

12.1.5. If (Y_i, \mathcal{G}_i) be \mathbb{L}^2 -MG_i, show that

$\forall i \leq j \leq k$, $\mathbb{E}[(Y_k - Y_j) Y_i] = 0$ and

$$\mathbb{E}((Y_k - Y_j)^2 | \mathcal{G}_i) = \mathbb{E}(Y_k^2 | \mathcal{G}_i) - \mathbb{E}(Y_j^2 | \mathcal{G}_i).$$

If $\exists K$, $\forall n$, $\mathbb{E} Y_n^2 \leq K$, show that $\{Y_n\}$ converge in \mathbb{L}^2 .

Pf.:

$$\mathbb{E}[(Y_k - Y_j) Y_i] = \mathbb{E}\left[Y_i \underbrace{\mathbb{E}(Y_k - Y_j | \mathcal{G}_i)}_{= 0}\right] = 0$$

$\mathbb{E}(Y_k - Y_j | \mathcal{G}_i) = 0$
(MG_i)

$$\begin{aligned}\mathbb{E}((Y_k - Y_j)^2 | \mathcal{G}_i) &= \mathbb{E}(Y_k^2 | \mathcal{G}_i) - 2 \underbrace{\mathbb{E}(Y_k Y_j | \mathcal{G}_i)}_{= 0} + \mathbb{E}(Y_j^2 | \mathcal{G}_i) \\ &\quad + \mathbb{E}[Y_j(Y_k - Y_j) | \mathcal{G}_i] \\ &= \mathbb{E}(Y_j^2 | \mathcal{G}_i) + \mathbb{E}\left(\mathbb{E}[Y_j(Y_k - Y_j) | \mathcal{G}_j]\right) \\ &= \mathbb{E}(Y_j^2 | \mathcal{G}_i) + \mathbb{E}\left(Y_j \cdot \underbrace{\mathbb{E}(Y_k - Y_j | \mathcal{G}_j)}_{= 0} \right) \\ &= \mathbb{E}(Y_j^2 | \mathcal{G}_i) \\ &= \mathbb{E}(Y_k^2 | \mathcal{G}_i) - \mathbb{E}(Y_j^2 | \mathcal{G}_i)\end{aligned}$$

If $\exists K$, $\forall n$, $|E Y_n^2| \leq K$, then $\sup_n |E Y_n^2| \leq K < \infty$,
 $\{Y_n\}$ converge in L^2 by MG L^2 -convergence thm.

12.1.: $\{Y_n\}$ sub-MG, u convex ↑ from \mathbb{R} to \mathbb{R} ,
show that $\{u(Y_n)\}$ is sub-MG if $|E|u(Y_n)| < \infty$
for $\forall n$. Show that Y_n^+ is sub-MG, but $|Y_n|$ is
not necessarily sub-MG.

Pf: $\{Y_n\}$ equipped with filtration $\{\mathcal{F}_n\}$.

Check:

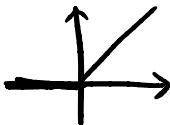
- ①: $\{u(Y_n)\}$ adapted to $\{\mathcal{F}_n\}$ ✓
- ②: $\forall n$, $|E|u(Y_n)| < \infty$ ✓
- ③: $|E[u(Y_{n+1})|\mathcal{F}_n] \geq u(|E(Y_{n+1}|\mathcal{F}_n)|)$

Jensen

$\geq u(Y_n)$

$u \uparrow$, sub-MG

so $\{u(Y_n)\}$ is sub-MG.

$u(x) = \max(x, 0)$  convex, ↑ so $\{Y_n^+\}$ is sub-MG

Counter-e.g.: $\{S_n\}$ SRW, up w.p. p , down w.p. q ,
 $|E(|S_{n+1}| |\mathcal{F}_n)| = p \cdot |S_n + 1| + q \cdot |S_n - 1|$

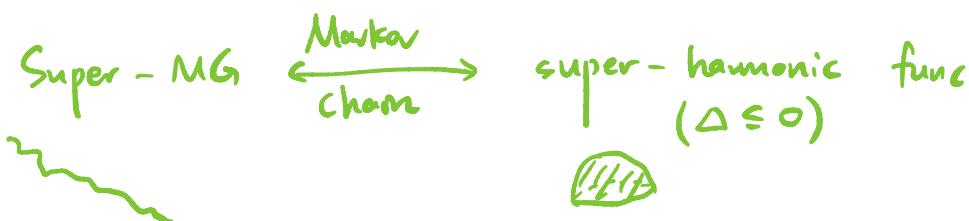
Now consider the case $p < \frac{1}{2}$, $S_n \geq 1$, then

$$\begin{aligned} \mathbb{E}(|S_{n+1}| \mid \mathcal{G}_n) &= p \cdot (|S_n| + 1) + q \cdot (|S_n| - 1) \\ &= |S_n| + \underbrace{\frac{p-q}{2}}_{<0} < |S_n|. \end{aligned}$$

12.1.8: $\{X_n\}$ Markov chain, countable state space S with transition matrix P . If $\psi: S \rightarrow \mathbb{R}$ bounded and $\sum_{j \in S} P_{ij} \psi(j) \leq \lambda \cdot \psi(i)$ for some $\lambda > 0$ and $\forall i \in S$. Show that $\lambda^{-n} \cdot \psi(X_n)$ is a super-MG.

PF: Check:

- ①: $\{\lambda^{-n} \cdot \psi(X_n)\}$ adapted to (\mathcal{G}_n)
- ②: $\forall n, \mathbb{E} |\lambda^{-n} \cdot \psi(X_n)| = \lambda^{-n} \cdot \mathbb{E} |\psi(X_n)| < \infty$ since ψ is bounded
- ③: $\mathbb{E} (\lambda^{-n-1} \cdot \psi(X_{n+1}) \mid \mathcal{G}_n) = \lambda^{-n-1} \cdot \mathbb{E} (\psi(X_{n+1}) \mid X_n)$
 $= \lambda^{-n-1} \cdot \underbrace{\sum_{j \in S} \psi(j) \cdot P_{X_n, j}}_{\text{Markov}} \leq \lambda^{-n-1} \cdot \lambda \cdot \underbrace{\psi(X_n)}_{\text{Markov}}$
 $= \lambda^{-n} \cdot \psi(X_n)$ proves it's a super-MG.



12.1.9 : $G_n(s)$ is prob g.f. of Z_n (branching process)
 where $Z_0 = 1$, $\text{Var}(Z_1) > 0$. Let H_n be inverse of
 G_n as a func defined on $[0,1]$, show that
 $M_n = (H_n(s))^{Z_n}$ defines a MG w.r.t. the filtration
 generated by $\{Z_n\}$.

If: check: ① adapted ✓

$$\textcircled{2}: \forall n, |E|M_n| = |E(H_n(s))^{Z_n}|_{G[0,1]}^{Z_n} < \infty$$

$$\textcircled{3}: |E(M_{n+1}|G_n) = |E([H_{n+1}(s)]^{Z_{n+1}}|G_n)$$

$$= |E([H_{n+1}(s)]^{Z_{n+1}}|Z_n)$$

$\{Z_n\}$ is Markov

$$Z_{n+1} = \underbrace{\tau_1^{n+1} + \dots + \tau_{Z_n}^{n+1}}$$

i.i.d. follow

offspring dist, indep of Z_n

$$= |E([H_{n+1}(s)]^{\tau_1^{n+1}} \cdot \dots \cdot [H_{n+1}(s)]^{\tau_{Z_n}^{n+1}}|Z_n)$$

Consider

$$|E([H_{n+1}(s)]^{\tau_1^{n+1}} \cdot \dots \cdot [H_{n+1}(s)]^{\tau_k^{n+1}}|Z_n = k)$$

$$= |E([H_{n+1}(s)]^{\tau_1^{n+1}} \cdot \dots \cdot [H_{n+1}(s)]^{\tau_k^{n+1}})$$

$$= \left(|E([H_{n+1}(s)]^{\tau_1^{n+1}}) \right)^k = \left[\sum_{j=0}^{\infty} [H_{n+1}(s)]^j \cdot |P(\tau_1^{n+1} = j) \right]^k$$

$$= \underbrace{[G_1(H_{n+1}(s))]^k}_{\text{by def., } G_1(s) = \mathbb{E} s^{Z_1}} \\ = \sum_{j=0}^{\infty} s^j \cdot \underbrace{\mathbb{P}(Z_1=j)}_{\text{offspring dist}}$$

since $H_{n+1} = (G_{n+1})^{-1}$ and $G_{n+1} = \underbrace{G_1 \circ G_1 \circ \dots \circ G_1}_{(n+1) \text{ of } G_1}$
 so $H_{n+1} = (G_n \circ G_1)^{-1} = G_1^{-1} \circ G_n^{-1} = G_1^{-1} \circ H_n$

now $= [H_n(s)]^k$ so

$$\mathbb{E}(M_{n+1} | \mathcal{G}_n) = [H_n(s)]^{Z_n} = M_n \quad \checkmark$$

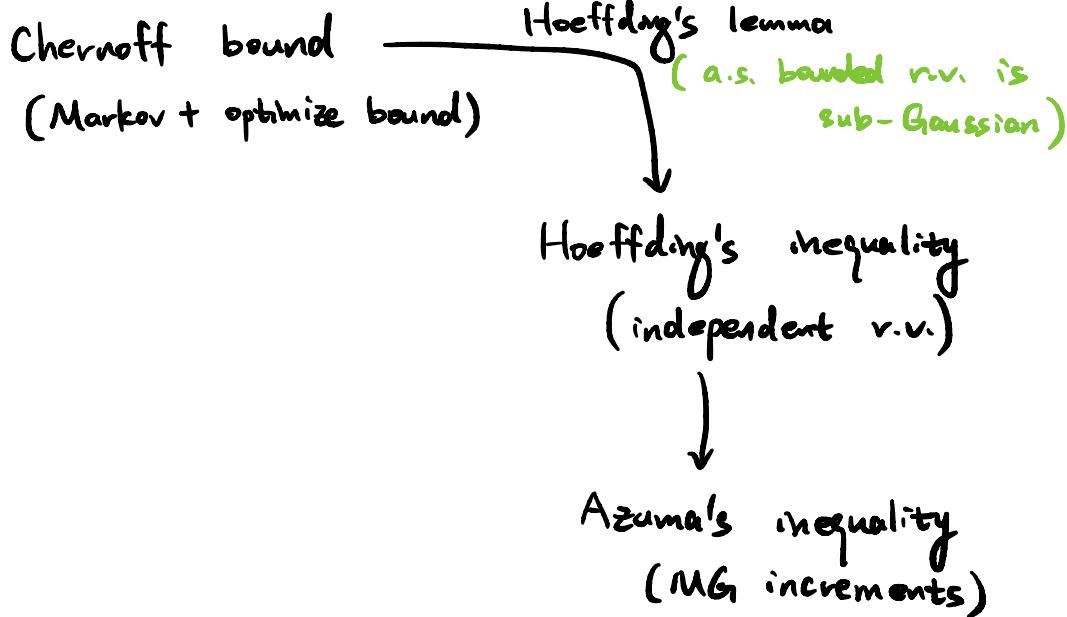
*: MG structure is hard to find typically, but if found, it tells a lot!

Try this question: $\{Z_n\}$ be branching process, $Z_0 = 1$, suppose offspring mean $\mu < 1$, $\mathbb{P}(Z_i \geq z) > 0$, show that $\mathbb{E}(\sup_n Z_n) \leq \frac{2}{\gamma - 1}$ where γ is the largest root of $x = G(x)$, G is g.f. of offspring dist

Hint: ①: check $\gamma > 1$ ②: $Y_n = \gamma^{Z_n}$ is positive MG

③: Doob's maximal inequality ④: Tail formula for expectation

Hoeffding & Azuma's inequality



Thm (Azuma): (Y, \mathcal{F}) is MG, if $\exists K_1, K_2, \dots$
such that $|Y_n - Y_{n-1}| \leq K_n$ a.s. for $\forall n$, then
 $\forall x > 0, \underbrace{\Pr(|Y_n - Y_0| \geq x)}_{\text{sum of MG increments}} \leq 2 \cdot e^{-\frac{x^2}{2 \cdot \sum_{i=1}^n K_i^2}}$

12.2.1 : (Knapsack)

n objects with volume V_i , worth W_i , (all indep & i.i.)
 $\exists M, V_i, W_i \in M$ a.s. Knapsack has volume C ,
maximize the worth packed in it within volume C .

Find $z_1, \dots, z_n \in \{0, 1\}$, $\begin{cases} \max \sum_i z_i W_i \\ \text{s.t. } \sum_i z_i V_i \leq C \end{cases}$

Let Z be maximal possible worth in knapsack, prove

$$\Pr(|Z - \mathbb{E}Z| \geq x) \leq 2e^{-\frac{x^2}{2nM^2}} \quad (\forall x > 0)$$

Pf: $\mathcal{G}_i = \sigma(V_1, \dots, V_i, W_1, \dots, W_i)$, set

$Y_i = \mathbb{E}(Z | \mathcal{G}_i)$ as a MG, wish to prove

$$|Y_j - Y_{j-1}| = |\mathbb{E}(Z | \mathcal{G}_j) - \mathbb{E}(Z | \mathcal{G}_{j-1})| \text{ is a.s. bounded}$$

Consider X_j as the maximal worth in knapsack

without j -th item, then $\mathbb{E}(X_j | \mathcal{G}_j) = \mathbb{E}(X_j | \mathcal{G}_{j-1})$,

since $X_j \leq Z \leq X_j + M$, we know connects \mathcal{G}_j and \mathcal{G}_{j-1}

$$\begin{cases} \mathbb{E}(X_j | \mathcal{G}_j) \leq Y_j \leq \mathbb{E}(X_j | \mathcal{G}_j) + M \\ \mathbb{E}(X_j | \mathcal{G}_{j-1}) \leq Y_{j-1} \leq \mathbb{E}(X_j | \mathcal{G}_{j-1}) + M \end{cases} \Rightarrow |Y_j - Y_{j-1}| \leq M$$

use Azuma, concludes the proof.

12.2.2 (Graph coloring)

n-vertex Erdos-Renyi graph (each edge prob p appear)

The chromatic number X is the minimal num of color needed such that each vertex colored differently from its neighbors. Show $P(|X - \mathbb{E}X| \geq x) \leq 2 \cdot e^{-\frac{x^2}{2n}}$ ($\forall x > 0$).

Of:

$$\mathcal{G}_i = \{(v_a, v_b) : 1 \leq a, b \leq i\}, Y_i = \mathbb{E}(X | \mathcal{G}_i) \text{ MG}$$

wish to prove $|Y_j - Y_{j-1}|$ is a.s. bounded.

Consider X_j as the minimal # of color needed in the graph with v_j deleted, then

$$X_j \leq X \leq X_j + 1, \quad \mathbb{E}(X_j | \mathcal{G}_j) = \mathbb{E}(X_j | \mathcal{G}_{j-1})$$

$$\text{now } \begin{cases} \mathbb{E}(X_j | \mathcal{G}_j) \leq Y_j \leq \mathbb{E}(X_j | \mathcal{G}_j) + 1 \\ \mathbb{E}(X_j | \mathcal{G}_{j-1}) \leq Y_{j-1} \leq \mathbb{E}(X_j | \mathcal{G}_{j-1}) + 1 \end{cases}$$

so $|Y_j - Y_{j-1}| \leq 1$, Azuma concludes the proof.