Notes on PSTAT 213

Haosheng Zhou

Sept, 2022

Simple Random Walk

$S_n = X_1 + ... + X_n$ is a SRW with X_i *i.i.d.* starting from $S_0 = 0$. T_b denotes the first hitting time of S_n to b, $\mathbb{P}(X_i = 1) = p, \mathbb{P}(X_i = -1) = q, p + q = 1$.

Theorem 1. *(Hitting Time Theorem) $\forall b \neq 0$ such that $\frac{n+b}{2} \in \{0, 1, ..., n\}$, $\mathbb{P}(T_b = n) = \frac{|b|}{n}\mathbb{P}(S_n = b) = \frac{|b|}{n}\binom{n}{\frac{n+b}{2}}p^{\frac{n+b}{2}}q^{\frac{n-b}{2}}$ $(n \geq 1)$*

Proof. Prove by counting paths. It's obvious that if $p = q$, each path consisting of points (t, S_t) $(t = 0, 1, ..., n)$ has same probability of appearing. Now p, q are not necessarily the same, so if a fixed path has a moving upward and $n - a$ moving downward, the probability of appearing is just

$$\frac{\binom{n}{a}}{2^n}p^a q^{n-a} \tag{1}$$

If now a path hits b at time n for the first time, it should first hit b at time n, which means that there are $\frac{n+b}{2}$ going upward and $\frac{n-b}{2}$ going downward. Each path that hits b at time n has same probability of appearing, which is $\frac{\binom{n}{\frac{n+b}{2}}}{2^n}p^{\frac{n+b}{2}}q^{\frac{n-b}{2}}$. As a result, the problem reduces to counting the number of all paths within those paths that have also hit b between time 0 to n.

We do a translation for all the paths such that now we start at $(0, -b)$ and want to count the number of paths that ends at $(n, 0)$ but has also hit 0 in between. This count is just the sum of the number of paths that starts at $(0, -b)$ and ends at $(n - 1, 1)$ but has also hit 0 in between and the number of paths that starts at $(0, -b)$ and ends at $(n - 1, -1)$ but has also hit 0 in between. Assume WLOG that $b > 0$, notice that the first count is

$$\binom{n-1}{\frac{n+b}{2}} \tag{2}$$

and the second count is due to reflection principle that it's just the number of paths that starts at $(0, b)$ and ends at $(n - 1, -1)$ which is

$$\binom{n-1}{\frac{n+b}{2}} \tag{3}$$

As a result, the sum should be

$$2\binom{n-1}{\frac{n+b}{2}} \tag{4}$$

The number of path that starts at $(0, b)$ and ends at $(n, 0)$ is

$$\binom{n}{\frac{n+b}{2}} \tag{5}$$

So if a path is conditioned on already starting at $(0, 0)$ and ending at (n, b), it has probability of hitting b in between as

$$\frac{2\binom{n-1}{\frac{n+b}{2}}}{\binom{n}{\frac{n+b}{2}}} = \frac{n-b}{n} \tag{6}$$

if a path is conditioned on already starting at $(0,0)$ and ending at (n,b), it has probability of not hitting b in between as

$$\frac{b}{n} \tag{7}$$

That's why $\mathbb{P}(T_b = n) = \frac{b}{n}\mathbb{P}(S_n = b)$ for $b > 0$ and the theorem is proved. The similar proof holds for $b < 0$. $\square$

Remark. *If we want to know the distribution of T_0, we also have to lift the time at 0 to the time at 1 (consider whether S_1 is 1 or -1) since reflection can't be applied for when the path starts or ends at 0.*

Theorem 2. *Set the **maximum process** $M_n = \max_{0\leq k\leq n} S_k$ for symmetric SRW S_n, then*

$$\forall r \geq 1, \mathbb{P}(M_n \geq r, S_n = v) = \begin{cases} \mathbb{P}(S_n = v) & v \geq r \\ \mathbb{P}(S_n = 2r - v) & v < r \end{cases} \tag{8}$$

Proof. If $v \geq r$, then $\mathbb{P}(M_n \geq r, S_n = v) = \mathbb{P}(S_n = v)$ naturally.

For the other case, let's count the number of paths. The number of paths from $(0,0)$ to $(n, 2r - v)$ is

$$\binom{n}{\frac{n+2r-v}{2}} \tag{9}$$

The number of paths from $(0,0)$ to (n,v) that has hit r in between is equal to the number of paths from $(0,-r)$ to $(n, v-r)$ that has hit 0 in between. By reflection principle, this is just the number of paths from $(0,r)$ to $(n, v-r)$, which is

$$\binom{n}{\frac{n+v-2r}{2}} \tag{10}$$

same to the count above, so it's proved.

Another Proof:

Since the SRW is Markov process and $T_r < \infty$ *a.s.*, strong Markov property tells us

$$S_n^{T_r} = S_{n+T_r} - S_{T_r} = S_{n+T_r} - r \tag{11}$$

is also a SRW and is independent of $\mathscr{F}_{T_r}$.

Let's then do calculations:

$$\begin{aligned}
\mathbb{P}\left(M_n \geq r, S_n = v\right) &= \mathbb{P}\left(T_r \leq n, S_n = v\right) && (12)\\
&= \mathbb{P}\left(T_r \leq n, S_{n-T_r}^{T_r} = v - r\right) && (13)\\
&= \mathbb{P}\left(T_r \leq n, -S_{n-T_r}^{T_r} = v - r\right) && (14)
\end{aligned}$$

the last step is due to the fact that $T_r \in \mathscr{F}_{T_r}, S_n^{T_r} \stackrel{d}{=} -S_n^{T_r}$ and that $S_n^{T_r}$ is independent of $\mathscr{F}_{T_r}$.

$$\begin{aligned}
\mathbb{P}\left(M_n \geq r, S_n = v\right) &= \mathbb{P}\left(T_r \leq n, -S_{n-T_r}^{T_r} = v - r\right) && (15)\\
&= \mathbb{P}\left(T_r \leq n, S_n = 2r - v\right) && (16)\\
&= \mathbb{P}\left(S_n = 2r - v\right) && (17)
\end{aligned}$$

□

Remark. *By this reflection principle, we see that for $r \geq 0$,*

$$\begin{aligned}
\mathbb{P}\left(M_n \geq r\right) &= \sum_{v=-n,-n+2,\ldots,n} \mathbb{P}\left(M_n \geq r, S_n = v\right) && (18)\\
&= \sum_{v<r} \mathbb{P}\left(S_n = 2r - v\right) + \sum_{v \geq r} \mathbb{P}\left(S_n = v\right) && (19)\\
&= \mathbb{P}\left(S_n = r\right) + \mathbb{P}\left(S_n \geq r+1\right) + \mathbb{P}\left(S_n = 2r + n\right) + \ldots + \mathbb{P}\left(S_n = 2r - r + 1\right) && (20)\\
&= \mathbb{P}\left(S_n = r\right) + \mathbb{P}\left(S_n \geq r+1\right) + \mathbb{P}\left(S_n \geq r+1\right) && (21)\\
&= \mathbb{P}\left(S_n = r\right) + 2\mathbb{P}\left(S_n \geq r+1\right) && (22)
\end{aligned}$$

that's why we get

$$\begin{aligned}
\mathbb{P}\left(M_n = r\right) &= \mathbb{P}\left(M_n \geq r\right) - \mathbb{P}\left(M_n \geq r+1\right) && (23)\\
&= \mathbb{P}\left(S_n = r\right) + 2\mathbb{P}\left(S_n \geq r+1\right) - \mathbb{P}\left(S_n = r+1\right) - 2\mathbb{P}\left(S_n \geq r+2\right) && (24)\\
&= \mathbb{P}\left(S_n = r\right) + 2\mathbb{P}\left(S_n = r+1\right) - \mathbb{P}\left(S_n = r+1\right) && (25)\\
&= \mathbb{P}\left(S_n = r\right) + \mathbb{P}\left(S_n = r+1\right) && (26)
\end{aligned}$$

To calculate probability like $\mathbb{P}\left(M_8 = 6\right)$, just use the formula to get

$$\begin{aligned}
\mathbb{P}\left(M_8 = 6\right) &= \mathbb{P}\left(S_8 = 6\right) + \mathbb{P}\left(S_8 = 7\right) && (27)\\
&= \frac{\binom{8}{1}}{2^8} = \frac{1}{32} && (28)
\end{aligned}$$

Generating Function of SRW

0 Hitting Time

Now in the general setting, p probability going upward and q going downward with $p+q=1$. Now

$$p_0(n) = \mathbb{P}(S_n = 0) \tag{29}$$

and

$$f_0(n) = \mathbb{P}(S_1 \neq 0, ..., S_{n-1} \neq 0, S_n = 0) \tag{30}$$

where $f_0(n)$ gives the probability mass of first hitting time T_0. There respective generating functions are denoted

$$P_0(s) = \sum_{n=0}^{\infty} p_0(n) s^n \tag{31}$$

$$F_0(s) = \sum_{n=0}^{\infty} f_0(n) s^n \tag{32}$$

$$\tag{33}$$

then since SRW is Markov, use the Markov property w.r.t. 1 unit of time translation to get

$$p_0(0) = 1, f_0(0) = 0 \tag{34}$$

$$\forall n \geq 1, p_0(n) = \mathbb{P}(S_n = 0) \tag{35}$$

$$= \sum_{k=1}^{n} \mathbb{P}(T_0 = k) \mathbb{P}(S_n = 0 | T_0 = k) \tag{36}$$

$$= \sum_{k=1}^{n} \mathbb{P}(T_0 = k) \mathbb{P}(S_{n-k} = 0) \tag{37}$$

$$= \sum_{k=1}^{n} f_0(k) p_0(n-k) \tag{38}$$

to compare the coefficient, proved that

$$P_0(s) = 1 + P_0(s) F_0(s) \tag{39}$$

Note that

$$\begin{aligned}
P_0(s) &= \sum_{n=0}^{\infty} \mathbb{P}\left(S_n = 0\right) s^n && (40)\\
&= \sum_{n=0,2,\dots} \binom{n}{\frac{n}{2}} (pq)^{\frac{n}{2}} s^n && (41)\\
&= \sum_{n=0}^{\infty} \binom{2n}{n} (pqs^2)^n && (42)\\
&= \sum_{n=0}^{\infty} \frac{(2n-1)!!2^n n!}{n!n!} (pqs^2)^n && (43)\\
&= \sum_{n=0}^{\infty} (-4)^n \binom{-\frac{1}{2}}{n} (pqs^2)^n && (44)\\
&= (1-4pqs^2)^{-\frac{1}{2}} && (45)
\end{aligned}$$

by the Taylor series.

As a result, plug in to get

$$\begin{aligned}
F_0(s) &= \frac{P_0(s)-1}{P_0(s)} && (46)\\
&= 1-(1-4pqs^2)^{\frac{1}{2}} && (47)
\end{aligned}$$

From this generating function, we can investigate whether T_0 is almost surely finite or has finite expectation for general SRW. It's easy to see that

$$\mathbb{P}\left(T_0 < \infty\right) = \sum_{n=1}^{\infty} \mathbb{P}\left(T_0 = n\right) = F_0(1) = 1 - |p-q| \tag{48}$$

as a result, $T_0 < \infty$ $a.s.$ **if and only if** $p = \frac{1}{2}$.

Taking derivative for $F_0(s)$ to get

$$\begin{aligned}
F_0'(s) &= 4pqs(1-4pqs^2)^{-\frac{1}{2}} && (49)\\
\mathbb{E}(T_0 \cdot \mathbb{I}_{T_0<\infty}) = F_0'(1) &= \frac{4pq}{|p-q|} && (50)
\end{aligned}$$

as a result, $\mathbb{E}(T_0 \cdot \mathbb{I}_{T_0<\infty}) < \infty$ **if and only if** $p = \frac{1}{2}$.

In the context above, we investigate all generating functions of the stopping time T_0 which is the hitting time of 0. One can notice that actually this gives us the generating function of the i-th hitting time to 0, denoted T_0^i. By

Markov property,

$$\mathbb{P}\left(T_0^i = k\right) = \sum_{j=0}^{k} \mathbb{P}\left(T_0^{i-1} = j\right) \cdot \mathbb{P}\left(T_0^i = k | T_0^{i-1} = j\right) \tag{51}$$

$$= \sum_{j=0}^{k} \mathbb{P}\left(T_0^{i-1} = j\right) \cdot \mathbb{P}\left(T_0 = k - j\right) \tag{52}$$

so if we denote the generating function of T_0^i by $F_0^i(s)$, then

$$F_0^i(s) = \sum_{k=0}^{\infty} \mathbb{P}\left(T_0^i = k\right) \cdot s^k \tag{53}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \mathbb{P}\left(T_0^{i-1} = j\right) \cdot \mathbb{P}\left(T_0 = k - j\right) \cdot s^k \tag{54}$$

$$= F_0^{i-1}(s) \cdot F_0(s) \tag{55}$$

$$= [F_0(s)]^i \tag{56}$$

it's then easy to see that

$$\mathbb{P}\left(T_0^i < \infty\right) = F_0^i(1) = [F_0(1)]^i = [1 - |p - q|]^i \tag{57}$$

so **SRW is recurrent if and only if** $p = \frac{1}{2}$. Naturally, let's investigate whether SRW is null recurrent when $p = \frac{1}{2}$.

$$\mathbb{E}(T_0^i \cdot \mathbb{I}_{T_0^i < \infty}) = \frac{d}{ds} F_0^i(s)|_{s=1} \tag{58}$$

$$= i[F_0(1)]^{i-1} \cdot F_0'(1) \tag{59}$$

$$= i[1 - |p - q|]^{i-1} \cdot \frac{4pq}{|p - q|} \tag{60}$$

so all states in SRW is null recurrent when $p = \frac{1}{2}$, which indicates a natural conclusion that there's no stationary distribution for symmetric SRW.

1 Hitting Time

One might find that generating functions for T_0 tells us nothing about the information of other hitting times, e.g. T_1. To get $F_1(s)$ as the generating function of T_1, we need to apply Markov property

$$\forall n > 1, \mathbb{P}\left(T_1 = n\right) = \mathbb{P}\left(T_1 = n | X_1 = 1\right) \cdot \mathbb{P}\left(X_1 = 1\right) + \mathbb{P}\left(T_1 = n | X_1 = -1\right) \cdot \mathbb{P}\left(X_1 = -1\right) \tag{61}$$

$$= q \cdot \mathbb{P}\left(T_1 = n | X_1 = -1\right) = q \cdot \mathbb{P}\left(T_2 = n - 1\right) \tag{62}$$

and it's obvious that $\mathbb{P}(T_1 = 1) = p$. To connect $F_1(s)$ with $F_2(s)$, it's natural to think of Markov property once more. Similar to what we have done for the i-th hitting time to 0, let's denote $F_i(1)$ as the generating function of T_i, the first hitting time to $i \geq 1$

$$\mathbb{P}(T_i = n) = \sum_{k=0}^{n} \mathbb{P}(T_i = n | T_1 = k) \cdot \mathbb{P}(T_1 = k) \tag{63}$$

$$= \sum_{k=0}^{n} \mathbb{P}(T_{i-1} = n-k) \cdot \mathbb{P}(T_1 = k) \tag{64}$$

here the strong Markov property is applied when $T_1 < \infty$ $a.s.$ w.r.t. $\mathscr{F}_{T_1}$, note that when $T_1 = \infty$, $T_i = \infty$ so such equation still holds. This is telling us that getting the generating function of T_1 is equivalent to getting the generating function of any hitting time T_i

$$F_i(s) = [F_1(s)]^i \tag{65}$$

Return to the previous question on $F_1(s)$, this provides connection between $\mathbb{P}(T_1 = n)$ and $\mathbb{P}(T_2 = n-1)$ that

$$F_1(s) = ps + \sum_{k=2}^{\infty} q \cdot \mathbb{P}(T_2 = k-1) s^k \tag{66}$$

$$= ps + qs \cdot F_2(s) \tag{67}$$

$$= ps + qs \cdot [F_1(s)]^2 \tag{68}$$

solve this quadratic equation w.r.t. $F_1(s)$ to get

$$F_1(s) = \frac{1 \pm \sqrt{1 - 4pqs^2}}{2qs} \tag{69}$$

notice that any generating function shall satisfy $F_1(0) = 0$, so we only take one appropriate root as the generating function

$$F_1(s) = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs} \tag{70}$$

naturally, one might calculate the quantity of one's interest that

$$\mathbb{P}(T_1 < \infty) = F_1(1) = \frac{1 - |p-q|}{2q} = \begin{cases} 1 & p \geq q \\ \frac{p}{q} & p < q \end{cases} \tag{71}$$

$$\mathbb{E}(T_1 \cdot \mathbb{I}_{T_1 < \infty}) = F_1'(1) = \frac{2p}{|p-q|} - \frac{1}{2q} + \frac{|p-q|}{2q} = \begin{cases} \frac{1}{p-q} & p > q \\ \frac{p}{q}\frac{1}{q-p} & p < q \\ \infty & p = q \end{cases} \tag{72}$$

in the more general case,

$$\mathbb{P}\left(T_i < \infty\right) = F_i(1) = \begin{cases} 1 & p \geq q \\ \left(\frac{p}{q}\right)^i & p < q \end{cases} \tag{73}$$

$$\mathbb{E}(T_i \cdot \mathbb{I}_{T_i < \infty}) = F_i'(1) = i[F_1(1)]^{i-1} \cdot F_1'(1) = \begin{cases} \frac{i}{p-q} & p > q \\ \left(\frac{p}{q}\right)^i \frac{i}{q-p} & p < q \\ \infty & p = q \end{cases} \tag{74}$$

as a result, $\mathbb{E}(T_i | T_i < \infty) = \frac{i}{|p-q|}$ holds generally.

Gambler's Ruin

Now for a general SRW, consider the exit time instead of the hitting time.

Law of Arcsine