Notes on MFG

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Contents

This note is mainly written based on the author's experience reading *Probability Theory of Mean Field Games with Applications* by *Rene Carmona, Francois Delarue* and is subject to errors and typos. You are welcome to read critically and carefully.

Basic Ideas of MFG

Notation and Assumption of Single-period MFG

We neglect the classical setting of stochastic differential games here and only focus on the special settings of mean field game (MFG) that are worth noting. For notation purpose, we clarify that x^{-i} denotes $(x^1,...,x^{i-1},x^{i+1},...,x^N)$, (x^i,x^{-i}) denotes $x=(x^1,...,x^{i-1},x^i,x^{i+1},...,x^N)$ and (x,x^{-i}) denotes $(x^1,...,x^{i-1},x,x^{i+1},...,x^N)$ with the i-th coordinate replaced with x. J^i always denotes the expected total cost of player i (altogether N players), with α^i denotes its control process taking value in A, the space of all admissible control process (assumed to be compact).

Nash equilibrium (NE) $(\hat{\alpha}^1,...,\hat{\alpha}^N)\in A^N$ is defined as the tuple of control such that for any player, when all other players' controls are frozen, this player has no motivation of deviating from it. In simple notations,

$$\forall i\in\{1,2,...,N\},\forall\alpha\in A, J^i(\hat{\alpha}^i,\hat{\alpha}^{-i})\leq J^i(\alpha,\hat{\alpha}^{-i}) \tag{1}$$

for theoretical analysis on NE, it's useful to represent it as a fixed point of the **best response function** $B:A^N\rightarrow A^N$ defined as

$$B(\alpha)=\beta,\ \beta^i\overset{def}{=}\arg\min_{\alpha} J^i(\alpha,\alpha^{-i}) \tag{2}$$

where β^i denotes the best reaction of player i given all other players' control.

MFG requires strong assumptions on symmetricity of players and the influence of each player diminishing. This requires us to define the **empirical measure**

$$\overline{\mu}^n_X\overset{def}{=}\frac{1}{n}\sum_{i=1}^n\delta_{x^i} \tag{3}$$

with $X=(x^1,...,x^n)\in E^n$ as given state of all players. The empirical measure is a probability measure on compact metric space E, we denote the space of all probability measures on E as $\mathscr{P}(E)$ and equip this space with the topology of weak convergence of measures (with a compatible metric ρ on $\mathscr{P}(E)$ so it's compact).

To see the assumptions required for MFG setting, we focus on adding restrictions to the cost functionals J^i when N gets large enough. The following technical lemma helps us figure out what assumptions to put up for MFG setting.

Lemma 1. *$\forall n$, if $u^n:E^n\rightarrow\mathbb{R}$ satisfies uniform boundedness condition*

$$\sup_n\sup_{X\in E^n}|u^n(X)|<\infty \tag{4}$$

and uniform Lipschitz continuity

$$\exists c>0,\forall n,\forall X,Y\in E^n,|u^n(X)-u^n(Y)|\leq c\rho(\overline{\mu}_X^n,\overline{\mu}_Y^n) \tag{5}$$

then exists subsequence u^{n_k} and Lipschitz continuous $U:\mathscr{P}(E)\to\mathbb{R}$ such that

$$\lim_{k\to\infty}\sup_{X\in E^{n_k}}|u^{n_k}(X)-U(\overline{\mu}_X^{n_k})|=0 \tag{6}$$

Remark. *Instead of showing the proof (which is an application of Arzela-Ascoli on some auxiliary function), we show the interpretation of the lemma. Notice that $\forall n, u^n$ must be* ***symmetric*** *and* ***continuous*** *if it satisfies the conditions above. The symmetricity is a consequence of uniform Lipschitz continuity (take Y whose components are permutations of those of X, this does not change the empirical measure, resulting in $u^n(X)=u^n(Y)$). On the other hand, if $X^{n_p}\to X^n$ $(p\to\infty)$, $\overline{\mu}^n_{X^{n_p}}\xrightarrow{w}\overline{\mu}^n_{X^n}$ resulting in $u^n(X^{n_p})\to u^n(X^n)$ $(p\to\infty)$ proves the continuity of u^n.*

As a result, those two conditions in the lemma have already contained strong assumptions on symmetricity. The conclusion is saying that at least along some subsequence, the function u^{n_k} can be uniformly well approximated as a regular function of the empirical measure if there are enough players.

Inspired by the lemma above, we put up the assumption called **large symmetric cost functional** that $\forall N,\exists J^N:A^N\to\mathbb{R}$ such that

$$\begin{cases}\forall N,\alpha\in A^N,J^{N,i}(\alpha)=J^N(\alpha^i,\alpha^{-i})\\ \sup_N\sup_{\alpha\in A^N}|J^N(\alpha)|<\infty\\ \exists c>0,\forall N,\forall\alpha,\beta\in A^N,|J^N(\alpha)-J^N(\beta)|\leq c\left[d_A(\alpha^1,\beta^1)+\rho(\overline{\mu}^{N-1}_{\alpha^{-1}},\overline{\mu}^{N-1}_{\beta^{-1}})\right]\end{cases} \tag{7}$$

where $J^{N,i}$ is the cost functional of player i, having the same meaning as J^i, d_A is the metric on A. Viewing $J^N(\alpha)=J^N(\alpha^1,\alpha^{-1})$ with the control of player 1 separated (due to the form of the third assumption in large symmetric cost functional), the same proving procedure tells us $\exists J:A\times\mathscr{P}(A)\to\mathbb{R}$ continuous and a subsequence N_k such that

$$\lim_{k\to\infty}\sup_{\alpha^{N_k}\in A^{N_k}}|J^{N_k}(\alpha^{N_k,1},...,\alpha^{N_k,N_k})-J(\alpha^{N_k,1},\overline{\mu}^{N_k-1}_{\alpha^{N_k,-1}})|=0 \tag{8}$$

in simple words, for each player (e.g. player 1), the cost functional can be uniformly well approximated in a way that all other players make a difference to the cost functional only through the empirical measure which is $\overline{\mu}^{N_k-1}_{\alpha^{N_k,-1}}=\frac{1}{N_k-1}\sum_{i=2}^{N_k}\delta_{\alpha^{N_k,i}}$.

So far, we have put up the large symmetric cost functional assumption providing us with a **limiting cost functional** J as an approximation to the cost functional of each player when $N\to\infty$. However, it should also be expected that MFG can be represented in terms of some population measure μ as some kind of weak limit of the sequence of empirical measures. If this is the case, MFG will have a simple structure since we only have to play with the limiting functional and population measure J,μ instead of a sequence of cost functional and empirical measures. The following theorem adopts such intuition and add some extra conditions to ensure the existence and good properties of population measure as weak limit.

Theorem 1. *Assume for* $\forall N$, $\hat{\alpha}^N = (\hat{\alpha}^{N,1}, ..., \hat{\alpha}^{N,N})$ *is NE for the game with cost functional* $J^{N,1}, ..., J^{N,N}$ *that satisfies the large symmetric cost functional assumption. Moreover, assume that*

$$\exists c > 0, \forall N, \alpha \in A, \mu \in \mathscr{P}(A), \rho\left(\mu, \frac{N-1}{N}\mu + \frac{1}{N}\delta_\alpha\right) \le \frac{c}{N} \tag{9}$$

then there exists subsequence N_k *and continuous* $J : A \times \mathscr{P}(A) \to \mathbb{R}$ *such that*

$$\overline{\mu}^{N_k}_{\hat{\alpha}^{N_k}} \stackrel{w}{\to} \hat{\mu} \in \mathscr{P}(A) \ (k \to \infty) \tag{10}$$

with

$$\lim_{k\to\infty} \sup_{\alpha^{N_k} \in A^{N_k}} |J^{N_k}(\alpha^{N_k,1}, ..., \alpha^{N_k,N_k}) - J(\alpha^{N_k,1}, \overline{\mu}^{N_k-1}_{\alpha^{N_k,-1}})| = 0 \tag{11}$$

and

$$\int_A J(\alpha, \hat{\mu})\hat{\mu}(d\alpha) = \inf_{\mu \in \mathscr{P}(A)} \int_A J(\alpha, \hat{\mu})\mu(d\alpha) \tag{12}$$

Remark. *The second conclusion has been proved using the technical lemma, and since* A *is compact,* $\overline{\mu}^{N_k}_{\hat{\alpha}^{N_k}} \in \mathscr{P}(A)$ *is tight, by Prokhorov's theorem, it has weak subsequential limit proves the existence of* $\hat{\mu}$*. The proof of the last conclusion requires the similar technique as the proof of the technical lemma above, omitted here.*

The interpretation of the last conclusion is crucial for the setting of MFG. Let's specify $\mu = \delta_{\alpha_0}$ *where* $\alpha_0 = \arg\min_\alpha J(\alpha, \hat{\mu})$ *to see*

$$J(\alpha_0, \hat{\mu}) = \inf_{\mu \in \mathscr{P}(A)} \int_A J(\alpha_0, \hat{\mu})\mu(d\alpha) \le \inf_{\mu \in \mathscr{P}(A)} \int_A J(\alpha, \hat{\mu})\mu(d\alpha) \le J(\alpha_0, \hat{\mu}) \tag{13}$$

so

$$\int_A J(\alpha, \hat{\mu})\hat{\mu}(d\alpha) = J(\alpha_0, \hat{\mu}) \tag{14}$$

now denote $A_{\hat{\mu}} = \{\alpha_0 \in A : \alpha_0 = \arg\min_\alpha J(\alpha, \hat{\mu})\}$ *as the collection of all controls minimizing the limiting cost functional at the population measure,*

$$\int_A J(\alpha, \hat{\mu})\hat{\mu}(d\alpha) = J(\alpha_0, \hat{\mu}) \cdot \hat{\mu}(A_{\hat{\mu}}) + \int_{A - A_{\hat{\mu}}} J(\alpha, \hat{\mu})\hat{\mu}(d\alpha) \le J(\alpha_0, \hat{\mu}) \cdot \hat{\mu}(A_{\hat{\mu}}) \tag{15}$$

proves $\hat{\mu}(A_{\hat{\mu}}) = 1$. *Conversely, if* $\hat{\mu}(A_{\hat{\mu}}) = 1$, $\int_A J(\alpha, \hat{\mu})\hat{\mu}(d\alpha) = J(\alpha_0, \hat{\mu})$ *immediately proves*

$$\int_A J(\alpha, \hat{\mu})\hat{\mu}(d\alpha) = \inf_{\mu \in \mathscr{P}(A)} \int_A J(\alpha, \hat{\mu})\mu(d\alpha) \tag{16}$$

as a result, this conclusion is saying that ***the support of population measure*** $\hat{\mu}$ ***is contained in the set of minimum*** $\arg\min_\alpha J(\alpha, \hat{\mu})$.

At this point, the setting of MFG shall be clear, which only depends on the limiting cost functional J and the population measure $\hat{\mu}$ of a single representative player. MFG is a game of N identical players as $N \to \infty$ and we care about the NE of MFG. Different from finite player game, there is an extra population measure in MFG, resulting in the fact that MFG has to take into consideration both optimality (minimizing the cost functional) and consistency (each player shall behave according to the population measure) conditions. **The basic strategy for MFG is to fix population measure as μ, solve out the set of best control for fixed μ:**

$$A_\mu = \arg\min_{\alpha \in A} J(\alpha, \mu) \tag{17}$$

and find a measure $\hat{\mu}$ that is concentrated on the arguments of the minimization $A_{\hat{\mu}}$. Notice that the restrictions on $\hat{\mu}$ directly comes from the conclusion in the theorem above that

$$\int_A J(\alpha, \hat{\mu})\hat{\mu}(d\alpha) = \inf_{\mu \in \mathscr{P}(A)} \int_A J(\alpha, \hat{\mu})\mu(d\alpha) \tag{18}$$

equivalently speaking,

$$supp(\hat{\mu}) \subset \arg\min_{\alpha \in A} J(\alpha, \hat{\mu}) \tag{19}$$

the **solution to MFG** is the population measure $\hat{\mu}$ and the control $\hat{\alpha}$ following the population measure $\hat{\mu}$ is a NE since it's the fixed point of the best response function (notice that all other players' control only make a difference through the population measure as N gets large enough).

Instead of existence of the solution, the uniqueness of the solution to this single-period MFG is not always guaranteed. One criterion for uniqueness is based on the strictly monotone property of J.

Theorem 2. *The solution to the single-period MFG is unique if J is strictly monotone, i.e.*

$$\forall \mu_1 \neq \mu_2, \int_A [J(\alpha, \mu_1) - J(\alpha, \mu_2)]\,[\mu_1 - \mu_2](d\alpha) > 0 \tag{20}$$

Proof. We prove it from definition that if μ_1, μ_2 are two different solutions to the MFG,

$$\int_A J(\alpha, \mu_1)\,\mu_1(d\alpha) \leq \int_A J(\alpha, \mu_1)\,\mu_2(d\alpha), \int_A J(\alpha, \mu_2)\,\mu_2(d\alpha) \leq \int_A J(\alpha, \mu_2)\,\mu_1(d\alpha) \tag{21}$$

sum up to find a contradiction with the strictly monotone condition. □

An Example of Mean Field Approximation

Consider the setting when a meeting is planned to start at deterministic time $t_{\geq}0$, player i has its control $\alpha^i = t_i$ as the time planned to attend the meeting. However, there is random effect so player i actually attends meeting at time X^i with

$$X^i = \alpha^i + \sigma^i \varepsilon^i \tag{22}$$

where $\varepsilon^1, \varepsilon^2, ... \overset{i.i.d.}{\sim} N(0,1), \sigma^1, \sigma^2, ... \overset{i.i.d.}{\sim} \nu$ with ν has its support on $(0,\infty)$ and the sequence ε^i is independent of the sequence σ^i. The cost functional of player i is

$$J^i(\alpha) = \mathbb{E}[a(X^i - t_0)^+ + b(X^i - t)^+ + c(t - X^i)_+] \tag{23}$$

when the meeting actually starts at time t instead of the original planned time t_0. The actual starting time t is actually determined based on the arrival time of all players, i.e.

$$t = \tau(\overline{\mu}_X^N) \tag{24}$$

for some deterministic function τ (e.g. start the meeting when a certain percentage of players arrive). This game is a one-period game so there's no SDE dynamics and the only interaction between players is through the empirical measure $\overline{\mu}_X^N$ (but this empirical measure is a measure on the state space, not control space).

If we treat this game as a finite player game, to get NE we need to do optimization

$$\forall i \in \{1, 2, ..., N\}, \inf_{\alpha^i} J^i(\alpha) \tag{25}$$

and the biggest trouble comes from the empirical measure that couples N optimization problems.

However, as long as we know that N is large enough, since ε^i, σ^i are *i.i.d.* sequence of random variables and the cost functional satisfies the large symmetric cost functional assumption for MFG, MFG approximation allows us to have $\overline{\mu}_X^N \overset{w}{\to} \mu$ $(N \to \infty)$ and to replace $t = \tau(\overline{\mu}_X^N)$ with $t = \tau(\mu)$ (since MFG approximation happens simultaneously for the empirical measure and the cost functional). At this point, we just need to solve the optimization for a representative player with cost functional

$$J(\alpha, \mu) = \mathbb{E}[a(X - t_0)^+ + b(X - t)^+ + c(t - X)^+] \ (a, b, c > 0) \tag{26}$$

where $t = \tau(\mu)$, $X = \alpha + \sigma\varepsilon$ with α as the control. For fixed μ, i.e. fixed t, let's do the minimization

$$A_\mu = \arg\min_\alpha J(\alpha, \mu) \tag{27}$$

take weak derivative for J w.r.t. α to find

$$\frac{\partial J}{\partial \alpha} = a\mathbb{P}\left(\alpha + \sigma\varepsilon - t_0 > 0\right) + b\mathbb{P}\left(\alpha + \sigma\varepsilon - t > 0\right) - c\mathbb{P}\left(-\alpha - \sigma\varepsilon + t > 0\right) \tag{28}$$

if we denote $Z = \sigma\varepsilon$ and F the CDF of Z, since Z has symmetric distribution around zero, $F(z) + F(-z) = 1$ so

$$A_\mu = \{\alpha \geq 0 : aF(\alpha - t_0) + (b + c)F(\alpha - t) = c\} \tag{29}$$

is determined implicitly by the equation on α. Since zero is not in the support of ν where $\sigma \sim \nu$, F is strictly positive, strictly increasing and continuous, it's then obvious that A_μ only contains a single point for fixed μ (the equation has unique solution).

That's all the work for the first optimization step, and the next step is to find the population measure that satisfies the consistency condition, that is to find measure $\hat{\mu} \overset{d}{=} \hat{\alpha} + \sigma\varepsilon$ such that $\hat{\alpha}$ is consistent with $\hat{\mu}$. It's clear that $\hat{\mu}$ is a measure on $\mathbb{R}_+$ induced by CDF $F(z - \hat{\alpha})$ so we denote the measure $\hat{\mu}$ as $F(\cdot - \hat{\alpha})$. When the representative player takes NE control $\hat{\alpha}$ and the population measure is $\hat{\mu}$, we know that $\hat{\alpha} \in A_{\hat{\mu}}$ resulting in

$$\begin{cases} aF(\hat{\alpha} - t_0) + (b + c)F(\hat{\alpha} - t) = c \\ t = \tau(F(\cdot - \hat{\alpha})) \end{cases} \tag{30}$$

as an equation w.r.t. $\hat{\alpha}$ (notice that τ maps a measure to a real number). With come constraints on τ added, one is able to ensure the existence and uniqueness of $\hat{\alpha}$ as the solution to the equations above and **the solution to MFG** is just $\hat{\mu} = F(\cdot - \hat{\alpha})$.

Remark. *Although it's not relevant to MFG, I would like to show the method of arguing the existence and uniqueness of $\hat{\alpha}$ by adding constraints on τ since it's a typical and crucial application of the contraction mapping theorem.*

Let's assume τ always takes value no less than t_0, is monotone, i.e. $\forall \alpha \geq 0$, if $\mu([0,\alpha]) \leq \mu'([0,\alpha])$ then $\tau(\mu) \geq \tau(\mu')$ and has sub-additivity, i.e. $\forall \alpha \geq 0, \tau(\mu(\cdot - \alpha)) \leq \tau(\mu) + \alpha$.

The proof starts from ***building $G : \mathbb{R}_+ \to \mathbb{R}_+$ mapping α to $G(\alpha)$ such that $\hat{\alpha}$ is the fixed point of G****. Function G first maps α to a measure $F(\cdot - \alpha)$ then to a real number $t = \tau(F(\cdot - \alpha))$ then to $\alpha(t)$ determined by the equation $aF(\alpha - t_0) + (b + c)F(\alpha - t) = c$ for given t. It's quite obvious that if we input $\hat{\alpha}$, the output will be $G(\hat{\alpha}) = \hat{\alpha}$ so it's the fixed point of G.*

The next step is to ***show that G is a contraction mapping****. Take $x, y \in \mathbb{R}_+, x < y$, by monotonicity $\tau(F(\cdot - x)) \leq \tau(F(\cdot - y))$ and by sub-additivity $\tau(F(\cdot - y)) \leq \tau(F(\cdot - x)) + (y - x)$ provides bounds for the first two mapping steps (map real number α to a measure and to a real number t again). The remaining step is the last step implicitly mapping t to $\alpha(t)$. Implicit function theorem is thus necessary. Assume*

$$H(\alpha, t) = aF(\alpha - t_0) + (b + c)F(\alpha - t) - c \tag{31}$$

and check $\frac{\partial H}{\partial \alpha} = aF'(\alpha - t_0) + (b+c)F'(\alpha - t) > 0$ since F is strictly increasing and $a, b, c > 0$, H is also continuous *with continuous partial derivatives, so* $H(\alpha, t) = 0$ *determines* C^1 *function* $\alpha(t)$ *with*

$$\frac{d\alpha(t)}{dt} = \frac{(b+c)F'(\alpha(t)-t)}{aF'(\alpha(t)-t_0)+(b+c)F'(\alpha(t)-t)} \tag{32}$$

since $F' > 0$, *the derivative only takes value in* $(0, C)$ *with constant* $0 < C < 1$. *Now we denote*

$$\begin{cases} t_x = \tau(F(\cdot - x)) \\ t_y = \tau(F(\cdot - y)) \end{cases} \tag{33}$$

and calculate

$$|G(y) - G(x)| = |\alpha(t_y) - \alpha(t_x)| \le C|t_y - t_x| \le C|y - x| \tag{34}$$

with $0 < C < 1$ *proves that* G *is contraction mapping proves the existence and uniqueness of* $\hat{\alpha}$ *as the fixed point.*

The difficulty here is that a real number is first mapped to a measure then to a real number again under some implicit function. However, by following the spirit of fixed point iteration, it's not hard to construct G ***and show that it's a contraction mapping by adding necessary conditions.***

This example shows from the beginning how we approximate finite player game with MFG for one-period game and how to solve one-period MFG following the strategy mentioned above. The continuous-time MFG will be much harder to solve, but the approximation and solution follows the similar logic that we always care about optimality and consistency conditions.

For more details and interpretations on the mean field approximation for single-period game, please refer to my handwritten notes *Notes on Mean Field Approximation of Single-Period Games* on my personal website.

Probabilistic Approach to Finite Player Game

Finite Player Game and Different Types of NE

Here we quickly review the setting of finite player game and the probabilistic approach to solving finite player games. Most of the details are omitted and only notations are introduced. In the continuous time setting, different players' states are coupled through SDE dynamics (a Markovian diffusion)

$$\begin{cases} dX_t = B(t, X_t, \alpha_t)\, dt + \Sigma(t, X_t, \alpha_t)\, dW_t \\ X_0 = x_0 \end{cases} \tag{35}$$

where X takes value in $\mathbb{R}^N$ denoting the states of all players. Written in terms of each player,

$$\begin{cases} dX_t^i = b^i(t, X_t, \alpha_t)\, dt + \sigma^i(t, X_t, \alpha_t)\, dW_t^i + \sigma^0(t, X_t, \alpha_t)\, dW_t^0 \\ X_0^i = x_0^i \end{cases} \tag{36}$$

where W^0 is a BM denoting the common noise shared by all players (often we neglect this) and the time horizon of the game is always assumed to be $[0, T]$. We add **assumptions called Game** here to ensure the existence and uniqueness of the solution to the coupled SDE system and to guarantee measurability. It's not very interesting to investigate those assumptions carefully so they are just listed here for the purpose of completeness

$$\begin{cases} \forall S \in [0, T], (t, \omega, x, \alpha) \mapsto (B, \Sigma)(t, \omega, x, \alpha) \quad \text{progressive measurable} \\ \exists c > 0, \forall t \in [0, T], \omega, x, x', \alpha, \alpha', |B(t, \omega, x, \alpha) - B(t, \omega, x', \alpha')| + |\Sigma(t, \omega, x, \alpha) - \Sigma(t, \omega, x', \alpha')| \leq c(|x - x'| + |\alpha - \alpha'|) \\ \forall \alpha, \mathbb{E} \int_0^T |B(t, 0, \alpha_t)|^2 + |\Sigma(t, 0, \alpha_t)|^2\, dt < \infty \\ \forall S \in [0, T], (t, \omega, x, \alpha) \mapsto f^i(t, \omega, x, \alpha) \quad \text{progressive measurable} \\ (\omega, x) \mapsto g^i(\omega, x) \quad \text{measurable} \end{cases} \tag{37}$$

here f^i is the running cost while g^i is the terminal cost, together they provide the cost functional of player i

$$J^i(\alpha) = \mathbb{E}\left[\int_0^T f^i(s, X_s, \alpha_s)\, ds + g^i(X_T)\right] \tag{38}$$

Next we introduce different types of NE. The **open loop NE** allows each player's control to depend on current time, initial state and the whole BM trajectory so far

$$\alpha_t^i = \phi^i(t, X_0, W_{[0,t]}) \tag{39}$$

where ϕ^i is a deterministic function. In special case where α_t^i has no dependence on W, we call it an open loop deterministic NE.

Remark. *The crucial point to understand here is that open loop NE admits no feedback, i.e. even after player i changes its strategy from $\hat{\alpha}^i$ to α^i, $\hat{\alpha}^{-i}$ maintains its original trajectory.*

On the other hand, **closed loop NE** allows the dependence of control on the current time and the whole state trajectory so far

$$\alpha_t^i = \phi^i(t, X_{[0,t]}) \tag{40}$$

notice that X depends on α in its dynamics so it's actually an implicit equation w.r.t. α. Closed loop NE has feedback effect in that the change of control of player i results in the change of control of all other players (α_t^i affects X_t and affects α_t^j $(j \neq i)$).

The final useful notion of NE is the **Markovian NE** allowing the control to depend on current time, initial state, and current state

$$\alpha_t^i = \phi^i(t, X_0, X_t) \tag{41}$$

this NE also has feedback effect but it's different from the closed loop NE in that the control cannot depend on past states. Generally, open loop NE is the easiest to derive but not very practically useful, closed loop NE is the hardest to derive but the closest to reality. As a trade-off between those two, Markovian NE is not so hard to derive and also has practical interpretations.

Classical Approach Solving Finite Player Game

It should be familiar that PDE approach and BSDE approach are the two most useful approaches solving NE for finite player game. The PDE approach is based on dynamic programming principle (DPP) and can only deal with Markovian NE while the BSDE approach is based on Pontryagin's maximum principle and can deal with all different kinds of NE. The proof of those approaches is neglected and can be found in the book as a generalization to that of single player game.

PDE approach solves **Markovian NE** by putting up the **value function of player** i thinking backwardly

$$V^i(t,x) = \inf_{\alpha^i} \mathbb{E}\left[\int_t^T f^i(s, X_s, (\alpha_s^i, \alpha_s^{-i}))\,ds + g^i(X_T)\middle| X_t = x\right] \tag{42}$$

where $\alpha_s^{-i} = \phi^{*,-i}(s, X_s)$ since the NE is Markovian. DPP then provides the Hamilton-Jacobi-Bellman equation (HJBE) that describes the evolution of the value function of player i

$$\begin{cases} \partial_t V^i + \inf_{\alpha^i \in A^i} \left\{\partial_x V^i \cdot B(t,x,\alpha) + \frac{1}{2}\mathrm{Tr}(\partial_{xx} V^i \cdot \Sigma(t,x,\alpha) \cdot \Sigma^T(t,x,\alpha)) + f^i(t,x,\alpha)\right\} = 0 \\ V^i(T,x) = g^i(x) \end{cases} \tag{43}$$

Remark. *Notice that* ***in the HJBE of player*** i *we have*

$$\alpha = (\alpha^i, \phi^{*,-i}(t,x)) \tag{44}$$

since player i can only manage to determine α^i, ***all α^{-i} shall be treated as $\alpha^{-i}(t,x)$, a function of current time t and current state x that has feedback effects.***

For linear-quadratic games, when there's sufficient symmetricity it's always possible to derive the closed-form solution. Typically the first step is to solve the inf in the HJBE of player i to get the Markovian NE $\hat{\alpha}^i$ (an expression containing $V^i, t, x, \alpha^{-i}(t,x)$) for $i \in \{1,2,...,N\}$, plug it back into HJBE and raise an ansatz (typically a quadratic form) to get N coupled Ricatti equations. Solve the Ricatti equations and plug back into the expression for $\hat{\alpha}^i$ to solve the Markovian NE.

It's quite obvious that HJBE is a necessary condition to satisfy for value functions, so rigorously speaking verification steps are required to argue that the solution to the HJBE must be the value function (sufficiency). In practice, however, verification step is always skipped when the finite player game is well-posed. Here we list a set of **assumptions called N-Nash system** that ensures the uniqueness of the solution to the Nash system that is smooth enough. This set of assumptions makes sure that skipping the verification step causes no problems. Although it's also not interesting to look into those assumptions, I list them here for reference.

$$\begin{cases} A^{(N)} \text{ is bounded} \\ B \text{ bounded, uniformly Lipschitz in } \alpha \\ \Sigma \text{ uncontrolled (free of } \alpha\text{), bounded cts, uniformly Lipschitz in } x, \Sigma\Sigma^T \text{ uniformly nondegenerate} \\ f^i \text{ bounded, uniformly Lipschitz in } \alpha, g^i \text{ bounded Lipschitz} \\ \text{Minimizer in Issac condition uniformly Lipschitz in } y \end{cases} \tag{45}$$

To introduce the Issac condition, we need the notion of **Hamiltonian of player** i which is also crucial in Pontryagin's maximum principle defined as

$$H^i(t,x,y^i,z^i,\alpha) = B(t,x,\alpha)\cdot y^i + \mathrm{Tr}(\Sigma^T(t,x,\alpha)\cdot z^i) + f^i(t,x,\alpha) \tag{46}$$

Issac condition holds if $\exists\hat{\alpha}(t,x,y,z)$ such that $\forall i, \forall t\in[0,T], x,y,z, \forall \alpha_i$,

$$H^i(t,x,y^i,z^i,\hat{\alpha}(t,x,y,z)) \le H^i(t,x,y^i,z^i,(\alpha^i,\hat{\alpha}^{-i}(t,x,y,z))) \tag{47}$$

Let's recall that Potryagin's maximum principle in single player game, telling us to minimize the Hamiltonian in minimization problems w.r.t. α, resulting in

$$\hat{\alpha}(t,x,y,z) = \arg\min_{\alpha} H(t,x,y,z,\alpha) \tag{48}$$

and put up the adjoint BSDE together with the FSDE dynamics to get FBSDE systems

$$\begin{cases} dX_t = b(t,X_t,\hat{\alpha}(t,X_t,Y_t,Z_t))\,dt + \sigma(t,X_t,\hat{\alpha}(t,X_t,Y_t,Z_t))\,dW_t \\ X_0 = x_0 \\ dY_t = -\partial_x H(t,X_t,Y_t,Z_t,\hat{\alpha}(t,X_t,Y_t,Z_t))\,dt + Z_t\,dW_t \\ Y_T = \partial_x g(X_T) \end{cases} \tag{49}$$

the solution to the system gives optimal control $\hat{\alpha}$. This method is basically the same in multi player game with the only exception to separate by cases when solving different kinds of NE.

We start with the **BSDE approach for open loop NE**. In this case, there are no feedback effects so let's simply minimize Hamiltonian of each player

$$\hat{\alpha}^i(t,x,y^i,z^i) = \arg\min_{\alpha^i} H^i(t,x,y^i,z^i,(\alpha^i,\alpha^{-i})) \tag{50}$$

and put up the adjoint BSDE together with the FSDE dynamics to get FBSDE systems

$$\begin{cases} dX_t^i = b^i(t, X_t, \hat{\alpha}^i(t, X_t, Y_t^i, Z_t^i))\,dt + \sigma^i(t, X_t, \hat{\alpha}^i(t, X_t, Y_t^i, Z_t^i))\,dW_t^i \\ X_0 = x_0 \\ dY_t^i = -\partial_x H^i(t, X_t, Y_t^i, Z_t^i, \hat{\alpha}(t, X_t, Y_t^i, Z_t^i))\,dt + Z_t^i\,dW_t \\ Y_T^i = \partial_x g^i(X_T) \end{cases} \tag{51}$$

solve it to get open loop NE. The verification condition only requires convexity of H^i in (x, α^i) and the convexity of g^i that can be easily verified.

When it comes to the **BSDE approach for Markovian NE**, the only difference appears in the adjoint BSDE where $-\partial_x H^i$ makes a difference. In the open loop case, $H^i = H^i(t, x, y, z, (\alpha^i, \alpha^{-i}))$ with α^{-i} not depending on x since there's no feedback effect. However, in the Markovian case,

$$H^i = H^i(t, x, y^i, z^i, (\alpha^i, \alpha^{-i}(t, x))), \alpha^{-i}(t, x) = \phi^{*,-i}(t, x) \tag{52}$$

and this is definitely changing the expression of $\partial_x H^i$. Let's collect all dependencies on x of H^i denoting

$$H^{-i}(t, x, y^i, z^i, \alpha^i) = H^i(t, x, y^i, z^i, (\alpha^i, \phi^{*,-i}(t, x))) \tag{53}$$

the driver in the adjoint BSDE for player i is actually $\partial_{x_p} H^{-i}$, an easy application of chain rule shows

$$\partial_{x_p} H^{-i} = \partial_{x_p} H^i(t, x, y^i, z^i, (\alpha^i, \phi^{*,-i}(t, x))) + \sum_{j \neq i} \partial_{\alpha^j} H^i(t, x, y^i, z^i, (\alpha^i, \phi^{*,-i}(t, x))) \cdot \partial_{x_p} \alpha^j \tag{54}$$

where $\alpha^j = \phi^{*,j}(t, x)$ $(j \neq i)$ is a function in terms of t, x from the perspective of player i. As a result, there is **one more summation term in the driver of the adjoint BSDE** and following the same procedure we get the Markovian NE

$$\hat{\alpha}^i(t, x, y^i, z^i) = \arg\min_{\alpha^i} H^i(t, x, y^i, z^i, (\alpha^i, \alpha^{-i}(t, x))) \tag{55}$$

get FBSDE systems

$$\begin{cases} dX_t^i = b^i(t, X_t, \hat{\alpha}^i(t, X_t, Y_t^i, Z_t^i))\,dt + \sigma^i(t, X_t, \hat{\alpha}^i(t, X_t, Y_t^i, Z_t^i))\,dW_t^i \\ X_0 = x_0 \\ dY_t^i = -\left[\partial_x H^i + \sum_{j \neq i} \partial_{\alpha^j} H^i \cdot \partial_x \phi^j(t, X_t)\right]\,dt + Z_t^i\,dW_t \\ Y_T^i = \partial_x g^i(X_T) \end{cases} \tag{56}$$

the verification step still depends on the convexity of H^i and g^i same as that for the open loop NE.

Example: Linear Quadratic (LQ) Flocking Model

For detailed explanation on the systemic risk model, please refer to my notes on stochastic control or chapter 2.5 in the book.

In this model, there are N players (birds), with player i having position X_t^i at time t taking value in $\mathbb{R}^3$. Player i can determine its control α_t^i as the velocity at time t and the state dynamics is given as

$$dX_t^i = \alpha_t^i\,dt + \sigma\,dW_t^i \tag{57}$$

on time horizon $[0, T]$. Player i has cost functional

$$J^i(\alpha) = \mathbb{E}\int_0^T f^i(t, X_t, \alpha_t)\,dt \tag{58}$$

with no terminal cost and the running cost is

$$f^i(t, x, \alpha) = \frac{\kappa^2}{2}|x^i - \overline{x}|^2 + \frac{1}{2}|\alpha^i|^2 \tag{59}$$

with $|\cdot|$ denoting the vector ℓ_2 norm since x^i, α^i take values in $\mathbb{R}^3$. This is a LQ game with mean field interaction through $|x^i - \overline{x}|^2$ so we expect to be able to find the closed-form solution.

Let's first work for the **open loop NE**. Since the diffusion coefficient is constant in the dynamics, reduced Hamiltonian can be used

$$\begin{aligned} H^i(t, x, y^i, z^i, \alpha) &= \alpha \cdot y^i + f^i(t, x, \alpha) && (60)\\ &= \sum_{j=1}^N \alpha^j \cdot y^{i,j} + \frac{\kappa^2}{2}|x^i - \overline{x}|^2 + \frac{1}{2}|\alpha^i|^2 && (61)\end{aligned}$$

to clarify, $y^{i,1}, ..., y^{i,N}$ take values in $\mathbb{R}^3$. Minimize H^i w.r.t. α^i to see

$$\hat{\alpha}^i = -y^{i,i} \tag{62}$$

holds for $\forall i \in \{1, 2, ..., N\}$ by symmetricity. Now since W^i is a BM in $\mathbb{R}^3$, it's clear that player i has process $Z^{i,j,k}$ such that $i, j, k \in \{1, 2, ..., N\}$ and $Z^{i,j,k}$ takes value in $\mathbb{R}^3$. At this point, we write down the adjoint BSDE for player i

$$dY_t^{i,j} = -\partial_{x^j} H^i(t, X_t, Y_t^i, Z_t^i, \hat{\alpha}_t)\,dt + \sum_{k=1}^n Z_t^{i,j,k}\,dW_t^k \tag{63}$$

calculate $\partial_{x^j} H^i = \kappa^2(x^i - \overline{x})(\delta_{ij} - \frac{1}{N})$ and plug in to see the adjoint BSDE

$$dY_t^{i,j} = -\kappa^2(X_t^i - \overline{X}_t)\left(\delta_{ij} - \frac{1}{N}\right)dt + \sum_{k=1}^{n} Z_t^{i,j,k}\,dW_t^k \tag{64}$$

with terminal condition $Y_T^{i,j} = 0$ since $g^i \equiv 0$. We derive the FBSDE system by replacing α with $\hat{\alpha}$

$$\begin{cases} dX_t^i = -Y_t^{i,i}\,dt + \sigma\,dW_t^i \\ dY_t^{i,j} = -\kappa^2(X_t^i - \overline{X}_t)\left(\delta_{ij} - \frac{1}{N}\right)dt + \sum_{k=1}^n Z_t^{i,j,k}\,dW_t^k \\ Y_T^{i,j} = 0 \end{cases} \tag{65}$$

put up the affine ansatz (always used in LQ game) with deterministic η_t

$$Y_t^{i,j} = \eta_t(X_t^i - \overline{X}_t)\left(\delta_{ij} - \frac{1}{N}\right) \tag{66}$$

differentiate both sides

$$dY_t^{i,j} = \left(\delta_{ij} - \frac{1}{N}\right)\left[\dot{\eta}_t(X_t^i - \overline{X}_t)\,dt + \eta_t\,d(X_t^i - \overline{X}_t)\right] \tag{67}$$

it's not hard to figure out $d(X_t^i - \overline{X}_t)$ from the FSDE that

$$d(X_t^i - \overline{X}_t) = -\eta_t(X_t^i - \overline{X}_t)\left(1 - \frac{1}{N}\right)dt + \sigma(\,dW_t^i - \,d\overline{W}_t) \tag{68}$$

a comparison principle of BSDE shows that the dt, dW_t part must be equal correspondingly

$$\begin{cases} -\kappa^2(X_t^i - \overline{X}_t)\left(\delta_{ij} - \frac{1}{N}\right) = \left(\delta_{ij} - \frac{1}{N}\right)\left[\dot{\eta}_t(X_t^i - \overline{X}_t) - \eta_t^2(X_t^i - \overline{X}_t)(1 - \frac{1}{N})\right] \\ Z_t^{i,j,k} = \left(\delta_{ij} - \frac{1}{N}\right)\eta_t\sigma\left(\delta_{i,k} - \frac{1}{N}\right) \end{cases} \tag{69}$$

simplify to get

$$\begin{cases} \eta_t^2(1 - \frac{1}{N}) - \kappa^2 = \dot{\eta}_t \\ Z_t^{i,j,k} = \sigma\left(\delta_{ij} - \frac{1}{N}\right)\left(\delta_{i,k} - \frac{1}{N}\right)\eta_t \end{cases} \tag{70}$$

the ODE w.r.t. η_t has terminal condition $\eta_T = 0$ can be solved easily (Ricatti equation) gives the closed-form open loop NE to this game. The verification step is obvious since H^i is convex in (x, α^i).

Then, let's solve the **Markovian NE** through BSDE approach. The Hamiltonian remains the same while

$$H^i(t,x,y^i,z^i,\alpha) = \alpha^i \cdot y^{i,i} + \sum_{l=1,l\neq i}^{N} \phi^l(t,x)\cdot y^{i,l} + \frac{\kappa^2}{2}|x^i - \overline{x}|^2 + \frac{1}{2}|\alpha^i|^2 \tag{71}$$

since for player i, all other players' controls have feedback effects $\alpha^l = \phi^l(t,x)$ $(l \neq i)$. Minimize H^i w.r.t. α^i to see

$$\hat{\alpha}^i = -y^{i,i} \tag{72}$$

still holds for $\forall i \in \{1,2,...,N\}$ but if we calculate the derivative of H^i w.r.t. x^j to get

$$\partial_{x^j} H^i = \kappa^2(x^i - \overline{x})\left(\delta_{ij} - \frac{1}{N}\right) + \sum_{l=1,l\neq i}^{N} \partial_{x^j}\phi^l(t,x)\cdot y^{i,l} \tag{73}$$

the last summation depends on the specific form of ϕ^l. At this step, ansatz has to be raised prior to the construction of FBSDE in order to proceed. From the open loop NE procedure exhibited above, we naturally put up an ansatz for the feedback function

$$\phi^l(t,x) = \left(1 - \frac{1}{N}\right)(x^l - \overline{x})\mu_t \tag{74}$$

with deterministic μ_t. Simple calculation shows $\partial_{x^j}\phi^l(t,x) = \left(1 - \frac{1}{N}\right)\left(\delta_{jl} - \frac{1}{N}\right)\mu_t I$ and the FBSDE is provided as

$$\begin{cases} dX_t^i = -Y_t^{i,i}\,dt + \sigma\,dW_t^i \\ dY_t^{i,j} = -\left[\kappa^2(X_t^i - \overline{X}_t)\left(\delta_{ij} - \frac{1}{N}\right) + \left(1 - \frac{1}{N}\right)\mu_t\sum_{l=1,l\neq i}^{N}\left(\delta_{jl} - \frac{1}{N}\right)Y_t^{i,l}\right]dt + \sum_{k=1}^{n} Z_t^{i,j,k}\,dW_t^k \\ Y_T^{i,j} = 0 \end{cases} \tag{75}$$

to solve this FBSDE, put up the same ansatz as before

$$Y_t^{i,j} = \mu_t(X_t^i - \overline{X}_t)\left(\delta_{ij} - \frac{1}{N}\right) \tag{76}$$

again comparison principle of BSDE tells us

$$\begin{cases} \dot{\mu}_t = \left(1 - \frac{1}{N}\right)^2\mu_t^2 - \kappa^2 \\ Z_t^{i,j,k} = \left(\delta_{ij} - \frac{1}{N}\right)\mu_t\sigma\left(\delta_{i,k} - \frac{1}{N}\right) \end{cases} \tag{77}$$

after some simplifications. Together with $\mu_T = 0$, it's another Ricatti equation which can be easily solved to get the Markovian NE. This example shows us that open loop NE and Markovian NE are generally different although they have similar forms (ODE for η_t and μ_t).

Probabilistic Approach to MFG

Problem Setting

The problem setting of MFG has state dynamics for player i as

$$\begin{cases} dX_t^i = b(t, X_t^i, \overline{\mu}_{X_t^{-i}}^{N-1}, \alpha_t^i)\,dt + \sigma(t, X_t^i, \overline{\mu}_{X_t^{-i}}^{N-1}, \alpha_t^i)\,dW_t^i \\ X_0^i = \xi \end{cases} \tag{78}$$

on time horizon $[0, T]$. Notice that b, σ, ξ are identical among all players and other players' states affect player i's state only through the empirical measure. The cost functional of player i has form

$$J^i(\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t^i, \overline{\mu}_{X_t^{-i}}^{N-1}, \alpha_t^i)\,dt + g(X_T^i, \overline{\mu}_{X_T^{-i}}^{N-1})\right] \tag{79}$$

with f, g identical among all players such that the large symmetric cost functional assumption holds.

To solve a MFG, we only have to consider a representative player, fix the flow of probability measure $\mu = \{\mu_t\}_{0\le t\le T}$ and solve the stochastic control problem to get the optimal control $\hat{\alpha} = \hat{\alpha}(\mu)$ as a function of μ. Such $\hat{\alpha}(\mu)$ induces the state evolution $\hat{X}^\mu$ depending on μ, so we just need to find a flow μ such that

$$\forall t \in [0, T], \mathscr{L}\left(\hat{X}_t^\mu\right) = \mu_t \tag{80}$$

where $\mathscr{L}(\cdot)$ denotes the law/distribution of random variable. This condition ensures the consistency of state evolution and empirical measure, providing such μ as the solution to MFG. It's quite easy to understand that the first step solving stochastic control problem ensures **optimality** while the second step finding flow μ ensures **consistency**.

Remark. *In this setting of MFG, we don't distinguish between open loop NE and Markovian NE any longer since when $N \to \infty$, they are gonna be asymptotically the same (a nontrivial fact).*

For simplicity, we also assume that when fixing μ and solving out the optimal control $\hat{\alpha}(\mu)$, the minimizer is unique for any flow μ.

The Hamiltonian of this MFG is still defined as

$$H(t, x, \mu, y, z, \alpha) = b(t, x, \mu, \alpha) \cdot y + \sigma(t, x, \mu, \alpha) \cdot z + f(t, x, \mu, \alpha) \tag{81}$$

and when the diffusion coefficient is free of control, i.e. $\sigma = \sigma(t, x, \mu)$, reduced Hamiltonian

$$H(t, x, \mu, y, \alpha) = b(t, x, \mu, \alpha) \cdot y + f(t, x, \mu, \alpha) \tag{82}$$

can be used in place of the original one.

Analytic Approach to MFG

Recall that we first replace the empirical measure with a measure flow μ and freeze the flow μ to solve the optimal control $\hat{\alpha}(\mu)$ for the representative player. This problem is a standard single player game and can be transformed into solving HJBE in the Markovian case. With the definition of value function

$$V(t,x) = \mathbb{E}\left[\int_t^T f(s,X_s,\mu_s,\alpha_s)\,ds + g(X_T,\mu_T)\Big|X_t = x\right] \tag{83}$$

where X,α denotes the state and control of the representative player, HJBE tells us

$$\partial_t V + \inf_\alpha \left\{b(t,x,\mu_t,\alpha)\cdot\partial_x V + \frac{1}{2}\mathrm{Tr}(\sigma(t,x,\mu_t,\alpha)\sigma^T(t,x,\mu_t,\alpha)\partial_{xx}V) + f(t,x,\mu_t,\alpha)\right\} = 0 \tag{84}$$

with terminal condition $V(T,x) = g(x,\mu_T)$.

On the other hand, we want to describe the evolution of measure flow μ after solving $\hat{\alpha}(\mu)$ (from the HJBE listed above) such that $\forall t\in[0,T], \mu_t = \mathscr{L}\left(\hat{X}_t^\mu\right)$ where $\hat{X}^\mu$ is generated by taking the control $\hat{\alpha}(\mu)$. Recall that the propagation of measure flow is described by the Fokker-Planck equation with given initial condition. Since the state dynamics of the representative player now becomes

$$\begin{cases} d\hat{X}_t^\mu = b(t,\hat{X}_t^\mu,\mu_t,\hat{\alpha}_t(\mu))\,dt + \sigma(t,\hat{X}_t^\mu,\mu_t,\hat{\alpha}_t(\mu))\,dW_t \\ \hat{X}_0^\mu = \xi \end{cases} \tag{85}$$

the Fokker-Planck equation is

$$\begin{cases} \partial_t\mu_t - L^*\mu_t = 0 \\ \mu_0 = \mathscr{L}(\xi) \end{cases} \tag{86}$$

where L^* is the adjoint of the infinitesimal generator L with action

$$L^* f = -\mathrm{div}_x(b\cdot f) + \frac{1}{2}\mathrm{Tr}(\partial_{xx}(\sigma\sigma^T f)) \tag{87}$$

explicitly written out to get

$$\begin{cases} \partial_t\mu_t + \mathrm{div}_x(b(t,x,\mu_t,\hat{\alpha}_t(\mu))\cdot\mu_t) - \frac{1}{2}\mathrm{Tr}\left[\partial_{xx}(\sigma(t,x,\mu_t,\hat{\alpha}_t(\mu))\cdot\sigma^T(t,x,\mu_t,\hat{\alpha}_t(\mu))\cdot\mu_t)\right] = 0 \\ \mu_0 = \mathscr{L}(\xi) \end{cases} \tag{88}$$

the **HJBE coupled with the Fokker-Planck equation** provides the analytic approach to MFG. Notice that HJBE has given terminal condition and Fokker-Planck equation has given initial condition. Notice that $\mu = \mu(t,x)$ can be understood as the density of $\hat{X}_t^\mu$ (if density exists) so t is the time variable and x is the space variable, i.e. for each fixed time t, $\mu(t,\cdot)$ is a density function.

Remark. *div is the divergence operator, for vector field* $F = (F_1, ..., F_n)$, $div_x F = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} = \nabla \cdot F$.

One might be confused why the adjoint of infinitesimal generator contains the divergence operator, let's do a simple derivation here as review. From the definition of adjoint, $\langle Lf, g\rangle = \langle f, L^* g\rangle$ *where the inner product is the standard one defined on* L^2 *function space* $\langle f, g\rangle = \int f(x) \cdot g(x)\, dx$.

$$\langle Lf, g\rangle = \int Lf(x) \cdot g(x)\, dx \tag{89}$$

$$= \int (b \cdot \partial_x f) \cdot g + \frac{1}{2} Tr(\sigma\sigma^T \partial_{xx} f) \cdot g\, dx \tag{90}$$

here assume that $b = (b^1, ..., b^n)$ *takes value in* $\mathbb{R}^n$ *and* σ *takes value in* $\mathbb{R}^{n\times m}$*, do integration by parts to see*

$$\langle Lf, g\rangle = \sum_{i=1}^n \int b^i \cdot \partial_{x^i} f \cdot g\, dx + \frac{1}{2} Tr\left(\int g \cdot \sigma\sigma^T \partial_{xx} f\, dx\right) \tag{91}$$

$$= -\sum_{i=1}^n \int f \cdot \partial_{x^i}(b^i \cdot g)\, dx - \frac{1}{2} Tr\left[\int \partial_x\left(g \cdot \sigma\sigma^T\right) \cdot \partial_x f\, dx\right] \tag{92}$$

$$= \int \left[-\sum_{i=1}^n \partial_{x^i}(b^i \cdot g)\right] \cdot f\, dx + \frac{1}{2} Tr\left[\int \partial_{xx}\left(g \cdot \sigma\sigma^T\right) \cdot f\, dx\right] \tag{93}$$

$$= \int -div_x(b \cdot g) \cdot f\, dx + \int \frac{1}{2} Tr\left[\partial_{xx}\left(g \cdot \sigma\sigma^T\right)\right] \cdot f\, dx \tag{94}$$

we conclude

$$L^* g = -div_x(b \cdot g) + \frac{1}{2} Tr\left[\partial_{xx}\left(g \cdot \sigma\sigma^T\right)\right] \tag{95}$$

In special cases where **the diffusion coefficient of the dynamics is free of control**, i.e. $\sigma = \sigma(t, x, \mu)$, the HJBE together with the Fokker-Planck equation has a simpler representation. Consider minimizing the reduced Hamiltonian w.r.t. α to get

$$\hat{\alpha}(t, x, \mu, y) = \arg\min_\alpha H(t, x, \mu, y, \alpha) = \arg\min_\alpha \{b(t, x, \mu, \alpha) \cdot y + f(t, x, \mu, \alpha)\} \tag{96}$$

it's obvious that $\hat{\alpha}(t, x, \mu_t, \partial_x V)$ minimizes the expression inside the inf in the HJBE.

Remark. *Such minimizer* $\hat{\alpha}$ *is guaranteed to exist and is unique with sufficient regularity conditions stated in lemma 3.3 under a set of assumptions.*

At this point, HJBE can be reduced to

$$\partial_t V + \frac{1}{2}\text{Tr}(\sigma(t, x, \mu_t)\sigma^T(t, x, \mu_t)\partial_{xx} V) + H(t, x, \mu_t, \partial_x V, \hat{\alpha}(t, x, \mu_t, \partial_x V)) = 0 \tag{97}$$

and Fokker-Planck equation can be reduced to

$$\partial_t \mu_t + \mathrm{div}_x(b(t,x,\mu_t,\hat{\alpha}(t,x,\mu_t,\partial_x V))\cdot\mu_t) - \frac{1}{2}\mathrm{Tr}\left[\partial_{xx}(\sigma(t,x,\mu_t)\cdot\sigma^T(t,x,\mu_t)\cdot\mu_t)\right] = 0 \tag{98}$$

written together with the terminal condition and initial condition, **when a MFG has uncontrolled diffusion coefficient, the analytic approach is to minimize reduced Hamiltonian and solve HJBE together with the Fokker-Planck equation**

$$\begin{cases} \hat{\alpha}(t,x,\mu,y) = \arg\min_\alpha H(t,x,\mu,y,\alpha) = \arg\min_\alpha \{b(t,x,\mu,\alpha)\cdot y + f(t,x,\mu,\alpha)\} \\ \partial_t V + \frac{1}{2}\mathrm{Tr}(\sigma(t,x,\mu_t)\sigma^T(t,x,\mu_t)\partial_{xx}V) + H(t,x,\mu_t,\partial_x V,\hat{\alpha}(t,x,\mu_t,\partial_x V)) = 0 \\ V(T,x) = g(x,\mu_T) \\ \partial_t \mu_t + \mathrm{div}_x(b(t,x,\mu_t,\hat{\alpha}(t,x,\mu_t,\partial_x V))\cdot\mu_t) - \frac{1}{2}\mathrm{Tr}\left[\partial_{xx}(\sigma(t,x,\mu_t)\cdot\sigma^T(t,x,\mu_t)\cdot\mu_t)\right] = 0 \\ \mu_0 = \mathscr{L}(X_0) = \mathscr{L}(\xi) \end{cases} \tag{99}$$

Remark. *This is a typical two-point boundary problem, hard to solve. Moreover, Cauchy-Lipschitz like theory can only ensure the existence and uniqueness of the solution locally (typically in a small time interval near* 0*) but not globally. This phenomenon can directly be seen in the closed-form solution of the finite player LQ game on graph that I have derived.*

Idea of MKV-FBSDE Approach

The idea of solving MFG is that when $N \to \infty$, we expect to see all players' state processes become asymptotically independent (propagation of chaos). As a result, when we consider the stochastic control problem the representative player is facing

$$\inf_{\alpha} \mathbb{E}\left[\int_0^T f(t, X_t, \mu_t, \alpha_t)\, dt + g(X_T, \mu_T)\right] \tag{100}$$

with dynamics

$$dX_t = b(t, X_t, \mu_t, \alpha_t)\, dt + \sigma(t, X_t, \mu_t, \alpha_t)\, dW_t \tag{101}$$

where the empirical measure is replaced with the fixed measure flow μ, we expect to see that μ_t is actually a deterministic measure instead of a random measure. Recall that the empirical measure in player i's dynamics is defined as

$$\overline{\mu}^{N-1}_{X_t^{-i}} = \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} \delta_{X_t^j} \tag{102}$$

which puts probability mass $\frac{1}{N-1}$ on the realization of X_t^j for $\forall j \neq i$. However, X_t^j contains randomness so $\overline{\mu}^{N-1}_{X_t^{-i}}$ is a random measure. On the other hand, if the propagation of chaos brings with asymptotic independence of different players' state process, for a regular enough ϕ, $\mathbb{E}\left|\phi(\overline{\mu}^{N-1}_{X_t^{-i}}) - \phi(\mu_t)\right|^2 \to 0$ $(N \to \infty)$ where $\mu_t = \mathscr{L}\left(X_t^1\right)$ is a deterministic measure, i.e. the randomness in the measure is gone asymptotically. Therefore, it makes sense to replace μ_t in the dynamics of the representative player with $\mathscr{L}\left(X_t\right)$ to get an **SDE of McKean-Vlasov (MKV) type**

$$dX_t = b(t, X_t, \mathscr{L}\left(X_t\right), \alpha_t)\, dt + \sigma(t, X_t, \mathscr{L}\left(X_t\right), \alpha_t)\, dW_t \tag{103}$$

that's the essential nature of state dynamics in MFG.

Remark. *When common noise is present, the situation becomes more complicated and it's natural to expect that $\overline{\mu}^{N-1}_{X_t^{-i}}$ can no longer be replaced with a deterministic measure μ_t. Although μ_t would be a random measure in this case, it would be the conditional marginal distribution of X_t given the realization of the common noise so the state dynamics can still be turned essentially into an SDE of conditional MKV type. However, MFG with common noise is harder to solve since the randomness in μ_t causes measurability problems in using the filtration generated by the BM, causing the failure of stochastic integrals being MG. That's why MFG with common noise requires a different setting in a random environment.*

Talking about using FBSDE to solve control problems, there are typically two approaches. We still assume an **uncontrolled diffusion coefficient** σ. One of the approach is to set up the value function and the HJBE it has to satisfy, then apply generalized Feynman-Kac formula to view it as the solution of a BSDE (recall that any solution to a semi-linear PDE has probabilistic representation under BSDE). In more details, the BSDE is given by

$$dY_t = -f(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, [\sigma^T(t, X_t, \mu_t)]^{-1} Z_t))\, dt + Z_t \cdot dW_t \tag{104}$$

with terminal condition $Y_T = g(X_T, \mu_T)$. Here $\hat{\alpha}$ is still formed as the minimizer of the reduced Hamiltonian w.r.t. α and X is the state process generated by taking control $\hat{\alpha}$.

Remark. *The HJBE of the value function is exactly*

$$\partial_t V + \frac{1}{2} Tr(\sigma(t, x, \mu_t)\sigma^T(t, x, \mu_t)\partial_{xx} V) + H(t, x, \mu_t, \partial_x V, \hat{\alpha}(t, x, \mu_t, \partial_x V)) = 0 \tag{105}$$

as shown above. It's clear that the correspondence of semi-linear PDE and BSDE through generalized Feynman-Kac formula is formed as (recall the value function and Delta-hedging strategy interpretation of BSDE)

$$\begin{cases} Y_t = V(t, X_t) \\ Z_t = \sigma^T(t, X_t, \mu_t)\ \partial_x V(t, X_t) \end{cases} \tag{106}$$

computation under Ito formula tells us (where dots mean inner product)

$$dY_t = dV(t, X_t) = \partial_t V(t, X_t)\, dt + \partial_x V(t, X_t) \cdot\, dX_t + \frac{1}{2} Tr(\partial_{xx} V \sigma\sigma^T)\, dt \tag{107}$$

$$= \partial_t V\, dt + (\partial_x V \cdot b)\, dt + (\sigma^T\ \partial_x V) \cdot\, dW_t + \frac{1}{2} Tr(\partial_{xx} V \sigma\sigma^T)\, dt \tag{108}$$

with variables in function V, b, σ omitted. It's immediate that the coefficient of dW_t which is $\sigma^T\ \partial_x V$ corresponds to Z_t and the coefficient of dt is

$$\partial_t V + b \cdot \partial_x V + \frac{1}{2} Tr(\partial_{xx} V \sigma\sigma^T) = b \cdot \partial_x V - H = -f \tag{109}$$

according to the HJBE. This results in the BSDE stated above

$$dY_t = -f(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, [\sigma^T(t, X_t, \mu_t)]^{-1} Z_t))\, dt + Z_t \cdot dW_t \tag{110}$$

with $\partial_x V(t, X_t) = [\sigma^T(t, X_t, \mu_t)]^{-1} Z_t$.

Generally speaking, when the dimension of BM and the dimension of state is the same, σ takes value as a square matrix and we would typically ***require σ to be invertible*** *since then solving the FBSDE gives the solution $\hat{Z}_t$ and $\partial_x V(t, x)$ can then be recovered from $\hat{Z}_t$, providing the NE $\hat{\alpha}_t$. More generally, when the dimension of BM is higher than the dimension of the state, e.g. σ takes value as $N \times (N + 1)$ matrix, this approach can be applied when $\sigma\sigma^T$ is invertible, otherwise we call the problem to be degenerated and we avoid discussing it here.*

The second approach is not based on the value function, but the Pontryagin's maximum principle we are familiar with. The adjoint BSDE is naturally provided as

$$dY_t = -\partial_x H^{full}(t, X_t, \mu_t, Y_t, Z_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))\, dt + Z_t\, dW_t \tag{111}$$

with terminal condition $Y_T = \partial_x g(X_T, \mu_T)$. Here $\hat{\alpha}$ is still formed as the minimizer of the reduced Hamiltonian w.r.t. α but in the adjoint BSDE we have to use the full Hamiltonian

$$H^{full}(t, x, \mu, y, z) = H(t, x, \mu, y, \alpha) + \sigma(t, x, \mu) \cdot z \tag{112}$$

instead of the reduced Hamiltonian H since σ is not necessarily free of the state x.

Remark. *Just to clarify the difference between full Hamiltonian and reduced Hamiltonian, when the diffusion coefficient is free of control, we can minimize the reduced Hamiltonian but still need the full Hamiltonian to construct the adjoint BSDE (which many books forget to tell and is somewhat misleading). The only case one can always stick to the reduced Hamiltonian is when $\sigma = \sigma(t)$ has no dependence on x (or constant σ which often appears in literature). In short, we can actually always use the full Hamiltonian instead of the reduced one.*

This second approach does not require σ to be always invertible but requires differentiability of the coefficients.

Remark. *In short, when it comes to FBSDE approach in control problems, there are mainly two ways.*

The first approach is to ***build up the value function, derive the HJBE and use the correspondence between semi-linear PDE and BSDE*** *(generalized Feynman-Kac formula, form value function as the solution to BSDE) to get FBSDE. However, this approach* ***requires σ to be uncontrolled and invertible at all times****. One can recall that the first approach is what we do in the Deep BSDE algorithm to numerically solve stochastic control problems.*

The second approach is to ***apply Pontryagin's maximum principle*** *and derive FBSDE. This approach puts no restrictions on σ but* ***requires the differentiability of coefficients****.*

Both approach ends up in the same form of FBSDE given uncontrolled diffusion coefficient

$$\begin{cases} dX_t = B(t, X_t, \mu_t, Y_t, Z_t)\, dt + \Sigma(t, X_t, \mu_t)\, dW_t \\ dY_t = -F(t, X_t, \mu_t, Y_t, Z_t)\, dt + Z_t\, dW_t \\ Y_T = G(X_T, \mu_T) \end{cases} \tag{113}$$

but a different correspondence of B, Σ, F, G with the original dynamics. To be specific, for the generalized Feynman-Kac formula approach,

$$\begin{cases} B(t, x, \mu, y, z) = b(t, x, \mu, \hat{\alpha}(t, x, \mu, [\sigma^T(t, x, \mu)]^{-1} z)) \\ \Sigma(t, x, \mu) = \sigma(t, x, \mu) \\ F(t, x, \mu, y, z) = f(t, x, \mu, \hat{\alpha}(t, x, \mu, [\sigma^T(t, x, \mu)]^{-1} z)) \\ G(x, \mu) = g(x, \mu) \end{cases} \tag{114}$$

and for the Pontryagin's maximum principle approach,

$$
\begin{cases}
B(t,x,\mu,y,z) = b(t,x,\mu,\hat{\alpha}(t,x,\mu,y)) \\
\Sigma(t,x,\mu) = \sigma(t,x,\mu) \\
F(t,x,\mu,y,z) = \partial_x H^{full}(t,x,\mu,y,z,\hat{\alpha}(t,x,\mu,y)) \\
G(x,\mu) = \partial_x g(x,\mu)
\end{cases}
\tag{115}
$$

where $\hat{\alpha}(t,x,\mu,y)$ is always the minimizer of Hamiltonian w.r.t. α. Combined with the idea of MFG discussed above replacing μ_t with $\mathscr{L}(X_t)$, **solving MFG turns into the problem of solving FBSDE of McKean-Vlasov (MKV-FBSDE) type**

$$
\begin{cases}
dX_t = B(t,X_t,\mathscr{L}(X_t),Y_t,Z_t)\,dt + \Sigma(t,X_t,\mathscr{L}(X_t))\,dW_t \\
dY_t = -F(t,X_t,\mathscr{L}(X_t),Y_t,Z_t)\,dt + Z_t\,dW_t \\
Y_T = G(X_T,\mathscr{L}(X_T))
\end{cases}
\tag{116}
$$

so it's natural to investigate the solvability of MKV-FBSDE.

Solvability of MKV-FBSDE

Here we present some general ideas to investigate the solvability of MKV-FBSDE. We start from an example of coupled forward backward differential equations

$$\begin{cases} \dot{x}_t = -y_t \\ \dot{y}_t = 0 \\ x_0 \in [-2, 2] \\ y_T = G(x_T) \end{cases} \tag{117}$$

where x, y are deterministic functions in t and the time horizon is $[0, T], T = 1$. Consider G taken as $G(x) = (-1) \vee x \wedge 1$, since y is free of t, when $x_1 > 1, \forall t \in [0, 1], y_t = y_1 = 1$ resulting in $\dot{x}_t = -1, x_t = -t + x_0$ so it must be true that $x_1 = -1 + x_0 > 1, x_0 > 2$, a contradiction. Similarly, $x_1 < -1$ cannot be true so $\forall t \in [0, 1], y_t = y_1 = G(x_1) = x_1$ with $x_1 \in [-1, 1]$. Now $\dot{x}_t = -x_1, x_t = -x_1 t + x_0$ has to satisfy $x_1 = -x_1 + x_0, x_0 = 2x_1, x_1 \in [-1, 1]$ is consistent. So the solution exists and is unique, given by

$$\begin{cases} x_t = -\frac{x_0}{2} t + x_0 \\ y_t = \frac{x_0}{2} \end{cases} \tag{118}$$

on the other hand, consider G taken as $G(x) = -(-1) \vee x \wedge 1$, when $x_0 \neq 0$, we get the following unique solution from a similar argument

$$\begin{cases} x_t = x_0 + t \cdot sgn(x_0) \\ y_t = -sgn(x_0) \end{cases} \tag{119}$$

however, when $x_0 = 0$, there are infinitely many solutions

$$\forall a \in [-1, 1], \begin{cases} x_t = at \\ y_t = -a \end{cases} \tag{120}$$

so the uniqueness of the solution is destroyed despite a small sign change in G. In this example, the **monotonicity of G** plays a key role in the property of FBSDE. An analogue to the adjoint BSDE derived in Pontryagin's maximum principle tells us that we can compare the terminal condition $y_T = \partial_x g(X_T)$ with $y_T = G(x_T)$ so G can somewhat be understood as $\partial_x g$. Recall that the verification step of Pontryagin's maximum principle requires the convexity of g, i.e. $\partial_x g$ to be increasing, analogue to G being increasing. From the example above, when G is increasing, the existence and uniqueness holds, so it's somewhat consistent with the fact we already know that **the convexity in g (or the monotonicity in the terminal condition) plays a crucial role in the existence and uniqueness argument of FBSDE**.

Remark. *The example above has an interesting interpretation under Burgers' equation. Setting $y_t = u(t, x_t)$ as the* *form of value function, FBSDE can be transformed into a nonlinear PDE*

$$\partial_t u - u \cdot \partial_x u = 0 \tag{121}$$

with terminal condition $u(T,x) = G(x)$.

Do a time reversal $v(t,x) = u(T-t,x)$ to see that

$$\partial_t v + v \cdot \partial_x v = 0 \tag{122}$$

with initial condition $v(0,x) = G(x)$. This is called ***the inviscid Burgers' equation*** *describing the motion of fluid along a tunnel under the conservation law (velocity of the fluid is proportional to v under the interpretation of Burgers' equation that $u(t,x)$ can be understood as the probability density of the fluid at spatial coordinate x at time t)*

When G in increasing in x, the fluid is in the ***dilation regime*** *(at place farther away from the starting point, there's more fluid), so as the motion starts, there will be no shock as time goes by, everything is well-posed. However, when G is decreasing in x, the fluid is in the* ***compression regime*** *(at place farther away from the starting point, there's less fluid), as time goes by the fluid at the starting point has a higher velocity, it catches from behind, creating a shock (singularity), that's why the uniqueness of the solution to FBSDE is destroyed.*

This ***correspondence between FBSDE with deterministic coefficients and and the system of characteristics of nonlinear PDE*** *is very interesting since I always believe it's hard to find some correspondence in reality for FBSDE. It's also an important correspondence since it directly motivates the existence and uniqueness argument of FBSDE from an intuitive perspective. Of course, the reader has to do some calculations in Burgers' equation to believe in what I have stated above. I am definitely willing to write something on Burgers' equation in another note if time allows.*

Yet, the convexity of g or increasing property of $\partial_x g$ meets some trouble extending to the MFG setting because $g = g(x,\mu)$ is a function of state and the measure. This requires us to define monotonicity for a function that maps a measure to a real number which will be discussed in a later context.

On the other hand, the correspondence between FBSDE with deterministic coefficients and and the system of characteristics of nonlinear PDE mentioned above provides motivation for the **decoupling field of FBSDE**. Still take the example listed above with the correspondence to Burgers' equation, adding diffusion term $\frac{1}{2}\partial_{xx}u$ on LHS gives

$$\partial_t u - u \cdot \partial_x u + \frac{1}{2}\partial_{xx} u = 0 \tag{123}$$

results in the **viscous Burgers' equation** (diffusive). This PDE has no singularity in its solution because of the regularizing effect of the heat kernel. A same correspondence back to FBSDE tells us that this PDE corresponds to

$$\begin{cases} dX_t = -Y_t\,dt + dW_t \\ dY_t = Z_t\,dW_t \\ x_0 \in \mathbb{R} \\ Y_T = G(X_T) \end{cases} \tag{124}$$

with the BM naturally appearing (recall that BM has infinitesimal generator as the Laplacian) and randomness is introduced. **Well-posedness of the viscous Burgers' equation corresponds to the existence and uniqueness of the solution to this FBSDE**, moreover the solution **admits the representation**

$$\mathbb{P}\left(\forall t \in [0,1], Y_t = u(t, X_t)\right) = 1, Z_t = \partial_x u(t, X_t)\ \lambda \times \mathbb{P} - a.e. \tag{125}$$

that Y has the structure as a value function in terms of underlying state process X and Z is the differential of value function (typical interpretation of BSDE). In this case, u is called the decoupling field of this FBSDE since solving u solves the FBSDE.

Remark. *To check the correspondence, set $Y_t = u(t, X_t), Z_t = \partial_x u(t, X_t)$ and apply Ito formula*

$$dY_t = du(t, X_t) = \partial_t u\,dt + \partial_x u\,dX_t + \frac{1}{2}\partial_{xx} u\,d\langle X, X\rangle_t \tag{126}$$

$$= \partial_t u\,dt - \partial_x u \cdot Y_t\,dt + \partial_x u\,dW_t + \frac{1}{2}\partial_{xx} u\,dt \tag{127}$$

$$= \left(\partial_t u - u \cdot \partial_x u + \frac{1}{2}\partial_{xx} u\right)(t, X_t)\,dt + Z_t\,dW_t \tag{128}$$

the comparison principle of BSDE against $dY_t = Z_t\,dW_t$ tells us

$$\partial_t u - u \cdot \partial_x u + \frac{1}{2}\partial_{xx} u = 0 \tag{129}$$

resulting in the correspondence.

Finding decoupling field thus provides another approach to solving FBSDE by turning it into solving PDE and the existence and uniqueness of solution to FBSDE is transformed to that of PDE which can be investigated in the traditional framework. However, this method only works for a certain kind of FBSDE, not generally applicable.

Toward the existence of solution to MKV-FBSDE, **Schauder's fixed point theorem** is an important tool to use on the Wasserstein space $\mathscr{P}_2(E)$ that contains probability measures on E. The definition of p-Wasserstein distance is

$$W_p(\mu, \mu') = \inf_{\pi \in \Pi_p(\mu, \mu')} \left(\int_{E\times E} [d(x,y)]^p \pi(dx, dy)\right)^{\frac{1}{p}} \tag{130}$$

with d as the distance on E and $\Pi_p(\mu, \mu')$ as the set of all couplings of μ, μ' (the set of all measures with marginals as

$\mu, \mu')$. $\mathscr{P}_2(E)$ is just the Wasserstein space equipped with 2-Wasserstein distance. The usage of fixed point theorem will be stated later but one can expect that we will define a mapping from a measure to another measure in the Wasserstein space with $\mathscr{L}(X_t)$ as the fixed point so that the solution to MKV-FBSDE is formed as the fixed point of this mapping (similar to what we have done in the meeting time example).

MKV-FBSDE Approach to MFG

The ideas of two different MKV-FBSDE approach to MFG are presented above, one from HJB under the characterization of generalized Feynman-Kac formula and the other from Pontryagin's maximum principle. Notice that **different assumptions are placed on the diffusion coefficient** $\sigma(t, x, \mu, \alpha)$ **for simplicity and consistency with the book**, but the readers are welcome to think about whether those conditions can be relaxed (refer to the conditions presented in the chapter of the idea of MKV-FBSDE approach). In this case, we organize the results as theorems below to remind readers of our main results (the description is not completely rigorous, only the most important conditions as the differences between two approaches are mentioned).

Theorem 3 (Value Function Approach)**.** *Assume* $\sigma = \sigma(t, x)$ *is free of empirical measure and control. Let* $\hat{\alpha}(t, x, \mu, y)$ *be the unique minimizer of the reduced Hamiltonian* $H(t, x, \mu, y, \alpha)$ *and if* σ *is uniformly elliptic, i.e.*

$$\exists L > 0, \forall t \in [0, T], \forall x, \sigma(t, x) \cdot \sigma^T(t, x) \geq L^{-1} I \tag{131}$$

where $\geq$ *is in the sense of semi-positive definite between symmetric matrices, then the continuous flow of measures* $\mu : [0, T] \to \mathscr{P}_2(\mathbb{R}^d)$ *is MFG equilibrium iff* $\mu_t = \mathscr{L}\left(\hat{X}_t\right)$ *where* $(\hat{X}, \hat{Y}, \hat{Z})$ *solves the MKV-FBSDE*

$$\begin{cases} d\hat{X}_t = b\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), \hat{\alpha}\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), [\sigma^T(t, \hat{X}_t)]^{-1}\hat{Z}_t\right)\right) dt + \sigma\left(t, \hat{X}_t\right) dW_t \\ d\hat{Y}_t = -f\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), \hat{\alpha}\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), [\sigma^T(t, \hat{X}_t)]^{-1}\hat{Z}_t\right)\right) dt + \hat{Z}_t \cdot dW_t \\ \hat{X}_0 = \xi \\ \hat{Y}_T = g\left(\hat{X}_T, \mathscr{L}\left(\hat{X}_T\right)\right) \end{cases} \tag{132}$$

Theorem 4 (Pontryagin's Maximum Principle Approach)**.** *Assume* $\sigma \in \mathbb{R}$ *is a constant. Let* $\hat{\alpha}(t, x, \mu, y)$ *be the unique minimizer of the reduced Hamiltonian* $H(t, x, \mu, y, \alpha)$*, then the continuous flow of measures* $\mu : [0, T] \to \mathscr{P}_2(\mathbb{R}^d)$ *is MFG equilibrium iff* $\mu_t = \mathscr{L}\left(\hat{X}_t\right)$ *where* $(\hat{X}, \hat{Y}, \hat{Z})$ *solves the MKV-FBSDE*

$$\begin{cases} d\hat{X}_t = b\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), \hat{\alpha}\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), \hat{Y}_t\right)\right) dt + \sigma\, dW_t \\ d\hat{Y}_t = -\partial_x H\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), \hat{Y}_t, \hat{\alpha}\left(t, \hat{X}_t, \mathscr{L}\left(\hat{X}_t\right), \hat{Y}_t\right)\right) dt + \hat{Z}_t \cdot dW_t \\ \hat{X}_0 = \xi \\ \hat{Y}_T = \partial_x g\left(\hat{X}_T, \mathscr{L}\left(\hat{X}_T\right)\right) \end{cases} \tag{133}$$

where $f, g \in C^1$ *and* g *is convex.*

Lasry-Lions Monotonicity Condition

When it comes to the **uniqueness** of the solution to MKV-FBSDE, there are mainly three approaches: Cauchy-Lipschitz theory on a small enough time horizon (locally near time 0), adding monotonicity conditions, and using decoupling field. Here we consider the second approach, naturally requiring us to define the monotonicity when it comes to a function mapping a measure to a real number.

Define for $h : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ to be **(Lasry-Lions) monotone** if $\forall \mu \in \mathscr{P}_2(\mathbb{R}^d)$, $h(x,\mu)$ is at most of quadratic growth in x and

$$\forall \mu, \mu' \in \mathscr{P}_2(\mathbb{R}^d), \int_{\mathbb{R}^d} [h(x,\mu) - h(x,\mu')] (\mu - \mu')(dx) \geq 0 \tag{134}$$

which is directly motivated from the uniqueness of the solution to the mean field approximation of single-period game. Uniqueness of the MFG equilibrium is typically proved under the **Lasry-Lions monotonicity assumptions**

$$\begin{cases} b = b(t,x,\alpha), \sigma = \sigma(t,x,\alpha) \text{ free of empirical measure} \\ f = f_0(t,x,\mu) + f_1(t,x,\alpha) \text{ separated structure of dependence on } \mu, \alpha \\ \text{Quadratic growth condition on } f, g \\ f_0(t,\cdot,\cdot), g \text{ both Lasry-Lions monotone} \end{cases} \tag{135}$$

with the theorem provided below as the result.

Theorem 5 (Uniqueness of MFG Equilibrium)**.** *Assume the above Lasry-Lions monotonicity assumptions hold and μ is a deterministic continuous measure flow of measure, then for each fixed empirical measure μ, the optimality step in MFG (optimize the control) has unique minimizer $\hat{\alpha}^\mu$ inducing state process $\hat{X}^\mu$. There exists at most one flow of measure μ such that $\forall t \in [0,T], \mathscr{L}\left(\hat{X}^\mu_t\right) = \mu_t$ so there is at most one MFG equilibrium.*

The proof of this theorem is intuitive and does not need any extra explanation, the reader shall check the book for the proof on his/her own.

Remark. *The Lasry-Lions assumptions are strong assumptions since the state dynamics are required to not contain the empirical measure, which is often not the case in practice. As a result, uniqueness of MFG equilibrium typically fails and in most cases we don't care too much about the uniqueness argument.*

Remark. *It's important to see some examples of $h : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ that are Lasry-Lions monotone. Trivially, if $h = C|x|^2$ does not depend on μ or h does not depend on x, it must be Lasry-Lions monotone.*

In the setting of LQ-MFG, we may meet with $h(x,\mu) = a\int_{\mathbb{R}^d} y\,\mu(dy)\cdot x$ $(a > 0)$. In the setting of potential games, we may see $h(x,\mu) = \int_{\mathbb{R}^d} l(x-y)\,\mu(dy)$ for odd function l such that $|l(x)| \leq C(1+|x|^2)$. Bearing the exact same form of h, we can consider $l : \mathbb{R}^d \to \mathbb{R}_+$ such that $l(r) = \int_{\mathbb{R}^d} e^{ir\cdot s}\lambda(ds)$ where λ is a symmetric positive finite measure on $\mathbb{R}^d$ (e.g. the measure induced by Gaussian or Cauchy r.v.). With such interpretation, l is actually the characteristic function (Fourier transform) of the distribution λ, with λ to be symmetric in order to ensure that the

Fourier transform only takes real values. Actually,

$$h(x,\mu) = \int_{\mathbb{R}^d} l(x-y)\,\mu(dy) \tag{136}$$

is an ***important example for a class of Lasry-Lions monotone functions*** *(plugging in $l(x) = sgn(x), d = 1$ gives $h(x,y) = \mu((-\infty,x)) - \mu((x,+\infty))$ which is another example, proving that $h(x,\mu) = \mu((-\infty,x)) + \frac{1}{2}\mu(\{x\})$ is also Lasry-Lions monotone). Those examples can be verified through simple calculations.*

Actually, the uniqueness of MFG equilibrium has another sufficient condition coming from the notion of **L-monotonicity**. It turns out that L-monotonicity sometimes can deal with cases where Lasry-Lions monotonicity fails. However, since we are not much concerned about the uniqueness of MFG equilibrium, we skip it for now and may come back to it later.

General Approach to Solve LQ-MFG

Let's consider the linear quadratic MFG as the simplest example where

$$b(t,x,\mu,\alpha)=b_1(t)x+\overline{b}_1(t)\overline{\mu}+b_2(t)\alpha \tag{137}$$

where state variable x, control variable α take value in $\mathbb{R}^d$, BM is of dimension d and diffusion coefficient $\sigma\in\mathbb{R}^{d\times d}$ is a constant matrix. Here $\overline{\mu}=\int_{\mathbb{R}^d}x\mu(dx)$ denotes the expectation of the probability measure μ (a measure on $\mathbb{R}^d$) and $b_1,\overline{b}_1,b_2$ are deterministic continuous matrix-valued functions. The running cost is

$$f(t,x,\mu,\alpha)=\frac{1}{2}\left(x^Tq(t)x+(x-s(t)\overline{\mu})^T\overline{q}(t)(x-s(t)\overline{\mu})+\alpha^Tr(t)\alpha\right) \tag{138}$$

where $q,\overline{q},r$ are continuous functions and take value as symmetric PD matrix. The terminal cost is

$$g(x,\mu)=\frac{1}{2}\left(x^Tqx+(x-s\overline{\mu})^T\overline{q}(x-s\overline{\mu})\right) \tag{139}$$

where $q,\overline{q}$ here are symmetric PD matrices with no dependence on time.

To solve this MFG, notice that the cost functions are convex, a natural correspondence with Pontryagin's maximum principle to derive MKV-FBSDE. Since σ has no dependence on x and α, reduced Hamiltonian can be used throughout the context

$$H(t,x,\mu,y,\alpha)=[b_1(t)x+\overline{b}_1(t)\overline{\mu}+b_2(t)\alpha]\cdot y+\frac{1}{2}\left(x^Tq(t)x+(x-s(t)\overline{\mu})^T\overline{q}(t)(x-s(t)\overline{\mu})+\alpha^Tr(t)\alpha\right) \tag{140}$$

take a derivative w.r.t. α

$$\partial_\alpha H=[b_2(t)]^Ty+r(t)\alpha \tag{141}$$

set as zero to get the optimal control

$$\hat{\alpha}(t,x,\mu,y)=-[r(t)]^{-1}[b_2(t)]^Ty \tag{142}$$

now calculate the coefficients in the adjoint BSDE

$$\partial_x H=[b_1(t)]^Ty+q(t)x+\overline{q}(t)(x-s(t)\overline{\mu}) \tag{143}$$

$$\partial_x g=qx+\overline{q}(x-s\overline{\mu}) \tag{144}$$

to write down the MKV-FBSDE under the interpretation that when the number of players is large enough, we can replace the flow of empirical measure (random measure) with the flow of deterministic measure $\mu_t=\mathscr{L}(X_t)$ and replace the control α with $\hat{\alpha}(t, X_t, \mu_t, Y_t)$ simultaneously

$$\begin{cases} dX_t = \left[b_1(t)X_t + \bar{b}_1(t)\overline{\mathscr{L}(X_t)} - b_2(t)[r(t)]^{-1}[b_2(t)]^T Y_t\right] dt + \sigma\, dW_t \\ X_0 = \xi \\ dY_t = -\left([b_1(t)]^T Y_t + q(t)X_t + \overline{q}(t)(X_t - s(t)\overline{\mathscr{L}(X_t)})\right) dt + Z_t\, dW_t \\ Y_T = qX_T + \overline{q}(X_T - s\overline{\mathscr{L}(X_T)}) \end{cases} \tag{145}$$

now we want to investigate the solvability of this MKV-FBSDE containing both optimality and consistency conditions. Since the measure acts on the coefficients only through its mean, we denote

$$\overline{x}_t = \mathbb{E}X_t = \overline{\mathscr{L}(X_t)} \tag{146}$$

and simplify the MKV-FBSDE

$$\begin{cases} dX_t = \left[b_1(t)X_t + \bar{b}_1(t)\overline{x}_t - b_2(t)[r(t)]^{-1}[b_2(t)]^T Y_t\right] dt + \sigma\, dW_t \\ X_0 = \xi \\ dY_t = -\left([b_1(t)]^T Y_t + [q(t) + \overline{q}(t)]X_t - \overline{q}(t)s(t)\overline{x}_t\right) dt + Z_t\, dW_t \\ Y_T = (q + \overline{q})X_T - \overline{q}s\overline{x}_T \end{cases} \tag{147}$$

it's still hard to see the solvability but the presence of the mean of X_t in the equations leads us to think about taking expectation on both sides of the MKV-FBSDE to get the coupled ODE w.r.t. the mean of X_t and Y_t

$$\begin{cases} d\overline{x}_t = \left[[b_1(t) + \bar{b}_1(t)]\overline{x}_t - b_2(t)[r(t)]^{-1}[b_2(t)]^T \overline{y}_t\right] dt \\ \overline{x}_0 = \mathbb{E}\xi \\ d\overline{y}_t = -\left([b_1(t)]^T \overline{y}_t + [q(t) + \overline{q}(t) - \overline{q}(t)s(t)]\overline{x}_t\right) dt \\ Y_T = (q + \overline{q} - \overline{q}s)\overline{x}_T \end{cases} \tag{148}$$

where

$$\overline{y}_t = \mathbb{E}Y_t \tag{149}$$

interestingly, the following theorem turns the analysis on MKV-FBSDE completely into the analysis on this coupled ODE w.r.t. the mean of the process.

Theorem 6 (Existence and Uniqueness of the Solution to LQ-MFG)**.** *Existence and uniqueness of the solution to*

LQ-MFG holds iff existence and uniqueness of the solution to the ODE system

$$\begin{cases} d\overline{x}_t = \left[[b_1(t) + \overline{b}_1(t)]\overline{x}_t - b_2(t)[r(t)]^{-1}[b_2(t)]^T\overline{y}_t\right] \, dt \\ \overline{x}_0 = \mathbb{E}\xi \\ d\overline{y}_t = -\left([b_1(t)]^T\overline{y}_t + [q(t) + \overline{q}(t) - \overline{q}(t)s(t)]\overline{x}_t\right) \, dt \\ Y_T = (q + \overline{q} - \overline{q}s)\overline{x}_T \end{cases} \tag{150}$$

holds.

At this point, we can put up ansatz

$$\overline{y}_t = \overline{\eta}_t\overline{x}_t + \overline{\chi}_t \tag{151}$$

where $\overline{\eta}, \overline{\chi}$ are deterministic functions in t, plug into the coupled ODE system, collect coefficients to see the decoupled version as an ODE w.r.t. $\overline{\eta}, \overline{\chi}$

$$\begin{cases} \dot{\overline{\eta}}_t = \overline{\eta}_t b_2(t)[r(t)]^{-1}[b_2(t)]^T\overline{\eta}_t - \overline{\eta}_t[b_1(t) + \overline{b}_1(t)] - [b_1(t)]^T\overline{\eta}_t - [q(t) + \overline{q}(t) - \overline{q}(t)s(t)] \\ \dot{\overline{\chi}}_t = \overline{\eta}_t b_2(t)[r(t)]^{-1}[b_2(t)]^T\overline{\chi}_t - [b_1(t)]^T\overline{\chi}_t \\ \overline{\eta}_T = (q + \overline{q} - \overline{q}s)I \\ \overline{\chi}_T = 0 \end{cases} \tag{152}$$

where $\overline{\eta}_t$ is matrix-valued and $\overline{\chi}_t$ is vector-valued. It's clear that the existence and uniqueness of the solution completely depends on the equation for $\overline{\eta}_t$ (since plugging in the solution of $\overline{\eta}_t$ into the second equation always gives the trivial solution $\overline{\chi}_t \equiv 0$), which is a **matrix-valued Ricatti equation**. In other words, the solvability of LQ-MFG can eventually be turned into the solvability of a Ricatti equation (not trivial in general).

If we want to solve the MFG equilibrium, after solving this ODE to get $\overline{x}_t, \overline{y}_t$, we can try to plug in and solve the FBSDE (not a McKean-Vlasov type any more) with the affine ansatz for LQ game

$$Y_t = \eta_t X_t + \chi_t \tag{153}$$

similarly, calculate dY_t and collect coefficients, we get an ODE w.r.t. η_t, χ_t with given terminal conditions, which is once more a Ricatti equation to solve. Solving out η_t, χ_t to get the solution $(\hat{X}, \hat{Y}, \hat{Z})$ to the original MKV-FBSDE provides the MFG equilibrium.

Remark. *One has to realize that the ODE for $\overline{\eta}_t, \overline{\chi}_t$ is different from the ODE for η_t, χ_t. The previous one is a characterization of the evolution of the mean $\overline{x}_t, \overline{y}_t$ while the latter is a characterization of the solution to the FBSDE.*

To conclude, in LQ-MFG, the strategy after writing MKV-FBSDE is to first take expectation to get the evolution of the mean (possibly solving it) and to prove/disprove the existence and uniqueness of the MFG equilibrium (the point of setting up $\overline{\eta}_t, \overline{\chi}_t$). After that, MKV-FBSDE can be simplified into normal FBSDE which can be solved by putting up an ansatz (the point of setting up η_t, χ_t).

Example: Linear Quadratic (LQ) Flocking Model

In this model, there are N players (birds), with player i having position X_t^i at time t taking value in $\mathbb{R}^3$. Player i can determine its control α_t^i as the velocity at time t and the state dynamics is given as

$$dX_t^i = \alpha_t^i \, dt + \sigma \, dW_t^i \tag{154}$$

on time horizon $[0, T]$. Player i has cost functional

$$J^i(\alpha) = \mathbb{E} \int_0^T f^i(t, X_t, \alpha_t) \, dt \tag{155}$$

with no terminal cost and the running cost is

$$f^i(t, x, \alpha) = \frac{\kappa^2}{2} |x^i - \overline{x}|^2 + \frac{1}{2} |\alpha^i|^2 \tag{156}$$

with $|\cdot|$ denoting the vector ℓ_2 norm since x^i, α^i take values in $\mathbb{R}^3$. This is a LQ game with mean field interaction through $|x^i - \overline{x}|^2$.

Instead of solving it as a finite player game, let's **do mean field approximation and solve it as an LQ-MFG**. It can be easily seen that when $N \to \infty$ this game can be approximated as the game for a single representative player with state dynamics

$$dX_t = \alpha_t \, dt + \sigma \, dW_t \tag{157}$$

and cost functional

$$J(\alpha) = \mathbb{E} \int_0^T f(t, X_t, \mu_t, \alpha_t) \, dt \tag{158}$$

with no terminal cost and the running cost is

$$f(t, x, \mu, \alpha) = \frac{\kappa^2}{2} |x - \overline{\mu}|^2 + \frac{1}{2} |\alpha|^2 \tag{159}$$

since σ is constant, use reduced Hamiltonian

$$H(t, x, \mu, y, \alpha) = \alpha \cdot y + \frac{\kappa^2}{2} |x - \overline{\mu}|^2 + \frac{1}{2} |\alpha|^2 \tag{160}$$

differentiate w.r.t. α

$$\partial_\alpha H = y + \alpha \tag{161}$$

set it as zero to get the optimal control

$$\hat{\alpha}(t,x,\mu,y) = -y \tag{162}$$

now calculate the coefficients in the adjoint BSDE

$$\partial_x H = \kappa^2(x-\overline{\mu}) \tag{163}$$

$$\partial_x g = 0 \tag{164}$$

write out the FBSDE for fixed flow of measure $\{\mu_t\}$ that

$$\begin{cases} dX_t = -Y_t\,dt + \sigma\,dW_t \\ X_0 = \xi \\ dY_t = -\kappa^2(X_t - \overline{\mu_t})\,dt + Z_t\,dW_t \\ Y_T = 0 \end{cases} \tag{165}$$

replace μ_t with $\mathscr{L}(X_t)$ to get the MKV-FBSDE that characterizes the solution to MFG

$$\begin{cases} dX_t = -Y_t\,dt + \sigma\,dW_t \\ X_0 = \xi \\ dY_t = -\kappa^2(X_t - \overline{\mathscr{L}(X_t)})\,dt + Z_t\,dW_t \\ Y_T = 0 \end{cases} \tag{166}$$

take expectation on both sides to turn it into a coupled ODE w.r.t. the mean of X_t, Y_t that

$$\begin{cases} d\overline{x}_t = -\overline{y}_t\,dt \\ \overline{x}_0 = \mathbb{E}\xi \\ d\overline{y}_t = 0 \\ \overline{y}_T = 0 \end{cases} \tag{167}$$

due to the simplicity of this ODE, we don't even have to set up $\overline{\eta}_t, \overline{\chi}_t$ for decoupling purpose, a direct solution is

$$\begin{cases} \overline{x}_t = \mathbb{E}\xi \in \mathbb{R}^3 \\ \overline{y}_t = 0 \end{cases} \tag{168}$$

and is unique, proves the existence and uniqueness of the MFG equilibrium.

When it comes to solving out the closed-form solution of the MFG equilibrium, put up affine ansatz with deterministic η, χ

$$Y_t = \eta_t X_t + \chi_t \tag{169}$$

the MKV-FBSDE is now a normal FBSDE

$$\begin{cases} dX_t = -Y_t\,dt + \sigma\,dW_t \\ X_0 = \xi \\ dY_t = -\kappa^2(X_t - \mathbb{E}\xi)\,dt + Z_t\,dW_t \\ Y_T = 0 \end{cases} \tag{170}$$

collect coefficients and turn it into an ODE w.r.t. η_t, χ_t (check that $Z_t = \sigma\eta_t$ is deterministic, so it's adapted, the key condition to satisfy as the solution to BSDE)

$$\begin{cases} \dot{\eta}_t - \eta_t^2 + \kappa^2 I_3 = 0 \\ \dot{\chi}_t - \eta_t\chi_t - \kappa^2\mathbb{E}\xi = 0 \\ \eta_T = 0, \chi_T = 0 \end{cases} \tag{171}$$

it's quite obvious that the solution is

$$\eta_t = \kappa\frac{e^{2\kappa(T-t)} - 1}{e^{2\kappa(T-t)} + 1} I_3 \overset{def}{=} \eta_t^* I_3 \tag{172}$$

where η_t^* takes value as real number, and to solve χ_t, we need to tear apart different components

$$\forall i \in \{1, 2, ..., d\}, \chi_t^i = -\kappa^2(\mathbb{E}\xi)^i \int_t^T e^{\int_s^t \eta_u^*\,du}\,ds \tag{173}$$

where $(\mathbb{E}\xi)^i$ is the i-th component of $\mathbb{E}\xi$. This gives the solution to the original MKV-FBSDE and the state dynamics at equilibrium is

$$dX_t = -(\eta_t X_t + \chi_t)\,dt + \sigma\,dW_t = -\eta_t(X_t - \mathbb{E}\xi)\,dt + \sigma\,dW_t \tag{174}$$

a mean-reverting Gaussian dynamics.

Remark. *Notice that $\mathbb{E}X_t = \mathbb{E}\xi, \mathbb{E}Y_t = 0$, this tells us that $\eta_t\mathbb{E}\xi + \chi_t = 0$ always holds. In other words,*

$$\chi_t = -\eta_t\mathbb{E}\xi = -\kappa\frac{e^{2\kappa(T-t)} - 1}{e^{2\kappa(T-t)} + 1}\mathbb{E}\xi \tag{175}$$

an easier representation to use. One can check that indeed this is true by plugging this expression into the ODE for χ_t.

Example: Systemic Risk Without Common Noise

The systemic risk example with common risk has already been solved as an example of finite player game. However, as we have mentioned above, different from finite player game setting, common noise in MFG causes essential difficulties (the randomness in the measure is not gone asymptotically) so at this point we only consider solving this example as a MFG without common noise.

For the purpose of completeness, the setting is copied here. There are N players, with player i having state X_t^i at time t taking value in $\mathbb{R}$. Player i can determine its control α_t^i at time t and the state dynamics is given as

$$dX_t^i = [a(\overline{X}_t - X_t^i) + \alpha_t^i]\, dt + \sigma\, dW_t^i \tag{176}$$

on time horizon $[0, T]$. Player i has cost functional

$$J^i(\alpha) = \mathbb{E}\left[\int_0^T f^i(t, X_t, \alpha_t)\, dt + g^i(T, X_T)\right] \tag{177}$$

with running cost

$$f^i(t, x, \alpha) = \frac{\varepsilon}{2}|\overline{x} - x^i|^2 - q\alpha^i(\overline{x} - x^i) + \frac{1}{2}|\alpha^i|^2 \tag{178}$$

and terminal cost

$$g^i(t, x) = \frac{c}{2}(\overline{x} - x^i)^2 \tag{179}$$

it's a LQ game with mean field interaction through $|x^i - \overline{x}|^2$ and the parameters ε, q, c are positive with $q^2 \leq \varepsilon$ such that f^i is a convex function in (x, α^i), resulting in the joint convexity of the reduced Hamiltonian.

First let's do mean field approximation to turn it into a control problem for the representative player. It's clear that the representative player is having state dynamics

$$dX_t = [a(\overline{\mu_t} - X_t) + \alpha_t]\, dt + \sigma\, dW_t \tag{180}$$

with cost functional

$$J(\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t, \mu_t, \alpha_t)\, dt + g(T, X_T, \mu_T)\right] \tag{181}$$

running cost

$$f(t, x, \mu, \alpha) = \frac{\varepsilon}{2}|\overline{\mu} - x|^2 - q\alpha(\overline{\mu} - x) + \frac{1}{2}|\alpha|^2 \tag{182}$$

and terminal cost

$$g(t,x,\mu) = \frac{c}{2}(\overline{\mu} - x)^2 \tag{183}$$

where $\overline{\mu}$ is the mean of the probability distribution μ.

Let's apply Pontryagin's maximum principle. Write down the reduced Hamiltonian (since σ is a constant)

$$H(t,x,\mu,y,\alpha) = [a(\overline{\mu} - x) + \alpha]y + \frac{\varepsilon}{2}|\overline{\mu} - x|^2 - q\alpha(\overline{\mu} - x) + \frac{1}{2}|\alpha|^2 \tag{184}$$

minimize w.r.t. α to get

$$\hat{\alpha}(t,x,\mu,y) = -y + q(\overline{\mu} - x) \tag{185}$$

derive the derivatives in the adjoint BSDE

$$\partial_x H = -ay + \varepsilon(x - \overline{\mu}) + q\alpha \tag{186}$$

$$\partial_x g = c(x - \overline{\mu}) \tag{187}$$

write down the FBSDE with $\hat{\alpha}$ plugged in for fixed flow of measure $\{\mu_t\}$

$$\begin{cases} dX_t = [(a+q)(\overline{\mu_t} - X_t) - Y_t]\,dt + \sigma\,dW_t \\ X_0 = \xi \\ dY_t = [(a+q)Y_t - (\varepsilon - q^2)(X_t - \overline{\mu_t})]\,dt + Z_t\,dW_t \\ Y_T = c(X_T - \overline{\mu_T}) \end{cases} \tag{188}$$

the consistency condition tells us to replace μ_t with $\mathscr{L}(X_t)$ to get the MKV-FBSDE

$$\begin{cases} dX_t = [(a+q)(\overline{\mathscr{L}(X_t)} - X_t) - Y_t]\,dt + \sigma\,dW_t \\ X_0 = \xi \\ dY_t = [(a+q)Y_t - (\varepsilon - q^2)(X_t - \overline{\mathscr{L}(X_t)})]\,dt + Z_t\,dW_t \\ Y_T = c(X_T - \overline{\mathscr{L}(X_T)}) \end{cases} \tag{189}$$

solving this MKV-FBSDE provides the MFG equilibrium as the solution.

Alike what we have done in the last example, we take expectation on both sides of the FBSDE denoting

$\overline{x}_t = \mathbb{E}X_t, \overline{y}_t = \mathbb{E}Y_t$ to get a coupled ODE system

$$\begin{cases} d\overline{x}_t = -\overline{y}_t\, dt \\ \overline{x}_0 = \mathbb{E}\xi \\ d\overline{y}_t = (a+q)\overline{y}_t\, dt \\ \overline{y}_T = 0 \end{cases} \tag{190}$$

it can be directly solved to get the unique solution

$$\begin{cases} \overline{x}_t = \mathbb{E}\xi \\ \overline{y}_t = 0 \end{cases} \tag{191}$$

which implies the existence and uniqueness of MFG equilibrium.

At this point, let's plug $\overline{x}_t = \mathbb{E}\xi$ back into the MKV-FBSDE to reduce it to a normal FBSDE

$$\begin{cases} dX_t = [(a+q)(\mathbb{E}\xi - X_t) - Y_t]\, dt + \sigma\, dW_t \\ X_0 = \xi \\ dY_t = [(a+q)Y_t - (\varepsilon - q^2)(X_t - \mathbb{E}\xi)]\, dt + Z_t\, dW_t \\ Y_T = c(X_T - \mathbb{E}\xi) \end{cases} \tag{192}$$

with an affine ansatz raised as

$$Y_t = \eta_t(X_t - \mathbb{E}\xi) \tag{193}$$

this results in the following fact that

$$dY_t = [\dot{\eta}_t - (a+q)\eta_t - \eta_t^2](X_t - \mathbb{E}\xi)\, dt + \sigma\eta_t\, dW_t \tag{194}$$

compare the stochastic integral term to see $Z_t = \sigma\eta_t$ is adapted since η_t is deterministic. Compare the drift term to get the Ricatti equation with terminal condition

$$\begin{cases} \dot{\eta}_t = \eta_t^2 + 2(a+q)\eta_t - (\varepsilon - q^2) \\ \eta_T = c \end{cases} \tag{195}$$

this equation can be solved easily so

$$\hat{\alpha}_t = -(\eta_t + q)(X_t - \mathbb{E}\xi) \tag{196}$$

the MFG equilibrium state dynamics is given by

$$dX_t = -(a + \eta_t + q)(X_t - \mathbb{E}\xi)\, dt + \sigma\, dW_t \tag{197}$$

Remark. *Notice that if this model is solved as a finite player game with the ansatz*

$$Y_t^{i,j} = \eta_t(\overline{X}_t - X_t^i)\left(\frac{1}{N} - \delta_{i,j}\right) \tag{198}$$

in the open-loop case we have Ricatti equation

$$\dot{\eta}_t = \left(1 - \frac{1}{N}\right)\eta_t^2 + \left[2(a+q) - \frac{q}{N}\right]\eta_t - (\varepsilon - q^2) \tag{199}$$

while in the Markovian case we have the Ricatti equation

$$\dot{\eta}_t = \left(1 - \frac{1}{N^2}\right)\eta_t^2 + 2(a+q)\eta_t - (\varepsilon - q^2) \tag{200}$$

as $N \to \infty$, by propagation of chaos, $\overline{X}_t$ is close to $\mathbb{E}\xi$ so $Y_t^{i,i} \approx Y_t$ (ansatz in finite player game and MFG become close). On the other hand, in the Ricatti equations of finite payer game, the coefficient converges so we see the Ricatti equation for MFG as the limit. This observation tells us that MFG equilibrium, under some conditions, can be formed as the limit of the NE of finite player game as $N \to \infty$ and ***the difference between open-loop and Markovian case disappears****. This is the consequence of* ***folk theorem*** *and we will come back to this later.*

One might wonder if we can solve this MFG with the generalized Feynman-Kac formula. The answer is yes and not surprisingly we shall get the same MFG equilibrium. Recall that after solving out $\hat{\alpha}(t, x, \mu, y)$, since σ is constant (uncontrolled) and is a given positive real number, generalized Feynman-Kac formula characterizes the solution to the HJB equation as the solution to a BSDE. With the correspondence $Y_t = V(t, X_t), Z_t = \sigma \cdot \partial_x V(t, X_t)$ in mind, we write out the FBSDE for fixed flow of measure μ that

$$\begin{cases} dX_t = b\left(t, X_t, \mu_t, \hat{\alpha}\left(t, X_t, \mu_t, \frac{1}{\sigma}Z_t\right)\right)\, dt + \sigma\, dW_t \\ dY_t = -f\left(t, X_t, \mu_t, \hat{\alpha}\left(t, X_t, \mu_t, \frac{1}{\sigma}Z_t\right)\right)\, dt + Z_t \cdot dW_t \\ X_0 = \xi \\ Y_T = g\left(X_T, \mu_T\right) \end{cases} \tag{201}$$

plug in the coefficients to get

$$\begin{cases} dX_t = \left[(a+q)(\overline{\mu_t} - X_t) - \frac{1}{\sigma} Z_t\right] dt + \sigma\, dW_t \\ dY_t = -\left[\frac{\varepsilon - q^2}{2}(\overline{\mu_t} - X_t)^2 + \frac{Z_t^2}{2\sigma^2}\right] dt + Z_t \cdot dW_t \\ X_0 = \xi \\ Y_T = \frac{c}{2}(\overline{\mu_T} - X_T)^2 \end{cases} \tag{202}$$

replace μ_t with $\mathscr{L}(X_t)$ to get the MKV-FBSDE

$$\begin{cases} dX_t = \left[(a+q)(\overline{\mathscr{L}(X_t)} - X_t) - \frac{1}{\sigma} Z_t\right] dt + \sigma\, dW_t \\ dY_t = -\left[\frac{\varepsilon - q^2}{2}(\overline{\mathscr{L}(X_t)} - X_t)^2 + \frac{Z_t^2}{2\sigma^2}\right] dt + Z_t \cdot dW_t \\ X_0 = \xi \\ Y_T = \frac{c}{2}(\overline{\mathscr{L}(X_T)} - X_T)^2 \end{cases} \tag{203}$$

unfortunately, taking expectation on both sides does not work now since the BSDE contain the quadratic term in $\overline{\mathscr{L}(X_t)} - X_t$ and Z_t so it's not easy to directly solve out $\mathscr{L}(X_t)$ and turn the equations to a normal FBSDE. Despite the difficulty solving this MKV-FBSDE, we can verify that

$$\begin{cases} Y_t = \frac{1}{2}\mu_t(X_t - \mathbb{E}\xi)^2 \\ Z_t = \sigma\eta_t(X_t - \mathbb{E}\xi) \\ \overline{\mathscr{L}(X_t)} = \mathbb{E}\xi \end{cases} \tag{204}$$

is the solution to this MKV-FBSDE where η_t, μ_t are deterministic functions in t (so that Z_t is adapted). To check, plug in $\overline{\mathscr{L}(X_t)} = \mathbb{E}\xi$ to turn it into a normal FBSDE. Plug in the ansatz for Y_t, Z_t to see that

$$dY_t = (X_t - \mathbb{E}\xi)^2 \left[\frac{1}{2}\dot{\mu}_t - (a+q)\mu_t - \mu_t\eta_t\right] dt + \sigma\mu_t(X_t - \mathbb{E}\xi)\, dW_t \tag{205}$$

compare with the BSDE to see

$$\begin{cases} \frac{1}{2}\dot{\mu}_t - (a+q)\mu_t - \mu_t\eta_t = -\frac{\eta_t^2}{2} - \frac{\varepsilon - q^2}{2} \\ \mu_t = \eta_t \\ \mu_T = c \end{cases} \tag{206}$$

after some simplification, the ODE for $\eta_t = \mu_t$ is provided as

$$\begin{cases} \dot{\eta}_t = \eta_t^2 + 2(a+q)\eta_t - (\varepsilon - q^2) \\ \eta_T = c \end{cases} \tag{207}$$

which is exactly the same as that derived using Pontryagin maximum principle. At this point,

$$\hat{\alpha}_t = -\frac{Z_t}{\sigma} + q(\mathbb{E}\xi - X_t) = -(q + \eta_t)(X_t - \mathbb{E}\xi) \tag{208}$$

and the MFG equilibrium state dynamics is given by

$$dX_t = -(a + \eta_t + q)(X_t - \mathbb{E}\xi)\, dt + \sigma\, dW_t \tag{209}$$

exactly the same as that derived from Pontryagin maximum principle. At this point, we can go back and check that $\forall t \in [0, T], \mathbb{E}X_t = \mathbb{E}\xi$ since

$$X_t = \xi - \int_0^t (a + \eta_s + q)(X_s - \mathbb{E}\xi)\, ds + \sigma W_t \tag{210}$$

so $\mathbb{E}X_t - \mathbb{E}\xi = -\int_0^t (a + \eta_s + q)(\mathbb{E}X_s - \mathbb{E}\xi)\, ds$. Denote $h(t) = \mathbb{E}X_t - \mathbb{E}\xi$, it provides the ODE

$$\begin{cases} dh(t) = -(a + \eta_t + q)h(t)\, dt \\ h(0) = 0 \end{cases} \tag{211}$$

the solution is trivially $h(t) = 0$ concludes the verification.

Remark. *In this example, Pontryagin maximum principle and generalized Feynman-Kac formula can both be applied to solve the LQ-MFG, but obviously one approach is much easier than the other due to the linearity of coefficients.*

One interesting observation is that the Z_t in generalized Feynman-Kac formula is different from the Y_t in the Pontryagin maximum principle only by a constant σ. This is within our expectation since $Y_t = \partial_x V(t, X_t)$ is the derivative of the value function in the Pontryagin maximum principle but $Z_t = \sigma \cdot \partial_x V(t, X_t)$ in generalized Feynman-Kac formula. This phenomenon indicates that we have to be careful with the different correspondence and interpretation in generalized Feynman-Kac formula and Pontryagin maximum principle.

Remark. *One important point to state here is the intuition of the systemic risk model. This model tries to explain the existence of systemic risk that when a certain amount of banks default, all other banks will also choose to default, causing problems for the economy (for details please refer to the paper Mean Field Game and Systemic Risk). In other words, this model is interesting in the large deviation sense in that default rarely happens but makes a large difference to other banks' behavior.*

However, the MFG equilibrium state dynamics tells us

$$dX_t = -(a + \eta_t + q)(X_t - \mathbb{E}\xi)\, dt + \sigma\, dW_t \tag{212}$$

which is just a normal OU process w.r.t. $\mathbb{E}\xi$. In this sense, each player's state shall exhibit mean-reverting behavior around $\mathbb{E}\xi$ and all players' states are asymptotically independent. In this sense, we completely lose the large deviation behavior described above.

Notice that there's actually no contradiction between those two behaviors. MFG exhibits the SLLN behavior while the original finite player game model exhibits the large deviation behavior. Imagine that one has i.i.d. random variables, if SLLN is working, the almost sure limit is a deterministic real number so of course one loses large deviation arguments. The essential reason is that SLLN provides a too coarse limiting behavior argument. To recover the large deviation behavior, it's natural to think about tracing time back a little bit to the time when SLLN has not yet completely come into effect, i.e. replacing the expectation with the sample mean. As a result, one can actually recover the large deviation behavior by modifying the MFG equilibrium state dynamics a little bit into

$$dX_t = -(a + \eta_t + q) \left(X_t - \frac{1}{k} \sum_{i=1}^{k} X_t^i \right) dt + \sigma \, dW_t \tag{213}$$

replacing $\mathbb{E}\xi$ *with the sample mean of states among* k *players for some fixed integer* k *so that SLLN does not fully come into effect.*

Example: Aiyagari's Growth Model

Let's introduce the diffusion form of Aiyagari's growth model as a finite player game. There are N players in the economy, with each player maintaining his state $X_t^i = (Z_t^i, A_t^i)$ where Z_t denotes the labor productivity at time t and A_t denotes the wealth at time t. The labor productivity has dynamics

$$dZ_t^i = \mu_Z(Z_t^i)\,dt + \sigma_Z(Z_t^i)\,dW_t^i \tag{214}$$

where $\mu_Z, \sigma_Z : \mathbb{R} \to \mathbb{R}$ are given. Each player's labor productivity is a private state, i.e. there's no interaction with other players. On the other hand, players receive wages depend on their productivity, there is an interest rate in the economy so the player's wealth enjoys natural growth as time goes by. Each player maintains his control c_t^i as the consumption rate at time t in order to receive utility. As a result, the dynamics of the wealth process is given by

$$dA_t^i = [w_t^i Z_t^i + r_t A_t^i - c_t^i]\,dt \tag{215}$$

where w_t^i, r_t are two stochastic processes denoting the wage and the interest rate specified later in the context. Given the control of all players, the i-th player tries to maximize the expected utility

$$J^i(c) = \mathbb{E}\int_0^\infty e^{-\rho t} U(c_t^i)\,dt \tag{216}$$

on infinite time horizon with continuous-time discount rate $\rho > 0$. The utility function is often taken as the CRRA utility function

$$U(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}\ (\gamma > 0) \tag{217}$$

to determine the construction of w_t^i and r_t, we need a little bit knowledge from economy that a production function is typically used to describe the production process. The production function $Y = F(K, L)$ describes how total capital K and total labor L decides the total output of the production Y. Naturally, the total capital in this economy is the sum of all players' wealth, but for the purpose of simplicity (easier correspondence with mean field interaction later), we do normalization here to view this economy as having only one person so the total capital would be

$$K_t = \frac{1}{N}\sum_{i=1}^N A_t^i \tag{218}$$

and the total labor would be

$$L_t = 1 \tag{219}$$

at this point, it would be natural to determine w_t^i and r_t. The partial derivative of the total output of production F w.r.t. total labor L gives the marginal product of labor, which in a competitive equilibrium, shall be equal to the wage. By the same reasoning, the marginal product of capital shall be equal to the user cost of capital, which is the interest rate plus the rate of capital depreciation $\delta \geq 0$.

Remark. *Here we have to be a little bit familiar with the economics setting that under equilibrium all costs and profits equate. Imagine that we are running a company in the economy, whenever the marginal product of labor is more than the wage, we shall recruit more people since the value brought by more production exceeds the cost of paying the worker. For the same reason, whenever the marginal product of capital is more than the user cost of capital, we shall invest more capital into production since the value brought by more production exceeds the cost of investing in new equipment or borrowing money. Just to remind, the user cost of capital equals the interest rate (time value) plus the rate of capital depreciation (e.g. equipment damage in production process).*

At this point, it shall be very clear that

$$r_t = \partial_K F(K_t, L_t) - \delta \tag{220}$$

$$w_t = \partial_L F(K_t, L_t) \tag{221}$$

where the wage process of all players in the economy is identical. To clarify, all players interact only through K_t which is actually a function of the empirical measure of the wealth A_t^i. In economics, we often choose the Cobb-Douglass production function

$$F(K, L) = AK^\alpha L^{1-\alpha} \ (A > 0, 0 < \alpha < 1) \tag{222}$$

which results in

$$r_t = A\alpha K_t^{\alpha-1} - \delta \tag{223}$$

$$w_t = A(1-\alpha)K_t^\alpha \tag{224}$$

this concludes the setting of the diffusion form of Aiyagari's growth model.

Now let's make some extra assumptions and try to solve this model as a MFG. The first assumption to make is that this model now takes place on finite time horizon $[0, T]$ so the discount rate $\rho = 0$ is unnecessary. Since it's now a finite horizon game, we add the terminal reward term $g^i(T, X_T^i) = A_T^i$ saying that each player also wants to take into account the wealth he has at terminal time. Other than that, we take $\mu_Z(z) = 1 - z$ so the dynamics of Z_t is an OU process and $\sigma_Z \equiv \sigma > 0$ is specified as a constant, $A = 1$ is taken WLOG. Let's repeat all our settings for now. The state dynamics is given by

$$\begin{cases} dZ_t^i = (1 - Z_t^i)\,dt + \sigma\,dW_t^i \\ dA_t^i = [(1-\alpha)K_t^\alpha Z_t^i + (\alpha K_t^{\alpha-1} - \delta)A_t^i - c_t^i]\,dt \\ \mathbb{E}Z_0^i = 1 \end{cases} \tag{225}$$

and we try to minimize the negative expected utility

$$J^i(c) = -\mathbb{E}\left[\int_0^T U(c_t^i)\,dt + A_T^i\right] \tag{226}$$

Firstly, let's conduct mean field approximation to turn this game into a game only for the representative player that

$$\begin{cases} dZ_t = (1 - Z_t)\,dt + \sigma\,dW_t \\ dA_t = [(1-\alpha)\overline{\mu_t}^{\alpha} Z_t + (\alpha\overline{\mu_t}^{\alpha-1} - \delta)A_t - c_t]\,dt \\ \mathbb{E}Z_0 = 1 \end{cases} \tag{227}$$

and he tries to minimize the negative expected utility

$$J(c) = -\mathbb{E}\left[\int_0^T U(c_t)\,dt + A_T\right] \tag{228}$$

notice that K_t is exactly the mean of the empirical measure of A_t^i so it's replaced with $\overline{\mu_t}$, keep in mind that μ_t is not exactly the empirical measure of state (state consists of two parts, wealth and labor productivity, here the empirical measure is only built w.r.t. wealth).

Apply Pontryagin maximum principle for fixed flow of measure μ, first write the state dynamics in the vector form

$$\begin{bmatrix} dZ_t \\ dA_t \end{bmatrix} = \begin{bmatrix} 1 - Z_t \\ (1-\alpha)\overline{\mu_t}^{\alpha} Z_t + (\alpha\overline{\mu_t}^{\alpha-1} - \delta)A_t - c_t \end{bmatrix} dt + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} dW_t \tag{229}$$

so the coefficients have the form

$$b(t,(z,a),\mu,c) = \begin{bmatrix} 1 - z \\ (1-\alpha)\overline{\mu}^{\alpha} z + (\alpha\overline{\mu}^{\alpha-1} - \delta)a - c \end{bmatrix} \tag{230}$$

$$\sigma(t,(z,a),\mu,c) = \begin{bmatrix} \sigma \\ 0 \end{bmatrix} \tag{231}$$

$$f(t,(z,a),\mu,c) = -U(c) \tag{232}$$

$$g(t,(z,a),\mu) = -a \tag{233}$$

calculate reduced Hamiltonian (diffusion coefficient is constant), notice that we split y into tuples since the state is also a tuple

$$H(t,(z,a),\mu,(y_z,y_a),c) = y_z(1-z) + y_a[(1-\alpha)\overline{\mu}^{\alpha} z + (\alpha\overline{\mu}^{\alpha-1} - \delta)a - c] - U(c) \tag{234}$$

take derivative w.r.t. c to get the optimal control

$$\hat{c}(t,(z,a),\mu,(y_z,y_a)) = (-y_a)^{-\frac{1}{\gamma}} \tag{235}$$

calculate the coefficient of the adjoint BSDE

$$\partial_z H = -y_z + y_a(1-\alpha)\overline{\mu}^{\alpha} \tag{236}$$
$$\partial_a H = y_a(\alpha\overline{\mu}^{\alpha-1} - \delta) \tag{237}$$
$$\partial_z g = 0 \tag{238}$$
$$\partial_a g = -1 \tag{239}$$

we write down the FBSDE

$$\begin{cases} dZ_t = (1-Z_t)\,dt + \sigma\,dW_t \\ dA_t = [(1-\alpha)\overline{\mu_t}^{\alpha} Z_t + (\alpha\overline{\mu_t}^{\alpha-1} - \delta)A_t - (-Y_{a,t})^{-\frac{1}{\gamma}}]\,dt \\ \mathbb{E}Z_0 = 1 \\ dY_{a,t} = -Y_{a,t}(\alpha\overline{\mu_t}^{\alpha-1} - \delta)\,dt + Z_{a,t}\,dW_t \\ Y_{a,T} = -1 \\ dY_{z,t} = [Y_{z,t} - Y_{a,t}(1-\alpha)\overline{\mu_t}^{\alpha}]\,dt + Z_{z,t}\,dW_t \\ Y_{z,T} = 0 \end{cases} \tag{240}$$

it's quite obvious that we don't actually need the last BSDE w.r.t. $Y_{z,t}$ since it's decoupled from other FBSDE (one can solve $Y_{a,t}$ from other equations then plug in to know $Y_{z,t}$). On the other hand, the optimal control $\hat{c}$ has no dependence on $Y_{z,t}$ so we don't actually care about solving this BSDE. This observation reduces the FBSDE into ($\{Z_t\}$ is also a given process, its dynamics is not coupled with other equations in the FBSDE)

$$\begin{cases} dA_t = [(1-\alpha)\overline{\mu_t}^{\alpha} Z_t + (\alpha\overline{\mu_t}^{\alpha-1} - \delta)A_t - (-Y_{a,t})^{-\frac{1}{\gamma}}]\,dt \\ dY_{a,t} = -Y_{a,t}(\alpha\overline{\mu_t}^{\alpha-1} - \delta)\,dt + Z_{a,t}\,dW_t \\ Y_{a,T} = -1 \end{cases} \tag{241}$$

replace μ_t with $\mathscr{L}(A_t)$ to get MKV-FBSDE

$$\begin{cases} dA_t = [(1-\alpha)\overline{\mathscr{L}(A_t)}^{\alpha} Z_t + (\alpha\overline{\mathscr{L}(A_t)}^{\alpha-1} - \delta)A_t - (-Y_{a,t})^{-\frac{1}{\gamma}}]\,dt \\ dY_{a,t} = -Y_{a,t}(\alpha\overline{\mathscr{L}(A_t)}^{\alpha-1} - \delta)\,dt + Z_{a,t}\,dW_t \\ Y_{a,T} = -1 \end{cases} \tag{242}$$

and take expectation on both sides to get a coupled ODE with the notation $\overline{a}_t = \mathbb{E}A_t, \overline{y}_{a,t} = \mathbb{E}Y_{a,t}$ that

$$\begin{cases} d\overline{a}_t = [(1-\alpha)\overline{a}_t^\alpha + (\alpha\overline{a}_t^{\alpha-1} - \delta)\overline{a}_t - \mathbb{E}(-Y_{a,t})^{-\frac{1}{\gamma}}]\, dt \\ d\overline{y}_{a,t} = -\overline{y}_{a,t}(\alpha\overline{a}_t^{\alpha-1} - \delta)\, dt \\ \overline{y}_{a,T} = -1 \end{cases} \tag{243}$$

here we used the fact that $\mathbb{E}Z_t = 1$ from the dynamics of Z_t together with $\mathbb{E}Z_0 = 1$. It's hard to deal with the expectation $\mathbb{E}(-Y_{a,t})^{-\frac{1}{\gamma}}$ in the ODE since it cannot be well represented by $\overline{a}_t, \overline{y}_t$ so we have to think of using other properties of this MKV-FBSDE. One observation is that in the BSDE w.r.t. $Y_{a,t}$, the state process A_t, Z_t does not directly appear in the coefficients except the interaction through $\overline{\mathscr{L}(A_t)}$ which is a function in t. As a result, this MKV-FBSDE is actually decoupled and $Y_{a,t}$ is actually deterministic since by setting $Z_{a,t} = 0$, $Y_{a,t}$ is actually the solution to

$$\begin{cases} dY_{a,t} = -Y_{a,t}(\alpha\overline{\mathscr{L}(A_t)}^{\alpha-1} - \delta)\, dt \\ Y_{a,T} = -1 \end{cases} \tag{244}$$

that provides

$$Y_{a,t} = -e^{-\int_t^T (\alpha\overline{\mathscr{L}(A_s)}^{\alpha-1} - \delta)\, ds} \tag{245}$$

as a result, the coupled ODE turns into

$$\begin{cases} d\overline{a}_t = [\overline{a}_t^\alpha - \delta\overline{a}_t - (-Y_{a,t})^{-\frac{1}{\gamma}}]\, dt \\ Y_{a,t} = -e^{-\int_t^T (\alpha\overline{a}_s^{\alpha-1} - \delta)\, ds} \end{cases} \tag{246}$$

where $Y_{a,t} = \overline{y}_{a,t}$ since it's deterministic. The existence and uniqueness of the solution to this ODE is equivalent to the existence and uniqueness of the MFG equilibrium. Here the important point is to ensure that the solution satisfies $\forall t \in [0,T], \overline{\mathscr{L}(A_t)} > 0, Y_t < 0$ so some restrictions have to be added on the initial wealth A_0. From the ODE w.r.t. $\overline{a}_t$,

$$\overline{a}_t = e^{-\delta t}\left[\overline{a}_0 + \int_0^t e^{\delta s}\overline{a}_s^\alpha\, ds - \int_0^t e^{\delta s}(-Y_{a,s})^{\frac{1}{\gamma}}\, ds\right] \tag{247}$$

the solution can be expressed in terms of $Y_{a,t}$. From the expression of $Y_{a,t}$, it's easily seen that

$$-Y_{a,t} \geq e^{-\delta(T-t)} \tag{248}$$

so

$$\overline{a}_t \geq e^{-\delta t}\left[\overline{a}_0 - \int_0^t e^{\delta s} e^{\frac{\delta(T-s)}{\gamma}}\, ds\right] \tag{249}$$

it always happens that $\forall t \in [0,T], \overline{a}_t > 0$ if

$$\mathbb{E}A_0 = \overline{a}_0 > \int_0^t e^{\delta s} e^{\frac{\delta(T-s)}{\gamma}}\, ds \tag{250}$$

the MFG equilibrium exists and is unique given the initial wealth is not too little.

As the solution to MFG, we can solve out $Y_{a,t}$ and $\overline{a}_t$ simultaneously from numerical methods. After that, $\hat{c}_t, \overline{\mathscr{L}(A_t)}$ are known and we can simulate the trajectory of $\{Z_t\}$ and then simulate the trajectory of $\{A_t\}$ to see the MFG equilibrium state dynamics.

Game with a Continuum of Players

For the purpose of completeness, we mention the construction of a game with a continuum of players. The motivation here is that macroeconomics model typically requires each individual to have negligible amount of impact on the whole economy. On the other hand, the MFG setting is generalized from the finite player game taking the number of players to infinity. It seems that two ways of construction have no significant difference but one actually has to face some measurability issues when dealing with the game with a continuum of players.

Consider player index set $I = [0, 1]$ as an uncountable set equipped with sigma field $\mathscr{I}$ and probability measure λ. Assume that the random variable X^i takes value in the polish space E, then $\left\{X^i\right\}_{i\in I}$ is a family of random variables describing some certain attributes of all players. Consider the mapping $i \in I \mapsto X^i(\omega) \in E$, it's expected to be nowhere continuous and not measurable. As a result, when dealing with independence under the setting of a continuum of players, we don't want to create uncountably many independent copies of X^i and then pair them up. Instead, we would rather sacrifice some independence properties to exchange for measurability. The definition of essentially pairwise independence is introduced with this motivation in mind. A family of random variables $\left\{X^i\right\}_{i\in I}$ is **essentially pairwise independent** if for $\lambda - a.e.\ i \in I$, X^i is independent of X^j for $\lambda - a.e.\ j \in I$. The existence of essentially pairwise independent families turns out to be not trivial. In order to investigate that, **Fubini extension** is introduced.

We neglect the details and the definitions here with one thing in mind that the original product space $(I\times\Omega, \mathscr{I}\times\mathscr{F}, \lambda\times\mathbb{P})$ shall be extended to another space called Fubini's extension denoted $(I\boxtimes\Omega, \mathscr{I}\boxtimes\mathscr{F}, \lambda\boxtimes\mathbb{P})$. The main point to take here is that on the Fubini extension, the measurability and λ-integrability of the mapping $i \in I \mapsto X^i(\omega) \in E$ is ensured for $\lambda - a.e.\ i \in I$ and Fubini's theorem always applies for the integration w.r.t. $\mathbb{P}$ and λ (can change the order of integration as one wishes).

This Fubini extension is called **rich** if there exists a real valued $\mathscr{I}\boxtimes\mathscr{F}$-measurable essentially pairwise independent family $\left\{X^i\right\}_{i\in I}$ such that $\forall i \in I, X^i \sim U(0, 1)$. Intuitively, if uniform random variable family exists, random variable family following any distribution exists.

It turns out that a rich Fubini extension must have λ **as an atomless measure** (e.g. the Lebesgue measure). With E as a polish space and $\mu : I \to \mathscr{P}(E)$ to be $\mathscr{I}$-measurable, there exists E-valued $\mathscr{I}\boxtimes\mathscr{F}$-measurable essentially pairwise independent family $\left\{Y^i\right\}_{i\in I}$ such that $\forall\lambda - a.e.\ i \in I, Y^i$ induces probability measure μ_i. In other words, μ **is a mapping that assigns each player a probability distribution to follow. Under measurable condition of μ, one can always replicate all those distributions on a rich Fubini extension and keep essentially pairwise independence at the same time.** This is not surprising at all since rich Fubini extension already requires the replication of uniform distribution, from which it's easy to replicate any probability distribution. It turns out that the exact law of large number can also be established on the FUbini extension.

Theorem 7 (Exact LLN)**.** *$\left\{X^i\right\}_{i\in I}$ is a measurable square integrable family of r.v. on Fubini extension, then $\left\{X^i\right\}_{i\in I}$ are essentially pairwise uncorrelated iff $\forall A \in \mathscr{I}, \lambda(A) > 0$, for $\mathbb{P} - a.s. \forall\omega \in \Omega$,*

$$\int_A X^i(\omega)\, d\lambda(i) = \int_A \mathbb{E}X^i\, d\lambda(i) \tag{251}$$

Remark. *Again we skip the details but mention the intuition here. One might be wondering why this is the law of* *large numbers. Actually,* $\lambda(A) > 0$ *implies that* A *is large enough (countable sets* A *has measure* 0 *under atomless* λ*), at least uncountable so* $\int_A X^i(\omega)\,d\lambda(i)$ *is the mean of* X^i *among a large enough number of players. We expect this mean to converge to a deterministic quantity related to the expectation, which is* $\int_A \mathbb{E}X^i\,d\lambda(i)$*, the average of* $\mathbb{E}X^i$ *among a large enough number of players. This is exactly the spirit of all variants of law of large number.*

At this point, we shift gears to apply the concepts exhibited above under the setting of MFG, trying to generalize it to a game with a continuum of players. It's quite obvious that for MFG without common noise, the state dynamics contains a family of BM $\left\{W^i\right\}_{i\in I}$ to be essentially pairwise independent under the rich Fubini extension. Interestingly, it turns out that on the same rich Fubini extension we can solve the problem of the progressive measurable requirement of $\left\{\alpha_t^i\right\}_{t\in[0,T]}$ and $\left\{\mu_t^i\right\}_{t\in[0,T]}$ so MFG with a continuum of players is well-defined.

With the help of the exact LLN, $\forall t \in [0,T]$, we would expect to see that the empirical measure at time t is actually not a random measure and shall coincide with μ_t, same as what we have in the traditional MFG setting. As a result, it's intuitive to conclude that **a solution to the traditional MFG provides NE for the MFG with a continuum of players**.

Example: Almgren-Chriss Price Impact Model

There are N traders, each having number of shares X_t^i at time t with the rate of trading α_t^i as the control. The traders are identical with dynamics

$$dX_t^i = \alpha_t^i\, dt + \sigma\, dW_t^i \tag{252}$$

on finite time horizon $[0, T]$. The stock has price S_t and each trader hold cash K_t^i at time t. The cost of trading is modelled by an non-negative convex function c such that $c(0) = 0$. As a result, the change in cash is modelled by

$$dK_t^i = -(\alpha_t^i S_t + c(\alpha_t^i))\, dt. \tag{253}$$

With h being a deterministic function known to everyone, the price impact is modelled by

$$dS_t = \frac{1}{N}\sum_{i=1}^N h(\alpha_t^i)\, dt + \sigma_0\, dW_t^0, \tag{254}$$

where W^0 is another independent BM. Each trader has wealth $V_t^i = K_t^i + X_t^i S_t$ at time t and hopes to minimize its expected cost

$$J^i(\alpha) = \mathbb{E}\left(\int_0^T c_X(X_t^i)\, dt + g(X_T^i) - V_T^i\right). \tag{255}$$

Here c_X and g penalizes against holding more than enough inventories and the trader wants to hold as much wealth as possible.

We first conduct mean field approximation for this game. The mean field interaction in this finite player game is through the term $\frac{1}{N}\sum_{i=1}^N h(\alpha_t^i)$, which has something to do with the empirical measure on the action space. As a result, we write the game in terms of a representative trader that

$$dX_t = \alpha_t\, dt + \sigma\, dW_t, \tag{256}$$

$$dK_t = -(\alpha_t S_t + c(\alpha_t))\, dt, \tag{257}$$

$$dS_t = \int h\, d\theta_t\, dt + \sigma_0\, dW_t^0, \tag{258}$$

where θ_t denotes the population measure at time t, taking values in $\mathscr{P}(\mathscr{A})$. By Ito formula,

$$dV_t = -(\alpha_t S_t + c(\alpha_t))\, dt + X_t \int h\, d\theta_t\, dt + \sigma_0\, dW_t^0 + S_t\alpha_t\, dt + \sigma\, dW_t \tag{259}$$

$$= -c(\alpha_t)\, dt + X_t \int h\, d\theta_t\, dt + \sigma_0\, dW_t^0 + \sigma\, dW_t. \tag{260}$$

Rewrite the expected cost as

$$\mathbb{E}\left[\int_0^T c_X(X_t)\,dt + g(X_T) - \int_0^T \left(-c(\alpha_t)\,dt + X_t \int h\,d\theta_t\,dt + \sigma_0\,dW_t^0 + \sigma\,dW_t\right)\,dt\right] \tag{261}$$

$$= \mathbb{E}\left[\int_0^T \left[c_X(X_t) + c(\alpha_t) - X_t \langle h, \theta_t\rangle\right] dt + g(X_T)\right]. \tag{262}$$

We use the notation $\langle h, \theta_t\rangle = \int h\,d\theta_t$ to represent the integral of h w.r.t. the measure θ_t in a compact way. The trader has running cost

$$f(t, x, \theta, \alpha) = c_X(x) + c(\alpha) - x\langle h, \theta\rangle. \tag{263}$$

This is an example of an **extended MFG** since the empirical measure on the action space appears in the game. Recall that in a traditional MFG, only the empirical measure on the state space appears.

To derive a closed-form solution to this EMFG, we assume that $c(\alpha) = \frac{c_\alpha}{2}\alpha^2, c_X(x) = \frac{c_X}{2}x^2, g(x) = \frac{c_g}{2}x^2, h(\alpha) = \overline{h}\alpha$. This model is known as the **Almgren-Chriss linear price impact model** and it's a linear-quadratic EMFG.

The **probabilistic** approach: Let's first apply Pontryagin's maximum principle to solve this game. Write out the reduced Hamiltonian (constant diffusion coefficient):

$$H(t, x, y, z, \theta, \alpha) = \alpha y + \frac{c_X}{2}x^2 + \frac{c_\alpha}{2}\alpha^2 - x\langle h, \theta\rangle. \tag{264}$$

Fix the measure θ and take derivative w.r.t. α, set it to zero to get the equilibrium strategy:

$$\hat{\alpha} = -\frac{y}{c_\alpha}. \tag{265}$$

Compute the coefficient in the adjoint BSDE

$$\partial_x H = c_X x - \langle h, \theta\rangle. \tag{266}$$

Plugging in the equilibrium strategy gives the FBSDE for a fixed flow of measure $\{\theta_t\}$ that

$$\begin{cases} dX_t = -\frac{1}{c_\alpha}Y_t\,dt + \sigma\,dW_t \\ dY_t = (-c_X X_t + \langle h, \theta_t\rangle)\,dt + Z_t\,dW_t \\ Y_T = c_g X_T \end{cases}. \tag{267}$$

Noticing the consistency condition $\hat{\theta}_t = \mathscr{L}(\hat{\alpha}_t)$ enables us to write down the MKV-FBSDE:

$$\begin{cases} dX_t = -\frac{1}{c_\alpha} Y_t \, dt + \sigma \, dW_t \\ dY_t = -(c_X X_t + \frac{\overline{h}}{c_\alpha} \mathbb{E} Y_t) \, dt + Z_t \, dW_t \\ Y_T = c_g X_T \end{cases} . \tag{268}$$

Finding a closed-form solution to the MKV-FBSDE always requires first turning it into a normal FBSDE. Hence we hope to find $\mathbb{E}Y_t$). Taking expectation on both sides of the FSDE provides

$$d\overline{x}_t = d\mathbb{E}X_t = -\frac{1}{c_\alpha} \mathbb{E}Y_t \, dt = -\frac{1}{c_\alpha} \overline{y}_t \, dt. \tag{269}$$

Taking expectation on both sides of the BSDE provides

$$d\overline{y}_t = d\mathbb{E}Y_t = -(c_X \mathbb{E}X_t + \frac{\overline{h}}{c_\alpha} \mathbb{E}Y_t) \, dt = -(c_X \overline{x}_t + \frac{\overline{h}}{c_\alpha} \overline{y}_t) \, dt, \tag{270}$$

together with the terminal condition $\overline{y}_T = c_g \overline{x}_T$. Solve this ODE system through affine ansatz $\overline{y}_t = \overline{\eta}_t \overline{x}_t + \overline{\chi}_t$ to see that

$$\dot{\overline{\eta}}_t - \frac{1}{c_\alpha} \overline{\eta}_t^2 + \frac{\overline{h}}{c_\alpha} \overline{\eta}_t + c_X = 0, \quad \dot{\overline{\chi}}_t + \frac{\overline{h}}{c_\alpha} \overline{\chi}_t - \frac{1}{c_\alpha} \overline{\eta}_t \overline{\chi}_t = 0, \quad \overline{\eta}_T = c_g, \overline{\chi}_T = 0. \tag{271}$$

The Riccati equations for $\overline{\eta}_t, \overline{\chi}_t$ can be easily solved to turn the MKV-FBSDE into a normal FBSDE. Afterwards, put up the affine ansatz $Y_t = \eta_t X_t + \chi_t$ for the normal FBSDE once more, where η, χ are deterministic, to solve for the closed-form solution to this EMFG.

Actually we can also compute the closed-form solution using the **analytic** approach: Write the HJBE for a fixed flow of measure $\{\theta_t\}$ that

$$\partial_t v + \inf_\alpha \left\{ \alpha \partial_x v + \frac{\sigma^2}{2} \partial_{xx} v + \frac{c_X}{2} x^2 + \frac{c_\alpha}{2} \alpha^2 - x \langle h, \theta_t \rangle \right\} = 0, \quad v(T,x) = \frac{c_g}{2} x^2. \tag{272}$$

Get rid of the inf to see that the equilibrium strategy is

$$\hat{\alpha}(t,x) = -\frac{1}{c_\alpha} \partial_x v(t,x). \tag{273}$$

Plug back into HJBE to get

$$\partial_t v + \frac{\sigma^2}{2} \partial_{xx} v + \frac{c_X}{2} x^2 - \frac{1}{2c_\alpha} (\partial_x v)^2 - x \langle h, \theta_t \rangle = 0, \quad v(T,x) = \frac{c_g}{2} x^2. \tag{274}$$

Instead of writing out the Fokker Planck equation for θ_t, we use a small trick combining the HJBE with the McKean-Vlasov nature by requiring $\hat{\theta}_t = \mathscr{L}(\hat{\alpha}_t)$. This provides

$$\partial_t v + \frac{\sigma^2}{2}\partial_{xx} v + \frac{c_X}{2}x^2 - \frac{1}{2c_\alpha}(\partial_x v)^2 + \frac{\overline{h}}{c_\alpha} x \cdot \mathbb{E}\partial_x v(t, X_t^{\hat{\alpha}}) = 0, \quad v(T,x) = \frac{c_g}{2}x^2. \tag{275}$$

Use the ansatz $v(t,x) = \frac{1}{2}x^T\eta_t x + \chi_t x + \mu_t$ to get

$$\frac{1}{2}\dot{\eta}_t x^2 + \dot{\chi}_t x + \dot{\mu}_t + \frac{\sigma^2}{2}\eta_t + \frac{c_X}{2}x^2 - \frac{1}{2c_\alpha}(\eta_t x + \chi_t)^2 + \frac{\overline{h}}{c_\alpha} x \cdot (\eta_t \mathbb{E}X_t^{\hat{\alpha}} + \chi_t) = 0, \quad v(T,x) = \frac{c_g}{2}x^2. \tag{276}$$

Use the state dynamics with $\hat{\alpha}$ plugged in,

$$dX_t^{\hat{\alpha}} = -\frac{1}{c_\alpha}(\eta_t X_t^{\hat{\alpha}} + \chi_t)\, dt + \sigma\, dW_t. \tag{277}$$

Take expectation on both sides, $d\mathbb{E}X_t^{\hat{\alpha}} = -\frac{1}{c_\alpha}(\eta_t \mathbb{E}X_t^{\hat{\alpha}} + \chi_t)\, dt, \mathbb{E}X_0^{\hat{\alpha}} = \mathbb{E}X_0$ allows one to represent $\mathbb{E}X_t^{\hat{\alpha}}$ in terms of η_t, χ_t. Plug $\mathbb{E}X_t^{\hat{\alpha}}$ back into the HJBE, one is able to derive three decoupled ODEs for η_t, χ_t, μ_t. Solving the ODEs provides the solution to this EMFG.

Example: Cryptocurrency Mining

Refer to the paper ***A Mean Field Games Model for Cryptocurrency Mining by Zhongxi Li, et. al.*** for the details of the model. This model is also an extended mean field game but the interaction is more complicated. The mean of the population measure and the strategy both appear in λ_t, and λ_t works as the intensity of a time-inhomogeneous Poisson process N_t. Finally, the Levy process N_t appears in the state dynamics, based on which the expected utility is maximized.

This model has a closed-form solution for some particular utility functions and one needs to be able to derive the infinitemsimal generator of a jumped diffusion (refer to ***Calculating Infinitesimal Generators by Majnu John and Yihren Wu*** for details).

An Introduction to Master Equation

We provide a very brief introduction to master equation, one of the core topics in the theory of MFG, without rigorous proof but with plenty of intuition. One is welcome to refer to Daniel Lacker's notes *Mean Field Games and Interacting Particle Systems* and Francois Delarue's paper *Mean Field Games and Master Equation* for more details. Most of the following contents are from my PhD cohort Kalok Lam's notes. I want to thank Kalok for providing his great notes!

Motivation

As we have mentioned above, MFG is formed as a fixed point problem due to its competitive nature. In detail, one first fixes the flow of measure $\{\mu_t\}$ and solves out the optimal strategy $\hat{\alpha}$. Following this strategy we get the state dynamics X^μ and it's required that $\mathscr{L}\left(X_t^\mu\right) = \mu_t$ to satisfy the consistency condition. In the context above, we realized that the analytic approach of MFG consists of two parts, the optimality characterized by HJBE and the consistency characterized by Fokker-Planck equation. However, the analytic approach is not useful for solving the closed-form solution to MFG since it's essentially a **two-point boundary value problem** with one PDE running forward and the other running backward. As a result, the probabilistic approach setting up MKV-FBSDE stands out.

However, when it comes to the theory of MFG, e.g. MFG equilibrium as the limit of the NE of finite-player game, the probabilistic approach seems not to be working that well so one has to return to the analytic approach. Instead of characterizing the MFG equilibrium as the solution to a two-point boundary value problem, one wants to find another characterization which is easier to deal with, known as the master equation. The motivation of master equation lies in the fact that **it's often possible to decouple equations at a price of increasing the dimension**. Instead of the normal value function $v(t,x)$ that has time component t and state component x, why not consider $U(t,x,m)$ that also has a probability measure m as one of its components. The price we are paying for U is the existence of the extra measure component m but it's hopeful that we can find a single PDE w.r.t. U that characterizes the MFG equilibrium, circumventing the two-point boundary value structure.

Naturally speaking, the very first question we would ask is how shall we differentiate a real-valued function w.r.t. its measure component m. This leads to our discussion of the calculus on the space of measures.

Calculus on the Space of Measures

Based on different perspectives, there are several approach defining the derivative w.r.t. measure. One of them is the **L-derivatives** established by Lions, understanding the measure through the distribution of random variables. Another perspective is the **linear functional derivatives** whose motivation comes from Frechet derivatives on normed spaces. Last but not least, **W-derivatives** are built on Wasserstein spaces with metric geodesics, uncovering the connection with optimal transport.

For now we mainly describe the setting of the linear functional derivatives. Recall the definition of **Frechet derivative** that if V, W are normed spaces with $f : U \to W$ where U is open in V. f is called Frechet differentiable at x if $\exists A(x) : V \to W$ linear bounded such that

$$\lim_{||h||_V \to 0} \frac{||f(x+h) - f(x) - A(x)h||_W}{||h||_V} = 0 \tag{278}$$

$A(x)$ is just the Frechet derivative of f at x. The key takeaway here is that the derivative of f at x is the slope of the first-order approximation of f at x.

Organizing A such that the image of x under A which is $A(x) : V \to W$ denotes the Frechet derivative of $f : U \to W$ at a single point $x \in U$, then such a mapping acts like

$$A : U \times V \to W \tag{279}$$

$$\forall x \in U, A(x) : V \to W \tag{280}$$

without any confusion, we call such A the Frechet derivative of f in the following context.

When it comes to the master equation, we care about some function $U : \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$ that maps a probability measure on $\mathbb{R}^d$ to a real number in the most general setting. It's immediately clear that the derivative of U w.r.t. measure m shall be a mapping $\mathscr{P}(\mathbb{R}^d) \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$. However, we argue below through a simple example that the derivative $\frac{\delta U}{\delta m}$ can actually be simplified to a mapping $\mathscr{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$.

Remark. *As an analogue, $f(x) = x^2$ has derivative 2 at $x = 1$ but what we call as derivative is the pointwise action of f', i.e. the mapping $f'(x) = 2x$.*

From the definition of Frechet derivative above, it seems that f' shall be a mapping $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ but why is the derivative $f'(x) = 2x$ actually a mapping $\mathbb{R} \to \mathbb{R}$? The answer is that there is some simplification going on. Written in the language of Frechet derivative,

$$\lim_{h \to 0} \frac{|(x+h)^2 - x^2 - 2xh|}{h} = 0 \tag{281}$$

so $A(x)$ maps h to $2xh$, which means the Frechet derivative $A(x,t) = 2x \cdot t$ is linear in t. As a result, the action on t is neglected and only the coefficient $2x$ is maintained and formed as the derivative $f'(x) = 2x$ we are familiar with.

Pointwisely, $A(x)$ maps a single real number to another single real number, which can always be represented by a real number multiplication. Similarly, now that $U : \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$, we have $A(m) = \frac{\delta U}{\delta m}(m) : \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$ maps a single measure on $\mathbb{R}^d$ to a single real number pointwisely, which can always be represented by integration, i.e. integrating *some real-valued function on $\mathbb{R}^d$ w.r.t. this measure on $\mathbb{R}^d$! As a result, it's guaranteed that $A(m) = \frac{\delta U}{\delta m}(m)$ can be represented as a mapping $\mathbb{R}^d \to \mathbb{R}$. Thanks to this clever simplification, $\frac{\delta U}{\delta m}$ is no longer a function on the space of measures.*

In the following context, we shall always keep in mind that

$$\frac{\delta U}{\delta m} : \mathscr{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R} \tag{282}$$

$$\frac{\delta U}{\delta m}(m) : \mathbb{R}^d \to \mathbb{R} \tag{283}$$

$$\frac{\delta U}{\delta m}(m)(v) \in \mathbb{R} \tag{284}$$

where $m \in \mathscr{P}(\mathbb{R}^d)$ is the measure where the derivative is evaluated and $v \in \mathbb{R}^d$ is the variable of the function $\frac{\delta U}{\delta m}(m)$. Since $\frac{\delta U}{\delta m}(m) : \mathbb{R}^d \to \mathbb{R}$, the integration of this function w.r.t. a measure provides a mapping $\mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$ that

$$\mu \mapsto \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m)(v)\, \mu(dv) \overset{def}{=} \left\langle \frac{\delta U}{\delta m}(m), \mu \right\rangle \tag{285}$$

interpreted as the action of measure μ on the function $\frac{\delta U}{\delta m}(m)$ through integration.

Finally, we can write out the definition of linear functional derivative. If $U : \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$, then U **has a linear functional derivative if** $\forall m \in \mathscr{P}(\mathbb{R}^d), \exists \frac{\delta U}{\delta m}(m) : \mathbb{R}^d \to \mathbb{R}$ **such that**

$$\forall m' \in \mathscr{P}(\mathbb{R}^d), U(m') - U(m) = \int_0^1 \left\langle \frac{\delta U}{\delta m}(m + (m' - m)t), m' - m \right\rangle dt \tag{286}$$

recall that $\langle \cdot, \cdot \rangle$ denotes the action of the measure on the function through integration, so

$$\int_0^1 \left\langle \frac{\delta U}{\delta m}(m + (m' - m)t), m' - m \right\rangle dt = \int_0^1 \left[\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m + (m' - m)t)(v)\, (m' - m)(dv) \right] dt \tag{287}$$

Remark. *Written in the language of Frechet derivative, the derivative of U at measure m denoted $A(m) : \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$ acts on μ and maps it to $\langle \frac{\delta U}{\delta m}(m), \mu \rangle$ (through the simplification we have mentioned in the remark above) denoted*

$$A(m)(\mu) = \left\langle \frac{\delta U}{\delta m}(m), \mu \right\rangle \tag{288}$$

the definition of linear functional derivative is saying that

$$\forall m' \in \mathscr{P}(\mathbb{R}^d), U(m') - U(m) = \int_0^1 A(m + (m' - m)t)(m' - m)\, dt \tag{289}$$

*i.e. A works as the "slope" in the first-order approximation, the **same spirit as the Frechet derivative!** The convex combination $m + (m' - m)t$ appears as the perturbed version of m to ensure that it is still a legal probability measure. The definition of linear functional derivative is smartly designed such that the chain rule formally holds.*

There is an equivalent variational definition of the derivative $\frac{\delta U}{\delta m}$ that one shall also keep in mind since it often helps with the calculation of some important examples. The definition is through identifying $\frac{\delta U}{\delta m}(m)$ as the unique function on $\mathbb{R}^d$ that satisfies

$$\forall m' \in \mathscr{P}(\mathbb{R}^d), \left[\frac{d}{dt}U(m+(m'-m)t)\right]\Big|_{t=0} = \left\langle \frac{\delta U}{\delta m}(m), m'-m \right\rangle \tag{290}$$

it can be shown that two definitions are equivalent. Here we provide the calculation of one direction in the remark below.

Remark. *By the original definition of linear functional derivative,*

$$\begin{aligned}\left[\frac{d}{dt}U(m+(m'-m)t)\right]\Big|_{t=0} &= \lim_{h\to 0}\frac{U(m+(m'-m)h)-U(m)}{h} &(291)\\ &= \lim_{h\to 0}\frac{\int_0^1 \left\langle \frac{\delta U}{\delta m}(m+(m'-m)th), (m'-m)h\right\rangle dt}{h} &(292)\\ &= \lim_{h\to 0}\int_0^1 \left\langle \frac{\delta U}{\delta m}(m+(m'-m)th), m'-m\right\rangle dt &(293)\\ &= \left\langle \frac{\delta U}{\delta m}(m), m'-m\right\rangle &(294)\end{aligned}$$

proves that the variational definition holds.

In most cases the linear functional derivative $\frac{\delta U}{\delta m}(m,v)$ is regular in v so the **intrinsic derivative** can be defined

$$D_mU(m,v) \overset{def}{=} D_v\frac{\delta U}{\delta m}(m,v) \tag{295}$$

taking value in $\mathbb{R}^d$. It might not be clear at this moment why the intrinsic derivative is defined like this, but we shall see later that it naturally appears in the Ito-Krylov's formula and the master equation.

One can similarly define the **second intrinsic derivative** of U w.r.t. measure m as

$$D_m^2U(m,v,v') \overset{def}{=} D_m(D_mU(\cdot,v))(m,v') \tag{296}$$

Since the original function $U:\mathscr{P}(\mathbb{R}^d)\to\mathbb{R}$ and the intrinsic derivative $D_mU:\mathscr{P}(\mathbb{R}^d)\times\mathbb{R}^d\to\mathbb{R}^d$, it's clear that each time of intrinsic differentiation results in a new $\mathbb{R}^d$ component in its domain, that's why the component $v'\in\mathbb{R}^d$ appears in the definition of D_m^2U. It's clear from the definition that

$$D_m^2U(m,v,v') = D_{v'}\left(\frac{\delta D_mU(\cdot,v)}{\delta m}(m)\right)(v') \tag{297}$$

Example: Linear Functional Derivative and Intrinsic Derivative

Let's try to calculate some examples under $d = 1$. WLOG, those calculations also holds for general d with a slight generalization.

The simplest example is $U(m) = \langle \phi, m \rangle = \int_{\mathbb{R}} \phi(v)\, m(dv)$. Use the variational definition,

$$U(m + (m' - m)t) = U(m) + tU(m' - m) \tag{298}$$

taking derivative w.r.t. t evaluated at $t = 0$ gives $U(m' - m) = \langle \phi, m' - m \rangle$ so

$$\frac{\delta U}{\delta m}(m, v) = \phi(v), D_m U(m, v) = \phi'(v) \tag{299}$$

as a result, taking $\phi(v) = v^n$ results in U **mapping measure** m **to its** n**-th moment**, in this case

$$D_m U(m, v) = nv^{n-1} \tag{300}$$

On the other hand, we can also consider any function of $\langle \phi, m \rangle$ denoted $U(m) = F(\langle \phi, m \rangle)$ where $F : \mathbb{R} \to \mathbb{R}$ is C^1. Similarly,

$$U(m + (m' - m)t) = F[\langle \phi, m \rangle + t\langle \phi, m' - m \rangle] \tag{301}$$

taking derivative w.r.t. t gives

$$F'[\langle \phi, m \rangle + t\langle \phi, m' - m \rangle] \cdot \langle \phi, m' - m \rangle \tag{302}$$

evaluate at $t = 0$

$$F'[\langle \phi, m \rangle] \cdot \langle \phi, m' - m \rangle \tag{303}$$

to get

$$\frac{\delta U}{\delta m}(m, v) = F'[\langle \phi, m \rangle] \cdot \phi(v), D_m U(m, v) = F'[\langle \phi, m \rangle] \cdot \phi'(v) \tag{304}$$

as a result, if $F(x) = x^2, \phi(v) = v$ is specified, U **maps measure** m **to its first moment squared**, in this case

$$D_m U(m, v) = 2 \int_{\mathbb{R}} v\, m(dv) \tag{305}$$

naturally, due to linearity of differentiation, if U **maps measure m to its variance**, then

$$D_m U(m, v) = 2v - 2 \int_{\mathbb{R}} v\, m(dv), \langle D_m U(m, \cdot), m \rangle = 0 \tag{306}$$

those intrinsic derivatives always appear in the LQ setting.

Consider another example where

$$U(m) = \int \phi(x,y)\, m(dx)\, m(dy) = \langle \phi, m \times m \rangle \tag{307}$$

where $m \times m$ means the product measure. First calculate linear functional derivative

$$\begin{aligned} U(m + (m'-m)t) &= U(m) + \langle \phi, m \times (m'-m)t \rangle + \langle \phi, (m'-m)t \times m \rangle + \langle \phi, (m'-m)t \times (m'-m)t \rangle && (308)\\ &= U(m) + t\langle \phi, m \times (m'-m) \rangle + t\langle \phi, (m'-m) \times m \rangle + t^2 \langle \phi, (m'-m) \times (m'-m) \rangle && (309) \end{aligned}$$

differentiate w.r.t. t and evaluate at $t = 0$ to get

$$\begin{aligned} \langle \phi, m \times (m'-m) \rangle + \langle \phi, (m'-m) \times m \rangle &= \int \phi(x,y)\, m(dx)\, (m'-m)(dy) + \int \phi(x,y)\, (m'-m)(dx)\, m(dy) && (310)\\ &= \int \left[\int \phi(x,y)\, m(dx) + \int \phi(y,x)\, m(dx) \right] (m'-m)(dy) && (311)\\ &= \left\langle \int \left(\phi(x,\cdot) + \phi(\cdot,x) \right) m(dx), m'-m \right\rangle && (312) \end{aligned}$$

assuming that Fubini's theorem holds (by adding constraints on ϕ), this tells us the linear functional derivative

$$\frac{\delta U}{\delta m}(m,v) = \int \left(\phi(x,v) + \phi(v,x) \right) m(dx) \tag{313}$$

from which we calculate the intrinsic derivative

$$\begin{aligned} D_m U(m,v) &= D_v \left[\int \left(\phi(x,v) + \phi(v,x) \right) m(dx) \right] && (314)\\ &= \int \left(\phi_2'(x,v) + \phi_1'(v,x) \right) m(dx) && (315) \end{aligned}$$

where ϕ_1' is the partial derivative of ϕ w.r.t. its first component and ϕ_2' is the one w.r.t. its second component. Repeat this procedure, calculate the linear functional derivative of $D_m U(m,v)$ seen as a function in m, i.e. set $V(m) = D_m U(m,v)$ then

$$V(m + (m'-m)t) = V(m) + t \int \left(\phi_2'(x,v) + \phi_1'(v,x) \right) (m'-m)(dx) \tag{316}$$

differentiate w.r.t. t and evaluate at $t = 0$ to get

$$\int \left(\phi_2'(x,v) + \phi_1'(v,x) \right) (m'-m)(dx) = \langle \phi_2'(\cdot,v) + \phi_1'(v,\cdot), m'-m \rangle \tag{317}$$

with a new variable $v' \in \mathbb{R}^d$ introduced

$$\frac{\delta V}{\delta m}(m, v') = \phi_2'(v', v) + \phi_1'(v, v') \tag{318}$$

the second intrinsic derivative is calculated

$$D_m^2 U(m, v, v') = D_{v'} \frac{\delta V}{\delta m}(m, v') = \phi_{12}''(v', v) + \phi_{21}''(v, v') \tag{319}$$

where ϕ_{12}'' is the second-order partial derivative of ϕ first taking partial derivative w.r.t. the second component then w.r.t. the first component. This is an important example for the second-order intrinsic derivative of the function U **that maps measure m to $\mathbb{E}\phi(X, Y)$ where $X, Y \sim m$ are independent**.

Remark. *The linear functional derivative has **chain rule** to hold for the composition of U with $F : \mathbb{R} \to \mathbb{R}$. The proof can be easily written through the variational definition once again, so it's left to the readers. We just use an example calculated above to illustrate the idea of chain rule.*

Consider U that maps measure m to its mean and $F(x) = x^2$, clearly $F \circ U$ maps measure m to its mean squared. By chain rule, the outer layer derivative shall be the derivative of F evaluated at $U(m)$, which is $2U(m) = 2\int x\, m(dx)$. The inner layer linear functional derivative is just $\frac{\delta U}{\delta m}(m) = 1$. The product gives the linear functional derivative of the composition

$$\frac{\delta(F \circ U)}{\delta m}(m) = 2 \int x\, m(dx) \cdot 1 \tag{320}$$

Example: Mean Field Interaction

When it comes to solving MFG, mean field interaction comes into play through a function of the empirical measure. If there are N players with the state of player i given as $x^i \in \mathbb{R}^d$, the empirical measure on state space $\mathbb{R}^d$ is

$$\mu_x^N = \frac{1}{N}\sum_{i=1}^N \delta_{x^i} \tag{321}$$

mean field interaction can always be represented as

$$u(x) = U(\mu_x^N) \tag{322}$$

where $U : \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$ and $u : \mathbb{R}^{d\times N} \to \mathbb{R}$. Mean field interaction through u is always observed in the dynamics or the cost functional of a MFG. As a result, it's crucial to figure out $\partial_{x^i} u$ and $\partial_{x^j,x^i} u$. WLOG we assume $d = 1$, i.e. the state takes value as real number.

Our calculation starts from the definition of partial derivative and linear functional derivative

$$\begin{aligned} u(x+he_i) - u(x) &= U\left(\frac{1}{N}\left[\sum_{j=1,j\neq i}^N \delta_{x^j} + \delta_{x^i+h}\right]\right) - U\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i}\right) && (323)\\ &= \int_0^1 \left\langle \frac{\delta U}{\delta m}\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i} + \frac{1}{N}(\delta_{x^i+h} - \delta_{x^i})t\right), \frac{1}{N}(\delta_{x^i+h} - \delta_{x^i})\right\rangle dt && (324) \end{aligned}$$

let's calculate the bracket (the action of integration) that

$$\begin{aligned} &\left\langle \frac{\delta U}{\delta m}\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i} + \frac{1}{N}(\delta_{x^i+h} - \delta_{x^i})t\right), \frac{1}{N}(\delta_{x^i+h} - \delta_{x^i})\right\rangle && (325)\\ &= \frac{1}{N}\int \frac{\delta U}{\delta m}\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i} + \frac{1}{N}(\delta_{x^i+h} - \delta_{x^i})t\right)(v)\,(\delta_{x^i+h} - \delta_{x^i})(dv) && (326)\\ &= \frac{1}{N}\left[\frac{\delta U}{\delta m}\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i} + \frac{1}{N}(\delta_{x^i+h} - \delta_{x^i})t\right)(x^i+h) - \frac{\delta U}{\delta m}\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i} + \frac{1}{N}(\delta_{x^i+h} - \delta_{x^i})t\right)(x^i)\right] && (327) \end{aligned}$$

at this point, we see $x^i + h$ and x^i, which matches the difference quotient in terms of h, as a result,

$$\begin{aligned} &\frac{u(x+he_i) - u(x)}{h} && (328)\\ &= \frac{1}{N}\int_0^1 \frac{\left[\frac{\delta U}{\delta m}\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i} + \frac{1}{N}(\delta_{x^i+h} - \delta_{x^i})t\right)(x^i+h) - \frac{\delta U}{\delta m}\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i} + \frac{1}{N}(\delta_{x^i+h} - \delta_{x^i})t\right)(x^i)\right]}{h}\,dt && (329) \end{aligned}$$

let's denote $\nu(t,h) = \frac{1}{N}\sum_{i=1}^N \delta_{x^i} + \frac{1}{N}(\delta_{x^i+h} - \delta_{x^i})t$ for simplicity, when $h \to 0$, we expect to see $D_v \frac{\delta U}{\delta m}$, which is the intrinsic derivative $D_m U$, and also $\nu(t,h) \to \mu_x^N$ at the same time. Simple calculation tells us

$$\partial_{x^i} u(x) = \frac{1}{N}\int_0^1 D_m U(\mu_x^N)(x^i)\,dt = \frac{1}{N} D_m U(\mu_x^N)(x^i) \tag{330}$$

the **first-order partial derivative of mean field interaction w.r.t. state component**, together with its connection with intrinsic derivative w.r.t. measure.

Remark. *Previously, we fail to provide interpretation of the intrinsic derivative. However, through this example, one can see the natural appearance of the differentiation w.r.t. v.*

When it comes to the second-order partial derivative,

$$\partial_{x^i} u(x+he_j) - \partial_{x^i} u(x) = \frac{1}{N}\left(D_m U(\mu_{x+he_j}^N)(x^i + h\delta_{ij}) - D_m U(\mu_x^N)(x^i)\right) \tag{331}$$

$$= \frac{1}{N}\left(D_m U(\mu_{x+he_j}^N)(x^i + h\delta_{ij}) - D_m U(\mu_{x+he_j}^N)(x^i) + D_m U(\mu_{x+he_j}^N)(x^i) - D_m U(\mu_x^N)(x^i)\right) \tag{332}$$

here we abuse the notation, δ_{ij} denotes a real number which is 1 if $i=j$ and 0 otherwise (it does not stand for a measure), notice that $h\delta_{ij}$ makes a difference when $i=j$ happens. Let's first focus on the second difference $D_m U(\mu_{x+he_j}^N)(x^i) - D_m U(\mu_x^N)(x^i)$ by noticing that on fixing the variable $v = x^i$, $D_m U$ can once again be seen as a function in the measure component, and the definition of linear functional derivative applies

$$D_m U(\mu_{x+he_j}^N) - D_m U(\mu_x^N) \tag{333}$$

$$= \int_0^1 \left\langle \frac{\delta}{\delta m} D_m U(\mu_x^N + (\mu_{x+he_j}^N - \mu_x^N)t), \mu_{x+he_j}^N - \mu_x^N \right\rangle dt \tag{334}$$

$$= \int_0^1 \int \frac{\delta}{\delta m} D_m U(\mu_x^N + (\mu_{x+he_j}^N - \mu_x^N)t)(v')\,(\mu_{x+he_j}^N - \mu_x^N)(dv')\,dt \tag{335}$$

$$= \frac{1}{N}\int_0^1 \left[\frac{\delta}{\delta m} D_m U(\mu_x^N + (\mu_{x+he_j}^N - \mu_x^N)t)(x^j + h) - \frac{\delta}{\delta m} D_m U(\mu_x^N + (\mu_{x+he_j}^N - \mu_x^N)t)(x^j)\right] dt \tag{336}$$

here the variable $v = x^i$ is temporarily omitted and a new variable v' is introduced. The difference quotient structure in terms of v' appears again, take $h \to 0$ to see

$$\frac{D_m U(\mu_{x+he_j}^N) - D_m U(\mu_x^N)}{h} \tag{337}$$

$$\to \frac{1}{N}\int_0^1 \left[D_{v'} \frac{\delta}{\delta m} D_m U(\mu_x^N)(x^j)\right] dt \tag{338}$$

$$= \frac{1}{N}\int_0^1 D_m^2 U(\mu_x^N)(x^j)\,dt = \frac{1}{N} D_m^2 U(\mu_x^N)(x^j)\ (h \to 0) \tag{339}$$

don't forget that the variable correspondence is given by $m = \mu_x^N, v = x^i, v' = x^j$ in the limit (we omitted the variable v in the calculation, have to add it back), written in the form of the definition of the second-order intrinsic derivative $D_m^2 U(m)(v)(v')$, the limit is actually

$$\frac{1}{N} D_m^2 U(\mu_x^N)(x^i)(x^j) \tag{340}$$

finally let's consider the first part of the difference which may be non-zero only if $i = j$. When $i = j$,

$$\frac{D_m U(\mu_{x+he_j}^N)(x^i + h\delta_{ij}) - D_m U(\mu_{x+he_j}^N)(x^i)}{h} = \frac{D_m U(\mu_{x+he_i}^N)(x^i + h) - D_m U(\mu_{x+he_i}^N)(x^i)}{h} \tag{341}$$

$$\to D_v \left[D_m U(\mu_x^N) \right] (x^i) \ (h \to 0) \tag{342}$$

combine two parts to see

$$\partial_{x^j x^i} u(x) = \frac{1}{N^2} D_m^2 U(\mu_x^N)(x^i)(x^j) + \delta_{ij} \frac{1}{N} D_v \left[D_m U(\mu_x^N) \right] (x^i) \tag{343}$$

Remark. *The definition of the second-order intrinsic derivative is also natural. To clarify, the term $D_v \left[D_m U(\mu_x^N) \right] (x^i)$ means first calculating $D_m U(m, v)$, evaluate it at $m = \mu_x^N$ so that it becomes a function in terms of v and then take the partial derivative w.r.t. v and evaluate at $v = x^i$. The calculation above still holds when the state takes value in $\mathbb{R}^d$ for general d.*

To sum up, we proved the following theorem

Theorem 8 (Differential of Mean Field Interaction)**.** *Now that the state takes value in $\mathbb{R}^d$ in an N-player game, with $U : \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}, u : \mathbb{R}^{Nd} \to \mathbb{R}$ and $\mu_x^N = \frac{1}{N} \sum_{i=1}^N \delta_{x^i}$ as empirical measure on state space, $u(x) = U(\mu_x^N)$ as mean field interaction, then*

$$D_{x^i} u(x) = \frac{1}{N} D_m U(\mu_x^N)(x^i) \tag{344}$$

$$D_{x^j} D_{x^i} u(x) = \frac{1}{N^2} D_m^2 U(\mu_x^N)(x^i)(x^j) + \delta_{ij} \frac{1}{N} D_v \left[D_m U(\mu_x^N) \right] (x^i) \tag{345}$$

can be represented in terms of the intrinsic derivatives of U.

Ito-Krylov's Formula for a Flow of Measure

We have learnt how to differentiate the mean field interaction w.r.t. the state variable x in the example above. However, the state itself is a stochastic process in the setting of MFG, with the empirical measure induced to be a flow of random measure. To be specific, assume that the state dynamics is given by

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t \tag{346}$$

we are curious about $dU(\mu_{X_t}^N)$ where X_t is a random variable so $\mu_{X_t}^N$ is a random empirical measure $\mu_{X_t}^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$. Since the propagation of chaos in MFG ensures the asymptotic independence of each player's state $X^1, ..., X^N$, we would like to identify $X_t^1, ..., X_t^N$ as *i.i.d.* random variables following the given state dynamics as $N \to \infty$. At this point, apply Ito formula

$$dU(\mu_{X_t}^N) = \sum_{i=1}^N D_{x^i} U(\mu_{X_t}^N)\,dX_t^i + \frac{1}{2}\sum_{i,j=1}^N D_{x^j} D_{x^i} U(\mu_{X_t}^N)\,d\left\langle X^i, X^j \right\rangle_t \tag{347}$$

notice that $D_{x^i}U, D_{x^j}D_{x^i}U$ are calculated in the theorem above, plug in those conclusions together with the state dynamics to get

$$dU(\mu_{X_t}^N) = \frac{1}{N}\sum_{i=1}^N D_m U(\mu_{X_t}^N)(X_t^i)\,dX_t^i + \frac{1}{2}\sum_{i,j=1}^N \left[\frac{1}{N^2} D_m^2 U(\mu_{X_t}^N)(X_t^i)(X_t^j) + \delta_{ij}\frac{1}{N} D_v \left[D_m U(\mu_x^N)\right](X_t^i)\right] d\left\langle X^i, X^j \right\rangle_t \tag{348}$$

$$= \frac{1}{N}\sum_{i=1}^N D_m U(\mu_{X_t}^N)(X_t^i) \cdot b(t, X_t^i)\,dt \tag{349}$$

$$+ \frac{1}{2}\sum_{i=1}^N \mathrm{Tr}\left(\left[\frac{1}{N^2} D_m^2 U(\mu_{X_t}^N)(X_t^i)(X_t^i) + \frac{1}{N} D_v \left[D_m U(\mu_x^N)\right](X_t^i)\right] \cdot \sigma\sigma^T(t, X_t^i)\right) dt \tag{350}$$

$$+ \frac{1}{N}\sum_{i=1}^N D_m U(\mu_{X_t}^N)(X_t^i) \cdot \sigma(t, X_t^i)\,dW_t^i \tag{351}$$

taking $N \to \infty$, from the propagation of chaos once again, $\mu_{X_t}^N$ converges to $\mu_t = \mathscr{L}(X_t)$ in the setting of MFG, which is a deterministic flow of measure and the randomness is gone. Meanwhile, the drift term has its limit if there is enough regularity that

$$\frac{1}{N}\sum_{i=1}^{N} D_m U(\mu_{X_t}^N)(X_t^i)\cdot b(t,X_t^i) \to \mathbb{E}_{X_t\sim\mu_t}\left[D_m U(\mu_t)(X_t)\cdot b(t,X_t)\right] \tag{352}$$

$$= \int_{\mathbb{R}^d} \left[D_m U(\mu_t)(v)\cdot b(t,v)\right]\mu_t(dv) \tag{353}$$

$$\frac{1}{2}\sum_{i=1}^{N} \mathrm{Tr}\left(\frac{1}{N^2} D_m^2 U(\mu_{X_t}^N)(X_t^i)(X_t^i)\cdot\sigma\sigma^T(t,X_t^i)\right) \to 0 \tag{354}$$

$$\frac{1}{2}\sum_{i=1}^{N} \mathrm{Tr}\left(\frac{1}{N} D_v\left[D_m U(\mu_x^N)\right](X_t^i)\cdot\sigma\sigma^T(t,X_t^i)\right) \to \frac{1}{2}\mathbb{E}_{X_t\sim\mu_t}\left[\mathrm{Tr}\left(D_v\left[D_m U(\mu_t)\right](X_t)\cdot\sigma\sigma^T(t,X_t)\right)\right] \tag{355}$$

$$= \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{Tr}\left(D_v\left[D_m U(\mu_t)\right](v)\cdot\sigma\sigma^T(t,v)\right)\mu_t(dv) \tag{356}$$

here the second limit gives zero since there is N^2 on the denominator (of order larger than N), the type of convergence is not specified here but an SLLN-type theorem applies.

On the LHS of the equation, as $N\to\infty$, $dU(\mu_{X_t}^N)$ converges to $dU(\mu_t)$. Since $dU(\mu_t)$ shall be deterministic, heuristically the local-MG part in Ito formula above does not contribute. As a result, we get the Ito-Krylov's formula for a flow of measure.

Theorem 9 (Ito-Krylov's Formula for a Flow of Measure)**.** *Under certain regularity conditions, if $U:\mathscr{P}(\mathbb{R}^d)\to\mathbb{R}$ and X_t has dynamics in $\mathbb{R}^d$*

$$dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t \tag{357}$$

denote $\mu_t = \mathscr{L}(X_t)$ as a deterministic flow of measure, then

$$dU(\mu_t) = \left[\int_{\mathbb{R}^d}\left[D_m U(\mu_t)(v)\cdot b(t,v)\right]\mu_t(dv) + \frac{1}{2}\int_{\mathbb{R}^d} Tr\left(D_v\left[D_m U(\mu_t)\right](v)\cdot\sigma\sigma^T(t,v)\right)\mu_t(dv)\right]dt \tag{358}$$

Remark. *The normal Ito formula enables us to quantify the change in $f(X_t)$ w.r.t. time t where $\{X_t\}$ is a stochastic process. Ito-Krylov's formula for a flow of measure, on the other hand, enables us to quantify the change in the value of $U(\mu_t)$ w.r.t. time t. Luckily, due to the propagation of chaos in MFG, randomness in the empirical measure goes away as $N\to\infty$ so the flow of measure is deterministic asymptotically.*

Feynman-Kac Correspondence of Master Field

The previously mentioned analytic approach of MFG is based on HJBE for optimality, coupled with Fokker-Planck equation for consistency. On the optimality side, on fixing the flow of measure $\{\mu_t\}$, we put up value function and HJBE describes how the value function evolve. On the consistency side, plugging in the optimal control $\hat{\alpha}$ solved out from HJBE, we derive the state dynamics X_t^μ and require $\mu_t = \mathscr{L}(X_t^\mu)$ to hold.

As mentioned beforehand, the motivation of master equation is mainly to find another characterization for this two-point boundary value problem by putting up a value function that also has a measure as one of its components. Since Feynman-Kac formula tells us the connection between SDE and PDE through a value function construction, by constructing the master field $U(t, X_t, \mu_t)$ as a new value function, we expect to see a PDE w.r.t. U which we call the master equation.

For a finite-horizon MFG, assume that we already have the NE strategy $\hat{\alpha}$ plugged into the state dynamics, each representative player shall see the state dynamics

$$dX_t = b(t, X_t, \mu_t)\, dt + \sigma(t, X_t, \mu_t)\, dW_t \tag{359}$$

without control α but with flow of measure $\{\mu_t\}$. Here we assume that $\mu_t = \mathscr{L}(Y_t)$ comes from

$$dY_t = b'(t, Y_t)\, dt + \sigma'(t, Y_t)\, dW_t \tag{360}$$

i.e. the flow of measure $\{\mu_t\}$ is generated by another diffusion with different coefficients b', σ'. Through Feynman-Kac correspondence, the **master field** (value function) shall be naturally defined as

$$U(t, x, m) \stackrel{def}{=} \mathbb{E}\left[\int_t^T f(s, X_s, \mu_s)\, ds + g(X_T, \mu_T)\Big| X_t = x, \mu_t = m\right],\ (x \in \mathbb{R}^d, m \in \mathscr{P}(\mathbb{R}^d)) \tag{361}$$

where f is the running cost and g is the terminal cost.

Similar to the derivation of Kolmogorov's backward equation, apply the tower property for $\forall h > 0$

$$U(t, X_t, \mu_t) = \mathbb{E}\left[U(t+h, X_{t+h}, \mu_{t+h}) + \int_t^{t+h} f(s, X_s, \mu_s)\, ds\Big| X_t, \mu_t\right] \tag{362}$$

and then the Ito formula, notice that when it comes to the measure component, the Ito-Krylov's formula for a flow of measure applies

$$U(t, X_t, \mu_t) = U(t, X_t, \mu_t) \tag{363}$$

$$+ \mathbb{E}\left[\int_t^{t+h} \left(\partial_t U(s, X_s, \mu_s)\, ds + \partial_x U(s, X_s, \mu_s)\, dX_s + \frac{1}{2}\partial_{xx} U(s, X_s, \mu_s)\, d\left\langle X, X\right\rangle_s + f(s, X_s, \mu_s)\right) ds \middle| X_t\right] \tag{364}$$

$$+ \int_t^{t+h} \left[\int_{\mathbb{R}^d} [D_m U(s, X_s)(\mu_s)(v) \cdot b'(s, v)]\, \mu_s(dv) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{Tr}\left(D_v\left[D_m U(s, X_s)(\mu_s)\right](v) \cdot \sigma'(\sigma')^T(s, v)\right)\, \mu_s(dv)\right] ds \tag{365}$$

the term on the second row on the RHS corresponds to the contribution of the change in t, X_t (from Ito formula) and the term on the third row on the RHS corresponds to the contribution of the change in measure component μ_t (from Ito-Krylov's formula). Be careful that here $\mu_t = \mathscr{L}(Y_t)$ so the coefficients are b', σ' in the third row on the RHS.

It's obvious that the second row on the RHS has a representation in terms of the infinitesimal generator L. After a standard procedure plugging in the dynamics of X_t, taking expectation on both sides to get rid of the local-MG term, divide by h and set $h \to 0$, the equation implies the **backward PDE**

$$\begin{cases} \partial_t U(t, x, m) + LU(t, x, m) + \overline{L}U(t, x, m) + f(t, x, m) = 0 \\ U(T, x, m) = g(x, m) \end{cases} \tag{366}$$

where the **operator $\overline{L}$ is defined to match the contribution of the measure component from Ito-Krylov's formula**, i.e.

$$\overline{L}U(t, x, m) \overset{def}{=} \int_{\mathbb{R}^d} [D_m U(t, x)(m)(v) \cdot b'(t, v)]\, m(dv) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{Tr}\left(D_v\left[D_m U(t, x)(m)\right](v) \cdot \sigma'(\sigma')^T(t, v)\right)\, m(dv) \tag{367}$$

where b', σ' are the drift and diffusion coefficients of the dynamics whose law is the flow of measure. That's all for the Feynman-Kac correspondence of the master field.

Remark. *In the correspondence above, L is defined through the dynamics of X_t while $\overline{L}$ is defined through the dynamics of Y_t. Since we are not yet entering the setting of MFG, L and $\overline{L}$ has no connection at this moment.*

To clarify the notation once more, in the definition of $\overline{L}$, the term $D_m U(t, x)(m)(v)$ means that when t, x are fixed, U can be seen as a function on the space of measures so its intrinsic derivative $D_m U(t, x)(m, v)$ can be calculated. Evaluate it at measure m, it becomes a function on $\mathbb{R}^d$ so one can integrate it w.r.t. a measure on $\mathbb{R}^d$. Similarly, $D_v\left[D_m U(t, x)(m)\right](v)$ means to fix t, x, calculate intrinsic derivative $D_m U(t, x)(m, v)$, evaluate at measure m, differentiate w.r.t. v and evaluate at v.

MFG and Master Equation

When it comes to MFG, where the state dynamics of the representative player is controlled, given by

$$dX_t = b(t, X_t, \mu_t, \alpha_t)\, dt + \sigma(t, X_t, \mu_t, \alpha_t)\, dW_t \tag{368}$$

together with running cost f and terminal cost g, obviously the first step is to calculate NE strategy $\hat{\alpha}$ through HJBE. Set up the **McKean-Vlasov Hamiltonian** matching the term in HJBE

$$H(t, x, \mu, \alpha, y, z) = b(t, x, \mu, \alpha) \cdot y + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(t, x, \mu, \alpha) \cdot z) + f(t, x, \mu, \alpha) \tag{369}$$

the NE strategy $\hat{\alpha}(t, x, \mu, y, z)$ is given by

$$\hat{\alpha}(t, x, \mu, y, z) = \arg\max_{\alpha} H(t, x, \mu, \alpha, y, z) \tag{370}$$

plug back $\hat{\alpha}$ into the state dynamics and the running cost, so the game is no longer controlled and we return to the setting where the state dynamics has its coefficients only depending on t, x, μ, the same as what we have in the Feynman-Kac correspondence above.

Keep in mind the correspondence of HJBE that y stands for the state derivative of value function and z for the second-order state derivative of value function. Since the master field is constructed in the similar way to the value function, we expect y to stand for $\partial_x U$ and z to stand for $\partial_{xx} U$ in the current context.

Noticing the fact that MFG is a fixed point problem, we expect to see $\mu_t = \mathscr{L}(X_t)$ at equilibrium, i.e. a McKean-Vlasov state dynamics, which means that L **and** $\overline{L}$ **shall both be built upon the same dynamics in MFG equilibrium**! As a result, the backward PDE given by the Feynman-Kac correspondence shall characterize the MFG equilibrium. The following theorem formalizes the argument above and provides the master equation as a characterization of MFG equilibrium.

Theorem 10 (Master Equation as a Characterization of MFG Equilibrium)**.** *For a given MFG under some regularity condition, put up* ***McKean-Vlasov Hamiltonian***

$$H(t, x, \mu, \alpha, y, z) = b(t, x, \mu, \alpha) \cdot y + \frac{1}{2}Tr(\sigma\sigma^T(t, x, \mu, \alpha) \cdot z) + f(t, x, \mu, \alpha) \tag{371}$$

derive the NE strategy

$$\hat{\alpha}(t, x, \mu, y, z) = \arg\max_{\alpha} H(t, x, \mu, \alpha, y, z) \tag{372}$$

plug back into the state dynamics and the running cost to get an uncontrolled game with dynamics

$$dX_t = b(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, \partial_x U(t, X_t, \mu_t), \partial_{xx} U(t, X_t, \mu_t)))\, dt + \sigma(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, \partial_x U(t, X_t, \mu_t), \partial_{xx} U(t, X_t, \mu_t)))\, dW_t \tag{373}$$

and running cost

$$f(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, \partial_x U(t, X_t, \mu_t), \partial_{xx} U(t, X_t, \mu_t))) \tag{374}$$

based on which operators $L, \overline{L}$ *are calculated. The* ***master equation*** *is given by*

$$\begin{cases} \partial_t U(t,x,m) + LU(t,x,m) + \overline{L}U(t,x,m) + f(t,x,m) = 0 \\ U(T,x,m) = g(x,m) \end{cases} \tag{375}$$

where the f *is the master equation is the running cost with* $\hat{\alpha}$ *plugged in. The solution* $U(t,x,m)$ *gives the MFG equilibrium* $\hat{\mu}_t = U(t, X_t, \mathscr{L}(X_t))$.

Remark. *One of the regularity condition required is that* ***the game has unique MFG equilibrium starting from any*** (t,m) ***pair*** *(at any time* t *and at any population measure* m*). This condition can be implied by the Lasry-Lions monotonicity condition mentioned in the previous chapters.*

In the last section we mentioned that generally L *and* $\overline{L}$ *are constructed based on different dynamics. It's because of the fixed point problem structure of MFG that* L *and* $\overline{L}$ *matches at equilibrium.*

Example: Linear Quadratic Flocking Model

Let's calculate one example to show that master equation recovers the same MFG equilibrium derived above through the probabilistic approach. The representative player has state dynamics

$$dX_t = \alpha_t \, dt + \sigma \, dW_t \tag{376}$$

in three-dimensional space, i.e. $d = 3$, and cost functional

$$J(\alpha) = \mathbb{E} \int_0^T f(t, X_t, \mu_t, \alpha_t) \, dt \tag{377}$$

with no terminal cost, the running cost is

$$f(t, x, \mu, \alpha) = \frac{\kappa^2}{2} |x - \overline{\mu}|^2 + \frac{1}{2} |\alpha|^2 \tag{378}$$

where $\overline{\mu}$ denotes the mean of measure μ.

Set up the McKean-Vlasov Hamiltonian,

$$H(t, x, \mu, \alpha, y, z) = \alpha \cdot y + \frac{3\sigma^2}{2} + \frac{\kappa^2}{2} |x - \overline{\mu}|^2 + \frac{1}{2} |\alpha|^2 \tag{379}$$

minimize w.r.t. α

$$\hat{\alpha}(t, x, \mu, y, z) = -y \tag{380}$$

plug back $\hat{\alpha}(t, X_t, \mu_t, \partial_x U, \partial_{xx} U)$ to get the uncontrolled state dynamics

$$dX_t = -\partial_x U(t, X_t, \mu_t) \, dt + \sigma \, dW_t \tag{381}$$

and the uncontrolled running cost

$$f(t, x, m) = \frac{\kappa^2}{2} |x - \overline{m}|^2 + \frac{1}{2} |\partial_x U(t, x, m)|^2 \tag{382}$$

The operators $L, \overline{L}$ are given by

$$LU = -|\partial_x U|^2 + \frac{\sigma^2}{2} \Delta U \tag{383}$$

$$\overline{L}U = -\int_{\mathbb{R}^d} [D_m U(t, x)(m)(v) \cdot \partial_x U(t, v, m)] \, m(dv) + \frac{\sigma^2}{2} \int_{\mathbb{R}^d} \mathrm{Tr} \left(D_v \left[D_m U(t, x)(m) \right] (v) \right) \, m(dv) \tag{384}$$

write down the master equation

$$\partial_t U - |\partial_x U|^2 + \frac{\sigma^2}{2}\Delta U + \frac{\kappa^2}{2}|x-\overline{m}|^2 + \frac{1}{2}|\partial_x U|^2 \tag{385}$$

$$- \int_{\mathbb{R}^d} [D_m U(t,x)(m)(v)\cdot \partial_x U(t,v,m)]\, m(dv) + \frac{\sigma^2}{2}\int_{\mathbb{R}^d} \mathrm{Tr}\left(D_v\left[D_m U(t,x)(m)\right](v)\right) m(dv) = 0 \tag{386}$$

with terminal condition $U(T,x,m)=0$.

Put up ansatz

$$U(t,x,m) = F(t,x,\overline{m}) \tag{387}$$

based on the fact that in LQ game the measure makes a difference only through its mean. At this point, intrinsic derivatives of U can be calculated

$$D_m U(t,x)(m,v) = D_v\left[\frac{\delta U(t,x,\cdot)}{\delta m}(m,v)\right] = D_v\left[\partial_{\overline{m}} F\cdot \frac{\delta \overline{m}}{\delta m}(m,v)\right](t,x,\overline{m}) = D_v\left[\partial_{\overline{m}} F\cdot v\right](t,x,\overline{m}) = \partial_{\overline{m}} F(t,x,\overline{m}) \tag{388}$$

where we use the chain rule of linear functional derivative and the well-known fact that $U(m)=\overline{m}$ has $\frac{\delta U}{\delta m}(m,v)=v$. Since $D_m U(t,x)(m)$ does not contain v, the last term $D_v[D_m U(t,x)(m)]$ in the master equation is zero. Now the equation becomes

$$\partial_t F - \frac{1}{2}|\partial_x F|^2 + \frac{\sigma^2}{2}\Delta F + \frac{\kappa^2}{2}|x-\overline{m}|^2 - \int_{\mathbb{R}^d} [\partial_{\overline{m}} F(t,x,\overline{m})\cdot \partial_x F(t,v,\overline{m})]\, m(dv) = 0 \tag{389}$$

with terminal condition $F(T,x,\overline{m})=0$.

Put up another ansatz according to the fact that in LQ game the value function is quadratic and has mean-reversion term included

$$F(t,x,\overline{m}) = \frac{1}{2} f_t |x-\overline{m}|^2 + g_t \tag{390}$$

where f,g are deterministic. Now that

$$\partial_t F = \frac{1}{2}\dot{f}_t|x-\overline{m}|^2 + \dot{g}_t \tag{391}$$

$$\partial_x F = f_t(x-\overline{m}) \tag{392}$$

$$\Delta F = 3f_t \tag{393}$$

$$\partial_{\overline{m}} F = f_t(\overline{m}-x) \tag{394}$$

plug into the master equation, the integral on $\mathbb{R}^d$ vanishes and we are left with

$$\frac{1}{2}\dot{f}_t|x-\overline{m}|^2+\dot{g}_t-\frac{1}{2}f_t^2|x-\overline{m}|^2+\frac{\sigma^2}{2}3f_t+\frac{\kappa^2}{2}|x-\overline{m}|^2=0 \tag{395}$$

set coefficients equal to zero respectively to get the Ricatti equation

$$\begin{cases}\frac{1}{2}\dot{f}_t-\frac{1}{2}f_t^2+\frac{\kappa^2}{2}=0\\ f_T=0\\ \dot{g}_t+\frac{3\sigma^2}{2}f_t=0\\ g_T=0\end{cases} \tag{396}$$

so

$$f_t=\kappa\frac{e^{2\kappa(T-t)}-1}{e^{2\kappa(T-t)}+1} \tag{397}$$

solves out the the optimal control

$$\hat{\alpha}_t=-\partial_x U(t,X_t,\mathscr{L}(X_t))=-f_t(X_t-\overline{\mathscr{L}(X_t)})=-f_t(X_t-\mathbb{E}X_t) \tag{398}$$

which is the same MFG equilibrium as what we have solved from the probabilistic approach.

Mean Field Control (MFC) Problems

In this section, we introduce the setting and the way to solve men field control problems. Different from mean field games where the population measure is assumed to be not immediately affected by the strategy, mean field control has the population measure to be a function of the strategy. This leads to a completely different setting from mean field game since it's no longer a fixed point problem. However, it turns out that this setting can be interpreted as a cooperative game, working as a supplementary part of mean field game.

The introduction starts with going back to the calculus of variation on the space of measures, which is an essential technical tool. Different from the linear functional derivative approach introduced above, we set up the framework from the perspective of L-derivatives, which is introduced in Carmona and Delarue's book for mean field game.

L-derivative

The idea of L-derivative is simple. It's hard to perturb a probability measure, but there exists a random variable whose law is the probability measure. Therefore, one can lift the function of measure into a function of a random variable and perturb the random variable instead. In this sense, we can use the definition of Frechet derivative to calculate the derivative w.r.t. a random variable and try to come back to the derivative w.r.t. a measure component.

We start with considering the measure $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ to be in the 2-Wasserstein space and a square-integrable random variable $X \in L^2(\mathbb{R}^d)$ is said to correspond to μ if $\mathscr{L}(X) = \mu$. For any function $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, there exists a lifting $\tilde{u} : L^2(\mathbb{R}^d) \to \mathbb{R}$ such that $u(\mathscr{L}(X)) = \tilde{u}(X)$. This is the first step turning the variation in probability measures into the variation in random variables.

We say u is L-differentiable at $\mu_0 = \mathscr{L}(X_0)$ iff $\tilde{u}$ is Frechet differentiable at X_0, and denote the Frechet derivative as $D\tilde{u}(X_0)$. Clearly by the definition of Frechet derivative, $D\tilde{u}(X_0) : L^2(\mathbb{R}^d) \to \mathbb{R}$ maps a random variable to a real number, which indicates that $D\tilde{u}(X_0) \in [L^2(\mathbb{R}^d)]^* \cong L^2(\mathbb{R}^d)$. As a result, we denote $D\tilde{u}(X_0) = Y \in L^2(\mathbb{R}^d)$ such that it maps any random variable $Z \in L^2(\mathbb{R}^d)$ to the action of Y onto Z, i.e., $\langle Y, Z\rangle = \mathbb{E}(Y \cdot Z)$. At this point, we need to check that this definition of L-derivative is intrinsic, meaning that the law of $D\tilde{u}(X_0)$, as a random variable in $L^2(\mathbb{R}^d)$, does not depend on the choice of the random variable X_0 such that $\mathscr{L}(X_0) = \mu_0$. This technical proof is presented in the book and we do not provide details here.

After checking that the L-derivative is well-defined, we would like to simplify the structure of $D\tilde{u}(X_0)$ by noticing that $D\tilde{u}(X) \in \sigma(X)$, $\forall X \in L^2(\mathbb{R}^d)$. This implies the existence of $\xi_\mu : \mathbb{R}^d \to \mathbb{R}^d$ such that $D\tilde{u}(X) = \xi_\mu(X)$ $a.s.$, $\forall X \in L^2(\mathbb{R}^d)$, given μ such that $\mu = \mathscr{L}(X)$. We refer the readers to the book for the proof. It's worth noting that this simplifies $D\tilde{u}$ from a mapping $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ to a mapping $\mathbb{R}^d \to \mathbb{R}^d$, and that such ξ_μ is uniform in X (intrinsic) and only depends on the selection of μ.

As a result, it's natural to set $\partial_\mu u(\mu_0)(X_0) := D\tilde{u}(X_0) = \xi_{\mu_0}(X_0)$, where $\mu_0 = \mathscr{L}(X_0)$ and $\partial_\mu u(\mu_0)(X_0)$ is the derivative of u w.r.t. measure component μ, evaluated at μ_0, acting on X_0. By considering only the action of

$\partial_\mu u(\mu_0)$ instead of the image, the L-derivative is defined as:

$$\partial_\mu u(\mu_0) : \mathbb{R}^d \to \mathbb{R}^d \tag{399}$$

$$x \mapsto \xi_{\mu_0}(x). \tag{400}$$

Remark. *Deriving the L-derivative $\partial_\mu u(\mu_0)$ is typically separated into two steps: (i) write out the lifting $\tilde{u}$ and calculate its Frechet derivative $D\tilde{u}(X_0), \mu_0 = \mathscr{L}(X_0)$. (ii) Find out the action $\xi_{\mu_0} : \mathbb{R}^d \to \mathbb{R}^d$ such that $D\tilde{u}(X_0) = \xi_{\mu_0}(X_0)$, and write the L-derivative at μ_0 as a deterministic function $\partial_\mu u(\mu_0)(x) = \xi_{\mu_0}(x)$. For the purpose of clarity, I recommend always writing down both components $\mu_0,\ x$ to avoid mistakes in calculations.*

The Frechet derivative $D\tilde{u}(X_0) \in L^2(\mathbb{R}^d)$ provides the first-order expansion at $\mu_0 = \mathscr{L}(X_0)$ that

$$u(\mu) = u(\mu_0) + \mathbb{E}[D\tilde{u}(X_0) \cdot (X - X_0)] + o(\|X - X_0\|). \tag{401}$$

Writing in the notation of the L-derivative, the first order expansion at $\mu_0 = \mathscr{L}(X_0)$ provides

$$u(\mu) = u(\mu_0) + \mathbb{E}[\partial_\mu u(\mathscr{L}(X_0))(X_0) \cdot (X - X_0)] + o(\|X - X_0\|). \tag{402}$$

Example: Calculation of L-Derivative

Let's check some important examples for the calculation of the L-derivative. Those examples shall give exactly the same answer as calculated through the linear functional derivative approach, but in a more intuitive and natural way.

Consider $u(\mu) = \int h(x)\, d\mu(x) = \langle h, \mu \rangle$ with its lifting $\tilde{u}(X) = \mathbb{E}h(X),\ \mathscr{L}(X) = \mu$.

$$\tilde{u}(X + \varepsilon U) - \tilde{u}(X) = \mathbb{E}[h(X + \varepsilon U) - h(X)] = \mathbb{E}h'(X)\varepsilon U + o(\varepsilon \|U\|). \tag{403}$$

Matching this equation with the first-order expansion provides $D\tilde{u}(\mu)(X) = h'(X)$, so

$$\partial_\mu u(\mu)(x) = h'(x). \tag{404}$$

Remark. *Counterintuitively, u can be written as the action of h on μ but the L-derivative is not h but h'!*

Another example: for even function h, $u(\mu) = \langle h * \mu, \mu \rangle = \int\int h(x - y)\, d\mu(y)\, d\mu(x)$ with its lifting $\tilde{u}(X) = \int \mathbb{E}h(x - X)\, d\mu(x),\ \mathscr{L}(X) = \mu$. To stress the dependence of μ on X, we denote $\mu = \mathbb{P}_X$.

$$\tilde{u}(X + \varepsilon U) - \tilde{u}(X) = \int \mathbb{E}h(x - X - \varepsilon U)\, d\mathbb{P}_{X+\varepsilon U}(x) - \int \mathbb{E}h(x - X)\, d\mathbb{P}_X(x) \tag{405}$$

$$= \int \mathbb{E}[h(x - X - \varepsilon U) - h(x - X)]\, d\mathbb{P}_{X+\varepsilon U}(x) + \int \mathbb{E}h(x - X)\, d(\mathbb{P}_{X+\varepsilon U} - \mathbb{P}_X)(x). \tag{406}$$

The first integral has first-order term

$$-\varepsilon \int \mathbb{E}h'(x-X)U\, d\mathbb{P}_X(x), \tag{407}$$

while the second integral needs an application of Fubini

$$\int \mathbb{E}h(x-X)\, d\mathbb{P}_{X+\varepsilon U}(x) = \int\int h(x-y)\, d\mathbb{P}_{X+\varepsilon U}(x)\, d\mathbb{P}_X(y) = \int \mathbb{E}h(X+\varepsilon U-x)\, d\mathbb{P}_X(x) \tag{408}$$

so the second integral is

$$\int \mathbb{E}h(X+\varepsilon U-x)\, d\mathbb{P}_X(x) - \int \mathbb{E}h(x-X)\, d\mathbb{P}_X(x) = \int \mathbb{E}h(X+\varepsilon U-x)\, d\mathbb{P}_X(x) - \int \mathbb{E}h(X-x)\, d\mathbb{P}_X(x) \tag{409}$$

$$\approx \int \mathbb{E}h'(X-x)\varepsilon U\, d\mathbb{P}_X(x), \tag{410}$$

providing the first-order terms. Adding up to see that $\tilde{u}(X+\varepsilon U) - \tilde{u}(X)$ has first-order terms

$$\mathbb{E}\int h'(X-x)\varepsilon U\, d\mathbb{P}_X(x) - \mathbb{E}\int h'(x-X)\varepsilon U\, d\mathbb{P}_X(x) = \mathbb{E}\left[2\int h'(X-x)\, d\mathbb{P}_X(x)\cdot \varepsilon U\right], \tag{411}$$

since h' is an odd function. As a result, $D\tilde{u}(X) = 2\int h'(X-x)\, d\mathbb{P}_X(x)$, and

$$\partial_\mu u(\mu)(x) = 2(h' * \mu)(x). \tag{412}$$

Another example: $u(\mu) = \int v(x,\mu)\, d\mu(x)$ with its lifting $\tilde{u}(X) = \mathbb{E}_{X\sim\mu} v(X,\mu)$, $\mathscr{L}(X) = \mu$.

$$\tilde{u}(X+\varepsilon U) - \tilde{u}(X) = \mathbb{E}_{X\sim\mu}[v(X+\varepsilon U, \mathscr{L}(X+\varepsilon U)) - v(X, \mathscr{L}(X))] \tag{413}$$

$$= \mathbb{E}_{X\sim\mu}[v(X+\varepsilon U, \mathscr{L}(X+\varepsilon U)) - v(X, \mathscr{L}(X+\varepsilon U))] \tag{414}$$

$$+ \mathbb{E}_{X\sim\mu}[v(X, \mathscr{L}(X+\varepsilon U)) - v(X, \mathscr{L}(X))]. \tag{415}$$

The first expectation has first-order terms $\mathbb{E}_{X\sim\mu}[\partial_x v(X,\mu)\cdot \varepsilon U]$. The second expectation requires more work. Since there is a variation in the measure component, we apply the first-order characterization of $\partial_\mu v(x,\mu)$ to get

$$v(x, \mathscr{L}(X+\varepsilon U)) = v(x, \mathscr{L}(X)) + \mathbb{E}_{X\sim\mu,U}[\partial_\mu v(x, \mathscr{L}(X))(X)\cdot \varepsilon U] + o(\varepsilon\|U\|). \tag{416}$$

Since the equation holds for $\forall x \in \mathbb{R}^d$, it also holds if we change the variable x to a random variable X. Taking expectation on both sides gives

$$\mathbb{E}[v(X, \mathscr{L}(X+\varepsilon U)) - v(X, \mathscr{L}(X))] \approx \mathbb{E}_{Y\sim\mu,U}\mathbb{E}_{X\sim\mu}[\partial_\mu v(X, \mathscr{L}(X))(Y)\cdot \varepsilon U]. \tag{417}$$

At this point, it is clear that $\tilde{u}(X+\varepsilon U)-\tilde{u}(X)$ has first-order terms

$$\mathbb{E}_{X\sim\mu,U}\left[\partial_x v(X,\mu)\cdot\varepsilon U+\int\partial_\mu v(x',\mathscr{L}(X))(X)\,d\mu(x')\cdot\varepsilon U\right]. \tag{418}$$

As a result, $D\tilde{u}(X)=\partial_x v(X,\mu)+\int\partial_\mu v(x',\mu)(X)\,d\mu(x')$. We have shown that

$$\partial_\mu u(\mu)(x)=\partial_x v(x,\mu)+\int\partial_\mu v(x',\mu)(x)\,d\mu(x'). \tag{419}$$

As an extension, we set $v(x,\mu)=\int g(x,x')\,d\mu(x')$ such that $u(\mu)=\int\int g(x,x')\,d\mu(x')\,d\mu(x)=\langle g,\mu\times\mu\rangle$. Obviously,

$$\partial_x v(x,\mu)=\int\partial_x g(x,x')\,d\mu(x'). \tag{420}$$

The L-derivative of v w.r.t. μ can be easily calculated using previous conclusions by viewing x as fixed:

$$\partial_\mu v(x,\mu)(y)=\partial_{x'}g(x,y). \tag{421}$$

Combining both facts yields

$$\partial_\mu u(\mu)(x)=\int\partial_x g(x,y)\,d\mu(y)+\int\partial_{x'}g(y,x)\,d\mu(y). \tag{422}$$

This expression intuitively makes sense since it's symmetric in the two variables of g.

Example: L-Derivative of the Shannon Entropy

The shannon entropy for a probability measure μ (whose density function is assumed to exist) is defined as

$$H(\mu):=\mathbb{E}[-\log p_X(X)], \tag{423}$$

where $\mathscr{L}(X)=\mu$ and p_X denotes the density function of X, whose distribution is μ. Rewrite the Shannon entropy:

$$H(\mu)=-\int\log p_X(x)\,d\mu(x)=\int v(x,\mu)\,d\mu(x), \tag{424}$$

where $v(x,\mu):=-\log p_X(x)$ and the dependence of v on μ is through p_X. From previous conclusions, $\partial_\mu H(\mu)(x)=\partial_x v(x,\mu)+\int\partial_\mu v(x',\mu)(x)\,d\mu(x')$ and $\partial_x v(x,\mu)=-\nabla_x\log p_X(x)$, so it suffices to calculate $\partial_\mu v(x,\mu)$. Let's try to calculate this L-derivative again through the Gateaux derivative on fixing x.

$$v(x,\mathscr{L}(X+\varepsilon U))-v(x,\mathscr{L}(X))=-\log\frac{p_{X+\varepsilon U}(x)}{p_X(x)}=-\log\frac{\int p_X(y)p_{\varepsilon U}(x-y)\,dy}{p_X(x)}, \tag{425}$$

by assuming that U is independent of X. To facilitate calculations, we further assume $U \sim U(-1,1)$ WLOG so that $\varepsilon U \sim U(-\varepsilon,\varepsilon)$, implying $p_{\varepsilon U}(x-y) = \frac{1}{2\varepsilon}\mathbb{I}_{(-\varepsilon,\varepsilon)}(x-y)$. Since we only care about first-order terms, we apply Taylor expansion for $p_X(y)$ at $y=x$ to get $p_X(y) \approx p_X(x) + p_X'(x)(y-x) + \frac{p_X''(x)}{2}(y-x)^2$ (we maintain terms up to the second-order for a specific reason). Combining all those approximations yields

$$\int p_X(y)p_{\varepsilon U}(x-y)\,dy \approx \frac{1}{2\varepsilon}\left(2\varepsilon p_X(x) + \frac{1}{3}p_X''(x)\varepsilon^3\right). \tag{426}$$

As a result,

$$v(x, \mathscr{L}(X+\varepsilon U)) - v(x, \mathscr{L}(X)) = -\log\left(1 + \frac{p_X''(x)}{6p_X(x)}\varepsilon^2\right) \approx -\frac{p_X''(x)}{6p_X(x)}\varepsilon^2. \tag{427}$$

If we try to match it with $\mathbb{E}[D\tilde{v}(X)\cdot\varepsilon U]$, we see $D\tilde{v}(X) \equiv 0$ since the difference above contains a quadratic term in ε. As a result, it might seems a little bit counter-intuitive but we get

$$\partial_\mu v(x,\mu) \equiv 0. \tag{428}$$

Plugging back to the equation, we see that

$$\partial_\mu H(\mu)(x) = -\nabla_x \log p_X(x),\ \mathscr{L}(X) = \mu. \tag{429}$$

Remark. *We remark that the derivative w.r.t. the measure can also be calculated through the linear functional derivative. Consider the perturbation in p_X introduced in terms of another density q, the difference is*

$$-\int (p_X(x)+\varepsilon q(x))\log(p_X(x)+\varepsilon q(x))\,dx + \int p_X(x)\log p_X(x)\,dx \tag{430}$$

$$\approx -\varepsilon\int q(x)\log p_X(x)\,dx - \varepsilon\int p_X(x)\frac{q(x)}{p_X(x)}\,dx + o(\varepsilon), \tag{431}$$

since $\log\frac{p_X(x)+\varepsilon q(x)}{p_X(x)} \approx \varepsilon\frac{q(x)}{p_X(x)}$ provides the first-order terms. Writing the difference as $\left\langle \frac{\delta H}{\delta p_X}(\mu), \varepsilon q\right\rangle$ results in

$$\frac{\delta H}{\delta p_X}(\mu)(x) = -\log p_X(x) - 1. \tag{432}$$

Notice that taking a derivative w.r.t. x yields the intrinsic derivative $\partial_\mu H(\mu)(x) = \partial_x\left(\frac{\delta H}{\delta p_X}(\mu)\right)(x) = -\nabla_x \log p_X(x)$, which aligns with the L-derivative derived above.

Linear Functional Derivative and L-Derivative

In previous context, the intrinsic derivative is defined as the derivative w.r.t. x of the linear functional derivative $\frac{\delta u}{\delta \mu}(\mu)(x)$. However, L-derivative is by itself intrinsic, so we expect that the definitions provided by two different ways are actually the same, which is also implied in the remark above. By the definition of linear functional derivative,

$$\tilde{u}(X+\varepsilon U)-\tilde{u}(X)=u(\mathscr{L}(X+\varepsilon U))-u(\mathscr{L}(X)) \tag{433}$$

$$=\int_0^1\int \frac{\delta u}{\delta \mu}(\mu_t^\varepsilon)(x)\,d[\mathscr{L}(X+\varepsilon U)-\mathscr{L}(X)](x)\,dt, \tag{434}$$

where $\mu_t^\varepsilon := t\mathscr{L}(X+\varepsilon U)+(1-t)\mathscr{L}(X)$ provides a flow of interpolated probability measures. To connect with the representation in terms of $D\tilde{u}$, we wish to write this difference as $\mathbb{E}[D\tilde{u}(X)\cdot \varepsilon U]$. Hence:

$$\int_0^1\int \frac{\delta u}{\delta \mu}(\mu_t^\varepsilon)(x)\,d[\mathscr{L}(X+\varepsilon U)-\mathscr{L}(X)](x)\,dt \tag{435}$$

$$=\int_0^1 \mathbb{E}_{X,U}\left[\frac{\delta u}{\delta \mu}(\mu_t^\varepsilon)(X+\varepsilon U)-\frac{\delta u}{\delta \mu}(\mu_t^\varepsilon)(X)\right]dt \tag{436}$$

$$=\mathbb{E}_{X,U}\int_0^1 \left[\frac{\delta u}{\delta \mu}(\mu_t^\varepsilon)(X+\varepsilon U)-\frac{\delta u}{\delta \mu}(\mu_t^\varepsilon)(X)\right]dt \tag{437}$$

$$\approx \mathbb{E}_{X,U}\left(\int_0^1\left[\left(\partial_x\frac{\delta u}{\delta \mu}(\mu_t^\varepsilon)\right)(X)\right]dt\cdot \varepsilon U\right)\ \ (\varepsilon\to 0). \tag{438}$$

This provides an interesting relationship:

$$D\tilde{u}(X)=\lim_{\varepsilon\to 0}\int_0^1\left[\left(\partial_x\frac{\delta u}{\delta \mu}(\mu_t^\varepsilon)\right)(X)\right]dt, \tag{439}$$

which implies that $D\tilde{u}(X)=\partial_x\left[\frac{\delta u}{\delta \mu}(\mathscr{L}(X))\right](X)$. As a result,

$$\partial_\mu u(\mu)(x)=\partial_x\left[\frac{\delta u}{\delta \mu}(\mu)\right](x). \tag{440}$$

This proves that the L-derivative is the space derivative of the linear functional derivative! At this point, we finally understand why it's necessary to define the intrinsic derivative by differentiating w.r.t. x once more when introducing the linear functional derivative approach.

At this point, all the conclusions we established through the linear functional derivative approach still holds, including the Ito-Krylov's formula for a flow of measure and the Feynman-Kac correspondence for the master field. We refer the readers to the previous context for the details.

Remark. *The Lasry-Lions condition ensures the uniqueness of the equilibrium of a MFG. From the perspective of measure derivative, it actually means that the cost functionals f, g are L-convex in the measure component μ, and convexity always has a close connection with the uniqueness of the minimizer. Similarly, the propagation of chaos in the McKean-Vlasov SDE can be explained using Ito-Krylov's formula as an application (chapter 5.7).*

MFC Problem Setup

Consider a finite player game with N players and the interactions being mean field interactions. The state dynamics of player i is given by:

$$dX_t^i = b(t, X_t^i, \overline{\mu}_{X_t}^N, \alpha_t^i)\,dt + \sigma(t, X_t^i, \overline{\mu}_{X_t}^N, \alpha_t^i)\,dW_t^i, \tag{441}$$

where $\overline{\mu}_{X_t}^N$ denotes the empirical measure. Each player uses a public feedback function $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ acting on the private states, such that $\alpha_t^i = \phi(t, X_t^i)$. Each player aims to minimize the expected cost:

$$J^i(\phi) := \mathbb{E}\left[\int_0^T f(t, X_t^i, \overline{\mu}_{X_t}^N, \alpha_t^i)\,dt + g(X_T^i, \overline{\mu}_{X_T}^N)\right], \tag{442}$$

for some public running cost f and terminal cost g.

Similar to MFG, in MFC, we require a large number of identical players to conduct the mean field approximation to turn the problem into a game only for the representative player (single-agent). However, different from MFG where we first fix a flow of measure to solve the equilibrium strategy, in the setting of MFC, we first take a limit in the sense of the mean field regime. A propagation of chaos statement ensures that the empirical measure (random measure) has the weak limit as a deterministic measure as $N \to \infty$. As a result, the state dynamics, in the mean field regime, is a controlled McKean-Vlasov SDE,

$$dX_t = b(t, X_t, \mathscr{L}(X_t), \alpha_t)\,dt + \sigma(t, X_t, \mathscr{L}(X_t), \alpha_t)\,dW_t, \quad \alpha_t = \phi(t, X_t). \tag{443}$$

The representative player aims to minimize the expected cost in the mean field regime:

$$J(\phi) := \mathbb{E}\left[\int_0^T f(t, X_t, \mathscr{L}(X_t), \alpha_t)\,dt + g(X_T, \mathscr{L}(X_T))\right]. \tag{444}$$

MFG is a fixed point problem, it assumes that the population measure does not immediately change as each player changes her strategy, which resembles the competitive game (one first makes a decision then observe the change of the population since everybody competes with each other). On the other hand, MFC is an optimization problem, since the population measure is always exactly equal to the marginal law $\mathscr{L}(X_t)$. This means that the population measure changes immediately as the players' strategies change, which resembles the situation of a cooperative game (players work towards a same goal, if the strategy changes, everybody immediately knows how the population measure is going to change). It turns out that those two schemes generally give totally different equilibrium as an approximation to that of the finite player game with mean field interaction.

Nevertheless, it does not necessarily mean that MFG and MFC are completely different topics. In the probabilistic approach solving MFG introduced in previous context, we see how to solve a MFG by transforming it into a McKean-Vlasov FBSDE, which has close connections with MFC problems.

The Probabilistic Approach

Due to the McKean-Vlasov nature, we decide to view the measure $\mathscr{L}(X_t)$ also as one component of the state variable, i.e., formulating the state of the system at time t as $(X_t, \mathscr{L}(X_t))$. Adopting the idea of L-derivative lifting a function in measures to a function in random variables, the modification of the Hamiltonian naturally follows:

$$H(t,x,\tilde{X},y,z,\alpha) := b(t,x,\mu,\alpha)\cdot y + \sigma(t,x,\mu,\alpha)\cdot z + f(t,x,\mu,\alpha), \quad \mu = \mathscr{L}(X), \tag{445}$$

replacing the μ in H, the normal Hamiltonian for MFG with the lifting provided by $\tilde{X}$, an independent copy of X on a cloned probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$. The **MFC Hamiltonian** $\mathscr{H}$ is defined as:

$$\mathscr{H}(t,X,Y,Z,\beta) := \mathbb{E}H(t,X,\tilde{X},Y,Z,\beta), \tag{446}$$

where $X, Y, Z, \tilde{X}, \beta$ are random variables instead of the corresponding deterministic variables and the expectation is taken w.r.t. X, Y, Z, β. In Pontryagin maximum principle, we need the differential of $\mathscr{H}$ w.r.t. X. Notice that the variation in X results in the change of $\mathscr{H}$ in two ways: (i) through the variation in X component and the expectation; (ii) through the variation in $\mu = \mathscr{L}(X)$ component. Simple calculations using the first-order characterization of the L-derivative show that

$$\mathscr{H}(t, X+\varepsilon U, Y, Z, \beta) - \mathscr{H}(t,X,Y,Z,\beta) \tag{447}$$
$$= \mathbb{E}[H(t, X+\varepsilon U, \mathscr{L}(X+\varepsilon U), Y, Z, \beta) - H(t, X, \mathscr{L}(X), Y, Z, \beta)] \tag{448}$$
$$\approx \mathbb{E}[\partial_x H(t, X, \mathscr{L}(X), Y, Z, \beta)\cdot \varepsilon U] \tag{449}$$
$$+ \mathbb{E}[H(t, X, \mathscr{L}(X+\varepsilon U), Y, Z, \beta) - H(t, X, \mathscr{L}(X), Y, Z, \beta)], \tag{450}$$

the second expectation term containing the variation in the measure component seems hard to deal with. However, notice that

$$\mathbb{E}[H(t, X, \mathscr{L}(X+\varepsilon U), Y, Z, \beta) - H(t, X, \mathscr{L}(X), Y, Z, \beta)] \tag{451}$$
$$= \mathbb{E}_{Y,Z,\beta}\left[\int H(t, x, \mathscr{L}(X+\varepsilon U), Y, Z, \beta)\, d\mu(x) - \int H(t, x, \mathscr{L}(X), Y, Z, \beta)\, d\mu(x)\right], \quad \mu = \mathscr{L}(X) \tag{452}$$
$$\approx \mathbb{E}_{Y,Z,\beta}\left[\int \mathbb{E}_{X,U}\left(\partial_\mu H(t, x, \mathscr{L}(X), Y, Z, \beta)(X)\cdot\varepsilon U\right)\, d\mu(x)\right] \tag{453}$$
$$= \tilde{\mathbb{E}}\left[\mathbb{E}_{X,U}\left(\partial_\mu H(t, \tilde{X}, \mathscr{L}(X), \tilde{Y}, \tilde{Z}, \tilde{\beta})(X)\cdot\varepsilon U\right)\right] \tag{454}$$
$$= \mathbb{E}_{X,U}\left[\tilde{\mathbb{E}}\left(\partial_\mu H(t, \tilde{X}, \mathscr{L}(X), \tilde{Y}, \tilde{Z}, \tilde{\beta})(X)\right)\cdot\varepsilon U\right]. \tag{455}$$

To clarify, we have used Fubini's theorem and the fact that $H(t, x, \mathscr{L}(X+\varepsilon U), Y, Z, \beta) - H(t, x, \mathscr{L}(X), Y, Z, \beta) \approx \mathbb{E}_{X,U}\left[\partial_\mu H(t, x, \mathscr{L}(X), Y, Z, \beta)(X)\cdot\varepsilon U\right]$ in the first-order sense. Here $\tilde{Y}, \tilde{Z}, \tilde{\beta}$ are all random variables on the cloned probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$, they are independent of their original versions but have exactly the same law. The notation $\mathbb{E}$ denotes the expectation on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, while $\tilde{\mathbb{E}}$ denotes the expectation on the cloned probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$. To sum up, we have derived the first-order approximation

$$\mathscr{H}(t, X+\varepsilon U, Y, Z, \beta) - \mathscr{H}(t, X, Y, Z, \beta) \tag{456}$$

$$\approx \mathbb{E}\left[\left(\partial_x H(t, X, \mathscr{L}(X), Y, Z, \beta) + \tilde{\mathbb{E}}\left(\partial_\mu H(t, \tilde{X}, \mathscr{L}(X), \tilde{Y}, \tilde{Z}, \tilde{\beta})(X)\right)\right) \cdot \varepsilon U\right]. \tag{457}$$

This provides **the state derivative of the MFC Hamiltonian we will be using in the adjoint equation:**

$$D_X \mathscr{H}(t, X, Y, Z, \beta) = \partial_x H(t, X, \mathscr{L}(X), Y, Z, \beta) + \tilde{\mathbb{E}}\left(\partial_\mu H(t, \tilde{X}, \mathscr{L}(X), \tilde{Y}, \tilde{Z}, \tilde{\beta})(X)\right). \tag{458}$$

At this point, we define the adjoint processes and the adjoint equations, as a preparation for the pontryagin principle for MFC problems. The couple (Y, Z) is called the adjoint process of X if they satisfy the **MKV-BSDE (adjoint equations):**

$$\begin{cases} dY_t = -\left[\partial_x H(t, X_t, \mathscr{L}(X_t), Y_t, Z_t, \alpha_t) + \tilde{\mathbb{E}}\left(\partial_\mu H(t, \tilde{X}_t, \mathscr{L}(X_t), \tilde{Y}_t, \tilde{Z}_t, \tilde{\alpha}_t)(X_t)\right)\right] dt + Z_t\, dW_t \\ Y_T = \partial_x g(X_T, \mathscr{L}(X_T)) + \tilde{\mathbb{E}}\left(\partial_\mu g(\tilde{X}_T, \mathscr{L}(X_T))(X_T)\right) \end{cases}. \tag{459}$$

Remark. *As expected, the adjoint BSDE is very similar to the normal one in the stochastic control problems. Differently, $\tilde{\mathbb{E}}$ terms are added both to the BSDE and to the terminal condition. This is completely due to the McKean-Vlasov nature of the MFC problems. In MFG, the flow of measure is fixed so the L-derivatives are neglected, whereas in MFC, the variation in the measure needs to be taken into consideration.*

At this point, we provide the Pontryagin maximum principle for solving MFC problems without a proof. The proof uses the convexity of the Hamiltonian and the cost functionals, which is similar to the Pontryagin principle for stochastic control problems.

Theorem 11. *Let $\hat{\alpha}$ be the minimizer*

$$\hat{\alpha}(t, x, \mu, y, z) := \arg\inf_\alpha H(t, x, \mu, y, z, \alpha). \tag{460}$$

Let (X, Y, Z) be the solution to the following MKV-FBSDE:

$$\begin{cases} dX_t = & b(t, X_t, \mathscr{L}(X_t), \hat{\alpha}(t, X_t, \mathscr{L}(X_t), Y_t, Z_t))\, dt + \sigma(t, X_t, \mathscr{L}(X_t), \hat{\alpha}(t, X_t, \mathscr{L}(X_t), Y_t, Z_t))\, dW_t \\ dY_t = & -\partial_x H(t, X_t, \mathscr{L}(X_t), Y_t, Z_t, \hat{\alpha}(t, X_t, \mathscr{L}(X_t), Y_t, Z_t))\, dt \\ & -\tilde{\mathbb{E}}\left(\partial_\mu H(t, \tilde{X}_t, \mathscr{L}(X_t), \tilde{Y}_t, \tilde{Z}_t, \hat{\alpha}(t, \tilde{X}_t, \mathscr{L}(X_t), \tilde{Y}_t, \tilde{Z}_t))(X_t)\right) dt + Z_t\, dW_t \\ Y_T = & \partial_x g(X_T, \mathscr{L}(X_T)) + \tilde{\mathbb{E}}\left(\partial_\mu g(\tilde{X}_T, \mathscr{L}(X_T))(X_T)\right) \end{cases}. \tag{461}$$

Then under the condition that g is convex in (x, μ) and H is convex in (x, μ, α), the equilibrium strategy is given by $\hat{\alpha}(t, X_t, \mathscr{L}(X_t), Y_t, Z_t)$.

Example: Ayagari's Growth Model

Let's come back to the Ayagari's growth model, solving it as a MFC problem instead of a MFG.

We skip the interpretation of the model and directly formulate the game. Viewed as a finite player game, the state dynamics is given by

$$\begin{cases} dZ_t^i = (1 - Z_t^i)\, dt + \sigma\, dW_t^i \\ dA_t^i = [(1-\alpha)K_t^\alpha Z_t^i + (\alpha K_t^{\alpha-1} - \delta)A_t^i - c_t^i]\, dt \\ \mathbb{E}Z_0^i = 1 \end{cases}, \tag{462}$$

where $K_t = \frac{1}{N}\sum_{i=1}^N A_t^i$ and c_t^i is the strategy process. The player tries to minimize the negative expected utility

$$J^i(c) = -\mathbb{E}\left[\int_0^T U(c_t^i)\, dt + A_T^i\right]. \tag{463}$$

Firstly, let's conduct mean field approximation to turn this game into a MFC problem only for the representative player:

$$\begin{cases} dZ_t = (1 - Z_t)\, dt + \sigma\, dW_t \\ dA_t = [(1-\alpha)\overline{\mathscr{L}(A_t)}^\alpha Z_t + (\alpha \overline{\mathscr{L}(A_t)}^{\alpha-1} - \delta)A_t - c_t]\, dt \\ \mathbb{E}Z_0 = 1 \end{cases}. \tag{464}$$

Notice that we are not keeping the population measure through the notation μ_t, since we never fix the flow of measure in MFC problem. The state dynamics of the MFC problem is given by a controlled McKean-Vlasov SDE. The player tries to minimize:

$$J(c) = -\mathbb{E}\left[\int_0^T U(c_t)\, dt + A_T\right] \tag{465}$$

Firstly, write the state dynamics in the vector form

$$\begin{bmatrix} dZ_t \\ dA_t \end{bmatrix} = \begin{bmatrix} 1 - Z_t \\ (1-\alpha)\overline{\mathscr{L}(A_t)}^\alpha Z_t + (\alpha \overline{\mathscr{L}(A_t)}^{\alpha-1} - \delta)A_t - c_t \end{bmatrix} dt + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} dW_t. \tag{466}$$

The coefficients have the form

$$b(t,(z,a),\mu,c) = \begin{bmatrix} 1-z \\ (1-\alpha)\overline{\mu}^{\alpha} z + (\alpha\overline{\mu}^{\alpha-1} - \delta)a - c \end{bmatrix}, \tag{467}$$

$$\sigma(t,(z,a),\mu,c) = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}, \tag{468}$$

$$f(t,(z,a),\mu,c) = -U(c), \tag{469}$$

$$g((z,a),\mu) = -a. \tag{470}$$

Calculate the reduced Hamiltonian (diffusion coefficient is constant):

$$H(t,(z,a),\mu,(y_z,y_a),c) = y_z(1-z) + y_a[(1-\alpha)\overline{\mu}^{\alpha} z + (\alpha\overline{\mu}^{\alpha-1} - \delta)a - c] - U(c), \tag{471}$$

take derivative w.r.t. c to get the equilibrium strategy

$$\hat{c}(t,(z,a),\mu,(y_z,y_a)) = (-y_a)^{-\frac{1}{\gamma}}. \tag{472}$$

calculate the coefficient of the adjoint BSDE

$$\partial_z H = -y_z + y_a(1-\alpha)\overline{\mu}^{\alpha}, \quad \partial_a H = y_a(\alpha\overline{\mu}^{\alpha-1} - \delta), \tag{473}$$

$$\partial_z g = 0, \quad \partial_a g = -1, \quad \partial_\mu g = 0. \tag{474}$$

Keep in mind that for the MFC case we need to also calculate the L-derivative of the Hamiltonian:

$$\partial_\mu H(\mu)(x') = y_a(1-\alpha)z\alpha\overline{\mu}^{\alpha-1} + \alpha a y_a(\alpha-1)\overline{\mu}^{\alpha-2}, \tag{475}$$

since $\partial_\mu \overline{\mu} \equiv 1$. Write down the MKV-FBSDE:

$$\begin{cases} dZ_t = (1-Z_t)\,dt + \sigma\,dW_t \\ dA_t = [(1-\alpha)(\mathbb{E}A_t)^{\alpha} Z_t + [\alpha(\mathbb{E}A_t)^{\alpha-1} - \delta]A_t - (-Y_{a,t})^{-\frac{1}{\gamma}}]\,dt \\ \mathbb{E}Z_0 = 1 \\ dY_{a,t} = -Y_{a,t}[\alpha(\mathbb{E}A_t)^{\alpha-1} - \delta]\,dt - \alpha(1-\alpha)(\mathbb{E}A_t)^{\alpha-1}\mathbb{E}(Y_{a,t}Z_t)\,dt - \alpha(\alpha-1)(\mathbb{E}A_t)^{\alpha-2}\mathbb{E}(A_tY_{a,t})\,dt + Z_{a,t}\,dW_t \\ Y_{a,T} = -1 \end{cases}. \tag{476}$$

We don't present the BSDE w.r.t. $Y_{z,t}$ since it does not appear in the equilibrium strategy.

Remark. *As a comparison, we provide the MKV-FBSDE in the probabilistic approach solving for MFG equilibrium:*

$$\begin{cases} dA_t = [(1-\alpha)(\mathbb{E}A_t)^\alpha Z_t + (\alpha(\mathbb{E}A_t)^{\alpha-1} - \delta)A_t - (-Y_{a,t})^{-\frac{1}{\gamma}}]\,dt \\ dY_{a,t} = -Y_{a,t}(\alpha(\mathbb{E}A_t)^{\alpha-1} - \delta)\,dt + Z_{a,t}\,dW_t \\ Y_{a,T} = -1 \end{cases} . \tag{477}$$

The difference is in the driver of the adjoint BSDE, where the one for MFC has extra terms provided by $\tilde{\mathbb{E}}$ terms in the Pontryagin maximum principle.

Notice that the solution to the BSDE is deterministic since $Z_{a,t}$ does not appear elsewhere, $\mathbb{E}(A_tY_{a,t}) = Y_{a,t}\mathbb{E}A_t, \mathbb{E}(Y_{a,t}Z_t) = Y_{a,t}$ since $\mathbb{E}Z_t = 1$. This results in the MKV-FBSDE being:

$$\begin{cases} dA_t = [(1-\alpha)(\mathbb{E}A_t)^\alpha Z_t + [\alpha(\mathbb{E}A_t)^{\alpha-1} - \delta]A_t - (-Y_{a,t})^{-\frac{1}{\gamma}}]\,dt \\ dY_{a,t} = -Y_{a,t}[\alpha(\mathbb{E}A_t)^{\alpha-1} - \delta]\,dt + Z_{a,t}\,dW_t \\ Y_{a,T} = -1 \end{cases} , \tag{478}$$

which is the same as the MKV-FBSDE for MFG!

As a result, Aiyagari's growth model provides an example where the MFC and the MFG equilibria coincide! However, we emphasize that this is NOT the general case. As we can see, the MKV-FBSDE systems are actually different when first written down. It is with the help of the special structure of $Y_{a,t}$ being deterministic (which is not generally true) that the systems become identical.

Example: Linear-Quadratic MFG

Now we calculate the MFC equilibrium. On finite time horizon $[0,T]$, the state dynamics:

$$dX_t = \alpha_t\,dt + \sigma\,dW_t, \tag{479}$$

the running cost and terminal cost:

$$f(x,\alpha,\mu) = \frac{1}{2}\alpha^2 + c_1(x-\overline{\mu})^2 + c_2x^2 + c_3\overline{\mu}^2, \quad g(x,\mu) = c_4(x-\overline{\mu})^2. \tag{480}$$

Consider the reduced Hamiltonian:

$$H(t,x,\mu,y,z,\alpha) = \alpha y + \frac{1}{2}\alpha^2 + c_1(x-\overline{\mu})^2 + c_2x^2 + c_3\overline{\mu}^2. \tag{481}$$

Simple calculations show the equilibrium strategy as a function in $\{\mu_t\}$:

$$\hat{\alpha}(t,x,\mu,y,z) = -y. \tag{482}$$

Compute the coefficients in the adjoint BSDE:

$$\partial_x H = 2c_1(x-\overline{\mu}) + 2c_2x, \quad \partial_x g = 2c_4(x-\overline{\mu}). \tag{483}$$

Write down the FBSDE for a given flow of measure $\{\mu_t\}$:

$$\begin{cases} dX_t = -Y_t\,dt + \sigma\,dW_t \\ dY_t = -[2c_1(X_t-\overline{\mu}_t) + 2c_2X_t]\,dt + Z_t\,dW_t \\ Y_T = 2c_4(X_T-\overline{\mu}_T) \end{cases}. \tag{484}$$

The consistency condition requires $\mu_t = \mathscr{L}(X_t)$, which results in the MKV-FBSDE:

$$\begin{cases} dX_t = -Y_t\,dt + \sigma\,dW_t \\ dY_t = -[2c_1(X_t-\mathbb{E}X_t) + 2c_2X_t]\,dt + Z_t\,dW_t \\ Y_T = 2c_4(X_T-\mathbb{E}X_T) \end{cases}. \tag{485}$$

Example: Linear-Quadratic MFC

On finite time horizon $[0, T]$, the state dynamics:

$$dX_t = \alpha_t \, dt + \sigma \, dW_t, \quad \mathbb{E}X_0 = 0, \tag{486}$$

the running cost and terminal cost:

$$f(x, \alpha, \mu) = \frac{1}{2}\alpha^2 + c_1(x - \overline{\mu})^2 + c_2 x^2 + c_3 \overline{\mu}^2, \quad g(x, \mu) = c_4(x - \overline{\mu})^2. \tag{487}$$

We first solve it as a MFG through the probabilistic approach in this section. Fix the flow of measure $\{\mu_t\}$ and consider the reduced Hamiltonian:

$$H(t, x, \mu, y, z, \alpha) = \alpha y + \frac{1}{2}\alpha^2 + c_1(x - \overline{\mu})^2 + c_2 x^2 + c_3 \overline{\mu}^2. \tag{488}$$

Simple calculations show the equilibrium strategy as a function in $\{\mu_t\}$:

$$\hat{\alpha}(t, x, \mu, y, z) = -y. \tag{489}$$

Compute the coefficients in the adjoint BSDE:

$$\partial_x H = 2c_1(x - \overline{\mu}) + 2c_2 x, \quad \partial_x g = 2c_4(x - \overline{\mu}). \tag{490}$$

In the MFC regime, L-derivatives are needed:

$$\partial_\mu H(\mu)(x') = 2c_1(\overline{\mu} - x) + 2c_3 \overline{\mu}, \quad \partial_\mu g(\mu)(x') = 2c_4(\overline{\mu} - x), \tag{491}$$

since $(\partial_\mu \overline{\mu})(\mu)(x') \equiv 1$. Write down the MKV-FBSDE:

$$\begin{cases} dX_t = -Y_t \, dt + \sigma \, dW_t \\ dY_t = -[2c_1(X_t - \mathbb{E}X_t) + 2c_2 X_t] \, dt - 2c_3 \mathbb{E}X_t \, dt + Z_t \, dW_t \\ Y_T = 2c_4(X_T - \mathbb{E}X_T) \end{cases} . \tag{492}$$

This is because

$$\tilde{\mathbb{E}}[2c_1(\overline{\mathscr{L}(X_t)} - \tilde{X}_t) + 2c_3 \mathbb{E}X_t] = 2c_3 \mathbb{E}X_t, \quad \tilde{\mathbb{E}}[2c_4(\overline{\mathscr{L}(X_T)} - \tilde{X}_T)] = 0. \tag{493}$$

Hence the MKV-FBSDE in the MFC regime:

$$\begin{cases} dX_t = -Y_t\,dt + \sigma\,dW_t \\ dY_t = -[2(c_1+c_2)X_t - 2(c_1-c_3)\mathbb{E}X_t]\,dt + Z_t\,dW_t \\ Y_T = 2c_4(X_T - \mathbb{E}X_T) \end{cases} . \tag{494}$$

Remark. *To see the clear difference, the MKV-FBSDE in the MFG regime:*

$$\begin{cases} dX_t = -Y_t\,dt + \sigma\,dW_t \\ dY_t = -[2(c_1+c_2)X_t - 2c_1\mathbb{E}X_t]\,dt + Z_t\,dW_t \\ Y_T = 2c_4(X_T - \mathbb{E}X_T) \end{cases} . \tag{495}$$

When $c_3 \neq 0$ (penalty coefficient for large population mean) and $\mathbb{E}X_t \neq 0$, there is a significant difference between MFG and MFC equilibrium. On the other hand, the values of c_1, c_2, c_4 (penalty coefficient for deviation from the mean and for a large state) does not cause the difference.

Example: Reinforcement Learning with Mean Field Interaction

We consider a simple RL problem with state space $\mathscr{S} = \{0, 1\}$, action space $\mathscr{A} = \{s := \text{'stay'}, m := \text{'move'}\}$. Whenever one chooses to stay, state s transits to itself with probability $1-p$ and transits to the other state with probability $p << 1$. Whenever one chooses to move, state s transits to itself with probability p and transits to the other state with probability $1-p$. Here p denotes the level of uncertainty contained in the transition kernel and is assumed to be close to zero. The reward function $r(s, \mu) = \varepsilon s - \overline{\mu}$, i.e., at time n one gets reward $r(S_n, \mu)$, solely based on the current state and the limiting distribution of the Markov chain $\{S_n\}$, where $0 < \varepsilon << 1$. Notice that here we adopt the asymptotic formulation of mean field interactions, i.e. instead of maintaining a flow of measure, we maintain a single measure μ as the limiting distribution of the state process $\{S_n\}_\pi$, given the policy π. This MDP model is a time-discretized game with discrete state, action spaces, and the mean field interaction only appears in the reward function but not the transition kernel.

We want to show that in MFG and MFC regimes, the optimal policy are actually the opposite. Let's first check the MFG case where the limiting measure μ is fixed. In this case, we can ignore the $\overline{\mu}$ in the reward function since it takes the same value regardless of the action we are taking. This problem now becomes a normal MDP, which we know has a deterministic optimal policy π_* such that

$$\pi_* = \arg\max_\pi \mathbb{E}_\pi \left[\sum_{n=0}^\infty \gamma^n r(S_n, \mu)\right]. \tag{496}$$

When p is small enough and γ is close enough to 1, we have enough reason to believe that the optimal policy is given by $\pi_*(0) = m, \pi_*(1) = s$. This is because the reward now solely depends on the state and state 1 results in a higher reward. (One can calculate the q^* function using Bellman optimality equation to see why it holds). Such π_* is the optimal policy on fixing the measure μ. Now we match the limiting distribution of $\{S_n\}_{\pi_*}$ and the measure μ to meet the consistency condition. Clearly, the Markov chain $\{S_n\}_{\pi_*}$ has transition matrix $\begin{bmatrix} p & 1-p \\ p & 1-p \end{bmatrix}$, which results in the limiting distribution $\mu(\{0\}) = p, \mu(\{1\}) = 1-p$. This concludes the calculation for the MFG equilibrium.

On the other hand, in the case of MFC, the limiting distribution μ changes immediately as the policy π changes. When ε is close enough to zero, $-\overline{\mu}$ dominates the reward function, so the player would like to minimize $\overline{\mu}$ in order to look for a higher reward. As a result, the optimal policy is given by $\pi_*(0) = s, \pi_*(1) = m$, which results in the equilibrium limiting measure $\mu(\{0\}) = 1-p, \mu(\{1\}) = p$.

Clearly, two limiting regimes provide totally opposite answers. The delay in the response of the population measure w.r.t. the change in the policy causes the equilibrium measure to concentration on state 1 in MFG, even if a large mean of the measure results in a low reward (c.f. prisoner's dilemma). The immediate response of the population measure w.r.t. the change in the policy causes the equilibrium measure to concentration on state 0 in MFC. The players are able to understand the negative effect of a large mean of the measure and successfully achieves the Pareto optimality. Through this simple example, we can better understand why MFG and MFC stand for competitive and cooperative games respectively.

Analytic Approach for MFC

The analytic approach turns the McKean-Vlasov optimal control problem into a normal optimal control problem where the state variable takes values as measures. In the setting of feedback strategies represented by the feedback function ϕ, denote $\mu_t := \mathscr{L}\left(X_t^\phi\right)$ so that the Fokker-Planck equation holds:

$$\partial_t \mu_t(t,x) + \mathrm{div}_x(b(t,x,\mu_t,\phi(t,x)) \cdot \mu_t) - \frac{1}{2}\mathrm{Tr}(\partial_{xx}[\sigma\sigma^T(t,x,\mu_t,\phi(t,x)) \cdot \mu_t]) = 0. \tag{497}$$

The objective is to figure out the optimal ϕ to minimize

$$\mathbb{E}\left[\int_0^T f(t,X_t,\mu_t,\phi(t,X_t))\,dt + g(X_T,\mu_T)\right] = \int_0^T \langle f(t,\cdot,\mu_t,\phi(t,\cdot)),\mu_t\rangle\,dt + \langle g(\cdot,\mu_T),\mu_T\rangle\,. \tag{498}$$

The problem becomes an optimal control problem with no randomness if we recognize the Fokker-Planck equation as the state dynamics with the state variable taking values as measures instead of vectors. At the cost of having the state variable taking values as infinite dimensional objects, the problem is reduced to a deterministic control problem.

We introduce the dual variable $u : \mathbb{R}^d \to \mathbb{R}$ of the measure $\mu_t \in \mathscr{P}(\mathbb{R}^d)$, which enables us to write down the Hamiltonian

$$\mathscr{H}(t,\mu,u,\beta) := \left\langle u, -\mathrm{div}_x(b(t,\cdot,\mu,\beta(\cdot)) \cdot \mu) + \frac{1}{2}\mathrm{Tr}(\partial_{xx}[\sigma\sigma^T(t,\cdot,\mu,\beta(\cdot)) \cdot \mu])\right\rangle + \langle f(t,\cdot,\mu,\beta(\cdot)),\mu\rangle\,, \tag{499}$$

where β stands for the feedback function at time t. By integration by parts,

$$\mathscr{H}(t,\mu,u,\beta) = \left\langle b(t,\cdot,\mu,\beta(\cdot)) \cdot \partial_x u(\cdot) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T(t,\cdot,\mu,\beta(\cdot)) \cdot \partial_{xx}u(\cdot)],\mu\right\rangle + \langle f(t,\cdot,\mu,\beta(\cdot)),\mu\rangle\,. \tag{500}$$

An analogue of the normal Pontryagin maximum principle tells us to first minimize $\mathscr{H}$ w.r.t. β by setting $\frac{\delta\mathscr{H}}{\delta\beta} = 0$. Since this functional derivative depends on the specific setting of the problem, we denote

$$\mathscr{H}^*(t,\mu,u) := \inf_\beta \mathscr{H}(t,\mu,u,\beta), \tag{501}$$

as the Hamiltonian with the optimal feedback function $\beta^* = \beta^*(t,\mu,u)$ plugged in (if such a minimizer exists).

Afterwards, one need the state derivative of the Hamiltonian to be appearing in the adjoint equation. As a result, we calculate the first variation at measure μ:

$$\delta\mathscr{H}(\mu)(\nu) = \langle b(t,\cdot,\mu,\beta(\cdot))\cdot\partial_x u(\cdot),\nu\rangle + \int\left\langle \frac{\delta b}{\delta\mu}(t,x,\mu,\beta(x))(\cdot),\nu\right\rangle \partial_x u(x)\,d\mu(x) \tag{502}$$

$$+\left\langle \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T(t,\cdot,\mu,\beta(\cdot))\cdot\partial_{xx}u(\cdot)],\nu\right\rangle + \frac{1}{2}\mathrm{Tr}\int\left\langle \frac{\delta(\sigma\sigma^T)}{\delta\mu}(t,x,\mu,\beta(x))(\cdot),\nu\right\rangle \partial_{xx}u(x)\,d\mu(x) \tag{503}$$

$$+\langle f(t,\cdot,\mu,\beta(\cdot)),\nu\rangle + \int\left\langle \frac{\delta f}{\delta\mu}(t,x,\mu,\beta(x))(\cdot),\nu\right\rangle d\mu(x). \tag{504}$$

This implies the following functional derivative

$$\frac{\delta\mathscr{H}}{\delta\mu}(\mu)(z) = b(t,z,\mu,\beta(z))\cdot\partial_x u(z) + \int\frac{\delta b}{\delta\mu}(t,x,\mu,\beta(x))(z)\cdot\partial_x u(x)\,d\mu(x) \tag{505}$$

$$+\frac{1}{2}\mathrm{Tr}[\sigma\sigma^T(t,z,\mu,\beta(z))\cdot\partial_{xx}u(z)] + \frac{1}{2}\mathrm{Tr}\int\frac{\delta(\sigma\sigma^T)}{\delta\mu}(t,x,\mu,\beta(x))(z)\cdot\partial_{xx}u(x)\,d\mu(x) \tag{506}$$

$$+f(t,z,\mu,\beta(z)) + \int\frac{\delta f}{\delta\mu}(t,x,\mu,\beta(x))(z)\,d\mu(x). \tag{507}$$

Here z is the variable of the function $\frac{\delta\mathscr{H}}{\delta\mu}(\mu)$. At this point, we write down the adjoint equation, with $u = u(t,x)$ (one $u(x)$ for each time t) that

$$\partial_t u(t,x) = -\frac{\delta\mathscr{H}}{\delta\mu}(\mu,\beta^*)(t,x). \tag{508}$$

The terminal condition of the adjoint equation is

$$u(T,x) = \frac{\delta\langle g(\cdot,\mu),\mu\rangle}{\delta\mu}(\mu_T)(x) = g(x,\mu_T) + \int\frac{\delta g}{\delta\mu}(y,\mu_T)(x)\,d\mu_T(y). \tag{509}$$

At this point, the PDE for u and the PDE for μ together characterizes the solution of the MFC problem. We rewrite those two PDEs and merge them:

$$\begin{cases} 0 = \partial_t u + \mathscr{H}^*(t,\mu_t,u) + \int\frac{\delta\mathscr{H}}{\delta\mu}(t,z,\mu_t,\beta^*(t,\mu_t,u))(x)\cdot\mu(t,z)\,dz \\ u(T,x) = g(x,\mu_T) + \int\frac{\delta g}{\delta\mu}(y,\mu_T)(x)\cdot\mu(T,y)\,dy \\ 0 = \partial_t\mu_t + \mathrm{div}_x(b(t,x,\mu_t,\beta^*(t,\mu_t,u))\cdot\mu_t) - \frac{1}{2}\mathrm{Tr}(\partial_{xx}[\sigma\sigma^T(t,x,\mu_t,\beta^*(t,\mu_t,u))\cdot\mu_t]) \\ \mu_0 = \rho \end{cases}. \tag{510}$$

which concludes the analytic approach to MFC. It consists of one forward and one backward PDE. Notice that the z is an integration variable, which matches with the $\tilde{\mathbb{E}}$ term in the probabilistic approach of MFC and is interpreted as the change caused by the measure component. We remark that the PDE for u seems like an HJB equation, but it's not exactly an HJB equation due to the presence of the functional derivative. In the MFC setting, the PDE for u is essentially an adjoint equation and the master equation is essentially the HJB equation.

Infinite-horizon MFG and MFC

In the context of infinite time horizon, we restrict ourselves to the special case, in which the coefficients and cost functionals are time-homogeneous, i.e. $b(t,x,\mu,\alpha) = b(x,\mu,\alpha)$, $f(t,x,\mu,\alpha) = f(x,\mu,\alpha)$. For a stochastic control problem under this setting, the control and the value function turn out to be time-homogeneous, i.e., only showing dependence on the state variable x. In the case of MFG and MFC, however, those restrictions do not suffice to ensure the time-homogeneity of the equilibrium strategy. The problem lies in the time-dependence of the flow of measure $\{\mu_t\}$, as will be shown in the examples below.

Remark. *If one adopts the asymptotic formulation of MFG and MFC, i.e., the coefficients and cost functionals only depend on a single measure μ, which is the limiting distribution of the state process, instead of a flow of measure, then the equilibrium strategy turns out to be time-homogeneous.*

In the first example, we have the state dynamics:

$$dX_t = \alpha_t \, dt + \sigma \, dW_t, \tag{511}$$

the running cost:

$$f(x,\alpha,\mu) = \frac{1}{2}\alpha^2 + c_1(x-\overline{\mu})^2, \tag{512}$$

and one wishes to minimize the expected cost:

$$J(\alpha) := \mathbb{E}\left[\int_0^\infty e^{-rt} f(X_t,\alpha_t,\mu_t)\, dt\right]. \tag{513}$$

We solve this problem by first solving the finite-horizon version with a terminal cost added:

$$g(x,\mu) = c_4(x-\overline{\mu})^2, \tag{514}$$

and identifying a new running cost

$$h(t,x,\alpha,\mu) = e^{-rt} f(x,\alpha,\mu). \tag{515}$$

In this case, one wishes to minimize the expected cost:

$$J(\alpha) := \mathbb{E}\left[\int_0^T h(t,X_t,\alpha_t,\mu_t)\, dt + g(X_T,\mu_T)\right]. \tag{516}$$

After solving for the equilibrium strategy, we set $c_4 \to 0, T \to \infty$ to approximate the equilibrium strategy of the infinite-horizon problem. Notice that all dependencies on the measure component μ are through $(x-\overline{\mu})^2$, hence MFC and MFG equilibrium coincide.

Example 1 (MFG)**.** *Consider the reduced Hamiltonian:*

$$H(t,x,\mu,y,z,\alpha) = \alpha y + e^{-rt}\left[\frac{1}{2}\alpha^2 + c_1(x-\overline{\mu})^2\right]. \tag{517}$$

Simple calculations show the equilibrium strategy as a function in $\{\mu_t\}$:

$$\hat{\alpha}(t,x,\mu,y,z) = -e^{rt}y. \tag{518}$$

Compute the coefficients in the adjoint BSDE:

$$\partial_x H = 2c_1 e^{-rt}(x-\overline{\mu}), \quad \partial_x g = 2c_4(x-\overline{\mu}). \tag{519}$$

Write down the FBSDE for a given flow of measure $\{\mu_t\}$:

$$\begin{cases} dX_t = -e^{rt}Y_t\,dt + \sigma\,dW_t \\ dY_t = -2c_1 e^{-rt}(X_t - \overline{\mu}_t)\,dt + Z_t\,dW_t \\ Y_T = 2c_4(X_T - \overline{\mu}_T) \end{cases}. \tag{520}$$

The consistency condition requires $\mu_t = \mathscr{L}(X_t)$, which results in the MKV-FBSDE:

$$\begin{cases} dX_t = -e^{rt}Y_t\,dt + \sigma\,dW_t \\ dY_t = -2c_1 e^{-rt}(X_t - \mathbb{E}X_t)\,dt + Z_t\,dW_t \\ Y_T = 2c_4(X_T - \mathbb{E}X_T) \end{cases}. \tag{521}$$

Taking expectation on both sides provides

$$\mathbb{E}Y_t \equiv 0, \quad \mathbb{E}X_t \equiv \mathbb{E}X_0. \tag{522}$$

Using affine ansatz $Y_t = \eta_t X_t + \mu_t$ yields

$$\begin{cases} \dot{\eta}_t - e^{rt}\eta_t^2 + 2c_1 e^{-rt} = 0, \quad \eta_T = 2c_4 \\ \dot{\mu}_t - e^{rt}\eta_t\mu_t - 2c_1 e^{-rt}\mathbb{E}X_0 = 0, \quad \mu_T = -2c_4\mathbb{E}X_0 \end{cases}. \tag{523}$$

Setting $c_4 \to 0, T \to \infty$ gives

$$\eta_t \to \eta_t^*, \quad \mu_t \to \mu_t^*, \tag{524}$$

where

$$\dot{\eta}_t^* - e^{rt}(\eta_t^*)^2 + 2c_1 e^{-rt} = 0, \quad \eta_\infty = 0, \tag{525}$$

$$\dot{\mu}_t^* - e^{rt}\eta_t^*\mu_t^* - 2c_1 e^{-rt}\mathbb{E}X_0 = 0, \quad \mu_\infty = 0. \tag{526}$$

Solve the ODE with ansatz $\eta_t^* = e^{-rt}\xi_t$ *to get*

$$\xi_t = \frac{-r + \sqrt{r^2 + 8c_1}}{2}, \tag{527}$$

$$\eta_t^* = e^{-rt}\frac{-r + \sqrt{r^2 + 8c_1}}{2}. \tag{528}$$

Plug η^* *into the ODE of* μ^* *to yield*

$$\mu_t^* = -\frac{2c_1\mathbb{E}X_0}{r + \frac{-r+\sqrt{r^2+8c_1}}{2}} e^{-rt}. \tag{529}$$

As a result, the equilibrium strategy is given by

$$\hat{\alpha}(t,x) = -e^{rt}(\eta_t^* x + \mu_t^*) = -\frac{-r + \sqrt{r^2 + 8c_1}}{2}x + \frac{2c_1\mathbb{E}X_0}{r + \frac{-r+\sqrt{r^2+8c_1}}{2}}. \tag{530}$$

Remark. *This example provides an equilibrium strategy that is time-homogeneous. Although* μ_t *changes as* t *changes, its mean is constant is time while its variance is time-dependent. However, the mean field interaction is only through the mean of the measure, which results in the time-homogeneous equilibrium strategy.*

As another example, we have the state dynamics:

$$dX_t = (\alpha_t - \overline{\mu}_t)\,dt + \sigma\,dW_t, \tag{531}$$

the running cost:

$$f(x, \alpha, \mu) = \frac{1}{2}\alpha^2 + c_1(x - \overline{\mu})^2, \tag{532}$$

and one wishes to minimize the expected cost:

$$J(\alpha) := \mathbb{E}\left[\int_0^\infty e^{-rt} f(X_t, \alpha_t, \mu_t)\,dt\right]. \tag{533}$$

We solve this problem by first solving the finite-horizon version with a terminal cost added:

$$g(x, \mu) = c_4(x - \overline{\mu})^2, \tag{534}$$

and identifying a new running cost

$$h(t, x, \alpha, \mu) = e^{-rt} f(x, \alpha, \mu). \tag{535}$$

In this case, one wishes to minimize the expected cost:

$$J(\alpha) := \mathbb{E}\left[\int_0^T h(t, X_t, \alpha_t, \mu_t)\,dt + g(X_T, \mu_T)\right]. \tag{536}$$

After solving for the equilibrium strategy, we set $c_4 \to 0, T \to \infty$ to approximate the equilibrium strategy of the infinite-horizon problem.

Example 2 (MFG)**.** *Consider the reduced Hamiltonian:*

$$H(t, x, \mu, y, z, \alpha) = (\alpha - \overline{\mu})y + e^{-rt}\left[\frac{1}{2}\alpha^2 + c_1(x - \overline{\mu})^2\right]. \tag{537}$$

Simple calculations show the equilibrium strategy as a function in $\{\mu_t\}$:

$$\hat{\alpha}(t, x, \mu, y, z) = -e^{rt} y. \tag{538}$$

Compute the coefficients in the adjoint BSDE:

$$\partial_x H = 2c_1 e^{-rt}(x - \overline{\mu}), \quad \partial_x g = 2c_4(x - \overline{\mu}). \tag{539}$$

Write down the FBSDE for a given flow of measure $\{\mu_t\}$:

$$\begin{cases} dX_t = (-\overline{\mu}_t - e^{rt} Y_t)\,dt + \sigma\,dW_t \\ dY_t = -2c_1 e^{-rt}(X_t - \overline{\mu}_t)\,dt + Z_t\,dW_t \\ Y_T = 2c_4(X_T - \overline{\mu}_T) \end{cases}. \tag{540}$$

The consistency condition requires $\mu_t = \mathscr{L}(X_t)$, which results in the MKV-FBSDE:

$$\begin{cases} dX_t = (-\mathbb{E}X_t - e^{rt} Y_t)\,dt + \sigma\,dW_t \\ dY_t = -2c_1 e^{-rt}(X_t - \mathbb{E}X_t)\,dt + Z_t\,dW_t \\ Y_T = 2c_4(X_T - \mathbb{E}X_T) \end{cases}. \tag{541}$$

Taking expectation on both sides provides

$$\mathbb{E}Y_t \equiv 0, \quad \mathbb{E}X_t = e^{-t}\mathbb{E}X_0. \tag{542}$$

Using affine ansatz $Y_t = \eta_t X_t + \mu_t$ yields

$$\begin{cases} \dot{\eta}_t - e^{rt}\eta_t^2 + 2c_1 e^{-rt} = 0, \quad \eta_T = 2c_4 \\ \dot{\mu}_t - e^{rt}\eta_t\mu_t - 2c_1 e^{-rt}e^{-t}\mathbb{E}X_0 - \eta_t e^{-t}\mathbb{E}X_0 = 0, \quad \mu_T = -2c_4 e^{-T}\mathbb{E}X_0 \end{cases}. \tag{543}$$

Setting $c_4 \to 0, T \to \infty$ gives

$$\eta_t \to \eta_t^*, \quad \mu_t \to \mu_t^*, \tag{544}$$

where

$$\dot{\eta}_t^* - e^{rt}(\eta_t^*)^2 + 2c_1 e^{-rt} = 0, \quad \eta_\infty = 0, \tag{545}$$

$$\dot{\mu}_t^* - e^{rt}\eta_t^*\mu_t^* - 2c_1 e^{-rt}e^{-t}\mathbb{E}X_0 - \eta_t^* e^{-t}\mathbb{E}X_0 = 0, \quad \mu_\infty = 0. \tag{546}$$

Solve the ODE with ansatz $\eta_t^* = e^{-rt}\xi_t$ to get

$$\xi_t = \frac{-r + \sqrt{r^2 + 8c_1}}{2}, \tag{547}$$

$$\eta_t^* = e^{-rt}\frac{-r + \sqrt{r^2 + 8c_1}}{2}. \tag{548}$$

Plug η^* into the ODE of μ^* to yield

$$\mu_t^* = -\frac{(2c_1 + \frac{-r+\sqrt{r^2+8c_1}}{2})\mathbb{E}X_0}{r + 1 + \frac{-r+\sqrt{r^2+8c_1}}{2}} e^{-(r+1)t}. \tag{549}$$

As a result, the equilibrium strategy is given by

$$\hat{\alpha}(t, x) = -e^{rt}(\eta_t^* x + \mu_t^*) = -\frac{-r + \sqrt{r^2 + 8c_1}}{2} x + \frac{(2c_1 + \frac{-r+\sqrt{r^2+8c_1}}{2})\mathbb{E}X_0}{r + 1 + \frac{-r+\sqrt{r^2+8c_1}}{2}} e^{-t}. \tag{550}$$

Remark. *This is the general case where the equilibrium strategy is time-inhomogeneous. As a remark,*

$$\partial_\mu H(\mu)(x') = 2c_1(\overline{\mu} - x) - y, \tag{551}$$

so that $\tilde{\mathbb{E}}\left(\partial_\mu H(t, \tilde{X}_t, \mathscr{L}(X_t), \tilde{Y}_t, \tilde{Z}_t, \hat{\alpha}(t, \tilde{X}_t, \mathscr{L}(X_t), \tilde{Y}_t, \tilde{Z}_t))(X_t)\right) = 0$ since $\mathbb{E}Y_t \equiv 0$. This implies that the MFG and MFC equilibrium still coincide.