Recitation Notes for PSTAT 170

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This note is based on the contents I have taught on the recitation class of PSTAT 170. The notes may be subject to typos, and you are welcome to provide you advice or ask questions at hzhou593@ucsb.edu.

Week 1

Mean Return and Volatility

For a given period of time, keep track of the stock price. The price at the beginning of the period is denoted P_s and the price at the end of the period is denoted P_e. The return rate during this period is defined as

$$r = \frac{P_e - P_s}{P_s} \tag{1}$$

Let's assume that we have kept track of the stock price for n periods, naturally, we would have $r_1, r_2, ..., r_n$ as the return rates for each period calculated. The **mean return** would be defined as

$$r_{mean} = \frac{\sum_{i=1}^{n} r_i}{n} \tag{2}$$

and the **volatility** would be defined as the standard deviation of the sequence $r_1, ..., r_n$.

$$\sigma = sd(r_1, ..., r_n) \tag{3}$$

Remark. *There's many kinds of averages and one can also apply the geometric average instead of the arithmetic average mentioned above, i.e.*

$$r_{mean} = \left(\prod_{i=1}^{n}(1+r_i)\right)^{\frac{1}{n}} - 1 \tag{4}$$

The geometric average also has a natural interpretation as the accumulation of interest. If we have 1 dollar at the very beginning, after n periods the amount would accumulate to $\prod_{i=1}^{n}(1+r_i)$. On the other hand, if r_{mean} is adopted for each period, the amount would accumulate to $(1+r_{mean})^n$. The geometric mean return is the return rate such that these two amounts are the same.

Remark. *There's many selections for the time period of the return rate. For example, it can be on a daily basis, or on a yearly basis etc. In practice, we would not use something like "annualized daily mean return" since the error would be huge. One may refer to the two real datasets I have posted to observe that the close price on one trading day is generally not equal to the open price on the next trading day because of after-hour tradings. That's why in practice we typically choose the time period to be 3 months, 1 year etc.*

In this course, you should be more likely to get in touch with the return rate on a yearly basis and the arithmetic mean return.

Remark. *You can run the program* ***stock.py*** *to see some of the plots and calculations I have made (it's written in Python). Note that there should be two datasets, one called* ***AMC.csv****, the other called* ***GS.csv****. You can change the file name in the ninth line of the code, i.e.* ***price = pd.read_csv('AMC.csv')*** *to run it on different datasets.*

You should observe that AMC's stock price has much more fluctuation and results in a much higher volatility. Also note the difference from daily mean return and yearly mean return. (The daily mean return here is not annualized, you have to annualize it in order to compare with the yearly mean return)

Stock Indices

The stock indices tells you what is happening on the whole stock market. The two main stock indices of consideration would be the DJIA (Dow Jones Industrial Average) and the SP500 (Standard & Poor's 500). The main difference in these two constructions is that DJIA is **dollar-weighted** but SP500 is **market capitalization weighted**.

The construction of DJIA only depends on the stock price of the component companies (30 large companies selected including Apple, Microsoft, Goldman Sachs etc.):

$$DJIA = \frac{\sum_i P_i}{Dow\ Index} \tag{5}$$

where P_i stands for the stock price of a component company and the Dow index is a fixed constant (currently 0.152 approximately). As a result, if the stock price of a component company rise 1, then $DJIA$ is going to rise $\frac{1}{0.152} = 6.59$ points.

The construction of SP500, however, takes market capitalization into consideration.

$$SP500 = \frac{\sum_i P_i Q_i}{Divisor} \tag{6}$$

where P_i stands for the stock price of a company and Q_i stands for the number of shares publicly available of a company and the divisor is a fixed constant.

One fact to notice is that we can use market captalization weights to simplify our calculations. Since the market capitalization weights are proportional to Q_i (actually the market capitalization weights are formed as $\frac{Q_i}{\sum_j Q_j} \propto Q_i$), if we know that a company has market capitalization weight w_i and its stock price increases by α, then the SP 500 index should increase by $w_i\alpha$ (the percentage of increase). By doing so, it's possible to compute the SP 500 index without knowing the value of Q_i and the value of the divisor.

Example

Let's use an example to illustrate these points (provided by Professor Michael). Now Microsoft is having market capitalization weight 5.72% and Goldman Sachs is having market capitalization weight 0.33%, and the SP 500 index now is 3655.

It's known that the stock price of Microsoft is changing from 237 to 239 with the stock prices of all the other companies fixed. Since Microsoft is one of the DJIA component companies, the DJIA will rise $2 \times 6.59 = 13.18$ points.

As stated above, since the stock price is rising by $\frac{2}{237}$, the SP 500 index will increase by $5.72\% \times \frac{2}{237} = 0.048\%$, thus resulting the SP 500 index to increase $3655 \times 0.048\% = 1.76$ points.

It's known that the stock price of Goldman Sachs is changing from 294 to 296 with the stock prices of all the other companies fixed. Since Goldman Sachs is one of the DJIA component companies, the DJIA will also rise $2 \times 6.59 = 13.18$ points. As stated above, since the stock price is rising by $\frac{2}{294}$, the SP 500 index will increase $0.33\% \times \frac{2}{294} \times 3655 = 0.08$ points.

As we can see, although the DJIA index is having the same amount of change, the stock price of Microsoft has a much larger impact on SP 500 index than Goldman Sachs. Actually, Microsoft has about 1.77 trillion dollars market capitalization and Goldman Sachs only has about 100 billion dollars market capitalization. It also tells us that the stock price does not necessarily reflect the value of the company. In this case, Goldman Sachs is having a higher stock price but Microsoft is a more valuable company.

Week 2

Collar

A **collar** is to buy a put and to sell a call at the same time, for which the call option has a higher strike price and both options share the same time to maturity. A **zero-cost collar** is the collar with zero net premium.

Example

See the problem 3.12 in the textbook, where one invest 1000 in the index, buy 950-strike put and sell 1107-strike call. The interest rate for 6 months is 2% and the 6-month forward price is 1020. The premium of the 950-strike put is 51.777 and the premium of the 1107-strike call is 51.873.

Let's draw the profit diagram for this position. Firstly it's clear that longing 1 unit of the index brings with profit

$$P - 1000 \times 1.02 = P - 1020 \tag{7}$$

where P stands for the future index after 6 months.

Buying the 950-put option has profit

$$\max\{0, 950 - P\} - 51.777 \tag{8}$$

Selling the 1107-strike call option has profit

$$-\max\{0, P - 1107\} + 51.873 \tag{9}$$

As a result, the profit (future value) for this position should be

$$P - 1020 + \max\{0, 950 - P\} - 51.777 \times 1.02 - \max\{0, P - 1107\} + 51.873 \times 1.02 \tag{10}$$

$$= \begin{cases} -69.90208 & P < 950 \\ P - 1019.90208 & 950 \leq P \leq 1107 \\ 87.09792 & P > 1107 \end{cases} \tag{11}$$

Remark. *Don't forget to multiply the premium by* 1.02 *to be consistent with other future values.*

Note that the net premium (present value) for this collar is

$$51.777 - 51.873 = -0.096 \tag{12}$$

so this is really close to a zero-cost collar.

Example

See the following figure for an SOA problem on options.

17.

The current price for a stock index is 1,000. The following premiums exist for various options to buy or sell the stock index six months from now:

Strike Price	Call Premium	Put Premium
950	120.41	51.78
1,000	93.81	74.20
1,050	71.80	101.21

Strategy I is to buy the 1,050-strike call and to sell the 950-strike call.

Strategy II is to buy the 1,050-strike put and to sell the 950-strike put.

Strategy III is to buy the 950-strike call, sell the 1,000-strike call, sell the 950-strike put, and buy the 1,000-strike put.

Assume that the price of the stock index in 6 months will be between 950 and 1,050.

Determine which, if any, of the three strategies will have greater payoffs in six months for lower prices of the stock index than for relatively higher prices.

(A) None

(B) I and II only

(C) I and III only

(D) II and III only

(E) The correct answer is not given by (A), (B), (C), or (D)

Figure 1: The SOA problem

To solve this, we can write out the payoff of all three strategies (the premium makes no difference here). For strategy I, the payoff is

$$\max\{0, P-1050\} - \max\{0, P-950\} = \begin{cases} 0 & P < 950 \\ 950 - P & 950 \leq P \leq 1050 \\ -100 & P > 1050 \end{cases} \tag{13}$$

it's actually a bear spread.

For strategy II, the payoff is

$$\max\{0, 1050-P\} - \max\{0, 950-P\} = \begin{cases} 100 & P < 950 \\ 1050 - P & 950 \leq P \leq 1050 \\ 0 & P > 1050 \end{cases} \tag{14}$$

it's also a bear spread.

For strategy III, the payoff is

$$\max\{0, P-950\} + \max\{0, 1000-P\} - \max\{0, P-1000\} - \max\{0, 950-P\} = 50 \tag{15}$$

it's actually the combination of two box spreads. The first one is buying 950-strike call and selling 950-strike put. The second one is buying 1000-strike put and selling 1000-strike call. The consequence is that one will always buy at price 950 and sell at price 1000, that's why the payoff is always $1000 - 950 = 50$.

Butterfly

Butterfly is a combination that bets on the volatility of the market. A typical symmetric butterfly can be made by buying one $(K+a)$-strike call, buying one $(K-a)$-stike call and selling two K-strike call (actually can also use put to construct). So the payoff will be

$$\max\{0, P-(K+a)\} + \max\{0, P-(K-a)\} - 2\max\{0, P-K\} \tag{16}$$

$$= \begin{cases} 0 & P < K-a \\ P-K+a & K-a \le P \le K \\ -P+K+a & K < P \le K+a \\ 0 & P > K+a \end{cases} \tag{17}$$

Then what if the strike prices are not equally distributed? For example, we have $A < B < C$ and A, B, C-strike call options to build a butterfly. In such situation, we would have to build an asymmetric one with a special proportion.

How to figure out the proportion of each call option? Note that **butterfly should have insured tails on both sides**, i.e. the payoff is always 0 when $P < A$ or $P > C$. For call options, the left tail is always insured because no call will be exercised when the stock price is too low. However, we would have to balance the right tail of the combination. Assume we are buying n_A, n_C number of call options with strike price A, C and selling n_B number of call options with strike price B. When the stock price $P > C$, the payoff would be

$$n_A(P-A) - n_B(P-B) + n_C(P-C) \tag{18}$$

setting it as 0 for any $P > C$ gives the condition

$$\begin{cases} n_A - n_B + n_C = 0 \\ An_A - Bn_B + Cn_C = 0 \end{cases} \tag{19}$$

This condition comes from the fact that the coefficient of P must be 0 (hold for any $P > C$) and the remaining constant must also be 0 (the expression equals to 0). Solving this equation gives feasible n_A, n_B, n_C.

See the lower graph on P83 of the textbook, where there are 35,43,45-strike call options to make an asymmetric butterfly. Just plug in $A = 35, B = 43, C = 45$ gives the equations

$$\begin{cases} n_A - n_B + n_C = 0 \\ 35n_A - 43n_B + 45n_C = 0 \end{cases} \tag{20}$$

solve these equations (obviously the solution is not unique) to get:

$$\begin{cases} n_B = 5n_A \\ n_C = 4n_A \end{cases} \tag{21}$$

That's why the textbook uses $n_A = 2, n_B = 10, n_C = 8$ to construct the asymmetric butterfly (which is one of the solutions to these equations).

Example

See the problem 3.17 in textbook. Construct an asymmetric butterfly using 950, 1020, 1050-strike options. Since I have already shown how to use call options to construct asymmetric butterfly above, let me show you how to use put options to construct (actually the same). Let's assume that all the options here are put options.

When the stock price is higher than 1050, the payoff is always 0, no exercising of put. As a result, we only need to ensure that the left tail always has 0 payoff. Assume the quantities of these three options to trade are n_1, n_2, n_3 respectively.

$$\forall P < 950, n_1(950 - P) - n_2(1020 - P) + n_3(1050 - P) = 0 \tag{22}$$

that's why we conclude

$$\begin{cases} 950n_1 - 1020n_2 + 1050n_3 = 0 \\ n_1 - n_2 + n_3 = 0 \end{cases} \tag{23}$$

solve to get:

$$\begin{cases} n_2 = \frac{10}{3}n_1 \\ n_3 = \frac{7}{3}n_1 \end{cases} \tag{24}$$

To make all of them to be integers, take $n_1 = 3, n_2 = 10, n_3 = 7$, so we buy 3 portions of 950-strike put, sell 10 portions of 1020-strike put and buy 7 portions of 1050-strike put.

Let's compute the payoff of this combination:

$$
\begin{aligned}
&3\max\{0,950-P\}+7\max\{0,1050-P\}-10\max\{0,1020-P\} &(25)\\
&=\begin{cases} 0 & P<950\\ 3P-2850 & 950\le P\le 1020\\ -7P+7350 & 1020<P\le 1050\\ 0 & P>1050 \end{cases} &(26)
\end{aligned}
$$

it's really an asymmetric butterfly.

Remark. *In general, if we only want to construct a butterfly, there will be 3 unknown variables* n_1, n_2, n_3 *but only 2 equations. This is because any multiple of these weights, i.e.* $n_1' = kn_1, n_2' = kn_2, n_3' = kn_3$ *is still a butterfly, as if we are buying* k *portions of the butterfly constructed with weights* n_1, n_2, n_3.

However, if the payoff/profit diagram is given, then the slopes of the lines in the butterfly are known (know exactly how many portions we are buying), we would be able to write down the third equation w.r.t. n_1, n_2, n_3 *and uniquely solve out the equations, i.e. solve out the* k *above.*

Week 3

Example

A producer of gold has expenses of 800 per ounce of gold produced. Assume that the cost of all other production-related expenses is negligible and that the producer will be able to sell all gold produced at the market price. In one year, the market price of gold will be one of three possible prices, corresponding to the following probability table:

Gold Price in one year	Probability
750 per ounce	0.2
850 per ounce	0.5
950 per ounce	0.3

The producer hedges the price of gold by buying a 1-year put option with an exercise price of 900 per ounce. The option costs 100 per ounce now, and the continuously compounded annual risk-free interest rate is 6%.

Calculate the expected 1-year profit per ounce of gold produced.

Figure 2: The problem

Firstly, the position of the producer is longing gold (he produces gold) and longing a put option. Assume the future price of gold per ounce is P after 1 year, then the overall payoff (future value) would be

$$payoff = P + \max\{0, 900 - P\} \tag{27}$$

and the overall cost comes from the production of gold and the premium of the option (note that we have to change them into future values)

$$cost = (100 + 800) \times e^{0.06} \tag{28}$$

the profit is just the difference

$$profit = payoff - cost = P + \max\{0, 900 - P\} - (100 + 800) \times e^{0.06} \tag{29}$$

Now there's three possible situations in the future, with P possibly taking values from $\{750, 850, 950\}$. Calculate the profit in these 3 situations:

$$profit = \begin{cases} -55.65 & P = 750 \\ -55.65 & P = 850 \\ -5.65 & P = 950 \end{cases} \tag{30}$$

to see that the expected 1-year profit per ounce gold will be

$$0.2 \times -55.65 + 0.5 \times -55.65 + 0.3 \times -5.65 = -40.65 \tag{31}$$

Remark. *Here we accumulate the cost of production 800 into future values. The explanation for this is that the future values of 800 should be the **opportunity cost**. If the producer chooses not to use 800 to produce 1 ounce of gold but chooses to buy risk-free bond with 800, he/she could have got* $800 \times e^{0.06}$ *after 1 year. That's why here the future value of the cost caused by production is actually* $800 \times e^{0.06}$.

In contrast, calculate the expected 1-year profit per ounce gold without the put option. The profit would then be

$$profit = P - 800 \times e^{0.06} \tag{32}$$

$$profit = \begin{cases} -99.47 & P = 750 \\ 0.53 & P = 850 \\ 100.53 & P = 950 \end{cases} \tag{33}$$

the expected profit is

$$0.2 \times -99.47 + 0.5 \times 0.53 + 0.3 \times 100.53 = 10.53 \tag{34}$$

Why are we even getting much lower profit with an insurance held? That's due to the relatively optimistic future state and the cost of the premium. However, if there's some probability for the gold price to be 500 instead of 750, then the insurance may make a large difference since with a long position of put option, the payoff is always fixed at 900 on the left tail no matter how low the gold price becomes.

Remark. *If one is more familiar with probability theory, one might notice that what we are doing is just first writing the profit as some function of price $f(P)$, with price P as a random variable with given distribution. To calculate the expected profit, it's just $\mathbb{E}f(P)$ with*

$$\mathbb{E}f(P) = \sum_k f(k) \cdot \mathbb{P}(P = k) \tag{35}$$

if P is a discrete random variable and

$$\mathbb{E}f(P) = \int_0^\infty f(x)p(x)\,dx \tag{36}$$

if P is a continuous random variable with density $p(x)$.

No Arbitrage Criterion

The core concept in derivative pricing is the no arbitrage criterion, which means that **there's no risk-free arbitrage opportunities with no initial endowment in the market**. The underlying logic of this criterion is that the existence of risk-free arbitrage causes the changes in the supply and demand on the financial markets, leading to the close of the risk-free arbitrage gaps. Actually, the pricing of forward contracts for stocks with continuously paid dividends can be derived from such perspective.

Assume the continuous-time compounded annual effective dividend rate is δ, with the continuous-time compounded annual effective risk-free interest rate r. The forward contract has time to maturity T with price $F_{0,T}$. The stock price now is S_0.

The forward contract works in a way that we shall pay for it and receive the stock both at time T. Consider the arbitrage strategy **borrowing** $S_0 \cdot e^{-\delta T}$ **money** and longing $e^{-\delta T}$ share of stock and selling a forward contract for 1 share of stock, such strategy has payoff (future value)

$$S_T - S_T = 0 \tag{37}$$

the reason we only buy $e^{-\delta T}$ share of stock is that by reinvesting all dividends back into the stock, at time T there's eventually $e^{-\delta T}e^{\delta T} = 1$ share of stock.

The cost (present value) of such strategy is

$$-e^{-rT}F_{0,T} + e^{-\delta T}S_0 \tag{38}$$

note that the price of the forward contract is paid in the future so it has to be discounted.

The risk-free arbitrage has profit (future value)

$$e^{rT}(e^{-rT}F_{0,T} - e^{-\delta T}S_0) \tag{39}$$

note that "risk-free" refers to the fact that such strategy does not depend on the future price of the stock, i.e. S_T.

Since the existence of risk-free arbitrage is not allowed, set

$$e^{rT}(e^{-rT}F_{0,T} - e^{-\delta T}S_0) = 0 \tag{40}$$

to solve out the price of the forward contract

$$F_{0,T} = S_0 e^{(r-\delta)T} \tag{41}$$

The strategy constructed above is called **cash-and-carry arbitrage**, where we buy the underlying asset and sell its forward contract, and it's **risk-free with no initial endowment**. Of course, by selling the underlying asset and buying its forward contract, one would get the **reverse cash-and-carry arbitrage**.

Remark. *The risk-free condition is crucial. Arbitrages with risk (uncertainty) are perfectly fine to exist on financial markets. I bolded "borrow $S_0 \cdot e^{-\delta T}$ money" to emphasize that the arbitrage strategy has to have no initial endowment. In the situation above, we are using the same interest rate for borrowing money and investment so it doesn't matter whether we have initial endowment or not. However, in the case where* ***the borrowing interest rate is different from the investment interest rate (the case in real life), the no initial endowment condition makes a difference!***

Example

Let's look at problem 5.19 in the textbook, another example of arbitrage with different currency. Given the spot exchange rate 0.008\$/¥, continuous-time compounded annual effective risk-free rate yen-denominated is 1% and continuous-time compounded annual effective risk-free rate dollar-denominated is 5%. 1 year forward exchange rate is 0.0084\$/¥. We would like to build an arbitrage strategy with no initial cost, no risk and calculate the profit after 1 year in ¥.

Remark. *The spot exchange rate is the exchange rate that works at present while the forward exchange rate is the exchange rate publicly announced at present but works in the future.*

Notice that the forward exchange rate announced today for 1-year period is totally different from the spot exchange rate after 1 year. The forward exchange rate announced today enables one to exchange a certain amount of currency with this rate after a certain time period (similar to forward contract but is known and fixed at present), while the spot exchange rate in the future is totally random at present.

To check whether arbitrage exists in such case, one can think about deriving the critical forward exchange rate F\$/¥ when no risk-free arbitrage exists. Consider shorting the yen-denominated risk-free bond to get 1¥ and turning it into 0.008\$ at present to buy the dollar-denominated risk-free bond. Also tell the bank that we would like to change dollar for yen with the forward exchange rate in 1 year.

This arbitrage strategy consists of no initial costs. At the end of 1 year, the yen-denominated risk-free bond requires us to pay $e^{0.01}$¥ while we would get $0.008 \times e^{0.05}$\$ from the dollar-denominated risk-free bond and change it using forward exchange rate into $\frac{0.008\times e^{0.05}}{F}$¥. As a result, the profit at the end of 1 year in yen is:

$$-e^{0.01} + \frac{0.008 \times e^{0.05}}{F} = 0 \tag{42}$$

to ensure that there's no risk-free arbitrage opportunities. Solve to get

$$F = 0.008326 \tag{43}$$

Remark. *If a risk-free arbitrage strategy has negative profit, it is also indicating the existence of risk-free arbitrage since one can hold the opposite position to make positive profit, i.e. any risk-free combination of derivatives with non-zero profit violates the no arbitrage criterion.*

This tells us that under no-arbitrage condition, the forward exchange rate should have been 0.008326\$/¥. However, now the real rate is higher than this, telling us that holding yen in our hands would be more profitable. Therefore, the arbitrage strategy should go like:

- Short the dollar-denominated risk-free bond to get 0.008\$
- Turn it into 1¥ at present
- Use 1¥ to buy the yen-denominated risk-free bond

- Tell the bank that we would like to exchange $\frac{0.008\times e^{0.05}}{0.0084}$ yen for dollar at the forward exchange rate after 1 year (only change the amount of yen into dollar such that it's enough to cover the debt in dollar)

By doing this, at the end of 1 year, we get $e^{0.01}$¥ as the payoff from the yen-denominated risk-free bond and we have to pay $\frac{0.008\times e^{0.05}}{0.0084}$¥ for shorting the dollar-denominated risk-free bond. The overall profit is

$$e^{0.01} - \frac{0.008 \times e^{0.05}}{0.0084} = 0.00884¥ \tag{44}$$

By applying the arbitrage strategy above, for each 1¥ invested, we receive risk-free profit 0.00884¥.

If now the real forward exchange rate is changed as 0.0083\$/¥, then it's lower than 0.008326 computed. As a result, holding dollars in our hand would be more profitable, so our position for arbitrage should be flipped into:

- Short the yen-denominated risk-free bond to get 1¥
- Turn it into 0.008\$ at present
- Use 0.008\$ to buy the dollar-denominated risk-free bond
- Tell the bank that we would like to exchange $0.008 \times e^{0.05}$ dollar for yen at the forward exchange rate after 1 year (turn all the dollars you get into yen since the final profit is to be calculated in yen)

The profit after 1 year would be

$$-e^{0.01} + \frac{0.008 \times e^{0.05}}{0.0083} = 0.00322¥ \tag{45}$$

By applying such arbitrage strategy, for each 1¥ invested, we receive risk-free profit 0.00322¥.

This arbitrage strategy is called **covered interest arbitrage** and it shows the determination of forward exchange rate using spot exchange rate and the risk-free rate of respective currency-denominated bond.

Remark. *The risk-free in the context of covered interest arbitrage refers to the fact that the payoff is* ***independent of the spot exchange rate in the future****.*

One must ***clear his/her position in the future date*** *to ensure that the arbitrage has no risk. That is to say, one can never operate with any spot rates in the future. For example, one can never first invest in dollar-denominated risk-free bonds and change the dollar into yen in the future (affected by spot exchange rate in future, which is different from the forward exchange rate announced today).*

Week 4

Reminders on No Arbitrage Criterion

Last week we talked about the no arbitrage criterion and this criterion was stated as "There is no risk-free arbitrage in financial markets." However, I missed a very important point which is that other than being risk-free, such arbitrage also has to **have no initial endowment**. I have already fixed the notes for last week and bolded all the parts that I want you to notice. You are welcome to go over last week's notes (the part of pricing forward contract) with this extra condition and I will also explain it here.

The meaning of "no initial endowment" is that one shall imagine himself having no money available at the beginning of constructing an arbitrage, i.e. one has to borrow money or short assets before longing assets or buying financial derivatives. For example, one can never long a stock index before shorting something else or borrowing money as the first step when constructing an arbitrage strategy. The covered interest arbitrage (last example) we have talked about last week shows exactly the correct formulation of a risk-free arbitrage with no initial endowment.

Next, let me briefly talk about why we are still getting the correct conclusion last week without this "no initial cost" condition. Actually, we have been adopting the assumption that there's **only one interest rate** in the financial market in last week's setting, which means that borrowing and investment share the same interest rate. However, in your homework problems, you will have to deal with the case where these two interest rates differ, and it's also true in practice that interest rates for different purposes differ. **Under the condition that there's only one interest rate in the financial market, one can see that this "no initial cost" condition can actually be removed.** When the investment interest rate r_i is lower than the borrowing interest rate r_b, which is often the case in practice, one cannot return all the money without extra endowment by borrowing a certain amount of money and investing in zero-coupon bond.

Commodity Forward

Compared to stock forward, the commodity forward mainly has two differences: the continuous-times annual effective storage cost λ and the continuous-times annual effective convenience yield c. λ works as negative dividend for commodity holders and increases the forward price. On the other hand, c works as the benefit for commodity holders as positive dividend and reduces the forward price.

Remark. *To understand how λ works, imagine the case where we have 1 unit of corn in the factory and because of storage conditions, corn goes bad continuously at an annual effective rate of λ. This means that for each small time interval Δt (in the unit of year), the amount of truly usable corn drops from x to $x\left(1-\frac{\lambda}{\Delta t}\right)$. Lasting T years, the amount of truly usable corn will be*

$$\left(1-\frac{\lambda}{\Delta t}\right)^{T\Delta t} \tag{46}$$

by assuming that the time partition is fine enough, the limit goes like

$$\lim_{\Delta t\to 0}\left(1-\frac{\lambda}{\Delta t}\right)^{T\Delta t}=e^{-\lambda T} \tag{47}$$

That's why 1 unit of corn turn into $e^{-\lambda T}$ unit of truly usable corn in T years. The same logic holds for continuous-time dividend yield and convenience yield.

Thinking about the following arbitrage strategy, where r is the continuous-time annual effective risk-free interest rate and S_0 is the price of the commodity now.

- Borrow $S_0 \cdot e^{(\lambda-c)T}$ to buy $e^{(\lambda-c)T}$ unit of commodity
- Sell a T-year mature forward contract on 1 unit of commodity (price $F_{0,T}$)

so the profit is

$$F_{0,T} - S_0 \cdot e^{(\lambda-c)T}e^{rT} \tag{48}$$

and this strategy has neither initial endowment nor risk. According to the no arbitrage criterion

$$F_{0,T} = S_0 \cdot e^{(\lambda-c+r)T} \tag{49}$$

is the price of this commodity forward contract. Here we denote $\delta = c-\lambda$ as the **lease rate** and the forward price is given by

$$F_{0,T} = S_0 \cdot e^{(r-\delta)T} \tag{50}$$

exactly the same form as that for stock forward contract.

Swap

Swaps are forward contracts replacing fluctuating future payments with fixed amount future payments. Swap buyers typically pay a same fixed amount X on several future dates $T_1, ..., T_n$ and receive 1 unit of specified asset on each date. Pricing swap is just about finding the appropriate X such that no arbitrage criterion holds in the financial market.

The pricing starts from viewing the swap as combinations of forward contracts. It's easy to see that such swap consists of a T_1-mature forward contract, a T_2-mature forward contract,..., a T_n-mature forward contract. To eliminate risk-free arbitrage opportunities with no initial endowment, the sum of the present values of the price of all those forward contracts should be equal to the present value of the annuity with X being paid at time $T_1, ..., T_n$. Denote F_{0,T_i} as the price of the T_i-mature forward contract and r_i as the continuous-times annual effective yield to maturity of T_i-mature zero-coupon bond. The equality of present values gives

$$\sum_{i=1}^{n} X \cdot e^{-r_i T_i} = \sum_{i=1}^{n} F_{0,T_i} \cdot e^{-r_i T_i} \tag{51}$$

$$X = \frac{\sum_{i=1}^{n} F_{0,T_i} \cdot e^{-r_i T_i}}{\sum_{i=1}^{n} e^{-r_i T_i}} \tag{52}$$

Remark. *The T-mature zero-coupon bond yield to maturity r_T (continuous-time annual effective) just works as the discount rate to calculate present values.*

In practice, we might observe that the continuous-time annual effective 1-year yield to maturity of zero-coupon bond is 5% while the continuous-time annual effective 2-year yield to maturity of zero-coupon bond is 5.5%. This is telling us that 1\$ after 1 year has present value $e^{-0.05}$ while 1\$ after 2 years has present value $e^{-0.055\times 2}$. Note that if we use the annual discount factor $e^{-0.05}$ to figure out the present value of 1\$ after 2 years, it would be $e^{-0.05\times 2}$ instead. Be careful with the difference

$$e^{-0.055\times 2} \neq e^{-0.05\times 2} \tag{53}$$

*This is a natural phenomenon called **the term structure of interest rates**.*

Example

The following table summarizes forward prices for gold:

Years to Maturity	Forward Price	Zero-Coupon Bond Yield
1	1600	5.00%
2	1700	5.50%
3	1800	6.00%

Note: bond yield is quoted in continuously-compounded units.

Part A: If today's price is $S_0 = 1525$, find the implied (continuously compounded) lease rate on gold for the next year.

Part B: Find the swap price for receiving 1 ounce of gold for the next 3 years (i.e. each year exchange 1 ounce for the fixed swap price).

For Part A, recall that the forward price is

$$F = S_0 \cdot e^{(r-\delta)T} \tag{54}$$

for lease rate δ. Plug in all the numbers for 1-year maturity to see

$$1600 = 1525 \times e^{(0.05-\delta)} \tag{55}$$
$$\delta = 0.002 \tag{56}$$

For Part B, recall the rule of swap pricing that the present value of the sequence of forward contracts should equal the present value of the sequence of fixed amount payments.

$$X \cdot e^{-0.05} + X \cdot e^{-0.055\times 2} + X \cdot e^{-0.06\times 3} = 1600 \times e^{-0.05} + 1700 \times e^{-0.055\times 2} + 1800 \times e^{-0.06\times 3} \tag{57}$$
$$X = 1695.68 \tag{58}$$

Put Call Parity

It's obvious that options are risky financial derivatives since the future stock price, which is random, appears in the payoff. As a result, no arbitrage criterion can never be used to price a single put/call option. However, by considering a combination of put and call options, we would be able to get the put call parity from no arbitrage criterion.

Let S_T denote the stock price in the future at time T, K denote the strike price of options and the options all have time to maturity T. The continuous-time annual effective risk-free interest rate is r. Call option has payoff

$$\max\{0, S_T - K\} \tag{59}$$

and put option has payoff

$$\max\{0, K - S_T\} \tag{60}$$

to find that

$$\max\{0, S_T - K\} - \max\{0, K - S_T\} = S_T - K \tag{61}$$

To fully eliminate the risk, also sell a forward contract on stock with time to maturity T and price $F_{0,T}$ with profit $F_{0,T} - S_T$. Combining these two positions and use C, P for upfront premium of such call and put option, the overall profit (FV) of this position is deterministic

$$F_{0,T} - S_T + S_T - K - (C-P)e^{rT} = F_{0,T} - K - (C-P)e^{rT} \tag{62}$$

According to no arbitrage criterion, this profit should be 0, providing the **put call parity**

$$C - P = e^{-rT}(F_{0,T} - K) \tag{63}$$

Remark. *One can also think intuitively that by longing a call option and shorting a put option, no matter what future stock price is, one always has to buy 1 unit of stock in the future. By shorting an extra forward contract to sell out this 1 unit of stock in the future, one can fully eliminate the risk.*

Put Call Parity with Dividends

The same logic holds for put call parity with dividends. The only thing to do is to **replace the forward price** $F_{0,T}$.

Let's first focus on discrete dividend payments. According to the pricing of forward contract, it should be clear that (stated in previous classes)

$$F_{0,T} = S_0 e^{rT} - FV(Div) \tag{64}$$

$$e^{-rT} \cdot F_{0,T} = S_0 - PV(Div) \tag{65}$$

replace the $F_{0,T}$ in the put call parity to get the formula 9.3 in the textbook, **the put call parity with discrete dividend payments**

$$C - P = S_0 - PV(Div) - e^{-rT}K \tag{66}$$

If the dividend is paid in a continuous way with continuous-time annual effective dividend rate δ, the forward contract has price

$$F_{0,T} = S_0 \cdot e^{(r-\delta)T} \tag{67}$$

so we get **the put call parity with continuous dividend payments**

$$C - P = e^{-rT}(S_0 \cdot e^{(r-\delta)T} - K) = S_0 \cdot e^{-\delta T} - K \cdot e^{-rT} \tag{68}$$

Example

Let's take a look at problem 9.3 in the textbook.

Stock price is 800 now and continuous-time annual effective risk-free interest rate is 5% with no dividend yield. Term is 1 year, with 815-call premium 75 and 815-put premium 45. Now the position is longing the stock, selling the 815-call, and buying the 815-put.

(a). What is the rate of return on this position held until the expiration of the options?

(b). What is the arbitrage implied by your answer to (a)?

(c). What difference between the call and put prices would eliminate arbitrage?

(d). What difference between the call and put prices eliminates arbitrage for strike prices of 780, 800, 820, and 840?

(a): The long position in stock gives profit (FV)

$$S_1 - 800 \times e^{0.05} \tag{69}$$

the short position in 815-call gives profit (FV)

$$-\max\{0, S_1 - 815\} + 75 \times e^{0.05} \tag{70}$$

the long position in 815-put gives profit (FV)

$$\max\{0, 815 - S_1\} - 45 \times e^{0.05} \tag{71}$$

the overall profit (FV) is

$$S_1 - 800 \times e^{0.05} - \max\{0, S_1 - 815\} + 75 \times e^{0.05} + \max\{0, 815 - S_1\} - 45 \times e^{0.05} = 5.52 \tag{72}$$

The overall upfront cost is $800 - 75 + 45 = 770$ and the amount of money we have in the future is exactly 815, so the rate of return is

$$\frac{815 - 770}{770} = 0.058 \tag{73}$$

(b): Since we are buying the underlying asset and at the same time selling a forward contract (that's what selling a call and buying a put essentially do), this arbitrage is a cash-and-carry arbitrage.

(c): By put call parity

$$C - P = e^{-rT}(F_{0,T} - K) \tag{74}$$

where the price of the forward contract should be $800 \times e^{0.05}$. Plug in all the numbers to find

$$C - P = e^{-0.05}(800 \times e^{0.05} - 815) = 24.75 \tag{75}$$

So under no arbitrage criterion, the premium gap should be 24.75, but now the actual gap is $75 - 45 = 30 > 24.75$. That's why risk-free arbitrage with no initial endowment exists in the financial market and such arbitrage strategy sells the call and buys the put.

(d): Just plug in different numbers in the put call parity:

$$C - P = e^{-0.05}(800 \times e^{0.05} - 780) = 58.04 \tag{76}$$

$$C - P = e^{-0.05}(800 \times e^{0.05} - 800) = 39.02 \tag{77}$$

$$C - P = e^{-0.05}(800 \times e^{0.05} - 820) = 20 \tag{78}$$

$$C - P = e^{-0.05}(800 \times e^{0.05} - 840) = 0.97 \tag{79}$$

Example

For put call parity with continuous dividend payments, refer to problem 9.1 in the textbook.

The current stock price is 32 and 6-month 35-call has premium 2.27. The continuous-time annual effective risk-free interest rate is 4% and the continuous-time annual effective dividend yield is 6%. What's the premium of the 6-month 35-put option?

In this setting of continuous dividend payments, apply the formula

$$C - P = S_0 \cdot e^{-\delta T} - K \cdot e^{-rT} \tag{80}$$

and plug in numbers to get

$$2.27 - P = 32 \times e^{-0.06\times 0.5} - 35 \times e^{-0.04\times 0.5} \tag{81}$$

$$P = 5.52 \tag{82}$$

Week 5

Option Price Relationship

Assume all the options talked below have the same date to maturity T but with different strike price $K_1 < K_2 < K_3$. The call options with those strike prices have price C_1, C_2, C_3 and the put options with those strike prices have price P_1, P_2, P_3, the continuous-time annual effective interest rate is 0. Let's figure out the relationship between those prices. Note that all relationships shall ensure that **no arbitrage exists (note that here risk-free only refers to being free of the risk of the negative profit while uncertainty is acceptable)**.

The first rule is that **call options with lower strike price have higher prices, put options with higher strike price have higher prices**, i.e.

$$C_1 \geq C_2 \tag{83}$$

$$P_1 \leq P_2 \tag{84}$$

Consider the strategy buying K_1-call and selling K_2-call.

$$payoff = \max\{0, S_T - K_1\} - \max\{0, S_T - K_2\} \tag{85}$$

$$cost = C_1 - C_2 \tag{86}$$

the profit is

$$profit = \max\{0, S_T - K_1\} - \max\{0, S_T - K_2\} - (C_1 - C_2) \tag{87}$$

$$= \begin{cases} -(C_1 - C_2) & S_T \leq K_1 \\ S_T - K_1 - (C_1 - C_2) & K_1 < S_T \leq K_2 \\ K_2 - K_1 - (C_1 - C_2) & S_T > K_2 \end{cases} \tag{88}$$

If now $C_1 < C_2$, then for all possible future stock price, the profit is always positive.

Remark. *Although risk still exists for such strategy, the risk only reflects the uncertainty of profit being positive. There's no way for one to take the risk of the profit being negative under the condition that $C_1 < C_2$. Intuitively, such arbitrage shall also not exist in the financial markets, that's why $C_1 \geq C_2$ holds. The relationship of option prices are derived by **the construction of spread and butterfly** (these two combinations have insured tails) and the new criterion used here is that **the minimum possible profit of any strategy shall be negative**.*

Note that if one creates a strategy where the profit is always negative under all circumstances and there is only one interest rate in the economy, that is also an arbitrage since one can hold the opposite position and the profit will always be positive.

The second rule is that **the difference in premium shall always be less than the difference in strike**

price, i.e.

$$C_1 - C_2 \leq K_2 - K_1 \tag{89}$$
$$P_2 - P_1 \leq K_2 - K_1 \tag{90}$$

Consider the strategy selling K_1-call and buying K_2-call.

$$payoff = -\max\{0, S_T - K_1\} + \max\{0, S_T - K_2\} \tag{91}$$
$$cost = C_2 - C_1 \tag{92}$$

the profit is

$$profit = -\max\{0, S_T - K_1\} + \max\{0, S_T - K_2\} - (C_2 - C_1) \tag{93}$$
$$= \begin{cases} -(C_2 - C_1) & S_T \leq K_1 \\ K_1 - S_T - (C_2 - C_1) & K_1 < S_T \leq K_2 \\ K_1 - K_2 - (C_2 - C_1) & S_T > K_2 \end{cases} \tag{94}$$

Now that $C_1 \geq C_2$ holds, so $-(C_2 - C_1) \geq 0$. However, if $C_1 - C_2 > K_2 - K_1$ then $K_1 - K_2 - (C_2 - C_1) > 0$ and no matter what future stock price is, the profit is always positive. As a result, we have proved the second rule.

The third rule is the **convexity**, i.e.

$$\frac{C_1 - C_2}{K_2 - K_1} \geq \frac{C_2 - C_3}{K_3 - K_2} \tag{95}$$
$$\frac{P_2 - P_1}{K_2 - K_1} \leq \frac{P_3 - P_2}{K_3 - K_2} \tag{96}$$

Consider the asymmetric butterfly buying 1 portion of K_1-call and $\frac{K_2 - K_1}{K_3 - K_2}$ portions of K_3-call while selling $\frac{K_3 - K_1}{K_3 - K_2}$ portions of K_2-call.

Remark. *If one does not know how those portions come from, please refer the contents above for the method of asymmetric butterfly construction. The key equations here are*

$$\begin{cases} n_1 - n_2 + n_3 = 0 \\ n_1 K_1 - n_2 K_2 + n_3 K_3 = 0 \end{cases} \tag{97}$$

for buying n_1 portions of K_1-call, selling n_2 portions of K_2-call, buying n_3 portions of K_3-call.

$$payoff = \max\{0, S_T - K_1\} - \frac{K_3 - K_1}{K_3 - K_2}\max\{0, S_T - K_2\} + \frac{K_2 - K_1}{K_3 - K_2}\max\{0, S_T - K_3\} \tag{98}$$

$$cost = C_1 - \frac{K_3 - K_1}{K_3 - K_2}C_2 + \frac{K_2 - K_1}{K_3 - K_2}C_3 \tag{99}$$

the profit is

$$profit = \max\{0, S_T - K_1\} - \frac{K_3 - K_1}{K_3 - K_2}\max\{0, S_T - K_2\} + \frac{K_2 - K_1}{K_3 - K_2}\max\{0, S_T - K_3\} - \left(C_1 - \frac{K_3 - K_1}{K_3 - K_2}C_2 + \frac{K_2 - K_1}{K_3 - K_2}C_3\right) \tag{100}$$

Since the expression is a little bit complicated, let's just think about the lowest possible profit of this asymmetric butterfly. Obviously the lowest profit is gained when $S_T < K_1$ and in such situation the profit is

$$-C_1 + \frac{K_3 - K_1}{K_3 - K_2}C_2 - \frac{K_2 - K_1}{K_3 - K_2}C_3 \tag{101}$$

Of course we have to set the lowest possible profit to be negative such that no arbitrage exists. As a result,

$$-C_1 + \frac{K_3 - K_1}{K_3 - K_2}C_2 - \frac{K_2 - K_1}{K_3 - K_2}C_3 \leq 0 \tag{102}$$

$$-(K_3 - K_2)C_1 + (K_3 - K_1)C_2 - (K_2 - K_1)C_3 \leq 0 \tag{103}$$

and one can verify that this inequality is equivalent to the convexity condition

$$\frac{C_1 - C_2}{K_2 - K_1} \geq \frac{C_2 - C_3}{K_3 - K_2} \tag{104}$$

Remark. *One might be interested in the relationships when there exists continuous-time interest rate $r > 0$. If an interest rate exists, the first and third relationship won't change. The first relationship is comparing two PV and the third relationship is comparing two ratio of PV and FV, so they won't be affected by interest rate.*

However, the second relationship changes since it's comparing PV with FV. It would be natural for one to modify the second relationship into

$$C_1 - C_2 \leq e^{-rT}(K_2 - K_1) \tag{105}$$

$$P_2 - P_1 \leq e^{-rT}(K_2 - K_1) \tag{106}$$

such that both sides are PV and the relationship holds in the general case. One can also construct arbitrage strategy to prove this, the construction is left as an exercise for the readers.

Remark. ***Geometric interpretations of these relationships*** *(only read this remark if you are interested)*

Let's only state for call options. The interpretations for put options are left as exercises for the readers. Denote $C = f(K)$ as a function with the input as the strike price of a call option and the output as the premium of a call *option, so that $C_i = f(K_i)$. Assume the time to maturity T is fixed and there's no interest rate exist. Let's consider what properties shall this function f satisfy according to the relationships above (in real-life situations, option prices would have much more constraints than that stated here).*

Let's assume for simplicity that f is continuous and smooth enough (derivative of any order exists)

$$C_1 \geq C_2 \tag{107}$$

*tells us that f shall be **monotone decreasing***

$$C_1 - C_2 \leq K_2 - K_1 \tag{108}$$

tells us that f is Lipschitz with constant 1 and

$$\frac{f(K_1) - f(K_2)}{K_2 - K_1} \leq 1 \tag{109}$$

$$\lim_{K_2 \to K_1} \frac{f(K_1) - f(K_2)}{K_1 - K_2} \geq -1 \tag{110}$$

$$f'(K_1) \geq -1 \tag{111}$$

the derivative at any point is no less than -1

$$\frac{C_1 - C_2}{K_2 - K_1} \geq \frac{C_2 - C_3}{K_3 - K_2} \tag{112}$$

tells us that for $\forall K_1 < K_2 < K_3 < K_4$

$$\frac{f(K_1) - f(K_2)}{K_2 - K_1} \geq \frac{f(K_2) - f(K_3)}{K_3 - K_2} \geq \frac{f(K_3) - f(K_4)}{K_4 - K_3} \tag{113}$$

$$\lim_{K_1 \to K_2} \frac{f(K_1) - f(K_2)}{K_1 - K_2} \leq \lim_{K_4 \to K_3} \frac{f(K_3) - f(K_4)}{K_3 - K_4} \tag{114}$$

$$f'(K_2) \leq f'(K_3) \tag{115}$$

$$f''(K) \geq 0 \tag{116}$$

*the function is **convex** (this explains why this relationship is called convexity).*

As a result, f shall satisfy the conditions that it's continuous and smooth enough, with

$$\forall K, f'(K) \in [-1, 0], f''(K) \geq 0 \tag{117}$$

a reasonable function under such criterion would be $f(K) = e^{-K}$ $(K \geq 0)$ or $f(K) = -\log K$ $(K \geq 1)$, so it's actually not a tight restriction on option price.

Example

Problem 9.12 (c) in the textbook. There are 90,100,105-strike call options with premium 15,10,6 respectively and there is no interest rate.

Let's construct an asymmetric butterfly buying 1 portion of 90-call and 2 portions of 105-call while selling 3 portions of 100-call. (plug numbers into the strategy above)

$$payoff = \max\{0, S_T - 90\} - 3\max\{0, S_T - 100\} + 2\max\{0, S_T - 105\} \tag{118}$$

$$cost = 15 - 3 \times 10 + 2 \times 6 = -3 \tag{119}$$

so the profit is

$$profit = payoff + 3 \tag{120}$$

$$= \begin{cases} 3 & S_T \le 90 \\ S_T - 87 & 90 < S_T \le 100 \\ -2S_T + 213 & 100 < S_T \le 105 \\ 3 & S_T > 105 \end{cases} \tag{121}$$

so one earns at least 3 from this arbitrage strategy.