Notes on PSTAT 213

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Simple Random Walk

$S_n = X_1 + ... + X_n$ is a SRW with X_i *i.i.d.* starting from $S_0 = 0$. T_b denotes the first hitting time of S_n to b, $\mathbb{P}(X_i = 1) = p, \mathbb{P}(X_i = -1) = q, p + q = 1$.

Theorem 1. *(Hitting Time Theorem) $\forall b \neq 0$ such that $\frac{n+b}{2} \in \{0, 1, ..., n\}$,*

$$\mathbb{P}(T_b = n) = \frac{|b|}{n}\mathbb{P}(S_n = b) = \frac{|b|}{n}\binom{n}{\frac{n+b}{2}}p^{\frac{n+b}{2}}q^{\frac{n-b}{2}} \ \ (n \geq 1) \tag{1}$$

Proof. Prove by counting paths. It's obvious that if $p = q$, each path consisting of points (t, S_t) $(t = 0, 1, ..., n)$ has same probability of appearing. Now p, q are not necessarily the same, so if a fixed path has a moving upward and $n - a$ moving downward, the probability of appearing is just

$$\frac{\binom{n}{a}}{2^n}p^a q^{n-a} \tag{2}$$

If now a path hits b at time n for the first time, it should first hit b at time n, which means that there are $\frac{n+b}{2}$ going upward and $\frac{n-b}{2}$ going downward. Each path that hits b at time n has same probability of appearing, which is $\frac{\binom{n}{\frac{n+b}{2}}}{2^n}p^{\frac{n+b}{2}}q^{\frac{n-b}{2}}$. As a result, the problem reduces to counting the number of all paths within those paths that have also hit b between time 0 to n.

We do a translation for all the paths such that now we start at $(0, -b)$ and want to count the number of paths that ends at $(n, 0)$ but has also hit 0 in between. This count is just the sum of the number of paths that starts at $(0, -b)$ and ends at $(n - 1, 1)$ but has also hit 0 in between and the number of paths that starts at $(0, -b)$ and ends at $(n - 1, -1)$ but has also hit 0 in between. Assume WLOG that $b > 0$, notice that the first count is

$$\binom{n-1}{\frac{n+b}{2}} \tag{3}$$

and the second count is due to reflection principle that it's just the number of paths that starts at $(0, b)$ and ends at $(n - 1, -1)$ which is

$$\binom{n-1}{\frac{n+b}{2}} \tag{4}$$

As a result, the sum should be

$$2\binom{n-1}{\frac{n+b}{2}} \tag{5}$$

The number of path that starts at $(0, b)$ and ends at $(n, 0)$ is

$$\binom{n}{\frac{n+b}{2}} \tag{6}$$

So if a path is conditioned on already starting at $(0,0)$ and ending at (n,b), it has probability of hitting b in between as

$$\frac{2\binom{n-1}{\frac{n+b}{2}}}{\binom{n}{\frac{n+b}{2}}} = \frac{n-b}{n} \tag{7}$$

if a path is conditioned on already starting at $(0,0)$ and ending at (n,b), it has probability of not hitting b in between as

$$\frac{b}{n} \tag{8}$$

That's why $\mathbb{P}(T_b = n) = \frac{b}{n}\mathbb{P}(S_n = b)$ for $b > 0$ and the theorem is proved. The similar proof holds for $b < 0$. □

Remark. *If we want to know the distribution of T_0, we also have to lift the time at 0 to the time at 1 (consider whether S_1 is 1 or -1) since reflection can't be applied for when the path starts or ends at 0.*

Theorem 2. *Set the **maximum process** $M_n = \max_{0\le k\le n} S_k$ for symmetric SRW S_n, then*

$$\forall r \ge 1, \mathbb{P}(M_n \ge r, S_n = v) = \begin{cases} \mathbb{P}(S_n = v) & v \ge r \\ \mathbb{P}(S_n = 2r - v) & v < r \end{cases} \tag{9}$$

Proof. If $v \ge r$, then $\mathbb{P}(M_n \ge r, S_n = v) = \mathbb{P}(S_n = v)$ naturally.

For the other case, let's count the number of paths. The number of paths from $(0,0)$ to $(n, 2r-v)$ is

$$\binom{n}{\frac{n+2r-v}{2}} \tag{10}$$

The number of paths from $(0,0)$ to (n,v) that has hit r in between is equal to the number of paths from $(0,-r)$ to $(n, v-r)$ that has hit 0 in between. By reflection principle, this is just the number of paths from $(0,r)$ to $(n, v-r)$, which is

$$\binom{n}{\frac{n+v-2r}{2}} \tag{11}$$

same to the count above, so it's proved.

Another Proof:

Since the SRW is Markov process and $T_r < \infty$ $a.s.$, strong Markov property tells us

$$S_n^{T_r} = S_{n+T_r} - S_{T_r} = S_{n+T_r} - r \tag{12}$$

is also a SRW and is independent of $\mathscr{F}_{T_r}$.

Let's then do calculations:

$$\begin{aligned}
\mathbb{P}\left(M_n \geq r, S_n = v\right) &= \mathbb{P}\left(T_r \leq n, S_n = v\right) && (13)\\
&= \mathbb{P}\left(T_r \leq n, S^{T_r}_{n-T_r} = v - r\right) && (14)\\
&= \mathbb{P}\left(T_r \leq n, -S^{T_r}_{n-T_r} = v - r\right) && (15)
\end{aligned}$$

the last step is due to the fact that $T_r \in \mathscr{F}_{T_r}, S_n^{T_r} \overset{d}{=} -S_n^{T_r}$ and that $S_n^{T_r}$ is independent of $\mathscr{F}_{T_r}$.

$$\begin{aligned}
\mathbb{P}\left(M_n \geq r, S_n = v\right) &= \mathbb{P}\left(T_r \leq n, -S^{T_r}_{n-T_r} = v - r\right) && (16)\\
&= \mathbb{P}\left(T_r \leq n, S_n = 2r - v\right) && (17)\\
&= \mathbb{P}\left(S_n = 2r - v\right) && (18)
\end{aligned}$$

$\square$

Remark. *By this reflection principle, we see that for $r \geq 0$,*

$$\begin{aligned}
\mathbb{P}\left(M_n \geq r\right) &= \sum_{v=-n,-n+2,\ldots,n} \mathbb{P}\left(M_n \geq r, S_n = v\right) && (19)\\
&= \sum_{v<r} \mathbb{P}\left(S_n = 2r - v\right) + \sum_{v \geq r} \mathbb{P}\left(S_n = v\right) && (20)\\
&= \mathbb{P}\left(S_n = r\right) + \mathbb{P}\left(S_n \geq r+1\right) + \mathbb{P}\left(S_n = 2r+n\right) + \ldots + \mathbb{P}\left(S_n = 2r - r + 1\right) && (21)\\
&= \mathbb{P}\left(S_n = r\right) + \mathbb{P}\left(S_n \geq r+1\right) + \mathbb{P}\left(S_n \geq r+1\right) && (22)\\
&= \mathbb{P}\left(S_n = r\right) + 2\mathbb{P}\left(S_n \geq r+1\right) && (23)
\end{aligned}$$

that's why we get

$$\begin{aligned}
\mathbb{P}\left(M_n = r\right) &= \mathbb{P}\left(M_n \geq r\right) - \mathbb{P}\left(M_n \geq r+1\right) && (24)\\
&= \mathbb{P}\left(S_n = r\right) + 2\mathbb{P}\left(S_n \geq r+1\right) - \mathbb{P}\left(S_n = r+1\right) - 2\mathbb{P}\left(S_n \geq r+2\right) && (25)\\
&= \mathbb{P}\left(S_n = r\right) + 2\mathbb{P}\left(S_n = r+1\right) - \mathbb{P}\left(S_n = r+1\right) && (26)\\
&= \mathbb{P}\left(S_n = r\right) + \mathbb{P}\left(S_n = r+1\right) && (27)
\end{aligned}$$

To calculate probability like $\mathbb{P}\left(M_8 = 6\right)$, just use the formula to get

$$\begin{aligned}
\mathbb{P}\left(M_8 = 6\right) &= \mathbb{P}\left(S_8 = 6\right) + \mathbb{P}\left(S_8 = 7\right) && (28)\\
&= \frac{\binom{8}{1}}{2^8} = \frac{1}{32} && (29)
\end{aligned}$$

Generating Function of SRW

0 Hitting Time

Now in the general setting, p probability going upward and q going downward with $p+q=1$. Now

$$p_0(n) = \mathbb{P}\left(S_n = 0\right) \tag{30}$$

and

$$f_0(n) = \mathbb{P}\left(S_1 \neq 0, ..., S_{n-1} \neq 0, S_n = 0\right) \tag{31}$$

where $f_0(n)$ gives the probability mass of first hitting time T_0. There respective generating functions are denoted

$$P_0(s) = \sum_{n=0}^{\infty} p_0(n) s^n \tag{32}$$

$$F_0(s) = \sum_{n=0}^{\infty} f_0(n) s^n \tag{33}$$

$$\tag{34}$$

then since SRW is Markov, use the Markov property w.r.t. 1 unit of time translation to get

$$p_0(0) = 1, f_0(0) = 0 \tag{35}$$

$$\forall n \geq 1, p_0(n) = \mathbb{P}\left(S_n = 0\right) \tag{36}$$

$$= \sum_{k=1}^{n} \mathbb{P}\left(T_0 = k\right) \mathbb{P}\left(S_n = 0 | T_0 = k\right) \tag{37}$$

$$= \sum_{k=1}^{n} \mathbb{P}\left(T_0 = k\right) \mathbb{P}\left(S_{n-k} = 0\right) \tag{38}$$

$$= \sum_{k=1}^{n} f_0(k) p_0(n-k) \tag{39}$$

to compare the coefficient, proved that

$$P_0(s) = 1 + P_0(s) F_0(s) \tag{40}$$

Note that

$$
\begin{aligned}
P_0(s) &= \sum_{n=0}^{\infty} \mathbb{P}\left(S_n = 0\right) s^n && (41)\\
&= \sum_{n=0,2,\ldots} \binom{n}{\frac{n}{2}} (pq)^{\frac{n}{2}} s^n && (42)\\
&= \sum_{n=0}^{\infty} \binom{2n}{n} (pqs^2)^n && (43)\\
&= \sum_{n=0}^{\infty} \frac{(2n-1)!!2^n n!}{n!n!} (pqs^2)^n && (44)\\
&= \sum_{n=0}^{\infty} (-4)^n \binom{-\frac{1}{2}}{n} (pqs^2)^n && (45)\\
&= (1-4pqs^2)^{-\frac{1}{2}} && (46)
\end{aligned}
$$

by the Taylor series.

As a result, plug in to get

$$
\begin{aligned}
F_0(s) &= \frac{P_0(s) - 1}{P_0(s)} && (47)\\
&= 1 - (1-4pqs^2)^{\frac{1}{2}} && (48)
\end{aligned}
$$

From this generating function, we can investigate whether T_0 is almost surely finite or has finite expectation for general SRW. It's easy to see that

$$
\mathbb{P}\left(T_0 < \infty\right) = \sum_{n=1}^{\infty} \mathbb{P}\left(T_0 = n\right) = F_0(1) = 1 - |p-q| \tag{49}
$$

as a result, $T_0 < \infty$ $a.s.$ **if and only if** $p = \frac{1}{2}$.

Taking derivative for $F_0(s)$ to get

$$
\begin{aligned}
F_0'(s) &= 4pqs(1-4pqs^2)^{-\frac{1}{2}} && (50)\\
\mathbb{E}(T_0 \cdot \mathbb{I}_{T_0<\infty}) = F_0'(1) &= \frac{4pq}{|p-q|} && (51)
\end{aligned}
$$

as a result, $\mathbb{E}(T_0 \cdot \mathbb{I}_{T_0<\infty}) < \infty$ **if and only if** $p = \frac{1}{2}$.

In the context above, we investigate all generating functions of the stopping time T_0 which is the hitting time of 0. One can notice that actually this gives us the generating function of the i-th hitting time to 0, denoted T_0^i. By

Markov property,

$$\mathbb{P}\left(T_0^i = k\right) = \sum_{j=0}^{k} \mathbb{P}\left(T_0^{i-1} = j\right) \cdot \mathbb{P}\left(T_0^i = k | T_0^{i-1} = j\right) \tag{52}$$

$$= \sum_{j=0}^{k} \mathbb{P}\left(T_0^{i-1} = j\right) \cdot \mathbb{P}\left(T_0 = k - j\right) \tag{53}$$

so if we denote the generating function of T_0^i by $F_0^i(s)$, then

$$F_0^i(s) = \sum_{k=0}^{\infty} \mathbb{P}\left(T_0^i = k\right) \cdot s^k \tag{54}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \mathbb{P}\left(T_0^{i-1} = j\right) \cdot \mathbb{P}\left(T_0 = k - j\right) \cdot s^k \tag{55}$$

$$= F_0^{i-1}(s) \cdot F_0(s) \tag{56}$$

$$= [F_0(s)]^i \tag{57}$$

it's then easy to see that

$$\mathbb{P}\left(T_0^i < \infty\right) = F_0^i(1) = [F_0(1)]^i = [1 - |p - q|]^i \tag{58}$$

so **SRW is recurrent if and only if** $p = \frac{1}{2}$. Naturally, let's investigate whether SRW is null recurrent when $p = \frac{1}{2}$.

$$\mathbb{E}(T_0^i \cdot \mathbb{I}_{T_0^i < \infty}) = \frac{d}{ds} F_0^i(s)|_{s=1} \tag{59}$$

$$= i[F_0(1)]^{i-1} \cdot F_0'(1) \tag{60}$$

$$= i[1 - |p - q|]^{i-1} \cdot \frac{4pq}{|p - q|} \tag{61}$$

so all states in SRW is null recurrent when $p = \frac{1}{2}$, which indicates a natural conclusion that there's no stationary distribution for symmetric SRW.

1 Hitting Time

One might find that generating functions for T_0 tells us nothing about the information of other hitting times, e.g. T_1. To get $F_1(s)$ as the generating function of T_1, we need to apply Markov property

$$\forall n > 1, \mathbb{P}\left(T_1 = n\right) = \mathbb{P}\left(T_1 = n | X_1 = 1\right) \cdot \mathbb{P}\left(X_1 = 1\right) + \mathbb{P}\left(T_1 = n | X_1 = -1\right) \cdot \mathbb{P}\left(X_1 = -1\right) \tag{62}$$

$$= q \cdot \mathbb{P}\left(T_1 = n | X_1 = -1\right) = q \cdot \mathbb{P}\left(T_2 = n - 1\right) \tag{63}$$

and it's obvious that $\mathbb{P}(T_1 = 1) = p$. To connect $F_1(s)$ with $F_2(s)$, it's natural to think of Markov property once more. Similar to what we have done for the i-th hitting time to 0, let's denote $F_i(1)$ as the generating function of T_i, the first hitting time to $i \geq 1$

$$\mathbb{P}(T_i = n) = \sum_{k=0}^{n} \mathbb{P}(T_i = n | T_1 = k) \cdot \mathbb{P}(T_1 = k) \tag{64}$$

$$= \sum_{k=0}^{n} \mathbb{P}(T_{i-1} = n-k) \cdot \mathbb{P}(T_1 = k) \tag{65}$$

here the strong Markov property is applied when $T_1 < \infty$ $a.s.$ w.r.t. $\mathscr{F}_{T_1}$, note that when $T_1 = \infty$, $T_i = \infty$ so such equation still holds. This is telling us that getting the generating function of T_1 is equivalent to getting the generating function of any hitting time T_i

$$F_i(s) = [F_1(s)]^i \tag{66}$$

Return to the previous question on $F_1(s)$, this provides connection between $\mathbb{P}(T_1 = n)$ and $\mathbb{P}(T_2 = n-1)$ that

$$F_1(s) = ps + \sum_{k=2}^{\infty} q \cdot \mathbb{P}(T_2 = k-1)\, s^k \tag{67}$$

$$= ps + qs \cdot F_2(s) \tag{68}$$

$$= ps + qs \cdot [F_1(s)]^2 \tag{69}$$

solve this quadratic equation w.r.t. $F_1(s)$ to get

$$F_1(s) = \frac{1 \pm \sqrt{1 - 4pqs^2}}{2qs} \tag{70}$$

notice that any generating function shall satisfy $F_1(0) = 0$, so we only take one appropriate root as the generating function

$$F_1(s) = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs} \tag{71}$$

naturally, one might calculate the quantity of one's interest that

$$\mathbb{P}(T_1 < \infty) = F_1(1) = \frac{1 - |p-q|}{2q} = \begin{cases} 1 & p \geq q \\ \frac{p}{q} & p < q \end{cases} \tag{72}$$

$$\mathbb{E}(T_1 \cdot \mathbb{I}_{T_1 < \infty}) = F_1'(1) = \frac{2p}{|p-q|} - \frac{1}{2q} + \frac{|p-q|}{2q} = \begin{cases} \frac{1}{p-q} & p > q \\ \frac{p}{q}\frac{1}{q-p} & p < q \\ \infty & p = q \end{cases} \tag{73}$$

in the more general case,

$$\mathbb{P}\left(T_i < \infty\right) = F_i(1) = \begin{cases} 1 & p \geq q \\ \left(\frac{p}{q}\right)^i & p < q \end{cases} \tag{74}$$

$$\mathbb{E}(T_i \cdot \mathbb{I}_{T_i<\infty}) = F_i'(1) = i[F_1(1)]^{i-1} \cdot F_1'(1) = \begin{cases} \frac{i}{p-q} & p > q \\ \left(\frac{p}{q}\right)^i \frac{i}{q-p} & p < q \\ \infty & p = q \end{cases} \tag{75}$$

as a result, $\mathbb{E}(T_i|T_i < \infty) = \frac{i}{|p-q|}$ holds generally.

A slight generalization is still the j-th hitting time to i, denoted T_i^j. To get its generating function $F_i^j(s)$, notice that

$$\mathbb{P}\left(T_i^j = n\right) = \sum_{k=0}^{n} \mathbb{P}\left(T_i^1 = k\right) \cdot \mathbb{P}\left(T_i^j = n | T_i^1 = k\right) \tag{76}$$

$$= \sum_{k=0}^{n} \mathbb{P}\left(T_i^1 = k\right) \cdot \mathbb{P}\left(T_0^{j-1} = n-k\right) \tag{77}$$

by Markov property, since after hitting i for the first time we are restarting the SRW from i and hitting 0 after restarting is equivalent to hitting i from the very start. As a result, $F_i^j(s) = F_i(s) \cdot F_0^{j-1}(s)$, by previous proofs, we know that $F_i(s) = [F_1(s)]^i$ and $F_0^{j-1}(s) = [F_0(s)]^{j-1}$, so

$$F_i^j(s) = [F_1(s)]^i \cdot [F_0(s)]^{j-1} \tag{78}$$

One is also able to calculate the probability and expectations one care about.

$$\mathbb{P}\left(T_i^j < \infty\right) = F_i^j(1) = [F_1(1)]^i \cdot [F_0(1)]^{j-1} \tag{79}$$

$$= \left(\frac{1-|p-q|}{2q}\right)^i \cdot (1-|p-q|)^{j-1} \tag{80}$$

and for the expectation

$$\mathbb{E}(T_i^j \cdot \mathbb{I}_{T_i^j<\infty}) = \frac{d}{ds} F_i^j(s)\Big|_{s=1} \tag{81}$$

$$= i[F_1(1)]^{i-1} \cdot F_1'(1) \cdot [F_0(1)]^{j-1} + [F_1(1)]^i \cdot (j-1)[F_0(1)]^{j-2} \cdot F_0'(1) \tag{82}$$

Remark. *The only important thing here is the* ***Markov property****. By selecting appropriate translation of time, one can always transform all j-th hitting time problems into the first hitting time of 0 and 1.*

Gambler's Ruin

Now for a general SRW, consider the exit time instead of the hitting time. Assume now the SRW starts at x and $T_{a,b}$ denotes the stopping time when SRW hits either a or b with $a < x < b$. It's quite clear that $T_{a,b} = T_a \wedge T_b$. This is telling us that if $p > q$ then $T_b < \infty$ $a.s.$, if $p < q$ then $T_a < \infty$ $a.s.$, if $p = q$ then $T_a, T_b < \infty$ $a.s.$. As a result, $T_{a,b} < \infty$ $a.s.$ is almost surely finite.

As a result, a natural question to ask is that what's the probability that the SRW is exiting from a. Since $T_{a,b} < \infty$ $a.s.$,

$$\mathbb{P}_x \left(S_{T_{a,b}} = a \right) + \mathbb{P}_x \left(S_{T_{a,b}} = b \right) = 1 \tag{83}$$

where $\mathbb{P}_x$ means the probability measure of the SRW starting from x. Set

$$r(x) = \mathbb{P}_x \left(S_{T_{a,b}} = a \right) \tag{84}$$

and apply the Markov property to consider the first step

$$\begin{aligned} r(x) &= p \cdot \mathbb{P}_x \left(S_{T_{a,b}} = a | X_1 = 1 \right) + q \cdot \mathbb{P}_x \left(S_{T_{a,b}} = a | X_1 = -1 \right) && (85)\\ &= p \cdot \mathbb{P}_{x+1} \left(S_{T_{a,b}} = a \right) + q \cdot \mathbb{P}_{x-1} \left(S_{T_{a,b}} = a \right) && (86)\\ &= p \cdot r(x+1) + q \cdot r(x-1) && (87) \end{aligned}$$

here $\mathbb{P}_x \left(S_{T_{a,b}} = a | X_1 = 1 \right) = \mathbb{P}_{x+1} \left(S_{T_{a,b}} = a \right)$ is due to the fact that we can stop the SRW at time 1 and restart it as if it starts from $x + 1$ at time 0. The boundary condition is $r(a) = 1, r(b) = 0$.

Use the characteristic equation to solve the recurrence relationship:

$$\begin{aligned} \lambda &= p\lambda^2 + q && (88)\\ \lambda &= 1 \ or \ \frac{q}{p} && (89) \end{aligned}$$

we have to discuss whether $p = q$ since there might be roots with multiplicity.

If $p = q$, $\lambda = 1$ has multiplicity 2 so

$$r(x) = (C_1 x + C_2) \cdot 1^x \tag{90}$$

for some constant C_1, C_2, plug in boundary condition to solve out

$$C_1 = \frac{1}{a-b}, C_2 = -\frac{b}{a-b} \tag{91}$$

so we conclude

$$\mathbb{P}_x \left(S_{T_{a,b}} = a \right) = \frac{x-b}{a-b} \tag{92}$$

when $p = q = \frac{1}{2}$ in the symmetric case.

Now if $p \neq q$, there are two different roots and

$$r(x) = C_1 \cdot 1^x + C_2 \cdot \left(\frac{q}{p}\right)^x \tag{93}$$

for some constant C_1, C_2, plug in boundary condition to solve out

$$C_1 = -\frac{\left(\frac{q}{p}\right)^b}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b}, C_2 = \frac{1}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b} \tag{94}$$

so we conclude

$$\mathbb{P}_x \left(S_{T_{a,b}} = a\right) = \frac{\left(\frac{q}{p}\right)^x - \left(\frac{q}{p}\right)^b}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b} \tag{95}$$

when $p \neq q$ in the asymmetric case.

Law of Arcsine

The law of arcsine describes the asymptotic distribution of the **last hitting time to 0 and the overall time above 0** for **symmetric SRW**. The setting of the problem is that the last hitting time to 0 in time interval $[0, 2n]$ is defined as

$$L_{2n} = \sup \{m \leq 2n : S_m = 0\} \tag{96}$$

note that if the time is not bounded above, such random variable would not even be a stopping time (prove using strong Markov property by contradiction). Consider $0 \leq \frac{L_{2n}}{2n} \leq 1$, we would prove that such quotient has the law of arcsine (SRW starts from 0).

Let's start by observing that

$$\begin{aligned} \forall 0 \leq k \leq n, k \in \mathbb{N}, \mathbb{P}\left(L_{2n} = 2k\right) &= \mathbb{P}\left(S_{2k} = 0\right) \cdot \mathbb{P}\left(L_{2n} = 2k | S_{2k} = 0\right) && (97) \\ &= \mathbb{P}\left(S_{2k} = 0\right) \cdot \mathbb{P}\left(S_1 \neq 0, S_2 \neq 0, ..., S_{2n-2k} \neq 0\right) && (98) \end{aligned}$$

by Markov property that we stop SRW at time $2k$ and restart it as if it starts from 0 at time 0. Due to former calculations, $\mathbb{P}\left(S_1 \neq 0, S_2 \neq 0, ..., S_{2n-2k} \neq 0\right) = \mathbb{P}\left(S_{2n-2k} = 0\right)$ so

$$\forall 0 \leq k \leq n, k \in \mathbb{N}, \mathbb{P}\left(L_{2n} = 2k\right) = \mathbb{P}\left(S_{2k} = 0\right) \cdot \mathbb{P}\left(S_{2n-2k} = 0\right) \tag{99}$$

now since

$$\mathbb{P}(S_{2k}=0)\cdot\mathbb{P}(S_{2n-2k}=0)=\frac{\binom{2k}{k}\cdot\binom{2n-2k}{n-k}}{2^{2n}} \tag{100}$$

$$\sim\frac{\sqrt{2k}\sqrt{(2n-2k)}}{2\pi k(n-k)}\ \ (n\to\infty) \tag{101}$$

$$=\frac{1}{\pi}\frac{1}{\sqrt{k(n-k)}}\ \ (n\to\infty) \tag{102}$$

by Stirling's formula, as a result, if $\frac{k}{n}\to x\ (n\to\infty)$

$$n\cdot\mathbb{P}(L_{2n}=2k)\to\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}} \tag{103}$$

which provides the main thought of the law of arcsine

$$\forall 0<a\le b<1,\mathbb{P}\left(a\le\frac{L_{2n}}{2n}\le b\right)\to\int_a^b\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}\,dx\ \ (n\to\infty) \tag{104}$$

the details can be verified by proving $\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}$ is the uniform limit on any compact set $[a,b]$. The "arcsine" comes from the fact that

$$\int_a^b\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}\,dx=\frac{2}{\pi}\arcsin\sqrt{x}\Big|_{(a,b)}=\frac{2}{\pi}\arcsin\sqrt{b}-\frac{2}{\pi}\arcsin\sqrt{a} \tag{105}$$

Remark. *The law of arcsine is interesting if we think of the following bet that we are having 0 money at first and by tossing the coin we can get 1 or -1 for the same probability $\frac{1}{2}$, which means that this is a totally fair bet.*

However, by the law of arcsine,

$$\mathbb{P}\left(a\le\frac{L_{2n}}{2n}\le\frac{1}{2}\right)=\frac{1}{2}-\frac{2}{\pi}\arcsin\sqrt{a}\to\frac{1}{2}\ \ (a\to 0,n\to\infty) \tag{106}$$

$$\mathbb{P}(L_{2n}\le n)\to\frac{1}{2}\ \ (n\to\infty) \tag{107}$$

which means that if we are keeping betting until time $2n$ where n is a large enough time, we have $\frac{1}{2}$ probability seeing that we are always having positive amount of money or negative amount of money after time n. So the asymptotic behavior of this fair bet model is now clear. If we are keeping betting until time $2n$ where n is a large enough time, we have $\frac{1}{4}$ probability of becoming a "winner", who always enjoys positive return in the latter half of the bet; we have $\frac{1}{4}$ probability of becoming a "loser", who always suffers from negative return in the latter half of the bet; we have $\frac{1}{2}$ probability of becoming a "normal person", whose return fluctuates up and down around 0.

This is telling us that even in totally fair games, the accumulation in time matters and presents ***concentration*** *phenomenon. This can be seen from the density $\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}$ that the likelihood is lowest at $\frac{1}{2}$ but goes to ∞ at $0,1$, which means that extreme values of $\frac{L_{2n}}{2n}$ are far more likely to be observed (either never hits 0 or always hits 0).*

Eventually, one might notice that for a symmetric SRW starting from 0, the overall time it spends above 0 also has the law of arcsine.

$$\pi_{2n} = \#\left\{(t, S_t) : 0 \leq t \leq 2n, S_t \geq 0\right\} \tag{108}$$

be the overall time during $[0, 2n]$ such that SRW takes positive values. Then

$$\forall 0 < a \leq b < 1, \mathbb{P}\left(a \leq \frac{\pi_{2n}}{2n} \leq b\right) \to \int_a^b \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}\, dx \ (n \to \infty) \tag{109}$$

One might notice that actually $\pi_{2n} \overset{d}{=} L_{2n}$, the reason is that we can break up the event according to when the SRW first hits 0 and whether the SRW before the first hitting time to 0 is positive or negative

$$\mathbb{P}\left(\pi_{2n} = 2k\right) = \sum_{m=1}^{n} \mathbb{P}\left(\pi_{2n} = 2k | T_0 = 2m, S_{0\to T_0} \geq 0\right) \cdot \mathbb{P}\left(T_0 = 2m, S_{0\to T_0} \geq 0\right) \tag{110}$$

$$+ \sum_{m=1}^{n} \mathbb{P}\left(\pi_{2n} = 2k | T_0 = 2m, S_{0\to T_0} \leq 0\right) \cdot \mathbb{P}\left(T_0 = 2m, S_{0\to T_0} \leq 0\right) \tag{111}$$

$$= \frac{1}{2}\sum_{m=1}^{k} \mathbb{P}\left(\pi_{2n} = 2k | T_0 = 2m, S_{0\to T_0} \geq 0\right) \cdot \mathbb{P}\left(T_0 = 2m\right) \tag{112}$$

$$+ \frac{1}{2}\sum_{m=1}^{n-k} \mathbb{P}\left(\pi_{2n} = 2k | T_0 = 2m, S_{0\to T_0} \leq 0\right) \cdot \mathbb{P}\left(T_0 = 2m\right) \tag{113}$$

$$= \frac{1}{2}\sum_{m=1}^{k} \mathbb{P}\left(\pi_{2n-2m} = 2k - 2m\right) \cdot \mathbb{P}\left(T_0 = 2m\right) + \frac{1}{2}\sum_{m=1}^{n-k} \mathbb{P}\left(\pi_{2n-2m} = 2k\right) \cdot \mathbb{P}\left(T_0 = 2m\right) \tag{114}$$

by Markov property. When the segment before $T_0 = 2m$ is positive, we just need another $2k - 2m$ to be positive in the remaining $2n - 2m$ time by restarting the SRW from 0. When the segment before $T_0 = 2m$ is negative, there's no contribution to π_{2n}, so we still need $2k$ to be positive in the remaining $2n - 2m$ time by restarting the SRW from 0.

Now notice that

$$\mathbb{P}\left(\pi_{2n} = 2n\right) = \mathbb{P}\left(S_1, ..., S_{2n} \geq 0\right) \tag{115}$$

$$= 2\mathbb{P}\left(S_1, ..., S_{2n} > 0\right) \tag{116}$$

$$= \mathbb{P}\left(S_1, ..., S_{2n} \neq 0\right) \tag{117}$$

$$= \mathbb{P}\left(S_{2n} = 0\right) \tag{118}$$

where the second equation comes from the reflection principle and the last equation is the property we have proved.

Now apply backward induction, the conclusion

$$\mathbb{P}\left(\pi_{2n}=2k\right)=\mathbb{P}\left(S_{2k}=0\right)\cdot\mathbb{P}\left(S_{2n-2k}=0\right) \tag{119}$$

holds for $k=n$. Assume that it's true for $k+1, k+2, ..., n$, let's see whether it's true for k

$$\begin{aligned}
\mathbb{P}\left(\pi_{2n}=2k\right) &= \frac{1}{2}\sum_{m=1}^{k}\mathbb{P}\left(\pi_{2n-2m}=2k-2m\right)\cdot\mathbb{P}\left(T_0=2m\right)+\frac{1}{2}\sum_{m=1}^{n-k}\mathbb{P}\left(\pi_{2n-2m}=2k\right)\cdot\mathbb{P}\left(T_0=2m\right) && (120)\\
&= \frac{1}{2}\sum_{m=1}^{k}\mathbb{P}\left(S_{2k-2m}=0\right)\cdot\mathbb{P}\left(S_{2n-2k}=0\right)\cdot\mathbb{P}\left(T_0=2m\right)+\frac{1}{2}\sum_{m=1}^{n-k}\mathbb{P}\left(S_{2k}=0\right)\cdot\mathbb{P}\left(S_{2n-2m-2k}=0\right)\cdot\mathbb{P}\left(T_0=2m\right) && (121)\\
&= \frac{1}{2}\mathbb{P}\left(S_{2n-2k}=0\right)\mathbb{P}\left(S_{2k}=0\right)+\frac{1}{2}\mathbb{P}\left(S_{2k}=0\right)\mathbb{P}\left(S_{2n-2k}=0\right) && (122)\\
&= \mathbb{P}\left(S_{2k}=0\right)\cdot\mathbb{P}\left(S_{2n-2k}=0\right) && (123)
\end{aligned}$$

where we used another Markov property that $\mathbb{P}\left(S_{2k}=0\right)=\sum_{m=1}^{k}\mathbb{P}\left(T_0=2m\right)\cdot\mathbb{P}\left(S_{2k-2m}=0\right)$. As a result, we have proved that

$$\pi_{2n}\overset{d}{=}L_{2n} \tag{124}$$

so the law of arcsine also holds.