Recitation Notes for PSTAT 170

Haosheng Zhou

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This note is based on the contents I have taught on the recitation class of PSTAT 170. The notes may be subject to typos, and you are welcome to provide you advice or ask questions at hzhou593@ucsb.edu.

Week 1

Mean Return and Volatility

For a given period of time, keep track of the stock price. The price at the beginning of the period is denoted P_s and the price at the end of the period is denoted P_e. The return rate during this period is defined as

$$r = \frac{P_e - P_s}{P_s} \tag{1}$$

Let's assume that we have kept track of the stock price for n periods, naturally, we would have $r_1, r_2, ..., r_n$ as the return rates for each period calculated. The **mean return** would be defined as

$$r_{mean} = \frac{\sum_{i=1}^{n} r_i}{n} \tag{2}$$

and the **volatility** would be defined as the standard deviation of the sequence $r_1, ..., r_n$.

$$\sigma = sd(r_1, ..., r_n) \tag{3}$$

Remark. *There's many kinds of averages and one can also apply the geometric average instead of the arithmetic average mentioned above, i.e.*

$$r_{mean} = \left(\prod_{i=1}^{n}(1 + r_i)\right)^{\frac{1}{n}} - 1 \tag{4}$$

The geometric average also has a natural interpretation as the accumulation of interest. If we have 1 dollar at the very beginning, after n periods the amount would accumulate to $\prod_{i=1}^{n}(1 + r_i)$. On the other hand, if r_{mean} is adopted for each period, the amount would accumulate to $(1+r_{mean})^n$. The geometric mean return is the return rate such that these two amounts are the same.

Remark. *There's many selections for the time period of the return rate. For example, it can be on a daily basis, or on a yearly basis etc. In practice, we would not use something like "annualized daily mean return" since the error would be huge. One may refer to the two real datasets I have posted to observe that the close price on one trading day is generally not equal to the open price on the next trading day because of after-hour tradings. That's why in practice we typically choose the time period to be 3 months, 1 year etc.*

In this course, you should be more likely to get in touch with the return rate on a yearly basis and the arithmetic mean return.

Remark. *You can run the program **stock.py** to see some of the plots and calculations I have made (it's written in Python). Note that there should be two datasets, one called **AMC.csv**, the other called **GS.csv**. You can change the file name in the ninth line of the code, i.e. **price = pd.read_csv('AMC.csv')** to run it on different datasets.*

You should observe that AMC's stock price has much more fluctuation and results in a much higher volatility. Also note the difference from daily mean return and yearly mean return. (The daily mean return here is not annualized, you have to annualize it in order to compare with the yearly mean return)

Stock Indices

The stock indices tells you what is happening on the whole stock market. The two main stock indices of consideration would be the DJIA (Dow Jones Industrial Average) and the SP500 (Standard & Poor's 500). The main difference in these two constructions is that DJIA is **dollar-weighted** but SP500 is **market capitalization weighted**.

The construction of DJIA only depends on the stock price of the component companies (30 large companies selected including Apple, Microsoft, Goldman Sachs etc.):

$$DJIA = \frac{\sum_i P_i}{Dow\ Index} \tag{5}$$

where P_i stands for the stock price of a component company and the Dow index is a fixed constant (currently 0.152 approximately). As a result, if the stock price of a component company rise 1, then $DJIA$ is going to rise $\frac{1}{0.152} = 6.59$ points.

The construction of SP500, however, takes market capitalization into consideration.

$$SP500 = \frac{\sum_i P_i Q_i}{Divisor} \tag{6}$$

where P_i stands for the stock price of a company and Q_i stands for the number of shares publicly available of a company and the divisor is a fixed constant.

One fact to notice is that we can use market captalization weights to simplify our calculations. Since the market capitalization weights are proportional to Q_i (actually the market capitalization weights are formed as $\frac{Q_i}{\sum_j Q_j} \propto Q_i$), if we know that a company has market capitalization weight w_i and its stock price increases by α, then the SP 500 index should increase by $w_i\alpha$ (the percentage of increase). By doing so, it's possible to compute the SP 500 index without knowing the value of Q_i and the value of the divisor.

Example

Let's use an example to illustrate these points (provided by Professor Michael). Now Microsoft is having market capitalization weight 5.72% and Goldman Sachs is having market capitalization weight 0.33%, and the SP 500 index now is 3655.

It's known that the stock price of Microsoft is changing from 237 to 239 with the stock prices of all the other companies fixed. Since Microsoft is one of the DJIA component companies, the DJIA will rise $2 \times 6.59 = 13.18$ points.

As stated above, since the stock price is rising by $\frac{2}{237}$, the SP 500 index will increase by $5.72\% \times \frac{2}{237} = 0.048\%$, thus resulting the SP 500 index to increase $3655 \times 0.048\% = 1.76$ points.

It's known that the stock price of Goldman Sachs is changing from 294 to 296 with the stock prices of all the other companies fixed. Since Goldman Sachs is one of the DJIA component companies, the DJIA will also rise $2 \times 6.59 = 13.18$ points. As stated above, since the stock price is rising by $\frac{2}{294}$, the SP 500 index will increase $0.33\% \times \frac{2}{294} \times 3655 = 0.08$ points.

As we can see, although the DJIA index is having the same amount of change, the stock price of Microsoft has a much larger impact on SP 500 index than Goldman Sachs. Actually, Microsoft has about 1.77 trillion dollars market capitalization and Goldman Sachs only has about 100 billion dollars market capitalization. It also tells us that the stock price does not necessarily reflect the value of the company. In this case, Goldman Sachs is having a higher stock price but Microsoft is a more valuable company.

Week 2

Collar

A **collar** is to buy a put and to sell a call at the same time, for which the call option has a higher strike price and both options share the same time to maturity. A **zero-cost collar** is the collar with zero premium.

Example

See the problem 3.12 in the textbook, where one invest 1000 in the index, buy 950-strike put and sell 1107-strike call. The interest rate for 6 months is 2% and the 6-month forward price is 1020. The premium of the 950-strike put is 51.777 and the premium of the 1107-strike call is 51.873.

Let's draw the profit diagram for this position. Firstly it's clear that longing 1 unit of the index brings with profit

$$P - 1000 \times 1.02 = P - 1020 \tag{7}$$

where P stands for the future index after 6 months.

Buying the 950-put option has profit

$$\max\{0, 950 - P\} - 51.777 \tag{8}$$

Selling the 1107-strike call option has profit

$$-\max\{0, P - 1107\} + 51.873 \tag{9}$$

As a result, the profit for this position should be

$$P - 1020 + \max\{0, 950 - P\} - 51.777 - \max\{0, P - 1107\} + 51.873 \tag{10}$$

$$= \begin{cases} -69.904 & P < 950 \\ P - 1019.904 & 950 \leq P \leq 1107 \\ 87.096 & P > 1107 \end{cases} \tag{11}$$

Note that the net premium for this collar is

$$51.777 - 51.873 = -0.096 \tag{12}$$

so this is really close to a zero-cost collar.

Example

See the following figure for an SOA problem on options.

17.

The current price for a stock index is 1,000. The following premiums exist for various options to buy or sell the stock index six months from now:

Strike Price	Call Premium	Put Premium
950	120.41	51.78
1,000	93.81	74.20
1,050	71.80	101.21

Strategy I is to buy the 1,050-strike call and to sell the 950-strike call.

Strategy II is to buy the 1,050-strike put and to sell the 950-strike put.

Strategy III is to buy the 950-strike call, sell the 1,000-strike call, sell the 950-strike put, and buy the 1,000-strike put.

Assume that the price of the stock index in 6 months will be between 950 and 1,050.

Determine which, if any, of the three strategies will have greater payoffs in six months for lower prices of the stock index than for relatively higher prices.

(A) None
(B) I and II only
(C) I and III only
(D) II and III only
(E) The correct answer is not given by (A), (B), (C), or (D)

Figure 1: The SOA problem

To solve this, we can write out the payoff of all three strategies (the premium makes no difference here). For strategy I, the payoff is

$$\max\{0, P-1050\} - \max\{0, P-950\} = \begin{cases} 0 & P < 950 \\ 950 - P & 950 \leq P \leq 1050 \\ -100 & P > 1050 \end{cases} \tag{13}$$

it's actually a bear spread.

For strategy II, the payoff is

$$\max\{0, 1050-P\} - \max\{0, 950-P\} = \begin{cases} 100 & P < 950 \\ 1050 - P & 950 \leq P \leq 1050 \\ 0 & P > 1050 \end{cases} \tag{14}$$

it's also a bear spread.

For strategy III, the payoff is

$$\max\{0, P-950\} + \max\{0, 1000-P\} - \max\{0, P-1000\} - \max\{0, 950-P\} = 50 \tag{15}$$

it's actually the combination of two box spreads. The first one is buying 950-strike call and selling 950-strike put. The second one is buying 1000-strike put and selling 1000-strike call. The consequence is that one will always buy at price 950 and sell at price 1000, that's why the payoff is always $1000 - 950 = 50$.

Butterfly

Butterfly is a combination that bets on the volatility of the market. A typical symmetric butterfly can be made by buying one $(K+a)$-strike call, buying one $(K-a)$-stike call and selling two K-strike call (actually can also use put to construct). So the payoff will be

$$\max\{0, P-(K+a)\} + \max\{0, P-(K-a)\} - 2\max\{0, P-K\} \tag{16}$$

$$= \begin{cases} 0 & P < K-a \\ P-K+a & K-a \leq P \leq K \\ -P+K+a & K < P \leq K+a \\ 0 & P > K+a \end{cases} \tag{17}$$

Then what if the strike prices are not equally distributed? For example, we have $A < B < C$ and A, B, C-strike call options to build a butterfly. In such situation, we would have to build an asymmetric one with a special proportion.

How to figure out the proportion of each call option? Note that **butterfly should have insured tails on both sides**, i.e. the payoff is always 0 when $P < A$ or $P > C$. For call options, the left tail is always insured because no call will be exercised when the stock price is too low. However, we would have to balance the right tail of the combination. Assume we are buying n_A, n_C number of call options with strike price A, C and selling n_B number of call options with strike price B. When the stock price $P > C$, the payoff would be

$$n_A(P-A) - n_B(P-B) + n_C(P-C) \tag{18}$$

setting it as 0 for any $P > C$ gives the condition

$$\begin{cases} n_A - n_B + n_C = 0 \\ An_A - Bn_B + Cn_C = 0 \end{cases} \tag{19}$$

This condition comes from the fact that the coefficient of P must be 0 (hold for any $P > C$) and the remaining constant must also be 0 (the expression equals to 0). Solving this equation gives feasible n_A, n_B, n_C.

See the lower graph on P83 of the textbook, where there are 35,43,45-strike call options to make an asymmetric butterfly. Just plug in $A = 35, B = 43, C = 45$ gives the equations

$$\begin{cases} n_A - n_B + n_C = 0 \\ 35n_A - 43n_B + 45n_C = 0 \end{cases} \tag{20}$$

solve these equations (obviously the solution is not unique) to get:

$$\begin{cases} n_B = 5n_A \\ n_C = 4n_A \end{cases} \tag{21}$$

That's why the textbook uses $n_A = 2, n_B = 10, n_C = 8$ to construct the asymmetric butterfly (which is one of the solutions to these equations).

Example

See the problem 3.17 in textbook. Construct an asymmetric butterfly using 950, 1020, 1050-strike options. Since I have already shown how to use call options to construct asymmetric butterfly above, let me show you how to use put options to construct (actually the same). Let's assume that all the options here are put options.

When the stock price is higher than 1050, the payoff is always 0, no exercising of put. As a result, we only need to ensure that the left tail always has 0 payoff. Assume the quantities of these three options to trade are n_1, n_2, n_3 respectively.

$$\forall P < 950, n_1(950 - P) - n_2(1020 - P) + n_3(1050 - P) = 0 \tag{22}$$

that's why we conclude

$$\begin{cases} 950n_1 - 1020n_2 + 1050n_3 = 0 \\ n_1 - n_2 + n_3 = 0 \end{cases} \tag{23}$$

solve to get:

$$\begin{cases} n_2 = \frac{10}{3}n_1 \\ n_3 = \frac{7}{3}n_1 \end{cases} \tag{24}$$

To make all of them to be integers, take $n_1 = 3, n_2 = 10, n_3 = 7$, so we buy 3 portions of 950-strike put, sell 10 portions of 1020-strike put and buy 7 portions of 1050-strike put.

Let's compute the payoff of this combination:

$$3\max\{0, 950 - P\} + 7\max\{0, 1050 - P\} - 10\max\{0, 1020 - P\} \tag{25}$$

$$= \begin{cases} 0 & P < 950 \\ 3P - 2850 & 950 \leq P \leq 1020 \\ -7P + 7350 & 1020 < P \leq 1050 \\ 0 & P > 1050 \end{cases} \tag{26}$$

it's really an asymmetric butterfly.