

g.f. application in SRW: $S_0 = 0$,

$$S_n = \sum_{i=1}^n X_i, \quad X_i = \begin{cases} 1 & \text{w.p. } P \\ -1 & \text{w.p. } q = 1 - p \end{cases}$$

$$P_o(s) \stackrel{\Delta}{=} \sum_{n=0}^{\infty} \underline{p_o(n)} \cdot s^n, \quad F_o(s) \stackrel{\Delta}{=} \sum_{n=0}^{\infty} \underline{f_o(n)} \cdot s^n$$

$\Pr(S_n=0)$

$\Pr(S_1 \neq 0, \dots,$

$S_{n-1} \neq 0, S_n = 0)$
first hitting to zero
happens at time n

then:

$$\begin{cases} P_o(s) = (1 - 4pq s^2)^{-\frac{1}{2}} \\ F_o(s) = 1 - (1 - 4pq s^2)^{\frac{1}{2}} \end{cases}$$

similarly, $F_r(s) \stackrel{\Delta}{=} \sum_{n=0}^{\infty} \underline{f_r(n)} \cdot s^n$

$\Pr(S_1 \neq r, \dots, S_{n-1} \neq r, S_n = r)$

then $\underline{F_r(s)} = \underline{[F_1(s)]^r} = \frac{1 - (1 - 4pq s^2)^{\frac{1}{2}}}{2qs}$

Markov property

e.g: (5.3.2) For SSRW, show

$$(a): 2k f_0(2k) = \mathbb{P}(S_{2k-2} = 0) \text{ for } k \geq 1$$

Def: $F_0(s) = \sum_{n=0}^{\infty} f_0(n) \cdot s^n = \sum_{k=0}^{\infty} f_0(2k) \cdot s^{2k} =$
 $(f_0(n)=0 \text{ if } n \text{ is odd}) \quad 1 - (1-4pq s^2)^{\frac{1}{2}}$

diff w.r.t. s , interchange diff and summation:

$$\sum_{k=1}^{\infty} 2k \cdot f_0(2k) s^{2k-1} = \frac{4pq s}{\sqrt{1-4pq s^2}}$$

|| Taylor

$$4pq s \cdot \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \cdot (-4pq s^2)^n$$

$$4pq s \cdot \sum_{n=0}^{\infty} \frac{\overset{||}{(2n-1)!!}}{2^n \cdot n!} (4pq)^n \cdot s^{2n}$$

$$\sum_{n=0}^{\infty} \frac{\overset{||}{(2n-1)!!}}{2^n \cdot n!} (4pq)^{n+1} s^{2n+1}$$

identify $2k-1$ as $2n+1$:

$$2k f_0(2k) = \frac{(2k-3)!!}{2^{k-1} \cdot (k-1)!} = \frac{2^{k-1} \cdot \frac{(2k-3)!!}{(k-1)!}}{2^{2k-2}} = \frac{\frac{(2k-2)!}{[(k-1)!]^2}}{2^{2k-2}}$$

$$= \frac{\binom{2k-2}{k-1}}{2^{2k-2}} = \mathbb{P}(S_{2k-2} = 0) \text{ for } \forall k \geq 1.$$

$$(b): \text{IP}(S_1, S_2, \dots, S_{2n} \neq 0) = \text{IP}(S_{2n} = 0) \text{ for } n \geq 1$$

if: connect with first hitting time to 0, notice that it's possible to hit 0 only at even time
 $LHS = \text{IP}(S_1, \dots, S_{2n} \neq 0, S_{2n+2} = 0)$

$$+ \text{IP}(S_1, \dots, S_{2n}, S_{2n+1}, S_{2n+2} \neq 0)$$

$$= f_0(z_{n+2}) + \text{IP}(S_1, \dots, S_n, S_{n+1}, S_{n+2} \neq 0)$$

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$$= \sum_{\substack{k=2n+2 \\ k \text{ even}}}^{\infty} f_0(k) = \sum_{k=n+1}^{\infty} f_0(2k)$$

Only need to prove: $\sum_{n=0}^{\infty} \text{IP}(S_1, S_2, \dots, S_{2n} \neq 0) \cdot s^{2n}$

$$= \sum_{n=0}^{\infty} \text{IP}(S_{2n} = 0) \cdot s^{2n} = P_0(s)$$

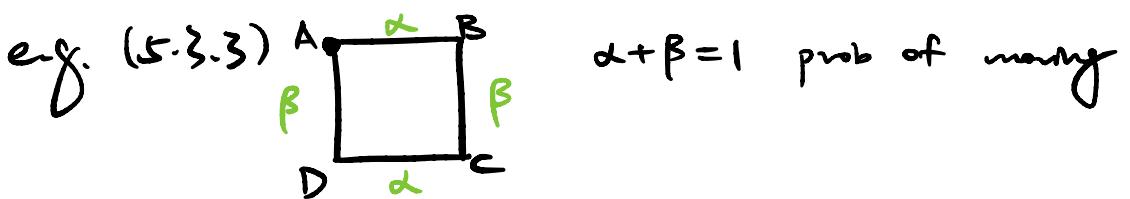
$$= (1 - 4pq\sigma^2)^{-\frac{1}{2}}$$

Calculate $\sum_{n=0}^{\infty} \text{IP}(S_1, S_2, \dots, S_{2n} \neq 0) \cdot s^{2n}$

$$= \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} f_0(2k) s^{2n} \stackrel{\text{Fub. hi}}{=} \sum_{k=1}^{\infty} f_0(2k) \sum_{n=0}^{k-1} s^{2n}$$

$$= \sum_{k=1}^{\infty} f_0(2k) \cdot \frac{1 - s^{2k}}{1 - s^2} \stackrel{\text{def of g.f.}}{=} \frac{1 - F_0(s)}{1 - s^2}$$

$$= \frac{\sqrt{1-s^2}}{1-s^2} = \frac{1}{\sqrt{1-s^2}} = P_0(s) \text{ when } p=q=\frac{1}{2}.$$



$G_A(s)$ is g.f. of $\{P_{AA}(n) : n \geq 0\}$, prob of particle starting at A is at A after n steps, calculate g.f. and find g.f. of first returning time to A.

Def: $X_n \triangleq$ loc of particle at time n, $X_0 = A$

$$P_{AA}(2n) = \underbrace{P(X_{2n} = A)}_{\text{non-trivial}} = P\left(\bigcup_{k=0}^n \left\{ \begin{array}{l} \text{2k times run through} \\ AB \text{ and } DC, \\ \text{only when } n \text{ is even} \end{array} \right. \right. \left. \left. \begin{array}{l} 2(n-k) \text{ times run through} \\ AD \text{ and } BC \end{array} \right\} \right)$$

$$= \sum_{k=0}^n \binom{2n}{2k} \alpha^{2k} \cdot \beta^{2(n-k)} = \beta^{2n} \cdot \sum_{k=0}^n \binom{2n}{2k} \left(\frac{\alpha}{\beta}\right)^{2k}$$

$$\left\{ \sum_{k=0}^n \binom{2n}{2k} \left(\frac{\alpha}{\beta}\right)^{2k} + \sum_{k=0}^{n-1} \binom{2n}{2k+1} \left(\frac{\alpha}{\beta}\right)^{2k+1} = \left(1 + \frac{\alpha}{\beta}\right)^{2n} \right.$$

$$\left. \sum_{k=0}^n \binom{2n}{2k} \left(\frac{\alpha}{\beta}\right)^{2k} - \sum_{k=0}^{n-1} \binom{2n}{2k+1} \left(\frac{\alpha}{\beta}\right)^{2k+1} = \left(1 - \frac{\alpha}{\beta}\right)^{2n} \right)$$

$$= \beta^{2n} \cdot \frac{1}{2} \cdot \left[\left(1 + \frac{\alpha}{\beta}\right)^{2n} + \left(1 - \frac{\alpha}{\beta}\right)^{2n} \right]$$

$$= \frac{1}{2} \left[(\alpha + \beta)^{2n} + (\beta - \alpha)^{2n} \right] = \frac{1}{2} \left[1 + (\beta - \alpha)^{2n} \right]$$

$$\begin{aligned}
 S_0: G_A(s) &= \sum_{n=0}^{\infty} P_{AA}(2n) s^{2n} \\
 &= \frac{1}{2} \cdot \left(\sum_{n=0}^{\infty} s^{2n} + \sum_{n=0}^{\infty} [(\beta - \alpha)s]^{2n} \right) \\
 &= \frac{1}{2} \left(\frac{1}{1-s^2} + \frac{1}{1-(\beta-\alpha)s^2} \right)
 \end{aligned}$$

Denote T as first returning time to A , and $f_A(n) = \text{IP}(T=n)$, we consider $F_A(s) \stackrel{\Delta}{=} \sum_{n=0}^{\infty} f_A(n) \cdot s^n$

By first-hitting-time decomposition,

$$\begin{aligned}
 \text{IP}(X_{2n}=A) &= \sum_{k=1}^n \text{IP}(T=2k) \cdot \underbrace{\text{IP}(X_{2n}=A | T=2k)}_{\Downarrow \text{Markov property}} \\
 &= \sum_{k=1}^n \text{IP}(T=2k) \cdot \underbrace{\text{IP}(X_{2n-2k}=A)}_{\text{Markov property}}
 \end{aligned}$$

$$S_0: P_{AA}(2n) = \sum_{k=1}^n f_A(2k) \cdot P_{AA}(2n-2k) \text{ for } \forall n \geq 1$$

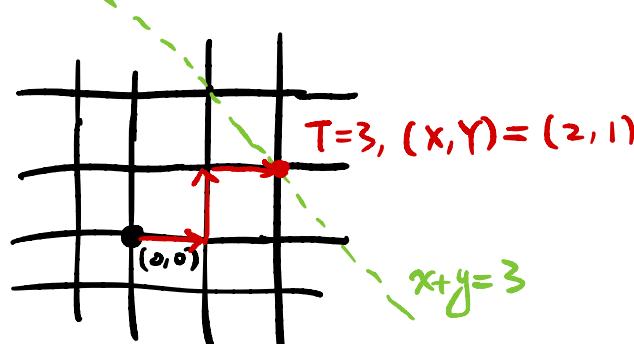
multiply by s^{2n} and sum w.r.t. n to get:

$$\begin{aligned}
 G_A(s) &= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n f_A(2k) s^{2k} \cdot P_{AA}(2n-2k) \cdot s^{2n-2k} \\
 &= 1 + \sum_{n=0}^{\infty} f_A(2n) s^{2n} \cdot \sum_{n=0}^{\infty} P_{AA}(2n) \cdot s^{2n} \\
 &= 1 + F_A(s) \cdot G_A(s)
 \end{aligned}$$

$$\text{So } F_A(s) = \frac{G_A(s) - 1}{G_A(s)} .$$

e.g. (5.2.4) particle, SSRW on \mathbb{Z}^2 , start at origin, each step length 1, with equal prob $\frac{1}{4}$ taking each direction. The particle reaches line $x+y=m$ at point (X, Y) at time T (first hitting time). Find g.f. of T and $X-Y$, specify convergence domain.

$\underline{\text{Def}}:$



At time n , has location (X_n, Y_n) so that

$$(X_{n+1}, Y_{n+1}) = (X_n, Y_n) + \begin{cases} (0, 1) \\ (1, 0) \\ (0, -1) \\ (-1, 0) \end{cases} \quad \begin{array}{l} \text{w.p. } \frac{1}{4} \\ \text{(indep movement)} \end{array}$$

check: $U_n \stackrel{\Delta}{=} X_n + Y_n$, $U_{n+1} = U_n + \begin{cases} 1 \\ -1 \end{cases} \quad \begin{array}{l} \text{w.p. } \frac{1}{2} \\ \text{(indep increment)} \end{array}$
 so U_n is SSRW on \mathbb{Z} .

$V_n \stackrel{\Delta}{=} X_n - Y_n$ is also SSRW on \mathbb{Z} .

$T = \inf\{n: U_n = m\}$ is first hitting time to m ,

$$G_T(s) = F_m(s) = \left[\frac{1 - \sqrt{1-s^2}}{s} \right]^m$$

$$X - Y = X_T - Y_T = V_T \quad \text{so}$$

$$G_{X-Y}(s) = \mathbb{E} s^{V_T} = \mathbb{E} [\mathbb{E}(s^{V_T} | T)]$$

$$\mathbb{E}(s^{V_T} | T=t) = \mathbb{E}(s^{V_t} | T=t) = \mathbb{E} s^{V_t} = G_{V_t}(s)$$

due to independence of U and V.

To see this:

$$\begin{aligned} & \text{increment of } (X_n, Y_n) \text{ as} \\ & \left\{ \begin{array}{l} (-1, 0) \\ (1, 0) \\ (0, -1) \\ (0, 1) \end{array} \right. \Rightarrow \end{aligned}$$

ξ_n^U	(increment of) U_n	(increment of) V_n
-1	-1	-1
1	1	1
-1	-1	1
1	1	-1

e.g:

$$\mathbb{P}(\xi_n^U = 1, \xi_n^V = 1) = \frac{1}{4} = \mathbb{P}(\xi_n^U = 1) \cdot \mathbb{P}(\xi_n^V = 1)$$

Also, $V_t = \sum_{k=1}^t \underbrace{\xi_k^V}_{\text{i.i.d.}}$, by property of g.f.,

$$\begin{aligned} G_{V_t}(s) &= [G_{\xi^V}(s)]^t = \left(\frac{1}{2}s + \frac{1}{2}\frac{1}{s} \right)^t \\ &= \mathbb{E} \left(\frac{s}{2} + \frac{1}{2s} \right)^T = \boxed{G_T \left(\frac{s}{2} + \frac{1}{2s} \right)} \end{aligned}$$

Convergence Domains:

$$G_T(s) = \left(\frac{1 - \sqrt{1-s^2}}{s} \right)^m \text{ well-defined for } s \in [-1, 1],$$

so for $G_{X-Y}(s)$, it's well-defined iff

$$\frac{s}{2} + \frac{1}{2s} \in [-1, 1] \Rightarrow s \in \underline{\underline{[-1, 1]}}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} \leq 1$$

$$\sqrt{\cos^2 - r \sin^2} \leq 1$$