

# Notes on PSTAT 213

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## Simple Random Walk

$S_n = X_1 + \dots + X_n$  is a SRW with  $X_i$  i.i.d. starting from  $S_0 = 0$ .  $T_b$  denotes the first hitting time of  $S_n$  to  $b$ ,  $\mathbb{P}(X_i = 1) = p, \mathbb{P}(X_i = -1) = q, p + q = 1$ .

**Theorem 1.** (*Hitting Time Theorem*)  $\forall b \neq 0$  such that  $\frac{n+b}{2} \in \{0, 1, \dots, n\}$ ,  $\mathbb{P}(T_b = n) = \frac{|b|}{n} \mathbb{P}(S_n = b) = \frac{|b|}{n} \binom{n}{\frac{n+b}{2}} p^{\frac{n+b}{2}} q^{\frac{n-b}{2}}$  ( $n \geq 1$ )

*Proof.* Prove by counting paths. It's obvious that if  $p = q$ , each path consisting of points  $(t, S_t)$  ( $t = 0, 1, \dots, n$ ) has same probability of appearing. Now  $p, q$  are not necessarily the same, so if a fixed path has  $a$  moving upward and  $n - a$  moving downward, the probability of appearing is just

$$\frac{\binom{n}{a}}{2^n} p^a q^{n-a} \quad (1)$$

If now a path hits  $b$  at time  $n$  for the first time, it should first hit  $b$  at time  $n$ , which means that there are  $\frac{n+b}{2}$  going upward and  $\frac{n-b}{2}$  going downward. Each path that hits  $b$  at time  $n$  has same probability of appearing, which is  $\frac{\binom{n}{\frac{n+b}{2}}}{2^n} p^{\frac{n+b}{2}} q^{\frac{n-b}{2}}$ . As a result, the problem reduces to counting the number of all paths within those paths that have also hit  $b$  between time 0 to  $n$ .

We do a translation for all the paths such that now we start at  $(0, -b)$  and want to count the number of paths that ends at  $(n, 0)$  but has also hit 0 in between. This count is just the sum of the number of paths that starts at  $(0, -b)$  and ends at  $(n-1, 1)$  but has also hit 0 in between and the number of paths that starts at  $(0, -b)$  and ends at  $(n-1, -1)$  but has also hit 0 in between. Assume WLOG that  $b > 0$ , notice that the first count is

$$\binom{n-1}{\frac{n+b}{2}} \quad (2)$$

and the second count is due to reflection principle that it's just the number of paths that starts at  $(0, b)$  and ends at  $(n-1, -1)$  which is

$$\binom{n-1}{\frac{n+b}{2}} \quad (3)$$

As a result, the sum should be

$$2 \binom{n-1}{\frac{n+b}{2}} \quad (4)$$

The number of path that starts at  $(0, b)$  and ends at  $(n, 0)$  is

$$\binom{n}{\frac{n+b}{2}} \quad (5)$$

So if a path is conditioned on already starting at  $(0, 0)$  and ending at  $(n, b)$ , it has probability of hitting  $b$  in

between as

$$\frac{2 \binom{n-1}{\frac{n+b}{2}}}{\binom{n}{\frac{n+b}{2}}} = \frac{n-b}{n} \quad (6)$$

if a path is conditioned on already starting at  $(0, 0)$  and ending at  $(n, b)$ , it has probability of not hitting  $b$  in between as

$$\frac{b}{n} \quad (7)$$

That's why  $\mathbb{P}(T_b = n) = \frac{b}{n} \mathbb{P}(S_n = b)$  for  $b > 0$  and the theorem is proved. The similar proof holds for  $b < 0$ .  $\square$

**Remark.** If we want to know the distribution of  $T_0$ , we also have to lift the time at 0 to the time at 1 (consider whether  $S_1$  is 1 or  $-1$ ) since reflection can't be applied for when the path starts or ends at 0.

**Theorem 2.** Set the *maximum process*  $M_n = \max_{0 \leq k \leq n} S_k$  for symmetric SRW  $S_n$ , then

$$\forall r \geq 1, \mathbb{P}(M_n \geq r, S_n = v) = \begin{cases} \mathbb{P}(S_n = v) & v \geq r \\ \mathbb{P}(S_n = 2r - v) & v < r \end{cases} \quad (8)$$

*Proof.* If  $v \geq r$ , then  $\mathbb{P}(M_n \geq r, S_n = v) = \mathbb{P}(S_n = v)$  naturally.

For the other case, let's count the number of paths. The number of paths from  $(0, 0)$  to  $(n, 2r - v)$  is

$$\binom{n}{\frac{n+2r-v}{2}} \quad (9)$$

The number of paths from  $(0, 0)$  to  $(n, v)$  that has hit  $r$  in between is equal to the number of paths from  $(0, -r)$  to  $(n, v - r)$  that has hit 0 in between. By reflection principle, this is just the number of paths from  $(0, r)$  to  $(n, v - r)$ , which is

$$\binom{n}{\frac{n+v-2r}{2}} \quad (10)$$

same to the count above, so it's proved.

**Another Proof:**

Since the SRW is Markov process and  $T_r < \infty$  a.s., strong Markov property tells us

$$S_n^{T_r} = S_{n+T_r} - S_{T_r} = S_{n+T_r} - r \quad (11)$$

is also a SRW and is independent of  $\mathcal{F}_{T_r}$ .

Let's then do calculations:

$$\mathbb{P}(M_n \geq r, S_n = v) = \mathbb{P}(T_r \leq n, S_n = v) \quad (12)$$

$$= \mathbb{P}(T_r \leq n, S_{n-T_r}^{T_r} = v - r) \quad (13)$$

$$= \mathbb{P}(T_r \leq n, -S_{n-T_r}^{T_r} = v - r) \quad (14)$$

the last step is due to the fact that  $T_r \in \mathcal{F}_{T_r}$ ,  $S_n^{T_r} \stackrel{d}{=} -S_{n-T_r}^{T_r}$  and that  $S_n^{T_r}$  is independent of  $\mathcal{F}_{T_r}$ .

$$\mathbb{P}(M_n \geq r, S_n = v) = \mathbb{P}(T_r \leq n, -S_{n-T_r}^{T_r} = v - r) \quad (15)$$

$$= \mathbb{P}(T_r \leq n, S_n = 2r - v) \quad (16)$$

$$= \mathbb{P}(S_n = 2r - v) \quad (17)$$

□

**Remark.** By this reflection principle, we see that for  $r \geq 0$ ,

$$\mathbb{P}(M_n \geq r) = \sum_{v=-n, -n+2, \dots, n} \mathbb{P}(M_n \geq r, S_n = v) \quad (18)$$

$$= \sum_{v < r} \mathbb{P}(S_n = 2r - v) + \sum_{v \geq r} \mathbb{P}(S_n = v) \quad (19)$$

$$= \mathbb{P}(S_n = r) + \mathbb{P}(S_n \geq r + 1) + \mathbb{P}(S_n = 2r + n) + \dots + \mathbb{P}(S_n = 2r - r + 1) \quad (20)$$

$$= \mathbb{P}(S_n = r) + \mathbb{P}(S_n \geq r + 1) + \mathbb{P}(S_n \geq r + 1) \quad (21)$$

$$= \mathbb{P}(S_n = r) + 2\mathbb{P}(S_n \geq r + 1) \quad (22)$$

that's why we get

$$\mathbb{P}(M_n = r) = \mathbb{P}(M_n \geq r) - \mathbb{P}(M_n \geq r + 1) \quad (23)$$

$$= \mathbb{P}(S_n = r) + 2\mathbb{P}(S_n \geq r + 1) - \mathbb{P}(S_n = r + 1) - 2\mathbb{P}(S_n \geq r + 2) \quad (24)$$

$$= \mathbb{P}(S_n = r) + 2\mathbb{P}(S_n = r + 1) - \mathbb{P}(S_n = r + 1) \quad (25)$$

$$= \mathbb{P}(S_n = r) + \mathbb{P}(S_n = r + 1) \quad (26)$$

To calculate probability like  $\mathbb{P}(M_8 = 6)$ , just use the formula to get

$$\mathbb{P}(M_8 = 6) = \mathbb{P}(S_8 = 6) + \mathbb{P}(S_8 = 7) \quad (27)$$

$$= \frac{\binom{8}{1}}{2^8} = \frac{1}{32} \quad (28)$$

## Generating Function of SRW

### 0 Hitting Time

Now in the general setting,  $p$  probability going upward and  $q$  going downward with  $p + q = 1$ . Now

$$p_0(n) = \mathbb{P}(S_n = 0) \quad (29)$$

and

$$f_0(n) = \mathbb{P}(S_1 \neq 0, \dots, S_{n-1} \neq 0, S_n = 0) \quad (30)$$

where  $f_0(n)$  gives the probability mass of first hitting time  $T_0$ . There respective generating functions are denoted

$$P_0(s) = \sum_{n=0}^{\infty} p_0(n) s^n \quad (31)$$

$$F_0(s) = \sum_{n=0}^{\infty} f_0(n) s^n \quad (32)$$

$$(33)$$

then since SRW is Markov, use the Markov property w.r.t. 1 unit of time translation to get

$$p_0(0) = 1, f_0(0) = 0 \quad (34)$$

$$\forall n \geq 1, p_0(n) = \mathbb{P}(S_n = 0) \quad (35)$$

$$= \sum_{k=1}^n \mathbb{P}(T_0 = k) \mathbb{P}(S_n = 0 | T_0 = k) \quad (36)$$

$$= \sum_{k=1}^n \mathbb{P}(T_0 = k) \mathbb{P}(S_{n-k} = 0) \quad (37)$$

$$= \sum_{k=1}^n f_0(k) p_0(n-k) \quad (38)$$

to compare the coefficient, proved that

$$P_0(s) = 1 + P_0(s)F_0(s) \quad (39)$$

Note that

$$P_0(s) = \sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) s^n \quad (40)$$

$$= \sum_{n=0,2,\dots} \binom{n}{\frac{n}{2}} (pq)^{\frac{n}{2}} s^n \quad (41)$$

$$= \sum_{n=0}^{\infty} \binom{2n}{n} (pq s^2)^n \quad (42)$$

$$= \sum_{n=0}^{\infty} \frac{(2n-1)!! 2^n n!}{n! n!} (pq s^2)^n \quad (43)$$

$$= \sum_{n=0}^{\infty} (-4)^n \binom{-\frac{1}{2}}{n} (pq s^2)^n \quad (44)$$

$$= (1 - 4pq s^2)^{-\frac{1}{2}} \quad (45)$$

by the Taylor series.

As a result, plug in to get

$$F_0(s) = \frac{P_0(s) - 1}{P_0(s)} \quad (46)$$

$$= 1 - (1 - 4pq s^2)^{\frac{1}{2}} \quad (47)$$

From this generating function, we can investigate whether  $T_0$  is almost surely finite or has finite expectation for general SRW. It's easy to see that

$$\mathbb{P}(T_0 < \infty) = \sum_{n=1}^{\infty} \mathbb{P}(T_0 = n) = F_0(1) = 1 - |p - q| \quad (48)$$

as a result,  $T_0 < \infty$  *a.s.* **if and only if**  $p = \frac{1}{2}$ .

Taking derivative for  $F_0(s)$  to get

$$F'_0(s) = 4pq s (1 - 4pq s^2)^{-\frac{1}{2}} \quad (49)$$

$$\mathbb{E}(T_0 \cdot \mathbb{I}_{T_0 < \infty}) = F'_0(1) = \frac{4pq}{|p - q|} \quad (50)$$

as a result,  $\mathbb{E}(T_0 \cdot \mathbb{I}_{T_0 < \infty}) < \infty$  **if and only if**  $p = \frac{1}{2}$ .

In the context above, we investigate all generating functions of the stopping time  $T_0$  which is the hitting time of 0. One can notice that actually this gives us the generating function of the  $i$ -th hitting time to 0, denoted  $T_0^i$ . By

Markov property,

$$\mathbb{P}(T_0^i = k) = \sum_{j=0}^k \mathbb{P}(T_0^{i-1} = j) \cdot \mathbb{P}(T_0^i = k | T_0^{i-1} = j) \quad (51)$$

$$= \sum_{j=0}^k \mathbb{P}(T_0^{i-1} = j) \cdot \mathbb{P}(T_0 = k - j) \quad (52)$$

so if we denote the generating function of  $T_0^i$  by  $F_0^i(s)$ , then

$$F_0^i(s) = \sum_{k=0}^{\infty} \mathbb{P}(T_0^i = k) \cdot s^k \quad (53)$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^k \mathbb{P}(T_0^{i-1} = j) \cdot \mathbb{P}(T_0 = k - j) \cdot s^k \quad (54)$$

$$= F_0^{i-1}(s) \cdot F_0(s) \quad (55)$$

$$= [F_0(s)]^i \quad (56)$$

it's then easy to see that

$$\mathbb{P}(T_0^i < \infty) = F_0^i(1) = [F_0(1)]^i = [1 - |p - q|]^i \quad (57)$$

so **SRW is recurrent if and only if**  $p = \frac{1}{2}$ . Naturally, let's investigate whether SRW is null recurrent when  $p = \frac{1}{2}$ .

$$\mathbb{E}(T_0^i \cdot \mathbb{1}_{T_0^i < \infty}) = \frac{d}{ds} F_0^i(s) |_{s=1} \quad (58)$$

$$= i[F_0(1)]^{i-1} \cdot F_0'(1) \quad (59)$$

$$= i[1 - |p - q|]^{i-1} \cdot \frac{4pq}{|p - q|} \quad (60)$$

so all states in SRW is null recurrent when  $p = \frac{1}{2}$ , which indicates a natural conclusion that there's no stationary distribution for symmetric SRW.

## 1 Hitting Time

One might find that generating functions for  $T_0$  tells us nothing about the information of other hitting times, e.g.  $T_1$ . To get  $F_1(s)$  as the generating function of  $T_1$ , we need to apply Markov property

$$\forall n > 1, \mathbb{P}(T_1 = n) = \mathbb{P}(T_1 = n | X_1 = 1) \cdot \mathbb{P}(X_1 = 1) + \mathbb{P}(T_1 = n | X_1 = -1) \cdot \mathbb{P}(X_1 = -1) \quad (61)$$

$$= q \cdot \mathbb{P}(T_1 = n | X_1 = -1) = q \cdot \mathbb{P}(T_2 = n - 1) \quad (62)$$

and it's obvious that  $\mathbb{P}(T_1 = 1) = p$ . To connect  $F_1(s)$  with  $F_2(s)$ , it's natural to think of Markov property once more. Similar to what we have done for the  $i$ -th hitting time to 0, let's denote  $F_i(1)$  as the generating function of  $T_i$ , the first hitting time to  $i \geq 1$

$$\mathbb{P}(T_i = n) = \sum_{k=0}^n \mathbb{P}(T_i = n | T_1 = k) \cdot \mathbb{P}(T_1 = k) \quad (63)$$

$$= \sum_{k=0}^n \mathbb{P}(T_{i-1} = n - k) \cdot \mathbb{P}(T_1 = k) \quad (64)$$

here the strong Markov property is applied when  $T_1 < \infty$  a.s. w.r.t.  $\mathcal{F}_{T_1}$ , note that when  $T_1 = \infty$ ,  $T_i = \infty$  so such equation still holds. This is telling us that getting the generating function of  $T_1$  is equivalent to getting the generating function of any hitting time  $T_i$

$$F_i(s) = [F_1(s)]^i \quad (65)$$

Return to the previous question on  $F_1(s)$ , this provides connection between  $\mathbb{P}(T_1 = n)$  and  $\mathbb{P}(T_2 = n - 1)$  that

$$F_1(s) = ps + \sum_{k=2}^{\infty} q \cdot \mathbb{P}(T_2 = k - 1) s^k \quad (66)$$

$$= ps + qs \cdot F_2(s) \quad (67)$$

$$= ps + qs \cdot [F_1(s)]^2 \quad (68)$$

solve this quadratic equation w.r.t.  $F_1(s)$  to get

$$F_1(s) = \frac{1 \pm \sqrt{1 - 4pqs^2}}{2qs} \quad (69)$$

notice that any generating function shall satisfy  $F_1(0) = 0$ , so we only take one appropriate root as the generating function

$$F_1(s) = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs} \quad (70)$$

naturally, one might calculate the quantity of one's interest that

$$\mathbb{P}(T_1 < \infty) = F_1(1) = \frac{1 - |p - q|}{2q} = \begin{cases} 1 & p \geq q \\ \frac{p}{q} & p < q \end{cases} \quad (71)$$

$$\mathbb{E}(T_1 \cdot \mathbb{I}_{T_1 < \infty}) = F_1'(1) = \frac{2p}{|p - q|} - \frac{1}{2q} + \frac{|p - q|}{2q} = \begin{cases} \frac{1}{p - q} & p > q \\ \frac{p}{q} \frac{1}{q - p} & p < q \\ \infty & p = q \end{cases} \quad (72)$$



in the more general case,

$$\mathbb{P}(T_i < \infty) = F_i(1) = \begin{cases} 1 & p \geq q \\ \left(\frac{p}{q}\right)^i & p < q \end{cases} \quad (73)$$

$$\mathbb{E}(T_i \cdot \mathbb{I}_{T_i < \infty}) = F'_i(1) = i[F_1(1)]^{i-1} \cdot F'_1(1) = \begin{cases} \frac{i}{p-q} & p > q \\ \left(\frac{p}{q}\right)^i \frac{i}{q-p} & p < q \\ \infty & p = q \end{cases} \quad (74)$$

as a result,  $\mathbb{E}(T_i | T_i < \infty) = \frac{i}{|p-q|}$  holds generally.

A slight generalization is still the  $j$ -th hitting time to  $i$ , denoted  $T_i^j$ . To get its generating function  $F_i^j(s)$ , notice that

$$\mathbb{P}(T_i^j = n) = \sum_{k=0}^n \mathbb{P}(T_i^1 = k) \cdot \mathbb{P}(T_i^j = n | T_i^1 = k) \quad (75)$$

$$= \sum_{k=0}^n \mathbb{P}(T_i^1 = k) \cdot \mathbb{P}(T_0^j = n - k) \quad (76)$$

by Markov property, since after hitting  $i$  for the first time we are restarting the SRW from  $i$  and hitting 0 after restarting is equivalent to hitting  $i$  from the very start. As a result,  $F_i^j(s) = F_i(s) \cdot F_0^j(s)$ , by previous proofs, we know that  $F_i(s) = [F_1(s)]^i$  and  $F_0^j(s) = [F_0(s)]^j$ , so

$$F_i^j(s) = [F_1(s)]^i \cdot [F_0(s)]^j \quad (77)$$

One is also able to calculate the probability and expectations one care about.

$$\mathbb{P}(T_i^j < \infty) = F_i^j(1) = [F_1(1)]^i \cdot [F_0(1)]^j \quad (78)$$

$$= \left(\frac{1 - |p - q|}{2q}\right)^i \cdot (1 - |p - q|)^j \quad (79)$$

and for the expectation

$$\mathbb{E}(T_i^j \cdot \mathbb{I}_{T_i^j < \infty}) = \frac{d}{ds} F_i^j(s) \Big|_{s=1} \quad (80)$$

$$= i[F_1(1)]^{i-1} \cdot F'_1(1) \cdot [F_0(1)]^j + [F_1(1)]^i \cdot j[F_0(1)]^{j-1} \cdot F'_0(1) \quad (81)$$

**Remark.** The only important thing here is the **Markov property**. By selecting appropriate translation of time, one can always transform all  $j$ -th hitting time problems into the first hitting time of 0 and 1.

## Gambler's Ruin

Now for a general SRW, consider the exit time instead of the hitting time. Assume now the SRW starts at  $x$  and  $T_{a,b}$  denotes the stopping time when SRW hits either  $a$  or  $b$  with  $a < x < b$ . It's quite clear that  $T_{a,b} = T_a \wedge T_b$ . This is telling us that if  $p > q$  then  $T_b < \infty$  a.s., if  $p < q$  then  $T_a < \infty$  a.s., if  $p = q$  then  $T_a, T_b < \infty$  a.s.. As a result,  $T_{a,b} < \infty$  a.s. is almost surely finite.

As a result, a natural question to ask is that what's the probability that the SRW is exiting from  $a$ . Since  $T_{a,b} < \infty$  a.s.,

$$\mathbb{P}_x(S_{T_{a,b}} = a) + \mathbb{P}_x(S_{T_{a,b}} = b) = 1 \quad (82)$$

where  $\mathbb{P}_x$  means the probability measure of the SRW starting from  $x$ . Set

$$r(x) = \mathbb{P}_x(S_{T_{a,b}} = a) \quad (83)$$

and apply the Markov property to consider the first step

$$r(x) = p \cdot \mathbb{P}_x(S_{T_{a,b}} = a | X_1 = 1) + q \cdot \mathbb{P}_x(S_{T_{a,b}} = a | X_1 = -1) \quad (84)$$

$$= p \cdot \mathbb{P}_{x+1}(S_{T_{a,b}} = a) + q \cdot \mathbb{P}_{x-1}(S_{T_{a,b}} = a) \quad (85)$$

$$= p \cdot r(x+1) + q \cdot r(x-1) \quad (86)$$

here  $\mathbb{P}_x(S_{T_{a,b}} = a | X_1 = 1) = \mathbb{P}_{x+1}(S_{T_{a,b}} = a)$  is due to the fact that we can stop the SRW at time 1 and restart it as if it starts from  $x+1$  at time 0. The boundary condition is  $r(a) = 1, r(b) = 0$ .

Use the characteristic equation to solve the recurrence relationship:

$$\lambda = p\lambda^2 + q \quad (87)$$

$$\lambda = 1 \text{ or } \frac{q}{p} \quad (88)$$

we have to discuss whether  $p = q$  since there might be roots with multiplicity.

If  $p = q$ ,  $\lambda = 1$  has multiplicity 2 so

$$r(x) = (C_1x + C_2) \cdot 1^x \quad (89)$$

for some constant  $C_1, C_2$ , plug in boundary condition to solve out

$$C_1 = \frac{1}{a-b}, C_2 = -\frac{b}{a-b} \quad (90)$$

so we conclude

$$\mathbb{P}_x(S_{T_{a,b}} = a) = \frac{x-b}{a-b} \quad (91)$$

when  $p = q = \frac{1}{2}$  in the symmetric case.

Now if  $p \neq q$ , there are two different roots and

$$r(x) = C_1 \cdot 1^x + C_2 \cdot \left(\frac{q}{p}\right)^x \quad (92)$$

for some constant  $C_1, C_2$ , plug in boundary condition to solve out

$$C_1 = -\frac{\left(\frac{q}{p}\right)^b}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b}, C_2 = \frac{1}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b} \quad (93)$$

so we conclude

$$\mathbb{P}_x(S_{T_{a,b}} = a) = \frac{\left(\frac{q}{p}\right)^x - \left(\frac{q}{p}\right)^b}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b} \quad (94)$$

when  $p \neq q$  in the asymmetric case.

## Law of Arcsine

The law of arcsine describes the asymptotic distribution of the **last hitting time to 0 and the overall time above 0** for **symmetric SRW**. The setting of the problem is that the last hitting time to 0 in time interval  $[0, 2n]$  is defined as

$$L_{2n} = \sup \{m \leq 2n : S_m = 0\} \quad (95)$$

note that if the time is not bounded above, such random variable would not even be a stopping time (prove using strong Markov property by contradiction). Consider  $0 \leq \frac{L_{2n}}{2n} \leq 1$ , we would prove that such quotient has the law of arcsine (SRW starts from 0).

Let's start by observing that

$$\forall 0 \leq k \leq n, k \in \mathbb{N}, \mathbb{P}(L_{2n} = 2k) = \mathbb{P}(S_{2k} = 0) \cdot \mathbb{P}(L_{2n} = 2k | S_{2k} = 0) \quad (96)$$

$$= \mathbb{P}(S_{2k} = 0) \cdot \mathbb{P}(S_1 \neq 0, S_2 \neq 0, \dots, S_{2n-2k} \neq 0) \quad (97)$$

by Markov property that we stop SRW at time  $2k$  and restart it as if it starts from 0 at time 0. Due to former calculations,  $\mathbb{P}(S_1 \neq 0, S_2 \neq 0, \dots, S_{2n-2k} \neq 0) = \mathbb{P}(S_{2n-2k} = 0)$  so

$$\forall 0 \leq k \leq n, k \in \mathbb{N}, \mathbb{P}(L_{2n} = 2k) = \mathbb{P}(S_{2k} = 0) \cdot \mathbb{P}(S_{2n-2k} = 0) \quad (98)$$

now since

$$\mathbb{P}(S_{2k} = 0) \cdot \mathbb{P}(S_{2n-2k} = 0) = \frac{\binom{2k}{k} \cdot \binom{2n-2k}{n-k}}{2^{2n}} \quad (99)$$

$$\sim \frac{\sqrt{2k} \sqrt{(2n-2k)}}{2\pi k(n-k)} \quad (n \rightarrow \infty) \quad (100)$$

$$= \frac{1}{\pi} \frac{1}{\sqrt{k(n-k)}} \quad (n \rightarrow \infty) \quad (101)$$

by Stirling's formula, as a result, if  $\frac{k}{n} \rightarrow x$  ( $n \rightarrow \infty$ )

$$n \cdot \mathbb{P}(L_{2n} = 2k) \rightarrow \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}} \quad (102)$$

which provides the main thought of the law of arcsine

$$\forall 0 < a \leq b < 1, \mathbb{P}\left(a \leq \frac{L_{2n}}{2n} \leq b\right) \rightarrow \int_a^b \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}} dx \quad (n \rightarrow \infty) \quad (103)$$

the details can be verified by proving  $\frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}$  is the uniform limit on any compact set  $[a, b]$ . The "arcsine" comes from the fact that

$$\int_a^b \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}} dx = \frac{2}{\pi} \arcsin \sqrt{x} \Big|_{(a,b)} = \frac{2}{\pi} \arcsin \sqrt{b} - \frac{2}{\pi} \arcsin \sqrt{a} \quad (104)$$

**Remark.** The law of arcsine is interesting if we think of the following bet that we are having 0 money at first and by tossing the coin we can get 1 or -1 for the same probability  $\frac{1}{2}$ , which means that this is a totally fair bet.

However, by the law of arcsine,

$$\mathbb{P}\left(a \leq \frac{L_{2n}}{2n} \leq \frac{1}{2}\right) = \frac{1}{2} - \frac{2}{\pi} \arcsin \sqrt{a} \rightarrow \frac{1}{2} \quad (a \rightarrow 0, n \rightarrow \infty) \quad (105)$$

$$\mathbb{P}(L_{2n} \leq n) \rightarrow \frac{1}{2} \quad (n \rightarrow \infty) \quad (106)$$

which means that if we are keeping betting until time  $2n$  where  $n$  is a large enough time, we have  $\frac{1}{2}$  probability seeing that we are always having positive amount of money or negative amount of money after time  $n$ . So the asymptotic behavior of this fair bet model is now clear. If we are keeping betting until time  $2n$  where  $n$  is a large enough time, we have  $\frac{1}{4}$  probability of becoming a "winner", who always enjoys positive return in the latter half of the bet; we have  $\frac{1}{4}$  probability of becoming a "loser", who always suffers from negative return in the latter half of the bet; we have  $\frac{1}{2}$  probability of becoming a "normal person", whose return fluctuates up and down around 0.

This is telling us that even in totally fair games, the accumulation in time matters and presents **concentration** phenomenon. This can be seen from the density  $\frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}$  that the likelihood is lowest at  $\frac{1}{2}$  but goes to  $\infty$  at 0, 1, which means that extreme values of  $\frac{L_{2n}}{2n}$  are far more likely to be observed (either never hits 0 or always hits 0).

Eventually, one might notice that for a symmetric SRW starting from 0, the overall time it spends above 0 also has the law of arcsine.

$$\pi_{2n} = \# \{(t, S_t) : 0 \leq t \leq 2n, S_t \geq 0\} \quad (107)$$

be the overall time during  $[0, 2n]$  such that SRW takes positive values. Then

$$\forall 0 < a \leq b < 1, \mathbb{P}\left(a \leq \frac{\pi_{2n}}{2n} \leq b\right) \rightarrow \int_a^b \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}} dx \quad (n \rightarrow \infty) \quad (108)$$

One might notice that actually  $\pi_{2n} \stackrel{d}{=} L_{2n}$ , the reason is that we can break up the event according to when the SRW first hits 0 and whether the SRW before the first hitting time to 0 is positive or negative

$$\mathbb{P}(\pi_{2n} = 2k) = \sum_{m=1}^n \mathbb{P}(\pi_{2n} = 2k | T_0 = 2m, S_{0 \rightarrow T_0} \geq 0) \cdot \mathbb{P}(T_0 = 2m, S_{0 \rightarrow T_0} \geq 0) \quad (109)$$

$$+ \sum_{m=1}^n \mathbb{P}(\pi_{2n} = 2k | T_0 = 2m, S_{0 \rightarrow T_0} \leq 0) \cdot \mathbb{P}(T_0 = 2m, S_{0 \rightarrow T_0} \leq 0) \quad (110)$$

$$= \frac{1}{2} \sum_{m=1}^k \mathbb{P}(\pi_{2n} = 2k | T_0 = 2m, S_{0 \rightarrow T_0} \geq 0) \cdot \mathbb{P}(T_0 = 2m) \quad (111)$$

$$+ \frac{1}{2} \sum_{m=1}^{n-k} \mathbb{P}(\pi_{2n} = 2k | T_0 = 2m, S_{0 \rightarrow T_0} \leq 0) \cdot \mathbb{P}(T_0 = 2m) \quad (112)$$

$$= \frac{1}{2} \sum_{m=1}^k \mathbb{P}(\pi_{2n-2m} = 2k - 2m) \cdot \mathbb{P}(T_0 = 2m) + \frac{1}{2} \sum_{m=1}^{n-k} \mathbb{P}(\pi_{2n-2m} = 2k) \cdot \mathbb{P}(T_0 = 2m) \quad (113)$$

by Markov property. When the segment before  $T_0 = 2m$  is positive, we just need another  $2k - 2m$  to be positive in the remaining  $2n - 2m$  time by restarting the SRW from 0. When the segment before  $T_0 = 2m$  is negative, there's no contribution to  $\pi_{2n}$ , so we still need  $2k$  to be positive in the remaining  $2n - 2m$  time by restarting the SRW from 0.

Now notice that

$$\mathbb{P}(\pi_{2n} = 2n) = \mathbb{P}(S_1, \dots, S_{2n} \geq 0) \quad (114)$$

$$= 2\mathbb{P}(S_1, \dots, S_{2n} > 0) \quad (115)$$

$$= \mathbb{P}(S_1, \dots, S_{2n} \neq 0) \quad (116)$$

$$= \mathbb{P}(S_{2n} = 0) \quad (117)$$

where the second equation comes from the reflection principle and the last equation is the property we have proved.

Now apply backward induction, the conclusion

$$\mathbb{P}(\pi_{2n} = 2k) = \mathbb{P}(S_{2k} = 0) \cdot \mathbb{P}(S_{2n-2k} = 0) \quad (118)$$

holds for  $k = n$ . Assume that it's true for  $k + 1, k + 2, \dots, n$ , let's see whether it's true for  $k$

$$\mathbb{P}(\pi_{2n} = 2k) = \frac{1}{2} \sum_{m=1}^k \mathbb{P}(\pi_{2n-2m} = 2k - 2m) \cdot \mathbb{P}(T_0 = 2m) + \frac{1}{2} \sum_{m=1}^{n-k} \mathbb{P}(\pi_{2n-2m} = 2k) \cdot \mathbb{P}(T_0 = 2m) \quad (119)$$

$$= \frac{1}{2} \sum_{m=1}^k \mathbb{P}(S_{2k-2m} = 0) \cdot \mathbb{P}(S_{2n-2k} = 0) \cdot \mathbb{P}(T_0 = 2m) + \frac{1}{2} \sum_{m=1}^{n-k} \mathbb{P}(S_{2k} = 0) \cdot \mathbb{P}(S_{2n-2m-2k} = 0) \cdot \mathbb{P}(T_0 = 2m) \quad (120)$$

$$= \frac{1}{2} \mathbb{P}(S_{2n-2k} = 0) \mathbb{P}(S_{2k} = 0) + \frac{1}{2} \mathbb{P}(S_{2k} = 0) \mathbb{P}(S_{2n-2k} = 0) \quad (121)$$

$$= \mathbb{P}(S_{2k} = 0) \cdot \mathbb{P}(S_{2n-2k} = 0) \quad (122)$$

where we used another Markov property that  $\mathbb{P}(S_{2k} = 0) = \sum_{m=1}^k \mathbb{P}(T_0 = 2m) \cdot \mathbb{P}(S_{2k-2m} = 0)$ . As a result, we have proved that

$$\pi_{2n} \stackrel{d}{=} L_{2n} \quad (123)$$

so the law of arcsine also holds.