Notes on MATH 246

Haosheng Zhou

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Week 1

Notations

$\alpha = (\alpha_1, ..., \alpha_n)$ to be a tuple of non-neg integers as indices for differentiation, $|\alpha| = \sum_i \alpha_i$ is the order of index.

$u : \Omega \to \mathbb{R}$ with Ω as an open connected subset of $\mathbb{R}^n$ and $\partial^\alpha u = \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n} u$ with $D^k u = \left\{ \partial^\alpha u \middle| |\alpha| = k \right\}$. For example, $Du = \{u_{x_1}, ..., u_{x_n}\}$ and $D^2 u = \{u_{x_1,x_1}, ..., u_{x_n,x_n}\}$.

As a common practice, Du stands for the gradient of u.

$C^k(\Omega)$ is the set of function $u : \Omega \to \mathbb{R}$ such that for all $|\alpha| \leq k$, $\partial^\alpha u$ exists and is continuous. $C^\infty(\Omega)$ denotes the set of smooth functions, infinitely continuously differentiable. If Ω is bounded, $C^k(\overline{\Omega})$ is the set of all $C^k(\Omega)$ functions that are continuous up to boundary. It's easy to se that is a Banach space w.r.t. norm $||u||_{C^k} = \sum_{|\alpha| \leq k} \sup_\Omega |\partial^\alpha u|$.

Actually, when $n = 1$, this norms degenerates to the canonical norm defined on C^k space.

$$||u||_{C^k} = ||u||_\infty + ||u'||_\infty + ... + ||u^{(k)}||_\infty \tag{1}$$

A k-th order PDE is just the equation such that $F(x, u, Du, ..., D^k u) = 0$. A PDE is called linear if any linear combination of solutions is still a solution to the PDE.

Transport Equation

Denote function u as $u(x, t)$ where $x \in \mathbb{R}^n$ is the spacial variable and $t \in (0, +\infty)$ is the time variable.

The transport equation is a first order PDE:

$$u_t + b \cdot Du = 0 \tag{2}$$

with $b \in \mathbb{R}^n$ as a given vector.

If $n = 1$, $u_t + bu_x = 0$. The interpretation of this PDE is about **the transport of particles with conservation law**. Let $u(x, t)$ stands for the density of particles at location $x \in \mathbb{R}^n$ at time t in a fixed area Ω and the particles are moving with constant speed v in one direction.

Then at time t, the amount of particles within (x_1, x_2) is:

$$\int_{x_1}^{x_2} u(x, t)\, dx \tag{3}$$

At location x, the amount of particles within time (t_1, t_2) is:

$$\int_{t_1}^{t_2} v \cdot u(x, t)\, dt \tag{4}$$

The conservation law then tells us that the amount of particles within location (x_1, x_2) at time t_2 minus the amount of particles within location (x_1, x_2) at time t_1 should be equal to the net amount of particles flowing in this region during time (t_1, t_2), i.e. the amount flowing in at location x_1 minus the amount flowing out at location x_2. As a result, the following equation reflects **the conservation law**:

$$\int_{x_1}^{x_2} u(x,t_2)\,dx - \int_{x_1}^{x_2} u(x,t_1)\,dx = \int_{t_1}^{t_2} v\cdot u(x_1,t)\,dt - \int_{t_1}^{t_2} v\cdot u(x_2,t)\,dt \tag{5}$$

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} u_t(x,s)\,ds\,dx = -\int_{t_1}^{t_2}\int_{x_1}^{x_2} v\cdot u_x(y,s)\,dy\,ds \tag{6}$$

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} u_t + v\cdot u_x\,ds\,dy = 0 \tag{7}$$

If it's assumed that $u \in C^1$, then since such equation holds for any $t_1 \le t_2, x_1 \le x_2$, we get the transport equation $u_t + v\cdot u_x = 0$. As a result, the general transport equation is the generalization in the higher dimensional space.

Remark. *If the conservation law is broken, then the transport equation becomes **non-homogeneous**, and the right hand side would not be 0, meaning that there's actually a source or a sink (the source or sink might change depending on the time and location, i.e. the non-homogeneous transport equation is $u_t + v\cdot u_x = f(x,t)$).*

Remark. *Although here a constant speed v is assumed, such v can be modified to depend on x and t. It's easy to imagine that if v gets smaller when x gets larger, there will be "shocks" happening, i.e. the density u at some time point would be non-differentiable at a certain place because particles at a righter location are caught up with those at a lefter location that moves quicker. (This is the situation where "the law of equal area" holds)*

To solve the transport equation analytically, we can use the **method of characteristics**. The motivation comes from the fact that if we view the PDE along a certain characteristic (c.h.) curve $x = x(t)$, it would be much easier to solve the equation. Let's look at the homogeneous transport equation in 1 dimension by viewing the left hand side as a directional derivative.

$$\nabla u \cdot (b,1) = 0 \tag{8}$$

The directional derivative along vector $(b,1)$ is 0, so u is constant along $x = x(t) = \frac{1}{b}t + c$ (c.h. lines), and $u(x,t) = u(c,0) = f(c) = f(x-bt)$ to figure out the solution (here c is a varying constant, standing for a family of c.h. lines).

If now consider the transport equation in n dimension with an initial value condition $u(x,0) = g(x)$, the solution is constant along the line $(x+sb, t+s)$ $(s \in \mathbb{R})$ and define $z(s) = u(x+sb, t+s)$. By chain rule,

$$z'(s) = b\cdot u_x + u_t = 0 \tag{9}$$

As a result, $z(s)$ is constant with $\forall s, z(s) = z(-t) = u(x - tb, 0) = g(x - tb)$ and $\forall s, u(x + sb, t + s) = g(x - tb)$. Set $s = 0$ to find that

$$u(x,t) = g(x - tb) \tag{10}$$

The non-homogeneous version of transport equation is $u_t + b \cdot Du = f(x,t)$ with $u(x,0) = g(x)$. Similarly, by setting $z(s) = u(x + sb, t + s)$, the z will satisfy ODE $z'(s) = f(x + sb, t + s)$. Use integration to solve $z(-t)$:

$$z(0) - z(-t) = \int_{-t}^{0} z'(s)\, ds = \int_{-t}^{0} f(x + sb, t + s)\, ds \tag{11}$$

$$z(0) = u(x,t) = u(x - tb, 0) + \int_{-t}^{0} f(x + sb, t + s)\, ds \tag{12}$$

$$u(x,t) = g(x - tb) + \int_{-t}^{0} f(x + sb, t + s)\, ds \tag{13}$$

$$\tag{14}$$

With a change of variable for the integral, we get the final form of the solution

$$u(x,t) = g(x - tb) + \int_{0}^{t} f(x + (s - t)b, s)\, ds \tag{15}$$

First Order Linear Equation

The general form:

$$a(x,t)u_t + b(x,t)u_x = c(x,t)u + d(x,t) \tag{16}$$

still apply the method of characteristics to find a c.h. curve $(x(s), t(s))$ such that $z(s) = u(x(s), t(s))$ can be calculated easily. (z is actually **the version of u viewing along the c.h. curve**) Then it's obvious that

$$z'(s) = u_x \frac{dx}{ds} + u_t \frac{dt}{ds} \tag{17}$$

Compare this with the left hand side of PDE to get the **c.h. equation (ODE)**:

$$\frac{dt}{ds} = a(x,t) \tag{18}$$

$$\frac{dx}{ds} = b(x,t) \tag{19}$$

Note that these two equations contain derivative w.r.t. s, but we actually want to know the ch. curve (the relationship between t and x). Eliminate the s to get

$$\frac{dx}{dt} = \frac{b(x,t)}{a(x,t)} \tag{20}$$

the solution to this ODE is just the c.h. curve.

After solving it out, do as above to denote $z(s) = u(x(s), t(s))$ as the function u along the c.h. curve.

$$z'(s) = c(x(s), t(s))u(x(s), t(s)) + d(x(s), t(s)) \tag{21}$$

$$z'(s) = c(x(s), t(s))z(s) + d(x(s), t(s)) \tag{22}$$

if $t(s_0) = 0$, then

$$z(s_0) = u(x(s_0), 0) = g(x(s_0)) \tag{23}$$

It's an ODE of $z(s)$ with initial value condition. After solving it out and replace the s with t, x, the PDE would be solved.

Example

Let's do an example, derive the explicit formula for a function u solving the initial value problem with $b \in \mathbb{R}^n, c \in \mathbb{R}$:

$$\begin{cases} u_t + b \cdot Du + cu = 0 \\ u(x,0) = g(x) \end{cases} \tag{24}$$

Apply the method of characteristics to assume that the c.h. curve is parameterized as $(x(s), t(s))$ and consider $z(s) = u(x(s), t(s))$.

$$z'(s) = x'(s) \cdot Du + u_t \frac{dt}{ds} \tag{25}$$

As a result, the c.h. equation should be

$$x'(t) = b \tag{26}$$

and the c.h. curve should be

$$x(t) = tb + C \tag{27}$$

with $C \in \mathbb{R}^n$ as any constant vector.

As a result, the PDE is transformed into an ODE w.r.t. z with initial value condition as

$$\begin{cases} z'(s) = -c \cdot z(s) \\ z(0) = u(C, 0) = g(C) \end{cases} \tag{28}$$

Solve it to get that

$$z(s) = g(C) \cdot e^{-cs} \tag{29}$$

As a result, we have

$$u(sb + C, s) = g(C) \cdot e^{-cs} \tag{30}$$

$$t(s) = s \tag{31}$$

$$x(s) = sb + C \tag{32}$$

The solution to the PDE is

$$u(x, t) = g(x - tb) \cdot e^{-ct} \tag{33}$$

Week 2

Example

Solve the PDE

$$\begin{cases} u_t + x \cdot u_x = u \\ u(x,0) = x^2 \end{cases} \tag{34}$$

Let $z(s) = u(x(s), t(s))$, the c.h. ODE is

$$\frac{dx}{dt} = x \tag{35}$$

so the c.h. line is

$$x(t) = C \cdot e^t \ (C > 0) \tag{36}$$

Note that we can also infer an ODE with initial condition of $z(s)$:

$$\begin{cases} z'(s) = z(s) \\ z(0) = u(C,0) = C^2 \end{cases} \tag{37}$$

so the solution is

$$z(s) = C^2 e^s \tag{38}$$

As a result, $u(C \cdot e^t, t) = C^2 e^t$, note that $C = xe^{-t}$ and plug in to get the solution that

$$u(x,t) = x^2 e^{-t} \tag{39}$$

Solve the c.h. ODE, set up and solve the ODE w.r.t. z which is a parametrized curve of u along the direction of c.h. curve, at last replace everything with x, t

Cauchy Problem for General First-Order PDE

For the general setting, denote $u = u(x)$, $x \in \mathbb{R}^n$ (with the t coordinate merged into the x coordinate). $F(Du, u, x) = 0, x \in \Omega \subset \mathbb{R}^n$ with F assumed to be smooth in all its components. Note that for such $F = F(p, z, x)$, it's actually a function on $\mathbb{R}^{2n+1}$ (Du, x has n dimension and u has 1 dimension). Denote $D_x F \in \mathbb{R}^n$ as the gradient of F w.r.t. x, $D_p F \in \mathbb{R}^n$ as the gradient of F w.r.t. p, $D_z F \in \mathbb{R}$ as the derivative of F w.r.t. z.

The Cauchy problem is stated as

$$\begin{cases} F(Du,u,x)=0 \\ u(x)=g(x)\ (x\in\Gamma) \end{cases} \tag{40}$$

with the Γ as some curves in Ω.

The **general method of c.h.** should be applied to solve this PDE. For any point in Ω, find a curve (c.h. curve) such that the the point can go along this curve to get to Γ and the known function value on Γ will help. The question: how to find these curves?

Suppose u is a C^2 solution (assume stronger regularity condition, we will see the reason for this afterwards), $x(s)$ is the c.h. curve with $x(0)\in\Gamma$. Set

$$\begin{cases} z(s)=u(x(s)) \\ p(s)=Du(x(s)) \end{cases} \tag{41}$$

as **the version of u and Du along the c.h. curve $x(s)$.**

Note that

$$p_i(s)=u_{x_i}(x(s)) \tag{42}$$

$$p_i'(s)=\frac{d}{ds}\frac{\partial u(x(s))}{\partial x_i} \tag{43}$$

$$=\sum_{j=1}^n u_{x_ix_j}(x(s))\cdot x_j'(s)\ (i=1,2,...,n) \tag{44}$$

Differentiate the PDE w.r.t. x_i to get an equation w.r.t. x:

$$\sum_{j=1}^n \frac{\partial F(Du,u,x)}{\partial p_j}u_{x_ix_j}(x)+\frac{\partial F(Du,u,x)}{\partial z}u_{x_i}(x)+\frac{\partial F(Du,u,x)}{\partial x_i}=0 \tag{45}$$

restrict this PDE on the c.h. curve to get an equation w.r.t. s:

$$\sum_{j=1}^n \frac{\partial F(p(s),z(s),x(s))}{\partial p_j}u_{x_ix_j}(x(s))+\frac{\partial F(p(s),z(s),x(s))}{\partial z}u_{x_i}(x(s))+\frac{\partial F(p(s),z(s),x(s))}{\partial x_i}=0 \tag{46}$$

Further simplifications requires us to **assume that $x(s)$ satisfies**

$$\forall j=1,2,...,n,\ x_j'(s)=\frac{\partial F(p(s),z(s),x(s))}{\partial p_j} \tag{47}$$

This is because if such assumption holds,

$$p_i'(s) = \sum_{j=1}^n u_{x_ix_j}(x(s)) \cdot x_j'(s) \tag{48}$$
$$= \sum_{j=1}^n u_{x_ix_j}(x(s)) \cdot \frac{\partial F(p(s),z(s),x(s))}{\partial p_j} \tag{49}$$
$$= -\frac{\partial F(p(s),z(s),x(s))}{\partial z} p_i(s) - \frac{\partial F(p(s),z(s),x(s))}{\partial x_i} \tag{50}$$

with the last equation using 46 to turn back to an ODE w.r.t. $p(s)$.

Similarly, figure out the expression for $z'(s)$

$$z'(s) = \sum_{j=1}^n u_{x_j}(x(s)) \cdot x_j'(s) \tag{51}$$
$$= \sum_{j=1}^n p_j(s) \cdot x_j'(s) \tag{52}$$

To summarize,

$$\begin{cases} x'(s) = D_pF(p(s),z(s),x(s)) \\ z'(s) = D_pF(p(s),z(s),x(s)) \cdot p(s) \\ p'(s) = -D_xF(p(s),z(s),x(s)) - D_zF(p(s),z(s),x(s)) \cdot p(s) \end{cases} \quad (p,x \in \mathbb{R}^n, z \in \mathbb{R}) \tag{53}$$

The ODEs are called **the characteristic equations (altogether $2n+1$ equations)** for $F(Du,u,x) = 0$, with the dot in the equation of $z'(s)$ having the meaning of dot product. The inference above is concluded as a theorem.

Theorem 1. *If $u \in C^2$ solves the PDE $F(Du,u,x) = 0$ in $\Omega \subset \mathbb{R}^n$, assume that $x(s)$ solves the equation $x_j'(s) = \frac{\partial F(p(s),z(s),x(s))}{\partial p_j}$ then $p(s), z(s)$ solves the original PDE.*

This is a generalization of the method of c.h. for the linear first-order PDE (where we don't have to use function p). (verified afterwards)

Now to solve the c.h. ODE system, still have to **impose initial values** $x(0), z(0), p(0)$. The initial values for x, z can be found easily, while $p(0)$ **remains to be a problem**.

$$x(0) \in \Gamma \tag{54}$$
$$z(0) = u(x(0)) = g(x(0)) \tag{55}$$

Motivation for the General Method of Characteristics

The statements above are mathematically correct. However, it's necessary to explain why we are doing all the things here and to make these statements intuitive.

It can seen above that the most important thought of the method of c.h. is to find some c.h. curves along which the function value can be easily determined. For first-order linear PDE, this point is clear since the linear combination of u_x, u_t can always be written as a directional derivative (the direction may contain x but has nothing to do with u). In other words, if we write the equation

$$u_t + bu_x = u \tag{56}$$

in the form of general first-order equation, then

$$F(Du, u, x) = 0 \tag{57}$$

$$F(p, z, x) = p \cdot b - z \tag{58}$$

it's easy to see that D_pF **is actually the direction we hope to follow**. That's exactly why we set $x'(s) = D_pF$ in the c.h. ODE to capture this special direction. Then $z(s)$ as the function u along $x(s)$ has the natural structure as $D_pF \cdot p$.

The dealing with $p(s)$ is more subtle. Since p is the function Du along $x(s)$, its derivative has to have something to do with the second-order derivatives of u. However, by taking derivatives of the original PDE w.r.t. x, the second-order derivatives of u are cancelled and replaced with first-order derivatives of F. Such operation enables us to keep the equation as a first-order one, but as the price to pay, C^2 assumption is required (for the existence of second-order derivatives and that $u_{x_ix_j} = u_{x_jx_i}$).

Example

Let's see how this method works for first-order linear PDE

$$B(x) \cdot Du + c(x) \cdot u = 0, \ (B \in \mathbb{R}^n, c, d \in \mathbb{R}) \tag{59}$$

First write out the general form:

$$F(Du, u, x) = 0 \tag{60}$$

$$F(p, z, x) = B(x) \cdot p + c(x) \cdot z \tag{61}$$

To set up the c.h. equations, note that first we should capture the c.h. direction, i.e.

$$x'(s) = D_pF = B(x(s)) \tag{62}$$

Then, set up $z(s) = u(x(s))$ as

$$z'(s) = \sum_{j=1}^{n} u_{x_j}(x(s))x'_j(s) = p(s) \cdot B(x(s)) \tag{63}$$

At last, set up $p(s) = Du(x(s))$ by taking the derivative w.r.t. x of the original PDE

$$\sum_i \frac{\partial F}{\partial p_i} u_{x_i x_j} + D_z F \cdot u_{x_j} + \frac{\partial F}{\partial x_j} = 0 \tag{64}$$

$$p_j'(s) = \sum_{i=1}^{n} u_{x_i x_j}(x(s)) \cdot x_i'(s) \tag{65}$$

$$p'(s) + D_x F + D_z F \cdot p(s) = 0 \tag{66}$$

$$p'(s) + z(s) \cdot \nabla c(x(s)) + \sum_{j=1}^{n} p_j(s) \cdot \nabla B_j(x(s)) + c(x(s)) \cdot p(s) = 0 \tag{67}$$

As a result, we get the c.h. equations

$$\begin{cases} x'(s) = B(x(s)) \\ z'(s) = p(s) \cdot B(x(s)) \\ p'(s) + z(s) \cdot \nabla c(x(s)) + \sum_{j=1}^{n} p_j(s) \cdot \nabla B_j(x(s)) + c(x(s)) \cdot p(s) = 0 \end{cases} \tag{68}$$

This equations seems to be very complicated. However, we might notice that the original PDE is

$$B(x(s)) \cdot p(s) + c(x(s)) \cdot z(s) = 0 \tag{69}$$

It's so good to find that **the c.h. equations reduce and $p(s)$ is not necessary any longer!**

$$\begin{cases} x'(s) = B(x(s)) \\ z'(s) = -c(x(s)) \cdot z(s) \end{cases} \tag{70}$$

Note that this is exactly what we would expect to get from the method of c.h. of the first-order linear PDE, so these two methods are actually consistent! The simplicity of linearity can thus be explained as the simplicity of c.h. equations, where it avoids the dependency on the initial value of $p(s)$ which is hard to derive.

Example

Let's compute a more specific example using the general method of c.h.

$$\begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u \ (x = (x_1, x_2) \in U) \\ u(x) = g(x) \ (x \in \Gamma) \end{cases} \tag{71}$$

where U is the open first quadrant and $\Gamma = \{x_1 > 0, x_2 = 0\}$.

Let's write it in the general form

$$F(Du, u, x) = 0 \tag{72}$$
$$F(p, z, x) = B(x) \cdot p - z \ (p, x \in \mathbb{R}^2, z \in \mathbb{R}) \tag{73}$$
$$B(x) = (-x_2, x_1) \tag{74}$$

capture the c.h. direction

$$x'(s) = B(x) \tag{75}$$

form the equation of $z(s) = u(x(s))$

$$z'(s) = p(s) \cdot B(x) \tag{76}$$

and the equation of $p(s) = Du(x(s))$

$$p'(s) + D_zF \cdot p(s) + D_xF = 0 \tag{77}$$
$$D_zF = -1, D_xF = p_1 \nabla B_1 + p_2 \nabla B_2 \tag{78}$$

Now let's plug in $B(x)$ and derive the c.h. equations: (note that this is linear, so only equations for $x(s), z(s)$ are needed)

$$\begin{cases} x_1'(s) = -x_2 \\ x_2'(s) = x_1 \\ z'(s) = z(s) \end{cases} \tag{79}$$

Solve this out to get:

$$\begin{cases} x_1(s) = C \cos s \\ x_2(s) = C \sin s \\ z(s) = De^s \ (D > 0) \end{cases} \tag{80}$$

with C, D as any fixed constant.

Now notice that $x(0) \in \Gamma$, so $z(0) = u(x(0)) = g(x(0)) = g(C, 0)$. As a result, $D = g(C, 0)$. On knowing that

$$u(C \cos s, C \sin s) = g(C, 0)e^s \tag{81}$$

do the transformation such that $\begin{cases} x_1 = C\cos s \\ x_2 = C\sin s \end{cases}$ to get the **solution** that:

$$u(x_1, x_2) = g\left(\sqrt{x_1^2 + x_2^2}, 0\right) e^{\arctan \frac{x_2}{x_1}} \tag{82}$$

Let's do a final check to see whether this solution satisfies the c.h. equation of $p(s)$ which has been eliminated. The equation should be:

$$\begin{cases} p_1'(s) - p_1(s) + p_2(s) = 0 \\ p_2'(s) - p_2(s) - p_1(s) = 0 \end{cases} \tag{83}$$

Now we know $p(s) = Du(x(s))$, so

$$Du(x) = \begin{bmatrix} e^{\arctan \frac{x_2}{x_1}} \left[\frac{x_1}{\sqrt{x_1^2+x_2^2}} g_1'(\sqrt{x_1^2 + x_2^2}, 0) - \frac{x_2}{x_1^2+x_2^2} g(\sqrt{x_1^2 + x_2^2}, 0)\right] \\ e^{\arctan \frac{x_2}{x_1}} \left[\frac{x_2}{\sqrt{x_1^2+x_2^2}} g_1'(\sqrt{x_1^2 + x_2^2}, 0) + \frac{x_1}{x_1^2+x_2^2} g(\sqrt{x_1^2 + x_2^2}, 0)\right] \end{bmatrix} \tag{84}$$

$$p(s) = \begin{bmatrix} e^s \left[\cos s \cdot g_1'(C, 0) - \sin s \cdot g(C, 0)\right] \\ e^s \left[\sin s \cdot g_1'(C, 0) + \cos s \cdot g(C, 0)\right] \end{bmatrix} \tag{85}$$

It's then quite obvious to see that the c.h. equation for $p(s)$ actually holds (but we won't have to deal with it).

First-Order Quasi-Linear PDE

Actually, not only the first-order linear PDE can get rid of the function p, a special type of PDE called **first-order quasi-linear PDE** also has no dependency on p. The general form of this kind of PDE is

$$B(x, u) \cdot Du + c(x, u) = 0 \tag{86}$$

in that it's linear w.r.t. the highest order derivative of the unknown function (which is Du here).

Write it in the general form:

$$F(Du, u, x) = 0 \tag{87}$$

$$F(p, z, x) = B(x, z) \cdot p + c(x, z) \tag{88}$$

capture the c.h. direction $x(s)$:

$$x'(s) = B(x(s), z(s)) \tag{89}$$

get the equation for $z(s) = u(x(s))$:

$$z'(s) = B(x(s), z(s)) \cdot p(s) \tag{90}$$

use the original PDE to get the c.h. equations

$$\begin{cases} x'(s) = B(x(s), z(s)) \\ z'(s) = -c(x(s), z(s)) \end{cases} \tag{91}$$

which also has nothing to do with $p(s)$.

Example

Let's look at a first-order quasi-linear PDE:

$$\begin{cases} u_{x_1} + u_{x_2} = u^2 \ (x \in U = \{x_2 > 0\}) \\ u(x) = g(x) \ (x \in \Gamma = \partial U) \end{cases} \tag{92}$$

with the non-linearity lying in u^2 but still quasi-linear.

Let's follow the steps to use the method of c.h., write it in general form

$$F(Du, u, x) = 0 \tag{93}$$

$$F(p, z, x) = B \cdot p - z^2 \tag{94}$$

$$B = (1, 1) \tag{95}$$

capture the c.h. direction

$$x'(s) = B \tag{96}$$

and write out the equation for $z(s) = u(x(s))$

$$z'(s) = p(s) \cdot B = z^2(s) \tag{97}$$

the c.h. equations are

$$\begin{cases} x_1'(s) = 1 \\ x_2'(s) = 1 \\ z'(s) = z^2(s) \end{cases} \tag{98}$$

Solve it to know that

$$z(s) = -\frac{1}{s+C} \tag{99}$$

$$x_1(s) = s + D \tag{100}$$

$$x_2(s) = s + E \tag{101}$$

figure out the initial value that

$$x(-E) = (D-E, 0) \in \Gamma \tag{102}$$

$$z(-E) = \frac{1}{E-C} \tag{103}$$

$$z(-E) = u(x(-E)) = u(D-E, 0) = g(D-E, 0) \tag{104}$$

So we get the final solution

$$u(s+D, s+E) = -\frac{1}{s+E-\frac{1}{g(D-E,0)}} \tag{105}$$

$$x_1 - x_2 = D - E \tag{106}$$

$$u(x_1, x_2) = -\frac{1}{x_2 - \frac{1}{g(x_1-x_2,0)}} \tag{107}$$

Nonlinear First-Order PDE

When the equation is even not quasi-linear, $p(s)$ will be necessary for us to solve out the c.h. equations. Let's look at an example to illustrate **how to construct initial value for function** p.

$$\begin{cases} u_{x_1} \cdot u_{x_2} = u \ (x \in U = \{x_1 > 0\}) \\ u(x) = x_2^2 \ (x \in \Gamma = \partial U) \end{cases} \tag{108}$$

This equation is even not quasi-linear since there is the product of two partial derivatives. Let's turn it into the general form

$$F(Du, u, x) = 0 \tag{109}$$

$$F(p, z, x) = p_1 p_2 - z \tag{110}$$

capture the ch. direction

$$x'(s) = D_p F = (p_2, p_1) \tag{111}$$

set up the equation for $z(s)$ (The original PDE tells us that $z = p_1 p_2$)

$$z'(s) = 2p_1(s)p_2(s) = 2z(s) \tag{112}$$

and the equation for $p(s)$

$$p'(s) = -D_z F \cdot p(s) - D_x F \tag{113}$$
$$p'(s) = p(s) \tag{114}$$

so the c.h. equations are

$$\begin{cases} x_1'(s) = p_2(s) \\ x_2'(s) = p_1(s) \\ z'(s) = 2z(s) \\ p_1'(s) = p_1(s) \\ p_2'(s) = p_2(s) \end{cases} \tag{115}$$

Now we assume that $x(0) = (0, C) \in \Gamma$, then

$$z(0) = u(x(0)) = u(0, C) = C^2 \tag{116}$$

However, the problem here is that the simplest equation is the one w.r.t. $p'(s)$ but we know nothing about the initial value of p. Actually, the information for p is already given, but hidden in the other conditions. **Turn back to the original PDE** to get

$$p_1(0)p_2(0) = z(0) = C^2 \tag{117}$$
$$\tag{118}$$

The last remaining initial value condition can be figured out by **taking derivative on both sides of the initial value condition of the original PDE w.r.t.** x_2

$$u_{x_2}(x(0)) = 2x_2(0) \ (x(0) \in \Gamma) \tag{119}$$
$$u_{x_2}(x(0)) = p_2(0) = 2x_2(0) = 2C \tag{120}$$

As a result, now we all the c.h. equations and initial value conditions and these equations can be solved

$$\begin{cases} p_1(s) = \frac{C}{2}e^s \\ p_2(s) = 2Ce^s \\ z(s) = C^2e^{2s} \\ x_1(s) = 2C(e^s - 1) \\ x_2(s) = \frac{C}{2}(e^s + 1) \end{cases} \tag{121}$$

As a result, get the final solution by

$$\begin{cases} e^s = \frac{4x_2+x_1}{4x_2-x_1} \\ C = \frac{4x_2-x_1}{4} \end{cases} \tag{122}$$

$$u(x_1, x_2) = \frac{(x_1 + 4x_2)^2}{16} \tag{123}$$

General Theory for Characteristic Method

For general case, assume that Γ is parameterized by $x = f(y)$ with parameter $y \in D \subset \mathbb{R}^{n-1}$. Fix $y_0 \in D, x_0 = f(y_0), z_0 = u(x_0) = g(x_0) = g(f(y_0)) \overset{def}{=} h(y_0)$ (since $x_0 \in \Gamma$). The reason to change variable x into y is that x may have to be on some surface (it depends on how the Γ looks) but y has no such restrictions. In other words, if we deal with x directly, we may have to operate on manifolds while changing the variable as y flattens the boundary Γ so it's now analysis in the Euclidean space $\mathbb{R}^{n-1}$.

The PDE at x_0 now becomes

$$F(Du(x_0), u(x_0), x_0) = 0 \tag{124}$$

$$F(p_0, z_0, f(y_0)) = 0 \tag{125}$$

with $p_0 = Du(x_0) = Dg(x_0)$ since $x_0 \in \Gamma$.

Note that it's necessarily true that under such circumstance (in the following context, use **subscript for partial derivative** and **superscript for coordinates**)

$$h_{y_j}(y_0) = \frac{\partial(g \circ f)(y)}{\partial y_j}\Big|_{y=y_0} \tag{126}$$

$$= \sum_{i=1}^n \frac{\partial g(f(y))}{\partial f_i} \cdot \frac{\partial f_i(y)}{\partial y_j}\Big|_{y=y_0} \tag{127}$$

$$= \sum_{i=1}^n \frac{\partial g(x_0)}{\partial x_i} \cdot \frac{\partial f_i(y_0)}{\partial y_j} = \sum_{i=1}^n \frac{\partial g(x_0)}{\partial f_i} \cdot \frac{\partial f_i(y_0)}{\partial y_j} = \sum_{i=1}^n p_0^i \cdot f_{y_j}^i(y_0) \tag{128}$$

Actually, given $y_0 \in D$, if the following conditions are satisfied

$$\begin{cases} \forall j = 1, 2, ..., n-1, h_{y_j}(y_0) = \sum_{i=1}^n p_0^i \cdot f_{y_j}^i(y_0) \\ F(p_0, h(y_0), f(y_0)) = 0 \end{cases} \tag{129}$$

we say p_0 is **admissible** at y_0. It's quite obvious to see that there are in all n equations for the unknowns and these conditions are called **compatibility conditions**. They are the conditions that p_0 must satisfy for the PDE to hold at $y = y_0$. The admissible conditions are important since it provides the **minimum condition to satisfy for an appropriate initial condition for** p_0, y_0. (Imagine the situation where initial value p_0 is provided but not satisfy the compatibility conditions, there's no way for the PDE to have solution! That's why we always add this condition when constructing the solution.)

To simplify the notations, denote

$$\begin{cases} \mathscr{F}_j(p, y) = \sum_{i=1}^n f_{y_j}^i(y) p^i - h_{y_j}(y) \ (j = 1, ..., n-1) \\ \mathscr{F}_n(p, y) = F(p, h(y), f(y)) \end{cases} \qquad (p \in \mathbb{R}^n, y \in D \subset \mathbb{R}^{n-1}) \tag{130}$$

$$\mathscr{F} = (\mathscr{F}_1, ..., \mathscr{F}_n) : \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{R}^n \tag{131}$$

then (p_0, y_0) is admissible iff $\mathscr{F}(p_0, y_0) = 0$. (The natural compatibility condition provides us with n equations to satisfy, which will be important in future use)

Call an **admissible** (p_0, y_0) **to be non-characteristic** if $\det \left(\frac{\partial \mathscr{F}(p,y)}{\partial p} \right) \Big|_{(p_0, y_0)} \neq 0$, i.e.

$$\det \begin{bmatrix} f_{y_1}^1(y_0) & ... & f_{y_1}^n(y_0) \\ ... & ... & \\ f_{y_{n-1}}^1(y_0) & ... & f_{y_{n-1}}^n(y_0) \\ F_{p_1}(p_0, z_0, x_0) & ... & F_{p_n}(p_0, z_0, x_0) \end{bmatrix} \neq 0 \tag{132}$$

An explanation is required for this definition of non-characteristic. A direct one is based on the observation that

$$\begin{bmatrix} F_{p_1}(p_0, z_0, x_0) & ... & F_{p_n}(p_0, z_0, x_0) \end{bmatrix} \tag{133}$$

is the c.h. direction (tangent direction of c.h. curve) at $x_0 \in \Gamma$ (recall the c.h. direction $x'(s) = D_p F$ mentioned above). The other rows

$$\begin{bmatrix} f_{y_j}^1(y_0) & ... & f_{y_j}^n(y_0) \end{bmatrix} \tag{134}$$

typically stands for $\frac{\partial f}{\partial y_j}\Big|_{y=y_0}$, i.e. the tangent vectors for surface Γ at point x_0. A **geometric interpretation** for the non-c.h. property is that: **the tangent vector of the c.h. curve at** x_0 **is not in the tangent space of surface** Γ **at** x_0**.**

Nevertheless, there is also an **analytic interpretation** of this property. Note that by implicit function theorem, $\det\left(\frac{\partial \mathscr{F}(p,y)}{\partial p}\right)\Big|_{(p_0,y_0)} \neq 0$ tells us that locally near (p_0, y_0) (exists a neighborhood), there exists a function $p = q(y)$ such that

$$p_0 = q(y_0) \tag{135}$$

$$\mathscr{F}(q(y), y) = \mathscr{F}(p_0, y_0) = 0 \ (\forall y \textit{ close to } y_0) \tag{136}$$

providing the intuition that the non-c.h. condition enables us to **build a local functional relationship between** y_0 **and** p_0. Note that the equations given by the compatibility conditions $\mathscr{F}(p_0, y_0) = 0$ help construct the function relationship in the implicit function theorem. This point of view is important because we are trying to turn everything into the setting of y, e.g. $x = f(y)$, and if now p_0 can be written as a function of y_0, we will be only working with y locally.

Week 3

From now on, assume that (p_0, y_0) admissible and non-c.h. As stated above, exists a function q defined locally at y_0 such that $p_0 = q(y_0)$. Let $(p(s), z(s), x(s))$ be the solution of c.h. ODEs with initial values $p(0) = q(y), z(0) = h(y), x(0) = f(y)$ for $y \in U$, which is a neighborhood of y_0 where q is defined, we can write the solution as $(p(s, y), z(s, y), x(s, y))$ (the solution depends on both s and y).

Lemma 1. *Let (p_0, y_0) be admissible and non-c.h., $x_0 = f(y_0)$, then exists an open interval I of 0 and a neighborhood U of y_0, a neighborhood V of x_0, such that $\forall x \in V$, there exists unique $s = s(x) \in I, y = y(x) \in U$, such that $x = x(y, s)$. Moreover, $s(x), y(x)$ are C^2 in x.*

Proof. Want to apply the **inverse function theorem** to get the dependency of s, y on x. Note that the only condition to satisfy is the existence of the inverse of the Jacobian at $s = 0, y = y_0$. By the functional relationship between x, y and x, s, the Jacobian can be calculated as follows:

$$x = f(y), x = x(s) \tag{137}$$

$$\frac{\partial x_j}{\partial s} = x_j'(s)\Big|_{s=0,y=y_0} = F_{p_j}(p_0, z_0, x_0) \tag{138}$$

$$\frac{\partial x_j}{\partial y_i} = f^j_{y_i}(y_0) \tag{139}$$

$$\frac{\partial x(y, s)}{\partial (y, s)}\Big|_{s=0,y=y_0} = \left(\frac{\partial \mathscr{F}(p, y)}{\partial p}\right)^T\Big|_{y=y_0} \tag{140}$$

By non-c.h. property, the Jacobian is invertible, meaning that the inverse function theorem holds. That's why there exists unique $s = s(x), y = y(x)$ locally near $s = 0, y = y_0$ and the C^2 property is preserved.

$\square$

Note that originally $u(x) = z(y, s)$ (the parametrization of c.h. curve and the initial value condition) but now we can locally set up s, y as functions of x, so $z(y, s) = z(y(x), s(x))$, therefore we find the solution to the original PDE with condition on Γ.

Theorem 2. *Let (p_0, y_0) be admissible and non-c.h., then function u defined by $u(x) = z(s(x), y(x))$ solves the PDE $F(Du, u, x) = 0$ on V with the Cauchy condition $u(x) = g(x)$ on surface $\Gamma \cap V$.* ***(Local existence theorem)***

Proof. Fix y such that $f(y) \in \Gamma$ is close to x_0 and solve the c.h. ODEs to get the solutions $p(s) = p(y, s), z(s) = z(y, s), x(s) = x(y, s)$. Now if $f(y)$ is close enough to x_0, then by compatibility conditions,

$$F(p(y, 0), z(y, 0), x(y, 0)) = F(p_0, h(y_0), f(y_0)) = 0 \tag{141}$$

and also

$$\frac{\partial}{\partial s}F(p(y,s),z(y,s),x(y,s)) = \sum_{j=1}^{n} D_{p_j}F\cdot p_j'(s) + D_zF\cdot z'(s) + \sum_{j=1}^{n} D_{x_j}F\cdot x_j'(s) \tag{142}$$

$$= \sum_{j=1}^{n} D_{p_j}F\cdot(-D_zF\cdot p_j(s) - D_{x_j}F) + D_zF\cdot(\sum_{j=1}^{n} D_{p_j}F\cdot p_j(s)) + \sum_{j=1}^{n} D_{x_j}F\cdot D_{p_j}F \tag{143}$$

$$= 0 \tag{144}$$

by replacing terms using c.h. ODEs.

Now that $F(p(y,0),z(y,0),x(y,0)) = 0$, $\frac{\partial}{\partial s}F(p(y,s),z(y,s),x(y,s)) = 0$, it's natural to conclude that

$$\forall s\in I, F(p(y,s),z(y,s),x(y,s)) = 0 \tag{145}$$

By applying the local intervibility lemma above changing y,s into x, we conclude that

$$\forall x\in V, F(p(x),u(x),x) = 0 \tag{146}$$

Now to conclude that this is actually a solution to the original PDE $F(Du,u,x) = 0$, the only step left is to prove that $\forall x\in V, p(x) = Du(x)$.

$$\forall j, u_{x_j}(x) = \frac{\partial}{\partial x_j}z(s(x),y(x)) \tag{147}$$

$$= z_s\cdot s_{x_j} + \sum_{i=1}^{n-1} z_{y_i}y^i_{x_j} \tag{148}$$

It's clear to us that by c.h. ODEs,

$$z_s = \sum_{j=1}^{n} p^j(y,s)\cdot x^j_s(y,s) \tag{149}$$

If we can show that

$$z_{y_i} = \sum_{j=1}^{n} p^j(y,s)\cdot x^j_{y_i}(y,s) \tag{150}$$

then the theorem is proved since

$$\forall j, u_{x_j}(x) = \sum_{k=1}^{n} p^k \cdot x_s^k \cdot s_{x_j} + \sum_{i=1}^{n-1} \sum_{k=1}^{n} p^k \cdot x_{y_i}^k \cdot y_{x_j}^i \tag{151}$$

$$= \sum_{k=1}^{n} p^k \left(x_s^k \cdot s_{x_j} + \sum_{i=1}^{n-1} x_{y_i}^k \cdot y_{x_j}^i \right) \tag{152}$$

$$= \sum_{k=1}^{n} p^k x_{x_j}^k = \sum_{k=1}^{n} p^k \delta_{jk} = p^j \tag{153}$$

so $\forall x \in V, p(x) = Du(x)$ and $F(Du, u, x) = 0$. Note that here x_k is a function of s, y and s, y depends on x_j, by using the chain rule one would get

$$x_s^k \cdot s_{x_j} + \sum_{i=1}^{n-1} x_{y_i}^k \cdot y_{x_j}^i = x_{x_j}^k \overset{def}{=} \frac{\partial x_k}{\partial x_j} \tag{154}$$

This tells us that under such conditions, **the solution to the c.h. ODEs is the locally unique solution to the original PDE**.

The last thing to prove is the expression for z_{y_i}, the approach is to set $r^i(s) = z_{y_i} - \sum_{j=1}^{n} p^j(y, s) \cdot x_{y_i}^j(y, s)$ and prove that it's 0. The strategy is to form an ODE w.r.t. $r^i(s)$ and to argue that such ODE with given initial conditions must give the zero solution. (Evans P108 for specific calculations) The constructed ODE is

$$\frac{d}{ds} r^i(s) = -D_z F \cdot r^i(s) \tag{155}$$

$$r^i(0) = 0 \tag{156}$$

so $\forall s \in I, \forall i, r^i(s) = 0$, it's proved.

□

Remark. *Although the calculations are too much, we shall capture the spirit of the theorem. If there is a* ***first-order PDE*** *with compatible boundary value condition (****admissible*** *–* ***compatibility conditions****) and at a certain point* x_0 *on the boundary, the tangent vector at* x_0 *of the c.h. curve is not in the tangent space at* x_0 *of the boundary surface* Γ *(****non-characteristic****), then the* ***solution*** *to the PDE* ***locally uniquely exists around*** x_0*, and such solution can be derived by* ***solving the c.h. ODEs*** *(method of c.h.).*

Example

Let's look at some cases where the non-c.h. condition fails. The PDE is on $\mathbb{R}^2$

$$x \cdot u_x + y \cdot u_y = u \tag{157}$$

with the Cauchy condition given on the diagonal

$$\forall \tau, u(\tau,\tau) = \tau \tag{158}$$

This is a first-order linear PDE and we can write out the general form of this Cauchy problem:

$$\begin{aligned} F(Du, u, x) &= 0 && (159)\\ F(p,z,x) &= x_1p_1 + x_2p_2 - z && (160)\\ \Gamma &= \{(x,y)|x=y\} && (161) \end{aligned}$$

compute the c.h. ODEs

$$\begin{cases} x'(s) = x(s) \\ y'(s) = y(s) \\ z'(s) = z(s) \end{cases} \tag{162}$$

notice that $\frac{y(s)}{x(s)}$ is always a constant, so all possible c.h. curves are the lines passing through the origin. However, fix any point $x_0 \in \Gamma$, the c.h. curve is then fixed as $y = x$, which is exactly the same as Γ, which violates the non-c.h. condition! This is how we see that non-c.h. does not hold from the geometric point of view. From the analytic point of view, calculate the tangent vectors: ($x = f^1(\tau) = \tau, y = f^2(\tau) = \tau$ gives the parametrization of Γ)

$$\begin{aligned} \forall (a,a) \in \Gamma, \frac{\partial \mathscr{F}}{\partial p} &= \begin{bmatrix} f^1_\tau & f^2_\tau \\ F_{p_1} & F_{p_2} \end{bmatrix} && (163)\\ &= \begin{bmatrix} 1 & 1 \\ x & y \end{bmatrix} && (164)\\ \frac{\partial \mathscr{F}}{\partial p}\Big|_{(a,a)} &= \begin{bmatrix} 1 & 1 \\ a & a \end{bmatrix} && (165) \end{aligned}$$

It's obvious that the matrix is not invertible. By definition, non-c.h. condition fails.

So what's the consequence of the failure of non-c.h. condition? Solve the c.h. equations to see that the solution changes depending on the initial values. If consider $x(0) = 0, y(0) = 0, z(0) = u(0,0) = 0$ then the solution is trivial

$$\begin{cases} x(s) = 0 \\ y(s) = 0 \\ z(s) = 0 \end{cases} \tag{166}$$

If consider $x(a) = a, y(a) = a, z(a) = u(a, a) = a$ $(a > 0)$ then the solution is

$$\begin{cases} x(s) = a \cdot e^{s-a} \\ y(s) = a \cdot e^{s-a} \\ z(s) = a \cdot e^{s-a} \end{cases} \tag{167}$$

As a result,

$$u(a \cdot e^{s-a}, a \cdot e^{s-a}) = a \cdot e^{s-a} \ (\forall a \in \mathbb{R}) \tag{168}$$

$$u(x, y) = \alpha x + (1 - \alpha) y \ (\forall \alpha \in \mathbb{R}) \tag{169}$$

the solution is any linear combination of x and y. The local existence theorem fails (if it holds there should be a unique solution locally) and now there are **infinitely many solutions due to the failure of non-c.h. condition**.

Example

This example has exactly the same PDE as above, but the boundary value condition changes into

$$\forall \tau, u(\tau, \tau) = 1 \tag{170}$$

It's easy to see that the c.h. ODEs and the Γ all remain the same so non-c.h. condition still fails. However, if we still try to solve the c.h. ODEs, we will find that

$$\begin{cases} x(s) = C_x \cdot e^{s} \\ y(s) = C_y \cdot e^{s} \\ z(s) = C_z \cdot e^{s} \end{cases} \tag{171}$$

with $\forall s, z(s) = 1$, which is obviously impossible. As a result, there's no solution to this c.h. ODE. It's natural to assert that there should also be no solution to the original PDE, but it requires more work.

If there is a solution u to the original PDE, consider $h(x) = u(x, x)$ to be the function along the line $y = x$. As a result, $h'(x) = u_x(x, x) + u_y(x, x)$. Since the boundary value condition tells us $\forall x, h(x) = 1$, we know that $h'(x) \equiv 0$.

$$\forall x, u_x(x, x) + u_y(x, x) = 0 \tag{172}$$

Since u is a solution, $\forall x \neq 0, u_x(x, x) + u_y(x, x) = \frac{u(x,x)}{x}$, that is to say

$$\forall x \neq 0, u(x, x) = 0 \tag{173}$$

which is a contradiction with the boundary value condition that $u(x, x) = 1$!

Example

This example still has exactly the same PDE as above, but the boundary value condition changes into

$$u(x,0) = g(x) \tag{174}$$

Now the c.h. ODEs remain the same but the boundary curve changes into

$$\Gamma = \{(x,y)|y=0\} \tag{175}$$

which is the x-axis. So for any x_0 on this curve, the c.h. curve is always the x-axis, which violates the non-c.h. condition.

Solve the c.h. ODEs with initial value conditions $x(a) = a, y(a) = 0, z(a) = u(a,0) = g(a)$ to get

$$\begin{cases} x(s) = a \cdot e^{s-a} \\ y(s) = 0 \\ z(s) = g(a) \cdot e^{s-a} \end{cases} \tag{176}$$

so $u(a \cdot e^{s-a}, 0) = g(a) \cdot e^{s-a}$, get the solution:

$$u(x,y) = \frac{g(a)}{a} x \tag{177}$$

$$u(0,0) = g(0) \cdot e^s \tag{178}$$

This might seem weird, but when $a = 0$ we cannot eliminate the s contained in $u(0,0)$. As a result, we might guess that $g(0) = 0$ has to hold so that this PDE has a solution. Moreover, in order to get rid of $\frac{g(a)}{a}$ for any $a \neq 0$, we might guess that $\frac{g(a)}{a}$ has to be a fixed constant, i.e. g **has to be a linear function such that the PDE has solution**.

Let's check whether this condition makes sense. When g is linear, i.e. $g(x) = kx$, then

$$u(x,y) = kx \tag{179}$$

and we can check that this is a solution to the PDE.

On the other hand, if there exists u as the solution to the PDE, then consider $g(x) = u(x,0)$ along the x-axis,

$$g'(x) = u_x(x,0) \tag{180}$$

the original PDE tells us that

$$x \cdot u_x(x,0) = u(x,0) \tag{181}$$

to derive an ODE w.r.t. g

$$x \cdot g'(x) = g(x) \tag{182}$$

and it's obvious that all solutions to this ODE are linear function, so g has to be linear.

To sum up, we proved that this PDE with boundary value condition

$$\begin{cases} x \cdot u_x + y \cdot u_y = u \\ u(x,0) = g(x) \end{cases} \tag{183}$$

has solution if and only if g is a linear function.

Remark. *The three examples show us that when non-c.h. condition is violated, there might exists infinitely many solutions or no solution for a PDE. There also might be the case that the existence of solution depends on the properties of a certain function involved in the boundary value condition.*

Example

The last example is one in multi-dimensional space. Consider the PDE with boundary value conditions

$$\begin{cases} x \cdot u_x = \alpha u \ (x \in \mathbb{R}^n) \\ u(x_1, ..., x_{n-1}, 1) = h(x_1, ..., x_{n-1}) \end{cases} \tag{184}$$

To write in the general form,

$$F(Du, u, x) = 0 \tag{185}$$

$$F(p, z, x) = x \cdot p - \alpha z \tag{186}$$

$$\Gamma = \{x | x_n = 1\} \tag{187}$$

First judge whether the non-c.h. condition holds. The tangent vector of the c.h. curve is

$$D_p F = x \tag{188}$$

it's clear that for any $x_0 \in \Gamma$, x_0 won't be in the tangent space at x_0 of Γ (just draw a plot, easy to see), so the non-c.h. condition holds, and local unique existence of the solution is ensured! To solve out the locally unique solutions, the method of c.h. must work, ensured by the theorem stated above.

Write out the c.h. ODEs

$$\begin{cases} x'(s) = x(s) \\ z'(s) = p(s) \cdot x(s) = \alpha z(s) \end{cases} \tag{189}$$

and assume that $\exists s_0, x_n(s_0) = 1$, so $z(s_0) = h(x_1(s_0), ..., x_{n-1}(s_0))$. Solve out the solution to c.h. ODEs with those initial value conditions

$$\begin{cases} x_1(s) = x_1(s_0) \cdot e^{s-s_0} \\ ... \\ x_{n-1}(s) = x_{n-1}(s_0) \cdot e^{s-s_0} \\ x_n(s) = e^{s-s_0} \\ z(s) = h(x_1(s_0), ..., x_{n-1}(s_0)) \cdot e^{\alpha(s-s_0)} \end{cases} \tag{190}$$

Eliminate the s and replace with x to get the solution

$$u(x_1(s_0) \cdot e^{s-s_0}, ..., x_{n-1}(s_0) \cdot e^{s-s_0}, e^{s-s_0}) = h(x_1(s_0), ..., x_{n-1}(s_0)) \cdot e^{\alpha(s-s_0)} \tag{191}$$

$$u(x) = h\left(\frac{x_1}{x_n}, ..., \frac{x_{n-1}}{x_n}\right) \cdot (x_n)^\alpha \tag{192}$$

Actually we see that this is not only a **local** solution but also a **global** one for $\alpha \geq 0$. However, if $\alpha < 0$, there will be singularity for the solution $u(x)$ since it won't be defined at the origin so the local existence of the solution in a neighborhood of the origin will be a problem (the solution blows up).

We can actually see that when it comes to the local existence of the solution in a neighborhood of the origin, the only "nice" (not blows up) solution is the trivial solution $u \equiv 0$. (Since the method of c.h. must work because the PDE is non-c.h., eliminating the nontrivial solutions given above, ther's only the trivial zero solution left)

Laplace's Equation

Laplace's equation is

$$\Delta u = 0 \tag{193}$$

and **Poisson's equation** is the nonhomogeneous case

$$-\Delta u = f \tag{194}$$

It's natural to assume that $u \in C^2$ and to notice that $\Delta = div \cdot \nabla$ (the divergence of gradient is Laplacian).

Remark. *The negative sign in the Poisson's equation refers to the fact that if $u : \mathbb{R}^n \to \mathbb{R}$ is always 0 on the boundary of integration domain (no boundary terms),*

$$\int ||\nabla u||^2 \, dx = \int \sum_j [u_{x_j}(x)]^2 \, dx = \sum_j \int u_{x_j}(x) \, du(x_j) dx_1 ... \overline{dx_j} ... dx_n \tag{195}$$

$$= -\sum_j \int u(x) \cdot u_{x_j, x_j}(x) \, dx = -\int (\Delta u) \cdot u \, dx \tag{196}$$

There are two main types of boundary value problems considered for these equations, **Dirichlet problem** and **Neumann problem**. In the setting of Dirichlet problem, look for $u : \Omega \to \mathbb{R}, u \in C^2(\Omega), u \in C(\overline{\Omega})$ with boundary value condition $\forall x \in \partial\Omega, u(x) = g(x)$. Here $f \in C(\Omega), g \in C(\partial\Omega)$. For the Neumann problem, look for $u : \Omega \to \mathbb{R}, u \in C^2(\Omega), u \in C^1(\overline{\Omega})$ with boundary value condition $\forall x \in \partial\Omega, \frac{\partial u}{\partial \nu}(x) = g(x)$. Here $f \in C(\Omega), g \in C(\partial\Omega)$. The notation $\frac{\partial u}{\partial \nu}$ stands for the directional derivative of u along direction ν, where ν is the unit outward normal vector on the boundary $\partial\Omega$.

Property of Harmonic Function

Notations: $B_r(x)$ for the ball centered at x with radius r. $U \subset\subset V$ means U **is relatively compact in** V, i.e. $\overline{U}$ is compact and $\overline{U} \subset V$. $|\cdot|$ means the volume of a certain area and dS means the surface integration.

Before stating the theorem, let's review some important theorems in integration theory.

Theorem 3. *(Divergence theorem) For vector field $F \in C^1$ on compact oriented n-manifold with boundary M, n as the outward unit normal vector and $\mu_{\partial M}$ as the volume form of ∂M,*

$$\int_M div(F)\, dx = \int_{\partial M} (F \cdot n)\mu_{\partial M} \tag{197}$$

Lemma 2. *(Volume of the ball and sphere in Euclidean space) Set $V_n(r)$ as the volume for n-dim ball with radius r, $A_n(r)$ as the volume for the boundary of n-dim ball with radius r (which is a $n-1$-dim sphere), then $A_n(1) = nV_n(1), A_n(r) = \frac{\partial V_n(r)}{\partial r}$.*

Theorem 4. *$u \in C^2(\Omega)$ and harmonic with $B_r(x) \subset\subset \Omega$, then*

$$\forall x \in \Omega, u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u\, dx = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u\, dS \tag{198}$$

Proof. Assume ν is the outward unit normal vector on the $n-1$-dim sphere, dS is the volume form of the $n-1$-dim sphere, then by the divergence theorem,

$$\int_{B_r(x)} \Delta u\, dx = \int_{\partial B_r(x)} \nabla u \cdot \nu\, dS \tag{199}$$

$$= \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}\, dS \tag{200}$$

$$= r^{n-1}\int_{\partial B_1(0)} \frac{\partial u}{\partial \nu}(x+ry)\, dS(y) \tag{201}$$

notice that for sphere $\frac{\partial u}{\partial \nu}(x+ry) = \frac{\partial u}{\partial r}(x+ry)$ since $\nu(x+ry) = \frac{y}{||y||_2}$

$$\int_{B_r(x)} \Delta u\, dx = r^{n-1}\int_{\partial B_1(0)} \frac{\partial u}{\partial r}(x+ry)\, dS(y) \tag{202}$$

$$= r^{n-1}\frac{\partial}{\partial r}\int_{\partial B_1(0)} u(x+ry)\, dS(y) \tag{203}$$

change the integration domain back into $B_r(x)$ and make use of the property that $A_n(1) = nV_n(1)$:

$$\frac{\partial}{\partial r}\left(\frac{1}{A_n(r)}\int_{\partial B_r(x)} u\,dS\right) = \frac{\partial}{\partial r}\left(\frac{r^{n-1}}{A_n(r)}\int_{\partial B_1(0)} u(x+ry)\,dS(y)\right) \tag{204}$$

$$= \frac{1}{nV_n(1)}\frac{\partial}{\partial r}\left(\int_{\partial B_1(0)} u(x+ry)\,dS(y)\right) \tag{205}$$

$$= \frac{r}{nV_n(r)}\int_{B_r(x)} \Delta u\,dx = 0 \tag{206}$$

This is telling us that $\frac{1}{A_n(r)}\int_{\partial B_r(x)} u\,dS$ is independent of r! Set $r \to 0$ then the Lebesgue differentiation theorem tells us that

$$\frac{1}{A_n(r)}\int_{\partial B_r(x)} u\,dS \to u(x)\ (r \to 0) \tag{207}$$

so we proved

$$\forall x, u(x) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u\,dS \tag{208}$$

To prove the other part, note that we can integrate on the sphere to get the integral on the ball and use the property that $A_n(r) = \frac{\partial V_n(r)}{\partial r}$

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} u\,dx = \frac{1}{|B_r(x)|}\int_0^r \int_{\partial B_t(x)} u\,dS\,dt \tag{209}$$

$$= \frac{1}{|B_r(x)|}\int_0^r |\partial B_t(x)| u(x)\,dt \tag{210}$$

$$= u(x) \tag{211}$$

ANOTHER PROOF:

Another proof can be provided as the one in textbook with the same thought but with more concise descriptions. Define $\phi(r)$ to be the mean value of u on a sphere with radius r

$$\phi(r) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,dS(y) \tag{212}$$

$$= \frac{1}{|\partial B_1(0)| r^{n-1}}\int_{\partial B_1(0)} u(x+rz) r^{n-1}\,dS(z) \tag{213}$$

$$= \frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} u(x+rz)\,dS(z) \tag{214}$$

now we hope to prove that $\phi(r) = u(x)$ for $B_r(x) \subset \Omega$. Since we would expect such $\phi(r)$ not to depend on r, let's calculate its derivative

$$\phi'(r) = \frac{d}{dr}\frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} u(x+rz)\,dS(z) \tag{215}$$
$$= \frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} \nabla u(x+rz)\cdot z\,dS(z) \tag{216}$$
$$= \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} \nabla u(y)\cdot \frac{y-x}{r}\,dS(y) \tag{217}$$
$$= \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} \nabla u\cdot \nu\,dS(y) \tag{218}$$
$$= \frac{1}{|\partial B_r(x)|}\int_{B_r(x)} \Delta u\,dy = 0 \tag{219}$$

here the interchange of derivative and integral is ensured by dominated convergence theorem and the last equation is by the divergence theorem. This proves that $\phi(r)$ is constant. To figure out what this constant is

$$\phi(r) = \lim_{p\to 0}\phi(p) = u(x) \tag{220}$$

by the Lebesgue differentiation theorem.

□

Remark. *The form of $\phi(r)$ is convenient to use and brings with other conclusions. For example, one can prove that for Poisson's equation on a ball with dimension $n \geq 3$,*

$$\begin{cases} -\Delta u = f \ (on\ B_r(0)) \\ u|_{\partial B_r(0)} = g \end{cases} \tag{221}$$

the equation that

$$u(0) = \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} g\,dS + \frac{1}{n(n-2)V_n(1)}\int_{B_r(0)} \left(||x||_2^{2-n} - r^{2-n}\right) f(x)\,dx \tag{222}$$

holds. Since (sketch of proof)

$$\frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} g\,dS - u(0) = \phi(r)-\phi(0) = \int_0^r \phi'(s)\,ds = \int_0^r ds\frac{1}{|\partial B_s(0)|}\int_{B_s(0)}\Delta u\,dy \tag{223}$$
$$= -\frac{1}{nV_n(1)}\int_0^r s^{1-n}\,ds\int_{B_s(0)} f(y)\,dy = -\frac{1}{nV_n(1)}\int_{B_r(0)} f(y)\,dy\int_{||y||_2}^r s^{1-n}\,ds \tag{224}$$
$$= -\frac{1}{nV_n(1)(2-n)}\int_{B_r(0)} (r^{2-n} - ||y||_2^{2-n}) f(y)\,dy \tag{225}$$

by Fubini theorem.

Actually, the converse of this theorem is also true, which means that for any continuous function if mean-value property holds on an area then u is smooth(C^∞) and harmonic. So mean-value property is the **characterization** of harmonic functions.

Theorem 5. *Suppose u continuous on Ω has mean value prop, then u is smooth and harmonic.*

The proof of this theorem would have to use the mollifier. A **mollifier** always has the following properties.

It's **non-negative, C^∞smooth and compactly supported satisfying $\int_{\mathbb{R}} \eta = 1$ is satisfied.**. A simple example of this would be

$$\eta(x) = \begin{cases} Ce^{-\frac{1}{(1-||x||^2)}} & ||x|| < 1 \\ 0 & ||x|| \geq 1 \end{cases} \tag{226}$$

where the C is selected such that $\int_{\mathbb{R}} \eta = 1$ is satisfied.

In practice we only care about the properties of the mollifier but not the specific form of the mollifier. Note that such mollifier η has support $[-1, 1]$ and we hope that the support would shrink to 0 in the sense that

$$\forall \varepsilon > 0, \eta^\varepsilon = \frac{\eta(\frac{x}{\varepsilon})}{\varepsilon^n} \tag{227}$$

All properties of mollifiers are maintained for η^ε with a fixed ε and this series of mollifiers has shrinking support as $\varepsilon \to 0$, called **standard mollifiers**. They are often used as a convolution with a known function in the approximation schemes.

Week 4

Suppose $f \in L_{loc}(\Omega)$ is locally integrable ($f \in L^1(\Omega'), \Omega' \subset\subset \Omega$), for $\varepsilon > 0$, define $\Omega^\varepsilon = \{x \in \Omega, dist(x, \partial\Omega) > \varepsilon\}$, so that any small enough open ball centered at any point in Ω^ε is in Ω.

Define the convolution

$$f^\varepsilon(x) = \int_\Omega \eta^\varepsilon(x-y) f(y)\, dy = \eta^\varepsilon * f \tag{228}$$

Theorem 6. *If $f \in L^1_{loc}(\Omega)$, then*

$$\forall \Omega^\varepsilon \subset\subset \Omega, \forall x \in \Omega^\varepsilon, f^\varepsilon \in C^\infty(\Omega^\varepsilon), f^\varepsilon(x) \overset{a.e., L^1}{\to} f(x)\ (\varepsilon \to 0) \tag{229}$$

Proof. First show the smoothness of the convolution. The differentiation w.r.t. x can interchange with integration w.r.t. y by dominated convergence theorem ($\frac{d}{dx}\eta^\varepsilon$ is bounded)

$$\frac{d}{dx} f^\varepsilon(x) = \int_\Omega \frac{d}{dx}\eta^\varepsilon(x-y) f(y)\, dy \tag{230}$$

and $\eta^\varepsilon \in C^\infty$ ensures that this function is still differentiable. As a result, the derivative of any order exists.

The *a.e.* convergence comes from the estimation that

$$\left| \int_\Omega \eta^\varepsilon(x-y) f(y)\, dy - f(x) \right| \le \int_\Omega \eta^\varepsilon(x-y) |f(y) - f(x)|\, dy \tag{231}$$

$$= \int_{B_\varepsilon(x)} \eta^\varepsilon(x-y) |f(y) - f(x)|\, dy \tag{232}$$

$$\le \frac{1}{\varepsilon^n} \int_{B_\varepsilon(x)} |f(y) - f(x)|\, dy \overset{a.e.}{\to} 0\ (\varepsilon \to 0) \tag{233}$$

by the boundedness of the mollifier η and the Lebesgue point theorem (almost every point is Lebesgue point of the locally integrable function).

□

Theorem 7. *If $u \in C(\Omega)$ has the mean-value property then it's smooth and harmonic.* ***(Mean-value property as characterization of harmonic function)***

Proof. Note that the definition of η only depends on the radial value $||x||$, so denote

$$\eta^\varepsilon(x) = \tilde{\eta}^\varepsilon(||x||) \tag{234}$$

If $B_\varepsilon(x) \subset\subset \Omega$, then

$$\begin{aligned}
(\eta^\varepsilon * u)(x) &= \int_{B_\varepsilon(0)} \eta^\varepsilon(y) u(x-y)\, dy && (235)\\
&= \int_0^\varepsilon \int_{\partial B_1(0)} \eta^\varepsilon(ry) u(x-ry)\, dS(y) r^{n-1}\, dr && (236)\\
&= |\partial B_1(0)| \int_0^\varepsilon \frac{1}{|\partial B_1(0)|} \int_{\partial B_1(0)} u(x-ry)\, dS(y) r^{n-1} \tilde{\eta}^\varepsilon(r)\, dr && (237)\\
&= |\partial B_1(0)| \int_0^\varepsilon u(x) r^{n-1} \tilde{\eta}^\varepsilon(r)\, dr && (238)\\
&= u(x) \int_{B_\varepsilon(0)} \eta^\varepsilon(y)\, dy = u(x) && (239)
\end{aligned}$$

here still use the trick to deparametrize w.r.t. y using radial value and make use of the mean-value property of u and change the variable back into the variable on sphere. This is telling us that u is smooth since the convolution must be smooth.

Going back to the equations in the proof of mean-value theorem, to see that $\int_{B_r(x)} \Delta u\, dx = 0$ on any ball $B_r(x)$. Since Δu is continuous, it must be true that $\Delta u = 0$ and it's harmonic.

$\square$

Remark. *Mean-value property says nothing about the boundary behavior of harmonic functions, but only talks about the behavior of harmonic functions in the interior. A counterexample is the real and imaginary part of $f(z) = \frac{1}{z}$ $(z \neq 0)$. These are harmonic functions, but can blow up at boundary because of the pole at 0.*

The definition of harmonic functions can be extended to sub-harmonic and super-harmonic functions. Suppose Ω open in $\mathbb{R}^n$, $u \in C^2$ is **sub-harmonic** if $\Delta u \geq 0$ and **super-harmonic** if $\Delta u \leq 0$.

Theorem 8. *Suppose Ω open, $B_r(x) \subset\subset \Omega$, $u \in C^2$, if u is sub-harmonic in Ω then*

$$u(x) \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy \tag{240}$$

$$u(x) \leq \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u\, dS \tag{241}$$

Remark. *The sub-harmonic and super-harmonic functions are defined similar to that for sub-MG and super-MG. Actually, the laplacian is some kind of "curvature" of u in a small neighborhood from the geometric point of view, so if the curvature is positive (locally convex), then the average near a point is greater. For sub-harmonic functions, we are underestimating the function value (just like for sub-MG, we are underestimating the value of the process at a fixed time).*

Actually, sub-harmonic functions and sub-MG has close connections under the setting of Markov chain. For

$p(x,y)$ as a Markov transition kernel, we can define f to be super-harmonic if

$$f(x) \geq \sum_y p(x,y) f(y) \tag{242}$$

equivalent to saying that $f(X_n)$ is a super-MG (X_n is a Markov process with transition kernel p). Actually, we can also see from the probability side that if p is irreducible, then p is recurrent if and only if every nonnegative super-harmonic function is constant.

Example

$u(x) = ||x||^4$ is sub-harmonic on $\mathbb{R}^n$ since

$$u(x) = \left(\sum_{i=1}^n x_i^2\right)^2 \tag{243}$$

$$\Delta u(x) = \sum_{i=1}^n \left(8x_1^2 + 4\sum_j x_j^2\right) \geq 0 \tag{244}$$

and we can see that it only obtains maximum on the boundary of $B_1(0)$.

Property of Harmonic Function

From the mean-value property, we can infer all other properties of harmonic functions. The following theorem shows that the partial derivative of harmonic function is still harmonic and can be bounded by its function value, which is a very strong statement for a real-value function.

Theorem 9. *If $u \in C^2(\Omega)$ is harmonic and $B_r(x) \subset\subset \Omega$, then $\forall i = 1, 2, ..., n, \partial_i u$ is harmonic and*

$$\forall i = 1, ..., n, |\partial_i u(x)| \leq \frac{n}{r} \max_{\overline{B}_r(x)} |u| \tag{245}$$

Proof. Notice that $\Delta = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$, so $\partial_i \Delta = \Delta \partial_i$ for harmonic functions (since it's C^∞), and that's why $\partial_i u$ is still harmonic. Use the mean-value property to get

$$\partial_i u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} \partial_i u(y)\, dy \tag{246}$$

To proceed, use the trick here to view the partial derivative as the divergence of the function $U = (0, 0, ..., 0, u, 0, ..., 0)$ with the i-th component to be u. ν denotes the outward unit normal vector at the point on $\partial B_r(x)$. The divergence theorem tells us

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} \partial_i u(y)\,dy = \frac{1}{|B_r(x)|}\int_{B_r(x)} div(U)(y)\,dy \tag{247}$$

$$= \frac{1}{|B_r(x)|}\int_{\partial B_r(x)} u\cdot \nu_i\,dS \tag{248}$$

So

$$|\partial_i u(x)| \leq \frac{1}{|B_r(x)|}\int_{\partial B_r(x)} |u\cdot \nu_i|\,dS \tag{249}$$

$$\leq \frac{1}{|B_r(x)|}\int_{\partial B_r(x)} \max_{\overline{B}_r(x)} |u|\,dS \tag{250}$$

$$= \frac{|\partial B_r(x)|}{|B_r(x)|}\max_{\overline{B}_r(x)} |u| \tag{251}$$

$$= \frac{n}{r}\max_{\overline{B}_r(x)} |u| \tag{252}$$

□

The direct consequence of this estimate is the Louville's theorem.

Theorem 10. *If $u\in C^2(\mathbb{R}^n)$ is harmonic and bounded on $\mathbb{R}^n$ then it's constant.* ***(Louville's theorem)***

Proof. The previous theorem tells us that $\forall i=1,...,n, |\partial_i u(x)|\leq \frac{n}{r}\max_{\overline{B}_r(x)}|u|$ with $\forall x\in\mathbb{R}^n, |u(x)|\leq M$

$$\forall i=1,...,n, |\partial_i u(x)|\leq \frac{nM}{r} \tag{253}$$

since u is harmonic on $\mathbb{R}^n$, can set $r\to\infty$ to get

$$\forall i=1,...,n, \partial_i u(x) = 0 \tag{254}$$

For any differentiable function, if all partial derivatives are 0 then it must be constant.

□

Similar to the conclusions in complex analysis, the estimation on derivatives can be generalized to all higher order derivatives.

Theorem 11. *If $u\in C^2(\Omega)$ is harmonic and $B_r(x)\subset\subset\Omega$, then for any multi-index α of order k, $\partial^\alpha u$ is harmonic and*

$$|\partial^\alpha u(x)|\leq \frac{n^k e^{k-1}k!}{r^k}\max_{\overline{B}_r(x)}|u| \tag{255}$$

Proof. It's obvious to see that $\partial^\alpha \Delta = \Delta \partial^\alpha$ for smooth functions, so $\partial^\alpha u$ is still harmonic. The theorem is already proved for $k = 1$. For larger k, use induction. Assume the conclusion holds for k, let's consider whether it still holds for $k + 1$. For multi-index β with order $k + 1$, WLOG assume $\partial^\beta = \partial^\gamma \partial_j$ for multi-index γ with order k. To get the $\max_{y \in \overline{B}_r(x)} |u(y)|$ in the final expression, we tear apart the radius r into pr and $(1 - p)r$ for some $p \in (0, 1)$ for two use of the estimation of partial derivatives. Here p is a fixed constant but is not specified and will be figured out later.

$$
\begin{aligned}
|\partial^\beta u(x)| &= |\partial^\gamma \partial_j u(x)| && (256)\\
&\leq \frac{n^k e^{k-1} k!}{(pr)^k} \max_{y \in \overline{B}_{pr}(x)} |\partial_j u(y)| && (257)\\
&\leq \frac{n^k e^{k-1} k!}{(pr)^k} \max_{y \in \overline{B}_{pr}(x)} \frac{n}{(1-p)r} \max_{z \in \overline{B}_{(1-p)r}(y)} |u(z)| && (258)\\
&\leq \frac{n^{k+1} e^{k-1} k!}{r^{k+1} p^k (1-p)} \max_{y \in \overline{B}_r(x)} |u(y)| && (259)
\end{aligned}
$$

now let's think about picking the best value for p, we want the bound to be as tight as possible, i.e. $f(p) = \frac{1}{p^k(1-p)}$ as small as possible. Consider $g(p) = \log f(p) = -k \log p - \log(1 - p)$ and take the derivative to get

$$
\begin{aligned}
g'(p) &= -\frac{k}{p} + \frac{1}{1-p} && (260)\\
p^* &= \frac{k}{k+1} && (261)
\end{aligned}
$$

This is telling us that when $p = \frac{k}{k+1}$, the bound would be the tightest. In such situation,

$$
f\left(\frac{k}{k+1}\right) = (k+1)\left(1 + \frac{1}{k}\right)^k \leq e(k+1) \tag{262}
$$

leading to the end of the proof that

$$
\frac{n^{k+1} e^{k-1} k!}{r^{k+1} p^k (1-p)} \max_{y \in \overline{B}_r(x)} |u(y)| \leq \frac{n^{k+1} e^k (k+1)!}{r^{k+1}} \max_{y \in \overline{B}_r(x)} |u(y)| \tag{263}
$$

□

The next theorem shows that harmonic functions are not only smooth but also analytic.

Theorem 12. *If $u \in C^2(\Omega)$ is harmonic, then it's real analytic in Ω, i.e. $\forall x \in \Omega, \exists r > 0, B_r(x) \subset\subset \Omega$, such that u is a convergent power series in $B_r(x)$.*

Proof. It's natural to consider the Taylor series at $\forall x \in \Omega$ and $\forall h > 0$ small enough such that $x + h \in \Omega$

$$u(x+h) = \sum_{0\le|\alpha|\le k-1} \frac{\partial^\alpha u(x)}{\alpha!} h^\alpha + R_k(x,h) \tag{264}$$

$$R_k(x,h) = \sum_{|\alpha|=k} \frac{\partial^\alpha u(x+\theta h)}{\alpha!} h^\alpha \tag{265}$$

where $\alpha! = \alpha_1!...\alpha_n!, h^\alpha = h_1^{\alpha_1}...h_n^{\alpha_n}$ and $R_k(x,h)$ is the Lagrange remainder with θ as some real number in $(0,1)$.

For $\forall\varepsilon > 0$, consider the case where $||h||_2 < \delta, k \to \infty$ and apply the estimation for the higher order partial derivatives

$$|R_k(x,h)| \le \sum_{|\alpha|=k} \frac{n^k e^{k-1} k!}{\alpha! r^k} \max_{y\in\overline{B}_r(x+\theta h)} |u(y)| \cdot h^\alpha \tag{266}$$

note that there exists some constant M such that $\max_{y\in\overline{B}_r(x+\theta h)} |u(y)| \le M$ (continuous on compact set) and $h^\alpha \le ||h||_2^k < \delta^k$

$$|R_k(x,h)| \le \sum_{|\alpha|=k} M \frac{n^k e^{k-1} k!}{\alpha! r^k} \cdot \delta^k \tag{267}$$

the multinomial expansion theorem gives

$$n^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \tag{268}$$

so for multi-index α with order k

$$k! \le n^k \alpha! \tag{269}$$

plug in this estimate to get

$$|R_k(x,h)| \le \frac{M}{e} \sum_{|\alpha|=k} \left(\frac{n^2 e\delta}{r}\right)^k \tag{270}$$

Now the last step is to notice that the number of terms in the sum is no more than n^k. So we wish to have $n^k \left(\frac{n^2 e\delta}{r}\right)^k \to 0$ $(k \to \infty)$. As a result, just take $\delta < \frac{r}{2^n n^3 e}$,

$$|R_k(x,h)| \le \frac{M}{e} n^k \frac{1}{2^{nk} n^k} = \frac{M}{e}\frac{1}{2^{nk}} \to 0 \ (k \to \infty) \tag{271}$$

proves that there exists a small enough δ such that the Taylor series is convergent in this open disk.

□

The analytic property tells us that all partial derivatives at a single point uniquely determines a harmonic function on a connected area.

Theorem 13. *If u, v are harmonic in connected open area $\Omega \subset \mathbb{R}^n$ and there exists $x_0 \in \Omega$ such that for all multi-index α,*

$$\partial^\alpha u(x_0) = \partial^\alpha v(x_0) \tag{272}$$

then $\forall x \in \Omega, u(x) = v(x)$. ***(All partials at a single point uniquely determines a harmonic function)***

Proof. Consider the set

$$F = \{x \in \Omega : \forall \alpha, \partial^\alpha u(x) = \partial^\alpha v(x)\} \tag{273}$$

then it's nonempty $x_0 \in F$ and it's closed since $\partial^\alpha u - \partial^\alpha v$ is still harmonic and continuous with

$$F = \bigcap_\alpha (\partial^\alpha u - \partial^\alpha v)^{-1}(\{0\}) \tag{274}$$

so F has to be closed. However, for $y \in F$, since $\partial^\alpha u - \partial^\alpha v$ is harmonic and takes value 0 at y, it's analytic and there exists $r > 0$ such that $\partial^\alpha u - \partial^\alpha v$ is a convergent power series in $B_r(y)$. This power series is just the Taylor series expanded at y, however, all partial derivatives at y of $\partial^\alpha u - \partial^\alpha v$ is 0 since $y \in F$. This is telling us that the power series is always 0 in the whole ball $B_r(y)$, and $B_r(y) \subset F$, so F is open.

Since Ω is connected, there are only trivial subsets which are open and close. Since F cannot be empty, $F = \Omega$ and $u = v$ on the whole area.

□

Remark. *Note that the* ***connected condition*** *of Ω is necessary since we need this topological condition to ensure that if a subset is both open and closed then it's trivial. If Ω is replaced with a general area, the theorem holds for any connected component of Ω.*

Remark. *One might think naturally that we would also have* ***isolation of zeros and the identity theorem (uniqueness)*** *to hold for harmonic functions just like that in complex analysis. However, this is* ***WRONG for harmonic functions!***

A counterexample is $u(x, y) = x$ which is obviously harmonic, and is zero on the whole y-axis (with accumulation points), but not identically zero.

The maximum principle is stated for sub-harmonic functions below. Similarly, super-harmonic functions and harmonic functions have their respective versions.

Theorem 14. *If u is sub-harmonic in connected open area $\Omega \subset \mathbb{R}^n$ and attains a maximum in Ω, then it's constant on Ω.* ***(Maximum principle for sub-harmonic)***

Proof. This proof is directly given by mean-value property. Assume the maximum is M with

$$F = \{x : u(x) = M\} \tag{275}$$

to be the set of points attaining the maximum. It's obvious that it's non-empty and closed (u continuous, $\{M\}$ closed). Let's consider $\forall x \in F$, then the mean-value property of sub-harmonic function tells us that

$$u(x) \leq \frac{1}{B_r(x)} \int_{B_r(x)} u(y)\, dy \leq \frac{1}{B_r(x)} \int_{B_r(x)} u(x)\, dy = u(x) \tag{276}$$

As a result, for any $r > 0$ such that $B_r(x) \subset\subset \Omega$

$$\frac{1}{B_r(x)} \int_{B_r(x)} u(y) - u(x)\, dy = 0 \tag{277}$$

since $u(y) - u(x) \leq 0$, conclude that $\forall y \in B_r(x), u(y) = u(x)$, so $B_r(x) \subset F$, F is open.

Since Ω is connected, the subset that is both open and closed has to be trivial, so $F = \Omega$ and u is constant.

□

Remark. *By the same reasoning, the* ***connectedness*** *of Ω can't be ignored and for general open area this theorem holds for each connected component.*

For super-harmonic functions, it's clear that if there exists minimum in the interior than the function must be constant.

Week 5

Property of Harmonic Function

Theorem 15. *Ω connected open, $u \in C^2(\Omega)$ is sup-harmonic, attains global minimum in interior, then it's constant.*

Theorem 16. *Ω connected open, $u \in C^2(\Omega)$ harmonic, attains global min/max in interior, then it's constant.*

Consider $u(x, y) = x^2 - y^2$ harmonic on $\mathbb{R}^2$. The origin is a critical point, but according to maximum principle, the origin must be a saddle point. The following weak maximum principles are equivalent to strong maximum principles.

Theorem 17. *Ω bounded connected open, $u \in C^2(\Omega), u \in C(\overline{\Omega})$ harmonic, then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$, the same argument holds for minimum.*

Theorem 18. *Ω bounded connected open, $f \in C(\Omega), g \in C(\partial\Omega)$, there is at most one solution to Dirichlet problem* $\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = g \end{cases}$ *with $u \in C^2(\Omega), u \in C(\overline{\Omega})$.*

Proof. Let u_1, u_2 be solutions to the Dirichlet problem and $v = u_1 - u_2$ so $\Delta v = 0, v|_{\partial\Omega} = 0$. So v must be constantly 0 by maximum principle.

□

Theorem 19. *(Harnack's inequality) $\Omega' \subset\subset \Omega$ is connected open set, then there exists C only depending on Ω, Ω' such that if $u \in C(\Omega), u \geq 0$ has mean-value property, then $\sup_{\Omega'} \leq C \inf_{\Omega'} u$. (Note: u is not necessarily harmonic, since it's not necessarily continuous)*

Proof. First prove it for open balls. Suppose $x \in \Omega'$ and $B_{4r}(x) \subset \Omega'$. Let u be non-neg function with mean-value property.

If $y \in B_r(x)$, then

$$u(y) = \frac{1}{|B_r(y)|} \int_{B_r(y)} u = \frac{2^n}{|B_{2r}(y)|} \int_{B_r(y)} u \tag{278}$$
$$\leq \frac{2^n}{|B_{2r}(y)|} \int_{B_{2r}(x)} u = 2^n u(x) \tag{279}$$

since $u \geq 0$ and $B_r(y) \subset B_{2r}(x)$.

Now take $z \in B_r(x)$, then

$$u(z) = \frac{1}{|B_{3r}(z)|} \int_{B_{3r}(z)} u = \left(\frac{2}{3}\right)^n \frac{1}{|B_{2r}(z)|} \int_{B_{3r}(z)} u \tag{280}$$
$$\geq \left(\frac{2}{3}\right)^n \frac{1}{|B_{2r}(z)|} \int_{B_{2r}(x)} u = \left(\frac{2}{3}\right)^n u(x) \tag{281}$$

since $u \geq 0$ and $B_{2r}(x) \subset B_{3r}(z)$.

Combine to get

$$\forall x \in \Omega', \forall y \in B_r(x), \left(\frac{2}{3}\right)^n u(x) \leq u(y) \leq 2^n u(x) \tag{282}$$

take inf and sup w.r.t. y to get

$$\forall x \in \Omega', \sup_{B_r(x)} u \leq 2^n u(x), \inf_{B_r(x)} u \geq \left(\frac{2}{3}\right)^n u(x) \tag{283}$$

so

$$\forall x \in \Omega', \sup_{B_r(x)} u \leq 3^n \inf_{B_r(x)} u \tag{284}$$

Now for general case, as long as there is at least $4r$ distance left between Ω' and $\partial\Omega$, i.e. $\exists r > 0, dist(\Omega', \partial\Omega) > 4r$, it should still hold. Since $\overline{\Omega'}$ is compact, it can be covered by finitely many (say, N) open balls with radius r, with N only depending on Ω, Ω'.

By connectedness and open set, $\forall x, y \in \Omega'$, there exists a path from x to y and can be covered by those open balls $B_r(x_i)$, $(i = 1, 2, ..., N)$ with $x_1 = x, x_N = y$. In each open ball, the conclusion above holds

$$\forall x \in \Omega', \sup_{B_r(x_i)} u \leq 3^n \inf_{B_r(x_i)} u \tag{285}$$

so the finiteness that $N < \infty$ gives

$$\forall x, y \in \Omega', u(x) = u(x_1) \leq 3^n u(x_2) \leq ... \leq 3^{nN} u(y) \tag{286}$$

$$\forall x, y \in \Omega', 3^{-nN} u(y) \leq u(x) \leq 3^{nN} u(y) \tag{287}$$

take the inf and sup to conclude.

□

Remark. *As we can see from the proof, generally there exists constant $C = C(\Omega, \Omega')$, such that*

$$\forall x, y \in \Omega', \frac{1}{C} u(y) \leq u(x) \leq C u(y) \tag{288}$$

the lower bound and upper bound can be given by the same constant C which does not depend on x, y.

Green's Identity and Energy Estimates

Theorem 20. *If Ω is bounded C^1 open set and $u, v \in C^2(\overline{\Omega})$, then*

$$\int_\Omega u \cdot \Delta v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} u \cdot \frac{\partial v}{\partial \nu} \, dS \tag{289}$$

$$\int_\Omega u \cdot \Delta v \, dx = \int_\Omega v \cdot \Delta u \, dx + \int_{\partial\Omega} \left(u \cdot \frac{\partial v}{\partial \nu} - v \cdot \frac{\partial u}{\partial \nu} \right) dS \tag{290}$$

Proof. Note the property that

$$div(u \cdot \nabla v) = u \cdot \Delta v + \nabla u \cdot \nabla v \tag{291}$$

and apply the divergence theorem for the unit outward normal vector ν of $\partial\Omega$

$$\int_\Omega (u \cdot \Delta v + \nabla u \cdot \nabla v) \, dx = \int_\Omega div(u \cdot \Delta v) \, dx \tag{292}$$

$$= \int_{\partial\Omega} (u \cdot \Delta v) \cdot \nu \, dS \tag{293}$$

$$= \int_{\partial\Omega} u \cdot \frac{\partial v}{\partial \nu} \, dS \tag{294}$$

To prove the second equation, apply the first one twice and subtract

$$\int_\Omega u \cdot \Delta v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} u \cdot \frac{\partial v}{\partial \nu} \, dS \tag{295}$$

$$\int_\Omega v \cdot \Delta u \, dx = -\int_\Omega \nabla v \cdot \nabla u \, dx + \int_{\partial\Omega} v \cdot \frac{\partial u}{\partial \nu} \, dS \tag{296}$$

$$\int_\Omega u \cdot \Delta v \, dx - \int_\Omega v \cdot \Delta u \, dx = \int_{\partial\Omega} \left(u \cdot \frac{\partial v}{\partial \nu} - v \cdot \frac{\partial u}{\partial \nu} \right) dS \tag{297}$$

□

Theorem 21. *Ω is C_1 bounded open (the boundary is a C_1 curve) with $f \in C(\overline{\Omega}), g \in C(\partial\Omega)$. If $u_1, u_2 \in C^2(\overline{\Omega})$ are solutions to the Dirichlet problem $\begin{cases} -\Delta = f \\ u|_{\partial\Omega} = g \end{cases}$, then $u_1 = u_2$.*

Theorem 22. *Ω is C_1 bounded open (the boundary is a C_1 curve) with $f \in C(\overline{\Omega}), g \in C(\partial\Omega)$. If $u_1, u_2 \in C^2(\overline{\Omega})$ are solutions to the Neumann problem $\begin{cases} -\Delta = f \\ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = g \end{cases}$, then $u_1 = u_2 + C$ for some constant C.*

Proof. For the Dirichlet problem, set $w = u_1 - u_2$ to be harmonic with $w|_{\partial\Omega} = 0$. Apply Green's idendity for $u = v = w$ to find

$$\int_\Omega w \cdot \Delta w \, dx + \int_\Omega \nabla w \cdot \nabla w \, dx = \int_{\partial\Omega} w \cdot \frac{\partial w}{\partial \nu} \, dS = 0 \tag{298}$$

so

$$\int_\Omega \nabla w \cdot \nabla w \, dx = 0 \tag{299}$$

$$\forall x \in \Omega, \nabla w(x) = 0 \tag{300}$$

$$\forall x \in \Omega, w(x) = 0 \tag{301}$$

For the Neumann problem, still apply the Green's identity to conclude

$$\int_\Omega \nabla w \cdot \nabla w \, dx = 0 \tag{302}$$

and the uniqueness is proved in the same way since now $\frac{\partial w}{\partial \nu}|_{\partial\Omega} = 0$.

$\square$

Remark. *The maximum principle for harmonic functions can also be applied to prove the uniqueness of the solution to the Dirichlet problem but cannot prove the uniqueness of the solution to the Neumann problem. The reason is that $\frac{\partial u}{\partial \nu}(x)$ contains the outward unit normal vector of $\partial\Omega$, which is ν that varies if $x \in \partial\Omega$ varies (so it's actually not a conventionally defined directional derivative).*

Fundamental Solution

The fundamental solution is defined as some special solutions to the Laplace equation. The motivation of the appearance of fundamental solutions is to first apply some symmetry conditions to get some special solutions and then use special solutions to construct general solutions to the Poisson's equation. Since Laplacian is invariant under rotation, it's natural to consider the special solutions that only depend on the radial variable $r = ||x||_2$.

To get such special solutions $u(x) = v(r)$, do a polar coordinate transformation and calculate the Laplacian under the polar coordinates to get:

$$\Delta u = \sum_i \partial_{x_i x_i} u \tag{303}$$

$$\partial_{x_i} u = \frac{\partial v}{\partial x_i} = v'(r) \cdot \frac{\partial r}{\partial x_i} \tag{304}$$

$$= v'(r) \cdot \frac{x_i}{r} \tag{305}$$

taking another partial to see

$$\partial_{x_i x_i} u = \frac{\partial \left[v'(r) \cdot \frac{x_i}{r} \right]}{\partial x_i} = v''(r) \cdot \frac{\partial r}{\partial x_i} \cdot \frac{x_i}{r} + v'(r) \cdot \frac{r - x_i \cdot \frac{\partial r}{\partial x_i}}{r^2} \tag{306}$$

$$= v''(r) \frac{x_i^2}{r^2} + v'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right) \tag{307}$$

As a result, the Laplacian equation turns into an ODE w.r.t. v

$$\Delta u = v''(r) + v'(r)\frac{n-1}{r} = 0 \tag{308}$$

Solve the ODE

$$v'(r) = C_1 r^{1-n} \text{ or } 0 \tag{309}$$

and

$$v(r) = \begin{cases} C_1 \log r + C_2 & n = 2 \\ C_1 r^{2-n} + C_2 & n \geq 3 \end{cases} \tag{310}$$

The **fundamental solution/free-space Green's function** Γ is defined as $v(r)$ for some specified C_1, C_2 that

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log ||x||_2 & n = 2 \\ \frac{1}{n(n-2)V_n(1)} ||x||_2^{2-n} & n \geq 3 \end{cases} \tag{311}$$

where $V_n(1)$ denotes the volume of the ball with radius 1 in $\mathbb{R}^n$.

The **properties** of fundamental solutions are stated as below:

The fundamental solution $\Gamma \in C^\infty(\mathbb{R} - \{0\})$ is **smooth at any point except the origin**, which is obvious since logarithm and power functions are smooth. It's also **harmonic** (solution to Laplacian equation) with estimates on the order of its partial derivatives

$$\partial_{x_i}\Gamma(x) = O(||x||^{1-n}), \partial_{x_i x_i}\Gamma(x) = O(||x||^{-n}) \ (x \neq 0) \tag{312}$$

since

$$\partial_{x_i}\Gamma(x) = \begin{cases} \frac{x_i}{2\pi ||x||_2^2} & n = 2 \\ -\frac{1}{nV_n(1)} \frac{x_i}{||x||_2^n} & n \geq 3 \end{cases} \tag{313}$$

$$\partial_{x_i x_i}\Gamma(x) = \begin{cases} \frac{||x||_2^2 - 2x_i^2}{2\pi ||x||_2^4} & n = 2 \\ -\frac{1}{nV_n(1)} \frac{||x||_2^n - n x_i^2 ||x||_2^{n-2}}{||x||_2^{2n}} & n \geq 3 \end{cases} \tag{314}$$

Consider for $n \geq 3$

$$\nabla\Gamma \cdot \frac{x}{||x||_2} = -\frac{1}{nV_n(1)} \frac{1}{||x||_2^{n-1}} \tag{315}$$

and for $n = 2$

$$\nabla\Gamma \cdot \frac{x}{||x||_2} = \frac{1}{2\pi}\frac{1}{||x||_2} \tag{316}$$

so

$$\int_{\partial B_r(0)} \nabla\Gamma \cdot \nu \, dS = -\frac{|\partial B_r(0)|}{nV_n(1)r^{n-1}} = -1 \ (n \geq 3) \tag{317}$$

$$\int_{\partial B_r(0)} \nabla\Gamma \cdot \nu \, dS = \frac{2\pi r}{2\pi r} = 1 \ (n = 2) \tag{318}$$

Remark. *Now we see why certain normalization constants are taken in the definition of the fundamental solution. The normalization constants are chosen such that the surface integral on the sphere of the fundamental solution's directional derivative along the outward unit normal vector is always normalized.*

Specially, the normalization constant is taken as 2π when $n = 2$ since unit sphere in 2-dim has volume 2π. This is also consistent with the Cauchy's integral formula in complex analysis. Notice that the fundamental solution is only increasing w.r.t. radius r when $n = 2$.

$$\int_{\partial B_r(0)} \frac{\partial\Gamma}{\partial\nu} \, dS = \int_{\partial B_r(0)} \nabla\Gamma \cdot \nu \, dS = \begin{cases} 1 & n = 2 \\ -1 & n \geq 3 \end{cases} \tag{319}$$

Remark. *In $\mathbb{R}^n$, it's clear that $\Gamma, \partial_{x_i}\Gamma$* ***are always locally integrable at 0 but $\partial_{x_i x_i}\Gamma$ is not****.*

$$\int_{B_1(0)} \partial_i\Gamma(x) \, dx \leq C \int_{B_1(0)} ||x||^{1-n} \, dx \tag{320}$$

$$= C \int_0^1 r^{n-1} \, dr \int_{\partial B_1(0)} r^{1-n} \, dS \tag{321}$$

$$= C|\partial B_1(0)| < \infty \tag{322}$$

while

$$\int_{B_1(0)} \partial_i\Gamma(x) \, dx \geq C \int_{B_1(0)} ||x||^{-n} \, dx \tag{323}$$

$$= C \int_0^1 r^{n-1} \, dr \int_{\partial B_1(0)} r^{-n} \, dS \tag{324}$$

$$= C|\partial B_1(0)| \int_0^1 \frac{1}{r} \, dr = \infty \tag{325}$$

since r^{n-1} is the Jacobian of polar coordinate transformation in $\mathbb{R}^n$.

This directly causes the failure of the interchange of intergal and Laplacian in the convolution

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy \tag{326}$$

for a good enough function f. This is because when $x-y$ is close to 0, there is singularity. Although such singularity does not affect the locally integrability of $\Gamma, \partial_{x_i}\Gamma$, it does affect the locally integrability of $\partial_{x_i x_i}\Gamma$, so we ***CANNOT*** *conclude $\Delta u = \int \Delta\Gamma \cdot f = 0$. Actually, this convolution satisfies*

$$-\Delta u = f \tag{327}$$

because of this singularity!

Theorem 23. *Let $f \in C_c^\infty(\mathbb{R}^n)$ and $u = \Gamma * f$, then $u \in C^\infty(\mathbb{R}^n), -\Delta u = f$.* ***(Solution to the Poisson's equation)***