Recitation Notes for PSTAT 170

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This note is based on the contents I have taught on the recitation class of PSTAT 170. The notes may be subject to typos, and you are welcome to provide you advice or ask questions at hzhou593@ucsb.edu.

Week 1

Mean Return and Volatility

For a given period of time, keep track of the stock price. The price at the beginning of the period is denoted P_s and the price at the end of the period is denoted P_e. The return rate during this period is defined as

$$r = \frac{P_e - P_s}{P_s} \tag{1}$$

Let's assume that we have kept track of the stock price for n periods, naturally, we would have $r_1, r_2, ..., r_n$ as the return rates for each period calculated. The **mean return** would be defined as

$$r_{mean} = \frac{\sum_{i=1}^{n} r_i}{n} \tag{2}$$

and the **volatility** would be defined as the standard deviation of the sequence $r_1, ..., r_n$.

$$\sigma = sd(r_1, ..., r_n) \tag{3}$$

Remark. *There's many kinds of averages and one can also apply the geometric average instead of the arithmetic average mentioned above, i.e.*

$$r_{mean} = \left(\prod_{i=1}^{n}(1 + r_i)\right)^{\frac{1}{n}} - 1 \tag{4}$$

The geometric average also has a natural interpretation as the accumulation of interest. If we have 1 dollar at the very beginning, after n periods the amount would accumulate to $\prod_{i=1}^{n}(1 + r_i)$. On the other hand, if r_{mean} is adopted for each period, the amount would accumulate to $(1 + r_{mean})^n$. The geometric mean return is the return rate such that these two amounts are the same.

Remark. *There's many selections for the time period of the return rate. For example, it can be on a daily basis, or on a yearly basis etc. In practice, we would not use something like "annualized daily mean return" since the error would be huge. One may refer to the two real datasets I have posted to observe that the close price on one trading day is generally not equal to the open price on the next trading day because of after-hour tradings. That's why in practice we typically choose the time period to be 3 months, 1 year etc.*

In this course, you should be more likely to get in touch with the return rate on a yearly basis and the arithmetic mean return.

Remark. *You can run the program **stock.py** to see some of the plots and calculations I have made (it's written in Python). Note that there should be two datasets, one called **AMC.csv**, the other called **GS.csv**. You can change the file name in the ninth line of the code, i.e. **price = pd.read_csv('AMC.csv')** to run it on different datasets.*

You should observe that AMC's stock price has much more fluctuation and results in a much higher volatility. Also note the difference from daily mean return and yearly mean return. (The daily mean return here is not annualized, you have to annualize it in order to compare with the yearly mean return)

Stock Indices

The stock indices tells you what is happening on the whole stock market. The two main stock indices of consideration would be the DJIA (Dow Jones Industrial Average) and the SP500 (Standard & Poor's 500). The main difference in these two constructions is that DJIA is **dollar-weighted** but SP500 is **market capitalization weighted**.

The construction of DJIA only depends on the stock price of the component companies (30 large companies selected including Apple, Microsoft, Goldman Sachs etc.):

$$DJIA = \frac{\sum_i P_i}{Dow\ Index} \tag{5}$$

where P_i stands for the stock price of a component company and the Dow index is a fixed constant (currently 0.152 approximately). As a result, if the stock price of a component company rise 1, then $DJIA$ is going to rise $\frac{1}{0.152} = 6.59$ points.

The construction of SP500, however, takes market capitalization into consideration.

$$SP500 = \frac{\sum_i P_i Q_i}{Divisor} \tag{6}$$

where P_i stands for the stock price of a company and Q_i stands for the number of shares publicly available of a company and the divisor is a fixed constant.

One fact to notice is that we can use market captalization weights to simplify our calculations. Since the market capitalization weights are proportional to Q_i (actually the market capitalization weights are formed as $\frac{Q_i}{\sum_j Q_j} \propto Q_i$), if we know that a company has market capitalization weight w_i and its stock price increases by α, then the SP 500 index should increase by $w_i\alpha$ (the percentage of increase). By doing so, it's possible to compute the SP 500 index without knowing the value of Q_i and the value of the divisor.

Example

Let's use an example to illustrate these points (provided by Professor Michael). Now Microsoft is having market capitalization weight 5.72% and Goldman Sachs is having market capitalization weight 0.33%, and the SP 500 index now is 3655.

It's known that the stock price of Microsoft is changing from 237 to 239 with the stock prices of all the other companies fixed. Since Microsoft is one of the DJIA component companies, the DJIA will rise $2 \times 6.59 = 13.18$ points.

As stated above, since the stock price is rising by $\frac{2}{237}$, the SP 500 index will increase by $5.72\% \times \frac{2}{237} = 0.048\%$, thus resulting the SP 500 index to increase $3655 \times 0.048\% = 1.76$ points.

It's known that the stock price of Goldman Sachs is changing from 294 to 296 with the stock prices of all the other companies fixed. Since Goldman Sachs is one of the DJIA component companies, the DJIA will also rise $2 \times 6.59 = 13.18$ points. As stated above, since the stock price is rising by $\frac{2}{294}$, the SP 500 index will increase $0.33\% \times \frac{2}{294} \times 3655 = 0.08$ points.

As we can see, although the DJIA index is having the same amount of change, the stock price of Microsoft has a much larger impact on SP 500 index than Goldman Sachs. Actually, Microsoft has about 1.77 trillion dollars market capitalization and Goldman Sachs only has about 100 billion dollars market capitalization. It also tells us that the stock price does not necessarily reflect the value of the company. In this case, Goldman Sachs is having a higher stock price but Microsoft is a more valuable company.

Week 2

Collar

A **collar** is to buy a put and to sell a call at the same time, for which the call option has a higher strike price and both options share the same time to maturity. A **zero-cost collar** is the collar with zero net premium.

Example

See the problem 3.12 in the textbook, where one invest 1000 in the index, buy 950-strike put and sell 1107-strike call. The interest rate for 6 months is 2% and the 6-month forward price is 1020. The premium of the 950-strike put is 51.777 and the premium of the 1107-strike call is 51.873.

Let's draw the profit diagram for this position. Firstly it's clear that longing 1 unit of the index brings with profit

$$P - 1000 \times 1.02 = P - 1020 \tag{7}$$

where P stands for the future index after 6 months.

Buying the 950-put option has profit

$$\max\{0, 950 - P\} - 51.777 \tag{8}$$

Selling the 1107-strike call option has profit

$$-\max\{0, P - 1107\} + 51.873 \tag{9}$$

As a result, the profit (future value) for this position should be

$$P - 1020 + \max\{0, 950 - P\} - 51.777 \times 1.02 - \max\{0, P - 1107\} + 51.873 \times 1.02 \tag{10}$$

$$= \begin{cases} -69.90208 & P < 950 \\ P - 1019.90208 & 950 \leq P \leq 1107 \\ 87.09792 & P > 1107 \end{cases} \tag{11}$$

Remark. *Don't forget to multiply the premium by* 1.02 *to be consistent with other future values.*

Note that the net premium (present value) for this collar is

$$51.777 - 51.873 = -0.096 \tag{12}$$

so this is really close to a zero-cost collar.

Example

See the following figure for an SOA problem on options.

17.

The current price for a stock index is 1,000. The following premiums exist for various options to buy or sell the stock index six months from now:

Strike Price	Call Premium	Put Premium
950	120.41	51.78
1,000	93.81	74.20
1,050	71.80	101.21

Strategy I is to buy the 1,050-strike call and to sell the 950-strike call.

Strategy II is to buy the 1,050-strike put and to sell the 950-strike put.

Strategy III is to buy the 950-strike call, sell the 1,000-strike call, sell the 950-strike put, and buy the 1,000-strike put.

Assume that the price of the stock index in 6 months will be between 950 and 1,050.

Determine which, if any, of the three strategies will have greater payoffs in six months for lower prices of the stock index than for relatively higher prices.

(A) None
(B) I and II only
(C) I and III only
(D) II and III only
(E) The correct answer is not given by (A), (B), (C), or (D)

Figure 1: The SOA problem

To solve this, we can write out the payoff of all three strategies (the premium makes no difference here). For strategy I, the payoff is

$$\max\{0, P-1050\} - \max\{0, P-950\} = \begin{cases} 0 & P < 950 \\ 950 - P & 950 \le P \le 1050 \\ -100 & P > 1050 \end{cases} \tag{13}$$

it's actually a bear spread.

For strategy II, the payoff is

$$\max\{0, 1050-P\} - \max\{0, 950-P\} = \begin{cases} 100 & P < 950 \\ 1050 - P & 950 \le P \le 1050 \\ 0 & P > 1050 \end{cases} \tag{14}$$

it's also a bear spread.

For strategy III, the payoff is

$$\max\{0, P-950\} + \max\{0, 1000-P\} - \max\{0, P-1000\} - \max\{0, 950-P\} = 50 \tag{15}$$

it's actually the combination of two box spreads. The first one is buying 950-strike call and selling 950-strike put. The second one is buying 1000-strike put and selling 1000-strike call. The consequence is that one will always buy at price 950 and sell at price 1000, that's why the payoff is always $1000-950=50$.

Butterfly

Butterfly is a combination that bets on the volatility of the market. A typical symmetric butterfly can be made by buying one $(K+a)$-strike call, buying one $(K-a)$-stike call and selling two K-strike call (actually can also use put to construct). So the payoff will be

$$\max\{0, P-(K+a)\} + \max\{0, P-(K-a)\} - 2\max\{0, P-K\} \tag{16}$$

$$= \begin{cases} 0 & P < K-a \\ P-K+a & K-a \le P \le K \\ -P+K+a & K < P \le K+a \\ 0 & P > K+a \end{cases} \tag{17}$$

Then what if the strike prices are not equally distributed? For example, we have $A < B < C$ and A, B, C-strike call options to build a butterfly. In such situation, we would have to build an asymmetric one with a special proportion.

How to figure out the proportion of each call option? Note that **butterfly should have insured tails on both sides**, i.e. the payoff is always 0 when $P < A$ or $P > C$. For call options, the left tail is always insured because no call will be exercised when the stock price is too low. However, we would have to balance the right tail of the combination. Assume we are buying n_A, n_C number of call options with strike price A, C and selling n_B number of call options with strike price B. When the stock price $P > C$, the payoff would be

$$n_A(P-A) - n_B(P-B) + n_C(P-C) \tag{18}$$

setting it as 0 for any $P > C$ gives the condition

$$\begin{cases} n_A - n_B + n_C = 0 \\ An_A - Bn_B + Cn_C = 0 \end{cases} \tag{19}$$

This condition comes from the fact that the coefficient of P must be 0 (hold for any $P > C$) and the remaining constant must also be 0 (the expression equals to 0). Solving this equation gives feasible n_A, n_B, n_C.

See the lower graph on P83 of the textbook, where there are 35,43,45-strike call options to make an asymmetric butterfly. Just plug in $A = 35, B = 43, C = 45$ gives the equations

$$\begin{cases} n_A - n_B + n_C = 0 \\ 35n_A - 43n_B + 45n_C = 0 \end{cases} \tag{20}$$

solve these equations (obviously the solution is not unique) to get:

$$\begin{cases} n_B = 5n_A \\ n_C = 4n_A \end{cases} \tag{21}$$

That's why the textbook uses $n_A = 2, n_B = 10, n_C = 8$ to construct the asymmetric butterfly (which is one of the solutions to these equations).

Example

See the problem 3.17 in textbook. Construct an asymmetric butterfly using 950, 1020, 1050-strike options. Since I have already shown how to use call options to construct asymmetric butterfly above, let me show you how to use put options to construct (actually the same). Let's assume that all the options here are put options.

When the stock price is higher than 1050, the payoff is always 0, no exercising of put. As a result, we only need to ensure that the left tail always has 0 payoff. Assume the quantities of these three options to trade are n_1, n_2, n_3 respectively.

$$\forall P < 950, n_1(950 - P) - n_2(1020 - P) + n_3(1050 - P) = 0 \tag{22}$$

that's why we conclude

$$\begin{cases} 950n_1 - 1020n_2 + 1050n_3 = 0 \\ n_1 - n_2 + n_3 = 0 \end{cases} \tag{23}$$

solve to get:

$$\begin{cases} n_2 = \frac{10}{3}n_1 \\ n_3 = \frac{7}{3}n_1 \end{cases} \tag{24}$$

To make all of them to be integers, take $n_1 = 3, n_2 = 10, n_3 = 7$, so we buy 3 portions of 950-strike put, sell 10 portions of 1020-strike put and buy 7 portions of 1050-strike put.

Let's compute the payoff of this combination:

$$3\max\{0, 950 - P\} + 7\max\{0, 1050 - P\} - 10\max\{0, 1020 - P\} \tag{25}$$

$$= \begin{cases} 0 & P < 950 \\ 3P - 2850 & 950 \leq P \leq 1020 \\ -7P + 7350 & 1020 < P \leq 1050 \\ 0 & P > 1050 \end{cases} \tag{26}$$

it's really an asymmetric butterfly.

Remark. *In general, if we only want to construct a butterfly, there will be 3 unknown variables* n_1, n_2, n_3 *but only 2 equations. This is because any multiple of these weights, i.e.* $n_1' = kn_1, n_2' = kn_2, n_3' = kn_3$ *is still a butterfly, as if we are buying* k *portions of the butterfly constructed with weights* n_1, n_2, n_3.

However, if the payoff/profit diagram is given, then the slopes of the lines in the butterfly are known (know exactly how many portions we are buying), we would be able to write down the third equation w.r.t. n_1, n_2, n_3 *and uniquely solve out the equations, i.e. solve out the* k *above.*

Week 3

Example

A producer of gold has expenses of 800 per ounce of gold produced. Assume that the cost of all other production-related expenses is negligible and that the producer will be able to sell all gold produced at the market price. In one year, the market price of gold will be one of three possible prices, corresponding to the following probability table:

Gold Price in one year	Probability
750 per ounce	0.2
850 per ounce	0.5
950 per ounce	0.3

The producer hedges the price of gold by buying a 1-year put option with an exercise price of 900 per ounce. The option costs 100 per ounce now, and the continuously compounded annual risk-free interest rate is 6%.

Calculate the expected 1-year profit per ounce of gold produced.

Figure 2: The problem

Firstly, the position of the producer is longing gold (he produces gold) and longing a put option. Assume the future price of gold per ounce is P after 1 year, then the overall payoff (future value) would be

$$payoff = P + \max\{0, 900 - P\} \tag{27}$$

and the overall cost comes from the production of gold and the premium of the option (note that we have to change them into future values)

$$cost = (100 + 800) \times e^{0.06} \tag{28}$$

the profit is just the difference

$$profit = payoff - cost = P + \max\{0, 900 - P\} - (100 + 800) \times e^{0.06} \tag{29}$$

Now there's three possible situations in the future, with P possibly taking values from $\{750, 850, 950\}$. Calculate the profit in these 3 situations:

$$profit = \begin{cases} -55.65 & P = 750 \\ -55.65 & P = 850 \\ -5.65 & P = 950 \end{cases} \tag{30}$$

to see that the expected 1-year profit per ounce gold will be

$$0.2 \times -55.65 + 0.5 \times -55.65 + 0.3 \times -5.65 = -40.65 \tag{31}$$

Remark. *Here we accumulate the cost of production 800 into future values. The explanation for this is that the future values of 800 should be the* ***opportunity cost****. If the producer chooses not to use 800 to produce 1 ounce of gold but chooses to buy risk-free bond with 800, he/she could have got* $800 \times e^{0.06}$ *after 1 year. That's why here the future value of the cost caused by production is actually* $800 \times e^{0.06}$.

In contrast, calculate the expected 1-year profit per ounce gold without the put option. The profit would then be

$$profit = P - 800 \times e^{0.06} \tag{32}$$

$$profit = \begin{cases} -99.47 & P = 750 \\ 0.53 & P = 850 \\ 100.53 & P = 950 \end{cases} \tag{33}$$

the expected profit is

$$0.2 \times -99.47 + 0.5 \times 0.53 + 0.3 \times 100.53 = 10.53 \tag{34}$$

Why are we even getting much lower profit with an insurance held? That's due to the relatively optimistic future state and the cost of the premium. However, if there's some probability for the gold price to be 500 instead of 750, then the insurance may make a large difference since with a long position of put option, the payoff is always fixed at 900 on the left tail no matter how low the gold price becomes.

Remark. *If one is more familiar with probability theory, one might notice that what we are doing is just first writing the profit as some function of price* $f(P)$*, with price* P *as a random variable with given distribution. To calculate the expected profit, it's just* $\mathbb{E}f(P)$ *with*

$$\mathbb{E}f(P) = \sum_k f(k) \cdot \mathbb{P}(P = k) \tag{35}$$

if P *is a discrete random variable and*

$$\mathbb{E}f(P) = \int_0^\infty f(x)p(x)\, dx \tag{36}$$

if P *is a continuous random variable with density* $p(x)$*.*

No Arbitrage Criterion

The core concept in derivative pricing is the no arbitrage criterion, which means that **there's no risk-free arbitrage opportunities with no initial endowment in the market**. The underlying logic of this criterion is that the existence of risk-free arbitrage causes the changes in the supply and demand on the financial markets, leading to the close of the risk-free arbitrage gaps. Actually, the pricing of forward contracts for stocks with continuously paid dividends can be derived from such perspective.

Assume the continuous-time compounded annual effective dividend rate is δ, with the continuous-time compounded annual effective risk-free interest rate r. The forward contract has time to maturity T with price $F_{0,T}$. The stock price now is S_0.

The forward contract works in a way that we shall pay for it and receive the stock both at time T. Consider the arbitrage strategy **borrowing $S_0 \cdot e^{-\delta T}$ money** and longing $e^{-\delta T}$ share of stock and selling a forward contract for 1 share of stock, such strategy has payoff (future value)

$$S_T - S_T = 0 \tag{37}$$

the reason we only buy $e^{-\delta T}$ share of stock is that by reinvesting all dividends back into the stock, at time T there's eventually $e^{-\delta T}e^{\delta T} = 1$ share of stock.

The cost (present value) of such strategy is

$$-e^{-rT}F_{0,T} + e^{-\delta T}S_0 \tag{38}$$

note that the price of the forward contract is paid in the future so it has to be discounted.

The risk-free arbitrage has profit (future value)

$$e^{rT}(e^{-rT}F_{0,T} - e^{-\delta T}S_0) \tag{39}$$

note that "risk-free" refers to the fact that such strategy does not depend on the future price of the stock, i.e. S_T.

Since the existence of risk-free arbitrage is not allowed, set

$$e^{rT}(e^{-rT}F_{0,T} - e^{-\delta T}S_0) = 0 \tag{40}$$

to solve out the price of the forward contract

$$F_{0,T} = S_0 e^{(r-\delta)T} \tag{41}$$

The strategy constructed above is called **cash-and-carry arbitrage**, where we buy the underlying asset and sell its forward contract, and it's **risk-free with no initial endowment**. Of course, by selling the underlying asset and buying its forward contract, one would get the **reverse cash-and-carry arbitrage**.

Remark. *The risk-free condition is crucial. Arbitrages with risk (uncertainty) are perfectly fine to exist on financial markets. I bolded "borrow $S_0 \cdot e^{-\delta T}$ money" to emphasize that the arbitrage strategy has to have no initial endowment. In the situation above, we are using the same interest rate for borrowing money and investment so it doesn't matter whether we have initial endowment or not. However, in the case where* ***the borrowing interest rate is different from the investment interest rate (the case in real life), the no initial endowment condition makes a difference!***

Example

Let's look at problem 5.19 in the textbook, another example of arbitrage with different currency. Given the spot exchange rate 0.008\$/¥, continuous-time compounded annual effective risk-free rate yen-denominated is 1% and continuous-time compounded annual effective risk-free rate dollar-denominated is 5%. 1 year forward exchange rate is 0.0084\$/¥. We would like to build an arbitrage strategy with no initial cost, no risk and calculate the profit after 1 year in ¥.

Remark. *The spot exchange rate is the exchange rate that works at present while the forward exchange rate is the exchange rate publicly announced at present but works in the future.*

Notice that the forward exchange rate announced today for 1-year period is totally different from the spot exchange rate after 1 year. The forward exchange rate announced today enables one to exchange a certain amount of currency with this rate after a certain time period (similar to forward contract but is known and fixed at present), while the spot exchange rate in the future is totally random at present.

To check whether arbitrage exists in such case, one can think about deriving the critical forward exchange rate F\$/¥ when no risk-free arbitrage exists. Consider shorting the yen-denominated risk-free bond to get 1¥ and turning it into 0.008\$ at present to buy the dollar-denominated risk-free bond. Also tell the bank that we would like to change dollar for yen with the forward exchange rate in 1 year.

This arbitrage strategy consists of no initial costs. At the end of 1 year, the yen-denominated risk-free bond requires us to pay $e^{0.01}$¥ while we would get $0.008 \times e^{0.05}$\$ from the dollar-denominated risk-free bond and change it using forward exchange rate into $\frac{0.008\times e^{0.05}}{F}$¥. As a result, the profit at the end of 1 year in yen is:

$$-e^{0.01} + \frac{0.008 \times e^{0.05}}{F} = 0 \tag{42}$$

to ensure that there's no risk-free arbitrage opportunities. Solve to get

$$F = 0.008326 \tag{43}$$

Remark. *If a risk-free arbitrage strategy has negative profit, it is also indicating the existence of risk-free arbitrage since one can hold the opposite position to make positive profit, i.e. any risk-free combination of derivatives with non-zero profit violates the no arbitrage criterion.*

This tells us that under no-arbitrage condition, the forward exchange rate should have been 0.008326\$/¥. However, now the real rate is higher than this, telling us that holding yen in our hands would be more profitable. Therefore, the arbitrage strategy should go like:

- Short the dollar-denominated risk-free bond to get 0.008\$
- Turn it into 1¥ at present
- Use 1¥ to buy the yen-denominated risk-free bond

- Tell the bank that we would like to exchange $\frac{0.008\times e^{0.05}}{0.0084}$ yen for dollar at the forward exchange rate after 1 year (only change the amount of yen into dollar such that it's enough to cover the debt in dollar)

By doing this, at the end of 1 year, we get $e^{0.01}$¥ as the payoff from the yen-denominated risk-free bond and we have to pay $\frac{0.008\times e^{0.05}}{0.0084}$¥ for shorting the dollar-denominated risk-free bond. The overall profit is

$$e^{0.01} - \frac{0.008 \times e^{0.05}}{0.0084} = 0.00884¥ \tag{44}$$

By applying the arbitrage strategy above, for each 1¥ invested, we receive risk-free profit 0.00884¥.

If now the real forward exchange rate is changed as 0.0083\$/¥, then it's lower than 0.008326 computed. As a result, holding dollars in our hand would be more profitable, so our position for arbitrage should be flipped into:

- Short the yen-denominated risk-free bond to get 1¥
- Turn it into 0.008\$ at present
- Use 0.008\$ to buy the dollar-denominated risk-free bond
- Tell the bank that we would like to exchange $0.008 \times e^{0.05}$ dollar for yen at the forward exchange rate after 1 year (turn all the dollars you get into yen since the final profit is to be calculated in yen)

The profit after 1 year would be

$$-e^{0.01} + \frac{0.008 \times e^{0.05}}{0.0083} = 0.00322¥ \tag{45}$$

By applying such arbitrage strategy, for each 1¥ invested, we receive risk-free profit 0.00322¥.

This arbitrage strategy is called **covered interest arbitrage** and it shows the determination of forward exchange rate using spot exchange rate and the risk-free rate of respective currency-denominated bond.

Remark. *The risk-free in the context of covered interest arbitrage refers to the fact that the payoff is* ***independent of the spot exchange rate in the future****.*

One must ***clear his/her position in the future date*** *to ensure that the arbitrage has no risk. That is to say, one can never operate with any spot rates in the future. For example, one can never first invest in dollar-denominated risk-free bonds and change the dollar into yen in the future (affected by spot exchange rate in future, which is different from the forward exchange rate announced today).*

Week 4

Reminders on No Arbitrage Criterion

Last week we talked about the no arbitrage criterion and this criterion was stated as "There is no risk-free arbitrage in financial markets." However, I missed a very important point which is that other than being risk-free, such arbitrage also has to **have no initial endowment**. I have already fixed the notes for last week and bolded all the parts that I want you to notice. You are welcome to go over last week's notes (the part of pricing forward contract) with this extra condition and I will also explain it here.

The meaning of "no initial endowment" is that one shall imagine himself having no money available at the beginning of constructing an arbitrage, i.e. one has to borrow money or short assets before longing assets or buying financial derivatives. For example, one can never long a stock index before shorting something else or borrowing money as the first step when constructing an arbitrage strategy. The covered interest arbitrage (last example) we have talked about last week shows exactly the correct formulation of a risk-free arbitrage with no initial endowment.

Next, let me briefly talk about why we are still getting the correct conclusion last week without this "no initial cost" condition. Actually, we have been adopting the assumption that there's **only one interest rate** in the financial market in last week's setting, which means that borrowing and investment share the same interest rate. However, in your homework problems, you will have to deal with the case where these two interest rates differ, and it's also true in practice that interest rates for different purposes differ. **Under the condition that there's only one interest rate in the financial market, one can see that this "no initial cost" condition can actually be removed.** When the investment interest rate r_i is lower than the borrowing interest rate r_b, which is often the case in practice, one cannot return all the money without extra endowment by borrowing a certain amount of money and investing in zero-coupon bond.

Commodity Forward

Compared to stock forward, the commodity forward mainly has two differences: the continuous-times annual effective storage cost λ and the continuous-times annual effective convenience yield c. λ works as negative dividend for commodity holders and increases the forward price. On the other hand, c works as the benefit for commodity holders as positive dividend and reduces the forward price.

Remark. *To understand how λ works, imagine the case where we have 1 unit of corn in the factory and because of storage conditions, corn goes bad continuously at an annual effective rate of λ. This means that for each small time interval Δt (in the unit of year), the amount of truly usable corn drops from x to $x\left(1-\frac{\lambda}{\Delta t}\right)$. Lasting T years, the amount of truly usable corn will be*

$$\left(1-\frac{\lambda}{\Delta t}\right)^{T\Delta t} \tag{46}$$

by assuming that the time partition is fine enough, the limit goes like

$$\lim_{\Delta t \to 0} \left(1 - \frac{\lambda}{\Delta t}\right)^{T\Delta t} = e^{-\lambda T} \tag{47}$$

That's why 1 unit of corn turn into $e^{-\lambda T}$ unit of truly usable corn in T years. The same logic holds for continuous-time dividend yield and convenience yield.

Thinking about the following arbitrage strategy, where r is the continuous-time annual effective risk-free interest rate and S_0 is the price of the commodity now.

- Borrow $S_0 \cdot e^{(\lambda-c)T}$ to buy $e^{(\lambda-c)T}$ unit of commodity
- Sell a T-year mature forward contract on 1 unit of commodity (price $F_{0,T}$)

so the profit is

$$F_{0,T} - S_0 \cdot e^{(\lambda-c)T} e^{rT} \tag{48}$$

and this strategy has neither initial endowment nor risk. According to the no arbitrage criterion

$$F_{0,T} = S_0 \cdot e^{(\lambda-c+r)T} \tag{49}$$

is the price of this commodity forward contract. Here we denote $\delta = c - \lambda$ as the **lease rate** and the forward price is given by

$$F_{0,T} = S_0 \cdot e^{(r-\delta)T} \tag{50}$$

exactly the same form as that for stock forward contract.

Swap

Swaps are forward contracts replacing fluctuating future payments with fixed amount future payments. Swap buyers typically pay a same fixed amount X on several future dates $T_1, ..., T_n$ and receive 1 unit of specified asset on each date. Pricing swap is just about finding the appropriate X such that no arbitrage criterion holds in the financial market.

The pricing starts from viewing the swap as combinations of forward contracts. It's easy to see that such swap consists of a T_1-mature forward contract, a T_2-mature forward contract,..., a T_n-mature forward contract. To eliminate risk-free arbitrage opportunities with no initial endowment, the sum of the present values of the price of all those forward contracts should be equal to the present value of the annuity with X being paid at time $T_1, ..., T_n$. Denote F_{0,T_i} as the price of the T_i-mature forward contract and r_i as the continuous-times annual effective yield to maturity of T_i-mature zero-coupon bond. The equality of present values gives

$$\sum_{i=1}^{n} X \cdot e^{-r_i T_i} = \sum_{i=1}^{n} F_{0,T_i} \cdot e^{-r_i T_i} \tag{51}$$

$$X = \frac{\sum_{i=1}^{n} F_{0,T_i} \cdot e^{-r_i T_i}}{\sum_{i=1}^{n} e^{-r_i T_i}} \tag{52}$$

Remark. *The T-mature zero-coupon bond yield to maturity r_T (continuous-time annual effective) just works as the discount rate to calculate present values.*

In practice, we might observe that the continuous-time annual effective 1-year yield to maturity of zero-coupon bond is 5% while the continuous-time annual effective 2-year yield to maturity of zero-coupon bond is 5.5%. This is telling us that 1\$ after 1 year has present value $e^{-0.05}$ while 1\$ after 2 years has present value $e^{-0.055\times 2}$. Note that if we use the annual discount factor $e^{-0.05}$ to figure out the present value of 1\$ after 2 years, it would be $e^{-0.05\times 2}$ instead. Be careful with the difference

$$e^{-0.055\times 2} \neq e^{-0.05\times 2} \tag{53}$$

This is a natural phenomenon called ***the term structure of interest rates****.*

Example

The following table summarizes forward prices for gold:

Years to Maturity	Forward Price	Zero-Coupon Bond Yield
1	1600	5.00%
2	1700	5.50%
3	1800	6.00%

Note: bond yield is quoted in continuously-compounded units.

Part A: If today's price is $S_0 = 1525$, find the implied (continuously compounded) lease rate on gold for the next year.

Part B: Find the swap price for receiving 1 ounce of gold for the next 3 years (i.e. each year exchange 1 ounce for the fixed swap price).

For Part A, recall that the forward price is

$$F = S_0 \cdot e^{(r-\delta)T} \tag{54}$$

for lease rate δ. Plug in all the numbers for 1-year maturity to see

$$1600 = 1525 \times e^{(0.05-\delta)} \tag{55}$$

$$\delta = 0.002 \tag{56}$$

For Part B, recall the rule of swap pricing that the present value of the sequence of forward contracts should equal the present value of the sequence of fixed amount payments.

$$X \cdot e^{-0.05} + X \cdot e^{-0.055\times 2} + X \cdot e^{-0.06\times 3} = 1600 \times e^{-0.05} + 1700 \times e^{-0.055\times 2} + 1800 \times e^{-0.06\times 3} \tag{57}$$

$$X = 1695.68 \tag{58}$$

Put Call Parity

It's obvious that options are risky financial derivatives since the future stock price, which is random, appears in the payoff. As a result, no arbitrage criterion can never be used to price a single put/call option. However, by considering a combination of put and call options, we would be able to get the put call parity from no arbitrage criterion.

Let S_T denote the stock price in the future at time T, K denote the strike price of options and the options all have time to maturity T. The continuous-time annual effective risk-free interest rate is r. Call option has payoff

$$\max\{0, S_T - K\} \tag{59}$$

and put option has payoff

$$\max\{0, K - S_T\} \tag{60}$$

to find that

$$\max\{0, S_T - K\} - \max\{0, K - S_T\} = S_T - K \tag{61}$$

To fully eliminate the risk, also sell a forward contract on stock with time to maturity T and price $F_{0,T}$ with profit $F_{0,T} - S_T$. Combining these two positions and use C, P for upfront premium of such call and put option, the overall profit (FV) of this position is deterministic

$$F_{0,T} - S_T + S_T - K - (C-P)e^{rT} = F_{0,T} - K - (C-P)e^{rT} \tag{62}$$

According to no arbitrage criterion, this profit should be 0, providing the **put call parity**

$$C - P = e^{-rT}(F_{0,T} - K) \tag{63}$$

Remark. *One can also think intuitively that by longing a call option and shorting a put option, no matter what future stock price is, one always has to buy 1 unit of stock in the future. By shorting an extra forward contract to sell out this 1 unit of stock in the future, one can fully eliminate the risk.*

Put Call Parity with Dividends

The same logic holds for put call parity with dividends. The only thing to do is to **replace the forward price** $F_{0,T}$.

Let's first focus on discrete dividend payments. According to the pricing of forward contract, it should be clear that (stated in previous classes)

$$F_{0,T} = S_0 e^{rT} - FV(Div) \tag{64}$$

$$e^{-rT} \cdot F_{0,T} = S_0 - PV(Div) \tag{65}$$

replace the $F_{0,T}$ in the put call parity to get the formula 9.3 in the textbook, **the put call parity with discrete dividend payments**

$$C - P = S_0 - PV(Div) - e^{-rT}K \tag{66}$$

If the dividend is paid in a continuous way with continuous-time annual effective dividend rate δ, the forward contract has price

$$F_{0,T} = S_0 \cdot e^{(r-\delta)T} \tag{67}$$

so we get **the put call parity with continuous dividend payments**

$$C - P = e^{-rT}(S_0 \cdot e^{(r-\delta)T} - K) = S_0 \cdot e^{-\delta T} - K \cdot e^{-rT} \tag{68}$$

Example

Let's take a look at problem 9.3 in the textbook.

Stock price is 800 now and continuous-time annual effective risk-free interest rate is 5% with no dividend yield. Term is 1 year, with 815-call premium 75 and 815-put premium 45. Now the position is longing the stock, selling the 815-call, and buying the 815-put.

(a). What is the rate of return on this position held until the expiration of the options?

(b). What is the arbitrage implied by your answer to (a)?

(c). What difference between the call and put prices would eliminate arbitrage?

(d). What difference between the call and put prices eliminates arbitrage for strike prices of 780, 800, 820, and 840?

(a): The long position in stock gives profit (FV)

$$S_1 - 800 \times e^{0.05} \tag{69}$$

the short position in 815-call gives profit (FV)

$$-\max\{0, S_1 - 815\} + 75 \times e^{0.05} \tag{70}$$

the long position in 815-put gives profit (FV)

$$\max\{0, 815 - S_1\} - 45 \times e^{0.05} \tag{71}$$

the overall profit (FV) is

$$S_1 - 800 \times e^{0.05} - \max\{0, S_1 - 815\} + 75 \times e^{0.05} + \max\{0, 815 - S_1\} - 45 \times e^{0.05} = 5.52 \tag{72}$$

The overall upfront cost is $800 - 75 + 45 = 770$ and the amount of money we have in the future is exactly 815, so the rate of return is

$$\frac{815 - 770}{770} = 0.058 \tag{73}$$

(b): Since we are buying the underlying asset and at the same time selling a forward contract (that's what selling a call and buying a put essentially do), this arbitrage is a cash-and-carry arbitrage.

(c): By put call parity

$$C - P = e^{-rT}(F_{0,T} - K) \tag{74}$$

where the price of the forward contract should be $800 \times e^{0.05}$. Plug in all the numbers to find

$$C - P = e^{-0.05}(800 \times e^{0.05} - 815) = 24.75 \tag{75}$$

So under no arbitrage criterion, the premium gap should be 24.75, but now the actual gap is $75 - 45 = 30 > 24.75$. That's why risk-free arbitrage with no initial endowment exists in the financial market and such arbitrage strategy sells the call and buys the put.

(d): Just plug in different numbers in the put call parity:

$$C - P = e^{-0.05}(800 \times e^{0.05} - 780) = 58.04 \tag{76}$$

$$C - P = e^{-0.05}(800 \times e^{0.05} - 800) = 39.02 \tag{77}$$

$$C - P = e^{-0.05}(800 \times e^{0.05} - 820) = 20 \tag{78}$$

$$C - P = e^{-0.05}(800 \times e^{0.05} - 840) = 0.97 \tag{79}$$

Example

For put call parity with continuous dividend payments, refer to problem 9.1 in the textbook.

The current stock price is 32 and 6-month 35-call has premium 2.27. The continuous-time annual effective risk-free interest rate is 4% and the continuous-time annual effective dividend yield is 6%. What's the premium of the 6-month 35-put option?

In this setting of continuous dividend payments, apply the formula

$$C - P = S_0 \cdot e^{-\delta T} - K \cdot e^{-rT} \tag{80}$$

and plug in numbers to get

$$2.27 - P = 32 \times e^{-0.06\times 0.5} - 35 \times e^{-0.04\times 0.5} \tag{81}$$

$$P = 5.52 \tag{82}$$

Week 5

Option Price Relationship

Assume all the options talked below have the same date to maturity T but with different strike price $K_1 < K_2 < K_3$. The call options with those strike prices have price C_1, C_2, C_3 and the put options with those strike prices have price P_1, P_2, P_3, the continuous-time annual effective interest rate is 0. Let's figure out the relationship between those prices. Note that all relationships shall ensure that **no arbitrage exists (note that here risk-free only refers to being free of the risk of the negative profit while uncertainty is acceptable)**.

The first rule is that **call options with lower strike price have higher prices, put options with higher strike price have higher prices**, i.e.

$$C_1 \geq C_2 \tag{83}$$

$$P_1 \leq P_2 \tag{84}$$

Consider the strategy buying K_1-call and selling K_2-call.

$$payoff = \max\{0, S_T - K_1\} - \max\{0, S_T - K_2\} \tag{85}$$

$$cost = C_1 - C_2 \tag{86}$$

the profit is

$$profit = \max\{0, S_T - K_1\} - \max\{0, S_T - K_2\} - (C_1 - C_2) \tag{87}$$

$$= \begin{cases} -(C_1 - C_2) & S_T \leq K_1 \\ S_T - K_1 - (C_1 - C_2) & K_1 < S_T \leq K_2 \\ K_2 - K_1 - (C_1 - C_2) & S_T > K_2 \end{cases} \tag{88}$$

If now $C_1 < C_2$, then for all possible future stock price, the profit is always positive.

Remark. *Although risk still exists for such strategy, the risk only reflects the uncertainty of profit being positive. There's no way for one to take the risk of the profit being negative under the condition that $C_1 < C_2$. Intuitively, such arbitrage shall also not exist in the financial markets, that's why $C_1 \geq C_2$ holds. The relationship of option prices are derived by* ***the construction of spread and butterfly*** *(these two combinations have insured tails) and the new criterion used here is that* ***the minimum possible profit of any strategy shall be negative****.*

Note that if one creates a strategy where the profit is always negative under all circumstances and there is only one interest rate in the economy, that is also an arbitrage since one can hold the opposite position and the profit will always be positive.

The second rule is that **the difference in premium shall always be less than the difference in strike**

price, i.e.

$$C_1 - C_2 \leq K_2 - K_1 \tag{89}$$
$$P_2 - P_1 \leq K_2 - K_1 \tag{90}$$

Consider the strategy selling K_1-call and buying K_2-call.

$$payoff = -\max\{0, S_T - K_1\} + \max\{0, S_T - K_2\} \tag{91}$$
$$cost = C_2 - C_1 \tag{92}$$

the profit is

$$profit = -\max\{0, S_T - K_1\} + \max\{0, S_T - K_2\} - (C_2 - C_1) \tag{93}$$
$$= \begin{cases} -(C_2 - C_1) & S_T \leq K_1 \\ K_1 - S_T - (C_2 - C_1) & K_1 < S_T \leq K_2 \\ K_1 - K_2 - (C_2 - C_1) & S_T > K_2 \end{cases} \tag{94}$$

Now that $C_1 \geq C_2$ holds, so $-(C_2 - C_1) \geq 0$. However, if $C_1 - C_2 > K_2 - K_1$ then $K_1 - K_2 - (C_2 - C_1) > 0$ and no matter what future stock price is, the profit is always positive. As a result, we have proved the second rule.

The third rule is the **convexity**, i.e.

$$\frac{C_1 - C_2}{K_2 - K_1} \geq \frac{C_2 - C_3}{K_3 - K_2} \tag{95}$$
$$\frac{P_2 - P_1}{K_2 - K_1} \leq \frac{P_3 - P_2}{K_3 - K_2} \tag{96}$$

Consider the asymmetric butterfly buying 1 portion of K_1-call and $\frac{K_2 - K_1}{K_3 - K_2}$ portions of K_3-call while selling $\frac{K_3 - K_1}{K_3 - K_2}$ portions of K_2-call.

Remark. *If one does not know how those portions come from, please refer the contents above for the method of asymmetric butterfly construction. The key equations here are*

$$\begin{cases} n_1 - n_2 + n_3 = 0 \\ n_1 K_1 - n_2 K_2 + n_3 K_3 = 0 \end{cases} \tag{97}$$

for buying n_1 portions of K_1-call, selling n_2 portions of K_2-call, buying n_3 portions of K_3-call.

$$payoff = \max\{0, S_T - K_1\} - \frac{K_3 - K_1}{K_3 - K_2}\max\{0, S_T - K_2\} + \frac{K_2 - K_1}{K_3 - K_2}\max\{0, S_T - K_3\} \tag{98}$$

$$cost = C_1 - \frac{K_3 - K_1}{K_3 - K_2}C_2 + \frac{K_2 - K_1}{K_3 - K_2}C_3 \tag{99}$$

the profit is

$$profit = \max\{0, S_T - K_1\} - \frac{K_3 - K_1}{K_3 - K_2}\max\{0, S_T - K_2\} + \frac{K_2 - K_1}{K_3 - K_2}\max\{0, S_T - K_3\} - \left(C_1 - \frac{K_3 - K_1}{K_3 - K_2}C_2 + \frac{K_2 - K_1}{K_3 - K_2}C_3\right) \tag{100}$$

Since the expression is a little bit complicated, let's just think about the lowest possible profit of this asymmetric butterfly. Obviously the lowest profit is gained when $S_T < K_1$ and in such situation the profit is

$$-C_1 + \frac{K_3 - K_1}{K_3 - K_2}C_2 - \frac{K_2 - K_1}{K_3 - K_2}C_3 \tag{101}$$

Of course we have to set the lowest possible profit to be negative such that no arbitrage exists. As a result,

$$-C_1 + \frac{K_3 - K_1}{K_3 - K_2}C_2 - \frac{K_2 - K_1}{K_3 - K_2}C_3 \leq 0 \tag{102}$$

$$-(K_3 - K_2)C_1 + (K_3 - K_1)C_2 - (K_2 - K_1)C_3 \leq 0 \tag{103}$$

and one can verify that this inequality is equivalent to the convexity condition

$$\frac{C_1 - C_2}{K_2 - K_1} \geq \frac{C_2 - C_3}{K_3 - K_2} \tag{104}$$

Remark. *One might be interested in the relationships when there exists continuous-time interest rate $r > 0$. If an interest rate exists, the first and third relationship won't change. The first relationship is comparing two PV and the third relationship is comparing two ratio of PV and FV, so they won't be affected by interest rate.*

However, the second relationship changes since it's comparing PV with FV. It would be natural for one to modify the second relationship into

$$C_1 - C_2 \leq e^{-rT}(K_2 - K_1) \tag{105}$$

$$P_2 - P_1 \leq e^{-rT}(K_2 - K_1) \tag{106}$$

such that both sides are PV and the relationship holds in the general case. One can also construct arbitrage strategy to prove this, the construction is left as an exercise for the readers.

Remark. ***Geometric interpretations of these relationships*** *(only read this remark if you are interested)*

Let's only state for call options. The interpretations for put options are left as exercises for the readers. Denote $C = f(K)$ as a function with the input as the strike price of a call option and the output as the premium of a call *option, so that* $C_i = f(K_i)$*. Assume the time to maturity* T *is fixed and there's no interest rate exist. Let's consider what properties shall this function* f *satisfy according to the relationships above (in real-life situations, option prices would have much more constraints than that stated here).*

Let's assume for simplicity that f *is continuous and smooth enough (derivative of any order exists)*

$$C_1 \geq C_2 \tag{107}$$

tells us that f *shall be* ***monotone decreasing***

$$C_1 - C_2 \leq K_2 - K_1 \tag{108}$$

tells us that f *is Lipschitz with constant 1 and*

$$\frac{f(K_1) - f(K_2)}{K_2 - K_1} \leq 1 \tag{109}$$

$$\lim_{K_2 \to K_1} \frac{f(K_1) - f(K_2)}{K_1 - K_2} \geq -1 \tag{110}$$

$$f'(K_1) \geq -1 \tag{111}$$

the derivative at any point is no less than -1

$$\frac{C_1 - C_2}{K_2 - K_1} \geq \frac{C_2 - C_3}{K_3 - K_2} \tag{112}$$

tells us that for $\forall K_1 < K_2 < K_3 < K_4$

$$\frac{f(K_1) - f(K_2)}{K_2 - K_1} \geq \frac{f(K_2) - f(K_3)}{K_3 - K_2} \geq \frac{f(K_3) - f(K_4)}{K_4 - K_3} \tag{113}$$

$$\lim_{K_1 \to K_2} \frac{f(K_1) - f(K_2)}{K_1 - K_2} \leq \lim_{K_4 \to K_3} \frac{f(K_3) - f(K_4)}{K_3 - K_4} \tag{114}$$

$$f'(K_2) \leq f'(K_3) \tag{115}$$

$$f''(K) \geq 0 \tag{116}$$

the function is ***convex*** *(this explains why this relationship is called convexity).*

As a result, f *shall satisfy the conditions that it's continuous and smooth enough, with*

$$\forall K, f'(K) \in [-1, 0], f''(K) \geq 0 \tag{117}$$

a reasonable function under such criterion would be $f(K) = e^{-K}$ $(K \geq 0)$ *or* $f(K) = -\log K$ $(K \geq 1)$*, so it's actually not a tight restriction on option price.*

Example

Problem 9.12 (c) in the textbook. There are 90,100,105-strike call options with premium 15,10,6 respectively and there is no interest rate.

Let's construct an asymmetric butterfly buying 1 portion of 90-call and 2 portions of 105-call while selling 3 portions of 100-call. (plug numbers into the strategy above)

$$payoff = \max\{0, S_T - 90\} - 3\max\{0, S_T - 100\} + 2\max\{0, S_T - 105\} \tag{118}$$
$$cost = 15 - 3 \times 10 + 2 \times 6 = -3 \tag{119}$$

so the profit is

$$profit = payoff + 3 \tag{120}$$
$$= \begin{cases} 3 & S_T \leq 90 \\ S_T - 87 & 90 < S_T \leq 100 \\ -2S_T + 213 & 100 < S_T \leq 105 \\ 3 & S_T > 105 \end{cases} \tag{121}$$

so one earns at least 3 from this arbitrage strategy.

Week 6

Law of One Price

The underlying logic of option pricing is the **law of one price**, i.e. if two positions have the same payoff they shall have the same cost in order to eliminate arbitrage opportunities. Otherwise, by buying the cheaper one and selling the more expensive one, one can always construct a risk-free strategy with no initial endowment with positive profit.

Binomial Tree Model for Stock Price

Under the binomial tree model, at each current state it's assumed that there are only two possible future states u, d with probability $p, 1-p$ respectively. In other words, if the stock price now is S_t, the stock price in the next term would be $S_{t+1} = uS_t$ with probability p and $S_{t+1} = dS_t$ with probability $1-p$. Moreover, the process is assumed to be time-homogeneous and at the end of each term the selections of future states (whether it's u or d) are independent. Described by mathematical language,

$$S_n = S_0 \cdot \prod_{j=1}^{n} \xi_j \tag{122}$$

$$\xi_1,...,\xi_n \ i.i.d. \tag{123}$$

$$\xi_j = \begin{cases} u & w.p. \ p \\ d & w.p. \ 1-p \end{cases} \tag{124}$$

where $w.p.$ refers to "with probability" and its connection with simple random walk can be seen after taking logarithm

$$X_n = \log S_n - \log S_0 = \sum_{j=1}^{n} \log \xi_j \tag{125}$$

that X_n has $i.i.d.$ increments, so it's a simple random walk starting from 0. Further simplification shows that

$$X_n = \sum_{j=1}^{n} \log\left(\frac{\xi_j}{d}\right) + n\log d \tag{126}$$

$$\log\left(\frac{\xi_j}{d}\right) = \begin{cases} \log\left(\frac{u}{d}\right) & w.p. \ p \\ 0 & w.p. \ 1-p \end{cases} \tag{127}$$

$$= \log\left(\frac{u}{d}\right) \cdot \begin{cases} 1 & w.p. \ p \\ 0 & w.p. \ 1-p \end{cases} \tag{128}$$

$$= \log\left(\frac{u}{d}\right) \cdot Z_j, \ Z_j \overset{i.i.d.}{\sim} B(1,p) \tag{129}$$

since independent binomial random variables have additivity, it's clear that

$$X_n = \log\left(\frac{u}{d}\right)\cdot\sum_{j=1}^{n} Z_j + n\log d \tag{130}$$

$$\stackrel{d}{=} \log\left(\frac{u}{d}\right)\cdot Z + n\log d,\ Z\sim B(n,p) \tag{131}$$

$$S_n \stackrel{d}{=} S_0\cdot d^n\cdot\left(\frac{u}{d}\right)^Z,\ Z\sim B(n,p) \tag{132}$$

where $\stackrel{d}{=}$ means equality in distribution. The stock price S_n can be written in the form as the exponential of a binomial random variable.

To calculate the expectation and the variance of the stock price S_n, one can make use of the form above to make easier calculations instead of brutal calculations.

$$\mathbb{E}S_n = S_0\cdot d^n\cdot\mathbb{E}\left(\frac{u}{d}\right)^Z \tag{133}$$

$$= S_0\cdot d^n\cdot\sum_{j=0}^{n}\left(\frac{u}{d}\right)^j\binom{n}{j}p^j(1-p)^{n-j} \tag{134}$$

$$= S_0\cdot d^n\cdot\sum_{j=0}^{n}\binom{n}{j}\left(\frac{pu}{d}\right)^j(1-p)^{n-j} \tag{135}$$

$$= S_0\cdot d^n\cdot\left(\frac{pu}{d}+1-p\right)^n \tag{136}$$

for the variance, since now the expectation is already known, the calculation of variance can be reduced to the calculation of $\mathbb{E}S_n^2$ noticing that $Var(S_n) = \mathbb{E}S_n^2 - \mathbb{E}^2 S_n$

$$\mathbb{E}S_n^2 = S_0^2\cdot d^{2n}\cdot\mathbb{E}\left(\frac{u}{d}\right)^{2Z} \tag{137}$$

$$= S_0^2\cdot d^{2n}\cdot\sum_{j=0}^{n}\left(\frac{u}{d}\right)^{2j}\binom{n}{j}p^j(1-p)^{n-j} \tag{138}$$

$$= S_0^2\cdot d^{2n}\cdot\sum_{j=0}^{n}\binom{n}{j}\left(\frac{pu^2}{d^2}\right)^j(1-p)^{n-j} \tag{139}$$

$$= S_0^2\cdot d^{2n}\cdot\left(\frac{pu^2}{d^2}+1-p\right)^n \tag{140}$$

Remark. *Here we are using the binomial expansion theorem that*

$$(a+b)^n = \sum_{j=0}^{\infty}\binom{n}{j}a^j b^{n-j} \tag{141}$$

Replicating Portfolio

Using the law of one price, if we can construct a portfolio such that in both future states the payoff of this portfolio is exactly the same as the payoff of the option (definition of **replicating portfolio**), then the price of the option is equal to the price of this portfolio.

Assume that there's a call option with strike price K and premium C, the value of the option in the future is C_u, C_d correspondent to future state u, d respectively. One period of a binomial tree refers to one time of branching of the tree and has time length h. The stock has current price S and continuous-time dividend yield δ, the continuous-time interest rate is r. Assume that the replicating portfolio consists of Δ **portions of the underlying stock and** B **as the amount of the money invested in risk-free bonds**.

If the future state is u which has a higher stock price, the future stock price is uS (note that u, d are percentages) and we will be holding $\Delta \cdot e^{\delta h}$ shares of stock. This brings with payoff

$$uS \cdot \Delta \cdot e^{\delta h} \tag{142}$$

the amount of money invested in risk-free bonds brings with payoff

$$B \cdot e^{rh} \tag{143}$$

as a result, we get the overall payoff of the replicating portfolio as

$$uS \cdot \Delta \cdot e^{\delta h} + B \cdot e^{rh} \tag{144}$$

Similarly, when the future state is d which has a lower stock price, the overall payoff of the replicating portfolio is

$$dS \cdot \Delta \cdot e^{\delta h} + B \cdot e^{rh} \tag{145}$$

By **law of one price**, one get the equations

$$\begin{cases} uS \cdot \Delta \cdot e^{\delta h} + B \cdot e^{rh} = C_u \\ dS \cdot \Delta \cdot e^{\delta h} + B \cdot e^{rh} = C_d \end{cases} \tag{146}$$

solve out Δ, B to get

$$\begin{cases} \Delta = e^{-\delta h} \frac{C_u - C_d}{S(u-d)} \\ B = e^{-rh} \frac{uC_d - dC_u}{u-d} \end{cases} \tag{147}$$

the construction of replicating portfolio. The price of the option is just the amount of the money we have to pay now in order to buy Δ shares of stock and to invest B into risk-free bonds.

$$C = \Delta \cdot S + B \tag{148}$$

Δ is the **sensitivity of the option to the change in stock price**. Each unit of change in stock price results in Δ unit of change in option price. When the option is mispriced, one will be able to use the replicating portfolio to construct arbitrage since it violates the law of one price. Selling a position with a higher price and buying the same position with a lower price will work.

Remark. *It's easy for one to calculate C_u, C_d since*

$$\begin{cases} C_u = \max\{0, uS - K\} \\ C_d = \max\{0, dS - K\} \end{cases} \tag{149}$$

by the payoff of call option in future states.

Risk-Neutral Probability Measure

Expand the option pricing formula we derive above to see

$$C = e^{-rh} \cdot \left[\left(\frac{e^{(r-\delta)h} - d}{u - d} \right) \cdot C_u + \left(\frac{u - e^{(r-\delta)h}}{u - d} \right) \cdot C_d \right] \tag{150}$$

one might find that the option price is the PV of the expectation of the payoff in future states C_u, C_d (these two values are FV) under probability measure $\mathbb{Q}$ such that

$$\begin{cases} \mathbb{Q}(\{u\}) = \frac{e^{(r-\delta)h} - d}{u-d} \\ \mathbb{Q}(\{d\}) = \frac{u - e^{(r-\delta)h}}{u-d} \end{cases} \tag{151}$$

and

$$C = e^{-rh} \cdot \mathbb{E}_{\mathbb{Q}} P \tag{152}$$

where P is the random payoff in the future states of this call option taking values C_u or C_d when the future state is u or d. Such probability measure $\mathbb{Q}$ is called the **risk-neutral measure**.

Remark. *The importance of risk-neutral measure is that it has already taken the risk premium into account and the price is just the PV of the expectation of future payoffs under this risk-neutral measure, which is not true in real life. So the risk-neutral measure is actually the probability measure in a virtual world where all investors are risk-neutral, so the price of any derivative in this world will always be equal to the PV of the expected payoff. Note that risk-neutral measure exists if and only if the market is free of arbitrage.*

One might be surprised to find that the option price has nothing to do with p, the probability in the real world *indicating the transition to the future states. However, this is not the case since p has already had implicit effects in the current stock price. For example, if the confidence in the financial market changes from* 0.5 *probability of future state being better into* 0.4 *probability of future state being better, the change would immediately result in the change of the stock price. In brief, option price has something to do with the real-world probability p, but p only affects option price through current stock price. As a result, when pricing an option, there's no need to get extra information of p having known the current stock price.*

Example

The binomial tree stock price model has $u = 1.04, d = 0.97, S_0 = 100$ with monthly steps (each branching in the binomial tree corresponds with a period of a month), $r = 0.01$ is the effective monthly interest rate. The real-world probability of stock price going up is $p = 0.75$.

1. Draw the tree for 3 periods, point out re-combining features
Now $S_0 = 100$, so

$$\begin{cases} S_u = uS_0 = 104 & w.p.\ 0.75 \\ S_d = dS_0 = 97 & w.p.\ 0.25 \end{cases} \tag{153}$$

gives the distribution of S_1

$$\begin{cases} S_{uu} = u^2 S_0 = 108.16 & w.p.\ 0.5625 \\ S_{ud} = udS_0 = 100.88 & w.p.\ 0.375 \\ S_{dd} = d^2 S_0 = 94.09 & w.p.\ 0.0625 \end{cases} \tag{154}$$

gives the distribution of S_2

$$\begin{cases} S_{uuu} = u^3 S_0 = 112.49 & w.p.\ 0.422 \\ S_{uud} = u^2 dS_0 = 104.92 & w.p.\ 0.422 \\ S_{udd} = ud^2 S_0 = 97.85 & w.p.\ 0.141 \\ S_{ddd} = d^3 S_0 = 91.27 & w.p.\ 0.016 \end{cases} \tag{155}$$

gives the distribution of S_3.

2. What are possible values of S_3? All possible values taken by S_n for general n?
It's easy to see that

$$S_3 \in \left\{u^3 S_0, u^2 dS_0, ud^2 S_0, d^3 S_0\right\} \tag{156}$$

for general n, all possible values are

$$S_n = \left\{S_0 u^k d^{n-k} : k = 0, 1, ..., n\right\} \tag{157}$$

with k ups and $n-k$ downs among those n operations.

3. Figure out $\mathbb{P}(S_2 > 100)$

$$\mathbb{P}(S_2 > 100) = \mathbb{P}(S_2 = 108.16) + \mathbb{P}(S_2 = 100.88) = 0.75^2 + 2 \times 0.75 \times 0.25 = 0.9375 \tag{158}$$

4. Find mean and variance of S_2 under $\mathbb{P}$, the real-world probability measure
If we do it by brutal force, it won't be realistic for large n

$$\mathbb{E}S_2 = 108.16 \times 0.5625 + 100.88 \times 0.375 + 94.09 \times 0.0625 = 104.55 \tag{159}$$

$$\mathbb{E}S_2^2 = 108.16^2 \times 0.5625 + 100.88^2 \times 0.375 + 94.09^2 \times 0.0625 = 10950.05 \tag{160}$$

$$Var(S_2) = \mathbb{E}S_2^2 - \mathbb{E}^2 S_2 = 19.35 \tag{161}$$

However, we can also compute this analytically as we have shown above

$$\mathbb{E}S_2 = 100 \times 0.97^2 \times \left(\frac{0.75 * 1.04}{0.97} + 1 - 0.75\right)^2 = 104.55 \tag{162}$$

$$\mathbb{E}S_2^2 = 100^2 \times 0.97^4 \times \left(\frac{0.75 * 1.04^2}{0.97^2} + 1 - 0.75\right)^2 = 10950.05 \tag{163}$$

$$Var(S_2) = \mathbb{E}S_2^2 - \mathbb{E}^2 S_2 = 19.35 \tag{164}$$

5. Consider 2 period binomial tree, consider a call option with strike $K = 100$. Write out the payoff C_{uu}, C_{ud}, C_{dd} in each scenario of S_2
For a call option, its payoff is equal to $\max\{0, S_2 - 100\}$, so

$$C_{uu} = \max\{0, 108.16 - 100\} = 8.16 \tag{165}$$

$$C_{ud} = \max\{0, 100.88 - 100\} = 0.88 \tag{166}$$

$$C_{dd} = \max\{0, 94.09 - 100\} = 0 \tag{167}$$

6. Find the replicating portfolio at $t = 1$ conditional on $S_1 = 104$
Assume holding Δ shares of stock and investing B in risk-free bonds. Conditional on $S_1 = 104$, the possible

future states are $S_2 \in \{108.16, 100.88\}$.

If $S_2 = 108.16$, the payoff of replicating portfolio is

$$108.16\Delta + B * 1.01 \tag{168}$$

if $S_2 = 100.88$, the payoff of replicating portfolio is

$$100.88\Delta + B * 1.01 \tag{169}$$

by the law of one price

$$\begin{cases} 108.16\Delta + B * 1.01 = 8.16 \\ 100.88\Delta + B * 1.01 = 0.88 \end{cases} \tag{170}$$

solve out to get

$$\begin{cases} \Delta = 1 \\ B = -99.01 \end{cases} \tag{171}$$

So the option price at this time point is

$$\Delta \cdot S_1 + B = 1 \times 104 - 99.01 = 4.99 \tag{172}$$

7. Find the replicating portfolio at $t = 1$ conditional on $S_1 = 97$

Assume holding Δ shares of stock and investing B in risk-free bonds. Conditional on $S_1 = 97$, the possible future states are $S_2 \in \{100.88, 94.09\}$.

If $S_2 = 100.88$, the payoff of replicating portfolio is

$$100.88\Delta + B * 1.01 \tag{173}$$

if $S_2 = 94.09$, the payoff of replicating portfolio is

$$94.09\Delta + B * 1.01 \tag{174}$$

by the law of one price

$$\begin{cases} 100.88\Delta + B * 1.01 = 0.88 \\ 94.09\Delta + B * 1.01 = 0 \end{cases} \tag{175}$$

solve out to get

$$\begin{cases} \Delta = 0.13 \\ B = -12.11 \end{cases} \tag{176}$$

So the option price at this time point is

$$\Delta \cdot S_1 + B = 0.13 \times 97 - 12.11 = 0.5 \tag{177}$$

Remark. *One might also try to calculate the risk-neutral probability to get that in the risk-neutral world the stock price rises with probability $\frac{4}{7}$ and drops with probability $\frac{3}{7}$. By using risk-neutral probability to calculate the option price (calculate the expected payoff of the option under this probability measure and turn into PV), one would get the same result. (We are going to talk about those next time)*

Week 7

Risk-Neutral Probability Measure

Last week we talked about constructing the replicating portfolio such that it has the same payoff as the call option. The replicating portfolio consists of holding Δ shares of stock and investing B in risk-free bonds. We solved out that in order to have no arbitrage in the market,

$$\begin{cases} \Delta = e^{-\delta h} \frac{C_u - C_d}{S(u-d)} \\ B = e^{-rh} \frac{uC_d - dC_u}{u-d} \end{cases} \tag{178}$$

By the law of one price, the call premium has to be $C = \Delta \cdot S + B$ such that no arbitrage exists.

Plug in the Δ, B and expand the option pricing formula we derive above to see

$$C = e^{-\delta h} \frac{C_u - C_d}{u-d} + e^{-rh} \frac{uC_d - dC_u}{u-d} \tag{179}$$

$$= e^{-rh} \cdot \left[\left(\frac{e^{(r-\delta)h} - d}{u-d} \right) \cdot C_u + \left(\frac{u - e^{(r-\delta)h}}{u-d} \right) \cdot C_d \right] \tag{180}$$

to find that the option price is **the PV of the expectation of the payoff / value of the option in future states C_u, C_d under probability measure** $\mathbb{Q}$ such that

$$\begin{cases} \mathbb{Q}(\{u\}) = \frac{e^{(r-\delta)h} - d}{u-d} \overset{def}{=} q \\ \mathbb{Q}(\{d\}) = \frac{u - e^{(r-\delta)h}}{u-d} \end{cases} \tag{181}$$

and

$$C = e^{-rh} \cdot \mathbb{E}_{\mathbb{Q}} P \tag{182}$$

where P is a random variable that stands for the value of the option in the future states. Such probability measure $\mathbb{Q}$ is called the **risk-neutral measure**. q is defined as **the probability of seeing a good future state under risk-neutral probability measure**, note that generally $p \neq q$.

Remark. *To clarify the notation of probability measure space, in this 1-period model, the sample space is*

$$\Omega = \{u, d\} \tag{183}$$

where u stands for the good future state (higher stock price) and d stands for the bad future state (lower stock price). The sigma algebra is taken as

$$\mathscr{F} = P(\Omega) \tag{184}$$

the power set of Ω which is a trivial one and now we are having two probability measures $\mathbb{P}$ and $\mathbb{Q}$

$$\begin{cases}\mathbb{P}(\{u\}) = p \\ \mathbb{P}(\{d\}) = 1-p\end{cases}, \begin{cases}\mathbb{Q}(\{u\}) = \frac{e^{(r-\delta)h}-d}{u-d} \\ \mathbb{Q}(\{d\}) = \frac{u-e^{(r-\delta)h}}{u-d}\end{cases} \tag{185}$$

thus we have two probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $(\Omega, \mathscr{F}, \mathbb{Q})$.

The payoff / value of the option, denoted P, is just a random variable on $(\Omega, \mathscr{F})$ such that

$$\begin{cases}P(u) = C_u = \max\{0, uS-K\} \\ P(d) = C_d = \max\{0, dS-K\}\end{cases} \tag{186}$$

this setting shall clarify the meaning of the expression

$$C = e^{-rh} \cdot \mathbb{E}_{\mathbb{Q}} P \tag{187}$$

Remark. *The importance of risk-neutral measure is that it has already taken the risk premium into account and the price is just the PV of the expectation of future payoffs under this risk-neutral measure, which is not true in real life (note that $C = e^{-rh} \cdot \mathbb{E}_{\mathbb{P}} P$ is generally* ***NOT TRUE****). So the risk-neutral measure is actually* ***the probability measure in a virtual world where all investors are risk-neutral****, so the price of any derivative in this world will always be equal to the PV of the expected payoff. Note that risk-neutral measure exists if and only if the market is free of arbitrage.*

One might be surprised to find that the option price has nothing to do with p, the probability in the real world indicating the transition to the future states. However, this is not the case since p ***has already had implicit effects in the current stock price****. For example, if the confidence in the financial market changes from 0.5 probability of future state being better into 0.4 probability of future state being better, the change would immediately result in the change of the stock price. In brief, option price has something to do with the real-world probability p, but p only affects option price through current stock price. As a result, when pricing an option, there's no need to get extra information of p if the current stock price can be observed.*

Example

Let's continue to present the example we have talked about last time.

The binomial tree stock price model has $u = 1.04, d = 0.97, S_0 = 100$ with monthly steps (each branching in the binomial tree corresponds with a period of a month), $r = 0.01$ is the effective monthly interest rate. The real-world probability of stock price going up is $p = 0.75$.

8. Compute the risk-neutral probability q

Use the formula with $h = 1$, change interest rate into continuous time rate $r = \log 1.01$ (monthly effective)

$$q = \frac{e^{(r-\delta)h} - d}{u - d} \tag{188}$$

$$= \frac{e^{\log 1.01 \times 1} - 0.97}{1.04 - 0.97} = \frac{4}{7} \tag{189}$$

as we can see, in the real world the good state happens with probability 0.75 but in the risk-neutral world the good state only happens with probability $\frac{4}{7}$.

9. Find the probability $\mathbb{Q}(S_2 > 100)$ and compare to $\mathbb{P}(S_2 > 100)$

We have computed that $\mathbb{P}(S_2 > 100) = 0.9375$. Now under risk-neutral probability measure,

$$\mathbb{Q}(S_2 > 100) = q^2 + 2q(1-q) \tag{190}$$

$$= \frac{16}{49} + \frac{24}{49} \tag{191}$$

$$= \frac{40}{49} = 0.8163 \tag{192}$$

since the stock price at time 2 will be larger than 100 if and only if there are two u evolution or one u and one d evolution from time 0 to 2.

Note that $\mathbb{Q}(S_2 > 100) < \mathbb{P}(S_2 > 100)$ is natural since in the risk-neutral world the probability of having a good state is lower than that in real life.

10. Find $\mathbb{E}_{\mathbb{Q}} S_2$, the expectation of S_2 under measure $\mathbb{Q}$.

Last time we have concluded that under probability measure $\mathbb{P}$

$$S_n \overset{d}{=} S_0 \cdot d^n \cdot \left(\frac{u}{d}\right)^Z, \ Z \sim B(n, p) \tag{193}$$

so it tells us that

$$\mathbb{E}_{\mathbb{Q}} S_n = S_0 \cdot d^n \cdot \mathbb{E}_{\mathbb{Q}} \left(\frac{u}{d}\right)^Z, \ Z \sim B(n, q) \tag{194}$$

note that one has to change the real probability p into the risk-neutral probability q since now our underlying probability space is $(\Omega, \mathscr{F}, \mathbb{Q})$.

A similar calculation gives us

$$\mathbb{E}_{\mathbb{Q}} S_n = S_0 \cdot d^n \cdot \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} \left(\frac{u}{d}\right)^k \tag{195}$$

$$= S_0 \cdot d^n \cdot \left(\frac{qu}{d} + 1 - q\right)^n \tag{196}$$

However, notice the special structure of q here to know

$$\frac{qu}{d}+1-q=\frac{\frac{u}{d}e^{(r-\delta)h}-u}{u-d}+\frac{u-e^{(r-\delta)h}}{u-d} \tag{197}$$
$$=\frac{e^{(r-\delta)h}}{d} \tag{198}$$

so

$$\mathbb{E}_{\mathbb{Q}}S_n=S_0\cdot e^{(r-\delta)nh} \tag{199}$$
$$=S_0\cdot e^{(r-\delta)T} \tag{200}$$

where T is just the time to maturity of the option. Plug in the numbers to see

$$\mathbb{E}_{\mathbb{Q}}S_2=100\times e^{\log 1.01\times 2}=100\times 1.01^2=102.01 \tag{201}$$

notice that $\mathbb{E}_{\mathbb{P}}S_2=104.55$ which is still consistent with our observation that under $\mathbb{Q}$ the future state is less likely to be good.

Remark. *One might notice that $S_0\cdot e^{(r-\delta)T}$ is just the pricing formula of the stock forward with continuous-time dividend yield δ. This makes sense because for risk-neutral investors, the price of the forward (FV) is just the expectation of its future payoff S_n under probability measure $\mathbb{Q}$.*

11. Fix 2-period binomial tree, $T=2$, consider call option with strike $K=100$. Find its premium under no arbitrage assumption

Just apply the formula mentioned above, which is

$$C=e^{-rT}\cdot\mathbb{E}_{\mathbb{Q}}P \tag{202}$$

the only difference is that now we are having 2 periods, so we have to figure out the value at each future state. The strategy is to **iterate in a backward way**. We know the exact value (which is just the payoff) of such option at time 2, so we are able to calculate the value of the option at time 1 by using the risk-neutral probability. After figuring out the value at each future state at time 1, we would be able to do the similar thing once again to compute the option value at time 0, which is just the option price.

It's first easy to see the value of the option at time 2 that

$$\begin{cases} C_{uu}=\max\{0,108.16-100\}=8.16\\ C_{ud}=\max\{0,100.88-100\}=0.88\\ C_{dd}=\max\{0,94.09-100\}=0 \end{cases} \tag{203}$$

then compute the option value at time 1

$$\begin{cases} C_u = e^{-\log 1.01 \times 1} \times \left(\frac{4}{7} \times 8.16 + \frac{3}{7} \times 0.88\right) = 4.99 \\ C_d = e^{-\log 1.01 \times 1} \times \left(\frac{4}{7} \times 0.88 + \frac{3}{7} \times 0\right) = 0.5 \end{cases} \tag{204}$$

to get the same result as we have got last time, but here we have to take one step further that

$$C = e^{-\log 1.01 \times 1} \times \left(\frac{4}{7} \times 4.99 + \frac{3}{7} \times 0.5\right) = 3.034 \tag{205}$$

as the price of such call option.

12. Find replicating portfolio of this call option at $t = 0$ and explain the self-financing hedging strategy if $S_0 = 100, S_1 = 104, S_2 = 100.88$

For the replicating portfolio at $t = 0$, of course we need to construct it using the value of option at time $t = 1$, $C_u = 4.99, C_d = 0.5$. Plug in number to the formula of Δ, B to get

$$\begin{cases} \Delta = \frac{4.99-0.5}{100\times(1.04-0.97)} = 0.64143 \\ B = e^{-\log 1.01 \times 1} \times \frac{1.04\times 0.5 - 0.97\times 4.99}{1.04-0.97} = -61.11 \end{cases} \tag{206}$$

so the replicating portfolio consists of longing 0.64143 shares of stock and borrowing 61.11 at risk-free interest rate (shorting zero-coupon bonds).

Remark. *This is telling us that we are doing the right thing for option pricing in problem 11 since*

$$\Delta \cdot S_0 + B = 0.64143 * 100 - 61.11 = 3.034 \tag{207}$$

gives us the same option price derived using risk-neutral probability measure.

When we observe the future state $S_0 = 100, S_1 = 104, S_2 = 100.88$, we have to do **self-financing hedging** for this replicating portfolio. The thought is rather natural: compute the replicating portfolio at time 1 seeing $S_1 = 104$ and change the replicating portfolio we already have.

Plug in number to the formula of Δ, B to get

$$\begin{cases} \Delta = \frac{8.16-0.88}{104\times(1.04-0.97)} = 1 \\ B = e^{-\log 1.01 \times 1} \times \frac{1.04\times 0.88 - 0.97\times 8.16}{1.04-0.97} = -99.01 \end{cases} \tag{208}$$

so the replicating portfolio consists of longing 1 shares of stock and borrowing 99.01 at risk-free interest rate (shorting zero-coupon bonds) at time 1 when $S_1 = 104$.

To modify our replicating portfolio from that at time 0 to that at time 1 when $S_1 = 104$, we have to hold

$$1 - 0.64143 = 0.3586 \tag{209}$$

more shares of stock that costs us

$$104 \times 0.3586 = 37.29 \tag{210}$$

and borrow more money by shorting zero-coupon bonds. It's clear that at time 1 the 61.11 money we got at time 0 becomes a debt of amount

$$61.11 * 1.01 = 61.72 \tag{211}$$

so at time 1 we have to borrow

$$99.01 - 61.72 = 37.29 \tag{212}$$

to reach the position of the replicating portfolio at time 1 for $S_1 = 104$.

In brief, the self-financing hedging strategy at time 1 is to short zero-coupon bonds to get 37.29 and use those money to buy 0.3586 shares of stock.

Remark. *The term* ***self-financing*** *refers to the fact that although we are changing our position, we are not having any money remaining or asking for other endowments, i.e. the* 37.29 *money we get is just equal to the amount of money needed to buy* 0.3586 *shares of stock at price* 104. *The change in the position of the Delta hedging w.r.t. time period requires no extra endowment and does not generate extra payoff, the hedging finances itself.*

13. For the same tree, find the price of a put with strike $K = 100$ and verify that put-call parity holds.

First write down the value of the option at time 2 that

$$\begin{cases} P_{uu} = \max\{0, 100 - 108.16\} = 0 \\ P_{ud} = \max\{0, 100 - 100.88\} = 0 \\ P_{dd} = \max\{0, 100 - 94.09\} = 5.91 \end{cases} \tag{213}$$

then compute the option value at time 1

$$\begin{cases} P_u = e^{-\log 1.01 \times 1} \times \left(\frac{4}{7} \times 0 + \frac{3}{7} \times 0\right) = 0 \\ P_d = e^{-\log 1.01 \times 1} \times \left(\frac{4}{7} \times 0 + \frac{3}{7} \times 5.91\right) = 2.508 \end{cases} \tag{214}$$

to get the same result as we have got last time, but here we have to take one step further that

$$P = e^{-\log 1.01\times 1} \times \left(\frac{4}{7}\times 0 + \frac{3}{7}\times 2.508\right) = 1.064 \tag{215}$$

as the price of such put option.

The put-call parity tells us that

$$C - P = e^{-rT}\cdot (F_{0,T} - K) \tag{216}$$

the left hand side is

$$C - P = 3.04 - 1.064 = 1.97 \tag{217}$$

the right hand side is

$$e^{-rT}\cdot (F_{0,T} - K) = e^{-\log 1.01\times 2} \times (100\times e^{\log 1.01\times 2} - 100) = 1.97 \tag{218}$$

so put-call parity holds. Note that **put-call parity is a model-free result**.

14. Find NPV of the forward contract with expiration $T = 2$ under measure $\mathbb{P}$ and find NPV of call option with strike $K = 99$ and $T = 3$

For the forward contract, the payoff is always $S_2 - F$ at time 2 where $F = S_0 \cdot e^{rT} = 100\times 1.01^2 = 102.01$ so at time 2, the value of this forward is

$$\begin{cases} V_{uu} = 108.16 - 102.01 = 6.15 \\ V_{ud} = 100.88 - 102.01 = -1.13 \\ V_{dd} = 94.09 - 102.01 = -7.92 \end{cases} \tag{219}$$

so under probability measure $\mathbb{P}$, the value of this forward at time 1 is

$$\begin{cases} V_u = e^{-\log 1.01\times 1} \times [6.15\times 0.75 + (-1.13)\times 0.25] = 4.287 \\ V_d = e^{-\log 1.01\times 1} \times [(-1.13)\times 0.75 + (-7.92)\times 0.25] = -2.802 \end{cases} \tag{220}$$

the NPV of this forward at time 0 is

$$V = e^{-\log 1.01\times 1} \times [4.287\times 0.75 + (-2.802)\times 0.25] = 2.49 \tag{221}$$

note that $F = 102.01$ shall be the forward price under no arbitrage assumption but the NPV of the forward under probability measure $\mathbb{P}$ is not zero! This is natural since in real world investors are not risk-neutral and the 2.49 is the **risk premium**, i.e. the extra payment made to investors for admitting risk.

Similarly, we can investigate the value of a call option with strike $K = 99$ and $T = 3$ under probability measure $\mathbb{P}$. First figure out the value of the option at time 3

$$\begin{cases} C_{uuu} = \max\{0, 112.49 - 99\} = 13.49 \\ C_{uud} = \max\{0, 104.92 - 99\} = 5.92 \\ C_{udd} = \max\{0, 97.85 - 99\} = 0 \\ C_{ddd} = \max\{0, 91.27 - 99\} = 0 \end{cases} \tag{222}$$

then the value at time 2 under $\mathbb{P}$

$$\begin{cases} C_{uu} = e^{-\log 1.01 \times 1} \times [13.49 \times 0.75 + 5.92 \times 0.25] = 11.485 \\ C_{ud} = e^{-\log 1.01 \times 1} \times [5.92 \times 0.75 + 0 \times 0.25] = 4.396 \\ C_{dd} = e^{-\log 1.01 \times 1} \times [0 \times 0.75 + 0 \times 0.25] = 0 \end{cases} \tag{223}$$

then the value at time 1 under $\mathbb{P}$

$$\begin{cases} C_{u} = e^{-\log 1.01 \times 1} \times [11.485 \times 0.75 + 4.396 \times 0.25] = 9.617 \\ C_{d} = e^{-\log 1.01 \times 1} \times [4.396 \times 0.75 + 0 \times 0.25] = 3.264 \end{cases} \tag{224}$$

then the value at time 0 under $\mathbb{P}$

$$C = e^{-\log 1.01 \times 1} \times [9.617 \times 0.75 + 3.264 \times 0.25] = 7.95 \tag{225}$$

note that **this is NOT the call premium under no arbitrage assumption** since such 7.95 contains risk premium!

15. Find the forward price with expiration $T = 2$ under no arbitrage assumption

Assume the forward price is F, then the value of the forward at time 2 is just $S_2 - F$

$$\begin{cases} V_{uu} = 108.16 - F \\ V_{ud} = 100.88 - F \\ V_{dd} = 94.09 - F \end{cases} \tag{226}$$

so under risk-neutral probability measure $\mathbb{Q}$, the value of this forward at time 1 is

$$\begin{cases} V_{u} = e^{-\log 1.01 \times 1} \times [(108.16 - F) \times \frac{4}{7} + (100.88 - F) \times \frac{7}{3}] = 104 - 0.99F \\ V_{d} = e^{-\log 1.01 \times 1} \times [(100.88 - F) \times \frac{4}{7} + (94.09 - F) \times \frac{3}{7}] = 97 - 0.99F \end{cases} \tag{227}$$

the NPV of this forward at time 0 is

$$V = e^{-\log 1.01 \times 1} \times \left[(104 - 0.99F) \times \frac{4}{7} + (97 - 0.99F) \times \frac{3}{7}\right] = 100 - 0.9803F \tag{228}$$

This NPV is the NPV in the risk-neutral world where the value / price of a risky asset is just its expected return. As a result, in order to eliminate arbitrage gaps, we shall set such NPV to 0

$$0.9803F = 100 \tag{229}$$

$$F = 102.01 \tag{230}$$

to get the price of the forward.

Remark. *The above examples show us that risk-neutral probability measure has taken risk premium into consideration. Note that the price of a risky asset is NEVER equal to the discounted expected return under $\mathbb{P}$, it should BE equal to the discounted expected return under $\mathbb{Q}$.*

One may verify that $100 \times 1.01^2 = 102.01$, this is telling us that **the forward pricing formula is also model-free**. The following relationship holds:

$$F = S_0 \cdot e^{(r-\delta)T} = \mathbb{E}_{\mathbb{Q}} S_n, \ (T = nh) \tag{231}$$

Estimating the Binomial Tree Model

In real life, the volatility is estimated as the scaled standard deviation of the log return. One might first get the series of log return $\log \frac{S_{t+1}}{S_t}$, then compute its standard deviation and multiply by the square root of the sample size to do the scaling and get an estimate for the volatility σ. One may also search for the risk-free interest rate and the dividend yield of a stock and turn them into continuous-time annual effective rates r, δ. At last, decide the time period h in the binomial tree model (annualized time). The upward and downward percentage u, d are then given by

$$\begin{cases} u = e^{(r-\delta)h + \sigma\sqrt{h}} \\ d = e^{(r-\delta)h - \sigma\sqrt{h}} \end{cases} \tag{232}$$

which helps us do the option pricing. Note that we don't need any information for the probability of u, d being the future state as we have mentioned above, such information has already been implicitly contained in the stock price.

See the Python code on my website with name "Python codes estimating the binomial tree model" for details, be sure to read my comments before running the code.

Week 8

Log-normal Stock Price in Continuous Time Setting

Now we are able to do option pricing on binomial trees, a natural idea is to set the number of periods $n \to \infty$ so that each period is thin enough. This binomial tree option pricing with much enough number of periods is expected to converge to the continuous-time model for option pricing.

Recall the binomial tree stock price model **under risk neutral measure** $\mathbb{Q}$ that

$$S_n = S_0 \cdot d^n \cdot \left(\frac{u}{d}\right)^Z, Z \sim B(n, \mathbf{q}) \tag{233}$$

$$\log\left(\frac{S_n}{S_0}\right) = n \log d + Z(\log u - \log d) \tag{234}$$

Take u, d as that in the **forward tree** construction, i.e.

$$\begin{cases} u = e^{(r-\delta)h + \sigma\sqrt{h}} \\ d = e^{(r-\delta)h - \sigma\sqrt{h}} \end{cases} \tag{235}$$

as the fluctuation around the 1-period forward price $S_0 \cdot e^{(r-\delta)h}$. Calculate the risk-neutral probability

$$q = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{\sigma\sqrt{h}} - 1}{e^{2\sigma\sqrt{h}} - 1} = \frac{1}{e^{\sigma\sqrt{h}} + 1} \tag{236}$$

and plug in u, d to see that

$$\log\left(\frac{S_n}{S_0}\right) = (r-\delta)nh + \sigma\sqrt{h}(2Z - n) \tag{237}$$

now for fixed time T, set $nh = T, n \to \infty$ to approximate continuous-time with binomial tree model to find that according to central limit theorem, Z is the *i.i.d.* sum of $B(1, q)$ random variables, so $\log\left(\frac{S_n}{S_0}\right)$ has to be Gaussian, so **the stock price S_n is log-normal**. Let's compute the mean and variance of the Gaussian random variable $\log\left(\frac{S_n}{S_0}\right)$ to find the limit distribution

$$\begin{aligned}
\mathbb{E} \log\left(\frac{S_n}{S_0}\right) &= (r-\delta)nh + \sigma\sqrt{h}(2qn - n) && (238)\\
&= (r-\delta)T + \sigma T \frac{2q-1}{\sqrt{h}} && (239)\\
&= (r-\delta)T + \sigma T \frac{1 - e^{\sigma\sqrt{h}}}{\sqrt{h}(1 + e^{\sigma\sqrt{h}})} && (240)\\
&\to (r-\delta)T - \frac{\sigma^2}{2} T \ (n \to \infty, h \to 0) && (241)
\end{aligned}$$

by an application of L'Hopital's Rule for the limit w.r.t. h. Let's also figure out the variance since expectation and variance provides all information of a Gaussian r.v.

$$Var \log\left(\frac{S_n}{S_0}\right) = \sigma^2 h \cdot Var(2Z-n) \tag{242}$$
$$= 4\sigma^2 h \cdot Var(Z) \tag{243}$$
$$= 4\sigma^2 h \cdot nq(1-q) \tag{244}$$
$$= 4\sigma^2 T \cdot \frac{e^{\sigma\sqrt{h}}}{(e^{\sigma\sqrt{h}}+1)^2} \tag{245}$$
$$\to \sigma^2 T \ (n\to\infty, h\to 0) \tag{246}$$

As a result, we can conclude that

$$\log\left(\frac{S_n}{S_0}\right) \xrightarrow{d} N\left((r-\delta)T - \frac{\sigma^2}{2}T, \sigma^2 T\right) \ (n\to\infty) \tag{247}$$

notice that here S_n denotes the stock price at the n-th period in the binomial tree model but since $nh = T$ for fixed T, it's actually just the stock price at time T and we can denote this stock price by S_T. Note that for any fixed $T > 0$ we can partition the time into n parts with equal time length and set n large enough such that

$$\log\left(\frac{S_T}{S_0}\right) \sim N\left((r-\delta)T - \frac{\sigma^2}{2}T, \sigma^2 T\right) \tag{248}$$

As a result, we have justified the **log-normal stock price assumption** adopted in Black Scholes model. Denote $Y \sim N\left((r-\delta)T - \frac{\sigma^2}{2}T, \sigma^2 T\right)$ such that

$$\log\left(\frac{S_T}{S_0}\right) \overset{d}{=} Y \tag{249}$$

then

$$S_T \overset{d}{=} S_0 \cdot e^Y \tag{250}$$
$$S_T \overset{d}{=} S_0 \cdot e^{\sigma\sqrt{T}G + (r-\delta-\frac{\sigma^2}{2})T}, G \sim N(0,1) \tag{251}$$

provides **the representation of the stock price S_T under $\mathbb{Q}$** (the information in $\mathbb{Q}$ has already been integrated in S_T according to the previous section). Here equality in distribution suffices for our purpose to do derivative pricing since the price is the discounted expected payoff under the risk-neutral measure.

Black Scholes Model

To derive the Black-Scholes formula for European call option, recall the previous conclusion that the price of the option can be calculated in a backward style on the binomial tree. At each state, denote the option value in the next period to be V, a random variable taking possible values C_u, C_d, the option value for the time being is always

$$C = e^{-rh} \cdot \mathbb{E}_{\mathbb{Q}} V \tag{252}$$

For European options, luckily, there is a closed-form formula for the value of the option at period 0. Since for each period we have to multiply the discount factor, we can extract all the discount factors and form it as $\left(e^{-rh}\right)^n = e^{-rT}$. For the expectation under risk-neutral measure $\mathbb{Q}$, denote $C_{u^i d^j}$ as the call option value at the state with i upward steps and j downward steps from the root of the binomial tree, then

$$\begin{aligned} C_{u^0 d^0} &= e^{-rT} \cdot \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} C_{u^k d^{n-k}} &(253)\\ &= e^{-rT} \cdot \mathbb{E}_{\mathbb{Q}} \max\{0, S_T - K\} &(254) \end{aligned}$$

where $C_{u^k d^{n-k}} = \max\{0, S_{u^k d^{n-k}} - K\}$ is the payoff of the call option at the time of maturity and S_T is a r.v. with distribution

$$\mathbb{P}\left(S_T = u^k d^{n-k} S_0\right) = \binom{n}{k} q^k (1-q)^{n-k} \ (k = 0, 1, ..., n) \tag{255}$$

Remark. *To see why $C_{u^0 d^0} = e^{-rT} \cdot \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} C_{u^k d^{n-k}}$ is true, notice that the expectation under $\mathbb{Q}$ is equivalent to replacing the probability on each upward edge in the binomial tree with q and replacing the probability on each downward edge in the binomial tree with $1-q$. Notice that the sum can be written exactly as the expectation under $\mathbb{Q}$ w.r.t. a function of the stock price S_T.*

As a result, one has already got **the compact form of Black Scholes formula** that

$$C = e^{-rT} \cdot \mathbb{E}_{\mathbb{Q}} \max\{0, S_T - K\} \tag{256}$$

by replacing the S_T in the binomial tree model (which is a discrete r.v.) with the log-normal distributed S_T (which is a continuous r.v.) as an approximation in the continuous time case when $n \to \infty$.

To get a more detailed Black-Scholes formula, one can plug $S_T \overset{d}{=} S_0 \cdot e^{\sigma\sqrt{T}G + (r-\delta-\frac{\sigma^2}{2})T}, G \sim N(0,1)$ into the formula above to get

$$\begin{aligned} C &= e^{-rT} \cdot \mathbb{E}_{\mathbb{Q}} \max\{0, S_T - K\} &(257)\\ &= e^{-rT} \cdot \mathbb{E} \max\left\{0, S_0 \cdot e^{\sigma\sqrt{T}G + (r-\delta-\frac{\sigma^2}{2})T} - K\right\} &(258)\\ &= e^{-rT} \cdot \mathbb{E}\left[\left(S_0 \cdot e^{\sigma\sqrt{T}G + (r-\delta-\frac{\sigma^2}{2})T} - K\right) \cdot \mathbb{I}_{G \geq \frac{\log\frac{K}{S_0} - (r-\delta-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}\right] &(259) \end{aligned}$$

Remark. *Note that on the second line the $\mathbb{Q}$ disappears since now the expectation is w.r.t. $G \sim N(0,1)$ which has nothing to do with $\mathbb{P}, \mathbb{Q}$. The information in the risk-neutral measure $\mathbb{Q}$ is already contained in $S_T \overset{d}{=} S_0 \cdot e^{\sigma\sqrt{T}G + (r-\delta-\frac{\sigma^2}{2})T}, G \sim N(0,1)$ since it only holds under risk-neutral measure $\mathbb{Q}$.*

naturally set

$$
\begin{cases} d_1 = \frac{\log \frac{S_0}{K} + (r-\delta+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ d_2 = \frac{\log \frac{S_0}{K} + (r-\delta-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \end{cases} \tag{260}
$$

and denote Φ as the CDF of standard Gaussian, so the option price turns into

$$
e^{-rT} \cdot \mathbb{E}\left[\left(S_0 \cdot e^{\sigma\sqrt{T}G + (r-\delta-\frac{\sigma^2}{2})T} - K\right) \cdot \mathbb{I}_{-G\le d_2}\right] \tag{261}
$$

$$
= e^{-rT} \cdot \left[S_0 \cdot \mathbb{E}\left(e^{\sigma\sqrt{T}G + (r-\delta-\frac{\sigma^2}{2})T} \cdot \mathbb{I}_{-G\le d_2}\right) - K \cdot \Phi(d_2)\right] \tag{262}
$$

The calculation reduces to computing the expectation and ϕ denotes the density of the standard Gaussian

$$
\begin{aligned}
\mathbb{E}\left(e^{\sigma\sqrt{T}G} \cdot \mathbb{I}_{-G\le d_2}\right) &= \int_{-d_2}^{\infty} e^{\sigma\sqrt{T}x} \cdot \phi(x)\, dx && (263)\\
&= \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + \sigma\sqrt{T}x}\, dx && (264)\\
&= \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2 + \frac{\sigma^2}{2}T}\, dx && (265)\\
&= e^{\frac{\sigma^2}{2}T} \cdot \int_{-d_2-\sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du \; (u = x - \sigma\sqrt{T}) && (266)\\
&= e^{\frac{\sigma^2}{2}T} \cdot \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du \; (u = x - \sigma\sqrt{T}) && (267)\\
&= e^{\frac{\sigma^2}{2}T} \cdot \Phi(d_1) && (268)
\end{aligned}
$$

Combining these results to get

$$
\begin{aligned}
C &= e^{-rT + (r-\delta-\frac{\sigma^2}{2})T} \cdot S_0 \cdot e^{\frac{\sigma^2}{2}T} \cdot \Phi(d_1) - e^{-rT} K \cdot \Phi(d_2) && (269)\\
&= e^{-\delta T} \cdot S_0 \cdot \Phi(d_1) - e^{-rT} \cdot K \cdot \Phi(d_2) && (270)
\end{aligned}
$$

the Black Scholes formula for pricing European call option with time to maturity T from now, strike price K, current stock price S_0, volatility σ, continuous-time interest rate r and dividend yield δ.

Remark. *One can try to infer the Black Scholes formula for pricing European put option, the compact form is obviously*

$$
P = e^{-rT} \cdot \mathbb{E}_{\mathbb{Q}} \max\{0, K - S_T\} \tag{271}
$$

by similar calculations one may get

$$
P = e^{-rT} \cdot K \cdot \Phi(-d_2) - e^{-\delta T} \cdot S_0 \cdot \Phi(-d_1) \tag{272}
$$

Remark. *One actually only need to remember the Black-Scholes formula for European call. The price of the European put can be given by the put-call parity with continuous-time dividend payments that*

$$C - P = S_0 \cdot e^{-\delta T} - K \cdot e^{-rT} \tag{273}$$

We can verify that the Black Scholes formula satisfies this model-free result in that

$$\begin{aligned} C - P &= e^{-\delta T} \cdot S_0 \cdot \Phi(d_1) - e^{-rT} \cdot K \cdot \Phi(d_2) - e^{-rT} \cdot K \cdot \Phi(-d_2) + e^{-\delta T} \cdot S_0 \cdot \Phi(-d_1) & (274) \\ &= S_0 \cdot e^{-\delta T} - K \cdot e^{-rT} & (275) \end{aligned}$$

since $\Phi(x) + \Phi(-x) = 1$.

Remark. *The American option does not have a closed-form Black Scholes formula since there is possibility for early exercising. However, similar reasoning for the log-normal limit distribution of S_T applies and one is able to solve out the price of the American option numerically.*

Delta Hedging in Continuous Time

Recall that in the binomial tree model, delta hedging is constructed by taking a position as a combination of longing Δ stock and B investment in risk-free bonds. The critical point to notice is that Δ has the interpretation as **the sensitivity of the option price w.r.t. the stock price**. This logic also holds here, for Black Scholes formula, an expression for the call premium in continuous time case, Δ is naturally defined as (here we assume the option is a call, for put similar calculations hold)

$$\Delta = \frac{\partial C}{\partial S_0} \tag{276}$$

plug in Black Scholes to get

$$\begin{aligned} \Delta &= \frac{\partial \left[e^{-\delta T} \cdot S_0 \cdot \Phi(d_1) - e^{-rT} \cdot K \cdot \Phi(d_2)\right]}{\partial S_0} & (277) \\ &= e^{-\delta T} \cdot \left(\Phi(d_1) + S_0 \cdot \frac{\partial \Phi(d_1)}{\partial S_0}\right) - e^{-rT} \cdot K \cdot \frac{\partial \Phi(d_2)}{\partial S_0} & (278) \\ &= e^{-\delta T} \cdot \left(\Phi(d_1) + S_0 \cdot \Phi'(d_1) \cdot \frac{1}{\sigma\sqrt{T} \cdot S_0}\right) - e^{-rT} \cdot K \cdot \Phi'(d_2) \cdot \frac{1}{\sigma\sqrt{T} \cdot S_0} & (279) \\ &= e^{-\delta T} \cdot \Phi(d_1) + \frac{e^{-\delta T} \cdot S_0 \cdot \Phi'(d_1) - e^{-rT} \cdot K \cdot \Phi'(d_2)}{\sigma\sqrt{T} \cdot S_0} & (280) \\ &= e^{-\delta T} \cdot \Phi(d_1) & (281) \end{aligned}$$

since

$$
\begin{aligned}
& e^{-\delta T} \cdot S_0 \cdot \Phi'(d_1) - e^{-rT} \cdot K \cdot \Phi'(d_2) && (282)\\
&= e^{-\delta T} \cdot S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} - e^{-rT} \cdot K \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} && (283)\\
&= \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{d_1^2}{2} - \delta T + \log S_0} - e^{-\frac{d_2^2}{2} - rT + \log K} \right) && (284)
\end{aligned}
$$

compare the power to find

$$
\begin{aligned}
-d_1^2 - 2\delta T + 2\log S_0 &= -(d_1 - \sigma\sqrt{T})^2 - 2\sigma\sqrt{T} d_1 + \sigma^2 T - 2\delta T + 2\log S_0 && (285)\\
&= -d_2^2 + (\sigma^2 - 2\delta)T - 2\log\frac{S_0}{K} - (2r - 2\delta + \sigma^2)T + 2\log S_0 && (286)\\
&= -d_2^2 - 2rT + 2\log K && (287)
\end{aligned}
$$

be careful that d_1, d_2 are both functions of S_0!

As a result, we have proved by calculations that **the Delta of the call option** is

$$
\Delta = e^{-\delta T} \cdot \Phi(d_1) \tag{288}
$$

then B can be computed through call premium and Δ that

$$
B = C - \Delta \cdot S_0 \tag{289}
$$

so the replicating portfolio can still be constructed. The only difference is that now we have to modify the replicating portfolio continuously as soon as the stock price changes.

For the Delta ratio of the put option with same strike price, one may take partial w.r.t. S_0 on both sides of put-call parity to get

$$
\begin{aligned}
& \frac{\partial C}{\partial S_0} - \frac{\partial P}{\partial S_0} = e^{-\delta T} && (290)\\
& \frac{\partial P}{\partial S_0} = e^{-\delta T} \cdot \Phi(d_1) - e^{-\delta T} = -e^{-\delta T} \cdot \Phi(-d_1) && (291)
\end{aligned}
$$

so **the Delta of the put option** is

$$
\Delta = -e^{-\delta T} \cdot \Phi(-d_1) \tag{292}
$$

Remark. *By put-call parity, we see that* ***the difference between call Delta and put Delta*** *is always $e^{-\delta T}$, which is exactly 1 in the case of the stock with no dividend payments.*

By noticing that $\Phi(d_1) \in [0, 1]$, one may conclude that ***the call Delta is always between 0 and 1 and the put Delta is always between -1 and 0****.*

Example

Consider still the same binomial tree setting as we have seen in the past two weeks. $S_0 = 100, u = 1.04, d = 0.97, p = 0.75$ with each period to be a month. $r = 0.01$ is the monthly effective interest rate so the continuous-time monthly effective rate is $\log 1.01$ and there's no dividend yield $\delta = 0$. We have already computed the risk-neutral probability $q = \frac{4}{7}$.

16. There's an American call with strike $K = 100$ and $T = 2$ (2 months). Check whether it's worthwhile to use early exercise.

Infer the value of the option backwardly. Let's first figure out the value at the time of maturity $t = 2$

$$\begin{cases} C_{uu} = \max\{0, S_{uu} - K\} = \max\{0, 108.16 - 100\} = 8.16 \\ C_{ud} = \max\{0, S_{ud} - K\} = \max\{0, 100.88 - 100\} = 0.88 \\ C_{uu} = \max\{0, S_{uu} - K\} = \max\{0, 94.09 - 100\} = 0 \end{cases} \tag{293}$$

then infer the value of the option at time $t = 1$. Notice that for American option, we have to calculate **the value of holding the option** (discounted expected future payoff under risk-neutral measure) and compare it with **the value of exercising the option immediately** and take the higher one.

$$\begin{cases} C_u = \max\left\{e^{-\log 1.01} \times (\frac{4}{7} \times 8.16 + \frac{3}{7} \times 0.88), \max\{0, S_u - K\}\right\} = \max\{4.99, 104 - 100\} = 4.99 \\ C_d = \max\left\{e^{-\log 1.01} \times (\frac{4}{7} \times 0.88 + \frac{3}{7} \times 0), \max\{0, S_d - K\}\right\} = \max\{0.5, 97 - 100\} = 0.5 \end{cases} \tag{294}$$

and infer the value of the option at time $t = 0$

$$C = \max\left\{e^{-\log 1.01} \times (\frac{4}{7} \times 4.99 + \frac{3}{7} \times 0.5), \max\{0, S - K\}\right\} = \max\{3.034, 100 - 100\} = 3.034 \tag{295}$$

so actually early exercise never happens.

One conclusion to bear in mind is that **American call option with no dividends is never early exercised**. This is because early exercise requires the payment of the strike price K. By holding these K until the expiration date, the option holder can save the interest on K. However, if there's dividend for such stock, the number of share of the stock may increase as time flows and holding the stock earlier may bring with extra benefits. On the other hand, **American put option with no dividends may be early exercised**.

Example

Refer to the following Figure 3 for the setting of the binomial tree. Now $S = 110, K = 100, \sigma = 0.3, r = 0.05, T = 1, \delta = 0.035, h = 0.333$ and we are having an American call option. Now the value of the option, Δ, B are all calculated. Note that if early exercise happens at a state, there's no need to hedge any longer so Δ, B won't be calculated.

Now assume that we have observed in reality that the state is going Up-Up-Up. We want to keep track of the replicating portfolio and the self-financing hedging strategy. In the following context, positive number in the parentheses mean receiving money and negative number mean paying money.

At time $t = 0$, the replicating portfolio consists of longing 0.691 shares of stock ($-0.691 \times 110 = -76.01$) and borrowing 57.408 from the bank ($+57.408$). Note that $-76.01 + 57.408 = -18.602$ which is the price of the call option.

At time $t = 1$ the stock price is $S_1 = 131.458$. Since there's continuous-time dividend payments on the stock, we are now holding $0.691 \times e^{0.035\times 0.333} = 0.699$ shares of stock but we hope to be holding 0.911 shares of stock so we have to buy $0.911 - 0.699 = 0.212$ shares of stock ($-0.212 \times 131.458 = -27.87$).We pay the bank an amount of $-57.408 \times e^{0.05\times 0.333} = -58.37$ borrowed at time $t = 0$. We borrow another $+86.185$ from the bank.

It can be seen that $-27.87 - 58.37 + 86.185 = -0.055$ very close to 0, so it's self-financing.

FIGURE 10.8

Binomial tree for pricing an American call option on a stock index; assumes S = \$110.00, K = \$100.00, $\sigma = 0.30$, $r = 0.05$, T = 1.00 years, $\delta = 0.035$, and $h = 0.333$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.

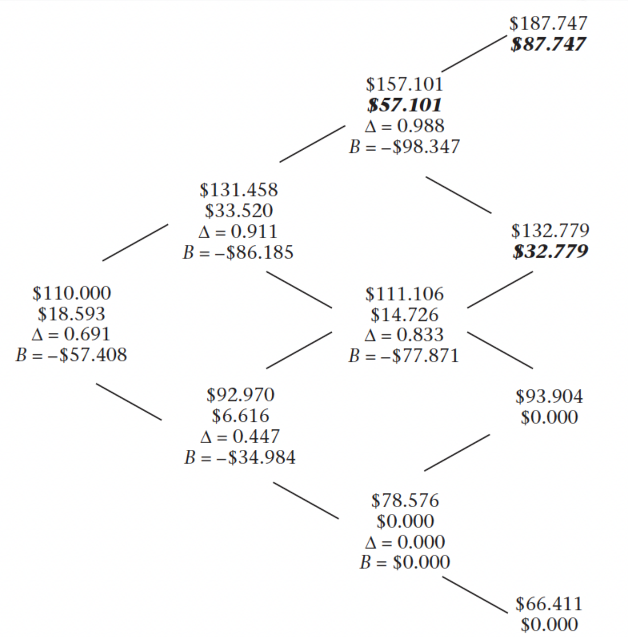

Figure 3: A plot for the Put Premium Gap of American and European Option

At time $t = 2$ the stock price is $S_2 = 157.101$. Since there's continuous-time dividend payments on the stock, we are now holding $0.911 \times e^{0.035\times 0.333} = 0.922$ shares of stock. We pay the bank an amount of $-86.185 \times e^{0.05\times 0.333} = -87.63$ borrowed at time $t = 1$.

However, at this state it's better to exercise immediately. Exercising the call option immediately enables us to collect $157.101 - 100 = +57.101$.

Week 9

Delta Hedging in Continuous Time

When it comes to continuous time, one actually needs to operate continuously on the number of share of the stock one is longing/shorting to construct the replicating portfolio and hedge the risk. However, this is not realistic in real life and one can only choose to change one's position discretely.

For example, at time 0, for a European call option one can calculate Δ_0 as the Delta ratio at time 0, i.e. the number of shares of stock one need to hold in order to create the replicating portfolio. One would also invest B_0 in zero-coupon bonds and sell the call option such that the hedging is self-financing with $C = \Delta_0 \cdot S_0 + B_0$.

However, after time $t < T$ (in annual basis) and the stock price at time t is S_t. Under such circumstances, one has to make sure that the replicating portfolio is modified such that one is now holding Δ_t shares of the stock. Naturally, one would have to spend

$$S_t \cdot (\Delta_t - \Delta_0 \cdot e^{\delta t}) \tag{296}$$

amount of money to change the position in the stock. However, one is able to plug in the maturity date as $T - t$ and the stock price as S_t to figure out the value of the call option at time t to be

$$C_t = e^{-\delta(T-t)} \cdot S_t \cdot \Phi(d_1) - e^{-r(T-t)} \cdot K \cdot \Phi(d_2) \tag{297}$$

it's easy to see that now $B_t = C_t - \Delta_t \cdot S_t$ with $\Delta_t = \frac{\partial C_t}{\partial S_t}$ to be the Delta at time t. It's necessary that one invest

$$B_t - e^{rt} \cdot B_0 \tag{298}$$

more amount of money in zero-coupon bonds.

One may find that

$$S_t \cdot (\Delta_t - \Delta_0 \cdot e^{\delta t}) + B_t - e^{rt} \cdot B_0 \neq 0 \tag{299}$$

which means the rebalancing process is not self-financing, contradictory to what we have seen in the discrete time case in the binomial tree model. This gap of self-financing is exactly caused by continuous time models.

Option Greeks

Option Greeks are defined as the partial derivative of the premium w.r.t. difference variables in the Black-Scholes formula (sensitivity w.r.t. the change in different variables). The list of definitions is presented below only for call premium

$$\Delta = \frac{\partial C}{\partial S_0} \tag{300}$$
$$\Gamma = \frac{\partial \Delta}{\partial S_0} = \frac{\partial^2 C}{\partial S_0^2} \tag{301}$$
$$Vega = \frac{\partial C}{\partial \sigma} \tag{302}$$
$$\theta = \frac{\partial C}{\partial T} \tag{303}$$
$$\rho = \frac{\partial C}{\partial r} \tag{304}$$
$$\Psi = \frac{\partial C}{\partial \delta} \tag{305}$$

where $C = e^{-\delta T} \cdot S_0 \cdot \Phi(d_1) - e^{-rT} \cdot K \cdot \Phi(d_2)$ is given by the Black-Scholes formula. Note that since option Greeks are all partial derivatives, the Greeks of a linear combination of different options are just the same linear combinations of the respective Greeks. This property will always be used in constructing hedging strategies.

Delta-Gamma Hedging

The **Delta hedging** as stated in the context above aims at creating a portfolio insensitive to underlying price changes with **0 net Delta**. To see this, notice that 1 option has Δ contribution of Delta, holding $-\Delta$ shares of stock has $-\Delta$ contribution of Delta and investing $-B$ has 0 contribution of Delta. So the overall Delta of the position is

$$\Delta - \Delta + 0 = 0 \tag{306}$$

However, one might notice that it is **only insensitive to infinitesimal changes in stock price but can't protect against large changes in stock price**. To see this fact, for call option, the Delta is

$$\Delta = e^{-\delta T} \cdot \Phi(d_1) \tag{307}$$

which is increasing in S_0 since $d_1 = \frac{\log \frac{S_0}{K} + (r - \delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$ is increasing in S_0 and Φ as CDF is also increasing. As a result,

$$\Gamma > 0 \tag{308}$$

and the call premium is convex w.r.t. S_0. This is telling us that when stock price increases from S_0 to $S_t > S_0$, denote $C(S_t)$ as the call premium when the initial stock price is S_t, by the first-order characterization of convex functions

$$C(S_t) - C(S_0) \geq \Delta \cdot (S_t - S_0) \tag{309}$$

As a result, if $S_t - S_0$ is a non-negligible large positive number, the call premium will increase to a large extent, actually much larger than $\Delta \cdot (S_t - S_0)$, so Delta hedging won't be working any longer.

This naturally gives rise to **Delta-Gamma Hedging**, a hedging strategy such that the position has **0 net Delta and 0 net Gamma**. To construct such hedging strategy, recall the things we have done in Delta hedging, which is to use stock and investment as a "basis" under linear combination to replicate the option payoff. Note that investment has 0 Gamma since it has 0 Delta and holding 1 share of stock also has 0 Gamma since Delta is a constant. As a result, we have to introduce another option such that it hedges the Γ we already have.

In detail, we consider buying 1 call with strike K_1 which is the initial financial derivative we are holding. Other than that, we choose to buy x calls with strike K_2, to hold y shares of stock and to invest z.

Remark. *Each instrument in this position has its own work to do! Strike K_2 call only hedges the Gamma of the strike K_1 call. Holding shares of stock hedges the Delta of the strike K_1 call and strike K_2 call. Cash investment hedges the cash flow of the strike K_1 call, the strike K_2 call and holding shares of stock.*

It's important to realize that what we are doing here is similar essentially to using the linear combination of basis vectors to represent a given vector in a vector space.

We would then get 3 equations for the **hedging of Gamma, Delta and cash flow** respectively.

$$\begin{cases} 1 \cdot \Gamma(K_1) + x \cdot \Gamma(K_2) + y \cdot 0 + z \cdot 0 = 0 \\ 1 \cdot \Delta(K_1) + x \cdot \Delta(K_2) + y \cdot 1 + z \cdot 0 = 0 \\ 1 \cdot C(K_1) + x \cdot C(K_2) + y \cdot S_0 + z \cdot 1 = 0 \end{cases} \tag{310}$$

solve these three equations and one would know how to construct the Delta-Gamma hedging.

Remark. *To get the formula of Γ, one just take another partial derivative of Δ w.r.t. S_0. For European call option,*

$$\Gamma = \frac{\partial \Delta}{\partial S_0} = \frac{\partial e^{-\delta T} \cdot \Phi(d_1)}{\partial S_0} \tag{311}$$

$$= e^{-\delta T} \cdot \Phi'(d_1) \cdot \frac{\partial d_1}{\partial S_0} \tag{312}$$

$$= e^{-\delta T} \cdot \varphi(d_1) \cdot \frac{1}{\sigma\sqrt{T} S_0} > 0 \tag{313}$$

where φ is the PDF of standard Gaussian. From the expression one can also see the convexity of C w.r.t. S_0. Note that ***put option has the same Gamma as call option by differentiating put-call parity****. (similar to what we have done last week)*

The first equation in the system is telling us that

$$x = \frac{-\Gamma(K_1)}{\Gamma(K_2)} \tag{314}$$

is the **static hedging for the Gamma**. Static refers to the fact that the number of options with strike K_2 does not change with time and is fixed as the negative Gamma ratio of the two options.

In the situation mentioned above, the portfolio consisting of holding y shares of stock and doing z cash investment in zero-coupon bonds is called the **Delta-Gamma neutral portfolio** since such portfolio clears up the overall Delta and Gamma if one is already holding 1 strike K_1 call.

Example

Let's see an example as a generalization of the concept of Delta-Gamma neutral portfolio but with the similar spirit.

Now in Black-Scholes model $\sigma = 0.3, S_0 = 50, r = 0.05, \delta = 0.03$ and we are buying 1 European put option with $T = 1, K = 48$. We want to use European put option with $T = 0.25$ and different strike prices $K \in \{46, 48, 50, 52, 54\}$ to construct a Delta-Vega-Theta neutral portfolio.

Firstly, Delta-Vega-Theta neutral portfolio is an analogue of the Delta-Gamma portfolio meaning that we want to make sure that the overall position of holding the $T = 1, K = 48$ put and the Delta-Vega-Theta neutral portfolio has 0 net Delta, 0 net Vega and 0 net Theta. As a result, we have to realize that we would have 3 equations to satisfy (Delta, Vega, Theta) and would need 3 "basis" in the vector space to represent the position uniquely. Actually, any 3 options with strike price picked from $K \in \{46, 48, 50, 52, 54\}$ would suffice.

Let's take put options with $T = 0.25$ and strike prices of 46,50,54 as an example to build this portfolio. Assume that we are buying x_1 strike 46 put, x_2 strike 50 put and x_3 strike 54 put. The linear system is given by

$$\begin{cases} 1 \cdot \Delta_* + x_1 \cdot \Delta_1 + x_2 \cdot \Delta_2 + x_3 \cdot \Delta_3 = 0 \\ 1 \cdot Vega_* + x_1 \cdot Vega_1 + x_2 \cdot Vega_2 + x_3 \cdot Vega_3 = 0 \\ 1 \cdot \theta_* + x_1 \cdot \theta_1 + x_2 \cdot \theta_2 + x_3 \cdot \theta_3 = 0 \end{cases} \tag{315}$$

where any option Greeks with subscript 1 is for the put with $K = 46, T = 0.25$, subscript 2 for the put with $K = 50, T = 0.25$, subscript 3 for the put with $K = 54, T = 0.25$ and subscript * for the put with $K = 48, T = 1$. Written in the matrix form

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ Vega_1 & Vega_2 & Vega_3 \\ \theta_1 & \theta_2 & \theta_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\Delta_* \\ -Vega_* \\ -\theta_* \end{bmatrix} \tag{316}$$

plug in the formula for option Greeks to solve out the linear system

$$x_1 = 39.8389, x_2 = -58.0230, x_3 = 25.93 \tag{317}$$

apply the Black-Scholes formula to get the premium of those four options and compute the total cost of this position to be \$ 20.81. The Delta-Vega-Theta neutral portfolio consists of buying 39.8389 46-puts, selling 58.0230 50-puts and buying 25.93 54-puts.

One might notice that the three numbers solved for constructing the portfolio are not reasonable. This shows that **blindly following hedging strategy would lead to messy hedging** and that **it's important to choose**

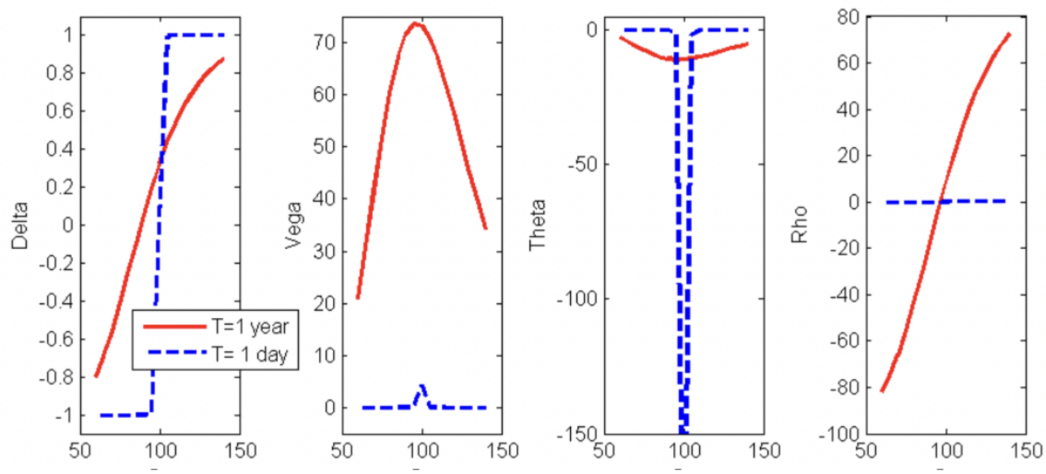

Figure 4: The change of option Greeks with respect to stock price for two expiration dates

appropriate instruments to do the hedging. In this example, choosing underlying stock and cash investments instead of put options would be a way simpler choice that can achieve the same objective.

Remark. *Here we are simply constructing the neutral portfolio but are not doing the* ***hedging*** *since one might notice that hedging requires us to* ***balance out the cash flows****. That's why in the Delta-Gamma hedging we are constructing above, although we only have to hedge net Delta and net Gamma (2 option Greeks) of the portfolio, we have to set up 3 equations!*

Example

Let's look at problem 12.16 in the textbook. Assume $r = 0.08, \sigma = 0.3, \delta = 0$, use a stock with price in 60 to 140, stock price increment as 5, two different times to expiration: 1 year and 1 day. Consider buying 100-strike straddle (buying 1 100-strike put and 1 100-strike call). Plot the Delta, Vega, Theta, Rho of the straddle w.r.t. stock price S_0 and explain why the graph looks like this.

When doing numerical calculations, one would never compute a true partial derivative, but to **use finite difference to approximate partial derivative**. For example, to compute the value of Vega which is the partial derivative w.r.t. σ when $\sigma = \sigma_0$ with all other variables in the Black-Scholes formula fixed,

$$Vega|_{\sigma=\sigma_0} \sim \frac{C(\sigma = \sigma_0 + \varepsilon) - C(\sigma = \sigma_0)}{\varepsilon} \tag{318}$$

so the only formula one has to implement is the Black-Scholes formula. Refer to Fig. 4 for the plot of the option Greeks.

The observations are stated as follows:

The Delta of the straddle is roughly linear in S_0, close to -1 when S_0 is small enough and close to 1 when S_0 is large enough. This is because when stock price is low enough, the call worth nothing and the put is deep in the money, so Delta is very close to -1, the lower bound of the Delta of a put option. It's the opposite case when S_0 is high enough. The longer maturity date the straddle has, the more tolerant it is for extreme low and high S_0 because longer time brings with more possibility.

The straddle Vega is big and positive for 1 year time to maturity. The reason is that more volatility results in larger payoff of the straddle in average (from the payoff diagram). The highest Vega always happens in the at-the-money case. This is because in this case both options have positive and equal Vegas. Vega is always positive since more volatility requires more risk premium to compensate.

The straddle has a negative Theta with moderate time-decay. When the time to maturity is very small, the change of T only has some effects when stock price is around 100, i.e. there's still some space for the fluctuation to work.

It's obvious that $T = 1$ curves have more curvature and are smoother since the log-normal distribution of S_T really shows through. In contrast, for one day maturity, the variance of S_T is so small that all actions are around $K = 100$. In detail, Vega and Rho are very small since there are no time for impact and Theta can be extreme while the option is at-the-money, Delta looks like the step function.

Example

Finally, let's look at a problem from the SOA exam. You're given that for some unknown call option its premium is $C_0 = 2.34, \Delta_0 = -0.181, \Gamma_0 = 0.035$. Initial S_0 is unknown, but after stock price changes (instantaneously) to $S = 86$, option value changes to 2.21. Use Delta-Gamma approximation find S_0.

One knows immediately that the Delta-Gamma approximation works as the second-order approximation of the option premium. Denote $C(S_t)$ as the value of the call option with initial stock price S_t

$$
\begin{aligned}
C(S_t) &= C(S_0) + \frac{\partial C}{\partial S_t}(S_0) \cdot (S_t - S_0) + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}(S_0) \cdot (S_t - S_0)^2 && (319) \\
&= C(S_0) + \Delta_0 \cdot (S_t - S_0) + \frac{1}{2}\Gamma_0 \cdot (S_t - S_0)^2 && (320)
\end{aligned}
$$

plug in the numbers to know

$$
\begin{aligned}
2.21 &= 2.34 - 0.181 \times (S_t - 86) + \frac{1}{2} \times 0.035 \times (S_t - 86)^2 && (321) \\
S_t &= 86.7765 && (322)
\end{aligned}
$$

Remark. *There are obviously two roots for this equation, one to be 86.7765 and the other is 95.566. The reason we are taking the previous root is that it's nearer to 86 because the Delta-Gamma approximation shall work for infinitesimal change in the variables.*