Notes on MFG

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Sep, 2023

Contents

This note is mainly written based on the author's experience reading *Probability Theory of Mean Field Games with Applications* by *Rene Carmona, Francois Delarue* and is subject to errors and typos. You are welcome to read critically and carefully.

Basic Ideas of MFG

Notation and Assumption of Single-period MFG

We neglect the classical setting of stochastic differential games here and only focus on the special settings of mean field game (MFG) that are worth noting. For notation purpose, we clarify that x^{-i} denotes $(x^1, ..., x^{i-1}, x^{i+1}, ..., x^N)$, (x^i, x^{-i}) denotes $x = (x^1, ..., x^{i-1}, x^i, x^{i+1}, ..., x^N)$ and (x, x^{-i}) denotes $(x^1, ..., x^{i-1}, x, x^{i+1}, ..., x^N)$ with the i-th coordinate replaced with x. J^i always denotes the expected total cost of player i (altogether N players), with α^i denotes its control process taking value in A, the space of all admissible control process (assumed to be compact).

Nash equilibrium (NE) $(\hat{\alpha}^1, ..., \hat{\alpha}^N) \in A^N$ is defined as the tuple of control such that for any player, when all other players' controls are frozen, this player has no motivation of deviating from it. In simple notations,

$$\forall i \in \{1, 2, ..., N\}, \forall \alpha \in A, J^i(\hat{\alpha}^i, \hat{\alpha}^{-i}) \leq J^i(\alpha, \hat{\alpha}^{-i}) \tag{1}$$

for theoretical analysis on NE, it's useful to represent it as a fixed point of the **best response function** $B : A^N \to A^N$ defined as

$$B(\alpha) = \beta,\ \beta^i \overset{def}{=} \arg\min_{\alpha} J^i(\alpha, \alpha^{-i}) \tag{2}$$

where β^i denotes the best reaction of player i given all other players' control.

MFG requires strong assumptions on symmetricity of players and the influence of each player diminishing. This requires us to define the **empirical measure**

$$\overline{\mu}^n_X \overset{def}{=} \frac{1}{n} \sum_{i=1}^n \delta_{x^i} \tag{3}$$

with $X = (x^1, ..., x^n) \in E^n$ as given state of all players. The empirical measure is a probability measure on compact metric space E, we denote the space of all probability measures on E as $\mathscr{P}(E)$ and equip this space with the topology of weak convergence of measures (with a compatible metric ρ on $\mathscr{P}(E)$ so it's compact).

To see the assumptions required for MFG setting, we focus on adding restrictions to the cost functionals J^i when N gets large enough. The following technical lemma helps us figure out what assumptions to put up for MFG setting.

Lemma 1. *$\forall n$, if $u^n : E^n \to \mathbb{R}$ satisfies uniform boundedness condition*

$$\sup_n \sup_{X \in E^n} |u^n(X)| < \infty \tag{4}$$

and uniform Lipschitz continuity

$$\exists c > 0, \forall n, \forall X, Y \in E^n, |u^n(X) - u^n(Y)| \leq c\rho(\overline{\mu}^n_X, \overline{\mu}^n_Y) \tag{5}$$

then exists subsequence u^{n_k} *and Lipschitz continuous* $U : \mathscr{P}(E) \to \mathbb{R}$ *such that*

$$\lim_{k\to\infty} \sup_{X\in E^{n_k}} |u^{n_k}(X) - U(\overline{\mu}^{n_k}_X)| = 0 \tag{6}$$

Remark. *Instead of showing the proof (which is an application of Arzela-Ascoli on some auxiliary function), we show the interpretation of the lemma. Notice that* $\forall n, u^n$ *must be **symmetric** and **continuous** if it satisfies the conditions above. The symmetricity is a consequence of uniform Lipschitz continuity (take* Y *whose components are permutations of those of* X*, this does not change the empirical measure, resulting in* $u^n(X) = u^n(Y)$*). On the other hand, if* $X^{n_p} \to X^n$ $(p \to \infty)$*,* $\overline{\mu}^n_{X^{n_p}} \xrightarrow{w} \overline{\mu}^n_{X^n}$ *resulting in* $u^n(X^{n_p}) \to u^n(X^n)$ $(p \to \infty)$ *proves the continuity of* u^n*.*

As a result, those two conditions in the lemma have already contained strong assumptions on symmetricity. The conclusion is saying that at least along some subsequence, the function u^{n_k} *can be uniformly well approximated as a regular function of the empirical measure if there are enough players.*

Inspired by the lemma above, we put up the assumption called **large symmetric cost functional** that $\forall N, \exists J^N : A^N \to \mathbb{R}$ such that

$$\begin{cases} \forall N, \alpha \in A^N, J^{N,i}(\alpha) = J^N(\alpha^i, \alpha^{-i}) \\ \sup_N \sup_{\alpha\in A^N} |J^N(\alpha)| < \infty \\ \exists c > 0, \forall N, \forall \alpha, \beta \in A^N, |J^N(\alpha) - J^N(\beta)| \leq c \left[d_A(\alpha^1, \beta^1) + \rho(\overline{\mu}^{N-1}_{\alpha^{-1}}, \overline{\mu}^{N-1}_{\beta^{-1}}) \right] \end{cases} \tag{7}$$

where $J^{N,i}$ is the cost functional of player i, having the same meaning as J^i, d_A is the metric on A. Viewing $J^N(\alpha) = J^N(\alpha^1, \alpha^{-1})$ with the control of player 1 separated (due to the form of the third assumption in large symmetric cost functional), the same proving procedure tells us $\exists J : A \times \mathscr{P}(A) \to \mathbb{R}$ continuous and a subsequence N_k such that

$$\lim_{k\to\infty} \sup_{\alpha^{N_k}\in A^{N_k}} |J^{N_k}(\alpha^{N_k,1}, ..., \alpha^{N_k,N_k}) - J(\alpha^{N_k,1}, \overline{\mu}^{N_k-1}_{\alpha^{N_k,-1}})| = 0 \tag{8}$$

in simple words, for each player (e.g. player 1), the cost functional can be uniformly well approximated in a way that all other players make a difference to the cost functional only through the empirical measure which is $\overline{\mu}^{N_k-1}_{\alpha^{N_k,-1}} = \frac{1}{N_k-1}\sum_{i=2}^{N_k} \delta_{\alpha^{N_k,i}}$.

So far, we have put up the large symmetric cost functional assumption providing us with a **limiting cost functional** J as an approximation to the cost functional of each player when $N \to \infty$. However, it should also be expected that MFG can be represented in terms of some population measure μ as some kind of weak limit of the sequence of empirical measures. If this is the case, MFG will have a simple structure since we only have to play with the limiting functional and population measure J, μ instead of a sequence of cost functional and empirical measures. The following theorem adopts such intuition and add some extra conditions to ensure the existence and good properties of population measure as weak limit.

Theorem 1. *Assume for $\forall N$, $\hat{\alpha}^N = (\hat{\alpha}^{N,1}, ..., \hat{\alpha}^{N,N})$ is NE for the game with cost functional $J^{N,1}, ..., J^{N,N}$ that satisfies the large symmetric cost functional assumption. Moreover, assume that*

$$\exists c > 0, \forall N, \alpha \in A, \mu \in \mathscr{P}(A), \rho\left(\mu, \frac{N-1}{N}\mu + \frac{1}{N}\delta_\alpha\right) \leq \frac{c}{N} \tag{9}$$

then there exists subsequence N_k and continuous $J : A \times \mathscr{P}(A) \to \mathbb{R}$ such that

$$\overline{\mu}^{N_k}_{\hat{\alpha}^{N_k}} \xrightarrow{w} \hat{\mu} \in \mathscr{P}(A) \ (k \to \infty) \tag{10}$$

with

$$\lim_{k\to\infty} \sup_{\alpha^{N_k} \in A^{N_k}} |J^{N_k}(\alpha^{N_k,1}, ..., \alpha^{N_k,N_k}) - J(\alpha^{N_k,1}, \overline{\mu}^{N_k-1}_{\alpha^{N_k,-1}})| = 0 \tag{11}$$

and

$$\int_A J(\alpha, \hat{\mu})\hat{\mu}(d\alpha) = \inf_{\mu \in \mathscr{P}(A)} \int_A J(\alpha, \hat{\mu})\mu(d\alpha) \tag{12}$$

Remark. *The second conclusion has been proved using the technical lemma, and since A is compact, $\overline{\mu}^{N_k}_{\hat{\alpha}^{N_k}} \in \mathscr{P}(A)$ is tight, by Prokhorov's theorem, it has weak subsequential limit proves the existence of $\hat{\mu}$. The proof of the last conclusion requires the similar technique as the proof of the technical lemma above, omitted here.*

The interpretation of the last conclusion is crucial for the setting of MFG. Let's specify $\mu = \delta_{\alpha_0}$ where $\alpha_0 = \arg\min_\alpha J(\alpha, \hat{\mu})$ to see

$$J(\alpha_0, \hat{\mu}) = \inf_{\mu \in \mathscr{P}(A)} \int_A J(\alpha_0, \hat{\mu})\mu(d\alpha) \leq \inf_{\mu \in \mathscr{P}(A)} \int_A J(\alpha, \hat{\mu})\mu(d\alpha) \leq J(\alpha_0, \hat{\mu}) \tag{13}$$

so

$$\int_A J(\alpha, \hat{\mu})\hat{\mu}(d\alpha) = J(\alpha_0, \hat{\mu}) \tag{14}$$

now denote $A_{\hat{\mu}} = \{\alpha_0 \in A : \alpha_0 = \arg\min_\alpha J(\alpha, \hat{\mu})\}$ as the collection of all controls minimizing the limiting cost functional at the population measure,

$$\int_A J(\alpha, \hat{\mu})\hat{\mu}(d\alpha) = J(\alpha_0, \hat{\mu}) \cdot \hat{\mu}(A_{\hat{\mu}}) + \int_{A - A_{\hat{\mu}}} J(\alpha, \hat{\mu})\hat{\mu}(d\alpha) \leq J(\alpha_0, \hat{\mu}) \cdot \hat{\mu}(A_{\hat{\mu}}) \tag{15}$$

proves $\hat{\mu}(A_{\hat{\mu}}) = 1$. *Conversely, if* $\hat{\mu}(A_{\hat{\mu}}) = 1$, $\int_A J(\alpha,\hat{\mu})\hat{\mu}(d\alpha) = J(\alpha_0,\hat{\mu})$ *immediately proves*

$$\int_A J(\alpha,\hat{\mu})\hat{\mu}(d\alpha) = \inf_{\mu\in\mathscr{P}(A)} \int_A J(\alpha,\hat{\mu})\mu(d\alpha) \tag{16}$$

as a result, this conclusion is saying that ***the support of population measure*** $\hat{\mu}$ ***is contained in the set of minimum*** $\arg\min_\alpha J(\alpha,\hat{\mu})$.

At this point, the setting of MFG shall be clear, which only depends on the limiting cost functional J and the population measure $\hat{\mu}$ of a single representative player. MFG is a game of N identical players as $N\to\infty$ and we care about the NE of MFG. Different from finite player game, there is an extra population measure in MFG, resulting in the fact that MFG has to take into consideration both optimality (minimizing the cost functional) and consistency (each player shall behave according to the population measure) conditions. **The basic strategy for MFG is to fix population measure as μ, solve out the set of best control for fixed μ:**

$$A_\mu = \arg\min_{\alpha\in A} J(\alpha,\mu) \tag{17}$$

and find a measure $\hat{\mu}$ that is concentrated on the arguments of the minimization $A_{\hat{\mu}}$. Notice that the restrictions on $\hat{\mu}$ directly comes from the conclusion in the theorem above that

$$\int_A J(\alpha,\hat{\mu})\hat{\mu}(d\alpha) = \inf_{\mu\in\mathscr{P}(A)} \int_A J(\alpha,\hat{\mu})\mu(d\alpha) \tag{18}$$

equivalently speaking,

$$supp(\hat{\mu}) \subset \arg\min_{\alpha\in A} J(\alpha,\hat{\mu}) \tag{19}$$

the **solution to MFG** is the population measure $\hat{\mu}$ and the control $\hat{\alpha}$ following the population measure $\hat{\mu}$ is a NE since it's the fixed point of the best response function (notice that all other players' control only make a difference through the population measure as N gets large enough).

Instead of existence of the solution, the uniqueness of the solution to this single-period MFG is not always guaranteed. One criterion for uniqueness is based on the strictly monotone property of J.

Theorem 2. *The solution to the single-period MFG is unique if J is strictly monotone, i.e.*

$$\forall \mu_1\neq\mu_2, \int_A [J(\alpha,\mu_1) - J(\alpha,\mu_2)]\,[\mu_1-\mu_2](d\alpha) > 0 \tag{20}$$

Proof. We prove it from definition that if μ_1,μ_2 are two different solutions to the MFG,

$$\int_A J(\alpha,\mu_1)\,\mu_1(d\alpha) \le \int_A J(\alpha,\mu_1)\,\mu_2(d\alpha), \int_A J(\alpha,\mu_2)\,\mu_2(d\alpha) \le \int_A J(\alpha,\mu_2)\,\mu_1(d\alpha) \tag{21}$$

sum up to find a contradiction with the strictly monotone condition. □

An Example of Mean Field Approximation

Consider the setting when a meeting is planned to start at deterministic time $t_{\geq}0$, player i has its control $\alpha^i = t_i$ as the time planned to attend the meeting. However, there is random effect so player i actually attends meeting at time X^i with

$$X^i = \alpha^i + \sigma^i \varepsilon^i \tag{22}$$

where $\varepsilon^1, \varepsilon^2, ... \overset{i.i.d.}{\sim} N(0,1), \sigma^1, \sigma^2, ... \overset{i.i.d.}{\sim} \nu$ with ν has its support on $(0, \infty)$ and the sequence ε^i is independent of the sequence σ^i. The cost functional of player i is

$$J^i(\alpha) = \mathbb{E}[a(X^i - t_0)^+ + b(X^i - t)^+ + c(t - X^i)_+] \tag{23}$$

when the meeting actually starts at time t instead of the original planned time t_0. The actual starting time t is actually determined based on the arrival time of all players, i.e.

$$t = \tau(\overline{\mu}_X^N) \tag{24}$$

for some deterministic function τ (e.g. start the meeting when a certain percentage of players arrive). This game is a one-period game so there's no SDE dynamics and the only interaction between players is through the empirical measure $\overline{\mu}_X^N$ (but this empirical measure is a measure on the state space, not control space).

If we treat this game as a finite player game, to get NE we need to do optimization

$$\forall i \in \{1, 2, ..., N\}, \inf_{\alpha^i} J^i(\alpha) \tag{25}$$

and the biggest trouble comes from the empirical measure that couples N optimization problems.

However, as long as we know that N is large enough, since ε^i, σ^i are *i.i.d.* sequence of random variables and the cost functional satisfies the large symmetric cost functional assumption for MFG, MFG approximation allows us to have $\overline{\mu}_X^N \overset{w}{\to} \mu$ $(N \to \infty)$ and to replace $t = \tau(\overline{\mu}_X^N)$ with $t = \tau(\mu)$ (since MFG approximation happens simultaneously for the empirical measure and the cost functional). At this point, we just need to solve the optimization for a representative player with cost functional

$$J(\alpha, \mu) = \mathbb{E}[a(X - t_0)^+ + b(X - t)^+ + c(t - X)^+] \ (a, b, c > 0) \tag{26}$$

where $t = \tau(\mu)$, $X = \alpha + \sigma\varepsilon$ with α as the control. For fixed μ, i.e. fixed t, let's do the minimization

$$A_\mu = \arg\min_\alpha J(\alpha, \mu) \tag{27}$$

take weak derivative for J w.r.t. α to find

$$\frac{\partial J}{\partial \alpha} = a\mathbb{P}\left(\alpha + \sigma\varepsilon - t_0 > 0\right) + b\mathbb{P}\left(\alpha + \sigma\varepsilon - t > 0\right) - c\mathbb{P}\left(-\alpha - \sigma\varepsilon + t > 0\right) \tag{28}$$

if we denote $Z = \sigma\varepsilon$ and F the CDF of Z, since Z has symmetric distribution around zero, $F(z) + F(-z) = 1$ so

$$A_\mu = \{\alpha \geq 0 : aF(\alpha - t_0) + (b + c)F(\alpha - t) = c\} \tag{29}$$

is determined implicitly by the equation on α. Since zero is not in the support of ν where $\sigma \sim \nu$, F is strictly positive, strictly increasing and continuous, it's then obvious that A_μ only contains a single point for fixed μ (the equation has unique solution).

That's all the work for the first optimization step, and the next step is to find the population measure that satisfies the consistency condition, that is to find measure $\hat{\mu} \stackrel{d}{=} \hat{\alpha} + \sigma\varepsilon$ such that $\hat{\alpha}$ is consistent with $\hat{\mu}$. It's clear that $\hat{\mu}$ is a measure on $\mathbb{R}_+$ induced by CDF $F(z - \hat{\alpha})$ so we denote the measure $\hat{\mu}$ as $F(\cdot - \hat{\alpha})$. When the representative player takes NE control $\hat{\alpha}$ and the population measure is $\hat{\mu}$, we know that $\hat{\alpha} \in A_{\hat{\mu}}$ resulting in

$$\begin{cases} aF(\hat{\alpha} - t_0) + (b + c)F(\hat{\alpha} - t) = c \\ t = \tau(F(\cdot - \hat{\alpha})) \end{cases} \tag{30}$$

as an equation w.r.t. $\hat{\alpha}$ (notice that τ maps a measure to a real number). With come constraints on τ added, one is able to ensure the existence and uniqueness of $\hat{\alpha}$ as the solution to the equations above and **the solution to MFG** is just $\hat{\mu} = F(\cdot - \hat{\alpha})$.

Remark. *Although it's not relevant to MFG, I would like to show the method of arguing the existence and uniqueness of $\hat{\alpha}$ by adding constraints on τ since it's a typical and crucial application of the contraction mapping theorem.*

Let's assume τ always takes value no less than t_0, is monotone, i.e. $\forall \alpha \geq 0$, if $\mu([0, \alpha]) \leq \mu'([0, \alpha])$ then $\tau(\mu) \geq \tau(\mu')$ and has sub-additivity, i.e. $\forall \alpha \geq 0, \tau(\mu(\cdot - \alpha)) \leq \tau(\mu) + \alpha$.

*The proof starts from **building $G : \mathbb{R}_+ \to \mathbb{R}_+$ mapping α to $G(\alpha)$ such that $\hat{\alpha}$ is the fixed point of G**. Function G first maps α to a measure $F(\cdot - \alpha)$ then to a real number $t = \tau(F(\cdot - \alpha))$ then to $\alpha(t)$ determined by the equation $aF(\alpha - t_0) + (b + c)F(\alpha - t) = c$ for given t. It's quite obvious that if we input $\hat{\alpha}$, the output will be $G(\hat{\alpha}) = \hat{\alpha}$ so it's the fixed point of G.*

*The next step is to **show that G is a contraction mapping**. Take $x, y \in \mathbb{R}_+, x < y$, by monotonicity $\tau(F(\cdot - x)) \leq \tau(F(\cdot - y))$ and by sub-additivity $\tau(F(\cdot - y)) \leq \tau(F(\cdot - x)) + (y - x)$ provides bounds for the first two mapping steps (map real number α to a measure and to a real number t again). The remaining step is the last step implicitly mapping t to $\alpha(t)$. Implicit function theorem is thus necessary. Assume*

$$H(\alpha, t) = aF(\alpha - t_0) + (b + c)F(\alpha - t) - c \tag{31}$$

and check $\frac{\partial H}{\partial \alpha} = aF'(\alpha - t_0) + (b + c)F'(\alpha - t) > 0$ since F is strictly increasing and $a, b, c > 0$, H is also continuous *with continuous partial derivatives, so* $H(\alpha, t) = 0$ *determines* C^1 *function* $\alpha(t)$ *with*

$$\frac{d\alpha(t)}{dt} = \frac{(b+c)F'(\alpha(t)-t)}{aF'(\alpha(t)-t_0)+(b+c)F'(\alpha(t)-t)} \tag{32}$$

since $F' > 0$*, the derivative only takes value in* $(0, C)$ *with constant* $0 < C < 1$*. Now we denote*

$$\begin{cases} t_x = \tau(F(\cdot - x)) \\ t_y = \tau(F(\cdot - y)) \end{cases} \tag{33}$$

and calculate

$$|G(y) - G(x)| = |\alpha(t_y) - \alpha(t_x)| \leq C|t_y - t_x| \leq C|y - x| \tag{34}$$

with $0 < C < 1$ *proves that* G *is contraction mapping proves the existence and uniqueness of* $\hat{\alpha}$ *as the fixed point.*

The difficulty here is that a real number is first mapped to a measure then to a real number again under some implicit function. However, by following the spirit of fixed point iteration, it's not hard to construct G ***and show that it's a contraction mapping by adding necessary conditions.***

This example shows from the beginning how we approximate finite player game with MFG for one-period game and how to solve one-period MFG following the strategy mentioned above. The continuous-time MFG will be much harder to solve, but the approximation and solution follows the similar logic that we always care about optimality and consistency conditions.

For more details and interpretations on the mean field approximation for single-period game, please refer to my handwritten notes *Notes on Mean Field Approximation of Single-Period Games* on my personal website.

Probabilistic Approach to Finite Player Game

Finite Player Game and Different Types of NE

Here we quickly review the setting of finite player game and the probabilistic approach to solving finite player games. Most of the details are omitted and only notations are introduced. In the continuous time setting, different players' states are coupled through SDE dynamics (a Markovian diffusion)

$$\begin{cases} dX_t = B(t, X_t, \alpha_t)\, dt + \Sigma(t, X_t, \alpha_t)\, dW_t \\ X_0 = x_0 \end{cases} \tag{35}$$

where X takes value in $\mathbb{R}^N$ denoting the states of all players. Written in terms of each player,

$$\begin{cases} dX_t^i = b^i(t, X_t, \alpha_t)\, dt + \sigma^i(t, X_t, \alpha_t)\, dW_t^i + \sigma^0(t, X_t, \alpha_t)\, dW_t^0 \\ X_0^i = x_0^i \end{cases} \tag{36}$$

where W^0 is a BM denoting the common noise shared by all players (often we neglect this) and the time horizon of the game is always assumed to be $[0, T]$. We add **assumptions called Game** here to ensure the existence and uniqueness of the solution to the coupled SDE system and to guarantee measurability. It's not very interesting to investigate those assumptions carefully so they are just listed here for the purpose of completeness

$$\begin{cases} \forall S \in [0, T], (t, \omega, x, \alpha) \mapsto (B, \Sigma)(t, \omega, x, \alpha) \ \ \text{progressive measurable} \\ \exists c > 0, \forall t \in [0, T], \omega, x, x', \alpha, \alpha', |B(t, \omega, x, \alpha) - B(t, \omega, x', \alpha')| + |\Sigma(t, \omega, x, \alpha) - \Sigma(t, \omega, x', \alpha')| \le c(|x - x'| + |\alpha - \alpha'|) \\ \forall \alpha, \mathbb{E} \int_0^T |B(t, 0, \alpha_t)|^2 + |\Sigma(t, 0, \alpha_t)|^2\, dt < \infty \\ \forall S \in [0, T], (t, \omega, x, \alpha) \mapsto f^i(t, \omega, x, \alpha) \ \ \text{progressive measurable} \\ (\omega, x) \mapsto g^i(\omega, x) \ \ \text{measurable} \end{cases} \tag{37}$$

here f^i is the running cost while g^i is the terminal cost, together they provide the cost functional of player i

$$J^i(\alpha) = \mathbb{E}\left[\int_0^T f^i(s, X_s, \alpha_s)\, ds + g^i(X_T)\right] \tag{38}$$

Next we introduce different types of NE. The **open loop NE** allows each player's control to depend on current time, initial state and the whole BM trajectory so far

$$\alpha_t^i = \phi^i(t, X_0, W_{[0,t]}) \tag{39}$$

where ϕ^i is a deterministic function. In special case where α_t^i has no dependence on W, we call it an open loop deterministic NE.

Remark. *The crucial point to understand here is that open loop NE admits no feedback, i.e. even after player i changes its strategy from $\hat{\alpha}^i$ to α^i, $\hat{\alpha}^{-i}$ maintains its original trajectory.*

On the other hand, **closed loop NE** allows the dependence of control on the current time and the whole state trajectory so far

$$\alpha_t^i = \phi^i(t, X_{[0,t]}) \tag{40}$$

notice that X depends on α in its dynamics so it's actually an implicit equation w.r.t. α. Closed loop NE has feedback effect in that the change of control of player i results in the change of control of all other players (α_t^i affects X_t and affects α_t^j $(j \neq i)$).

The final useful notion of NE is the **Markovian NE** allowing the control to depend on current time, initial state, and current state

$$\alpha_t^i = \phi^i(t, X_0, X_t) \tag{41}$$

this NE also has feedback effect but it's different from the closed loop NE in that the control cannot depend on past states. Generally, open loop NE is the easiest to derive but not very practically useful, closed loop NE is the hardest to derive but the closest to reality. As a trade-off between those two, Markovian NE is not so hard to derive and also has practical interpretations.

Classical Approach Solving Finite Player Game

It should be familiar that PDE approach and BSDE approach are the two most useful approaches solving NE for finite player game. The PDE approach is based on dynamic programming principle (DPP) and can only deal with Markovian NE while the BSDE approach is based on Pontryagin's maximum principle and can deal with all different kinds of NE. The proof of those approaches is neglected and can be found in the book as a generalization to that of single player game.

PDE approach solves **Markovian NE** by putting up the **value function of player** i thinking backwardly

$$V^i(t,x) = \inf_{\alpha^i} \mathbb{E}\left[\int_t^T f^i(s, X_s, (\alpha_s^i, \alpha_s^{-i}))\, ds + g^i(X_T)\middle| X_t = x\right] \tag{42}$$

where $\alpha_s^{-i} = \phi^{*,-i}(s, X_s)$ since the NE is Markovian. DPP then provides the Hamilton-Jacobi-Bellman equation (HJBE) that describes the evolution of the value function of player i

$$\begin{cases} \partial_t V^i + \inf_{\alpha^i \in A^i} \left\{\partial_x V^i \cdot B(t,x,\alpha) + \frac{1}{2}\mathrm{Tr}(\partial_{xx} V^i \cdot \Sigma(t,x,\alpha)\cdot \Sigma^T(t,x,\alpha)) + f^i(t,x,\alpha)\right\} = 0 \\ V^i(T,x) = g^i(x) \end{cases} \tag{43}$$

Remark. *Notice that* ***in the HJBE of player*** i *we have*

$$\alpha = (\alpha^i, \phi^{*,-i}(t,x)) \tag{44}$$

since player i *can only manage to determine* α^i*,* ***all*** α^{-i} ***shall be treated as*** $\alpha^{-i}(t,x)$***, a function of current time*** t ***and current state*** x ***that has feedback effects.***

For linear-quadratic games, when there's sufficient symmetricity it's always possible to derive the closed-form solution. Typically the first step is to solve the inf in the HJBE of player i to get the Markovian NE $\hat{\alpha}^i$ (an expression containing $V^i, t, x, \alpha^{-i}(t,x)$) for $i \in \{1, 2, ..., N\}$, plug it back into HJBE and raise an ansatz (typically a quadratic form) to get N coupled Ricatti equations. Solve the Ricatti equations and plug back into the expression for $\hat{\alpha}^i$ to solve the Markovian NE.

It's quite obvious that HJBE is a necessary condition to satisfy for value functions, so rigorously speaking verification steps are required to argue that the solution to the HJBE must be the value function (sufficiency). In practice, however, verification step is always skipped when the finite player game is well-posed. Here we list a set of **assumptions called** N**-Nash system** that ensures the uniqueness of the solution to the Nash system that is smooth enough. This set of assumptions makes sure that skipping the verification step causes no problems. Although it's also not interesting to look into those assumptions, I list them here for reference.

$$\begin{cases} A^{(N)} \text{ is bounded} \\ B \text{ bounded, uniformly Lipschitz in } \alpha \\ \Sigma \text{ uncontrolled (free of } \alpha\text{), bounded cts, uniformly Lipschitz in } x, \Sigma\Sigma^T \text{ uniformly nondegenerate} \\ f^i \text{ bounded, uniformly Lipschitz in } \alpha, g^i \text{ bounded Lipschitz} \\ \text{Minimizer in Issac condition uniformly Lipschitz in } y \end{cases} \tag{45}$$

To introduce the Issac condition, we need the notion of **Hamiltonian of player** i which is also crucial in Pontryagin's maximum principle defined as

$$H^i(t,x,y^i,z^i,\alpha) = B(t,x,\alpha)\cdot y^i + \mathrm{Tr}(\Sigma^T(t,x,\alpha)\cdot z^i) + f^i(t,x,\alpha) \tag{46}$$

Issac condition holds if $\exists \hat{\alpha}(t,x,y,z)$ such that $\forall i, \forall t \in [0,T], x, y, z, \forall \alpha_i$,

$$H^i(t,x,y^i,z^i,\hat{\alpha}(t,x,y,z)) \le H^i(t,x,y^i,z^i,(\alpha^i,\hat{\alpha}^{-i}(t,x,y,z))) \tag{47}$$

Let's recall that Potryagin's maximum principle in single player game, telling us to minimize the Hamiltonian in minimization problems w.r.t. α, resulting in

$$\hat{\alpha}(t,x,y,z) = \arg\min_{\alpha} H(t,x,y,z,\alpha) \tag{48}$$

and put up the adjoint BSDE together with the FSDE dynamics to get FBSDE systems

$$\begin{cases} dX_t = b(t,X_t,\hat{\alpha}(t,X_t,Y_t,Z_t))\,dt + \sigma(t,X_t,\hat{\alpha}(t,X_t,Y_t,Z_t))\,dW_t \\ X_0 = x_0 \\ dY_t = -\partial_x H(t,X_t,Y_t,Z_t,\hat{\alpha}(t,X_t,Y_t,Z_t))\,dt + Z_t\,dW_t \\ Y_T = \partial_x g(X_T) \end{cases} \tag{49}$$

the solution to the system gives optimal control $\hat{\alpha}$. This method is basically the same in multi player game with the only exception to separate by cases when solving different kinds of NE.

We start with the **BSDE approach for open loop NE**. In this case, there are no feedback effects so let's simply minimize Hamiltonian of each player

$$\hat{\alpha}^i(t,x,y^i,z^i) = \arg\min_{\alpha^i} H^i(t,x,y^i,z^i,(\alpha^i,\alpha^{-i})) \tag{50}$$

and put up the adjoint BSDE together with the FSDE dynamics to get FBSDE systems

$$\begin{cases} dX_t^i = b^i(t, X_t, \hat{\alpha}^i(t, X_t, Y_t^i, Z_t^i))\, dt + \sigma^i(t, X_t, \hat{\alpha}^i(t, X_t, Y_t^i, Z_t^i))\, dW_t^i \\ X_0 = x_0 \\ dY_t^i = -\partial_x H^i(t, X_t, Y_t^i, Z_t^i, \hat{\alpha}(t, X_t, Y_t^i, Z_t^i))\, dt + Z_t^i\, dW_t \\ Y_T^i = \partial_x g^i(X_T) \end{cases} \tag{51}$$

solve it to get open loop NE. The verification condition only requires convexity of H^i in (x, α^i) and the convexity of g^i that can be easily verified.

When it comes to the **BSDE approach for Markovian NE**, the only difference appears in the adjoint BSDE where $-\partial_x H^i$ makes a difference. In the open loop case, $H^i = H^i(t, x, y, z, (\alpha^i, \alpha^{-i}))$ with α^{-i} not depending on x since there's no feedback effect. However, in the Markovian case,

$$H^i = H^i(t, x, y^i, z^i, (\alpha^i, \alpha^{-i}(t, x))), \alpha^{-i}(t, x) = \phi^{*,-i}(t, x) \tag{52}$$

and this is definitely changing the expression of $\partial_x H^i$. Let's collect all dependencies on x of H^i denoting

$$H^{-i}(t, x, y^i, z^i, \alpha^i) = H^i(t, x, y^i, z^i, (\alpha^i, \phi^{*,-i}(t, x))) \tag{53}$$

the driver in the adjoint BSDE for player i is actually $\partial_{x_p} H^{-i}$, an easy application of chain rule shows

$$\partial_{x_p} H^{-i} = \partial_{x_p} H^i(t, x, y^i, z^i, (\alpha^i, \phi^{*,-i}(t, x))) + \sum_{j \neq i} \partial_{\alpha^j} H^i(t, x, y^i, z^i, (\alpha^i, \phi^{*,-i}(t, x))) \cdot \partial_{x_p} \alpha^j \tag{54}$$

where $\alpha^j = \phi^{*,j}(t, x)$ $(j \neq i)$ is a function in terms of t, x from the perspective of player i. As a result, there is **one more summation term in the driver of the adjoint BSDE** and following the same procedure we get the Markovian NE

$$\hat{\alpha}^i(t, x, y^i, z^i) = \arg\min_{\alpha^i} H^i(t, x, y^i, z^i, (\alpha^i, \alpha^{-i}(t, x))) \tag{55}$$

get FBSDE systems

$$\begin{cases} dX_t^i = b^i(t, X_t, \hat{\alpha}^i(t, X_t, Y_t^i, Z_t^i))\, dt + \sigma^i(t, X_t, \hat{\alpha}^i(t, X_t, Y_t^i, Z_t^i))\, dW_t^i \\ X_0 = x_0 \\ dY_t^i = -\left[\partial_x H^i + \sum_{j \neq i} \partial_{\alpha^j} H^i \cdot \partial_x \phi^j(t, X_t)\right] dt + Z_t^i\, dW_t \\ Y_T^i = \partial_x g^i(X_T) \end{cases} \tag{56}$$

the verification step still depends on the convexity of H^i and g^i same as that for the open loop NE.

Example: Linear Quadratic (LQ) Flocking Model

For detailed explanation on the systemic risk model, please refer to my notes on stochastic control or chapter 2.5 in the book.

In this model, there are N players (birds), with player i having position X_t^i at time t taking value in $\mathbb{R}^3$. Player i can determine its control α_t^i as the velocity at time t and the state dynamics is given as

$$dX_t^i = \alpha_t^i \, dt + \sigma \, dW_t^i \tag{57}$$

on time horizon $[0, T]$. Player i has cost functional

$$J^i(\alpha) = \mathbb{E} \int_0^T f^i(t, X_t, \alpha_t) \, dt \tag{58}$$

with no terminal cost and the running cost is

$$f^i(t, x, \alpha) = \frac{\kappa^2}{2} |x^i - \overline{x}|^2 + \frac{1}{2} |\alpha^i|^2 \tag{59}$$

with $|\cdot|$ denoting the vector ℓ_2 norm since x^i, α^i take values in $\mathbb{R}^3$. This is a LQ game with mean field interaction through $|x^i - \overline{x}|^2$ so we expect to be able to find the closed-form solution.

Let's first work for the **open loop NE**. Since the diffusion coefficient is constant in the dynamics, reduced Hamiltonian can be used

$$H^i(t, x, y^i, z^i, \alpha) = \alpha \cdot y^i + f^i(t, x, \alpha) \tag{60}$$

$$= \sum_{j=1}^N \alpha^j \cdot y^{i,j} + \frac{\kappa^2}{2} |x^i - \overline{x}|^2 + \frac{1}{2} |\alpha^i|^2 \tag{61}$$

to clarify, $y^{i,1}, ..., y^{i,N}$ take values in $\mathbb{R}^3$. Minimize H^i w.r.t. α^i to see

$$\hat{\alpha}^i = -y^{i,i} \tag{62}$$

holds for $\forall i \in \{1, 2, ..., N\}$ by symmetricity. Now since W^i is a BM in $\mathbb{R}^3$, it's clear that player i has process $Z^{i,j,k}$ such that $i, j, k \in \{1, 2, ..., N\}$ and $Z^{i,j,k}$ takes value in $\mathbb{R}^3$. At this point, we write down the adjoint BSDE for player i

$$dY_t^{i,j} = -\partial_{x^j} H^i(t, X_t, Y_t^i, Z_t^i, \hat{\alpha}_t) \, dt + \sum_{k=1}^n Z_t^{i,j,k} \, dW_t^k \tag{63}$$

calculate $\partial_{x^j} H^i = \kappa^2(x^i - \overline{x})(\delta_{ij} - \frac{1}{N})$ and plug in to see the adjoint BSDE

$$dY_t^{i,j} = -\kappa^2(X_t^i - \overline{X}_t)\left(\delta_{ij} - \frac{1}{N}\right)\,dt + \sum_{k=1}^n Z_t^{i,j,k}\,dW_t^k \tag{64}$$

with terminal condition $Y_T^{i,j} = 0$ since $g^i \equiv 0$. We derive the FBSDE system by replacing α with $\hat{\alpha}$

$$\begin{cases} dX_t^i = -Y_t^{i,i}\,dt + \sigma\,dW_t^i \\ dY_t^{i,j} = -\kappa^2(X_t^i - \overline{X}_t)\left(\delta_{ij} - \frac{1}{N}\right)\,dt + \sum_{k=1}^n Z_t^{i,j,k}\,dW_t^k \\ Y_T^{i,j} = 0 \end{cases} \tag{65}$$

put up the affine ansatz (always used in LQ game) with deterministic η_t

$$Y_t^{i,j} = \eta_t(X_t^i - \overline{X}_t)\left(\delta_{ij} - \frac{1}{N}\right) \tag{66}$$

differentiate both sides

$$dY_t^{i,j} = \left(\delta_{ij} - \frac{1}{N}\right)\left[\dot{\eta}_t(X_t^i - \overline{X}_t)\,dt + \eta_t\,d(X_t^i - \overline{X}_t)\right] \tag{67}$$

it's not hard to figure out $d(X_t^i - \overline{X}_t)$ from the FSDE that

$$d(X_t^i - \overline{X}_t) = -\eta_t(X_t^i - \overline{X}_t)\left(1 - \frac{1}{N}\right)\,dt + \sigma(\,dW_t^i - \,d\overline{W}_t) \tag{68}$$

a comparison principle of BSDE shows that the dt, dW_t part must be equal correspondingly

$$\begin{cases} -\kappa^2(X_t^i - \overline{X}_t)\left(\delta_{ij} - \frac{1}{N}\right) = \left(\delta_{ij} - \frac{1}{N}\right)\left[\dot{\eta}_t(X_t^i - \overline{X}_t) - \eta_t^2(X_t^i - \overline{X}_t)(1 - \frac{1}{N})\right] \\ Z_t^{i,j,k} = \left(\delta_{ij} - \frac{1}{N}\right)\eta_t\sigma\left(\delta_{i,k} - \frac{1}{N}\right) \end{cases} \tag{69}$$

simplify to get

$$\begin{cases} \eta_t^2(1 - \frac{1}{N}) - \kappa^2 = \dot{\eta}_t \\ Z_t^{i,j,k} = \sigma\left(\delta_{ij} - \frac{1}{N}\right)\left(\delta_{i,k} - \frac{1}{N}\right)\eta_t \end{cases} \tag{70}$$

the ODE w.r.t. η_t has terminal condition $\eta_T = 0$ can be solved easily (Ricatti equation) gives the closed-form open loop NE to this game. The verification step is obvious since H^i is convex in (x, α^i).

Then, let's solve the **Markovian NE** through BSDE approach. The Hamiltonian remains the same while

$$H^i(t, x, y^i, z^i, \alpha) = \alpha^i \cdot y^{i,i} + \sum_{l=1, l\neq i}^N \phi^l(t,x) \cdot y^{i,l} + \frac{\kappa^2}{2}|x^i - \overline{x}|^2 + \frac{1}{2}|\alpha^i|^2 \tag{71}$$

since for player i, all other players' controls have feedback effects $\alpha^l = \phi^l(t,x)$ $(l \neq i)$. Minimize H^i w.r.t. α^i to see

$$\hat{\alpha}^i = -y^{i,i} \tag{72}$$

still holds for $\forall i \in \{1,2,...,N\}$ but if we calculate the derivative of H^i w.r.t. x^j to get

$$\partial_{x^j} H^i = \kappa^2(x^i - \overline{x})\left(\delta_{ij} - \frac{1}{N}\right) + \sum_{l=1,l\neq i}^{N} \partial_{x^j}\phi^l(t,x)\cdot y^{i,l} \tag{73}$$

the last summation depends on the specific form of ϕ^l. At this step, ansatz has to be raised prior to the construction of FBSDE in order to proceed. From the open loop NE procedure exhibited above, we naturally put up an ansatz for the feedback function

$$\phi^l(t,x) = \left(1 - \frac{1}{N}\right)(x^l - \overline{x})\mu_t \tag{74}$$

with deterministic μ_t. Simple calculation shows $\partial_{x^j}\phi^l(t,x) = \left(1 - \frac{1}{N}\right)\left(\delta_{jl} - \frac{1}{N}\right)\mu_t I$ and the FBSDE is provided as

$$\begin{cases} dX_t^i = -Y_t^{i,i}\,dt + \sigma\,dW_t^i \\ dY_t^{i,j} = -\left[\kappa^2(X_t^i - \overline{X}_t)\left(\delta_{ij} - \frac{1}{N}\right) + \left(1 - \frac{1}{N}\right)\mu_t \sum_{l=1,l\neq i}^{N}\left(\delta_{jl} - \frac{1}{N}\right)Y_t^{i,l}\right]dt + \sum_{k=1}^{n} Z_t^{i,j,k}\,dW_t^k \\ Y_T^{i,j} = 0 \end{cases} \tag{75}$$

to solve this FBSDE, put up the same ansatz as before

$$Y_t^{i,j} = \mu_t(X_t^i - \overline{X}_t)\left(\delta_{ij} - \frac{1}{N}\right) \tag{76}$$

again comparison principle of BSDE tells us

$$\begin{cases} \dot{\mu}_t = \left(1 - \frac{1}{N}\right)^2 \mu_t^2 - \kappa^2 \\ Z_t^{i,j,k} = \left(\delta_{ij} - \frac{1}{N}\right)\mu_t\sigma\left(\delta_{i,k} - \frac{1}{N}\right) \end{cases} \tag{77}$$

after some simplifications. Together with $\mu_T = 0$, it's another Ricatti equation which can be easily solved to get the Markovian NE. This example shows us that open loop NE and Markovian NE are generally different although they have similar forms (ODE for η_t and μ_t).

Probabilistic Approach to MFG

Problem Setting

The problem setting of MFG has state dynamics for player i as

$$\begin{cases} dX_t^i = b(t, X_t^i, \overline{\mu}_{X_t^{-i}}^{N-1}, \alpha_t^i)\, dt + \sigma(t, X_t^i, \overline{\mu}_{X_t^{-i}}^{N-1}, \alpha_t^i)\, dW_t^i \\ X_0^i = \xi \end{cases} \tag{78}$$

on time horizon $[0, T]$. Notice that b, σ, ξ are identical among all players and other players' states affect player i's state only through the empirical measure. The cost functional of player i has form

$$J^i(\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t^i, \overline{\mu}_{X_t^{-i}}^{N-1}, \alpha_t^i)\, dt + g(X_T^i, \overline{\mu}_{X_T^{-i}}^{N-1})\right] \tag{79}$$

with f, g identical among all players such that the large symmetric cost functional assumption holds.

To solve a MFG, we only have to consider a representative player, fix the flow of probability measure $\mu = \{\mu_t\}_{0\le t\le T}$ and solve the stochastic control problem to get the optimal control $\hat{\alpha} = \hat{\alpha}(\mu)$ as a function of μ. Such $\hat{\alpha}(\mu)$ induces the state evolution $\hat{X}^\mu$ depending on μ, so we just need to find a flow μ such that

$$\forall t \in [0, T], \mathscr{L}\left(\hat{X}_t^\mu\right) = \mu_t \tag{80}$$

where $\mathscr{L}(\cdot)$ denotes the law/distribution of random variable. This condition ensures the consistency of state evolution and empirical measure, providing such μ as the solution to MFG. It's quite easy to understand that the first step solving stochastic control problem ensures **optimality** while the second step finding flow μ ensures **consistency**.

Remark. *In this setting of MFG, we don't distinguish between open loop NE and Markovian NE any longer since when $N \to \infty$, they are gonna be asymptotically the same (a nontrivial fact).*

For simplicity, we also assume that when fixing μ and solving out the optimal control $\hat{\alpha}(\mu)$, the minimizer is unique for any flow μ.

The Hamiltonian of this MFG is still defined as

$$H(t, x, \mu, y, z, \alpha) = b(t, x, \mu, \alpha) \cdot y + \sigma(t, x, \mu, \alpha) \cdot z + f(t, x, \mu, \alpha) \tag{81}$$

and when the diffusion coefficient is free of control, i.e. $\sigma = \sigma(t, x, \mu)$, reduced Hamiltonian

$$H(t, x, \mu, y, \alpha) = b(t, x, \mu, \alpha) \cdot y + f(t, x, \mu, \alpha) \tag{82}$$

can be used in place of the original one.

Analytic Approach to MFG

Recall that we first replace the empirical measure with a measure flow μ and freeze the flow μ to solve the optimal control $\hat{\alpha}(\mu)$ for the representative player. This problem is a standard single player game and can be transformed into solving HJBE in the Markovian case. With the definition of value function

$$V(t,x) = \mathbb{E}\left[\int_t^T f(s,X_s,\mu_s,\alpha_s)\,ds + g(X_T,\mu_T)\Big|X_t = x\right] \tag{83}$$

where X,α denotes the state and control of the representative player, HJBE tells us

$$\partial_t V + \inf_\alpha\left\{b(t,x,\mu_t,\alpha)\cdot\partial_x V + \frac{1}{2}\mathrm{Tr}(\sigma(t,x,\mu_t,\alpha)\sigma^T(t,x,\mu_t,\alpha)\partial_{xx}V) + f(t,x,\mu_t,\alpha)\right\} = 0 \tag{84}$$

with terminal condition $V(T,x) = g(x,\mu_T)$.

On the other hand, we want to describe the evolution of measure flow μ after solving $\hat{\alpha}(\mu)$ (from the HJBE listed above) such that $\forall t\in[0,T], \mu_t = \mathscr{L}\left(\hat{X}_t^\mu\right)$ where $\hat{X}^\mu$ is generated by taking the control $\hat{\alpha}(\mu)$. Recall that the propagation of measure flow is described by the Fokker-Planck equation with given initial condition. Since the state dynamics of the representative player now becomes

$$\begin{cases} d\hat{X}_t^\mu = b(t,\hat{X}_t^\mu,\mu_t,\hat{\alpha}_t(\mu))\,dt + \sigma(t,\hat{X}_t^\mu,\mu_t,\hat{\alpha}_t(\mu))\,dW_t \\ \hat{X}_0^\mu = \xi \end{cases} \tag{85}$$

the Fokker-Planck equation is

$$\begin{cases} \partial_t\mu_t - L^*\mu_t = 0 \\ \mu_0 = \mathscr{L}(\xi) \end{cases} \tag{86}$$

where L^* is the adjoint of the infinitesimal generator L with action

$$L^* f = -\mathrm{div}_x(b\cdot f) + \frac{1}{2}\mathrm{Tr}(\partial_{xx}(\sigma\sigma^T f)) \tag{87}$$

explicitly written out to get

$$\begin{cases} \partial_t\mu_t + \mathrm{div}_x(b(t,x,\mu_t,\hat{\alpha}_t(\mu))\cdot\mu_t) - \frac{1}{2}\mathrm{Tr}\left[\partial_{xx}(\sigma(t,x,\mu_t,\hat{\alpha}_t(\mu))\cdot\sigma^T(t,x,\mu_t,\hat{\alpha}_t(\mu))\cdot\mu_t)\right] = 0 \\ \mu_0 = \mathscr{L}(\xi) \end{cases} \tag{88}$$

the **HJBE coupled with the Fokker-Planck equation** provides the analytic approach to MFG. Notice that HJBE has given terminal condition and Fokker-Planck equation has given initial condition. Notice that $\mu = \mu(t,x)$ can be understood as the density of $\hat{X}_t^\mu$ (if density exists) so t is the time variable and x is the space variable, i.e. for each fixed time t, $\mu(t,\cdot)$ is a density function.

Remark. *div is the divergence operator, for vector field* $F = (F_1, ..., F_n)$, $div_x F = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} = \nabla \cdot F$.

One might be confused why the adjoint of infinitesimal generator contains the divergence operator, let's do a simple derivation here as review. From the definition of adjoint, $\langle Lf, g\rangle = \langle f, L^* g\rangle$ *where the inner product is the standard one defined on* L^2 *function space* $\langle f, g\rangle = \int f(x) \cdot g(x)\, dx$.

$$\langle Lf, g\rangle = \int Lf(x) \cdot g(x)\, dx \tag{89}$$

$$= \int (b \cdot \partial_x f) \cdot g + \frac{1}{2} Tr(\sigma\sigma^T \partial_{xx} f) \cdot g\, dx \tag{90}$$

here assume that $b = (b^1, ..., b^n)$ *takes value in* $\mathbb{R}^n$ *and* σ *takes value in* $\mathbb{R}^{n\times m}$, *do integration by parts to see*

$$\langle Lf, g\rangle = \sum_{i=1}^n \int b^i \cdot \partial_{x^i} f \cdot g\, dx + \frac{1}{2} Tr\left(\int g \cdot \sigma\sigma^T \partial_{xx} f\, dx\right) \tag{91}$$

$$= -\sum_{i=1}^n \int f \cdot \partial_{x^i}(b^i \cdot g)\, dx - \frac{1}{2} Tr\left[\int \partial_x \left(g \cdot \sigma\sigma^T\right) \cdot \partial_x f\, dx\right] \tag{92}$$

$$= \int \left[-\sum_{i=1}^n \partial_{x^i}(b^i \cdot g)\right] \cdot f\, dx + \frac{1}{2} Tr\left[\int \partial_{xx}\left(g \cdot \sigma\sigma^T\right) \cdot f\, dx\right] \tag{93}$$

$$= \int -div_x(b \cdot g) \cdot f\, dx + \int \frac{1}{2} Tr\left[\partial_{xx}\left(g \cdot \sigma\sigma^T\right)\right] \cdot f\, dx \tag{94}$$

we conclude

$$L^* g = -div_x(b \cdot g) + \frac{1}{2} Tr\left[\partial_{xx}\left(g \cdot \sigma\sigma^T\right)\right] \tag{95}$$

In special cases where **the diffusion coefficient of the dynamics is free of control**, i.e. $\sigma = \sigma(t, x, \mu)$, the HJBE together with the Fokker-Planck equation has a simpler representation. Consider minimizing the reduced Hamiltonian w.r.t. α to get

$$\hat{\alpha}(t, x, \mu, y) = \arg\min_\alpha H(t, x, \mu, y, \alpha) = \arg\min_\alpha \{b(t, x, \mu, \alpha) \cdot y + f(t, x, \mu, \alpha)\} \tag{96}$$

it's obvious that $\hat{\alpha}(t, x, \mu_t, \partial_x V)$ minimizes the expression inside the inf in the HJBE.

Remark. *Such minimizer* $\hat{\alpha}$ *is guaranteed to exist and is unique with sufficient regularity conditions stated in lemma 3.3 under a set of assumptions.*

At this point, HJBE can be reduced to

$$\partial_t V + \frac{1}{2}\mathrm{Tr}(\sigma(t, x, \mu_t)\sigma^T(t, x, \mu_t)\partial_{xx} V) + H(t, x, \mu_t, \partial_x V, \hat{\alpha}(t, x, \mu_t, \partial_x V)) = 0 \tag{97}$$

and Fokker-Planck equation can be reduced to

$$\partial_t \mu_t + \mathrm{div}_x(b(t,x,\mu_t,\hat{\alpha}(t,x,\mu_t,\partial_x V)) \cdot \mu_t) - \frac{1}{2}\mathrm{Tr}\left[\partial_{xx}(\sigma(t,x,\mu_t) \cdot \sigma^T(t,x,\mu_t) \cdot \mu_t)\right] = 0 \tag{98}$$

written together with the terminal condition and initial condition, **when a MFG has uncontrolled diffusion coefficient, the analytic approach is to minimize reduced Hamiltonian and solve HJBE together with the Fokker-Planck equation**

$$\begin{cases} \hat{\alpha}(t,x,\mu,y) = \arg\min_\alpha H(t,x,\mu,y,\alpha) = \arg\min_\alpha \{b(t,x,\mu,\alpha)\cdot y + f(t,x,\mu,\alpha)\} \\ \partial_t V + \frac{1}{2}\mathrm{Tr}(\sigma(t,x,\mu_t)\sigma^T(t,x,\mu_t)\partial_{xx}V) + H(t,x,\mu_t,\partial_x V,\hat{\alpha}(t,x,\mu_t,\partial_x V)) = 0 \\ V(T,x) = g(x,\mu_T) \\ \partial_t \mu_t + \mathrm{div}_x(b(t,x,\mu_t,\hat{\alpha}(t,x,\mu_t,\partial_x V)) \cdot \mu_t) - \frac{1}{2}\mathrm{Tr}\left[\partial_{xx}(\sigma(t,x,\mu_t)\cdot\sigma^T(t,x,\mu_t)\cdot\mu_t)\right] = 0 \\ \mu_0 = \mathscr{L}(X_0) = \mathscr{L}(\xi) \end{cases} \tag{99}$$

Remark. *This is a typical two-point boundary problem, hard to solve. Moreover, Cauchy-Lipschitz like theory can only ensure the existence and uniqueness of the solution locally (typically in a small time interval near* 0*) but not globally. This phenomenon can directly be seen in the closed-form solution of the finite player LQ game on graph that I have derived.*

Idea of MKV-FBSDE Approach

The idea of solving MFG is that when $N \to \infty$, we expect to see all players' state processes become asymptotically independent (propagation of chaos). As a result, when we consider the stochastic control problem the representative player is facing

$$\inf_{\alpha} \mathbb{E}\left[\int_0^T f(t, X_t, \mu_t, \alpha_t)\, dt + g(X_T, \mu_T)\right] \tag{100}$$

with dynamics

$$dX_t = b(t, X_t, \mu_t, \alpha_t)\, dt + \sigma(t, X_t, \mu_t, \alpha_t)\, dW_t \tag{101}$$

where the empirical measure is replaced with the fixed measure flow μ, we expect to see that μ_t is actually a deterministic measure instead of a random measure. Recall that the empirical measure in player i's dynamics is defined as

$$\overline{\mu}^{N-1}_{X_t^{-i}} = \frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \delta_{X_t^j} \tag{102}$$

which puts probability mass $\frac{1}{N-1}$ on the realization of X_t^j for $\forall j \neq i$. However, X_t^j contains randomness so $\overline{\mu}^{N-1}_{X_t^{-i}}$ is a random measure. On the other hand, if the propagation of chaos brings with asymptotic independence of different players' state process, for a regular enough ϕ, $\mathbb{E}\left|\phi(\overline{\mu}^{N-1}_{X_t^{-i}}) - \phi(\mu_t)\right|^2 \to 0$ $(N \to \infty)$ where $\mu_t = \mathscr{L}\left(X_t^1\right)$ is a deterministic measure, i.e. the randomness in the measure is gone asymptotically. Therefore, it makes sense to replace μ_t in the dynamics of the representative player with $\mathscr{L}\left(X_t\right)$ to get an **SDE of McKean-Vlasov (MKV) type**

$$dX_t = b(t, X_t, \mathscr{L}\left(X_t\right), \alpha_t)\, dt + \sigma(t, X_t, \mathscr{L}\left(X_t\right), \alpha_t)\, dW_t \tag{103}$$

that's the essential nature of state dynamics in MFG.

Remark. *When common noise is present, the situation becomes more complicated and it's natural to expect that $\overline{\mu}^{N-1}_{X_t^{-i}}$ can no longer be replaced with a deterministic measure μ_t. Although μ_t would be a random measure in this case, it would be the conditional marginal distribution of X_t given the realization of the common noise so the state dynamics can still be turned essentially into an SDE of conditional MKV type. However, MFG with common noise is harder to solve since the randomness in μ_t causes measurability problems in using the filtration generated by the BM, causing the failure of stochastic integrals being MG. That's why MFG with common noise requires a different setting in a random environment.*

Talking about using FBSDE to solve control problems, there are typically two approaches. We still assume an **uncontrolled diffusion coefficient** σ. One of the approach is to set up the value function and the HJBE it has to satisfy, then apply generalized Feynman-Kac formula to view it as the solution of a BSDE (recall that any solution to a semi-linear PDE has probabilistic representation under BSDE). In more details, the BSDE is given by

$$dY_t = -f(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, [\sigma^T(t, X_t, \mu_t)]^{-1} Z_t))\, dt + Z_t \cdot dW_t \tag{104}$$

with terminal condition $Y_T = g(X_T, \mu_T)$. Here $\hat{\alpha}$ is still formed as the minimizer of the reduced Hamiltonian w.r.t. α and X is the state process generated by taking control $\hat{\alpha}$.

Remark. *The HJBE of the value function is exactly*

$$\partial_t V + \frac{1}{2} Tr(\sigma(t, x, \mu_t)\sigma^T(t, x, \mu_t)\partial_{xx} V) + H(t, x, \mu_t, \partial_x V, \hat{\alpha}(t, x, \mu_t, \partial_x V)) = 0 \tag{105}$$

as shown above. It's clear that the correspondence of semi-linear PDE and BSDE through generalized Feynman-Kac formula is formed as (recall the value function and Delta-hedging strategy interpretation of BSDE)

$$\begin{cases} Y_t = V(t, X_t) \\ Z_t = \sigma^T(t, X_t, \mu_t)\ \partial_x V(t, X_t) \end{cases} \tag{106}$$

computation under Ito formula tells us (where dots mean inner product)

$$dY_t = dV(t, X_t) = \partial_t V(t, X_t)\, dt + \partial_x V(t, X_t) \cdot\ dX_t + \frac{1}{2} Tr(\partial_{xx} V \sigma\sigma^T)\, dt \tag{107}$$

$$= \partial_t V\, dt + (\partial_x V \cdot b)\, dt + (\sigma^T\ \partial_x V) \cdot\ dW_t + \frac{1}{2} Tr(\partial_{xx} V \sigma\sigma^T)\, dt \tag{108}$$

with variables in function V, b, σ omitted. It's immediate that the coefficient of dW_t which is $\sigma^T\ \partial_x V$ corresponds to Z_t and the coefficient of dt is

$$\partial_t V + b \cdot \partial_x V + \frac{1}{2} Tr(\partial_{xx} V \sigma\sigma^T) = b \cdot \partial_x V - H = -f \tag{109}$$

according to the HJBE. This results in the BSDE stated above

$$dY_t = -f(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, [\sigma^T(t, X_t, \mu_t)]^{-1} Z_t))\, dt + Z_t \cdot dW_t \tag{110}$$

with $\partial_x V(t, X_t) = [\sigma^T(t, X_t, \mu_t)]^{-1} Z_t$.

It's not hard to notice that this approach requires the diffusion coefficient σ to be invertible at all times.

The second approach is not based on the value function, but the Pontryagin's maximum principle we are familiar with. The adjoint BSDE is naturally provided as

$$dY_t = -\partial_x H^{full}(t, X_t, \mu_t, Y_t, Z_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))\, dt + Z_t\, dW_t \tag{111}$$

with terminal condition $Y_T = \partial_x g(X_T, \mu_T)$. Here $\hat{\alpha}$ is still formed as the minimizer of the reduced Hamiltonian w.r.t.

α but in the adjoint BSDE we have to use the full Hamiltonian

$$H^{full}(t, x, \mu, y, z) = H(t, x, \mu, y, \alpha) + \sigma(t, x, \mu) \cdot z \tag{112}$$

instead of the reduced Hamiltonian H since σ is not necessarily free of the state x.

Remark. *Just to clarify the difference between full Hamiltonian and reduced Hamiltonian, when the diffusion coefficient is free of control, we can minimize the reduced Hamiltonian but still need the full Hamiltonian to construct the adjoint BSDE (which many books forget to tell and is somewhat misleading). The only case one can always stick to the reduced Hamiltonian is when $\sigma = \sigma(t)$ has no dependence on x (or constant σ which often appears in literature). In short, we can actually always use the full Hamiltonian instead of the reduced one.*

This second approach does not require σ to be always invertible but requires differentiability of the coefficients.

Remark. *In short, when it comes to FBSDE approach in control problems, there are mainly two ways.*

The first approach is to ***build up the value function, derive the HJBE and use the correspondence between semi-linear PDE and BSDE*** *(generalized Feynman-Kac formula, form value function as the solution to BSDE) to get FBSDE. However, this approach* ***requires σ to be uncontrolled and invertible at all times****. One can recall that the first approach is what we do in the Deep BSDE algorithm to numerically solve stochastic control problems.*

The second approach is to ***apply Pontryagin's maximum principle*** *and derive FBSDE. This approach puts no restrictions on σ but* ***requires the differentiability of coefficients****.*

Both approach ends up in the same form of FBSDE given uncontrolled diffusion coefficient

$$\begin{cases} dX_t = B(t, X_t, \mu_t, Y_t, Z_t)\, dt + \Sigma(t, X_t, \mu_t)\, dW_t \\ dY_t = -F(t, X_t, \mu_t, Y_t, Z_t)\, dt + Z_t\, dW_t \\ Y_T = G(X_T, \mu_T) \end{cases} \tag{113}$$

but a different correspondence of B, Σ, F, G with the original dynamics. To be specific, for the generalized Feynman-Kac formula approach,

$$\begin{cases} B(t, x, \mu, y, z) = b(t, x, \mu, \hat{\alpha}(t, x, \mu, [\sigma^T(t, x, \mu)]^{-1} z)) \\ \Sigma(t, x, \mu) = \sigma(t, x, \mu) \\ F(t, x, \mu, y, z) = f(t, x, \mu, \hat{\alpha}(t, x, \mu, [\sigma^T(t, x, \mu)]^{-1} z)) \\ G(x, \mu) = g(x, \mu) \end{cases} \tag{114}$$

and for the Pontryagin's maximum principle approach,

$$\begin{cases} B(t,x,\mu,y,z) = b(t,x,\mu,\hat{\alpha}(t,x,\mu,y)) \\ \Sigma(t,x,\mu) = \sigma(t,x,\mu) \\ F(t,x,\mu,y,z) = \partial_x H^{full}(t,x,\mu,y,z,\hat{\alpha}(t,x,\mu,y)) \\ G(x,\mu) = \partial_x g(x,\mu) \end{cases} \tag{115}$$

where $\hat{\alpha}(t,x,\mu,y)$ is always the minimizer of Hamiltonian w.r.t. α. Combined with the idea of MFG discussed above replacing μ_t with $\mathscr{L}(X_t)$, **solving MFG turns into the problem of solving FBSDE of McKean-Vlasov (MKV-FBSDE) type**

$$\begin{cases} dX_t = B(t,X_t,\mathscr{L}(X_t),Y_t,Z_t)\,dt + \Sigma(t,X_t,\mathscr{L}(X_t))\,dW_t \\ dY_t = -F(t,X_t,\mathscr{L}(X_t),Y_t,Z_t)\,dt + Z_t\,dW_t \\ Y_T = G(X_T,\mathscr{L}(X_T)) \end{cases} \tag{116}$$

so it's natural to investigate the solvability of MKV-FBSDE.

Solvability of MKV-FBSDE

Here we present some general ideas to investigate the solvability of MKV-FBSDE. We start from an example of coupled forward backward differential equations

$$\begin{cases} \dot{x}_t = -y_t \\ \dot{y}_t = 0 \\ x_0 \in [-2, 2] \\ y_T = G(x_T) \end{cases} \tag{117}$$

where x, y are deterministic functions in t and the time horizon is $[0, T], T = 1$. Consider G taken as $G(x) = (-1) \vee x \wedge 1$, since y is free of t, when $x_1 > 1, \forall t \in [0, 1], y_t = y_1 = 1$ resulting in $\dot{x}_t = -1, x_t = -t + x_0$ so it must be true that $x_1 = -1 + x_0 > 1, x_0 > 2$, a contradiction. Similarly, $x_1 < -1$ cannot be true so $\forall t \in [0, 1], y_t = y_1 = G(x_1) = x_1$ with $x_1 \in [-1, 1]$. Now $\dot{x}_t = -x_1, x_t = -x_1 t + x_0$ has to satisfy $x_1 = -x_1 + x_0, x_0 = 2x_1, x_1 \in [-1, 1]$ is consistent. So the solution exists and is unique, given by

$$\begin{cases} x_t = -\frac{x_0}{2} t + x_0 \\ y_t = \frac{x_0}{2} \end{cases} \tag{118}$$

on the other hand, consider G taken as $G(x) = -(-1) \vee x \wedge 1$, when $x_0 \neq 0$, we get the following unique solution from a similar argument

$$\begin{cases} x_t = x_0 + t \cdot sgn(x_0) \\ y_t = -sgn(x_0) \end{cases} \tag{119}$$

however, when $x_0 = 0$, there are infinitely many solutions

$$\forall a \in [-1, 1], \begin{cases} x_t = at \\ y_t = -a \end{cases} \tag{120}$$

so the uniqueness of the solution is destroyed despite a small sign change in G. In this example, the **monotonicity of** G plays a key role in the property of FBSDE. An analogue to the adjoint BSDE derived in Pontryagin's maximum principle tells us that we can compare the terminal condition $y_T = \partial_x g(X_T)$ with $y_T = G(x_T)$ so G can somewhat be understood as $\partial_x g$. Recall that the verification step of Pontryagin's maximum principle requires the convexity of g, i.e. $\partial_x g$ to be increasing, analogue to G being increasing. From the example above, when G is increasing, the existence and uniqueness holds, so it's somewhat consistent with the fact we already know that **the convexity in** g **(or the monotonicity in the terminal condition) plays a crucial role in the existence and uniqueness argument of FBSDE**.

Remark. *The example above has an interesting interpretation under Burgers' equation. Setting* $y_t = u(t, x_t)$ *as the* *form of value function, FBSDE can be transformed into a nonlinear PDE*

$$\partial_t u - u \cdot \partial_x u = 0 \tag{121}$$

with terminal condition $u(T, x) = G(x)$.

Do a time reversal $v(t, x) = u(T - t, x)$ to see that

$$\partial_t v + v \cdot \partial_x v = 0 \tag{122}$$

with initial condition $v(0, x) = G(x)$. This is called ***the inviscid Burgers' equation*** *describing the motion of fluid along a tunnel under the conservation law (velocity of the fluid is proportional to v under the interpretation of Burgers' equation that $u(t, x)$ can be understood as the probability density of the fluid at spatial coordinate x at time t)*

When G in increasing in x, the fluid is in the ***dilation regime*** *(at place farther away from the starting point, there's more fluid), so as the motion starts, there will be no shock as time goes by, everything is well-posed. However, when G is decreasing in x, the fluid is in the* ***compression regime*** *(at place farther away from the starting point, there's less fluid), as time goes by the fluid at the starting point has a higher velocity, it catches from behind, creating a shock (singularity), that's why the uniqueness of the solution to FBSDE is destroyed.*

This ***correspondence between FBSDE with deterministic coefficients and and the system of characteristics of nonlinear PDE*** *is very interesting since I always believe it's hard to find some correspondence in reality for FBSDE. It's also an important correspondence since it directly motivates the existence and uniqueness argument of FBSDE from an intuitive perspective. Of course, the reader has to do some calculations in Burgers' equation to believe in what I have stated above. I am definitely willing to write something on Burgers' equation in another note if time allows.*

Yet, the convexity of g or increasing property of $\partial_x g$ meets some trouble extending to the MFG setting because $g = g(x, \mu)$ is a function of state and the measure. This requires us to define monotonicity for a function that maps a measure to a real number which will be discussed in a later context.

On the other hand, the correspondence between FBSDE with deterministic coefficients and and the system of characteristics of nonlinear PDE mentioned above provides motivation for the **decoupling field of FBSDE**. Still take the example listed above with the correspondence to Burgers' equation, adding diffusion term $\frac{1}{2}\partial_{xx}u$ on LHS gives

$$\partial_t u - u \cdot \partial_x u + \frac{1}{2}\partial_{xx} u = 0 \tag{123}$$

results in the **viscous Burgers' equation** (diffusive). This PDE has no singularity in its solution because of the regularizing effect of the heat kernel. A same correspondence back to FBSDE tells us that this PDE corresponds to

$$\begin{cases} dX_t = -Y_t \, dt + dW_t \\ dY_t = Z_t \, dW_t \\ x_0 \in \mathbb{R} \\ Y_T = G(X_T) \end{cases} \tag{124}$$

with the BM naturally appearing (recall that BM has infinitesimal generator as the Laplacian) and randomness is introduced. **Well-posedness of the viscous Burgers' equation corresponds to the existence and uniqueness of the solution to this FBSDE**, moreover the solution **admits the representation**

$$\mathbb{P}\left(\forall t \in [0,1], Y_t = u(t, X_t)\right) = 1, Z_t = \partial_x u(t, X_t) \ \lambda \times \mathbb{P} - a.e. \tag{125}$$

that Y has the structure as a value function in terms of underlying state process X and Z is the differential of value function (typical interpretation of BSDE). In this case, u is called the decoupling field of this FBSDE since solving u solves the FBSDE.

Remark. *To check the correspondence, set $Y_t = u(t, X_t), Z_t = \partial_x u(t, X_t)$ and apply Ito formula*

$$dY_t = du(t, X_t) = \partial_t u \, dt + \partial_x u \, dX_t + \frac{1}{2}\partial_{xx} u \, d\langle X, X\rangle_t \tag{126}$$

$$= \partial_t u \, dt - \partial_x u \cdot Y_t \, dt + \partial_x u \, dW_t + \frac{1}{2}\partial_{xx} u \, dt \tag{127}$$

$$= \left(\partial_t u - u \cdot \partial_x u + \frac{1}{2}\partial_{xx} u\right)(t, X_t) \, dt + Z_t \, dW_t \tag{128}$$

the comparison principle of BSDE against $dY_t = Z_t \, dW_t$ tells us

$$\partial_t u - u \cdot \partial_x u + \frac{1}{2}\partial_{xx} u = 0 \tag{129}$$

resulting in the correspondence.

Finding decoupling field thus provides another approach to solving FBSDE by turning it into solving PDE and the existence and uniqueness of solution to FBSDE is transformed to that of PDE which can be investigated in the traditional framework. However, this method only works for a certain kind of FBSDE, not generally applicable.

Toward the existence of solution to MKV-FBSDE, **Schauder's fixed point theorem** is an important tool to use on the Wasserstein space $\mathscr{P}_2(E)$ that contains probability measures on E. The definition of p-Wasserstein distance is

$$W_p(\mu, \mu') = \inf_{\pi \in \Pi_p(\mu, \mu')} \left(\int_{E \times E} [d(x,y)]^p \pi(dx, dy)\right)^{\frac{1}{p}} \tag{130}$$

with d as the distance on E and $\Pi_p(\mu, \mu')$ as the set of all couplings of μ, μ' (the set of all measures with marginals as

$\mu, \mu')$. $\mathscr{P}_2(E)$ is just the Wasserstein space equipped with 2-Wasserstein distance. The usage of fixed point theorem will be stated later but one can expect that we will define a mapping from a measure to another measure in the Wasserstein space with $\mathscr{L}(X_t)$ as the fixed point so that the solution to MKV-FBSDE is formed as the fixed point of this mapping (similar to what we have done in the meeting time example).

MKV-FBSDE Approach to MFG

The ideas of two different MKV-FBSDE approach to MFG are presented above, one from HJB under the characterization of generalized Feynman-Kac formula and the other from Pontryagin's maximum principle. Notice that **different assumptions are placed on the diffusion coefficient** $\sigma(t, x, \mu, \alpha)$ **for simplicity and consistency with the book**, but the readers are welcome to think about whether those conditions can be relaxed (refer to the conditions presented in the chapter of the idea of MKV-FBSDE approach). In this case, we organize the results as theorems below to remind readers of our main results (the description is not completely rigorous, only the most important conditions as the differences between two approaches are mentioned).

Theorem 3 (Value Function Approach)**.** *Assume* $\sigma = \sigma(t, x)$ *is free of empirical measure and control. Let* $\hat{\alpha}(t, x, \mu, y)$ *be the unique minimizer of the reduced Hamiltonian* $H(t, x, \mu, y, \alpha)$ *and if* σ *is uniformly elliptic, i.e.*

$$\exists L > 0, \forall t \in [0, T], \forall x, \sigma(t, x) \cdot \sigma^T(t, x) \geq L^{-1} I \tag{131}$$

where $\geq$ *is in the sense of semi-positive definite between symmetric matrices, then the continuous flow of measures* $\mu : [0, T] \to \mathscr{P}_2(\mathbb{R}^d)$ *is MFG equilibrium iff* $\mu_t = \mathscr{L}\left(\hat{X}_t\right)$ *where* $(\hat{X}, \hat{Y}, \hat{Z})$ *solves the MKV-FBSDE*

$$\begin{cases} d\hat{X}_t = b\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), \hat{\alpha}\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), [\sigma^T(t, \hat{X}_t)]^{-1}\hat{Z}_t\right)\right) dt + \sigma\left(t, \hat{X}_t\right) dW_t \\ d\hat{Y}_t = -f\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), \hat{\alpha}\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), [\sigma^T(t, \hat{X}_t)]^{-1}\hat{Z}_t\right)\right) dt + \hat{Z}_t \cdot dW_t \\ \hat{X}_0 = \xi \\ \hat{Y}_T = g\left(\hat{X}_T, \mathscr{L}\left(\hat{X}_T\right)\right) \end{cases} \tag{132}$$

Theorem 4 (Pontryagin's Maximum Principle Approach)**.** *Assume* $\sigma \in \mathbb{R}$ *is a constant. Let* $\hat{\alpha}(t, x, \mu, y)$ *be the unique minimizer of the reduced Hamiltonian* $H(t, x, \mu, y, \alpha)$*, then the continuous flow of measures* $\mu : [0, T] \to \mathscr{P}_2(\mathbb{R}^d)$ *is MFG equilibrium iff* $\mu_t = \mathscr{L}\left(\hat{X}_t\right)$ *where* $(\hat{X}, \hat{Y}, \hat{Z})$ *solves the MKV-FBSDE*

$$\begin{cases} d\hat{X}_t = b\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), \hat{\alpha}\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), \hat{Y}_t\right)\right) dt + \sigma\, dW_t \\ d\hat{Y}_t = -\partial_x H\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), \hat{Y}_t, \hat{\alpha}\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), \hat{Y}_t\right)\right) dt + \hat{Z}_t \cdot dW_t \\ \hat{X}_0 = \xi \\ \hat{Y}_T = \partial_x g\left(\hat{X}_T, \mathscr{L}\left(\hat{X}_T\right)\right) \end{cases} \tag{133}$$

where $f, g \in C^1$ *and* g *is convex.*

Lasry-Lions Monotonicity Condition

When it comes to the **uniqueness** of the solution to MKV-FBSDE, there are mainly three approaches: Cauchy-Lipschitz theory on a small enough time horizon (locally near time 0), adding monotonicity conditions, and using decoupling field. Here we consider the second approach, naturally requiring us to define the monotonicity when it comes to a function mapping a measure to a real number.

Define for $h : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ to be **(Lasry-Lions) monotone** if $\forall \mu \in \mathscr{P}_2(\mathbb{R}^d)$, $h(x, \mu)$ is at most of quadratic growth in x and

$$\forall \mu, \mu' \in \mathscr{P}_2(\mathbb{R}^d), \int_{\mathbb{R}^d} [h(x, \mu) - h(x, \mu')] (\mu - \mu')(dx) \geq 0 \tag{134}$$

which is directly motivated from the uniqueness of the solution to the mean field approximation of single-period game. Uniqueness of the MFG equilibrium is typically proved under the **Lasry-Lions monotonicity assumptions**

$$\begin{cases} b = b(t, x, \alpha), \sigma = \sigma(t, x, \alpha) \text{ free of empirical measure} \\ f = f_0(t, x, \mu) + f_1(t, x, \alpha) \text{ separated structure of dependence on } \mu, \alpha \\ \text{Quadratic growth condition on } f, g \\ f_0(t, \cdot, \cdot), g \text{ both Lasry-Lions monotone} \end{cases} \tag{135}$$

with the theorem provided below as the result.

Theorem 5 (Uniqueness of MFG Equilibrium)**.** *Assume the above Lasry-Lions monotonicity assumptions hold and μ is a deterministic continuous measure flow of measure, then for each fixed empirical measure μ, the optimality step in MFG (optimize the control) has unique minimizer $\hat{\alpha}^\mu$ inducing state process $\hat{X}^\mu$. There exists at most one flow of measure μ such that $\forall t \in [0, T], \mathscr{L}\left(\hat{X}_t^\mu\right) = \mu_t$ so there is at most one MFG equilibrium.*

The proof of this theorem is intuitive and does not need any extra explanation, the reader shall check the book for the proof on his/her own.

Remark. *The Lasry-Lions assumptions are strong assumptions since the state dynamics are required to not contain the empirical measure, which is often not the case in practice. As a result, uniqueness of MFG equilibrium typically fails and in most cases we don't care too much about the uniqueness argument.*

Remark. *It's important to see some examples of $h : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ that are Lasry-Lions monotone. Trivially, if $h = C|x|^2$ does not depend on μ or h does not depend on x, it must be Lasry-Lions monotone.*

In the setting of LQ-MFG, we may meet with $h(x, \mu) = a \int_{\mathbb{R}^d} y \, \mu(dy) \cdot x$ $(a > 0)$. In the setting of potential games, we may see $h(x, \mu) = \int_{\mathbb{R}^d} l(x - y) \, \mu(dy)$ for odd function l such that $|l(x)| \leq C(1 + |x|^2)$. Bearing the exact same form of h, we can consider $l : \mathbb{R}^d \to \mathbb{R}_+$ such that $l(r) = \int_{\mathbb{R}^d} e^{ir \cdot s} \lambda(ds)$ where λ is a symmetric positive finite measure on $\mathbb{R}^d$ (e.g. the measure induced by Gaussian or Cauchy r.v.). With such interpretation, l is actually the characteristic function (Fourier transform) of the distribution λ, with λ to be symmetric in order to ensure that the

Fourier transform only takes real values. Actually,

$$h(x,\mu) = \int_{\mathbb{R}^d} l(x-y)\,\mu(dy) \tag{136}$$

*is an **important example for a class of Lasry-Lions monotone functions** (plugging in $l(x) = sgn(x), d = 1$ gives $h(x,y) = \mu((-\infty,x)) - \mu((x,+\infty))$ which is another example, proving that $h(x,\mu) = \mu((-\infty,x)) + \frac{1}{2}\mu(\{x\})$ is also Lasry-Lions monotone). Those examples can be verified through simple calculations.*

Actually, the uniqueness of MFG equilibrium has another sufficient condition coming from the notion of **L-monotonicity**. It turns out that L-monotonicity sometimes can deal with cases where Lasry-Lions monotonicity fails. However, since we are not much concerned about the uniqueness of MFG equilibrium, we skip it for now and may come back to it later.

General Approach to Solve LQ-MFG

Let's consider the linear quadratic MFG as the simplest example where

$$b(t,x,\mu,\alpha) = b_1(t)x + \overline{b}_1(t)\overline{\mu} + b_2(t)\alpha \tag{137}$$

where state variable x, control variable α take value in $\mathbb{R}^d$, BM is of dimension d and diffusion coefficient $\sigma \in \mathbb{R}^{d\times d}$ is a constant matrix. Here $\overline{\mu} = \int_{\mathbb{R}^d} x\mu(dx)$ denotes the expectation of the probability measure μ (a measure on $\mathbb{R}^d$) and $b_1, \overline{b}_1, b_2$ are deterministic continuous matrix-valued functions. The running cost is

$$f(t,x,\mu,\alpha) = \frac{1}{2}\left(x^T q(t)x + (x - s(t)\overline{\mu})^T \overline{q}(t)(x - s(t)\overline{\mu}) + \alpha^T r(t)\alpha\right) \tag{138}$$

where $q, \overline{q}, r$ are continuous functions and take value as symmetric PD matrix. The terminal cost is

$$g(x,\mu) = \frac{1}{2}\left(x^T qx + (x - s\overline{\mu})^T \overline{q}(x - s\overline{\mu})\right) \tag{139}$$

where $q, \overline{q}$ here are symmetric PD matrices with no dependence on time.

To solve this MFG, notice that the cost functions are convex, a natural correspondence with Pontryagin's maximum principle to derive MKV-FBSDE. Since σ has no dependence on x and α, reduced Hamiltonian can be used throughout the context

$$H(t,x,\mu,y,\alpha) = [b_1(t)x + \overline{b}_1(t)\overline{\mu} + b_2(t)\alpha]\cdot y + \frac{1}{2}\left(x^T q(t)x + (x - s(t)\overline{\mu})^T \overline{q}(t)(x - s(t)\overline{\mu}) + \alpha^T r(t)\alpha\right) \tag{140}$$

take a derivative w.r.t. α

$$\partial_\alpha H = [b_2(t)]^T y + r(t)\alpha \tag{141}$$

set as zero to get the optimal control

$$\hat{\alpha}(t,x,\mu,y) = -[r(t)]^{-1}[b_2(t)]^T y \tag{142}$$

now calculate the coefficients in the adjoint BSDE

$$\partial_x H = [b_1(t)]^T y + q(t)x + \overline{q}(t)(x - s(t)\overline{\mu}) \tag{143}$$

$$\partial_x g = qx + \overline{q}(x - s\overline{\mu}) \tag{144}$$

to write down the MKV-FBSDE under the interpretation that when the number of players is large enough, we can replace the flow of empirical measure (random measure) with the flow of deterministic measure $\mu_t = \mathscr{L}(X_t)$ and replace the control α with $\hat{\alpha}(t, X_t, \mu_t, Y_t)$ simultaneously

$$\begin{cases} dX_t = \left[b_1(t)X_t + \bar{b}_1(t)\overline{\mathscr{L}(X_t)} - b_2(t)[r(t)]^{-1}[b_2(t)]^T Y_t\right] dt + \sigma\, dW_t \\ X_0 = \xi \\ dY_t = -\left([b_1(t)]^T Y_t + q(t)X_t + \overline{q}(t)(X_t - s(t)\overline{\mathscr{L}(X_t)})\right) dt + Z_t\, dW_t \\ Y_T = qX_T + \overline{q}(X_T - s\overline{\mathscr{L}(X_T)}) \end{cases} \tag{145}$$

now we want to investigate the solvability of this MKV-FBSDE containing both optimality and consistency conditions. Since the measure acts on the coefficients only through its mean, we denote

$$\overline{x}_t = \mathbb{E}X_t = \overline{\mathscr{L}(X_t)} \tag{146}$$

and simplify the MKV-FBSDE

$$\begin{cases} dX_t = \left[b_1(t)X_t + \bar{b}_1(t)\overline{x}_t - b_2(t)[r(t)]^{-1}[b_2(t)]^T Y_t\right] dt + \sigma\, dW_t \\ X_0 = \xi \\ dY_t = -\left([b_1(t)]^T Y_t + [q(t) + \overline{q}(t)]X_t - \overline{q}(t)s(t)\overline{x}_t\right) dt + Z_t\, dW_t \\ Y_T = (q + \overline{q})X_T - \overline{q}s\overline{x}_T \end{cases} \tag{147}$$

it's still hard to see the solvability but the presence of the mean of X_t in the equations leads us to think about taking expectation on both sides of the MKV-FBSDE to get the coupled ODE w.r.t. the mean of X_t and Y_t

$$\begin{cases} d\overline{x}_t = \left[[b_1(t) + \bar{b}_1(t)]\overline{x}_t - b_2(t)[r(t)]^{-1}[b_2(t)]^T \overline{y}_t\right] dt \\ \overline{x}_0 = \mathbb{E}\xi \\ d\overline{y}_t = -\left([b_1(t)]^T \overline{y}_t + [q(t) + \overline{q}(t) - \overline{q}(t)s(t)]\overline{x}_t\right) dt \\ Y_T = (q + \overline{q} - \overline{q}s)\overline{x}_T \end{cases} \tag{148}$$

where

$$\overline{y}_t = \mathbb{E}Y_t \tag{149}$$

interestingly, the following theorem turns the analysis on MKV-FBSDE completely into the analysis on this coupled ODE w.r.t. the mean of the process.

Theorem 6 (Existence and Uniqueness of the Solution to LQ-MFG)**.** *Existence and uniqueness of the solution to*

LQ-MFG holds iff existence and uniqueness of the solution to the ODE system

$$\begin{cases} d\overline{x}_t = \left[[b_1(t) + \overline{b}_1(t)]\overline{x}_t - b_2(t)[r(t)]^{-1}[b_2(t)]^T\overline{y}_t\right] dt \\ \overline{x}_0 = \mathbb{E}\xi \\ d\overline{y}_t = -\left([b_1(t)]^T\overline{y}_t + [q(t) + \overline{q}(t) - \overline{q}(t)s(t)]\overline{x}_t\right) dt \\ Y_T = (q + \overline{q} - \overline{q}s)\overline{x}_T \end{cases} \tag{150}$$

holds.

At this point, we can put up ansatz

$$\overline{y}_t = \overline{\eta}_t\overline{x}_t + \overline{\chi}_t \tag{151}$$

where $\overline{\eta}, \overline{\chi}$ are deterministic functions in t, plug into the coupled ODE system, collect coefficients to see the decoupled version as an ODE w.r.t. $\overline{\eta}, \overline{\chi}$

$$\begin{cases} \dot{\overline{\eta}}_t = \overline{\eta}_t b_2(t)[r(t)]^{-1}[b_2(t)]^T\overline{\eta}_t - \overline{\eta}_t[b_1(t) + \overline{b}_1(t)] - [b_1(t)]^T\overline{\eta}_t - [q(t) + \overline{q}(t) - \overline{q}(t)s(t)] \\ \dot{\overline{\chi}}_t = \overline{\eta}_t b_2(t)[r(t)]^{-1}[b_2(t)]^T\overline{\chi}_t - [b_1(t)]^T\overline{\chi}_t \\ \overline{\eta}_T = (q + \overline{q} - \overline{q}s)I \\ \overline{\chi}_T = 0 \end{cases} \tag{152}$$

where $\overline{\eta}_t$ is matrix-valued and $\overline{\chi}_t$ is vector-valued. It's clear that the existence and uniqueness of the solution completely depends on the equation for $\overline{\eta}_t$ (since plugging in the solution of $\overline{\eta}_t$ into the second equation always gives the trivial solution $\overline{\chi}_t \equiv 0$), which is a **matrix-valued Ricatti equation**. In other words, the solvability of LQ-MFG can eventually be turned into the solvability of a Ricatti equation (not trivial in general).

If we want to solve the MFG equilibrium, after solving this ODE to get $\overline{x}_t, \overline{y}_t$, we can try to plug in and solve the FBSDE (not a McKean-Vlasov type any more) with the affine ansatz for LQ game

$$Y_t = \eta_t X_t + \chi_t \tag{153}$$

similarly, calculate dY_t and collect coefficients, we get an ODE w.r.t. η_t, χ_t with given terminal conditions, which is once more a Ricatti equation to solve. Solving out η_t, χ_t to get the solution $(\hat{X}, \hat{Y}, \hat{Z})$ to the original MKV-FBSDE provides the MFG equilibrium.

Remark. *One has to realize that the ODE for $\overline{\eta}_t, \overline{\chi}_t$ is different from the ODE for η_t, χ_t. The previous one is a characterization of the evolution of the mean $\overline{x}_t, \overline{y}_t$ while the latter is a characterization of the solution to the FBSDE.*

To conclude, in LQ-MFG, the strategy after writing MKV-FBSDE is to first take expectation to get the evolution of the mean (possibly solving it) and to prove/disprove the existence and uniqueness of the MFG equilibrium (the point of setting up $\overline{\eta}_t, \overline{\chi}_t$). After that, MKV-FBSDE can be simplified into normal FBSDE which can be solved by putting up an ansatz (the point of setting up η_t, χ_t).

Example: Linear Quadratic (LQ) Flocking Model

In this model, there are N players (birds), with player i having position X_t^i at time t taking value in $\mathbb{R}^3$. Player i can determine its control α_t^i as the velocity at time t and the state dynamics is given as

$$dX_t^i = \alpha_t^i \, dt + \sigma \, dW_t^i \tag{154}$$

on time horizon $[0, T]$. Player i has cost functional

$$J^i(\alpha) = \mathbb{E} \int_0^T f^i(t, X_t, \alpha_t) \, dt \tag{155}$$

with no terminal cost and the running cost is

$$f^i(t, x, \alpha) = \frac{\kappa^2}{2} |x^i - \overline{x}|^2 + \frac{1}{2} |\alpha^i|^2 \tag{156}$$

with $|\cdot|$ denoting the vector ℓ_2 norm since x^i, α^i take values in $\mathbb{R}^3$. This is a LQ game with mean field interaction through $|x^i - \overline{x}|^2$.

Instead of solving it as a finite player game, let's **do mean field approximation and solve it as an LQ-MFG**. It can be easily seen that when $N \to \infty$ this game can be approximated as the game for a single representative player with state dynamics

$$dX_t = \alpha_t \, dt + \sigma \, dW_t \tag{157}$$

and cost functional

$$J(\alpha) = \mathbb{E} \int_0^T f(t, X_t, \mu_t, \alpha_t) \, dt \tag{158}$$

with no terminal cost and the running cost is

$$f(t, x, \mu, \alpha) = \frac{\kappa^2}{2} |x - \overline{\mu}|^2 + \frac{1}{2} |\alpha|^2 \tag{159}$$

since σ is constant, use reduced Hamiltonian

$$H(t, x, \mu, y, \alpha) = \alpha \cdot y + \frac{\kappa^2}{2} |x - \overline{\mu}|^2 + \frac{1}{2} |\alpha|^2 \tag{160}$$

differentiate w.r.t. α

$$\partial_\alpha H = y + \alpha \tag{161}$$

set it as zero to get the optimal control

$$\hat{\alpha}(t, x, \mu, y) = -y \tag{162}$$

now calculate the coefficients in the adjoint BSDE

$$\partial_x H = \kappa^2(x - \overline{\mu}) \tag{163}$$

$$\partial_x g = 0 \tag{164}$$

and write out the MKV-FBSDE that characterizes the solution to MFG

$$\begin{cases} dX_t = -Y_t\,dt + \sigma\,dW_t \\ X_0 = \xi \\ dY_t = -\kappa^2(X_t - \overline{\mathscr{L}(X_t)})\,dt + Z_t\,dW_t \\ Y_T = 0 \end{cases} \tag{165}$$

take expectation on both sides to turn it into a coupled ODE w.r.t. the mean of X_t, Y_t that

$$\begin{cases} d\overline{x}_t = -\overline{y}_t\,dt \\ \overline{x}_0 = \mathbb{E}\xi \\ d\overline{y}_t = 0 \\ \overline{y}_T = 0 \end{cases} \tag{166}$$

due to the simplicity of this ODE, we don't even have to set up $\overline{\eta}_t, \overline{\chi}_t$ for decoupling purpose, a direct solution is

$$\begin{cases} \overline{x}_t = \mathbb{E}\xi \in \mathbb{R}^3 \\ \overline{y}_t = 0 \end{cases} \tag{167}$$

and is unique, proves the existence and uniqueness of the MFG equilibrium.

When it comes to solving out the closed-form solution of the MFG equilibrium, put up affine ansatz with deterministic η, χ

$$Y_t = \eta_t X_t + \chi_t \tag{168}$$

the MKV-FBSDE is now a normal FBSDE

$$\begin{cases} dX_t = -Y_t\,dt + \sigma\,dW_t \\ X_0 = \xi \\ dY_t = -\kappa^2(X_t - \mathbb{E}\xi)\,dt + Z_t\,dW_t \\ Y_T = 0 \end{cases} \tag{169}$$

collect coefficients and turn it into an ODE w.r.t. η_t, χ_t (check that $Z_t = \sigma\eta_t$ is deterministic, so it's adapted, the key condition to satisfy as the solution to BSDE)

$$\begin{cases} \dot{\eta}_t - \eta_t^2 + \kappa^2 I_3 = 0 \\ \dot{\chi}_t - \eta_t\chi_t - \kappa^2\mathbb{E}\xi = 0 \\ \eta_T = 0, \chi_T = 0 \end{cases} \tag{170}$$

it's quite obvious that the solution is

$$\eta_t = \kappa\frac{e^{2\kappa(T-t)} - 1}{e^{2\kappa(T-t)} + 1} I_3 \overset{def}{=} \eta_t^* I_3 \tag{171}$$

where η_t^* takes value as real number, and to solve χ_t, we need to tear apart different components

$$\forall i \in \{1, 2, ..., d\}\,, \chi_t^i = -\kappa^2(\mathbb{E}\xi)^i \int_t^T e^{\int_s^t \eta_u^*\,du}\,ds \tag{172}$$

where $(\mathbb{E}\xi)^i$ is the i-th component of $\mathbb{E}\xi$. This gives the solution to the original MKV-FBSDE and the state dynamics at equilibrium is

$$dX_t = -(\eta_t X_t + \chi_t)\,dt + \sigma\,dW_t = -\eta_t(X_t - \mathbb{E}\xi)\,dt + \sigma\,dW_t \tag{173}$$

a mean-reverting Gaussian dynamics.

Remark. *Notice that $\mathbb{E}X_t = \mathbb{E}\xi, \mathbb{E}Y_t = 0$, this tells us that $\eta_t\mathbb{E}\xi + \chi_t = 0$ always holds. In other words,*

$$\chi_t = -\eta_t\mathbb{E}\xi = -\kappa\frac{e^{2\kappa(T-t)} - 1}{e^{2\kappa(T-t)} + 1}\mathbb{E}\xi \tag{174}$$

an easier representation to use. One can check that indeed this is true by plugging this expression into the ODE for χ_t.