Section Notes for PSTAT 213C

Haosheng Zhou

Apr, 2025

Contents

Week 1

Review: Conditional Expectation

Exercise 1. *Let $\mathscr{F}_2 \subset \mathscr{F}_1 \subset \mathscr{F}$ be σ-fields on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For $X \in L^1$, show that*

$$\mathbb{E}(X - \mathbb{E}(X|\mathscr{F}_1))^2 \leq \mathbb{E}(X - \mathbb{E}(X|\mathscr{F}_2))^2. \tag{1}$$

If $X \in L^2$, what is the geometric interpretation?

Hints. Expand squares, it suffices to show that $\mathbb{E}[\mathbb{E}(X|\mathscr{F}_1)]^2 \geq \mathbb{E}[\mathbb{E}(X|\mathscr{F}_2)]^2$. Apply Jensen's inequality for the r.v. $\mathbb{E}(X|\mathscr{F}_1)$ under the conditional expectation $\mathbb{E}(\cdot|\mathscr{F}_2)$ yields $\mathbb{E}([\mathbb{E}(X|\mathscr{F}_1)]^2|\mathscr{F}_2) \geq [\mathbb{E}(\mathbb{E}(X|\mathscr{F}_1)|\mathscr{F}_2)]^2$. Taking expectations on both sides and using the tower property conclude the proof.

Geometrically, the distance from a fixed point to a vector space V (e.g., a plane) is no larger than the distance from the same point to a linear subspace of V (e.g., any line within the plane). □

Exercise 2. *Define the conditional independence $\mathscr{F}_1 \perp \mathscr{F}_2|\mathscr{G}$ to hold if and only if*

$$\mathbb{E}(X_1X_2|\mathscr{G}) = \mathbb{E}(X_1|\mathscr{G}) \cdot \mathbb{E}(X_2|\mathscr{G}),\ \forall X_1 \in \mathscr{F}_1, X_2 \in \mathscr{F}_2. \tag{2}$$

Show that:

(i): $\mathscr{F}_1 \perp \mathscr{F}_2|\mathscr{G}$ if and only if $(\mathscr{F}_1 \vee \mathscr{G}) \perp \mathscr{F}_2|\mathscr{G}$, where $\mathscr{F}_1 \vee \mathscr{G} := \sigma(\mathscr{F}_1 \cup \mathscr{G})$.

(ii): If $\mathscr{G} \subset \mathscr{F}_1$, then $\mathscr{F}_1 \perp \mathscr{F}_2|\mathscr{G}$ if and only if $\mathbb{E}(X_2|\mathscr{F}_1) \in \mathscr{G}$ for $\forall X_2 \in \mathscr{F}_2$.

Hints. (i): Direction $\Leftarrow$ is trivial. Direction $\Rightarrow$ requires $\pi - \lambda$ theorem (first prove the statement for $X_1\mathbb{I}_G$, where $X_1 \in \mathscr{F}_1$ and $G \in \mathscr{G}$).

(ii): Direction $\Leftarrow$ is trivial (tower property). Direction $\Rightarrow$ requires proving $\mathbb{E}(X_2|\mathscr{F}_1) = \mathbb{E}(X_2|\mathscr{G})$, using the definition of conditional expectation. □

Martingale

Exercise 3. *$\{X_n\}$ is a discrete-state Markov chain (with countable state space S) and transition probability matrix P. If there exists a bounded function $\psi : S \to \mathbb{R}$ such that*

$$\sum_{j\in S} P_{ij}\psi(j) = \psi(i),\ \forall i \in S, \tag{3}$$

check that $\{X_n\}$ is a martingale under the natural filtration.

A discrete-state Markov chain induces a difference operator L such that

$$(L\psi)(i) := \sum_{j\in S} P_{ij}\psi(j) - \psi(i),\ \forall i \in S. \tag{4}$$

What about a continuous-state Markov chain (e.g., Brownian motion)? That induces a differential operator on $\mathbb{R}^d$*, which is one half the Laplacian* $\Delta := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$*. With time evolution of the Markov chain introduced, this provides crucial connection between probability and PDE (generator, forward/backward equations).*

Exercise 4. $\{X_n\}$ *is a sequence of i.i.d. r.v. with an unknown density* f*. The density* f *is known to be either* p *or* q *(both are known and strictly positive), which results in a likelihood ratio test*

$$\begin{cases} H_0 : f = q \\ H_1 : f = p \end{cases}, \tag{5}$$

with a test statistic

$$Y_n := \prod_{i=1}^n \frac{p(X_i)}{q(X_i)}. \tag{6}$$

Show that $\{Y_n\}$ *is a martingale under the natural filtration of* $\{X_n\}$ *if* H_0 *is true.*

The rejection region of this test: $Y_n \geq a$*. The simplest example of sequential hypothesis testing (continue collecting evidence until a conclusion can be drawn).*

Does $\{Y_n\}$ *admit a limit? If one terminates the test only when a rejection is made, can we control the probability of rejecting* H_0 *if* H_0 *is true (type-I error)? Will be answered next week.*

Exercise 5. *A village contains* $N + 1$ *people, one of whom suffers from an infectious illness that cannot be cured.* S_t *is the number of susceptibles at time* t*,* I_t *is the number of infectives and* D_t *is the number of deads such that* $S_t + I_t + D_t = N + 1$*, while* $S_0 = N, I_0 = 1, D_0 = 0$*.*

Once a person is infected, his remaining lifespan is a random time that follows $\mathcal{E}(\mu)$*. Once a susceptible interacts with an infective, he gets infected after a random time that follows* $\mathcal{E}(\lambda)$*. It is assumed that at any fixed time, any two individuals within the system interact, and the random times are all independent.*

(i): Specify the state space S_X *and the dynamics of the continuous-time discrete-state Markov chain* (S_t, I_t)*.*

(ii): Specify the generator G *of* (S_t, I_t)*.*

(iii): If $G\psi = 0$ *for some* $\psi : S_X \to \mathbb{R}$*, show that* $Y_t := \psi(S_t, I_t)$ *is a martingale under the natural filtration of* $\{(S_t, I_t)\}$*.*

(iv): Find one such $\psi(s, i) = \alpha(s)\beta(i)$ *that reveals the martingale structure of the epidemic model.*

This epidemic model must result in one of the two situations: either everyone dies due to illness or the illness dies out itself before infecting everybody. How to calculate the probability of those situations happening? Will also be answered next week.

Hints. (i): $S_X = \{(s, i) : s + i \leq N + 1, s \geq 0, i \geq 0\}$. Given $(S_t, I_t) = (s, i)$, the next state transition is either a death $(s, i) \to (s, i - 1)$ or an infection $(s, i) \to (s - 1, i + 1)$.

The time until the next death is the minimum of the lifespan of each infective, which is the minimum of i independent $\mathcal{E}(\mu)$ r.v. Consequently, death rate $DR_{(s,i)} = i\mu$.

The time until the next infection is the minimum of is independent $\mathcal{E}(\lambda)$ r.v. (since each of the s susceptible interacts with all i infectives). Consequently, infection rate $IR_{(s,i)} = is\lambda$.

Therefore, the holding rate of state (s, i) is $i(\mu + s\lambda)$. Whenever a state transition happens, there is $\frac{i\mu}{i\mu+is\lambda}$ probability transiting to $(s, i-1)$, there is $\frac{is\lambda}{i\mu+is\lambda}$ probability transiting to $(s-1, i+1)$.

(ii): By definition,

$$G_{(s,i),(s,i-1)} = i\mu, \quad G_{(s,i),(s-1,i+1)} = is\lambda, \quad G_{(s,i),(s,i)} = -i(\mu + s\lambda), \tag{7}$$

while all other entries are zero.

(iii): Prove the martingale property within an infinitesimal interval $[t, t + \Delta t]$.

(iv): Assume $\alpha(N) = 1$ and solve $G\psi = 0$ to get

$$\psi(s, i) = \prod_{k=s+1}^{N} \frac{kB\lambda - (1-B)\mu}{k\lambda B^2} B^i, \tag{8}$$

which is a valid solution for an arbitrary $B > 0$. $\square$

Remark. *One obtains the freedom of choosing $B > 0$ in the example above that ensures*

$$Y_t = \prod_{k=S_t+1}^{N} \frac{kB\lambda - (1-B)\mu}{k\lambda B^2} B^{I_t} \tag{9}$$

being a martingale.

As we shall see later, martingales have very nice structures and are the easiest stochastic processes to investigate. Unfortunately, processes of general interests (e.g., branching processes, diffusions, asymmetric random walks) are typically not martingales. This example shows how one should discover the hidden martingale structure to be able to apply technical tools that are specifically designed for martingales (e.g., optional stopping theorem, maximal inequalities, etc.). ***The martingale structure is crucial but never for free!***

Week 2

Concentration Inequality

Theorem (Azuma-Hoeffding). *$\{Y_n\}$ is a martingale under filtration $\{\mathscr{F}_n\}$ and there exists a sequence of real numbers $\{K_n\}$ such that $|Y_n - Y_{n-1}| \le K_n$ a.s. for $\forall n$. Then*

$$\mathbb{P}\left(|Y_n - Y_0| \ge x\right) \le 2e^{-\frac{1}{2}\frac{x^2}{\sum_{i=1}^n K_i^2}},\ \forall x > 0. \tag{10}$$

Exercise 6. *Given n objects with independent identically distributed random sizes $X_1, ..., X_n$ on $[0,1]$, let B_n be the minimum number of bins (of size 1) required to pack $X_1, ..., X_n$. Show that*

$$\mathbb{P}\left(|B_n - \mathbb{E}B_n| \ge x\right) \le 2e^{-\frac{1}{2}\frac{x^2}{n}},\ \forall x > 0. \tag{11}$$

Hints. Set $\mathscr{F}_i := \sigma(X_1, ..., X_i)$ and $Y_i := \mathbb{E}(B_n|\mathscr{F}_i)$ as a martingale. Prove $|Y_{i+1} - Y_i| \le 1$ by considering the minimum number of bins (of size 1) required to pack $X_1, ..., X_n$ without packing X_i (leave-one-out). $\square$

Exercise 7. *Let $P_i := (U_i, V_i)$ such that $P_1, ..., P_n$ are independent and uniformly distributed points in $[0,1]^2$. Let D_n denote the length of the shortest tour that passes through each point exactly once and returns to the starting point. Show that there exists a constant $A > 0$, such that*

$$\mathbb{P}\left(|D_n - \mathbb{E}D_n| \ge x\right) \le 2e^{-\frac{Ax^2}{\log n}},\ \forall x > 0. \tag{12}$$

Hints. Set $\mathscr{F}_i := \sigma(P_1, ..., P_i)$ and $Y_i := \mathbb{E}(D_n|\mathscr{F}_i)$ as a martingale.

Consider the leave-P_i-out shortest path that has length $D_n(i)$. Note $D_n(i) \le D_n \le D_n(i) + 2Z_i$ for $i \le n-1$, where Z_i is the shortest distance from P_i to one of the points in $\{P_{i+1}, ..., P_n\}$ (why?). We derive $|Y_i - Y_{i-1}| \le 2\max\{\mathbb{E}(Z_i|\mathscr{F}_i), \mathbb{E}(Z_i|\mathscr{F}_{i-1})\}$. Now prove that $\max\{\mathbb{E}(Z_i|\mathscr{F}_i), \mathbb{E}(Z_i|\mathscr{F}_{i-1})\} \le \frac{C}{\sqrt{n-i}}$ for some constant $C > 0$. $\square$

Week 3

Application of OST

Exercise 8 (Doob's Maximal Inequality)**.** *Let $\{X_n\}$ be a sub-MG, prove that*

$$\mathbb{P}\left(\max_{1\le m\le n} X_m \ge \lambda\right) \le \frac{\mathbb{E}X_n^+ \mathbb{I}_{\max_{1\le m\le n} X_m \ge \lambda}}{\lambda} \le \frac{\mathbb{E}X_n^+}{\lambda},\ \forall \lambda > 0. \tag{13}$$

Hints. $T := \inf\{n : X_n \ge \lambda\}$, check that $\mathbb{P}(T \le n) \le \frac{\mathbb{E}X_T^+ \mathbb{I}_{T\le n}}{\lambda} = \frac{\mathbb{E}X_{T\wedge n}^+ - \mathbb{E}X_n^+ \mathbb{I}_{T>n}}{\lambda}$. By OST, $\mathbb{E}X_{T\wedge n}^+ \le \mathbb{E}X_n^+$. □

Exercise 9. *If $\{X_n\}$ is a simple symmetric random walk, prove that: (i). $\{X_n\}$ is a MG (ii). $\{X_n^2 - n\}$ is a MG (quadratic-MG) (iii). $\{\frac{e^{\lambda X_n}}{(\cosh\lambda)^n}\}$ is a MG for $\forall \lambda > 0$, where $\cosh\lambda := \frac{e^\lambda + e^{-\lambda}}{2}$ (exponential-MG).*

Let T be the first exit time of (a, b), where $a < 0, b > 0$ for $a, b \in \mathbb{Z}$, i.e., $T = \inf\{n : X_n = a$ or $X_n = b\}$. Using the MGs above, calculate $\mathbb{P}(X_T = a)$ and $\mathbb{E}T$.

Let T_1 be the first hitting time to 1, find the distribution of T_1.

Bonus: Can you find the distribution of T?

Hints. By OST, $\mathbb{P}(X_T = a) = \frac{b}{b-a}$ and $\mathbb{E}T = -ab$. PGF of T_1 is $\mathbb{E}s^{T_1} = \frac{1-\sqrt{1-s^2}}{s}$.

To find the distribution of T, consider the MG $Y_n = \frac{\cos\left(\lambda(X_n - \frac{b+a}{2})\right)}{\cos^n\lambda}$ (subtract the midpoint $\frac{b+a}{2}$ to symmetrize the exit time). By OST, $\mathbb{E}(\cos\lambda)^{-T} = \frac{\cos(\lambda\frac{b+a}{2})}{\cos(\lambda\frac{b-a}{2})}$. To verify the validity of OST, apply DCT and check $\mathbb{E}(\cos\lambda)^{-T} < \infty$ (prove by Fatou's lemma). □

Exercise 10. *$\{X_n\}$ is a simple asymmetric random walk with probability p stepping upward and probability $q = 1-p$ stepping downward. Let T be the first exit time of (a, b), where $a < 0, b > 0$ for $a, b \in \mathbb{Z}$, i.e., $T = \inf\{n : X_n = a$ or $X_n = b\}$. Calculate $\mathbb{P}(X_T = a)$ and $\mathbb{E}T$.*

Hints. $Y_n = \left(\frac{q}{p}\right)^{X_n}$ is a MG. $Z_n = X_n - (p-q)n$ is a MG. □

Exercise 11. *Consider the sequential hypothesis testing model in Example 4, show that $\{Y_n\}$ converges and identify the limit. If one terminates the test iff the first rejection is made, provide an upper bound for the probability of type-I error.*

Hints. The limit is almost surely 1 if $p = q$, otherwise it's 0 (Jensen).

Doob's maximal inequality: $\mathbb{P}(\sup_n Y_n \ge a | H_0) \le \frac{1}{a}$. □

Exercise 12. *Consider the epidemic model in Example 5, calculate the probability that eventually there are still people alive, i.e., the illness kills itself before infecting everyone.*

Hints. $T := \inf\{t : I_t = 0\}$. Use OST and find a clever way to specify B: consider B_r such that $rB_r\lambda - \mu(1-B_r) = 0$. Plug into the OST for $\forall 1 \le r \le N$ to get a system of equations. □

Week 4

Practice Problems for Midterm Revision

Exercise 13. *Let $f : \mathbb{Z}^2 \to \mathbb{R}_+$ be any non-negative function such that*

$$f(x, y) = \frac{f(x-1, y) + f(x+1, y) + f(x, y-1) + f(x, y+1)}{4}, \ \forall (x, y) \in \mathbb{Z}^2. \tag{14}$$

Show that f must be a constant function.

Hints. Construct a martingale and apply martingale convergence theorem. Use recurrence/transience properties of Markov chain if necessary. □

Exercise 14. *Let $\{X_n\}$ be a symmetric simple random walk starting at 0 and T be the first exit time of $(-a, a)$ for a positive integer a. Compute $\mathbb{E}T^2$.*

Hints. Recall the computation of $\mathbb{E}T$ where we use the square martingale $\{X_n^2 - n\}$ because the n term would become T when applying OST. Therefore, we hope to construct a martingale that looks like $f(X_n) + bn^2 + cn$ for some function f. Try the function $f(S_n) = S_n^4 - 6nS_n^2$ and figure out the values of b, c such that the martingale property holds. Apply OST to yield $\mathbb{E}T^2 = \frac{5a^4 - 2a^2}{3}$. □

Exercise 15. *Let $\{Z_n\}$ be a branching process with $Z_0 = 1$. The offspring distribution has mean μ and variance $\sigma^2 > 0$. We wish to investigate the maximum population $\sup_n Z_n$ that has ever appeared in the history when the branching process is in its subcritical phase, i.e., $\mu < 1$. For simplicity, we also assume that each individual has positive probability of giving birth to more than two children, i.e., $\mathbb{P}(Z_1 \geq 2) > 0$.*

(i): Let G denote the probability generating function of Z_1. Let η be the largest root of the equation $x = G(x)$. Show that $\eta > 1$.

(ii): Show that under the filtration $\mathscr{F}_n := \sigma(Z_0, ..., Z_n)$, $Y_n := \eta^{Z_n}$ is a martingale.

(iii): Show that $\mathbb{E}\sup_n Z_n \leq \frac{\eta}{\eta - 1}$.

Hints. For (iii), use the tail formula for expectation and apply Doob's maximal inequality. □

Exercise 16. *Let $\{X_n\}$ be an L^2 martingale, i.e., $X_n \in L^2$, $\forall n$ under some filtration $\{\mathscr{F}_n\}$ such that $X_0 = 0$. Clearly, $\{X_n^2\}$ is a sub-martingale and admits a unique Doob's decomposition*

$$X_n^2 = M_n + A_n, \tag{15}$$

where $\{M_n\}$ is a martingale and $\{A_n\}$ is a predictable increasing process with $A_0 = 0$.

(i): Show that $\mathbb{E}\sup_n |X_n|^2 \leq 4\mathbb{E}A_\infty$.

(ii): Fix any $a > 0$ and consider $T_a := \inf\{n : A_{n+1} > a^2\}$. Show that T_a is a stopping time w.r.t. $\{\mathscr{F}_n\}$.

(iii): Show that

$$\mathbb{P}\left(\sup_n |X_n| > a\right) \leq \mathbb{P}\left(A_\infty > a^2\right) + \mathbb{P}\left(\sup_n |X_{n \wedge T_a}| > a\right), \tag{16}$$

where

$$\mathbb{P}\left(\sup_n |X_{n\wedge T_a}| > a\right) \leq \frac{\mathbb{E}(A_\infty \wedge a^2)}{a^2}. \tag{17}$$

(iv): Use (iv) to show that $\mathbb{E}\sup_n |X_n| \leq 3\mathbb{E}\sqrt{A_\infty}$.

(v): Let T *be the first hitting time to* 1 *of a simple symmetric random walk that starts from* 0*. Prove that* $\mathbb{E}\sqrt{T} = \infty$.

Hints. For (i), apply Doob's L^2 inequality and use MCT.

For (iii), discuss if $\{T_a = \infty\}$ happens and use Doob's maximal inequality.

For (iv), use tail formula for expectation and apply Fubini's theorem.

For (v), consider Doob's decomposition of $\{X_{n\wedge T}\}$ and reach a contradiction by DCT. $\square$

Remark. *From optional stopping theorem, we prove that* $\mathbb{E}T = \infty$*, while the exercise above proves a stronger conclusion* $\mathbb{E}\sqrt{T} = \infty$*. Actually, the continuous limit of this stopping time is the first hitting time to* 1 *of Brownian motion, which follows an* α*-table law with* $\alpha = \frac{1}{2}$*. This agrees with the current observation* $\mathbb{E}\sqrt{T} = \infty$*.*

Week 6

Brownian Motion

Exercise 17. *For $0 \leq t_0 < t_1$, calculate the probability that a standard BM has at least one zero in the time interval (t_0, t_1).*

Hints. Use Markov property of BM and condition on W_{t_0}. Relate to the distribution of the first hitting time of BM. The answer is $\frac{2}{\pi} \arccos \sqrt{\frac{t_0}{t_1}}$. □

Exercise 18. *Prove that $\forall \varepsilon > 0$, $\sup_{s\in[0,\varepsilon]} W_s > 0, \inf_{s\in[0,\varepsilon]} W_s < 0$ holds almost surely.*

In addition, prove that a.s. $\forall \varepsilon > 0$, $\sup_{s\in[0,\varepsilon]} W_s > 0, \inf_{s\in[0,\varepsilon]} W_s < 0$. Think about the difference between two arguments!

Hints. Use the last conclusion, and the path continuity of BM. □

Exercise 19. *Prove that, a.s., $t \mapsto W_t$ is not monotone on any nontrivial interval.*

Hints. Use Markov property and the conclusion above. Be careful with the location of *a.s.* □

Exercise 20 (Brownian bridge)**.** *Consider $B_t := W_t - tW_1$ induced by a standard BM $\{W_t\}$ on $[0, 1]$. What would be a legal filtration that $\{B_t\}$ is adapted to? Prove that $\{B_t\}$ has the same finite-dimensional distribution as $\{W_t\}|_{W_1=0}$.*

Proof. Filtration $\mathscr{F}_t^B \equiv \mathscr{F}_1^W$, where $\mathscr{F}^W$ is the natural filtration of $\{W_t\}$. Use the Gaussian process characterization of BM. □

Exercise 21 (Law of iterated logarithm)**.** *Let $h(t) := \sqrt{2t \log \log t}$ and $\{W_t\}$ be a standard BM, with $S_t := \sup_{s\in[0,t]} W_s$ as the running supremum.*

(i): Show that $\forall t > 0$, $\mathbb{P}\left(S_t > u\sqrt{t}\right) \sim \frac{2}{u\sqrt{2\pi}} e^{-\frac{u^2}{2}}$ $(u \to \infty)$, where $\sim$ means asymptotic equivalence.

(ii): Let $r, c \in \mathbb{R}, 1 < r < c^2$. Consider $\mathbb{P}\left(S_{r^n} > ch(r^{n-1})\right)$ and show that $\limsup_{n\to\infty} \frac{W_t}{h(t)} \leq 1$ a.s.

(iii): Show that a.s. there exists infinitely many values of n such that

$$W_{r^n} - W_{r^{n-1}} \geq \sqrt{\frac{r-1}{r}} h(r^n). \tag{18}$$

(iv): Prove the law of iterated logarithm

$$\limsup_{n\to\infty} \frac{W_t}{h(t)} = 1, \quad \liminf_{n\to\infty} \frac{W_t}{h(t)} = -1 \ a.s. \tag{19}$$

Hints. Note that $S_t \overset{d}{=} |W_t|$ and use Borel-Cantelli lemma. □

Exercise 22. *Let T_a be the first hitting time to a of a standard BM. Check that $\mathbb{E}|T|^\alpha < \infty$ if and only if $\alpha < \frac{1}{2}$. Compare to the similar conclusion we have proved for SSRW. This is actually a consequence of Donsker's invariance principle.*

Week 7

BM and Stochastic Integration

Exercise 23. *For a simple process $\{X_t\}$ that is adapted to the BM natural filtration $\{\mathscr{F}_t\}$, show that for $Z_t := \int_0^t X_s\,dW_s$,*

$$\langle Z, Z\rangle_t = \int_0^t X_s^2\,ds. \tag{20}$$

Hints. Definition. □

Exercise 24. *For BM $\{W_t\}$, let $S_t := \sup_{s\in[0,t]} W_s$ be its running sup and define $T := \inf\{t \geq 0 : W_t = S_1\}$.*

(i): Use OST to prove that T is not a stopping time w.r.t. the BM natural filtration $\{\mathscr{F}_t\}$.

(ii): Use strong Markov property to prove that T is not a stopping time w.r.t. the BM natural filtration $\{\mathscr{F}_t\}$.

Hints. (i): Check $T \leq 1$ *a.s.* so that $\mathbb{E}S_1 = \mathbb{E}W_T = 0$ implies $S_1 = 0$ *a.s.*

(ii): Prove $\mathbb{P}(T = 1) = 0$. Use the fact that $B_t^T := W_{t+T} - W_T = W_{t+T} - S_1$ is a BM. □

Exercise 25 (Fake BM)**.** *Let $\{B_t\}, \{W_t\}$ be BMs, $G \sim N(0,1)$ and $G, \{B_t\}, \{W_t\}$ be independent. Set*

$$Y_t := \begin{cases} B_t & t \in [0,1] \\ \sqrt{t}[B_1\cos(W_{\log t}) + G\sin(W_{\log t})] & t \geq 1 \end{cases}, \tag{21}$$

which is adapted to the filtration

$$\mathscr{G}_t := \begin{cases} \sigma(B_s,\ \forall s \in [0,t]) & t \in [0,1] \\ \sigma(B_1, G, W_s,\ \forall s \in [0, \log t]) & t \geq 1 \end{cases}. \tag{22}$$

(i): Compute the marginal distribution of Y_t for each $t \geq 0$.

(ii): Prove that $\{Y_t\}$ is a continuous MG under $\{\mathscr{G}_t\}$.

(iii): Show that $\{Y_t\}$ is not a BM.

This is an example of a mimicking process, which replicates the marginal distribution but not the joint distribution of the process.

Hints. (i): Calculate c.f. $\phi_{Y_t}(s)$. $Y_t \sim N(0,t)$.

(ii): Definition. Use $\mathbb{E}\cos(W_{\log t}) = \frac{1}{\sqrt{t}}$ by taking real parts for the Gaussian c.f.

(iii): Prove by contradiction that $Y_e - Y_1$ is not Gaussian. Calculate c.f. $\phi_{Y_e - Y_1}(s) = e^{-\frac{1}{2}(e+1)s^2} \cdot \mathbb{E}e^{s^2\sqrt{e}\cos(W_1)}$. If $Y_e - Y_1$ is Gaussian, $\mathbb{E}e^{s^2\sqrt{e}\cos(W_1)} = e^{cs^2}$ for some $c \in \mathbb{R}$. Using Jensen's inequality (strict) for $\mathbb{E}\left(e^{\sqrt{e}\cos(W_1)}\right)^4$ produces a contradiction. □

Week 8

Itô's Formula

The introduction of stochastic integral allows us to quantify the change in $f(X_t)$ caused by a change in time t for some stochastic process $\{X_t\}$. This result is known as Itô's formula. Throughout the context, the filtration $\{\mathscr{F}_t\}$ denotes the natural filtration of a one-dimensional BM $\{W_t\}$.

Exercise 26 (Itô's formula)**.** *For any $f \in C^2$, show that*

$$f(W_t) - f(W_0) = \int_0^t f'(W_s)\,dW_s + \frac{1}{2}\int_0^t f''(W_s)\,d\langle W, W\rangle_s. \tag{23}$$

This yields one-dimensional Itô's formula for $f(W_t)$.

Hints. Write $f(W_{t_{i+1}})$ as a Taylor expansion of f at W_{t_i} (with a Lagrange-type remainder). Sum both sides w.r.t. i, given a partition $\Delta : 0 = t_0 < t_1 < ... < t_n = T$, and set $n \to \infty$. □

Exercise 27 (Itô's formula)**.** *For any $v \in C^{1,2}$, let $\{X_t\}$ be an Itô process given by $X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s$, where $\{b_t\}, \{\sigma_t\} \in \mathscr{L}^2_{\text{loc}}$. Show that*

$$v(t, X_t) - v(0, X_0) = \int_0^t \partial_t v(s, X_s)\,ds + \int_0^t \partial_x v(s, X_s)\,dX_s + \frac{1}{2}\int_0^t \partial_{xx} v(s, X_s)\,d\langle X, X\rangle_s, \tag{24}$$

where $d\langle X, X\rangle_s = \sigma_s^2\,ds$. This yields one-dimensional Itô's formula for $v(t, X_t)$.

Hints. Follow the same proof as above. Use the multi-dimensional Taylor expansion up to the second order. Prove that the $dt\,dt$ and $dt\,dW_t$ terms have no contributions. □

Remark. *For simpler notations, Itô's formula is always written in the differential form*

$$dv(t, X_t) = \partial_t v(t, X_t)\,dt + \partial_x v(t, X_t)\,dX_t + \frac{1}{2}\partial_{xx} v(t, X_t)\,d\langle X, X\rangle_t. \tag{25}$$

Exercise 28. *Use Itô's formula to calculate $\int_0^t W_s\,dW_s$, $\int_0^t e^{W_s}\,dW_s$.*

Hints. Consider W_t^2, e^{W_s}. □

Exercise 29. *If $\{Z_t\}$ satisfies $dZ_t = \frac{1}{Z_t}\,dt + dW_t$, check that $\frac{1}{Z_t}$ is a local MG (an Itô integral with no drift).*

Exercise 30. *Let $\{X_t\}$ be an Itô process given by $X_t = X_0 + \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dW_s$, where b, σ are deterministic bounded functions. Let $v \in C^{1,2}$ and $\partial_x v$ be bounded. Calculate $\mathbb{E}v(t, X_t)$. If b, σ are both constants and $v(t, x) = tx^2$, calculate the expectation.*

Week 9

Stochastic Differential Equation (SDE)

Exercise 31. *Solve the Ornstein-Uhlenbeck SDE*

$$dX_t = a(m - X_t)\,dt + \sigma\,dW_t,\ \ (a, \sigma > 0), \tag{26}$$

with a given initial condition X_0 that is independent of $\{W_t\}$. What are the limiting distribution and the stationary distribution? Do they coincide?

Hints. Use the ansatz $X_t = m + C_t e^{-at}$ (think about how to get this ansatz). Apply Itô's formula to solve $C_t = X_0 - m + \sigma\int_0^t e^{as}\,dW_s$. Clearly, X_t is Gaussian with mean $m(1-e^{-at}) + e^{-at}\mathbb{E}X_0$ and variance $e^{-2at}\text{Var}(X_0) + \sigma^2\frac{1-e^{-2at}}{2a}$.

Stationary distribution: $N(m, \frac{\sigma^2}{2a})$. Limiting distribution aligns with stationary distribution. □

Exercise 32. *Solve the geometric Brownian motion SDE (Black-Scholes model)*

$$dX_t = \mu X_t\,dt + \sigma X_t\,dW_t,\ \ (\sigma > 0) \tag{27}$$

with a given initial condition X_0 that is independent of $\{W_t\}$. What is the distribution of X_t? Discuss the asymptotic behavior of X_t.

Hints. Apply Itô's formula for $Y_t = \log X_t$ to get $X_t = X_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$. □

Exercise 33. *Consider Tanaka's equation*

$$dX_t = \text{sgn}(X_t)\,dW_t, \tag{28}$$

with a given initial condition $X_0 = 0$. Prove that if the strong solution $\{X_t\}$ exists, it must be a Brownian motion under the natural filtration of $\{W_t\}$.

Actually, the strong solution to Tanaka's equation does not exist! By setting $\{X_t\}$ as a BM, and $\tilde{B}_t := \int_0^t \text{sgn}(X_s)\,dX_s$, $dX_t = \text{sgn}(X_t)\,d\tilde{B}_t$. This implies that $(X, \tilde{B})$ admits a weak solution, where the BM path is not fixed a priori.

Hints. Levy's characterization of Brownian motion. □

Week 10

Extra Exercise for Stochastic Calculus

Exercise 34. *For $Y_t = W_t^4$, apply Itô's formula to calculate $\mathbb{E}W_t^4$.*

Proof. Apply Itô's formula for $g(x) = x^4$ to yield

$$dW_t^4 = 4W_t^3\,dW_t + \frac{1}{2}12W_t^2\,dt. \tag{29}$$

Integrating both sides yields

$$W_t^4 = 4\int_0^t W_s^3\,dW_s + 6\int_0^t W_s^2\,ds. \tag{30}$$

Taking expectations on both sides, since $\mathbb{E}\langle\int_0^\cdot W_s^3\,dW_s, \int_0^\cdot W_s^3\,dW_s\rangle_t = \int_0^t \mathbb{E}W_s^6\,ds < \infty$ for any fixed $t > 0$, $\int_0^t W_s^3\,dW_s$ is a martingale and $\mathbb{E}\int_0^t W_s^3\,dW_s = 0$. Hence,

$$\mathbb{E}W_t^4 = 6\int_0^t \mathbb{E}W_s^2\,ds = 6\int_0^t s\,ds = 3t^2. \tag{31}$$

□

Exercise 35. *For $Y_t = tW_t$, apply Itô's formula to calculate $\mathbb{E}Y_t^2$.*

Proof. Since $Y_t^2 = t^2W_t^2$, apply Itô's formula for $v(t,x) = t^2x^2$ to yield

$$dY_t^2 = 2tW_t^2\,dt + 2t^2W_t\,dW_t + t^2\,dt. \tag{32}$$

Integrating both sides yields

$$Y_t^2 = \int_0^t (2sW_s^2 + s^2)\,ds + 2\int_0^t s^2W_s\,dW_s. \tag{33}$$

Taking expectations on both sides, since $\mathbb{E}\langle\int_0^\cdot s^2W_s\,dW_s, \int_0^\cdot s^2W_s\,dW_s\rangle_t = \int_0^t s^4\mathbb{E}W_s^2\,ds = \int_0^t s^5\,ds < \infty$ for any fixed $t > 0$, $\int_0^t s^2W_s\,dW_s$ is a martingale and $\mathbb{E}\int_0^t s^2W_s\,dW_s = 0$. Hence,

$$\mathbb{E}Y_t^2 = \int_0^t \mathbb{E}(2sW_s^2 + s^2)\,ds = \int_0^t 3s^2\,ds = t^3. \tag{34}$$

□

Exercise 36. *Find the SDE (in diffusion form) satisfied by the BM on an ellipse: $X_t = a\cos W_t, Y_t = b\sin W_t$, where $ab \neq 0$.*

Proof. Apply Itô's formula for $g(x) = a\cos(x)$ to yield

$$dX_t = -a\sin W_t\,dW_t - \frac{a}{2}\cos W_t\,dt = -\frac{a}{b}Y_t\,dW_t - \frac{1}{2}X_t\,dt. \tag{35}$$

Similarly, apply Itô's formula for $h(x) = b\sin(x)$ to yield

$$dY_t = b\cos W_t\,dW_t - \frac{b}{2}\sin W_t\,dt = \frac{b}{a}X_t\,dW_t - \frac{1}{2}Y_t\,dt. \tag{36}$$

Remember to mention the initial condition: $X_0 = a, Y_0 = 0$. $\square$

Exercise 37. *Define $V_t := W_t - tW_1,\ \forall t \in [0,1]$ as the Brownian bridge induced by a BM $\{W_t\}$.*

(i): Find the mean and covariance kernel of $\{V_t\}$.

(ii): Show that $\{V_t\}$ has the same fdd as $\{W_t|_{W_1=0}\}$.

(iii): Show that $\{V_t\}$ has the same fdd as $U_t := (1-t)\int_0^t \frac{1}{1-s}\,dW_s$, and check that $\{U_t\}$ is the solution to the SDE

$$dU_t = -\frac{U_t}{1-t}\,dt + dW_t, \quad U_0 = 0. \tag{37}$$

What is the point of introducing such an SDE representation?

Hints. (i): Calculation, $\mathbb{E}V_t = 0, \operatorname{cov}(V_s, V_t) = s\wedge t - st$.

(ii): Prove both processes are Gaussian processes, it suffices to check $\mathbb{E}(W_t|W_1 = 0)$ and $\operatorname{cov}(W_t, W_s|W_1 = 0)$.

(iii): Itô's formula. The SDE representation makes $\{U_t\}$ adapted to the natural filtration of $\{W_t\}$, while $\{V_t\}$ is not. Markovianizing the process is a key technique in practical applications. $\square$

Exercise 38. *Let $X_t := 2 + \sqrt{2}W_t$ be a linearly scaled BM that starts from $X_0 = 2$. Let $L_1(t) = 2t+3$ and $L_2(t) = 2t-1$ be two straight lines, find the probability $\{X_t\}$ hits L_1 before hitting L_2.*

Proof. Denote T_1, T_2 as first hitting times of $\{X_t\}$ to L_1 and L_2. Then $T_1 = \inf\{t : W_t - \sqrt{2}t = \frac{1}{\sqrt{2}}\}$ and $T_2 = \inf\{t : W_t - \sqrt{2}t = \frac{-3}{\sqrt{2}}\}$. Naturally, we define $D_t := W_t - \sqrt{2}t$ so that the stopping times become the first hitting time of $\{D_t\}$ to constant levels.

To find the martingale structure, we motivate ourselves by the exponential MG of BM: $e^{\lambda W_t - \frac{1}{2}\lambda^2 t}$, where we have the freedom of selecting $\forall\lambda \in \mathbb{R}$. On the exponent, we have $\lambda(W_t - \frac{1}{2}\lambda t)$, which is hopefully proportional to D_t. This implies that we shall consider $\lambda = 2\sqrt{2}$, so that the exponential martingale $M_t := e^{2\sqrt{2}D_t}$ is the martingale we hope to consider.

Applying OST for $T := T_1 \wedge T_2$ yields

$$\mathbb{E}M_{t\wedge T} = \mathbb{E}M_0 = 1,\ \forall t \geq 0. \tag{38}$$

Since $\frac{-3}{\sqrt{2}} \leq D_{t\wedge T} \leq \frac{1}{\sqrt{2}}$ *a.s.*, setting $t \to \infty$ yields (by bounded convergence theorem)

$$1 = \mathbb{E}M_T = \mathbb{P}(T_1 < T_2)\,e^{2\sqrt{2}\frac{1}{\sqrt{2}}} + \mathbb{P}(T_1 > T_2)\,e^{2\sqrt{2}\frac{-3}{\sqrt{2}}}. \tag{39}$$

This implies $\mathbb{P}(T_1 < T_2) = \frac{e^6-1}{e^8-1}$. $\square$

Exercise 39. *Let $T := \inf\{t : W_t = a \text{ or } -b\}$ for some $a, b > 0$. Denote A_T as the expected signed area under the path of $\{W_t\}$ up to time T, where the area above the x-axis counts as positive, and below as negative. Compute $\mathbb{E}A_T$.*

Proof. Clearly $A_T = \int_0^T W_s\,ds$. We wish to know the time evolution of $A_t = \int_0^t W_s\,ds$ and evaluate it at the stopping time T, which naturally leads us to Itô's formula. In order to have A_t appear within Itô's formula, we decide to match ds with $d\langle W, W\rangle_s$ so that $g''(x) \propto x$, which points us towards $g(x) = x^3$.

Applying Itô's formula for $g(x) = x^3$ yields

$$dW_t^3 = 3W_t^2\,dW_t + 3W_t\,dt. \tag{40}$$

Integrating both sides yields

$$W_t^3 = 3\int_0^t W_s^2\,dW_s + 3A_t. \tag{41}$$

Denote $Z_t := \int_0^t W_s^2\,dW_s$. Since $\mathbb{E}\langle Z, Z\rangle_t = \int_0^t \mathbb{E}W_s^4\,ds < \infty$, $\{Z_t\}$ is a MG with mean zero and so does $\{Z_{t\wedge T}\}$ (by OST). Evaluating both sides at the stopping time T and taking expectations yield

$$\mathbb{E}W_T^3 = 3\mathbb{E}Z_T + 3\mathbb{E}A_T. \tag{42}$$

For the term $\mathbb{E}Z_T$, we know $Z_{t\wedge T} \overset{a.s.}{\to} Z_T$ $(t \to \infty)$. Consider the quadratic variation of the stopped MG $\{Z_{t\wedge T}\}$:

$$\mathbb{E}\langle Z_{\cdot\wedge T}, Z_{\cdot\wedge T}\rangle_\infty = \mathbb{E}\int_0^T W_s^4\,ds \le (a^4 \vee b^4)\mathbb{E}T, \tag{43}$$

where $-b \le W_{t\wedge T} \le a$ *a.s.* due to the continuity of BM sample paths. Next, we prove $\mathbb{E}T < \infty$. Apply OST for the square martingale $W_t^2 - t$ to see that

$$\mathbb{E}W_{t\wedge T}^2 = \mathbb{E}(t \wedge T),\ \forall t \ge 0. \tag{44}$$

Setting $t \to \infty$ yields (by bounded convergence theorem and monotone convergence theorem)

$$\mathbb{E}T = \mathbb{E}W_T^2 \le a^2 \vee b^2 < \infty. \tag{45}$$

As a result, $\mathbb{E}\langle Z_{\cdot\wedge T}, Z_{\cdot\wedge T}\rangle_\infty < \infty$, which implies that $\{Z_{t\wedge T}\}$ is an L^2-bounded MG. By the MG L^2 convergence theorem, it converges in L^2 and almost surely to Z_T. Therefore, $\mathbb{E}Z_T = \lim_{t\to\infty} \mathbb{E}Z_{t\wedge T} = 0$.

Finally, it suffices to calculate $\mathbb{E}W_T^3$, which reduces to the calculation of the probability that $\mathbb{P}(T = T_a)$, where T_a and T_{-b} denotes the first hitting time to a and $-b$ respectively. Using the BM $\{W_t\}$ itself as a MG,

$$\mathbb{E}W_{t\wedge T} = \mathbb{E}W_0 = 0,\ \forall t \ge 0. \tag{46}$$

Setting $t \to \infty$ yields (by bounded convergence theorem)

$$0 = \mathbb{E}W_T = a\mathbb{P}(T = T_a) - b\mathbb{P}(T = T_{-b}). \tag{47}$$

As a result,

$$\mathbb{P}(T = T_a) = \frac{b}{a+b}, \quad \mathbb{E}W_T^3 = a^3\frac{b}{a+b} - b^3\frac{a}{a+b} = ab(a-b). \tag{48}$$

Combining all previous conclusions yields

$$\mathbb{E}A_T = \frac{1}{3}\mathbb{E}W_T^3 = \frac{ab(a-b)}{3}. \tag{49}$$

$\square$

Challenging Problems

Don't read this part if you are not interested in stochastic calculus!!!

Exercise 40. *For a BM $\{B_t\}$, we want to compare the probabilities that the path of $\{B_t\}$ stays close to two different smooth paths. Let $h:\mathbb{R}\to\mathbb{R}, h\in C^2$, such that $h(0)=0$ and define*

$$M_t := e^{-\int_0^t h'(s)\,dB_s - \frac{1}{2}\int_0^t [h'(s)]^2\,ds} \tag{50}$$

as the stochastic exponential of $-\int_0^t h'(s)\,dB_s$.

(i): Show $\{M_t\}$ is a MG and calculate $\mathbb{E}M_1$.

(ii): Prove that, if $\|f\| := \sup_{t\in[0,1]}|f(t)|$ denotes the supremum norm for any continuous $f:[0,1]\to\mathbb{R}$, then

$$\mathbb{P}\left(\|B-h\|\le\varepsilon\right) = \mathbb{E}[M_1\mathbb{I}_{\|B\|\le\varepsilon}]. \tag{51}$$

(iii): Show that there exists $C>0$ such that

$$\left|\int_0^1 h'(s)\,dB_s\right| \le C\|B\|. \tag{52}$$

(iv): Deduce that

$$\lim_{\varepsilon\to 0}\frac{\mathbb{P}\left(\|B-h\|\le\varepsilon\right)}{\mathbb{P}\left(\|B\|\le\varepsilon\right)} = e^{-\frac{1}{2}\int_0^1[h'(s)]^2\,ds}. \tag{53}$$

(v): Assuming $h_1,h_2:\mathbb{R}\to\mathbb{R}$ are both C^2 and $h_1(0)=h_2(0)=0$, show that the limit exists and compute it:

$$\lim_{\varepsilon\to 0}\frac{\mathbb{P}\left(\|B-h_1\|\le\varepsilon\right)}{\mathbb{P}\left(\|B-h_2\|\le\varepsilon\right)}. \tag{54}$$

Proof. (i): By considering dM_t and applying Itô's formula, it is obvious that

$$dM_t = -M_t h'(t)\,dB_t, \tag{55}$$

which implies $\{M_t\}$ is a local-MG. Since $\{M_t\}$ is non-negative, it must be a super-MG, hence $\mathbb{E}e^{-\int_0^t h'(s)\,dB_s} \le e^{\frac{1}{2}\int_0^t[h'(s)]^2\,ds}$. Now that $\mathbb{E}\langle M,M\rangle_t = \int_0^t [h'(s)]^2\mathbb{E}M_s^2\,ds$, we need to upper bound $\mathbb{E}M_t^2$. Replace h with $2h$ to yield

$$\mathbb{E}\left(e^{-\int_0^t h'(s)\,dB_s}\right)^2 \le e^{\frac{1}{2}\int_0^t 4[h'(s)]^2\,ds}. \tag{56}$$

Multiplying both sides by $e^{-\int_0^t[h'(s)]^2\,ds}$ yields

$$\mathbb{E}M_t^2 \le e^{\int_0^t[h'(s)]^2\,ds}. \tag{57}$$

Therefore, $\mathbb{E}\langle M,M\rangle_t<\infty$ (continuous function on compact set is bounded) which proves that $\{M_t\}$ is a MG and $\mathbb{E}M_1 = \mathbb{E}M_0 = 1$. (One can also directly check Novikov's condition).

(ii): Consider $h(t) = \int_0^t h'(s)\,ds$ restricted on $t \in [0,1]$, which is within the Cameron-Martin space $\mathscr{H}$. Let $W(dw)$ denote the Wiener measure on the path space $C([0,1];\mathbb{R})$, the Cameron-Martin formula implies that for any non-negative path function $\Phi : C([0,1];\mathbb{R}) \to \mathbb{R}$,

$$\int \Phi(w-h)\,W(dw) = \int e^{-\int_0^1 h'(s)\,dw(s) - \frac{1}{2}\int_0^1 [h'(s)]^2\,ds}\Phi(w)\,W(dw). \tag{58}$$

Taking $\Phi(w) = \mathbb{I}_{\sup_{s\in[0,1]}|w(s)|\le\varepsilon}$ yields

$$\mathbb{P}\left(\sup_{s\in[0,1]}|B_s - h(s)| \le \varepsilon\right) = \mathbb{E}[e^{-\int_0^1 h'(s)\,dB_s - \frac{1}{2}\int_0^1 [h'(s)]^2\,ds}\mathbb{I}_{\sup_{s\in[0,1]}|B_s|\le\varepsilon}]. \tag{59}$$

Therefore,

$$\mathbb{P}\left(\|B-h\| \le \varepsilon\right) = \mathbb{E}[M_1\mathbb{I}_{\|B\|\le\varepsilon}]. \tag{60}$$

(iii): Apply Itô's formula for $h(t)B_t$:

$$\int_0^1 h'(s)\,dB_s = h(1)B_1 - \int_0^1 B_s h'(s)\,ds. \tag{61}$$

Since h, h' are continuous on $[0,1]$, they are bounded. Applying triangle inequality concludes the proof.

(iv): As implied by (ii),

$$\frac{\mathbb{P}\left(\|B-h\| \le \varepsilon\right)}{\mathbb{P}\left(\|B\| \le \varepsilon\right)} = \frac{\mathbb{E}[e^{-\int_0^1 h'(s)\,dB_s}\mathbb{I}_{\|B\|\le\varepsilon}]}{\mathbb{E}[\mathbb{I}_{\|B\|\le\varepsilon}]}e^{-\frac{1}{2}\int_0^1 [h'(s)]^2\,ds}. \tag{62}$$

According to (iii), for some constant $C > 0$, $e^{-C\|B\|} \le e^{-\int_0^1 h'(s)\,dB_s} \le e^{C\|B\|}$.

$$e^{-C\varepsilon} = \frac{\mathbb{E}[e^{-C\varepsilon}\mathbb{I}_{\|B\|\le\varepsilon}]}{\mathbb{E}[\mathbb{I}_{\|B\|\le\varepsilon}]} \le \frac{\mathbb{E}[e^{-\int_0^1 h'(s)\,dB_s}\mathbb{I}_{\|B\|\le\varepsilon}]}{\mathbb{E}[\mathbb{I}_{\|B\|\le\varepsilon}]} \le \frac{\mathbb{E}[e^{C\varepsilon}\mathbb{I}_{\|B\|\le\varepsilon}]}{\mathbb{E}[\mathbb{I}_{\|B\|\le\varepsilon}]} = e^{C\varepsilon}. \tag{63}$$

Noticing that both sides converge to 1 as $\varepsilon \to 0$ concludes the proof.

(v): Obviously, the limit exists

$$\lim_{\varepsilon\to 0}\frac{\mathbb{P}\left(\|B-h_1\| \le \varepsilon\right)}{\mathbb{P}\left(\|B-h_2\| \le \varepsilon\right)} = e^{-\frac{1}{2}\int_0^1 [h_1'(s)]^2 - [h_2'(s)]^2\,ds}. \tag{64}$$

□

Exercise 41. *For $n \ge 0$, the n-th degree Hermite polynomial is defined as*

$$H_n(x) := (-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}}. \tag{65}$$

Define

$$H_n(x,v) := v^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{v}}\right), \ \forall v > 0, \quad H_n(x,0) := x^n. \tag{66}$$

(i): Prove the generating function

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} u^n = e^{ux - \frac{u^2}{2}}, \ \forall u, x \in \mathbb{R}. \tag{67}$$

(ii): Show that $H_n(x)$ *satisfies the following ODE:*

$$H_n''(x) - xH_n'(x) + nH_n(x) = 0. \tag{68}$$

(iii): Define $M_t^n := H_n(B_t, t)$*. Show that for* $\forall n \geq 0$, $\{M_t^n\}$ *is a martingale.*

(iv): Show that this martingale is actually the n*-th order signature:*

$$M_t^n = n! \int_0^t dB_{t_1} \int_0^{t_1} dB_{t_2} ... \int_0^{t_{n-1}} dB_{t_n}. \tag{69}$$