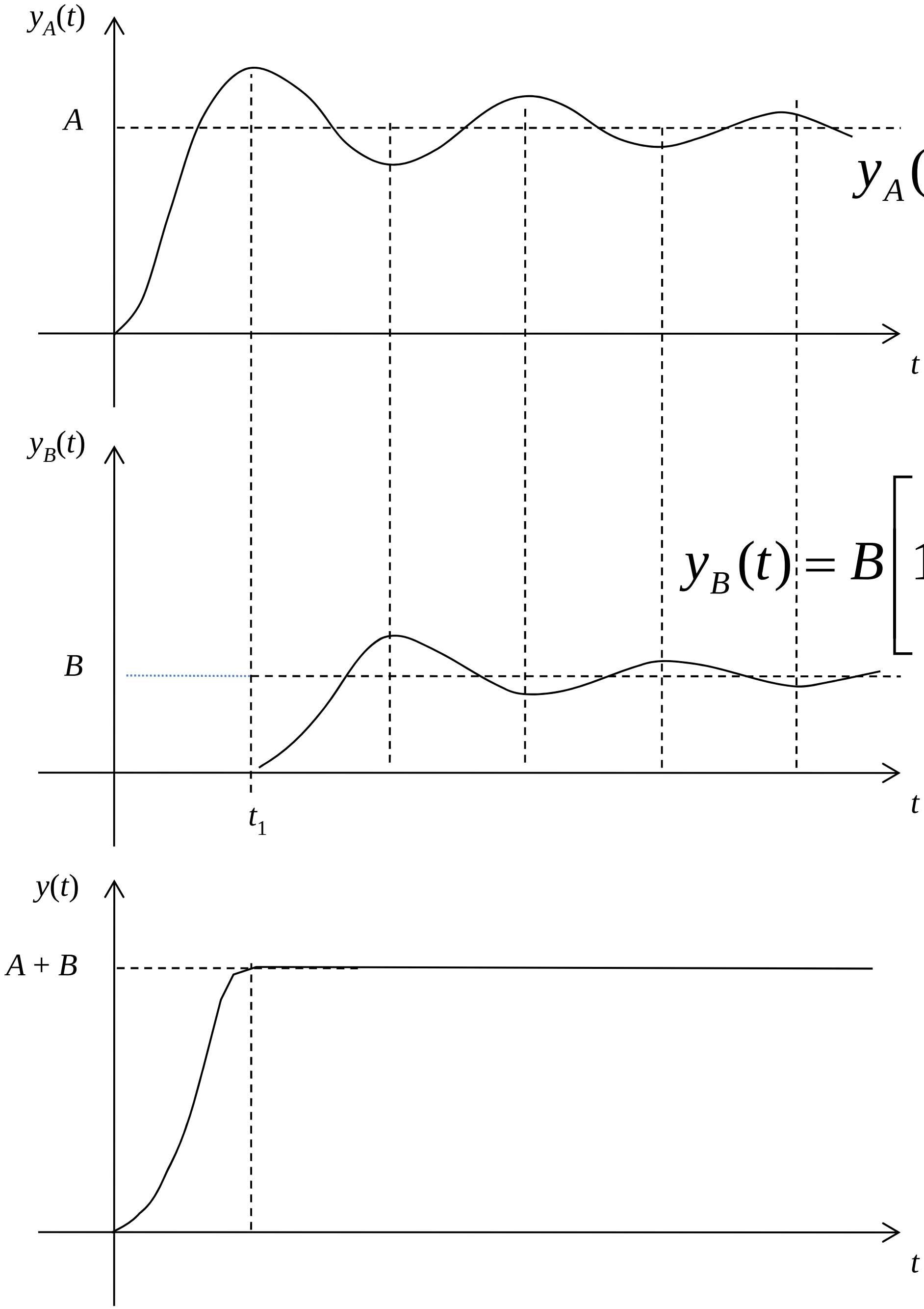
Now, $r(t) = A\,u(t) + B\,u(t - t_1)$, hence

$$y(t) = y_A(t) + y_B(t)$$



$$y_A(t) = A\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \theta\right)\right]u(t)$$

$$y_B(t) = B\left[1 - \frac{e^{-\zeta\omega_n (t-t_1)}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d (t-t_1) + \theta\right)\right]u(t-t_1)$$



$$y_A(t) = A\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \theta\right)\right]u(t)$$

$$y_B(t) = B\left[1 - \frac{e^{-\zeta\omega_n (t-t_1)}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d (t-t_1) + \theta\right)\right]u(t-t_1)$$

If $\zeta = 0.4$ and $\omega_n = 1.0$, then

Peak of $y_A(t)$ at $t_1 = t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad \Rightarrow t_1 = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 3.43$

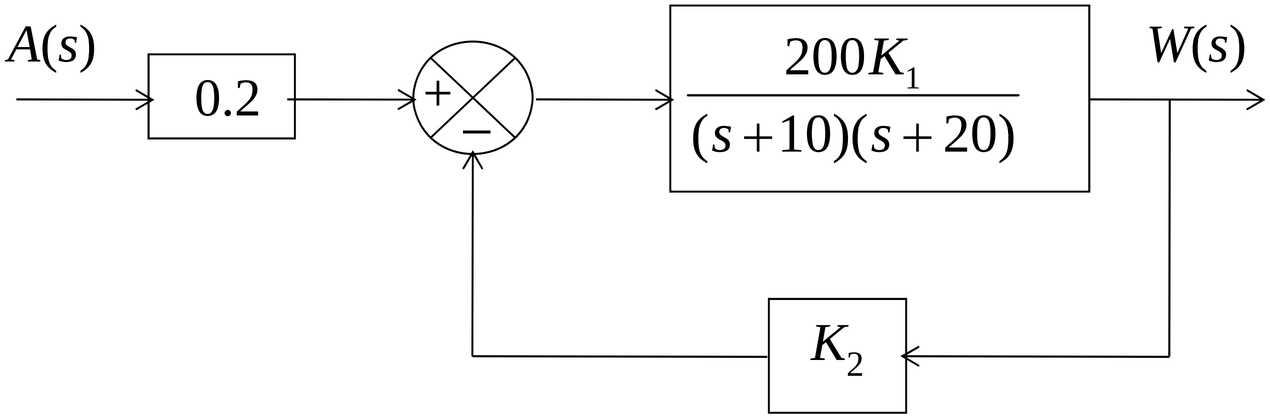
We also want $y_A(t_1) = H = 1$.

$$\therefore\ y(t_1) = y_A(t_1) = A\left[1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}\right] = H = 1$$

$$A = \frac{1}{1+e^{-\zeta\pi/\sqrt{1-\zeta^2}}} = \frac{1}{1+0.25} = 0.8$$

and $B = H - A = 0.2$

3.



$$\frac{W(s)}{A(s)} = \frac{40K_1}{(s+10)(s+20)+200K_1K_2}$$

$$= \frac{40K_1}{s^2+30s+200\left(1+K_1K_2\right)}$$

$$= \frac{a\cdot 200\left(1+K_1K_2\right)}{s^2+30s+200\left(1+K_1K_2\right)} \quad ; \qquad a = \frac{40K_1}{200\left(1+K_1K_2\right)}$$

$$\equiv \frac{a\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad ;$$

a is the amplitude of the steady-state gain

Equating coefficients, we have

$$2\zeta\omega_n = 30; \qquad 200(1 + K_1K_2) = \omega_n^2 \qquad\qquad \left(a = \frac{40K_1}{200\left(1+K_1K_2\right)}\right)$$

For $\zeta = 1$ & $a = 0.3$, we have

$$2\zeta\omega_n = 30 \Rightarrow \omega_n = 15$$

$$\therefore\ 200(1 + K_1K_2) = 15^2 \qquad \text{------ (1)}$$

$$\frac{40K_1}{200(1+K_1K_2)} = 0.3 \qquad \text{------ (2)}$$

Solving (1) and (2),

$K_1 = 1.69$, $K_2 = 0.074$

If $\zeta = 0.7$ and $a = 0.2$, then

$$2\zeta\omega_n = 30 \qquad \Rightarrow \qquad \omega_n = \frac{15}{0.7}$$

$$\therefore \quad 200\left(1+K_1K_2\right) = \left(\frac{15}{0.7}\right)^2 \qquad \text{----- (3)}$$

$$\frac{40K_1}{200\left(1+K_1K_2\right)} = 0.2 \qquad \text{------ (4)}$$

Solving (3) and (4),

$K_1 = 2.30$, $K_2 = 0.56$

4. Since $$\frac{9(\alpha s+1)}{s^2+6s+9}=\frac{3^2}{s^2+2\times 1\times 3s+3^2}+s\times\alpha\times\frac{3^2}{s^2+2\times 1\times 3s+3^2}$$

the output w.r.t. unit step input can be expressed as

$$Y(s)=Y_0(s)+\alpha\cdot s\cdot Y_0(s)\quad \text{where}\quad Y_0(s)=\frac{3^2}{s^2+2\cdot 1\cdot 3s+3^2}\times\frac{1}{s}$$

Hence, the unit step response is given by

$$y(t)=y_0(t)+\alpha\frac{dy_0(t)}{dt}=1-e^{-3t}\left(1+3t\right)+\alpha(9te^{-3t})$$

So, $y'(t)=9te^{-3t}+9\alpha e^{-3t}-27\alpha te^{-3t}=0$ if $t=\dfrac{\alpha}{3\alpha-1}$

We get positive t when α > 1/3, and there will be an overshoot.

NB: You’ll also get a positive t when $\alpha < 0$. Investigate this case!

At $t_p = \dfrac{\alpha}{3\alpha - 1}$, we have

$$y(t_p) = 1 - e^{-\frac{3\alpha}{3\alpha-1}}\left(1 + 3\frac{\alpha}{3\alpha-1}\right) + \alpha\left(9\frac{\alpha}{3\alpha-1}e^{-\frac{3\alpha}{3\alpha-1}}\right)$$

Hence the overshoot is

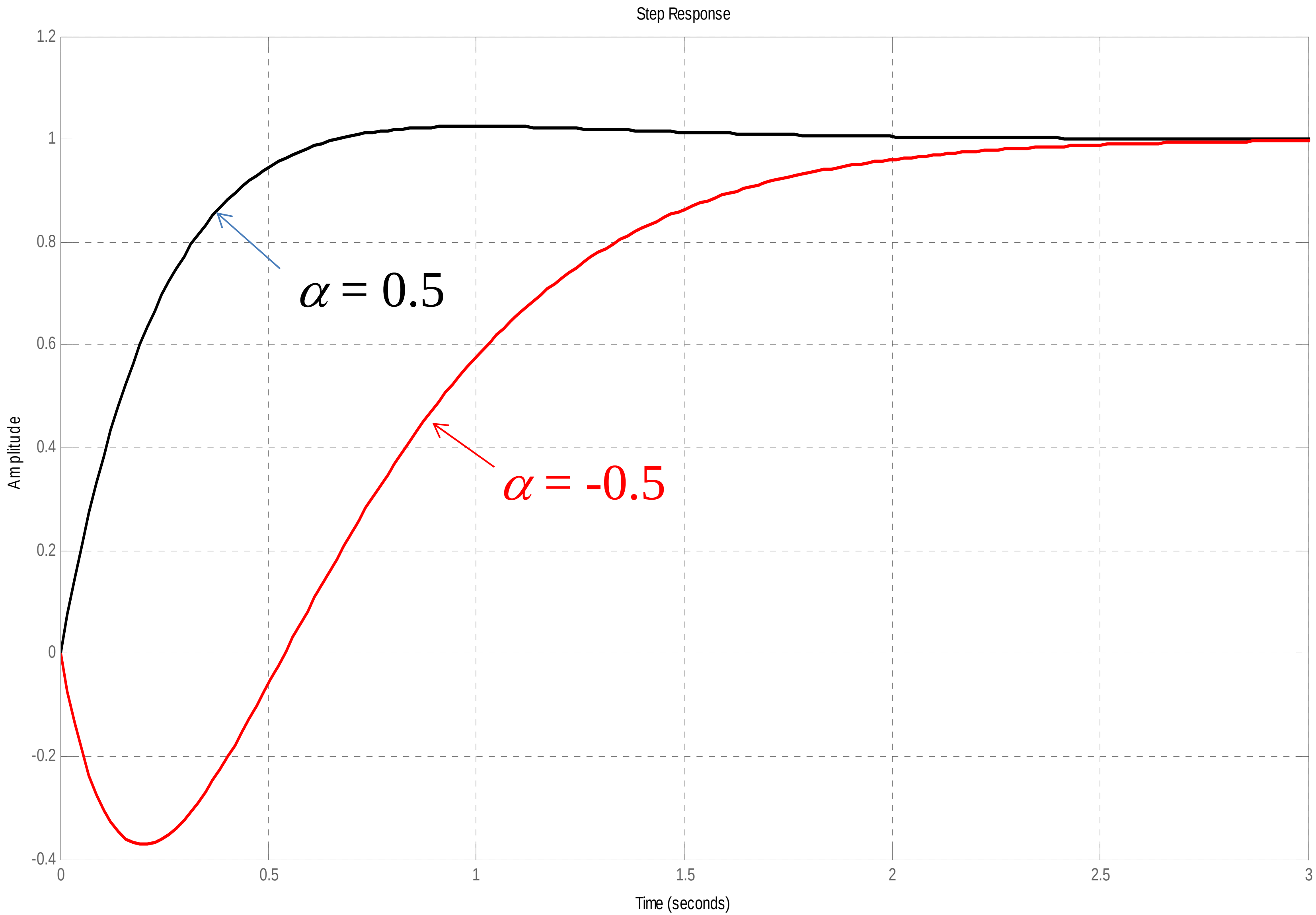
$$y(t_p) - 1 = \left(\frac{9\alpha^2}{3\alpha-1} - 1 - \frac{3\alpha}{3\alpha-1}\right)e^{-\frac{3\alpha}{3\alpha-1}} = \frac{1+9\alpha^2-6\alpha}{3\alpha-1}\times e^{-\frac{3\alpha}{3\alpha-1}} = (3\alpha-1)e^{-\frac{3\alpha}{3\alpha-1}}$$

To compute the rise time, we have

$$y(t_r) = 1 - e^{-3t_r}\left(1 + 3t_r\right) + \alpha(9t_r e^{-3t_r}) = 1$$

Thus, $t_r = \dfrac{1}{9\alpha - 3}$

NB: this is valid only when $\alpha > 1/3$


Step Response
Amplitude
Time (seconds)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
0.5
1
1.5
2
2.5
3
α = 0.5
α = -0.5

With $\dfrac{2(\alpha s+1)}{s^2+2s+2}=\dfrac{\sqrt{2}^2}{s^2+2\bullet\frac{1}{\sqrt{2}}\bullet\sqrt{2}s+\sqrt{2}^2}+s\alpha\dfrac{\sqrt{2}^2}{s^2+2\bullet\frac{1}{\sqrt{2}}\bullet\sqrt{2}s+\sqrt{2}^2}$

the output w.r.t. unit step input can be expressed as

$$Y(s)=Y_0(s)+\alpha\cdot s\cdot Y_0(s)\ \text{ where }\ Y_0(s)=\frac{\sqrt{2}^2}{s^2+2\cdot\frac{1}{\sqrt{2}}\cdot\sqrt{2}s+\sqrt{2}^2}\times\frac{1}{s}$$

Hence, the unit step response is given by

$$y(t)=y_0(t)+\alpha\frac{dy_0(t)}{dt}$$

where $y_0(t)=1-\dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t+\theta\right)=1-\sqrt{2}e^{-t}\sin\left(t+\theta\right)$

$$\frac{dy_0}{dt}=\sqrt{2}e^{-t}\sin\left(t+\theta\right)-\sqrt{2}e^{-t}\cos\left(t+\theta\right)$$

$$\because\quad \zeta=\frac{1}{\sqrt{2}},\ \omega_n=\sqrt{2},\ \theta=\cos^{-1}\zeta=\frac{\pi}{4},\ \omega_d=\sqrt{1-\zeta^2}\omega_n=1$$

So,

$$y(t) = 1 - \sqrt{2}e^{-t}\sin\left(t+\theta\right) + \alpha\sqrt{2}e^{-t}\sin\left(t+\theta\right) - \alpha\sqrt{2}e^{-t}\cos\left(t+\theta\right)$$

To calculate peak time:

$$\frac{dy}{dt} = \sqrt{2}e^{-t}\sin\left(t+\theta\right) - \sqrt{2}e^{-t}\cos\left(t+\theta\right) + 2\sqrt{2}\alpha e^{-t}\cos\left(t+\theta\right)$$

$$\frac{dy}{dt} = 0 \text{ if } \sin\left(t+\theta\right) + (2\alpha-1)\cos\left(t+\theta\right) = 0; \quad \theta = \frac{\pi}{4}$$

$$\Rightarrow \tan\left(t_p+\theta\right) = 1-2\alpha$$

Overshoot: $M_P = y(t_p) - 1 = \sqrt{2}e^{-t_p}\left((\alpha-1)\sin(t_p+\theta) - \alpha\cos(t_p+\theta)\right)$

Rise-time: $y(t_r) = 1 = 1 - \sqrt{2}e^{-t_r}\left((\alpha-1)\sin(t_r+\theta) - \alpha\cos(t_r+\theta)\right)$

$$\Rightarrow (\alpha-1)\sin\left(t_r+\theta\right) - \alpha\cos\left(t_r+\theta\right) = 0 \quad \Rightarrow \quad \tan\left(t_r+\theta\right) = \frac{\alpha}{\alpha-1}$$

$$\tan\left(t_p+\theta\right)=1-2\alpha \quad \Rightarrow \quad t_p=\pi+\tan^{-1}(1-2\alpha)-\frac{\pi}{4}$$

$$M_p=\sqrt{2}e^{-t_p}\left((\alpha-1)\sin(t_p+\theta)-\alpha\cos(t_p+\theta)\right)$$

$$\tan\left(t_r+\theta\right)=\frac{\alpha}{\alpha-1} \quad \Rightarrow \quad t_r=\pi+\tan^{-1}\left(\frac{\alpha}{\alpha-1}\right)-\frac{\pi}{4}$$

α	t_p	M_p	t_r
0	π	$e^{-\pi}$	$\frac{3\pi}{4}$
0.5	$\frac{3\pi}{4}$	$\frac{\sqrt{2}e^{-3\pi/4}}{2}$	$\frac{\pi}{2}$
1.0	$\frac{\pi}{2}$	$e^{-\pi/2}$	$\frac{\pi}{4}$