Notes on MATH 246

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Week 1

Notations

$\alpha = (\alpha_1, ..., \alpha_n)$ to be a tuple of non-neg integers as indices for differentiation, $|\alpha| = \sum_i \alpha_i$ is the order of index.

$u : \Omega \to \mathbb{R}$ with Ω as an open connected subset of $\mathbb{R}^n$ and $\partial^\alpha u = \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n} u$ with $D^k u = \left\{ \partial^\alpha u \middle| |\alpha| = k \right\}$. For example, $Du = \{u_{x_1}, ..., u_{x_n}\}$ and $D^2 u = \{u_{x_1,x_1}, ..., u_{x_n,x_n}\}$.

As a common practice, Du stands for the gradient of u.

$C^k(\Omega)$ is the set of function $u : \Omega \to \mathbb{R}$ such that for all $|\alpha| \leq k$, $\partial^\alpha u$ exists and is continuous. $C^\infty(\Omega)$ denotes the set of smooth functions, infinitely continuously differentiable. If Ω is bounded, $C^k(\overline{\Omega})$ is the set of all $C^k(\Omega)$ functions that are continuous up to boundary. It's easy to se that is a Banach space w.r.t. norm $||u||_{C^k} = \sum_{|\alpha| \leq k} \sup_\Omega |\partial^\alpha u|$.

Actually, when $n = 1$, this norms degenerates to the canonical norm defined on C^k space.

$$||u||_{C^k} = ||u||_\infty + ||u'||_\infty + ... + ||u^{(k)}||_\infty \tag{1}$$

A k-th order PDE is just the equation such that $F(x, u, Du, ..., D^k u) = 0$. A PDE is called linear if any linear combination of solutions is still a solution to the PDE.

Transport Equation

Denote function u as $u(x,t)$ where $x \in \mathbb{R}^n$ is the spacial variable and $t \in (0, +\infty)$ is the time variable.

The transport equation is a first order PDE:

$$u_t + b \cdot Du = 0 \tag{2}$$

with $b \in \mathbb{R}^n$ as a given vector.

If $n = 1$, $u_t + bu_x = 0$. The interpretation of this PDE is about **the transport of particles with conservation law**. Let $u(x,t)$ stands for the density of particles at location $x \in \mathbb{R}^n$ at time t in a fixed area Ω and the particles are moving with constant speed v to the right.

Then at time t, the amount of particles within (x_1, x_2) is:

$$\int_{x_1}^{x_2} u(x,t)\, dx \tag{3}$$

at location x, the amount of particles within time interval (t_1, t_2) is:

$$\int_{t_1}^{t_2} v \cdot u(x,t)\, dt \tag{4}$$

The conservation law then tells us that the amount of particles within location (x_1, x_2) at time t_2 minus the amount of particles within location (x_1, x_2) at time t_1 should be equal to the net amount of particles flowing in this region during time (t_1, t_2), i.e. the amount flowing in at location x_1 minus the amount flowing out at location x_2. As a result, the following equation reflects **the conservation law**:

$$\int_{x_1}^{x_2} u(x,t_2)\,dx - \int_{x_1}^{x_2} u(x,t_1)\,dx = \int_{t_1}^{t_2} v\cdot u(x_1,t)\,dt - \int_{t_1}^{t_2} v\cdot u(x_2,t)\,dt \tag{5}$$

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} u_t(x,s)\,ds\,dx = -\int_{t_1}^{t_2}\int_{x_1}^{x_2} v\cdot u_x(y,s)\,dy\,ds \tag{6}$$

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} u_t + v\cdot u_x\,ds\,dy = 0 \tag{7}$$

If it's assumed that $u \in C^1$, since such equation holds for any $t_1 \le t_2, x_1 \le x_2$, we get the transport equation $u_t + v\cdot u_x = 0$. As a result, the general transport equation is the generalization in the higher dimensional space and the ratio of the coefficient of the partial w.r.t. the spatial variable x and the coefficient of the partial w.r.t. the time variable t, which is v, stands for the moving speed of the particle.

Remark. *If the conservation law is broken, the transport equation becomes* ***non-homogeneous****, and the right hand side would not be 0, meaning that there's actually a source or a sink (the source or sink might change depending on the time and location, i.e. the non-homogeneous transport equation is $u_t + v\cdot u_x = f(x,t)$).*

Remark. *Although here a constant speed v is assumed, such v can be modified to depend on x and t. It's easy to imagine that if v gets smaller when x gets larger, there will be "shocks" happening, i.e. the density u at some time point would be non-differentiable at a certain place because particles at a righter location are caught up with those at a lefter location that moves quicker. (This is the situation where "the law of equal area" holds)*

To solve the transport equation analytically, we can use the **method of characteristics**. The motivation comes from the fact that if we view the PDE along a certain characteristic (c.h.) curve $x = x(t)$, it would be much easier to solve the equation. Let's look at the homogeneous transport equation in 1 dimension by viewing the left hand side as a directional derivative.

$$\nabla u \cdot (b, 1) = 0 \tag{8}$$

The directional derivative along vector $(b,1)$ is 0, so u is constant along $x = x(t) = \frac{1}{b}t + c$ (c.h. lines), and $u(x,t) = u(c,0) = f(c) = f(x - bt)$ to figure out the solution (here c is a varying constant, standing for a family of c.h. lines).

If now consider the transport equation in n dimension with an initial value condition $u(x,0) = g(x)$, the solution is constant along the line $(x + sb, t + s)$ $(s \in \mathbb{R})$ and define $z(s) = u(x + sb, t + s)$. By chain rule,

$$z'(s) = b \cdot u_x + u_t = 0 \tag{9}$$

As a result, $z(s)$ is constant with $\forall s, z(s) = z(-t) = u(x - tb, 0) = g(x - tb)$ and $\forall s, u(x + sb, t + s) = g(x - tb)$. Set $s = 0$ to find that

$$u(x, t) = g(x - tb) \tag{10}$$

The non-homogeneous version of transport equation is $u_t + b \cdot Du = f(x, t)$ with $u(x, 0) = g(x)$. Similarly, by setting $z(s) = u(x + sb, t + s)$, the z will satisfy ODE $z'(s) = f(x + sb, t + s)$. Use integration to solve $z(-t)$:

$$z(0) - z(-t) = \int_{-t}^{0} z'(s)\, ds = \int_{-t}^{0} f(x + sb, t + s)\, ds \tag{11}$$

$$z(0) = u(x, t) = u(x - tb, 0) + \int_{-t}^{0} f(x + sb, t + s)\, ds \tag{12}$$

$$u(x, t) = g(x - tb) + \int_{-t}^{0} f(x + sb, t + s)\, ds \tag{13}$$

$$\tag{14}$$

With a change of variable for the integral, we get the final form of the solution

$$u(x, t) = g(x - tb) + \int_{0}^{t} f(x + (s - t)b, s)\, ds \tag{15}$$

First Order Linear Equation

The general form:

$$a(x, t)u_t + b(x, t)u_x = c(x, t)u + d(x, t) \tag{16}$$

still apply the method of characteristics to find a c.h. curve $(x(s), t(s))$ such that $z(s) = u(x(s), t(s))$ can be calculated easily. (z is actually **the version of u viewing along the c.h. curve**) Then it's obvious that

$$z'(s) = u_x \frac{dx}{ds} + u_t \frac{dt}{ds} \tag{17}$$

Compare this with the left hand side of PDE to get the **c.h. equation (ODE)**:

$$\begin{cases} \frac{dt}{ds} = a(x,t) \\ \frac{dx}{ds} = b(x,t) \end{cases} \tag{18}$$

Note that these two equations contain derivative w.r.t. s, but we actually want to know the ch. curve (the relationship between t and x). Eliminate the s to get

$$\frac{dx}{dt} = \frac{b(x,t)}{a(x,t)} \tag{19}$$

the solution to this ODE is just the c.h. curve.

After solving it out, do as above to denote $z(s) = u(x(s), t(s))$ as the function u along the c.h. curve.

$$z'(s) = c(x(s), t(s))u(x(s), t(s)) + d(x(s), t(s)) \tag{20}$$

$$z'(s) = c(x(s), t(s))z(s) + d(x(s), t(s)) \tag{21}$$

if $t(s_0) = 0$, then

$$z(s_0) = u(x(s_0), 0) = g(x(s_0)) \tag{22}$$

It's an ODE of $z(s)$ with initial value condition. After solving it out and replace the s with t, x, the PDE would be solved.

Example

Let's do an example, derive the explicit formula for a function u solving the initial value problem with $b \in \mathbb{R}^n, c \in \mathbb{R}$:

$$\begin{cases} u_t + b \cdot Du + cu = 0 \\ u(x,0) = g(x) \end{cases} \tag{23}$$

Apply the method of characteristics to assume that the c.h. curve is parameterized as $(x(s), t(s))$ and consider $z(s) = u(x(s), t(s))$.

$$z'(s) = x'(s) \cdot Du + u_t \frac{dt}{ds} \tag{24}$$

As a result, the c.h. equation should be

$$x'(t) = b \tag{25}$$

and the c.h. curve should be

$$x(t) = tb + C \tag{26}$$

with $C \in \mathbb{R}^n$ as any constant vector.

As a result, the PDE is transformed into an ODE w.r.t. z with initial value condition as

$$\begin{cases} z'(s) = -c \cdot z(s) \\ z(0) = u(C, 0) = g(C) \end{cases} \tag{27}$$

solve it to get that

$$z(s) = g(C) \cdot e^{-cs} \tag{28}$$

as a result, we have

$$u(sb + C, s) = g(C) \cdot e^{-cs} \tag{29}$$

$$t(s) = s \tag{30}$$

$$x(s) = sb + C \tag{31}$$

the solution to the PDE is

$$u(x, t) = g(x - tb) \cdot e^{-ct} \tag{32}$$

Week 2

Example

Solve the PDE

$$\begin{cases} u_t + x \cdot u_x = u \\ u(x,0) = x^2 \end{cases} \tag{33}$$

Let $z(s) = u(x(s), t(s))$, the c.h. ODE is

$$\frac{dx}{dt} = x \tag{34}$$

so the c.h. line is

$$x(t) = C \cdot e^t \ (C > 0) \tag{35}$$

Note that we can also infer an ODE with initial condition of $z(s)$:

$$\begin{cases} z'(s) = z(s) \\ z(0) = u(C,0) = C^2 \end{cases} \tag{36}$$

so the solution is

$$z(s) = C^2 e^s \tag{37}$$

As a result, $u(C \cdot e^t, t) = C^2 e^t$, note that $C = xe^{-t}$ and plug in to get the solution that

$$u(x,t) = x^2 e^{-t} \tag{38}$$

Solve the c.h. ODE, set up and solve the ODE w.r.t. z which is a parametrized curve of u along the direction of c.h. curve, at last replace everything with x, t

Cauchy Problem for General First-Order PDE

For the general setting, denote $u = u(x)$, $x \in \mathbb{R}^n$ (with the t coordinate merged into the x coordinate). $F(Du, u, x) = 0, x \in \Omega \subset \mathbb{R}^n$ with F assumed to be smooth in all its components. Note that for such $F = F(p, z, x)$, it's actually a function on $\mathbb{R}^{2n+1}$ (Du, x has n dimension and u has 1 dimension). Denote $D_x F \in \mathbb{R}^n$ as the gradient of F w.r.t. x, $D_p F \in \mathbb{R}^n$ as the gradient of F w.r.t. p, $D_z F \in \mathbb{R}$ as the derivative of F w.r.t. z.

The Cauchy problem is stated as

$$\begin{cases} F(Du,u,x)=0 \\ u(x)=g(x)\ (x\in\Gamma) \end{cases} \tag{39}$$

with the Γ as some curves in Ω.

The **general method of c.h.** should be applied to solve this PDE. For any point in Ω, find a curve (c.h. curve) such that the the point can go along this curve to get to Γ and the known function value on Γ will help. The question: how to find these curves?

Suppose u is a C^2 solution (assume stronger regularity condition, we will see the reason for this afterwards), $x(s)$ is the c.h. curve with $x(0)\in\Gamma$. Set

$$\begin{cases} z(s)=u(x(s)) \\ p(s)=Du(x(s)) \end{cases} \tag{40}$$

as **the version of u and Du along the c.h. curve** $x(s)$.

Note that

$$p_i(s)=u_{x_i}(x(s)) \tag{41}$$

$$p_i'(s)=\frac{d}{ds}\frac{\partial u(x(s))}{\partial x_i} \tag{42}$$

$$=\sum_{j=1}^n u_{x_ix_j}(x(s))\cdot x_j'(s)\ (i=1,2,...,n) \tag{43}$$

differentiate the PDE w.r.t. x_i to get an equation w.r.t. x:

$$\sum_{j=1}^n \frac{\partial F(Du,u,x)}{\partial p_j}u_{x_ix_j}(x)+\frac{\partial F(Du,u,x)}{\partial z}u_{x_i}(x)+\frac{\partial F(Du,u,x)}{\partial x_i}=0 \tag{44}$$

restrict this PDE on the c.h. curve to get an equation w.r.t. s:

$$\sum_{j=1}^n \frac{\partial F(p(s),z(s),x(s))}{\partial p_j}u_{x_ix_j}(x(s))+\frac{\partial F(p(s),z(s),x(s))}{\partial z}u_{x_i}(x(s))+\frac{\partial F(p(s),z(s),x(s))}{\partial x_i}=0 \tag{45}$$

further simplifications requires us to **assume that $x(s)$ satisfies**

$$\forall j=1,2,...,n,\ x_j'(s)=\frac{\partial F(p(s),z(s),x(s))}{\partial p_j} \tag{46}$$

this is because if such assumption holds,

$$p_i'(s) = \sum_{j=1}^{n} u_{x_i x_j}(x(s)) \cdot x_j'(s) \tag{47}$$

$$= \sum_{j=1}^{n} u_{x_i x_j}(x(s)) \cdot \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} \tag{48}$$

$$= -\frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) - \frac{\partial F(p(s), z(s), x(s))}{\partial x_i} \tag{49}$$

with the last equation using 45 to turn back to an ODE w.r.t. $p(s)$.

Similarly, figure out the expression for $z'(s)$

$$z'(s) = \sum_{j=1}^{n} u_{x_j}(x(s)) \cdot x_j'(s) \tag{50}$$

$$= \sum_{j=1}^{n} p_j(s) \cdot x_j'(s) \tag{51}$$

To summarize,

$$\begin{cases} x'(s) = D_p F(p(s), z(s), x(s)) \\ z'(s) = D_p F(p(s), z(s), x(s)) \cdot p(s) \\ p'(s) = -D_x F(p(s), z(s), x(s)) - D_z F(p(s), z(s), x(s)) \cdot p(s) \end{cases} \qquad (p, x \in \mathbb{R}^n, z \in \mathbb{R}) \tag{52}$$

The ODEs are called **the characteristic equations (altogether $2n+1$ equations)** for $F(Du, u, x) = 0$, with the dot in the equation of $z'(s)$ having the meaning of dot product. The inference above is concluded as a theorem.

Theorem 1. *If $u \in C^2$ solves the PDE $F(Du, u, x) = 0$ in $\Omega \subset \mathbb{R}^n$, assume that $x(s)$ solves the equation $x_j'(s) = \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}$ then $p(s), z(s)$ solves the original PDE.*

This is a generalization of the method of c.h. for the linear first-order PDE (where we don't have to use function p). (verified afterwards)

Now to solve the c.h. ODE system, still have to **impose initial values** $x(0), z(0), p(0)$. The initial values for x, z can be found easily, while $p(0)$ **remains to be a problem**.

$$x(0) \in \Gamma \tag{53}$$

$$z(0) = u(x(0)) = g(x(0)) \tag{54}$$

Motivation for the General Method of Characteristics

The statements above are mathematically correct. However, it's necessary to explain why we are doing all the things here and to make these statements intuitive.

It can seen above that the most important thought of the method of c.h. is to find some c.h. curves along which the function value can be easily determined. For first-order linear PDE, this point is clear since the linear combination of u_x, u_t can always be written as a directional derivative (the direction may contain x but has nothing to do with u). In other words, if we write the equation

$$u_t + bu_x = u \tag{55}$$

in the form of general first-order equation, then

$$F(Du, u, x) = 0 \tag{56}$$
$$F(p, z, x) = p \cdot b - z \tag{57}$$

it's easy to see that D_pF **is actually the direction we hope to follow**. That's exactly why we set $x'(s) = D_pF$ in the c.h. ODE to capture this special direction. Then $z(s)$ as the function u along $x(s)$ has the natural structure as $D_pF \cdot p$.

The dealing with $p(s)$ is more subtle. Since p is the function Du along $x(s)$, its derivative has to have something to do with the second-order derivatives of u. However, by taking derivatives of the original PDE w.r.t. x, the second-order derivatives of u are cancelled and replaced with first-order derivatives of F. Such operation enables us to keep the equation as a first-order one, but as the price to pay, C^2 assumption is required (for the existence of second-order derivatives and that $u_{x_ix_j} = u_{x_jx_i}$).

Example

Let's see how this method works for first-order linear PDE

$$B(x) \cdot Du + c(x) \cdot u = 0, \ (B \in \mathbb{R}^n, c, d \in \mathbb{R}) \tag{58}$$

First write out the general form:

$$F(Du, u, x) = 0 \tag{59}$$
$$F(p, z, x) = B(x) \cdot p + c(x) \cdot z \tag{60}$$

To set up the c.h. equations, note that first we should capture the c.h. direction, i.e.

$$x'(s) = D_pF = B(x(s)) \tag{61}$$

Then, set up $z(s) = u(x(s))$ as

$$z'(s) = \sum_{j=1}^{n} u_{x_j}(x(s))x_j'(s) = p(s) \cdot B(x(s)) \tag{62}$$

At last, set up $p(s) = Du(x(s))$ by taking the derivative w.r.t. x of the original PDE

$$\sum_i \frac{\partial F}{\partial p_i} u_{x_i x_j} + D_z F \cdot u_{x_j} + \frac{\partial F}{\partial x_j} = 0 \tag{63}$$

$$p_j'(s) = \sum_{i=1}^n u_{x_i x_j}(x(s)) \cdot x_i'(s) \tag{64}$$

$$p'(s) + D_x F + D_z F \cdot p(s) = 0 \tag{65}$$

$$p'(s) + z(s) \cdot \nabla c(x(s)) + \sum_{j=1}^n p_j(s) \cdot \nabla B_j(x(s)) + c(x(s)) \cdot p(s) = 0 \tag{66}$$

As a result, we get the c.h. equations

$$\begin{cases} x'(s) = B(x(s)) \\ z'(s) = p(s) \cdot B(x(s)) \\ p'(s) + z(s) \cdot \nabla c(x(s)) + \sum_{j=1}^n p_j(s) \cdot \nabla B_j(x(s)) + c(x(s)) \cdot p(s) = 0 \end{cases} \tag{67}$$

This equations seems to be very complicated. However, we might notice that the original PDE is

$$B(x(s)) \cdot p(s) + c(x(s)) \cdot z(s) = 0 \tag{68}$$

It's so good to find that **the c.h. equations reduce and $p(s)$ is not necessary any longer!**

$$\begin{cases} x'(s) = B(x(s)) \\ z'(s) = -c(x(s)) \cdot z(s) \end{cases} \tag{69}$$

Note that this is exactly what we would expect to get from the method of c.h. of the first-order linear PDE, so these two methods are actually consistent! The simplicity of linearity can thus be explained as the simplicity of c.h. equations, where it avoids the dependency on the initial value of $p(s)$ which is hard to derive.

Example

Let's compute a more specific example using the general method of c.h.

$$\begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u \ (x = (x_1, x_2) \in U) \\ u(x) = g(x) \ (x \in \Gamma) \end{cases} \tag{70}$$

where U is the open first quadrant and $\Gamma = \{x_1 > 0, x_2 = 0\}$.

Let's write it in the general form

$$F(Du,u,x)=0 \tag{71}$$
$$F(p,z,x)=B(x)\cdot p-z\ (p,x\in\mathbb{R}^2,z\in\mathbb{R}) \tag{72}$$
$$B(x)=(-x_2,x_1) \tag{73}$$

capture the c.h. direction

$$x'(s)=B(x) \tag{74}$$

form the equation of $z(s)=u(x(s))$

$$z'(s)=p(s)\cdot B(x) \tag{75}$$

and the equation of $p(s)=Du(x(s))$

$$p'(s)+D_zF\cdot p(s)+D_xF=0 \tag{76}$$
$$D_zF=-1,D_xF=p_1\nabla B_1+p_2\nabla B_2 \tag{77}$$

Now let's plug in $B(x)$ and derive the c.h. equations: (note that this is linear, so only equations for $x(s),z(s)$ are needed)

$$\begin{cases}x_1'(s)=-x_2\\x_2'(s)=x_1\\z'(s)=z(s)\end{cases} \tag{78}$$

Solve this out to get:

$$\begin{cases}x_1(s)=C\cos s\\x_2(s)=C\sin s\\z(s)=De^s\ (D>0)\end{cases} \tag{79}$$

with C,D as any fixed constant.

Now notice that $x(0)\in\Gamma$, so $z(0)=u(x(0))=g(x(0))=g(C,0)$. As a result, $D=g(C,0)$. On knowing that

$$u(C\cos s,C\sin s)=g(C,0)e^s \tag{80}$$

do the transformation such that $\begin{cases} x_1 = C\cos s \\ x_2 = C\sin s \end{cases}$ to get the **solution** that:

$$u(x_1, x_2) = g\left(\sqrt{x_1^2 + x_2^2}, 0\right) e^{\arctan \frac{x_2}{x_1}} \tag{81}$$

Let's do a final check to see whether this solution satisfies the c.h. equation of $p(s)$ which has been eliminated. The equation should be:

$$\begin{cases} p_1'(s) - p_1(s) + p_2(s) = 0 \\ p_2'(s) - p_2(s) - p_1(s) = 0 \end{cases} \tag{82}$$

Now we know $p(s) = Du(x(s))$, so

$$Du(x) = \begin{bmatrix} e^{\arctan \frac{x_2}{x_1}} \left[\frac{x_1}{\sqrt{x_1^2+x_2^2}} g_1'(\sqrt{x_1^2+x_2^2}, 0) - \frac{x_2}{x_1^2+x_2^2} g(\sqrt{x_1^2+x_2^2}, 0)\right] \\ e^{\arctan \frac{x_2}{x_1}} \left[\frac{x_2}{\sqrt{x_1^2+x_2^2}} g_1'(\sqrt{x_1^2+x_2^2}, 0) + \frac{x_1}{x_1^2+x_2^2} g(\sqrt{x_1^2+x_2^2}, 0)\right] \end{bmatrix} \tag{83}$$

$$p(s) = \begin{bmatrix} e^s \left[\cos s \cdot g_1'(C, 0) - \sin s \cdot g(C, 0)\right] \\ e^s \left[\sin s \cdot g_1'(C, 0) + \cos s \cdot g(C, 0)\right] \end{bmatrix} \tag{84}$$

It's then quite obvious to see that the c.h. equation for $p(s)$ actually holds (but we won't have to deal with it).

First-Order Quasi-Linear PDE

Actually, not only the first-order linear PDE can get rid of the function p, a special type of PDE called **first-order quasi-linear PDE** also has no dependency on p. The general form of this kind of PDE is

$$B(x, u) \cdot Du + c(x, u) = 0 \tag{85}$$

in that it's linear w.r.t. the highest order derivative of the unknown function (which is Du here).

Write it in the general form:

$$F(Du, u, x) = 0 \tag{86}$$

$$F(p, z, x) = B(x, z) \cdot p + c(x, z) \tag{87}$$

capture the c.h. direction $x(s)$:

$$x'(s) = B(x(s), z(s)) \tag{88}$$

get the equation for $z(s) = u(x(s))$:

$$z'(s) = B(x(s), z(s)) \cdot p(s) \tag{89}$$

use the original PDE to get the c.h. equations

$$\begin{cases} x'(s) = B(x(s), z(s)) \\ z'(s) = -c(x(s), z(s)) \end{cases} \tag{90}$$

which also has nothing to do with $p(s)$.

Example

Let's look at a first-order quasi-linear PDE:

$$\begin{cases} u_{x_1} + u_{x_2} = u^2 \ (x \in U = \{x_2 > 0\}) \\ u(x) = g(x) \ (x \in \Gamma = \partial U) \end{cases} \tag{91}$$

with the non-linearity lying in u^2 but still quasi-linear.

Let's follow the steps to use the method of c.h., write it in general form

$$F(Du, u, x) = 0 \tag{92}$$

$$F(p, z, x) = B \cdot p - z^2 \tag{93}$$

$$B = (1, 1) \tag{94}$$

capture the c.h. direction

$$x'(s) = B \tag{95}$$

and write out the equation for $z(s) = u(x(s))$

$$z'(s) = p(s) \cdot B = z^2(s) \tag{96}$$

the c.h. equations are

$$\begin{cases} x_1'(s) = 1 \\ x_2'(s) = 1 \\ z'(s) = z^2(s) \end{cases} \tag{97}$$

Solve it to know that

$$z(s) = -\frac{1}{s+C} \tag{98}$$
$$x_1(s) = s + D \tag{99}$$
$$x_2(s) = s + E \tag{100}$$

figure out the initial value that

$$x(-E) = (D-E, 0) \in \Gamma \tag{101}$$
$$z(-E) = \frac{1}{E-C} \tag{102}$$
$$z(-E) = u(x(-E)) = u(D-E, 0) = g(D-E, 0) \tag{103}$$

So we get the final solution

$$u(s+D, s+E) = -\frac{1}{s+E-\frac{1}{g(D-E,0)}} \tag{104}$$
$$x_1 - x_2 = D - E \tag{105}$$
$$u(x_1, x_2) = -\frac{1}{x_2 - \frac{1}{g(x_1-x_2,0)}} \tag{106}$$

Nonlinear First-Order PDE

When the equation is even not quasi-linear, $p(s)$ will be necessary for us to solve out the c.h. equations. Let's look at an example to illustrate **how to construct initial value for function** p.

$$\begin{cases} u_{x_1} \cdot u_{x_2} = u \ (x \in U = \{x_1 > 0\}) \\ u(x) = x_2^2 \ (x \in \Gamma = \partial U) \end{cases} \tag{107}$$

This equation is even not quasi-linear since there is the product of two partial derivatives. Let's turn it into the general form

$$F(Du, u, x) = 0 \tag{108}$$
$$F(p, z, x) = p_1 p_2 - z \tag{109}$$

capture the ch. direction

$$x'(s) = D_p F = (p_2, p_1) \tag{110}$$

set up the equation for $z(s)$ (The original PDE tells us that $z = p_1 p_2$)

$$z'(s) = 2p_1(s)p_2(s) = 2z(s) \tag{111}$$

and the equation for $p(s)$

$$p'(s) = -D_z F \cdot p(s) - D_x F \tag{112}$$
$$p'(s) = p(s) \tag{113}$$

so the c.h. equations are

$$\begin{cases} x_1'(s) = p_2(s) \\ x_2'(s) = p_1(s) \\ z'(s) = 2z(s) \\ p_1'(s) = p_1(s) \\ p_2'(s) = p_2(s) \end{cases} \tag{114}$$

Now we assume that $x(0) = (0, C) \in \Gamma$. We are doing this because we need to fix the initial value condition of $p(s)$ in order to solve out $p(s)$ to proceed and to avoid having too many parameters

$$z(0) = u(x(0)) = u(0, C) = C^2 \tag{115}$$

However, the problem here is that the simplest equation is the one w.r.t. $p'(s)$ but we know nothing about the initial value of p. Actually, the information for p is already given, but hidden in the other conditions. **Turn back to the original PDE** to get

$$p_1(0)p_2(0) = z(0) = C^2 \tag{116}$$
$$\tag{117}$$

The last remaining initial value condition can be figured out by **taking derivative on both sides of the initial value condition of the original PDE w.r.t.** x_2

$$u_{x_2}(x(0)) = 2x_2(0) \ (x(0) \in \Gamma) \tag{118}$$
$$u_{x_2}(x(0)) = p_2(0) = 2x_2(0) = 2C \tag{119}$$

As a result, now we all the c.h. equations and initial value conditions and these equations can be solved

$$\begin{cases} p_1(s) = \frac{C}{2}e^s \\ p_2(s) = 2Ce^s \\ z(s) = C^2e^{2s} \\ x_1(s) = 2C(e^s-1) \\ x_2(s) = \frac{C}{2}(e^s+1) \end{cases} \tag{120}$$

As a result, get the final solution by

$$\begin{cases} e^s = \frac{4x_2+x_1}{4x_2-x_1} \\ C = \frac{4x_2-x_1}{4} \end{cases} \tag{121}$$

$$u(x_1,x_2) = \frac{(x_1+4x_2)^2}{16} \tag{122}$$

Remark. *When applying the method of characteristics in the non-linear case, always solve out $p(s)$ first since $x(s), z(s)$ depend on $p(s)$. If it's hard to find the initial value condition, assume that there exists some s_0 such that $x(s_0) \in \Gamma$ to make use of the initial value condition of the PDE and to simplify the calculations. The initial value condition of $p(s)$ always comes from the original PDE and taking derivatives w.r.t. the initial value condition of the PDE.*

General Theory for Characteristic Method

For general case, assume that Γ is parameterized by $x = f(y)$ with parameter $y \in D \subset \mathbb{R}^{n-1}$. Fix $y_0 \in D, x_0 = f(y_0), z_0 = u(x_0) = g(x_0) = g(f(y_0)) \overset{def}{=} h(y_0)$ (since $x_0 \in \Gamma$). The reason to change variable x into y is that x may have to be on some surface (it depends on how the Γ looks) but y has no such restrictions. In other words, if we deal with x directly, we may have to operate on manifolds while changing the variable as y flattens the boundary Γ so it's now the analysis in the Euclidean space $\mathbb{R}^{n-1}$.

The PDE at x_0 now becomes

$$F(Du(x_0), u(x_0), x_0) = 0 \tag{123}$$

$$F(p_0, z_0, f(y_0)) = 0 \tag{124}$$

with $p_0 = Du(x_0) = Dg(x_0)$ since $x_0 \in \Gamma$.

Note that it's necessarily true that under such circumstance (in the following context, use **subscript for partial**

derivative and **superscript for coordinates**)

$$h_{y_j}(y_0) = \frac{\partial (g\circ f)(y)}{\partial y_j}\Big|_{y=y_0} \tag{125}$$

$$= \sum_{i=1}^{n} \frac{\partial g(f(y))}{\partial f_i} \cdot \frac{\partial f_i(y)}{\partial y_j}\Big|_{y=y_0} \tag{126}$$

$$= \sum_{i=1}^{n} \frac{\partial g(x_0)}{\partial x_i} \cdot \frac{\partial f_i(y_0)}{\partial y_j} = \sum_{i=1}^{n} \frac{\partial g(x_0)}{\partial f_i} \cdot \frac{\partial f_i(y_0)}{\partial y_j} = \sum_{i=1}^{n} p_0^i \cdot f_{y_j}^i(y_0) \tag{127}$$

Actually, given $y_0 \in D$, if the following conditions are satisfied

$$\begin{cases} \forall j = 1, 2, ..., n-1, h_{y_j}(y_0) = \sum_{i=1}^n p_0^i \cdot f_{y_j}^i(y_0) \\ F(p_0, h(y_0), f(y_0)) = 0 \end{cases} \tag{128}$$

we say p_0 is **admissible** at y_0. It's quite obvious to see that there are in all n equations for the unknowns and these conditions are called **compatibility conditions**. They are the conditions that p_0 must satisfy for the PDE to hold at $y = y_0$. The admissible conditions are important since it provides the **minimum condition to satisfy for an appropriate initial condition for** p_0, y_0. (Imagine the situation where initial value p_0 is provided but not satisfy the compatibility conditions, there's no way for the PDE to have solution! That's why we always add this condition when constructing the solution.)

To simplify the notations, denote

$$\begin{cases} \mathscr{F}_j(p, y) = \sum_{i=1}^n f_{y_j}^i(y)p^i - h_{y_j}(y) \ (j = 1, ..., n-1) \\ \mathscr{F}_n(p, y) = F(p, h(y), f(y)) \end{cases} \quad (p \in \mathbb{R}^n, y \in D \subset \mathbb{R}^{n-1}) \tag{129}$$

$$\mathscr{F} = (\mathscr{F}_1, ..., \mathscr{F}_n) : \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{R}^n \tag{130}$$

then (p_0, y_0) is admissible iff $\mathscr{F}(p_0, y_0) = 0$. (The natural compatibility condition provides us with n equations to satisfy, which will be important in future use)

Call an **admissible** (p_0, y_0) **to be non-characteristic** if $\det\left(\frac{\partial \mathscr{F}(p,y)}{\partial p}\right)\Big|_{(p_0,y_0)} \neq 0$, i.e.

$$\det \begin{bmatrix} f_{y_1}^1(y_0) & ... & f_{y_1}^n(y_0) \\ ... & ... & \\ f_{y_{n-1}}^1(y_0) & ... & f_{y_{n-1}}^n(y_0) \\ F_{p_1}(p_0, z_0, x_0) & ... & F_{p_n}(p_0, z_0, x_0) \end{bmatrix} \neq 0 \tag{131}$$

An explanation is required for this definition of non-characteristic. A direct one is based on the observation that

$$\begin{bmatrix} F_{p_1}(p_0, z_0, x_0) & ... & F_{p_n}(p_0, z_0, x_0) \end{bmatrix} \tag{132}$$

is the c.h. direction (tangent direction of c.h. curve) at $x_0 \in \Gamma$ (recall the c.h. direction $x'(s) = D_pF$ mentioned above). The other rows

$$\begin{bmatrix} f^1_{y_j}(y_0) & ... & f^n_{y_j}(y_0) \end{bmatrix} \tag{133}$$

typically stands for $\frac{\partial f}{\partial y_j}\Big|_{y=y_0}$, i.e. the tangent vectors for surface Γ at point x_0. A **geometric interpretation** for the non-c.h. property is that: **the tangent vector of the c.h. curve at x_0 is not in the tangent space of surface Γ at x_0.** Note that $x = f(y), f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ gives the equation of the surface Γ.

Nevertheless, there is also an **analytic interpretation** of this property. Note that by implicit function theorem, $\det\left(\frac{\partial \mathscr{F}(p,y)}{\partial p}\right)\Big|_{(p_0,y_0)} \neq 0$ tells us that locally near (p_0, y_0) (exists a neighborhood), there exists a function $p = q(y)$ such that

$$p_0 = q(y_0) \tag{134}$$

$$\mathscr{F}(q(y), y) = \mathscr{F}(p_0, y_0) = 0 \ (\forall y \textit{ close to } y_0) \tag{135}$$

providing the intuition that the non-c.h. condition enables us to **build a local functional relationship between y_0 and p_0**. Note that the equations given by the compatibility conditions $\mathscr{F}(p_0, y_0) = 0$ help construct the function relationship in the implicit function theorem. This point of view is important because we are trying to turn everything into the setting of y, e.g. $x = f(y)$, and if now p_0 can be written as a function of y_0, we will be only working with y locally.

Week 3

From now on, assume that (p_0, y_0) admissible and non-c.h. As stated above, exists a function q defined locally at y_0 such that $p_0 = q(y_0)$. Let $(p(s), z(s), x(s))$ be the solution of c.h. ODEs with initial values $p(0) = q(y), z(0) = h(y), x(0) = f(y)$ for $y \in U$, which is a neighborhood of y_0 where q is defined, we can write the solution as $(p(s, y), z(s, y), x(s, y))$ (the solution depends on both s and y).

Lemma 1. *Let (p_0, y_0) be admissible and non-c.h., $x_0 = f(y_0)$, then exists an open interval I of 0 and a neighborhood U of y_0, a neighborhood V of x_0, such that $\forall x \in V$, there exists unique $s = s(x) \in I, y = y(x) \in U$, such that $x = x(y, s)$. Moreover, $s(x), y(x)$ are C^2 in x.*

Proof. Want to apply the **inverse function theorem** to get the dependency of s, y on x. Note that the only condition to satisfy is the existence of the inverse of the Jacobian at $s = 0, y = y_0$. By the functional relationship between x, y and x, s, the Jacobian can be calculated as follows:

$$x = f(y), x = x(s) \tag{136}$$

$$\frac{\partial x_j}{\partial s} = x_j'(s)\Big|_{s=0,y=y_0} = F_{p_j}(p_0, z_0, x_0) \tag{137}$$

$$\frac{\partial x_j}{\partial y_i} = f^j_{y_i}(y_0) \tag{138}$$

$$\frac{\partial x(y, s)}{\partial (y, s)}\Big|_{s=0,y=y_0} = \left(\frac{\partial \mathscr{F}(p, y)}{\partial p}\right)^T\Big|_{y=y_0} \tag{139}$$

By non-c.h. property, the Jacobian is invertible, meaning that the inverse function theorem holds. That's why there exists unique $s = s(x), y = y(x)$ locally near $s = 0, y = y_0$ and the C^2 property is preserved.

$\square$

Note that originally $u(x) = z(y, s)$ (the parametrization of c.h. curve and the initial value condition) but now we can locally set up s, y as functions of x, so $z(y, s) = z(y(x), s(x))$, therefore we find the solution to the original PDE with condition on Γ.

Theorem 2. ***(Local Unique Existence of Solution by the Method of C.h.)*** *Let (p_0, y_0) be admissible and non-c.h., then function u defined by $u(x) = z(s(x), y(x))$ solves the PDE $F(Du, u, x) = 0$ on V with the Cauchy condition $u(x) = g(x)$ on surface $\Gamma \cap V$.*

Proof. Fix y such that $f(y) \in \Gamma$ is close to x_0 and solve the c.h. ODEs to get the solutions $p(s) = p(y, s), z(s) = z(y, s), x(s) = x(y, s)$. Now if $f(y)$ is close enough to x_0, then by compatibility conditions,

$$F(p(y, 0), z(y, 0), x(y, 0)) = F(p_0, h(y_0), f(y_0)) = 0 \tag{140}$$

and also

$$\frac{\partial}{\partial s}F(p(y,s),z(y,s),x(y,s)) = \sum_{j=1}^n D_{p_j}F\cdot p_j'(s) + D_zF\cdot z'(s) + \sum_{j=1}^n D_{x_j}F\cdot x_j'(s) \tag{141}$$

$$= \sum_{j=1}^n D_{p_j}F\cdot(-D_zF\cdot p_j(s) - D_{x_j}F) + D_zF\cdot(\sum_{j=1}^n D_{p_j}F\cdot p_j(s)) + \sum_{j=1}^n D_{x_j}F\cdot D_{p_j}F \tag{142}$$

$$= 0 \tag{143}$$

by replacing terms using c.h. ODEs.

Now that $F(p(y,0),z(y,0),x(y,0)) = 0$, $\frac{\partial}{\partial s}F(p(y,s),z(y,s),x(y,s)) = 0$, it's natural to conclude that

$$\forall s\in I, F(p(y,s),z(y,s),x(y,s)) = 0 \tag{144}$$

By applying the local intervibility lemma above changing y,s into x, we conclude that

$$\forall x\in V, F(p(x),u(x),x) = 0 \tag{145}$$

Now to conclude that this is actually a solution to the original PDE $F(Du,u,x) = 0$, the only step left is to prove that $\forall x\in V, p(x) = Du(x)$.

$$\forall j, u_{x_j}(x) = \frac{\partial}{\partial x_j}z(s(x),y(x)) \tag{146}$$

$$= z_s\cdot s_{x_j} + \sum_{i=1}^{n-1} z_{y_i}y^i_{x_j} \tag{147}$$

It's clear to us that by c.h. ODEs,

$$z_s = \sum_{j=1}^n p^j(y,s)\cdot x^j_s(y,s) \tag{148}$$

If we can show that

$$z_{y_i} = \sum_{j=1}^n p^j(y,s)\cdot x^j_{y_i}(y,s) \tag{149}$$

then the theorem is proved since

$$\begin{aligned}
\forall j, u_{x_j}(x) &= \sum_{k=1}^{n} p^k \cdot x_s^k \cdot s_{x_j} + \sum_{i=1}^{n-1}\sum_{k=1}^{n} p^k \cdot x_{y_i}^k \cdot y_{x_j}^i && (150)\\
&= \sum_{k=1}^{n} p^k \left(x_s^k \cdot s_{x_j} + \sum_{i=1}^{n-1} x_{y_i}^k \cdot y_{x_j}^i \right) && (151)\\
&= \sum_{k=1}^{n} p^k x_{x_j}^k = \sum_{k=1}^{n} p^k \delta_{jk} = p^j && (152)
\end{aligned}$$

so $\forall x \in V, p(x) = Du(x)$ and $F(Du, u, x) = 0$. Note that here x_k is a function of s, y and s, y depends on x_j, by using the chain rule one would get

$$x_s^k \cdot s_{x_j} + \sum_{i=1}^{n-1} x_{y_i}^k \cdot y_{x_j}^i = x_{x_j}^k \overset{def}{=} \frac{\partial x_k}{\partial x_j} \tag{153}$$

This tells us that under such conditions, **the solution to the c.h. ODEs is the locally unique solution to the original PDE**.

The last thing to prove is the expression for z_{y_i}, the approach is to set $r^i(s) = z_{y_i} - \sum_{j=1}^{n} p^j(y,s) \cdot x_{y_i}^j(y,s)$ and prove that it's 0. The strategy is to form an ODE w.r.t. $r^i(s)$ and to argue that such ODE with given initial conditions must give the zero solution. (Evans P108 for specific calculations) The constructed ODE is

$$\begin{aligned}
\frac{d}{ds} r^i(s) &= -D_z F \cdot r^i(s) && (154)\\
r^i(0) &= 0 && (155)
\end{aligned}$$

so $\forall s \in I, \forall i, r^i(s) = 0$, it's proved.

□

Remark. *Although the calculations are too much, we shall capture the spirit of the theorem. If there is a* ***first-order PDE*** *with compatible boundary value condition (**admissible** – **compatibility conditions**) and at a certain point x_0 on the boundary, the tangent vector at x_0 of the c.h. curve is not in the tangent space at x_0 of the boundary surface Γ (**non-characteristic**), then the* ***solution*** *to the PDE* ***locally uniquely exists around*** *x_0, and such solution can be derived by* ***solving the c.h. ODEs*** *(method of c.h.).*

Example

Let's look at some cases where the non-c.h. condition fails. The PDE is on $\mathbb{R}^2$

$$x \cdot u_x + y \cdot u_y = u \tag{156}$$

with the Cauchy condition given on the diagonal

$$\forall \tau, u(\tau,\tau) = \tau \tag{157}$$

This is a first-order linear PDE and we can write out the general form of this Cauchy problem:

$$F(Du, u, x) = 0 \tag{158}$$
$$F(p, z, x) = x_1 p_1 + x_2 p_2 - z \tag{159}$$
$$\Gamma = \{(x, y) | x = y\} \tag{160}$$

compute the c.h. ODEs

$$\begin{cases} x'(s) = x(s) \\ y'(s) = y(s) \\ z'(s) = z(s) \end{cases} \tag{161}$$

notice that $\frac{y(s)}{x(s)}$ is always a constant, so all possible c.h. curves are the lines passing through the origin. However, fix any point $x_0 \in \Gamma$, the c.h. curve is then fixed as $y = x$, which is exactly the same as Γ, which violates the non-c.h. condition! This is how we see that non-c.h. does not hold from the geometric point of view. From the analytic point of view, calculate the tangent vectors: ($x = f^1(\tau) = \tau, y = f^2(\tau) = \tau$ gives the parametrization of Γ)

$$\forall (a, a) \in \Gamma, \frac{\partial \mathscr{F}}{\partial p} = \begin{bmatrix} f^1_\tau & f^2_\tau \\ F_{p_1} & F_{p_2} \end{bmatrix} \tag{162}$$
$$= \begin{bmatrix} 1 & 1 \\ x & y \end{bmatrix} \tag{163}$$
$$\frac{\partial \mathscr{F}}{\partial p}\Big|_{(a,a)} = \begin{bmatrix} 1 & 1 \\ a & a \end{bmatrix} \tag{164}$$

It's obvious that the matrix is not invertible. By definition, non-c.h. condition fails.

So what's the consequence of the failure of non-c.h. condition? Solve the c.h. equations to see that the solution changes depending on the initial values. If consider $x(0) = 0, y(0) = 0, z(0) = u(0, 0) = 0$ then the solution is trivial

$$\begin{cases} x(s) = 0 \\ y(s) = 0 \\ z(s) = 0 \end{cases} \tag{165}$$

If consider $x(a) = a, y(a) = a, z(a) = u(a,a) = a$ $(a > 0)$ then the solution is

$$\begin{cases} x(s) = a \cdot e^{s-a} \\ y(s) = a \cdot e^{s-a} \\ z(s) = a \cdot e^{s-a} \end{cases} \tag{166}$$

As a result,

$$u(a \cdot e^{s-a}, a \cdot e^{s-a}) = a \cdot e^{s-a} \ (\forall a \in \mathbb{R}) \tag{167}$$

$$u(x,y) = \alpha x + (1-\alpha)y \ (\forall \alpha \in \mathbb{R}) \tag{168}$$

the solution is any linear combination of x and y. The local existence theorem fails (if it holds there should be a unique solution locally) and now there are **infinitely many solutions due to the failure of non-c.h. condition**.

Example

This example has exactly the same PDE as above, but the boundary value condition changes into

$$\forall \tau, u(\tau, \tau) = 1 \tag{169}$$

It's easy to see that the c.h. ODEs and the Γ all remain the same so non-c.h. condition still fails. However, if we still try to solve the c.h. ODEs, we will find that

$$\begin{cases} x(s) = C_x \cdot e^{s} \\ y(s) = C_y \cdot e^{s} \\ z(s) = C_z \cdot e^{s} \end{cases} \tag{170}$$

with $\forall s, z(s) = 1$, which is obviously impossible. As a result, there's no solution to this c.h. ODE. It's natural to assert that there should also be no solution to the original PDE, but it requires more work.

If there is a solution u to the original PDE, consider $h(x) = u(x,x)$ to be the function along the line $y = x$. As a result, $h'(x) = u_x(x,x) + u_y(x,x)$. Since the boundary value condition tells us $\forall x, h(x) = 1$, we know that $h'(x) \equiv 0$.

$$\forall x, u_x(x,x) + u_y(x,x) = 0 \tag{171}$$

Since u is a solution, $\forall x \neq 0, u_x(x,x) + u_y(x,x) = \frac{u(x,x)}{x}$, that is to say

$$\forall x \neq 0, u(x,x) = 0 \tag{172}$$

which is a contradiction with the boundary value condition that $u(x,x) = 1$!

Example

This example still has exactly the same PDE as above, but the boundary value condition changes into

$$u(x, 0) = g(x) \tag{173}$$

Now the c.h. ODEs remain the same but the boundary curve changes into

$$\Gamma = \{(x, y)|y = 0\} \tag{174}$$

which is the x-axis. So for any x_0 on this curve, the c.h. curve is always the x-axis, which violates the non-c.h. condition.

Solve the c.h. ODEs with initial value conditions $x(a) = a, y(a) = 0, z(a) = u(a, 0) = g(a)$ to get

$$\begin{cases} x(s) = a \cdot e^{s-a} \\ y(s) = 0 \\ z(s) = g(a) \cdot e^{s-a} \end{cases} \tag{175}$$

so $u(a \cdot e^{s-a}, 0) = g(a) \cdot e^{s-a}$, get the solution:

$$u(x, y) = \frac{g(a)}{a} x \tag{176}$$

$$u(0, 0) = g(0) \cdot e^s \tag{177}$$

This might seem weird, but when $a = 0$ we cannot eliminate the s contained in $u(0, 0)$. As a result, we might guess that $g(0) = 0$ has to hold so that this PDE has a solution. Moreover, in order to get rid of $\frac{g(a)}{a}$ for any $a \neq 0$, we might guess that $\frac{g(a)}{a}$ has to be a fixed constant, i.e. g **has to be a linear function such that the PDE has solution**.

Let's check whether this condition makes sense. When g is linear, i.e. $g(x) = kx$, then

$$u(x, y) = kx \tag{178}$$

and we can check that this is a solution to the PDE.

On the other hand, if there exists u as the solution to the PDE, then consider $g(x) = u(x, 0)$ along the x-axis,

$$g'(x) = u_x(x, 0) \tag{179}$$

the original PDE tells us that

$$x \cdot u_x(x, 0) = u(x, 0) \tag{180}$$

to derive an ODE w.r.t. g

$$x \cdot g'(x) = g(x) \tag{181}$$

and it's obvious that all solutions to this ODE are linear function, so g has to be linear.

To sum up, we proved that this PDE with boundary value condition

$$\begin{cases} x \cdot u_x + y \cdot u_y = u \\ u(x, 0) = g(x) \end{cases} \tag{182}$$

has solution if and only if g is a linear function.

Remark. *The three examples show us that when non-c.h. condition is violated, there might exists infinitely many solutions or no solution for a PDE. There also might be the case that the existence of solution depends on the properties of a certain function involved in the boundary value condition.*

Example

The last example is one in multi-dimensional space. Consider the PDE with boundary value conditions

$$\begin{cases} x \cdot u_x = \alpha u \ (x \in \mathbb{R}^n) \\ u(x_1, ..., x_{n-1}, 1) = h(x_1, ..., x_{n-1}) \end{cases} \tag{183}$$

To write in the general form,

$$F(Du, u, x) = 0 \tag{184}$$

$$F(p, z, x) = x \cdot p - \alpha z \tag{185}$$

$$\Gamma = \{x | x_n = 1\} \tag{186}$$

First judge whether the non-c.h. condition holds. The tangent vector of the c.h. curve is

$$D_p F = x \tag{187}$$

it's clear that for any $x_0 \in \Gamma$, x_0 won't be in the tangent space at x_0 of Γ (just draw a plot, easy to see), so the non-c.h. condition holds, and local unique existence of the solution is ensured! To solve out the locally unique solutions, the method of c.h. must work, ensured by the theorem stated above.

Write out the c.h. ODEs

$$\begin{cases} x'(s) = x(s) \\ z'(s) = p(s) \cdot x(s) = \alpha z(s) \end{cases} \tag{188}$$

and assume that $\exists s_0, x_n(s_0) = 1$, so $z(s_0) = h(x_1(s_0), ..., x_{n-1}(s_0))$. Solve out the solution to c.h. ODEs with those initial value conditions

$$\begin{cases} x_1(s) = x_1(s_0) \cdot e^{s-s_0} \\ ... \\ x_{n-1}(s) = x_{n-1}(s_0) \cdot e^{s-s_0} \\ x_n(s) = e^{s-s_0} \\ z(s) = h(x_1(s_0), ..., x_{n-1}(s_0)) \cdot e^{\alpha(s-s_0)} \end{cases} \tag{189}$$

Eliminate the s and replace with x to get the solution

$$u(x_1(s_0) \cdot e^{s-s_0}, ..., x_{n-1}(s_0) \cdot e^{s-s_0}, e^{s-s_0}) = h(x_1(s_0), ..., x_{n-1}(s_0)) \cdot e^{\alpha(s-s_0)} \tag{190}$$

$$u(x) = h\left(\frac{x_1}{x_n}, ..., \frac{x_{n-1}}{x_n}\right) \cdot (x_n)^\alpha \tag{191}$$

Actually we see that this is not only a **local** solution but also a **global** one for $\alpha \geq 0$. However, if $\alpha < 0$, there will be singularity for the solution $u(x)$ since it won't be defined at the origin so the local existence of the solution in a neighborhood of the origin will be a problem (the solution blows up).

We can actually see that when it comes to the local existence of the solution in a neighborhood of the origin, the only "nice" (not blows up) solution is the trivial solution $u \equiv 0$. (Since the method of c.h. must work because the PDE is non-c.h., eliminating the nontrivial solutions given above, there's only the trivial zero solution left)

Property of the Solution

As we have seen in the context above, sometimes we want to prove the non-existence of the solution to the PDE or the equivalent condition such that solution to the PDE exists. We have so far proved that **for PDE**

$$x \cdot u_x + y \cdot u_y = u \tag{192}$$

when the boundary value condition is given as $\forall \tau, u(\tau, \tau) = 1$, it violates the non-c.h. condition and the solution does not exist while if the boundary value condition is given as $\forall x, u(x, 0) = g(x)$, it violates the non-c.h. condition and the solution exists if and only if g is a linear function.

Those conclusions are still natural to see from the method of characteristics, but most of the times the proof on the property of the solution without solving the PDE can be tricky. Let's list some examples below to remind the readers of some useful techniques.

Consider Ω as the closed unit ball in $\mathbb{R}^n$ and $u \in C^1(\Omega)$ as the solution to the PDE

$$B(x) \cdot Du(x) = -u \tag{193}$$

if $\forall x \in \partial\Omega, B(x) \cdot x > 0$ $(B \in \mathbb{R}^n)$, then $\forall x \in \Omega, u(x) = 0$.

Since u is continuous on the compact set Ω, the maximum and minimum exists. Consider where the max/min is attained. It's clear that since $u \in C^1$, if u attains maximum in the interior of Ω at $x_0 \in \Omega^\circ$ then $Du(x_0) = 0$. By the PDE, $u(x_0) = 0$.

Now let's consider the case where the maximum of u is attained on the boundary at $x_0 \in \partial\Omega$. Consider $g(t) = x_0 + tB(x_0) : \mathbb{R} \to \mathbb{R}^n$ (**perturbation inside** Ω). Note that since $B(x_0) \cdot x_0 > 0$, $g(t) \in \Omega$ if $t \to 0^-$. As a result, $h(t) = u(g(t))$ (**perturbed version of** u) is well-defined in a small enough left neighborhood of 0 with

$$||x_0 + tB(x_0)||^2 = ||x_0||^2 + t^2||B(x_0)||^2 + 2t(x_0 \cdot B(x_0)) \leq 1 \ (t \to 0^-) \tag{194}$$

compute the derivative and use the original PDE to find

$$\lim_{t \to 0^-} h'(t) = Du(x_0) \cdot B(x_0) = -u(x_0) \tag{195}$$

since x_0 attains maximum, $\lim_{t\to 0^-} h'(t) \geq 0$, that's why $u(x_0) \leq 0$.

To conclude, we have proved that $\max_{x\in\Omega} u(x) \leq 0$. For the same reason, $\min_{x\in\Omega} u(x) \geq 0$, that's why u is constantly 0.

Remark. *Intuitively, let's imagine that $x_0 \in \Omega$ is close enough to the boundary and the angle formed by $B(x_0)$ and x_0 is small enough (not only acute). When the gradient $Du(x_0)$ is pointing outward Ω, it forms an acute angle with $B(x_0)$ so $u(x_0) < 0$ and this is telling us that if a point near boundary has negative function value, going towards the boundary increases its function value. When the gradient $Du(x_0)$ is pointing inward Ω, it forms a blunt angle with $B(x_0)$ so $u(x_0) > 0$ and this is telling us that if a point near boundary has positive function value, going towards the boundary decreases its function value. As a result, one would expect to see that such u is constantly 0.*

The trick we have applied here is the classic perturbation trick, a very useful trick when we have conditions on the sign of the inner product.

Consider $a, h \in C^\infty(\mathbb{R})$ **and the quasilinear PDE**

$$\begin{cases} a(u) \cdot u_x + u_y = 0 \\ u(x, 0) = h(x) \end{cases} \tag{196}$$

the solution to the PDE becomes singular for some $y > 0$ **unless** $a(h(z))$ **is nondecreasing in** z**.**

First, let's figure out the solution to this PDE using the method of c.h. Write it in the general form

$$F(Du, u, x) = 0 \tag{197}$$

$$F(p, z, x) = a(z)p_1 + p_2 \tag{198}$$

the c.h. direction is then $x'(s) = D_pF = (a(z(s)), 1)$, with the function along the c.h. curve to be $z(s) = u(x(s))$ and the gradient to be $p(s) = Du(x(s))$. The ODE w.r.t. $z(s)$ should be

$$z'(s) = p(s) \cdot (a(z(s)), 1) = 0 \tag{199}$$

So the c.h. ODEs are

$$\begin{cases} x'(s) = a(z(s)) \\ y'(s) = 1 \\ z'(s) = 0 \end{cases} \tag{200}$$

now why don't we assume that $z(0) = C$, so $\forall s, z(s) = C$, a fixed constant. There exists s_0 such that $y(s_0) = 0$ so $z(s_0) = u(x(s_0), 0) = h(x(s_0)) = C$. As a result, the solution is

$$\begin{cases} x(s) = a(C) \cdot (s - s_0) + h^{-1}(C) \\ y(s) = s - s_0 \\ z(s) = C \end{cases} \tag{201}$$

by eliminating s_0, C, get that

$$u(x, y) = C \tag{202}$$

$$x = a(C) \cdot y + h^{-1}(C) \tag{203}$$

note that here h^{-1} is just a notation but not the rigorous inverse and there's no way to solve out C explicitly, so we form it as a functional equation

$$u = h(x - a(u) \cdot y) \tag{204}$$

Denote $s = x - a(u) \cdot y$ so the equation becomes

$$h(s) = u(s + a(u) \cdot y, y) \ \forall s, y \tag{205}$$

if $a \circ h$ violates the nondecreasing property, then there exists $s_1 < s_2$ such that $a(h(s_1)) > a(h(s_2))$. Note that there exists y_0 such that $s_1 + a(h(s_1)) \cdot y_0 = s_2 + a(h(s_2)) \cdot y_0$ (denote this value as x_0), i.e.

$$y_0 = -\frac{s_1 - s_2}{a(h(s_1)) - a(h(s_2))} > 0 \tag{206}$$

now the solution is singular at $y = y_0$ because

$$u(x_0, y_0) = u(s_1 + a(h(s_1)) \cdot y_0, y_0) = h(s_1) \tag{207}$$

$$u(x_0, y_0) = u(s_2 + a(h(s_2)) \cdot y_0, y_0) = h(s_2) \tag{208}$$

$h(s_1) = h(s_2)$, so $a(h(s_1)) = a(h(s_2))$, a contradiction!

Remark. *The trick we have applied here is to use the structure of the functional equation of u to find x_0, y_0 such* *that when u is replaced with h at that point, the functional equation still holds for h, so the solution can be represented by function h.*

Consider the PDE

$$u_y = (u_x)^3 \tag{209}$$

every C^∞ solution to this equation on the whole $\mathbb{R}^2$ must be linear function

$$u(x,y) = ax + by + c \tag{210}$$

To see this, let's first calculate an example with initial value condition $u(x,0) = 2x^{\frac{3}{2}}$. Turn into general form

$$F(Du, u, x) = 0 \tag{211}$$

$$F(p, z, x) = p_2 - p_1^3 \tag{212}$$

capture the c.h. direction $x'(s) = D_pF = (-3p_1^2, 1)$, set the function along the c.h. curve $z(s) = u(x(s), y(s))$ and the gradient $p(s) = Du(x(s), y(s))$. The ODE for $z(s)$ should be

$$z'(s) = p(s) \cdot (-3p_1^2(s), 1) = -3p_1^3(s) + p_2(s) \tag{213}$$

the ODE for $p(s)$ should be

$$p'(s) + D_zF \cdot p(s) + D_xF = 0 \tag{214}$$

$$p'(s) = 0 \tag{215}$$

So the c.h. ODEs are

$$\begin{cases} x'(s) = -3p_1^2(s) \\ y'(s) = 1 \\ z'(s) = -3p_1^3(s) + p_2(s) \\ p_1'(s) = 0 \\ p_2'(s) = 0 \end{cases} \tag{216}$$

first assume that exists s_0 such that $y(s_0) = 0$ and solve out p_1, p_2 adding initial conditions $p_2(s) = p_1^3(s)$ (coming from the original PDE) and $p_1(s_0) = \frac{\partial u}{\partial x}(x(s_0), y(s_0)) = 3\sqrt{x(s_0)}$ with $\frac{d}{dx}[u(x,0)] = 3\sqrt{x}$ (coming from taking the derivative of the initial value condition)

This c.h. ODEs are simplified as

$$\begin{cases} x'(s) = -27x(s_0) \\ y(s) = s - s_0 \\ z'(s) = -54(x(s_0))^{\frac{3}{2}} \\ p_1(s) = 3\sqrt{x(s_0)} \\ p_2(s) = 27(x(s_0))^{\frac{3}{2}} \end{cases} \tag{217}$$

the solution should be

$$\begin{cases} x(s) = -27x(s_0)(s - s_0) + x(s_0) \\ y(s) = s - s_0 \\ z(s) = -54(x(s_0))^{\frac{3}{2}}(s - s_0) + 2(x(s_0))^{\frac{3}{2}} \\ p_1(s) = 3\sqrt{x(s_0)} \\ p_2(s) = 27(x(s_0))^{\frac{3}{2}} \end{cases} \tag{218}$$

we get the final solution by eliminating $s, x(s_0)$ that

$$u(x, y) = 2x^{\frac{3}{2}}(1 - 27y)^{-\frac{1}{2}} \tag{219}$$

One shall now see that the solution has **singularity** at $y = \frac{1}{27}$ so it's not smooth on the whole $\mathbb{R}^2$. To prove the argument, we have to use the proposition we have just proved above for PDE of the form $a(u) \cdot u_x + u_y = 0$. To let such form appear, we think of differentiating both sides of the PDE. Differentiate the PDE w.r.t. x to get:

$$u_{xy} = 3u_x^2 \cdot u_{xx} \tag{220}$$

change the variables by $v = u_x$ to get

$$v_y = 3v^2 \cdot v_x \tag{221}$$

exactly the same form as that in problem 3 with $a(v) = -3v^2$. From the conclusion above, it's clear that in order to let u to be smooth on the whole plain, $a(v(x, 0))$ has to be nondecreasing, i.e.

$$-3v^2(x, 0) \tag{222}$$

is nondecreasing in x (this is the restriction that exists smooth solution on the whole upper plane). By looking at the existence of smooth solution on the lower half plane, we can argue that

$$-3v^2(x, 0) \tag{223}$$

is also nonincreasing in x. Note that since u is smooth, v has to be continuous, so $v(x, 0)$ must be **constant**. Now turn to the PDE w.r.t. v to see that the directional derivative of v along a fixed vector is always 0, so v is constant along such a vector. Since v is continuous, we can conclude that v has to be constant on the whole plane (otherwise if the constants on different directions are different, there will be discontinuity)! As a result, we proved that $v = u_x$ is constant on the whole plane.

As a result, $v_x = u_{xx} \equiv 0, v_y = u_{xy} \equiv 0$. Note that by differentiating the PDE for u w.r.t. y, we get

$$u_{yy} = 3u_x^2 \cdot u_{xy} \tag{224}$$

so $u_{yy} \equiv 0$, all second partial derivatives are 0 on the whole plane, so u must be linear.

Laplace's Equation

Laplace's equation is

$$\Delta u = 0 \tag{225}$$

and **Poisson's equation** is the nonhomogeneous case

$$-\Delta u = f \tag{226}$$

It's natural to assume that $u \in C^2$ and to notice that $\Delta = div(\nabla)$ (the divergence of gradient is Laplacian).

Remark. *The negative sign in the Poisson's equation refers to the fact that if $u : \mathbb{R}^n \to \mathbb{R}$ is always 0 on the boundary of integration domain (no boundary terms),*

$$\int ||\nabla u||^2 \, dx = \int \sum_j [u_{x_j}(x)]^2 \, dx = \sum_j \int u_{x_j}(x) \, du(x_j) dx_1 ... \overline{dx_j} ... dx_n \tag{227}$$

$$= -\sum_j \int u(x) \cdot u_{x_j, x_j}(x) \, dx = -\int (\Delta u) \cdot u \, dx \tag{228}$$

There are two main types of boundary value problems considered for these equations, **Dirichlet problem** and **Neumann problem**. In the setting of Dirichlet problem, look for $u : \Omega \to \mathbb{R}, u \in C^2(\Omega), u \in C(\overline{\Omega})$ with boundary value condition $\forall x \in \partial\Omega, u(x) = g(x)$. Here $f \in C(\Omega), g \in C(\partial\Omega)$. For the Neumann problem, look for $u : \Omega \to \mathbb{R}, u \in C^2(\Omega), u \in C^1(\overline{\Omega})$ with boundary value condition $\forall x \in \partial\Omega, \frac{\partial u}{\partial \nu}(x) = g(x)$. Here $f \in C(\Omega), g \in C(\partial\Omega)$. The notation $\frac{\partial u}{\partial \nu}$ stands for the directional derivative of u along direction ν, where ν is the unit outward normal vector on the boundary $\partial\Omega$.

Property of Harmonic Function

Notations: $B_r(x)$ for the ball centered at x with radius r. $U \subset\subset V$ means U **is relatively compact in** V, i.e. $\overline{U}$ is compact and $\overline{U} \subset V$. $|\cdot|$ means the volume of a certain area and dS means the surface integration.

Before stating the theorem, let's review some important theorems in integration theory.

Theorem 3. *(Divergence theorem) For vector field $F \in C^1$ on compact oriented n-manifold with boundary M, n as the outward unit normal vector and $\mu_{\partial M}$ as the volume form of ∂M,*

$$\int_M div(F) \, dx = \int_{\partial M} (F \cdot n) \mu_{\partial M} \tag{229}$$

Lemma 2. *(Volume of the ball and sphere in Euclidean space) Set $V_n(r)$ as the volume for n-dim ball with radius r, $A_n(r)$ as the volume for the boundary of n-dim ball with radius r (which is a $n-1$-dim sphere), then $A_n(1) = nV_n(1), A_n(r) = \frac{\partial V_n(r)}{\partial r}$.*

Theorem 4. ***(Mean-Value Property)*** *$u \in C^2(\Omega)$ and harmonic with $B_r(x) \subset\subset \Omega$, then*

$$\forall x \in \Omega, u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u\,dx = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u\,dS \tag{230}$$

Proof. Assume ν is the outward unit normal vector on the $n-1$-dim sphere, dS is the volume form of the $n-1$-dim sphere, then by the divergence theorem,

$$\int_{B_r(x)} \Delta u\,dx = \int_{\partial B_r(x)} \nabla u \cdot \nu\,dS \tag{231}$$

$$= \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}\,dS \tag{232}$$

$$= r^{n-1}\int_{\partial B_1(0)} \frac{\partial u}{\partial \nu}(x+ry)\,dS(y) \tag{233}$$

notice that for sphere $\frac{\partial u}{\partial \nu}(x+ry) = \frac{\partial u}{\partial r}(x+ry)$ since $\nu(x+ry) = \frac{y}{||y||_2}$

$$\int_{B_r(x)} \Delta u\,dx = r^{n-1}\int_{\partial B_1(0)} \frac{\partial u}{\partial r}(x+ry)\,dS(y) \tag{234}$$

$$= r^{n-1}\frac{\partial}{\partial r}\int_{\partial B_1(0)} u(x+ry)\,dS(y) \tag{235}$$

change the integration domain back into $B_r(x)$ and make use of the property that $A_n(1) = nV_n(1)$:

$$\frac{\partial}{\partial r}\left(\frac{1}{A_n(r)}\int_{\partial B_r(x)} u\,dS\right) = \frac{\partial}{\partial r}\left(\frac{r^{n-1}}{A_n(r)}\int_{\partial B_1(0)} u(x+ry)\,dS(y)\right) \tag{236}$$

$$= \frac{1}{nV_n(1)}\frac{\partial}{\partial r}\left(\int_{\partial B_1(0)} u(x+ry)\,dS(y)\right) \tag{237}$$

$$= \frac{r}{nV_n(r)}\int_{B_r(x)} \Delta u\,dx = 0 \tag{238}$$

This is telling us that $\frac{1}{A_n(r)}\int_{\partial B_r(x)} u\,dS$ is independent of r! Set $r \to 0$ then the Lebesgue differentiation theorem tells us that

$$\frac{1}{A_n(r)}\int_{\partial B_r(x)} u\,dS \to u(x)\ (r \to 0) \tag{239}$$

so we proved

$$\forall x, u(x) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u\,dS \tag{240}$$

To prove the other part, note that we can integrate on the sphere to get the integral on the ball and use the property that $A_n(r) = \frac{\partial V_n(r)}{\partial r}$

$$\begin{aligned}
\frac{1}{|B_r(x)|}\int_{B_r(x)} u\,dx &= \frac{1}{|B_r(x)|}\int_0^r \int_{\partial B_t(x)} u\,dS\,dt &(241)\\
&= \frac{1}{|B_r(x)|}\int_0^r |\partial B_t(x)|u(x)\,dt &(242)\\
&= u(x) &(243)
\end{aligned}$$

ANOTHER PROOF:

Another proof can be provided as the one in textbook with the same thought but with more concise descriptions. Define $\phi(r)$ to be the mean value of u on a sphere with radius r

$$\begin{aligned}
\phi(r) &= \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,dS(y) &(244)\\
&= \frac{1}{|\partial B_1(0)|r^{n-1}}\int_{\partial B_1(0)} u(x+rz)r^{n-1}\,dS(z) &(245)\\
&= \frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} u(x+rz)\,dS(z) &(246)
\end{aligned}$$

now we hope to prove that $\phi(r) = u(x)$ for $B_r(x) \subset \Omega$. Since we would expect such $\phi(r)$ not to depend on r, let's calculate its derivative

$$\begin{aligned}
\phi'(r) &= \frac{d}{dr}\frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} u(x+rz)\,dS(z) &(247)\\
&= \frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} \nabla u(x+rz)\cdot z\,dS(z) &(248)\\
&= \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} \nabla u(y)\cdot \frac{y-x}{r}\,dS(y) &(249)\\
&= \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} \nabla u\cdot \nu\,dS(y) &(250)\\
&= \frac{1}{|\partial B_r(x)|}\int_{B_r(x)} \Delta u\,dy = 0 &(251)
\end{aligned}$$

here the interchange of derivative and integral is ensured by dominated convergence theorem and the last equation is by the divergence theorem. This proves that $\phi(r)$ is constant. To figure out what this constant is

$$\phi(r) = \lim_{p\to 0}\phi(p) = u(x) \tag{252}$$

by the Lebesgue differentiation theorem.

□

Remark. *The form of $\phi(r)$ is convenient to use and brings with other conclusions. For example, one can prove that* *for Poisson's equation on a ball with dimension $n \geq 3$,*

$$\begin{cases} -\Delta u = f \ (on\ B_r(0)) \\ u|_{\partial B_r(0)} = g \end{cases} \tag{253}$$

the equation that

$$u(0) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} g\, dS + \frac{1}{n(n-2)V_n(1)} \int_{B_r(0)} \left(||x||_2^{2-n} - r^{2-n}\right) f(x)\, dx \tag{254}$$

holds. Since (sketch of proof)

$$\frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} g\, dS - u(0) = \phi(r) - \phi(0) = \int_0^r \phi'(s)\, ds = \int_0^r ds \frac{1}{|\partial B_s(0)|} \int_{B_s(0)} \Delta u\, dy \tag{255}$$

$$= -\frac{1}{nV_n(1)} \int_0^r s^{1-n}\, ds \int_{B_s(0)} f(y)\, dy = -\frac{1}{nV_n(1)} \int_{B_r(0)} f(y)\, dy \int_{||y||_2}^r s^{1-n}\, ds \tag{256}$$

$$= -\frac{1}{nV_n(1)(2-n)} \int_{B_r(0)} (r^{2-n} - ||y||_2^{2-n}) f(y)\, dy \tag{257}$$

by Fubini theorem.

Actually, the converse of this theorem is also true, which means that for any continuous function if mean-value property holds on an area then u is smooth(C^∞) and harmonic. So mean-value property is the **characterization** of harmonic functions.

Theorem 5. ***(Mean-value Property as Characterization of Harmonic Function)*** *Suppose u continuous on Ω has mean value prop, then u is smooth and harmonic.*

The proof of this theorem would have to use the mollifier. A **mollifier** always has the following properties.

It's **non-negative, C^∞smooth and compactly supported satisfying $\int_{\mathbb{R}} \eta = 1$ is satisfied.** An example would be

$$\eta(x) = \begin{cases} Ce^{-\frac{1}{(1-||x||^2)}} & ||x|| < 1 \\ 0 & ||x|| \geq 1 \end{cases} \tag{258}$$

where the C is selected such that $\int_{\mathbb{R}} \eta = 1$ is satisfied.

In practice we only care about the properties of the mollifier but not the specific form of the mollifier. Note that such mollifier η has support $[-1, 1]$ and we hope that the support would shrink to 0 in the sense that

$$\forall \varepsilon > 0, \eta^\varepsilon = \frac{\eta(\frac{x}{\varepsilon})}{\varepsilon^n} \tag{259}$$

All properties of mollifiers hold for η^ε with a fixed ε and this series of mollifiers has shrinking support as $\varepsilon \to 0$, called **standard mollifiers**. They are often used through convolution in the approximation schemes.

Week 4

Suppose $f \in L_{loc}(\Omega)$ is locally integrable ($f \in L^1(\Omega'), \Omega' \subset\subset \Omega$), for $\varepsilon > 0$, define $\Omega^\varepsilon = \{x \in \Omega, dist(x, \partial\Omega) > \varepsilon\}$, so that any small enough open ball centered at any point in Ω^ε is in Ω.

Define the convolution

$$f^\varepsilon(x) = \int_\Omega \eta^\varepsilon(x-y) f(y)\, dy = \eta^\varepsilon * f \tag{260}$$

Theorem 6. *If $f \in L^1_{loc}(\Omega)$, then*

$$\forall \Omega^\varepsilon \subset\subset \Omega, \forall x \in \Omega^\varepsilon, f^\varepsilon \in C^\infty(\Omega^\varepsilon), f^\varepsilon(x) \overset{a.e.}{\to} f(x)\ (\varepsilon \to 0) \tag{261}$$

Moreover, if $f \in L^p_{loc}(\Omega)$ for $1 \leq p < \infty$, then

$$f^\varepsilon(x) \overset{L^p_{loc}(\Omega_\varepsilon)}{\to} f(x)\ (\varepsilon \to 0) \tag{262}$$

Proof. First show the smoothness of the convolution. The differentiation w.r.t. x can interchange with integration w.r.t. y by dominated convergence theorem ($\frac{d}{dx}\eta^\varepsilon$ is bounded)

$$\frac{d}{dx} f^\varepsilon(x) = \int_\Omega \frac{d}{dx}\eta^\varepsilon(x-y) f(y)\, dy \tag{263}$$

and $\eta^\varepsilon \in C^\infty$ ensures that this function is still differentiable. As a result, the derivative of any order exists.

The *a.e.* convergence comes from the estimation that

$$\left| \int_\Omega \eta^\varepsilon(x-y) f(y)\, dy - f(x) \right| \leq \int_\Omega \eta^\varepsilon(x-y) |f(y) - f(x)|\, dy \tag{264}$$

$$= \int_{B_\varepsilon(x)} \eta^\varepsilon(x-y) |f(y) - f(x)|\, dy \tag{265}$$

$$\leq \frac{1}{\varepsilon^n} \int_{B_\varepsilon(x)} |f(y) - f(x)|\, dy \overset{a.e.}{\to} 0\ (\varepsilon \to 0) \tag{266}$$

by the boundedness of the mollifier η and the Lebesgue point theorem (almost every point is Lebesgue point of the locally integrable function).

To prove the $L^p_{loc}(\Omega)$ convergence, apply Holder's inequality for $p, q = \frac{p}{p-1}$ conjugate to make the L^p norm appear

$$|f^\varepsilon(x)| = \left| \int_{B_\varepsilon(x)} \eta^\varepsilon(x-y) f(y)\, dy \right| \tag{267}$$

$$\leq \left| \int_{B_\varepsilon(x)} [\eta^\varepsilon(x-y)]^{q(1-\frac{1}{p})}\, dy \right|^{\frac{1}{q}} \cdot \left| \int_{B_\varepsilon(x)} \eta^\varepsilon(x-y) |f(y)|^p\, dy \right|^{\frac{1}{p}} \tag{268}$$

notice that the first integral is just 1 by the definition of mollifiers, integrate the p-th power of both sides on Ω^ε to get

$$\int_{\Omega^\varepsilon} |f^\varepsilon(x)|^p \, dx \leq \int_{\Omega^\varepsilon} \int_{B_\varepsilon(x)} \eta^\varepsilon(x-y)|f(y)|^p \, dy \, dx \tag{269}$$

$$= \int_\Omega |f(y)|^p \int_{B_y(\varepsilon)} \eta^\varepsilon(x-y) \, dx \, dy \tag{270}$$

$$= \int_\Omega |f(y)|^p \, dy \tag{271}$$

so we have proved that

$$||f^\varepsilon||_{L^p(\Omega^\varepsilon)} \leq ||f||_{L^p(\Omega)} \tag{272}$$

to see the convergence, one first has to use the dense argument, for $\forall \delta > 0$, exists $g \in C(\Omega)$ such that

$$||f - g||_{L^p(\Omega)} < \delta \tag{273}$$

and notice that $\eta^\varepsilon * g \to g$ $(\varepsilon \to 0)$ converges uniformly on any compact subset of Ω. By the following estimation,

$$||f^\varepsilon - f||_{L^p(\Omega_\varepsilon)} \leq ||f^\varepsilon - g^\varepsilon||_{L^p(\Omega^\varepsilon)} + ||g^\varepsilon - g||_{L^p(\Omega^\varepsilon)} + ||g - f||_{L^p(\Omega^\varepsilon)} \tag{274}$$

$$\leq ||g^\varepsilon - g||_{L^p(\Omega^\varepsilon)} + 2||g - f||_{L^p(\Omega^\varepsilon)} \tag{275}$$

by the calculations above, note that by taking $\varepsilon \to 0$, one get

$$\overline{\lim_{\varepsilon \to 0}} ||f^\varepsilon - f||_{L^p(\Omega_\varepsilon)} = 0 \tag{276}$$

which proves the convergence.

$\square$

Theorem 7. ***(Mean-value property as characterization of harmonic function)*** *If $u \in C(\Omega)$ has the mean-value property then it's smooth and harmonic.*

Proof. Note that the definition of η only depends on the radial value $||x||$, so denote

$$\eta^\varepsilon(x) = \tilde{\eta}^\varepsilon(||x||) \tag{277}$$

If $B_\varepsilon(x) \subset\subset \Omega$, then

$$(\eta^\varepsilon * u)(x) = \int_{B_\varepsilon(0)} \eta^\varepsilon(y) u(x-y)\, dy \tag{278}$$

$$= \int_0^\varepsilon \int_{\partial B_1(0)} \eta^\varepsilon(ry) u(x-ry)\, dS(y) r^{n-1}\, dr \tag{279}$$

$$= |\partial B_1(0)| \int_0^\varepsilon \frac{1}{|\partial B_1(0)|} \int_{\partial B_1(0)} u(x-ry)\, dS(y) r^{n-1} \tilde{\eta}^\varepsilon(r)\, dr \tag{280}$$

$$= |\partial B_1(0)| \int_0^\varepsilon u(x) r^{n-1} \tilde{\eta}^\varepsilon(r)\, dr \tag{281}$$

$$= u(x) \int_{B_\varepsilon(0)} \eta^\varepsilon(y)\, dy = u(x) \tag{282}$$

here still use the trick to deparametrize w.r.t. y using radial value and make use of the mean-value property of u and change the variable back into the variable on sphere. This is telling us that u is smooth since the convolution must be smooth.

Going back to the equations in the proof of mean-value theorem, to see that $\int_{B_r(x)} \Delta u\, dx = 0$ on any ball $B_r(x)$. Since Δu is continuous, it must be true that $\Delta u = 0$ and it's harmonic.

□

Remark. *Using the notation what we have used to prove the mean-value property, $\phi(r)$ denotes the integral average of u over $\partial B_r(x)$ for fixed x. If mean-value property hold, then $\phi(r)$ is independent of r and naturally $\phi'(r) = 0$ so*

$$\forall x, \forall r, \int_{B_r(x)} \Delta u(y)\, dy = 0 \tag{283}$$

as a result, $\Delta u = 0$ if it's continuous. The only difficulty is to prove that $u \in C^2$ and this can be done by convoluting with mollifier. Since mean-value property holds, the convolution with mollifier as local smoothing shall have no effects. A corollary of this proof is that ***harmonic functions are smooth****.*

Remark. *Mean-value property says nothing about the boundary behavior of harmonic functions, but only talks about the behavior of harmonic functions in the interior. A counterexample is the real and imaginary part of $f(z) = \frac{1}{z}$ $(z \neq 0)$. These are harmonic functions, but can blow up at boundary because of the pole at 0.*

The definition of harmonic functions can be extended to sub-harmonic and super-harmonic functions. Suppose Ω open in $\mathbb{R}^n$, $u \in C^2$ is **sub-harmonic** if $\Delta u \geq 0$ and **super-harmonic** if $\Delta u \leq 0$.

Theorem 8. ***(Mean-value Property for Sub-harmonic Functions)*** *Suppose Ω open, $B_r(x) \subset\subset \Omega$, $u \in C^2$, if u is sub-harmonic in Ω then*

$$u(x) \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy \tag{284}$$

$$u(x) \leq \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u\, dS \tag{285}$$

Remark. *The sub-harmonic and super-harmonic functions are defined similar to that for sub-MG and super-MG. Actually, the laplacian is some kind of "curvature" of u in a small neighborhood from the geometric point of view, so if the curvature is positive (locally convex), then the average near a point is greater. For sub-harmonic functions, we are underestimating the function value (just like for sub-MG, we are underestimating the value of the process at a fixed time).*

Actually, sub-harmonic functions and sub-MG has close connections under the setting of Markov chain. For $p(x,y)$ as a Markov transition kernel, we can define f to be super-harmonic if

$$f(x) \geq \sum_y p(x,y)f(y) \tag{286}$$

equivalent to saying that $f(X_n)$ is a super-MG (X_n is a Markov process with transition kernel p). Actually, we can also see from the probability side that if p is irreducible, then p is recurrent if and only if every nonnegative super-harmonic function is constant.

Example

$u(x) = ||x||^4$ is sub-harmonic on $\mathbb{R}^n$ since

$$u(x) = \left(\sum_{i=1}^n x_i^2\right)^2 \tag{287}$$

$$\Delta u(x) = \sum_{i=1}^n \left(8x_1^2 + 4\sum_j x_j^2\right) \geq 0 \tag{288}$$

and we can see that it only obtains maximum on the boundary of $B_1(0)$.

Property of Harmonic Function

From the mean-value property, we can infer all other properties of harmonic functions. The following theorem shows that the partial derivative of harmonic function is still harmonic and can be bounded by its function value, which is a very strong statement for a real-value function.

Theorem 9. ***(Function Values of Harmonic Functions Controls Partial Derivatives)*** *If $u \in C^2(\Omega)$ is harmonic and $B_r(x) \subset\subset \Omega$, then $\forall i = 1,2,...,n, \partial_i u$ is harmonic and*

$$\forall i = 1,...,n, |\partial_i u(x)| \leq \frac{n}{r} \max_{\overline{B}_r(x)} |u| \tag{289}$$

Proof. Notice that $\Delta = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$, so $\partial_i \Delta = \Delta \partial_i$ for harmonic functions (since it's C^∞), and that's why $\partial_i u$ is still harmonic. Use the mean-value property to get

$$\partial_i u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} \partial_i u(y)\, dy \tag{290}$$

To proceed, use the trick here to view the partial derivative as the divergence of the function $U = (0, 0, ..., 0, u, 0, ..., 0)$ with the i-th component to be u. ν denotes the outward unit normal vector at the point on $\partial B_r(x)$. The divergence theorem tells us

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} \partial_i u(y)\, dy = \frac{1}{|B_r(x)|} \int_{B_r(x)} div(U)(y)\, dy \tag{291}$$

$$= \frac{1}{|B_r(x)|} \int_{\partial B_r(x)} u \cdot \nu_i \, dS \tag{292}$$

So

$$|\partial_i u(x)| \leq \frac{1}{|B_r(x)|} \int_{\partial B_r(x)} |u \cdot \nu_i| \, dS \tag{293}$$

$$\leq \frac{1}{|B_r(x)|} \int_{\partial B_r(x)} \max_{\overline{B}_r(x)} |u| \, dS \tag{294}$$

$$= \frac{|\partial B_r(x)|}{|B_r(x)|} \max_{\overline{B}_r(x)} |u| \tag{295}$$

$$= \frac{n}{r} \max_{\overline{B}_r(x)} |u| \tag{296}$$

□

The direct consequence of this estimate is the Louville's theorem.

Theorem 10. ***(Louville's Theorem)*** *If $u \in C^2(\mathbb{R}^n)$ is harmonic and bounded on $\mathbb{R}^n$ then it's constant.*

Proof. The previous theorem tells us that $\forall i = 1, ..., n, |\partial_i u(x)| \leq \frac{n}{r} \max_{\overline{B}_r(x)} |u|$ with $\forall x \in \mathbb{R}^n, |u(x)| \leq M$

$$\forall i = 1, ..., n, |\partial_i u(x)| \leq \frac{nM}{r} \tag{297}$$

since u is harmonic on $\mathbb{R}^n$, can set $r \to \infty$ to get

$$\forall i = 1, ..., n, \partial_i u(x) = 0 \tag{298}$$

For any differentiable function, if all partial derivatives are 0 then it must be constant.

□

Similar to the conclusions in complex analysis, the estimation on derivatives can be generalized to all higher order derivatives.

Theorem 11. ***(Estimation on Higher Order Partial Derivatives)*** *If $u \in C^2(\Omega)$ is harmonic and $B_r(x) \subset\subset \Omega$, then for any multi-index α of order k, $\partial^\alpha u$ is harmonic and*

$$|\partial^\alpha u(x)| \leq \frac{n^k e^{k-1} k!}{r^k} \max_{\overline{B}_r(x)} |u| \tag{299}$$

Proof. It's obvious to see that $\partial^\alpha \Delta = \Delta \partial^\alpha$ for smooth functions, so $\partial^\alpha u$ is still harmonic. The theorem is already proved for $k = 1$. For larger k, use induction. Assume the conclusion holds for k, let's consider whether it still holds for $k+1$. For multi-index β with order $k+1$, WLOG assume $\partial^\beta = \partial^\gamma \partial_j$ for multi-index γ with order k. To get the $\max_{y \in \overline{B}_r(x)} |u(y)|$ in the final expression, we tear apart the radius r into pr and $(1-p)r$ for some $p \in (0,1)$ for two use of the estimation of partial derivatives. Here p is a fixed constant but not specified now.

$$\begin{aligned}
|\partial^\beta u(x)| &= |\partial^\gamma \partial_j u(x)| && (300)\\
&\leq \frac{n^k e^{k-1} k!}{(pr)^k} \max_{y \in \overline{B}_{pr}(x)} |\partial_j u(y)| && (301)\\
&\leq \frac{n^k e^{k-1} k!}{(pr)^k} \max_{y \in \overline{B}_{pr}(x)} \frac{n}{(1-p)r} \max_{z \in \overline{B}_{(1-p)r}(y)} |u(z)| && (302)\\
&\leq \frac{n^{k+1} e^{k-1} k!}{r^{k+1} p^k (1-p)} \max_{y \in \overline{B}_r(x)} |u(y)| && (303)
\end{aligned}$$

now let's think about picking the best value for p, we want the bound to be as tight as possible, i.e. $f(p) = \frac{1}{p^k(1-p)}$ as small as possible. Consider $g(p) = \log f(p) = -k \log p - \log(1-p)$ and take the derivative to get

$$\begin{aligned}
g'(p) &= -\frac{k}{p} + \frac{1}{1-p} && (304)\\
p^* &= \frac{k}{k+1} && (305)
\end{aligned}$$

This is telling us that when $p = \frac{k}{k+1}$, the bound would be the tightest. In such situation,

$$f\left(\frac{k}{k+1}\right) = (k+1)\left(1 + \frac{1}{k}\right)^k \leq e(k+1) \tag{306}$$

leading to the end of the proof that

$$\frac{n^{k+1} e^{k-1} k!}{r^{k+1} p^k (1-p)} \max_{y \in \overline{B}_r(x)} |u(y)| \leq \frac{n^{k+1} e^k (k+1)!}{r^{k+1}} \max_{y \in \overline{B}_r(x)} |u(y)| \tag{307}$$

□

The next theorem shows that harmonic functions are not only smooth but also analytic.

Theorem 12. ***(Analyticity)*** *If $u \in C^2(\Omega)$ is harmonic, then it's real analytic in Ω, i.e. $\forall x \in \Omega, \exists r > 0, B_r(x) \subset\subset$*

Ω, such that u is a convergent power series in $B_r(x)$.

Proof. It's natural to consider the Taylor series at $\forall x \in \Omega$ and $\forall h > 0$ small enough such that $x + h \in \Omega$

$$u(x+h) = \sum_{0 \le |\alpha| \le k-1} \frac{\partial^\alpha u(x)}{\alpha!} h^\alpha + R_k(x,h) \tag{308}$$

$$R_k(x,h) = \sum_{|\alpha|=k} \frac{\partial^\alpha u(x+\theta h)}{\alpha!} h^\alpha \tag{309}$$

where $\alpha! = \alpha_1!...\alpha_n!, h^\alpha = h_1^{\alpha_1}...h_n^{\alpha_n}$ and $R_k(x,h)$ is the Lagrange remainder with θ as some real number in $(0,1)$.

For $\forall \varepsilon > 0$, consider the case where $||h||_2 < \delta, k \to \infty$ and apply the estimation for the higher order partial derivatives

$$|R_k(x,h)| \le \sum_{|\alpha|=k} \frac{n^k e^{k-1} k!}{\alpha! r^k} \max_{y \in \overline{B}_r(x+\theta h)} |u(y)| \cdot h^\alpha \tag{310}$$

note that there exists some constant M such that $\max_{y \in \overline{B}_r(x+\theta h)} |u(y)| \le M$ (continuous on compact set) and $h^\alpha \le ||h||_2^k < \delta^k$

$$|R_k(x,h)| \le \sum_{|\alpha|=k} M \frac{n^k e^{k-1} k!}{\alpha! r^k} \cdot \delta^k \tag{311}$$

the multinomial expansion theorem gives

$$n^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \tag{312}$$

so for multi-index α with order k

$$k! \le n^k \alpha! \tag{313}$$

plug in this estimate to get

$$|R_k(x,h)| \le \frac{M}{e} \sum_{|\alpha|=k} \left(\frac{n^2 e \delta}{r} \right)^k \tag{314}$$

Now the last step is to notice that the number of terms in the sum is no more than n^k. So we wish to have $n^k \left(\frac{n^2 e \delta}{r} \right)^k \to 0$ $(k \to \infty)$. As a result, just take $\delta < \frac{r}{2^n n^3 e}$,

$$|R_k(x,h)| \le \frac{M}{e} n^k \frac{1}{2^{nk} n^k} = \frac{M}{e} \frac{1}{2^{nk}} \to 0 \ (k \to \infty) \tag{315}$$

proves that there exists a small enough δ such that the Taylor series is convergent in this open disk.

□

The analytic property tells us that all partial derivatives at a single point uniquely determines a harmonic function on a connected area.

Theorem 13. ***(All Partials at a Single Point Determines Harmonic Function)*** *If u, v are harmonic in connected open area $\Omega \subset \mathbb{R}^n$ and there exists $x_0 \in \Omega$ such that for all multi-index α,*

$$\partial^\alpha u(x_0) = \partial^\alpha v(x_0) \tag{316}$$

then $\forall x \in \Omega, u(x) = v(x)$.

Proof. Consider the set

$$F = \{x \in \Omega : \forall \alpha, \partial^\alpha u(x) = \partial^\alpha v(x)\} \tag{317}$$

then it's nonempty $x_0 \in F$ and it's closed since $\partial^\alpha u - \partial^\alpha v$ is still harmonic and continuous with

$$F = \bigcap_\alpha (\partial^\alpha u - \partial^\alpha v)^{-1}(\{0\}) \tag{318}$$

so F has to be closed. However, for $y \in F$, since $\partial^\alpha u - \partial^\alpha v$ is harmonic and takes value 0 at y, it's analytic and there exists $r > 0$ such that $\partial^\alpha u - \partial^\alpha v$ is a convergent power series in $B_r(y)$. This power series is just the Taylor series expanded at y, however, all partial derivatives at y of $\partial^\alpha u - \partial^\alpha v$ is 0 since $y \in F$. This is telling us that the power series is always 0 in the whole ball $B_r(y)$, and $B_r(y) \subset F$, so F is open.

Since Ω is connected, there are only trivial subsets which are open and close. Since F cannot be empty, $F = \Omega$ and $u = v$ on the whole area.

□

Remark. *Note that the **connected condition** of Ω is necessary since we need this topological condition to ensure that if a subset is both open and closed then it's trivial. If Ω is replaced with a general area, the theorem holds for any connected component of Ω.*

Remark. *One might think naturally that we would also have **isolation of zeros and the identity theorem (uniqueness)** to hold for harmonic functions just like that in complex analysis. However, this is **WRONG for harmonic functions!***

A counterexample is $u(x, y) = x$ which is obviously harmonic, and is zero on the whole y-axis (with accumulation points), but not identically zero.

The maximum principle is stated for sub-harmonic functions below. Similarly, super-harmonic functions and harmonic functions have their respective versions.

Theorem 14. ***(Maximum Principle for Sub-harmonic Functions)*** *If u is sub-harmonic in connected open area $\Omega \subset \mathbb{R}^n$ and attains a maximum in Ω, then it's constant on Ω.*

Proof. This proof is directly given by mean-value property. Assume the maximum is M with

$$F = \{x : u(x) = M\} \tag{319}$$

to be the set of points attaining the maximum. It's obvious that it's non-empty and closed (u continuous, $\{M\}$ closed). Let's consider $\forall x \in F$, then the mean-value property of sub-harmonic function tells us that

$$u(x) \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} u(x)\, dy = u(x) \tag{320}$$

As a result, for any $r > 0$ such that $B_r(x) \subset\subset \Omega$

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) - u(x)\, dy = 0 \tag{321}$$

since $u(y) - u(x) \leq 0$, conclude that $\forall y \in B_r(x), u(y) = u(x)$, so $B_r(x) \subset F$, F is open.

Since Ω is connected, the subset that is both open and closed has to be trivial, so $F = \Omega$ and u is constant.

□

Remark. *By the same reasoning, the* ***connectedness*** *of Ω can't be ignored and for general open area this theorem holds for each connected component.*

For super-harmonic functions, it's clear that if there exists minimum in the interior than the function must be constant.

Week 5

Property of Harmonic Function

Theorem 15. *Ω connected open, $u \in C^2(\Omega)$ is sup-harmonic, attains global minimum in interior, then it's constant.*

Theorem 16. *Ω connected open, $u \in C^2(\Omega)$ harmonic, attains global min/max in interior, then it's constant.*

Consider $u(x, y) = x^2 - y^2$ harmonic on $\mathbb{R}^2$. The origin is a critical point, but according to maximum principle, the origin must be a saddle point. The following weak maximum principles are equivalent to strong maximum principles.

Theorem 17. ***(Maximum Principle of Harmonic Functions)*** *Ω bounded connected open, $u \in C^2(\Omega), u \in C(\overline{\Omega})$ harmonic, then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$, the same argument holds for minimum.*

Theorem 18. *Ω bounded connected open, $f \in C(\Omega), g \in C(\partial\Omega)$, there is at most one solution to Dirichlet problem* $\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = g \end{cases}$ *with $u \in C^2(\Omega), u \in C(\overline{\Omega})$.*

Proof. Let u_1, u_2 be solutions to the Dirichlet problem and $v = u_1 - u_2$ so $\Delta v = 0, v|_{\partial\Omega} = 0$. So v must be constantly 0 by maximum principle.

□

Remark. *There's* ***another proof for the maximum principle of harmonic functions*** *which is elementary. The trick is to do perturbations of a multiple of $||x||_2^2$ based on u.*

Set

$$u_\varepsilon(x) = u(x) + \varepsilon||x||_2^2 \ (\varepsilon > 0) \tag{322}$$

as a smooth function, with

$$\Delta u_\varepsilon = \Delta u + \varepsilon \cdot \Delta||x||_2^2 = 2n\varepsilon > 0 \tag{323}$$

so it's sub-harmonic and the maximum principle holds, u_ε cannot take maximum in the interior. As a result,

$$\max_{\overline{\Omega}} u \leq \max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon \leq \max_{\partial\Omega} u + \varepsilon \max_{\partial\Omega} ||x||_2^2 \tag{324}$$

by setting $\varepsilon \to 0^+$, we know that $\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u$, and $\max_{\overline{\Omega}} u \geq \max_{\partial\Omega} u$ is obvious, so we have proved the maximum principle.

Maximum Principle for the Solution to Dirichlet Problem

Actually the maximum principle does not only hold for harmonic functions, for the solution to Dirichlet problem, similar arguments can be built.

Let's first consider Ω **as a bounded open area and** u **as a smooth solution to**

$$\begin{cases} -\Delta u = f & in\ \Omega \\ u = g & on\ \partial\Omega \end{cases} \tag{325}$$

then **there exists a constant** $C = C(\Omega)$ **that only depends on the area** Ω **such that**

$$\max_{\overline{\Omega}} |u| \leq C(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f|) \tag{326}$$

The trick is still to **use** $||x||^2$ **to perturb** u **with a coefficient depending on** f **such that the perturbed version is still sub-harmonic.** Consider

$$v(x) = u(x) + \frac{\lambda}{2n}||x||_2^2 \tag{327}$$

with λ as a fixed constant but not specified yet. Let's compute its Laplacian

$$\Delta v = \Delta u + \lambda \tag{328}$$

to ensure that $\Delta v \geq 0$ and connect its maximum with the maximum of f, let's take $\lambda = \max_{\overline{\Omega}} |f|$ to find that maximum principle still holds for v. As a result,

$$\max_{\overline{\Omega}} u \leq \max_{\overline{\Omega}} v = \max_{\partial\Omega} v \leq \max_{\partial\Omega} g + \frac{\max_{\partial\Omega} ||x||_2^2}{2n} \max_{\overline{\Omega}} |f| \tag{329}$$

and $C(\Omega) = \max\left\{\frac{\max_{\partial\Omega} ||x||_2^2}{2n}, 1\right\}$ only depends on the bounded area Ω. By repeating the same operations on $-u$ one would be able to prove the proposition that bounds $\max_{\overline{\Omega}} |u|$.

More interestingly, when it comes to **areas with special shapes**, we are able to build **tighter bounds** of the same type. However, the thoughts and tricks applied are basically the same.

Consider Ω as the **ellipsoid** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ **in** $\mathbb{R}^3$ and let's try to build a tighter bound for the solution to the Dirichlet problem stated above.

Firstly, notice that on the boundary of the ellipsoid $1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, so it's natural to build the function

$$v(x) = u(x) - \lambda \max_{\overline{\Omega}} |f| \cdot \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \tag{330}$$

with constant λ not specified yet and notice that the Laplacian

$$\Delta v = -f + 2\lambda \max_{\overline{\Omega}} |f| \cdot \left(a^{-2} + b^{-2} + c^{-2}\right) \tag{331}$$

to ensure that v is sub-harmonic, the smallest λ that will work is $\lambda = \frac{1}{2(a^{-2}+b^{-2}+c^{-2})}$ so by the maximum principle,

$$\max_{\overline{\Omega}} |u| \le \max_{\overline{\Omega}} |v| + \frac{1}{2\left(a^{-2}+b^{-2}+c^{-2}\right)} \max_{\overline{\Omega}} |f| \cdot \max_{\overline{\Omega}} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \tag{332}$$

$$\le \max_{\overline{\Omega}} |v| + \frac{1}{2\left(a^{-2}+b^{-2}+c^{-2}\right)} \max_{\overline{\Omega}} |f| \tag{333}$$

$$= \max_{\partial\Omega} |g| + \frac{1}{2\left(a^{-2}+b^{-2}+c^{-2}\right)} \max_{\overline{\Omega}} |f| \tag{334}$$

Remark. *An example one can try to calculate is that* $a = \sqrt{2}, b = \sqrt{3}, c = 2$ *and one might find that by applying the estimate that holds for any bounded area, one get*

$$\max_{\overline{\Omega}} |u| \le \max_{\partial\Omega} |g| + \frac{\max_{\partial\Omega} ||x||_2^2}{2n} \max_{\overline{\Omega}} |f| \tag{335}$$

$$= \max_{\partial\Omega} |g| + \frac{2}{3} \max_{\overline{\Omega}} |f| \tag{336}$$

however, if one try to apply the estimate above, one may find that

$$\max_{\overline{\Omega}} |u| \le \max_{\partial\Omega} |g| + \frac{1}{2\left(a^{-2}+b^{-2}+c^{-2}\right)} \max_{\overline{\Omega}} |f| \tag{337}$$

$$= \max_{\partial\Omega} |g| + \frac{6}{13} \max_{\overline{\Omega}} |f| \tag{338}$$

gives a tighter bound thanks to the special structure of ellipsoid.

Harnack's Inequality

Theorem 19. ***(Harnack's inequality)*** $\Omega' \subset\subset \Omega$ *is connected open set, then there exists* C *only depending on* Ω, Ω' *such that if* $u \in C(\Omega), u \ge 0$ *has mean-value property, then* $\sup_{\Omega'} \le C \inf_{\Omega'} u$. *(Note:* u *is not necessarily harmonic, since it's not necessarily continuous)*

Proof. First prove it for open balls. Suppose $x \in \Omega'$ and $B_{4r}(x) \subset \Omega'$. Let u be non-neg function with mean-value property.

If $y \in B_r(x)$, then

$$u(y) = \frac{1}{|B_r(y)|} \int_{B_r(y)} u = \frac{2^n}{|B_{2r}(y)|} \int_{B_r(y)} u \tag{339}$$

$$\le \frac{2^n}{|B_{2r}(y)|} \int_{B_{2r}(x)} u = 2^n u(x) \tag{340}$$

since $u \ge 0$ and $B_r(y) \subset B_{2r}(x)$.

Now take $z \in B_r(x)$, then

$$u(z) = \frac{1}{|B_{3r}(z)|}\int_{B_{3r}(z)} u = \left(\frac{2}{3}\right)^n \frac{1}{|B_{2r}(z)|}\int_{B_{3r}(z)} u \tag{341}$$

$$\geq \left(\frac{2}{3}\right)^n \frac{1}{|B_{2r}(z)|}\int_{B_{2r}(x)} u = \left(\frac{2}{3}\right)^n u(x) \tag{342}$$

since $u \geq 0$ and $B_{2r}(x) \subset B_{3r}(z)$.

Combine to get

$$\forall x \in \Omega', \forall y \in B_r(x), \left(\frac{2}{3}\right)^n u(x) \leq u(y) \leq 2^n u(x) \tag{343}$$

take inf and sup w.r.t. y to get

$$\forall x \in \Omega', \sup_{B_r(x)} u \leq 2^n u(x), \inf_{B_r(x)} u \geq \left(\frac{2}{3}\right)^n u(x) \tag{344}$$

so

$$\forall x \in \Omega', \sup_{B_r(x)} u \leq 3^n \inf_{B_r(x)} u \tag{345}$$

Now for general case, as long as there is at least $4r$ distance left between Ω' and $\partial\Omega$, i.e. $\exists r > 0, dist(\Omega', \partial\Omega) > 4r$, it should still hold. Since $\overline{\Omega'}$ is compact, it can be covered by finitely many (say, N) open balls with radius r, with N only depending on Ω, Ω'.

By connectedness and open set, $\forall x, y \in \Omega'$, there exists a path from x to y and can be covered by those open balls $B_r(x_i),\ (i = 1, 2, ..., N)$ with $x_1 = x, x_N = y$. In each open ball, the conclusion above holds

$$\forall x \in \Omega', \sup_{B_r(x_i)} u \leq 3^n \inf_{B_r(x_i)} u \tag{346}$$

so the finiteness that $N < \infty$ gives

$$\forall x, y \in \Omega', u(x) = u(x_1) \leq 3^n u(x_2) \leq ... \leq 3^{nN} u(y) \tag{347}$$

$$\forall x, y \in \Omega', 3^{-nN} u(y) \leq u(x) \leq 3^{nN} u(y) \tag{348}$$

take the inf and sup to conclude.

$\square$

Remark. *As we can see from the proof, generally there exists constant $C = C(\Omega, \Omega')$, such that*

$$\forall x, y \in \Omega', \frac{1}{C} u(y) \leq u(x) \leq C u(y) \tag{349}$$

the lower bound and upper bound can be given by the same constant C which does not depend on x, y, a very strong argument!

One can get the following Harnack's principle describing the behavior of monotone harmonic function series by directly applying Harnack's inequality.

Theorem 20. ***(Harnack's Principle)*** *If u_n is a decreasing sequence of harmonic functions on Ω and there exists $x_0 \in \Omega$ such that $u_n(x_0)$ is bounded, then the series converges uniformly on compact subsets of Ω to a harmonic function.*

Proof. WLOG, prove the theorem for non-negative increasing harmonic function series u_n (one can consider $-u_n$, if it's negative then the case is trivial). Now let's first observe that since the sequence is increasing, the limit $u_n \to u$ $(n \to \infty)$ must exist.

Apply Harnack's inequality to know that there exists constant $C \geq 1$ such that

$$\forall x \in K, \frac{1}{C} u_n(x_0) \leq u_n(x) \leq C u_n(x_0) \tag{350}$$

since $u_n(x_0)$ is bounded by M, $u_n(x)$ is always bounded by CM, an upper bound uniform in n, so the limit is always finite $\forall x \in K, u(x) < \infty$ with

$$\sup_K |u_n(x) - u(x)| \leq 2CM < \infty \tag{351}$$

so the convergence is uniform.

To see that the limit is actually harmonic, notice that u_n is continuous, so the uniform limit u must be continuous. By uniform convergence, we can also interchange limit and integral of u_n,

$$\begin{aligned}
\forall x \in K, u(x) &= \lim_{n \to \infty} u_n(x) && (352)\\
&= \lim_{n \to \infty} \frac{1}{|B_r(x)|} \int_{B_r(x)} u_n(y)\, dy && (353)\\
&= \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy && (354)
\end{aligned}$$

so the mean-value property holds. Note that continuous function with mean-value property must be harmonic, so the limit u is harmonic. $\square$

Green's Identity and Energy Estimates

Theorem 21. ***(Green's Identity)*** *If Ω is bounded C^1 open set and $u, v \in C^2(\overline{\Omega})$, then*

$$\int_\Omega u \cdot \Delta v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} u \cdot \frac{\partial v}{\partial \nu} \, dS \tag{355}$$

$$\int_\Omega u \cdot \Delta v \, dx = \int_\Omega v \cdot \Delta u \, dx + \int_{\partial\Omega} \left(u \cdot \frac{\partial v}{\partial \nu} - v \cdot \frac{\partial u}{\partial \nu} \right) dS \tag{356}$$

Proof. Note the property that

$$div(u \cdot \nabla v) = u \cdot \Delta v + \nabla u \cdot \nabla v \tag{357}$$

and apply the divergence theorem for the unit outward normal vector ν of $\partial\Omega$

$$\int_\Omega (u \cdot \Delta v + \nabla u \cdot \nabla v) \, dx = \int_\Omega div(u \cdot \Delta v) \, dx \tag{358}$$

$$= \int_{\partial\Omega} (u \cdot \Delta v) \cdot \nu \, dS \tag{359}$$

$$= \int_{\partial\Omega} u \cdot \frac{\partial v}{\partial \nu} \, dS \tag{360}$$

To prove the second equation, apply the first one twice and subtract

$$\int_\Omega u \cdot \Delta v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} u \cdot \frac{\partial v}{\partial \nu} \, dS \tag{361}$$

$$\int_\Omega v \cdot \Delta u \, dx = -\int_\Omega \nabla v \cdot \nabla u \, dx + \int_{\partial\Omega} v \cdot \frac{\partial u}{\partial \nu} \, dS \tag{362}$$

$$\int_\Omega u \cdot \Delta v \, dx - \int_\Omega v \cdot \Delta u \, dx = \int_{\partial\Omega} \left(u \cdot \frac{\partial v}{\partial \nu} - v \cdot \frac{\partial u}{\partial \nu} \right) dS \tag{363}$$

□

Theorem 22. ***(Uniqueness of the Solution to Dirichlet Problem)*** *Ω is C_1 bounded open (the boundary is a C_1 curve) with $f \in C(\overline{\Omega}), g \in C(\partial\Omega)$. If $u_1, u_2 \in C^2(\overline{\Omega})$ are solutions to the Dirichlet problem $\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = g \end{cases}$, then $u_1 = u_2$.*

Theorem 23. ***(Uniqueness of the Solution to Neumann Problem)*** *Ω is C_1 bounded open (the boundary is a C_1 curve) with $f \in C(\overline{\Omega}), g \in C(\partial\Omega)$. If $u_1, u_2 \in C^2(\overline{\Omega})$ are solutions to the Neumann problem $\begin{cases} -\Delta u = f \\ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = g \end{cases}$, then $u_1 = u_2 + C$ for some constant C.*

Proof. For the Dirichlet problem, set $w = u_1 - u_2$ to be harmonic with $w|_{\partial\Omega} = 0$. Apply Green's idendity for

$u = v = w$ to find

$$\int_{\Omega} w \cdot \Delta w \, dx + \int_{\Omega} \nabla w \cdot \nabla w \, dx = \int_{\partial\Omega} w \cdot \frac{\partial w}{\partial \nu} \, dS = 0 \tag{364}$$

so

$$\int_{\Omega} \nabla w \cdot \nabla w \, dx = 0 \tag{365}$$

$$\forall x \in \Omega, \nabla w(x) = 0 \tag{366}$$

$$\forall x \in \Omega, w(x) = 0 \tag{367}$$

For the Neumann problem, still apply the Green's identity to conclude

$$\int_{\Omega} \nabla w \cdot \nabla w \, dx = 0 \tag{368}$$

and the uniqueness is proved in the same way since now $\frac{\partial w}{\partial \nu}|_{\partial\Omega} = 0$.

□

Remark. *The maximum principle for harmonic functions can also be applied to prove the uniqueness of the solution to the Dirichlet problem but cannot prove the uniqueness of the solution to the Neumann problem. The reason is that* $\frac{\partial u}{\partial \nu}(x)$ *contains the outward unit normal vector of* $\partial\Omega$*, which is* ν *that varies if* $x \in \partial\Omega$ *varies (so it's actually not a conventionally defined directional derivative).*

Fundamental Solution

The fundamental solution is defined as some special solutions to the Laplace equation. The motivation of the appearance of fundamental solutions is to first apply some symmetry conditions to get some special solutions and then use special solutions to construct general solutions to the Poisson's equation. Since Laplacian is invariant under rotation, it's natural to consider the special solutions that only depend on the radial variable $r = ||x||_2$.

To get such special solutions $u(x) = v(r)$, do a polar coordinate transformation and calculate the Laplacian under the polar coordinates to get:

$$\Delta u = \sum_i \partial_{x_i x_i} u \tag{369}$$

$$\partial_{x_i} u = \frac{\partial v}{\partial x_i} = v'(r) \cdot \frac{\partial r}{\partial x_i} \tag{370}$$

$$= v'(r) \cdot \frac{x_i}{r} \tag{371}$$

taking another partial to see

$$\partial_{x_i x_i} u = \frac{\partial\left[v'(r)\cdot\frac{x_i}{r}\right]}{\partial x_i} = v''(r)\cdot\frac{\partial r}{\partial x_i}\cdot\frac{x_i}{r} + v'(r)\cdot\frac{r - x_i\cdot\frac{\partial r}{\partial x_i}}{r^2} \tag{372}$$

$$= v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) \tag{373}$$

As a result, the Laplacian equation turns into an ODE w.r.t. v

$$\Delta u = v''(r) + v'(r)\frac{n-1}{r} = 0 \tag{374}$$

Solve the ODE

$$v'(r) = C_1 r^{1-n} \ or\ 0 \tag{375}$$

and

$$v(r) = \begin{cases} C_1 \log r + C_2 & n = 2 \\ C_1 r^{2-n} + C_2 & n \geq 3 \end{cases} \tag{376}$$

The **fundamental solution/free-space Green's function** Γ is defined as $v(r)$ for some specified C_1, C_2 that

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi}\log ||x||_2 & n = 2 \\ \frac{1}{n(n-2)V_n(1)}||x||_2^{2-n} & n \geq 3 \end{cases} \tag{377}$$

where $V_n(1)$ denotes the volume of the ball with radius 1 in $\mathbb{R}^n$.

The **properties** of fundamental solutions are stated as below:

The fundamental solution $\Gamma \in C^\infty(\mathbb{R} - \{0\})$ is **smooth at any point except the origin**, which is obvious since logarithm and power functions are smooth. It's also **harmonic in** $\mathbb{R}^n - \{0\}$ with estimates on the order of its partial derivatives

$$\partial_{x_i}\Gamma(x) = O(||x||^{1-n}), \partial_{x_i x_i}\Gamma(x) = O(||x||^{-n}) \ (x \neq 0) \tag{378}$$

since

$$\partial_{x_i}\Gamma(x) = \begin{cases} \frac{x_i}{2\pi ||x||_2^2} & n = 2 \\ -\frac{1}{nV_n(1)}\frac{x_i}{||x||_2^n} & n \geq 3 \end{cases} \tag{379}$$

$$\partial_{x_i x_i}\Gamma(x) = \begin{cases} \frac{||x||_2^2 - 2x_i^2}{2\pi ||x||_2^4} & n = 2 \\ -\frac{1}{nV_n(1)}\frac{||x||_2^n - nx_i^2||x||_2^{n-2}}{||x||_2^{2n}} & n \geq 3 \end{cases} \tag{380}$$

Consider for $n \geq 3$

$$\nabla\Gamma \cdot \frac{x}{||x||_2} = -\frac{1}{nV_n(1)}\frac{1}{||x||_2^{n-1}} \tag{381}$$

and for $n = 2$

$$\nabla\Gamma \cdot \frac{x}{||x||_2} = \frac{1}{2\pi}\frac{1}{||x||_2} \tag{382}$$

so

$$\int_{\partial B_r(0)} \nabla\Gamma \cdot \nu \, dS = -\frac{|\partial B_r(0)|}{nV_n(1)r^{n-1}} = -1 \ (n \geq 3) \tag{383}$$

$$\int_{\partial B_r(0)} \nabla\Gamma \cdot \nu \, dS = \frac{2\pi r}{2\pi r} = 1 \ (n = 2) \tag{384}$$

Remark. *Now we see why certain normalization constants are taken in the definition of the fundamental solution. The normalization constants are chosen such that the surface integral on the sphere of the fundamental solution's directional derivative along the outward unit normal vector is always normalized.*

Specially, the normalization constant is taken as 2π when $n = 2$ since unit sphere in 2-dim has volume 2π. This is also consistent with the Cauchy's integral formula in complex analysis. Notice that the fundamental solution is only increasing w.r.t. radius r when $n = 2$.

$$\int_{\partial B_r(0)} \frac{\partial\Gamma}{\partial\nu} \, dS = \int_{\partial B_r(0)} \nabla\Gamma \cdot \nu \, dS = \begin{cases} 1 & n = 2 \\ -1 & n \geq 3 \end{cases} \tag{385}$$

Remark. *Fundamental solutions are inspired by the radial value dependence and the harmonic property, but are **not true solutions to the Laplacian equation**. Note the singularity at 0 and the fact that fundamental solution contains no information of the boundary value condition $u|_{\partial\Omega} = g$.*

Remark. *In $\mathbb{R}^n$, it's clear that $\Gamma, \partial_{x_i}\Gamma$ **are always locally integrable at 0 but** $\partial_{x_ix_i}\Gamma$ **is not**.*

$$\int_{B_1(0)} \partial_{x_i}\Gamma(x) \, dx \leq C \int_{B_1(0)} ||x||^{1-n} \, dx \tag{386}$$

$$= C \int_0^1 r^{n-1} \, dr \int_{\partial B_1(0)} r^{1-n} \, dS \tag{387}$$

$$= C|\partial B_1(0)| < \infty \tag{388}$$

while

$$\int_{B_1(0)} \partial_{x_i x_i} \Gamma(x)\, dx \geq C \int_{B_1(0)} ||x||^{-n}\, dx \tag{389}$$

$$= C \int_0^1 r^{n-1}\, dr \int_{\partial B_1(0)} r^{-n}\, dS \tag{390}$$

$$= C|\partial B_1(0)| \int_0^1 \frac{1}{r}\, dr = \infty \tag{391}$$

since r^{n-1} is the Jacobian of polar coordinate transformation in $\mathbb{R}^n$.

This directly causes the failure of the interchange of intergal and Laplacian in the convolution

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy \tag{392}$$

for a good enough function f. This is because when $x - y$ is close to 0, there is singularity. Although such singularity does not affect the locally integrability of $\Gamma, \partial_{x_i}\Gamma$, it does affect the locally integrability of $\partial_{x_i x_i}\Gamma$, so we ***CANNOT*** *conclude $\Delta u = \int \Delta\Gamma \cdot f = 0$. Actually, this convolution satisfies*

$$-\Delta u = f \tag{393}$$

because of this singularity!

Theorem 24. ***(Recovering Laplacian by Fundamental Solution)*** *Let $f \in C_c^\infty(\mathbb{R}^n)$ and $u = \Gamma * f$, then $u \in C^\infty(\mathbb{R}^n), -\Delta u = f$.*

Week 6

Theorem 25. ***(Recovering Laplacian by Fundamental Solution)*** *Let $f \in C_c^\infty(\mathbb{R}^n)$ and $u = \Gamma * f$, then $u \in C^\infty(\mathbb{R}^n), -\Delta u = f$.*

Proof. Now that $f \in C_c^\infty, \Gamma \in L^1_{loc}$, the partial derivative of u can be derived by

$$\frac{\partial u}{\partial x_i} = \int \Gamma(y) \frac{\partial}{\partial x_i} f(x-y)\, dy \tag{394}$$

$$= \int \Gamma(y) \frac{\partial f}{\partial x_i}(x-y)\, dy \tag{395}$$

the interchange of derivative and integral is ensured by dominated convergence theorem, so it's easy to see that such iteration can be applied for infinitely many times, $u \in C^\infty(\mathbb{R}^n)$.

Then let's prove $-\Delta u = f$. Now we prefer to write the convolution as

$$u(x) = \int \Gamma(x-y) f(y)\, dy \tag{396}$$

to see that if $x \notin supp(f)$, then the singularity of fundamental solution at 0 actually makes no difference, so we can still interchange derivative and integral.

In detail, since $supp(f)$ is compact, if $x \notin supp(f)$, there exists an open set Ω such that $supp(f) \subset\subset \Omega, x \notin \Omega$, so by Green's identity, the boundary term disappears

$$\Delta u(x) = \int \Gamma(x-y) \Delta f(y)\, dy \tag{397}$$

$$= \int \Delta\Gamma(x-y) f(y)\, dy + \int_{\partial\Omega} \left(\Gamma \frac{\partial f}{\partial \nu} - f \frac{\partial \Gamma}{\partial \nu}\right) dS(y) \tag{398}$$

$$= \int \Delta\Gamma(x-y) f(y)\, dy = 0 \tag{399}$$

and $x \in \Omega^c$ is bounded away from $y \in supp(f) \subset\subset \Omega$ so Γ is harmonic at $x-y$. So outside the support of f, $-\Delta u = f$ is proved.

Now consider the case where $x \in supp(f)$, we have to deal with the singularity of Γ. The thought is to first remove the ball $B_r(x)$ around the singularity $y = x$ and shrink $r \to 0$ to get the limit. Denote $\Omega_r = \Omega - B_r(x)$ for $B_r(x) \subset\subset \Omega$.

$$\Delta u(x) = \int_\Omega \Gamma(x-y) \Delta f(y)\, dy \tag{400}$$

$$= \lim_{r \to 0} \int_{\Omega_r} \Gamma(x-y) \Delta f(y)\, dy \tag{401}$$

by dominated convergence theorem since Δf is bounded on $\overline{\Omega}$ and Γ is locally integrable at 0 even if it's a singularity.

For the integral on Ω_r, apply Green's identity (w.r.t. y) again to turn it into integrals on the boundary (let

$g(y) = \Gamma(x-y)$ and apply Green's identity for $g(y)$ may be clearer)

$$\int_{\Omega_r} \Gamma(x-y)\Delta f(y)\,dy = -\int_{\Omega_r} \Delta\Gamma(x-y)f(y)\,dy + \int_{\partial\Omega} \left(\Gamma(x-y)\frac{\partial f}{\partial \nu}(y) + f(y)\frac{\partial \Gamma}{\partial \nu}(x-y)\right) dS(y) \tag{402}$$

$$- \int_{\partial B_r(x)} \left(\Gamma(x-y)\frac{\partial f}{\partial \nu}(y) + f(y)\frac{\partial \Gamma}{\partial \nu}(x-y)\right) dS(y) \tag{403}$$

$$= -\int_{\partial B_r(x)} \Gamma(x-y)\frac{\partial f}{\partial \nu}(y)\,dS(y) - \int_{\partial B_r(x)} f(y)\frac{\partial \Gamma}{\partial \nu}(x-y)\,dS(y) \tag{404}$$

$$= -A_1 - A_2 \tag{405}$$

where the ν inside $\int_{\partial B_r(x)}$ means the outward unit normal vector of $\partial B_r(x)$ and note that the boundary of Ω_r consists of two parts with opposite orientations.

Let's estimate $A_1 = \int_{\partial B_r(x)} \Gamma(x-y)\frac{\partial f}{\partial \nu}(y)\,dS(y)$ first by noticing that if $n \geq 3$, $\Gamma(x) = O(||x||^{2-n})$ and ∇f is bounded on $\partial B_r(x)$, so $A_1 = O(r^{2-n}|\partial B_r(x)|) = O(r) \to 0$ $(r \to 0)$. If $n = 2$, $\Gamma(x) = O(\log ||x||)$ and ∇f is bounded on $\partial B_r(x)$, so $A_1 = O(\log r|\partial B_r(x)|) = O(r\log r) \to 0$ $(r \to 0)$. So A_1 vanishes when taking the limit.

Then estimate $A_2 = \int_{\partial B_r(x)} f(y)\frac{\partial \Gamma}{\partial \nu}(x-y)\,dS(y)$ by plugging in $\nu = \frac{y-x}{r}$

$$A_2 = \int_{\partial B_r(x)} f(y)\left(\nabla\Gamma(x-y)\cdot\nu\right) dS(y) \tag{406}$$

$$= \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} f(y)\,dS(y) \to f(x) \ (r \to 0) \tag{407}$$

by Lebesgue differentiation theorem and the fact that

$$\nabla\Gamma(x-y)\cdot\nu = \frac{1}{|\partial B_r(x)|} \tag{408}$$

this can be proved by calculations. For $n = 2$,

$$\nabla\Gamma(x-y)\cdot\nu = -\frac{1}{2\pi}\frac{x-y}{r^2}\cdot\frac{y-x}{r} = \frac{1}{2\pi r} = \frac{1}{|\partial B_r(x)|} \tag{409}$$

for $n \geq 3$

$$\nabla\Gamma(x-y)\cdot\nu = -\frac{1}{nV_n(1)}||x-y||^{-n}(x-y)\cdot\frac{y-x}{r} = \frac{1}{r^{n-1}nV_n(1)} = \frac{1}{|\partial B_r(x)|} \tag{410}$$

As a result, we have proved that $\forall x \in \mathbb{R}^n, \Delta u(x) = -f(x)$.

$\square$

Remark. *Here the notation $\nabla\Gamma(x-y)$ means the value of the gradient of Γ at the point $x-y$. One can also use the notation $\nabla_y[\Gamma(x-y)]$ which means the gradient w.r.t. y for the function $\Gamma(x-y)$. If one uses the latter notation, some signs in the Green's property shall be changed (including the sign before A_2) and one should derive that the limit of A_2 is actually $-f(x)$. Essentially, the result is the same and the proof is very similar.*

Remark. *Actually the converse of this theorem is also true. One can conclude that* ***for any*** $f \in C_c^\infty(\mathbb{R}^n)$***, it's always a convolution of*** $-\Delta f$ ***with the fundamental solution*** Γ*. To see this, consider the Poisson's equation* $-\Delta u = -\Delta f$*, it has* $g = \Gamma * (-\Delta f)$ *as one of its solutions and* f *as the other solution. However, note that* $f - g$ *is harmonic with compact support, so it would be harmonic and bounded on* $\mathbb{R}^n$*, thus constantly 0.*

$\Gamma * f$ *is actually called the* ***Newtonian potential*** *of* f*.*

Remark. *From the above theorem, one would be able to find the asymptotic order of growth of* $u = \Gamma * f$*.*

When $n = 2$*,*

$$u(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log ||x - y|| f(y) \, dy \tag{411}$$

$$\frac{u(x)}{\log ||x||} = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\log ||x - y||}{\log ||x||} f(y) \, dy \tag{412}$$

$$\to -\frac{1}{2\pi} \int_{\mathbb{R}^2} f(y) \, dy \ \ (||x|| \to \infty) \tag{413}$$

by the dominated convergence theorem. This is telling us that the ***Newtonian potential of*** f ***is growing logarithmically asymptotically in 2-dimension****.*

When $n \geq 3$*,*

$$u(x) = \frac{1}{n(n-2)V_n(1)} \int_{\mathbb{R}^n} ||x - y||^{2-n} f(y) \, dy \tag{414}$$

$$= \frac{1}{n(n-2)V_n(1)||x||^{n-2}} \int_{\mathbb{R}^n} \left(\frac{||x||}{||x - y||} \right)^{n-2} f(y) \, dy \tag{415}$$

$$\to \frac{1}{n(n-2)V_n(1)||x||^{n-2}} \int_{\mathbb{R}^n} f(y) \, dy \ \ (||x|| \to \infty) \tag{416}$$

by the dominated convergence theorem. This is telling us that the ***Newtonian potential of*** f ***is decaying at the same speed as*** $||x||^{2-n}$ ***asymptotically in higher dimension (bounded)****.*

Theorem 26. ***(Uniqueness of the Bounded Solution to Poisson's Equation with Compactly Supported Smooth Potential)*** *If* $n \geq 3$ *and* $f \in C_c^\infty(\mathbb{R}^n)$*, then any bounded solution of the Poisson's equation on* $\mathbb{R}^n$ *is given by* $u = \Gamma * f + C$ *for some constant* C*.*

Proof. If u is a bounded solution to Poisson's equation in dimension no less than 3, note that $v = \Gamma * f$ is also a bounded solution (since it's decaying asymptotically). $u - v$ is bounded and harmonic on $\mathbb{R}^n$. By Louville's theorem, $u - v$ is constant. □

Remark. *Note that such argument only holds for* $n \geq 3$ *since the boundedness of the convolution* $\Gamma * f$ *in no less than 3 dimension is ensured.*

Green's Function

Now we want to deal with the Dirichlet problem in subsets of $\mathbb{R}^n$, i.e. adding the boundary conditions to the equations and deal with general potential f which is not necessarily a compactly supported smooth function. Although the fact that $\Gamma * f$ has Laplacian $-f$ is only ensured for compactly supported smooth f, it tells us about the action of the convolution of the fundamental solution and will be of great help. Assume Ω is a C^1 bounded area. Now for any $x \in \Omega$, there always exists $0 < r < dist(x, \partial\Omega)$ such that $B_r(x) \subset\subset \Omega$. Denote $\Omega = \Omega - B_r(x)$.

In the last section, the Poisson equation on the whole space $\mathbb{R}^n$ with potential $f \in C_c^\infty(\mathbb{R}^n)$ is completely solved by taking the convolution of the fundamental solution and the potential to construct the Newtonian potential. However, in general region Ω, the boundary term in Green's identity won't go away, creating trouble for solving the Dirichlet problem.

To see this, by Green's identity, for $u \in C^2(\Omega)$

$$\int_{\Omega_r} \Gamma(y-x)\Delta u(y)\,dy = \int_{\Omega_r} \Delta\Gamma(y-x)u(y)\,dy + \int_{\partial\Omega} \left(\Gamma(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Gamma}{\partial \nu}(y-x)\right) dS(y) \tag{417}$$

$$- \int_{\partial B_r(x)} \left(\Gamma(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Gamma}{\partial \nu}(y-x)\right) dS(y) \tag{418}$$

where $\Delta\Gamma(y-x)$ is the gradient of Γ evaluated at $y-x$ and ν is the outward unit normal vector on the boundary of the integration region. Same calculations in the previous theorem tells us that

$$\int_{\partial B_r(x)} \left(\Gamma(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Gamma}{\partial \nu}(y-x)\right) dS(y) \to u(x)\ (r \to 0) \tag{419}$$

$$\int_{\Omega_r} \Delta\Gamma(y-x)u(y)\,dy \to 0\ (r \to 0) \tag{420}$$

but since f is not trivial on the boundary Ω_r any longer (which is assumed in the previous case since f has compact support), boundary terms won't vanish. This gives us **the formula for solving Dirichlet problems on general area Ω**

$$u(x) = \int_{\partial\Omega} \left(\Gamma(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Gamma}{\partial \nu}(y-x)\right) dS(y) - \int_{\Omega} \Gamma(y-x)\Delta u(y)\,dy \tag{421}$$

Notice that everything related to Γ is already known and for the Dirichlet problem

$$\begin{cases} -\Delta u = f\ (on\ \Omega) \\ u|_{\partial\Omega} = g \end{cases} \tag{422}$$

$\Delta u, u|_{\partial\Omega}$ are also known. The only unknown thing in the formula above is $\frac{\partial u}{\partial \nu}$**, the boundary normal derivative of** u. Unfortunately, any information of this term is not given in the Dirichlet problem and it seems impossible to know the boundary normal derivative without even solving out the solution u.

This is where the **Green's function** is introduced. The main spirit is to **add some terms to the fundamental**

solution that helps get rid of the boundary normal derivative of u but is still harmonic inside Ω. In detail, one adds the **corrector function** $h^x(y) \in C^2(\overline{\Omega})$ to the fundamental solution to get the Green's function

$$G(x,y) = \Gamma(y-x) - h^x(y) \ (x \in \Omega, y \in \overline{\Omega}, x \neq y) \tag{423}$$

Remark. *Under the problem setting, we are integrating w.r.t. y both on the boundary $\partial\Omega$ and inside the area Ω, and trying to derive the value of u at x inside the area Ω. That's why Green's function is defined for $x \in \Omega, y \in \overline{\Omega}$. The condition $x \neq y$ is due to the fact that fundamental solution has a singularity at 0 and $\Gamma(y-x)$ appears in the integrals.*

Such corrector function should satisfy the **condition** that

$$\Delta h^x(y) = 0 \ (\forall y \in \Omega) \tag{424}$$

$$h^x(y)|_{y \in \partial\Omega} = \Gamma(y-x) \tag{425}$$

where the first condition makes sure that the corrector function does not interfere with the functionality of fundamental solution and the second condition makes sure that the boundary normal derivative term disappears, i.e. $\forall y \in \partial\Omega, G(x,y) = \Gamma(y-x) - \Gamma(y-x) = 0$.

As expected, we show in the following theorem that replacing the fundamental solution with the Green's function provides a representation of the solution to the Dirichlet problem.

Theorem 27. ***(Representation Formula for Dirichlet Problem)*** *If $u \in C^2$ solves the Dirichlet problem $\begin{cases} -\Delta u = f \ (on\ \Omega) \\ u|_{\partial\Omega} = g \end{cases}$, then such u can be written as*

$$u(x) = -\int_{\partial\Omega} g(y) \frac{\partial G(x,y)}{\partial \nu_y} \, dS(y) + \int_\Omega G(x,y) f(y) \, dy \tag{426}$$

*where $k(x,y) = -\frac{\partial G(x,y)}{\partial \nu_y}$ is defined as **the Poisson kernel** and ν_y stands for the outward unit normal vector on $\partial\Omega$ w.r.t. variable y.*

Proof. As previously calculated, the formula for u inside Ω is provided by the Green's identity that

$$
\begin{aligned}
u(x) &= \int_{\partial\Omega}\left(\Gamma(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Gamma}{\partial \nu}(y-x)\right)\,dS(y) - \int_{\Omega}\Gamma(y-x)\Delta u(y)\,dy && (427)\\
&= -\int_{\partial\Omega} g(y)\frac{\partial \Gamma}{\partial \nu}(y-x)\,dS(y) + \int_{\Omega}\Gamma(y-x)f(y)\,dy + \int_{\partial\Omega}\Gamma(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) && (428)\\
&= -\int_{\partial\Omega} g(y)\frac{\partial G(x,y)}{\partial \nu_y}\,dS(y) + \int_{\Omega} G(x,y)f(y)\,dy - \int_{\partial\Omega} g(y)\frac{\partial h^x(y)}{\partial \nu_y}\,dS(y) && (429)\\
&+ \int_{\Omega} h^x(y)f(y)\,dy + \int_{\partial\Omega}\Gamma(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) && (430)\\
&= -\int_{\partial\Omega} g(y)\frac{\partial G(x,y)}{\partial \nu_y}\,dS(y) + \int_{\Omega} G(x,y)f(y)\,dy && (431)
\end{aligned}
$$

where the Green's identity applied for $h^x(y)$ and $u(y)$ tells us that

$$
\begin{aligned}
-\int_{\partial\Omega} g(y)\frac{\partial h^x(y)}{\partial \nu_y}\,dS(y) + \int_{\Omega} h^x(y)f(y)\,dy + \int_{\partial\Omega}\Gamma(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) &= -\int_{\Omega} u(y)\Delta h^x(y)\,dy && (432)\\
&= 0 && (433)
\end{aligned}
$$

$\square$

Remark. *The representation formula completely solves the Dirichlet problem under the condition that the Green's function is known. The solution consists of two parts:* ***the Poisson integral***

$$\int_{\partial\Omega} g(y)k(x,y)\,dS(y) \tag{434}$$

as the recovery of the boundary condition, and ***the integral w.r.t. the Green's function***

$$\int_{\Omega} G(x,y)f(y)\,dy \tag{435}$$

as the recovery of the Laplacian.

Remark. *The corrector function can be seen as the solution to another Dirichlet problem that for each fixed $x\in\Omega$,*

$$
\begin{cases}
\Delta h^x = 0 \ (on\ \Omega)\\
h^x(y) = \Gamma(y-x)\ (\forall y\in\partial\Omega)
\end{cases} \tag{436}
$$

As a result, the boundary condition of h^x kills the boundary normal derivative of u and the harmonic property of h^x ensures that replacing the fundamental solution $\Gamma(y-x)$ with Green's function $G(x,y)$ keeps the representation of u invariant.

Remark. *Notationally, one might also write the Green's function as the solution to a Dirichlet problem that for each* *fixed $x \in \Omega$, see $G(x,y) = G(x,\cdot)$ as a function of y*

$$\begin{cases} -\Delta G = -\delta_x \ (on\ \Omega) \\ G|_{\partial\Omega} = 0 \end{cases} \tag{437}$$

note that δ_x is the delta function at x. To understand this notation, note that $G(x,y) = \Gamma(y-x) - h^x(y)$ and $h^x(y)$ is harmonic for $\forall y \in \Omega$ while the fundamental solution is harmonic on $\mathbb{R}^n - \{0\}$. The trivial boundary condition of G is by the definition of the corrector function and is to kill the boundary normal derivative. For the Laplacian of G at $y = x$, refer to the representation formula above. The Green's identity applied for $G(x,\cdot)$ and u gives

$$\int_\Omega u(y)\Delta G(x,y)\,dy = \int_\Omega \Delta u(y) G(x,y)\,dy + \int_{\partial\Omega} \left(u(y)\frac{\partial G(x,y)}{\partial \nu_y} - G(x,y)\frac{\partial u}{\partial \nu}(y) \right) dS(y) \tag{438}$$

$$= -\int_\Omega f(y)G(x,y)\,dy + \int_{\partial\Omega} g(y)\frac{\partial G(x,y)}{\partial \nu_y}\,dS(y) \tag{439}$$

combine with the representation formula above to see

$$\int_\Omega u(y)\Delta G(x,y)\,dy = -u(x) = -\lim_{r\to 0}\frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,dy \tag{440}$$

as a result, it's reasonable to say that locally

$$\Delta G(x,y) \sim -\lim_{r\to 0}\frac{1}{|B_r(x)|}\mathbb{I}_{B_r(x)}(y) \tag{441}$$

*which is approximately the negative delta function $-\delta_x$. Actually Green's identity can't be applied to Green's function so the arguments above are just correct notationally. Nevertheless, it tells us an important fact that **intuitively Green's function is just constructed such that its Laplacian is a Dirac Delta function**.*

Theorem 28. ***(Symmetricity)*** *Green's function is symmetric, i.e. $\forall x, y \in \Omega, x \neq y, G(x,y) = G(y,x)$.*

Proof. Set $v(z) = G(x,z)$ to be a function defined on $\overline{\Omega}$ for fixed $x \in \Omega$ and $z \neq x$, $w(z) = G(y,z)$ to be a function defined on $\overline{\Omega}$ for fixed $y \in \Omega$ and $z \neq y$, we just want to prove that $v(y) = w(x)$.

By the definition of Green's function, $\Delta v = 0, \Delta w = 0$ are harmonic and satisfy the boundary condition that $\forall z \in \partial\Omega, v(z) = 0, w(z) = 0$. Apply the Green's identity for v, w on the region $\Omega_r = \Omega - B_r(x) - B_r(y)$ to get

$$\int_{\Omega_r} v(z)\Delta w(z)\,dz = \int_{\Omega_r} w(z)\Delta v(z)\,dz + \int_{\partial\Omega_r} \left(v(z)\frac{\partial w}{\partial\nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z) \right) dS(z) \tag{442}$$

$$0 = -\int_{\partial B_r(x)} \left(v(z)\frac{\partial w}{\partial\nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z) \right) dS(z) - \int_{\partial B_r(y)} \left(v(z)\frac{\partial w}{\partial\nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z) \right) dS(z) \tag{443}$$

where ν is the outward unit normal vector. Since w is smooth near x and v is smooth near y,

$$\left|\int_{\partial B_r(x)} v(z)\frac{\partial w}{\partial \nu}(z)\, dS(z)\right| \leq C \cdot |\partial B_r(x)| \cdot \sup_{\partial B_r(x)} |v| \to 0 \ (r \to 0) \tag{444}$$

$$\left|\int_{\partial B_r(y)} w(z)\frac{\partial v}{\partial \nu}(z)\, dS(z)\right| \leq C \cdot |\partial B_r(y)| \cdot \sup_{\partial B_r(y)} |w| \to 0 \ (r \to 0) \tag{445}$$

since $|\partial B_r(x)|$ is shrinking with order r^{n-1} and $\sup_{\partial B_r(x)} |v|$ is growing with order

$$\Gamma(z-x) = \frac{1}{n(n-2)V_n(1)}||z-x||^{2-n} \leq Cr^{2-n}, \ (z \in \partial B_r(x)) \tag{446}$$

so the product has order $o(r) \to 0$ $(r \to 0)$. Now the equality reduces under the limit $r \to 0$ to

$$\int_{\partial B_r(y)} v(z)\frac{\partial w}{\partial \nu}(z)\, dS(z) = \int_{\partial B_r(x)} w(z)\frac{\partial v}{\partial \nu}(z)\, dS(z) \tag{447}$$

by noticing that

$$\begin{aligned}\int_{\partial B_r(y)} v(z)\frac{\partial w}{\partial \nu}(z)\, dS(z) &= \int_{\partial B_r(y)} v(z)\frac{\partial \Gamma(z-y) - h^y(z)}{\partial \nu}\, dS(z) &(448)\\ &= \int_{\partial B_r(y)} \left(v(z)\frac{\partial \Gamma}{\partial \nu}(z-y) - \frac{\partial h^y}{\partial \nu}(z)\right) dS(z) &(449)\end{aligned}$$

with the corrector function at y which is h^y being smooth near y

$$\left|\int_{\partial B_r(y)} \frac{\partial h^y}{\partial \nu}(z)\, dS(z)\right| \leq C|\partial B_r(y)| \to 0 \ (r \to 0) \tag{450}$$

and by previous calculations

$$\int_{\partial B_r(y)} v(z)\frac{\partial \Gamma}{\partial \nu}(z-y)\, dS(z) \to -v(y) \ (r \to 0) \tag{451}$$

so

$$\int_{\partial B_r(y)} v(z)\frac{\partial w}{\partial \nu}(z)\, dS(z) \to -v(y) \ (r \to 0) \tag{452}$$

we conclude that

$$v(y) = w(x) \tag{453}$$

which ends the proof. $\square$

Remark. *The symmetric property of the Green's functions seems surprising at first glance since it's telling us that*

$$\forall x, y \in \Omega, \Gamma(y-x) - h^x(y) = \Gamma(x-y) - h^y(x) \tag{454}$$

where h^x is the corrector function at x and h^y is the corrector function at y. However, this is actually natural from the point of view that h^x, h^y are not local since they are the solutions to the Dirichlet problems with certain boundary conditions.

If we adopt previous understanding on the Green's function $G(x,y)$ as the solution to the Dirichlet problem such that its potential is $-\delta_x$ with trivial boundary condition and $G(y,x)$ as the solution to the Dirichlet problem such that its potential is $-\delta_y$ with trivial boundary condition, the dual structure is natural.

Remark. *One frequently used property of the fundamental solution in previous calculations of the representation formula is that* ***the boundary normal derivative of the fundamental solution at its singularity 0 recovers function value near a point through convolution for any function that is smooth near such point***

$$\int_{\partial B_r(x)} u(y) \frac{\partial \Gamma}{\partial \nu}(x-y)\, dS(y) \to u(x) \ (r \to 0) \tag{455}$$

Existence of Green's Function

Although the representation formula is very useful in finding the solutions to the Dirichlet problem, the Green's functions does not necessarily exist since it depends on the structure of the area Ω. However, knowing the existence of Green's function on a certain region may provide us with much information and enables us to make estimations. Let's use an example to illustrate how to use Green's function under the assumption that it exists.

Now assume that Ω is an ellipsoid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} \leq 1$ and we assume that the Green's function on such ellipsoid exists. For any $u \in C^2(\Omega)$,

$$\max_{\overline{\Omega}} |u| \leq \max_{\partial\Omega} |u| + \frac{1}{2\left(a^{-2} + b^{-2} + c^{-2}\right)} \max_{\overline{\Omega}} |\Delta u| \tag{456}$$

actually we have already proved this fact by constructing another sub-harmonic function in the context above, but here we will take a different approach with the use of Green's function.

By the representation formula, one immediately knows that

$$\forall x \in \Omega, u(x) = \int_{\partial\Omega} k(x,y) u(y)\, dS(y) + \int_\Omega G(x,y) \Delta u(y)\, dy \tag{457}$$

and the Poisson kernel integrates to 1, as a result, one get the estimate that

$$\max_{\overline{\Omega}} |u| \leq \max_{\partial\Omega} |u| + \max_{x\in\Omega} \left| \int_\Omega G(x,y)\, dy \right| \cdot \max_{\overline{\Omega}} |u| \tag{458}$$

Since we do not know the specific form of the Green's function, so it seems difficult to figure out its integral.

However, one can notice that the representation formula holds for any $u \in C^2(\Omega)$ and thus holds for the function

$$v(x) = -\frac{1}{2\left(a^{-2}+b^{-2}+c^{-2}\right)}\left(1-\frac{x_1^2}{a^2}-\frac{x_2^2}{b^2}-\frac{x_3^2}{c^2}\right) \tag{459}$$

the auxiliary function v is formed in this way such that it vanishes on the boundary $\partial\Omega$ and $\Delta v = 1$. Plug in the representation formula

$$v(x) = \int_\Omega G(x,y)\Delta v(y)\,dy = \int_\Omega G(x,y)\,dy \tag{460}$$

to know immediately that

$$\max_{x\in\Omega}\left|\int_\Omega G(x,y)\,dy\right| = \frac{1}{2\left(a^{-2}+b^{-2}+c^{-2}\right)} \tag{461}$$

and the proposition gets proved. **The inspirational point is that one can plug in appropriate auxiliary function in the representation formula to find properties of the Green's function.**

General Strategy to Apply Green's Function

The main idea is to find an appropriate corrector function. For fixed $x \in \Omega$, **consider the corrector function of the following form**

$$h^x(y) = \Gamma(b(x)\cdot[y-a(x)])\ (b\in\mathbb{R}, b\neq 0, a\in\mathbb{R}^n-\Omega) \tag{462}$$

such corrector function always satisfies the harmonic property since

$$\forall y\in\Omega, \Delta h^x(y) = b^2(x)\cdot\Delta\Gamma(b(x)\cdot[y-a(x)]) = 0 \tag{463}$$

since $a(x)\notin\Omega$, the singularity (origin) of the fundamental solution is always avoided.

However, such construction cannot naturally satisfy the boundary condition that $\forall y\in\partial\Omega, h^x(y) = \Gamma(y-x)$. To find the appropriate a, b such that this condition holds, there are some starting points. Since the fundamental solution only depends on the radial value, **an equivalent condition** is that

$$\forall x\in\Omega, y\in\partial\Omega, |b(x)|\cdot||y-a(x)|| = ||y-x|| \tag{464}$$

in other words, find the point $a(x)$ such that the distance between any boundary point y and the transformed point $a(x)$ is always a multiple $b(x)$ of the distance between y and x. Such a, b is not always easy to find, but for area Ω with good symmetricity, the selection would be obvious.

Green's Function for the Upper Half Plane

Now take $\Omega = \mathbb{R}^n_+$ as the open upper half plane with the boundary $\partial\Omega = \{x \in \mathbb{R}^n : x_n = 0\} \cong \mathbb{R}^{n-1}$. It's easy to think of the reflection transformation as $a(x)$ since if

$$a(x) = (x_1, ..., x_{n-1}, -x_n) \tag{465}$$

is the reflection w.r.t. the boundary, the quotient of the distance is

$$\forall x \in \Omega, y \in \partial\Omega, |b(x)| = \frac{||y-x||}{||y-a(x)||} = 1 \tag{466}$$

As a result, the corrector function can be chosen as

$$h^x(y) = \Gamma(y - a(x)) \tag{467}$$

and the Green's function is just

$$G(x,y) = \Gamma(y-x) - \Gamma(y-a(x)) \ (x \in \mathbb{R}^n_+, y \in \overline{\mathbb{R}^n_+}, x \neq y) \tag{468}$$

To get a simplified representation formula for the solution to the Dirichlet problem on the upper half plane, calculate the Poisson kernel

$$k(x,y) = -\frac{\partial G}{\partial \nu_y}(x,y) = \frac{\partial G}{\partial y_n}(x,y) \tag{469}$$

since the boundary unit normal vector $\nu_y = -e_n$ where e_n is the vector such that the only non-zero component is the n-th component and it's 1.

When $n = 2$, the Poisson kernel is

$$k(x,y) = \frac{x_2}{\pi||y-x||^2} \tag{470}$$

when $n \geq 3$, the Poisson kernel is

$$k(x,y) = \frac{2x_n}{nV_n(1)||y-x||^n} \tag{471}$$

and the Poisson kernel for $n \geq 3$ is actually consistent with that for $n = 2$. The following theorem concludes what we have done above with the area being the upper half plane.

Theorem 29. ***(Poisson's Formula for the Upper Half Plane)*** *Assume $g \in C(\partial\mathbb{R}^n_+) \cap L^\infty(\partial\mathbb{R}^n_+)$,*

$$\forall x \in \Omega, u(x) = \frac{2x_n}{nV_n(1)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{||y-x||^n}\, dS(y) \tag{472}$$

then such $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 \ (on\ \mathbb{R}^n_+) \\ u|_{\partial \mathbb{R}^n_+} = g \end{cases} \tag{473}$$

The boundary condition is satisfied in the sense of limit that

$$\forall x_0 \in \partial \mathbb{R}^n_+, \lim_{x \to x_0, x \in \mathbb{R}^n_+} u(x) = g(x_0) \tag{474}$$

Green's Function for the Unit Ball

Now take $\Omega = B_1(0)$ as the open unit ball centered at origin. It's easy to think of the conjugate transformation as $a(x)$ since if

$$a(x) = \frac{x}{||x||^2} \tag{475}$$

gives the conjugate point of x w.r.t. the sphere, the quotient of the distance is

$$\forall x \in \Omega, y \in \partial\Omega, |b(x)| = \frac{||y - x||}{||y - a(x)||} = ||x|| \tag{476}$$

As a result, the corrector function can be chosen as

$$h^x(y) = \Gamma(||x|| \cdot [y - a(x)]) \tag{477}$$

and the Green's function is just

$$G(x, y) = \begin{cases} \Gamma(y - x) - \Gamma(||x|| \cdot [y - a(x)]) & x \neq 0 \\ \Gamma(-y) - \Gamma(1) & x = 0 \end{cases} \ (x \in B_1(0), y \in \overline{B_1(0)}, x \neq y) \tag{478}$$

note that the conjugate point of 0 w.r.t. the sphere is ∞ so the Green's function at $x = 0$ has to be specified in another way (see the remark for details).

To get a simplified representation formula for the solution to the Dirichlet problem on the upper half plane, calculate the Poisson kernel

$$k(x, y) = -\frac{\partial G}{\partial \nu_y}(x, y) = -\nabla_y G(x, y) \cdot y \tag{479}$$

since the boundary unit normal vector $\nu_y = y$ on the sphere.

When $n = 2$, the Poisson kernel is

$$k(x, y) = \frac{1 - ||x||^2}{2\pi ||y - x||^2} \tag{480}$$

when $n \geq 3$, the Poisson kernel is

$$k(x, y) = \frac{1 - ||x||^2}{nV_n(1)||y - x||^n} \tag{481}$$

and the Poisson kernel for $n \geq 3$ is actually consistent with that for $n = 2$. The following theorem concludes what we have done above with the area being the unit ball centered at origin.

Theorem 30. ***(Poisson's Formula for the Standard Unit Ball)*** *Assume $g \in C(\partial B_0(1))$,*

$$\forall x \in \Omega, u(x) = \frac{1}{nV_n(1)} \int_{\partial B_1(0)} g(y) \frac{1 - ||x||^2}{||y - x||^n} \, dS(y) \tag{482}$$

then such $u \in C^\infty(B_1(0))$ solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 \ (\text{on } B_1(0)) \\ u|_{\partial B_1(0)} = g \end{cases} \tag{483}$$

The boundary condition is satisfied in the sense of limit that

$$\forall x_0 \in \partial B_1(0), \lim_{x \to x_0, x \in B_1(0)} u(x) = g(x_0) \tag{484}$$

Remark. *For the unit ball, the transformation $a(x) = \frac{x}{||x||^2}$ gives the conjugate point. However, the origin will be mapped to ∞, so the Green's function is not well-defined for $x = 0$.*

However, this problem can be solved by noticing the fact that Green's function is symmetric, so

$$G(0, y) = G(y, 0) = \Gamma(-y) - \Gamma\left(||y|| \cdot \frac{y}{||y||^2}\right) = \Gamma(-y) - \Gamma(1) \tag{485}$$

since the fundamental solution only depends on the radial value.

Remark. *When $n = 2$, we know that the Poisson kernel is*

$$k(x, y) = \frac{1 - ||x||^2}{2\pi ||y - x||^2} \tag{486}$$

plugging in $x = re^{i\theta}, y = e^{it}, r \in (0, 1), \theta, t \in [0, 2\pi)$ to turn it into the Poisson kernel on the complex plane $\mathbb{C}$ to find *that*

$$k(x,y) = \frac{1}{2\pi}\frac{1-r^2}{(e^{it}-re^{i\theta})(e^{-it}-re^{-i\theta})} \tag{487}$$

$$= \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2} \tag{488}$$

this gives the Poisson integration formula for the Dirichlet problem on standard unit disk in complex analysis that

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}g(e^{it})\,dt \tag{489}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-t)g(e^{it})\,dt \tag{490}$$

where $P_r(\theta) = \frac{1-r^2}{1-2r\cos(\theta)+r^2} = \sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta}$ *is the Poisson kernel in complex analysis and the convolution with such kernel is called Poisson transformation.*

Actually the Dirichlet problem on any ball $B_r(x_0)$ has also been solved since it's just a translation and stretching of the standard unit ball.

Theorem 31. ***(Poisson's Formula for General Balls)*** *Assume* $g \in C(\partial B_r(x_0))$*,*

$$\forall x \in B_r(x_0), u(x) = \frac{r^2-||x-x_0||^2}{rnV_n(1)}\int_{\partial B_r(x_0)}\frac{g(y)}{||y-x||^n}\,dS(y) \tag{491}$$

then such $u \in C^\infty(B_r(x_0))$ *solves the Dirichlet problem*

$$\begin{cases} \Delta u = 0 \ (on\ B_r(x_0)) \\ u|_{\partial B_r(x_0)} = g \end{cases} \tag{492}$$

The boundary condition is satisfied in the sense of limit that

$$\forall y_0 \in \partial B_r(x_0), \lim_{y\to y_0, y\in B_r(x_0)} u(y) = g(y_0) \tag{493}$$

Proof. Consider $\tilde{u}(z) = u(x_0+rz)$ satisfying the Dirichlet problem

$$\begin{cases} \Delta \tilde{u} = 0 \ (on\ B_1(0)) \\ \tilde{u}|_{\partial B_1(0)} = \tilde{g} \end{cases} \tag{494}$$

where $\tilde{g}(z) = g(x_0+rz)$. Apply the Poisson's formula on standard unit ball to get

$$\forall z \in B_1(0), \tilde{u}(z) = \frac{1}{nV_n(1)}\int_{\partial B_1(0)}\tilde{g}(y)\frac{1-||z||^2}{||y-z||^n}\,dS(y) \tag{495}$$

so

$$\forall z \in B_1(0), u(x_0 + rz) = \frac{r}{nV_n(1)} \int_{\partial B_r(x_0)} g(y) \frac{1 - ||z||^2}{||y - x_0 - rz||^n} \, dS(y) \tag{496}$$

$$\forall x \in B_r(x_0), u(x) = \frac{r^2 - ||x - x_0||^2}{rnV_n(1)} \int_{\partial B_r(x_0)} \frac{g(y)}{||y - x||^n} \, dS(y) \tag{497}$$

proved.

□

Application of Representation Formula

Green's function and Poisson's kernel may allow us to figure out the structure of the solution without explicitly solving it. For example, **consider the Dirichlet problem on the upper half plane**

$$\begin{cases} \Delta u = 0 \ in \ \mathbb{R}^n_+ \\ u = g \ on \ \partial \mathbb{R}^n_+ \end{cases} \tag{498}$$

now assume g is bounded and $g(x) = ||x||$ for $\partial \mathbb{R}^n_+ \cap \{x : ||x|| \leq 1\}$, it can be inferred that Du is not bounded near the origin.

By Poisson formula on the upper half plane, the solution can be represented as

$$\forall x \in \Omega, u(x) = \frac{2x_n}{nV_n(1)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{||y - x||^n} \, dS(y) \tag{499}$$

consider $\frac{u(\lambda e_n) - u(0)}{\lambda}$, the directional derivative along the last component when λ is small enough, note that $u(0) = 0$ and

$$u(\lambda e_n) = \frac{2\lambda}{nV_n(1)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{||y - \lambda e_n||^n} \, dS(y) \tag{500}$$

plug in to know that

$$\frac{u(\lambda e_n) - u(0)}{\lambda} = \frac{2}{nV_n(1)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{||y - \lambda e_n||^n} \, dS(y) \tag{501}$$

to check whether such thing is bounded when $\lambda \to 0$, note that g is bounded, one just need to check the boundedness of the following integral near 0

$$\int_{\partial\mathbb{R}^n_+\cap B_1(0)} \frac{g(y)}{||y-\lambda e_n||^n}\,dS(y) = \int_{\partial\mathbb{R}^n_+\cap B_1(0)} \frac{||y||}{||y-\lambda e_n||^n}\,dS(y) \tag{502}$$

$$= \int_{\partial\mathbb{R}^n_+\cap B_1(0)} \frac{\sqrt{y_1^2+...+y_{n-1}^2}}{\left(\sqrt{y_1^2+...+y_{n-1}^2+\lambda^2}\right)^n}\,dS(y) \tag{503}$$

actually this integral is on the unit ball in $\mathbb{R}^{n-1}$ which is $B = \left\{y : y_1^2+...+y_{n-1}^2 \le 1\right\}$ and this surface integral degenerates to a normal integral w.r.t. $z = (y_1,...,y_{n-1}) \in \mathbb{R}^{n-1}$

$$\int_B \frac{||z||}{\left(\sqrt{||z||^2+\lambda^2}\right)^n}\,dz = |\partial B|\cdot\int_0^1 \frac{r}{(r^2+\lambda^2)^{\frac{n}{2}}} r^{n-2}\,dr \tag{504}$$

by using polar coordinates in $\mathbb{R}^{n-1}$ so the Jacobian is r^{n-2}, the problem finally turns into figuring out the boundedness of the integral

$$\int_0^1 \frac{r^{n-1}}{(r^2+\lambda^2)^{\frac{n}{2}}}\,dr \ge 2^{-\frac{n}{2}}\int_0^1 \frac{r^{n-1}}{r^n+\lambda^n}\,dr \tag{505}$$

by using $(a+b)^n \le 2^n(a^n+b^n)$, and the integral on RHS is obviously unbounded as $\lambda \to 0$ since

$$\int_0^1 \frac{r^{n-1}}{r^n+\lambda^n}\,dr = \frac{1}{n}\log(r^n+\lambda^n)|_{r=0\to1} = \frac{\log(\frac{\lambda^n+1}{\lambda^n})}{n} \to \infty\ (\lambda\to 0) \tag{506}$$

As a result, we finally conclude that for the solution u to this problem, Du is not bounded near 0 since the directional derivative along e_n is not bounded.

Actually the representation formula allows us to prove more things about the Poisson kernel. By setting u as the solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = 1 \end{cases} \tag{507}$$

we know that

$$v(x) = \int_{\partial\Omega} k(x,y)\,dS(y) \tag{508}$$

is a solution to this problem. However, by the maximum principle of harmonic functions, such u attains maximum and minimum on the boundary so it has to be constantly 1. This is telling us that

$$\int_{\partial\Omega} k(x,y)\,dS(y) = 1 \tag{509}$$

The integral of the Poisson kernel on the boundary of a region is always 1.

Similarly, one can show the almost everywhere non-negativity of the Poisson kernel. Let's prove by contradiction and assume that the Poisson kernel is strictly negative on a positive measure set in $\partial\Omega$ under the Lebesgue measure on $\partial\Omega$, this is telling us that there exists compact set $K \subset \partial\Omega$ such that the Poisson kernel is strictly negative on K. One may construct a function $g : \partial\Omega \to \mathbb{R}$ such that it takes value 1 on K and shrinks continuously to 0 rapidly. Consider the solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases} \tag{510}$$

it's easy to see that since K has positive measure, $u(x) = \int_{\partial\Omega} g(y)k(x,y)\,dS(y)$ is strictly negative. However, by maximum principle of the harmonic function, u takes values in $[0,1]$ on Ω and this is a contradiction! Thus we have proved that **Poisson kernel is always positive for** $\forall x \in \Omega, y \in \partial\Omega$.

At last, let's show that **Green's function is non-negative for** $\forall x \in \Omega, y \in \overline{\Omega}, y \neq x$. First fix $x \in \Omega$, we only want to show that $f(y) = \Gamma(y-x) - h^x(y) \geq 0$. Notice that the fundamental solution blows up to $+\infty$ when $y \to x$ and $h^x \in C^2(\overline{\Omega})$ so we can definitely find $r > 0$ such that $f \geq 1$ on $\partial B_r(x)$. For $y \in B_r(x)$, of course f is non-negative. For $y \in \Omega - B_r(x)$, note that $\Delta f = 0$ and $\forall y \in \partial\Omega, h^x(y) = \Gamma(y-x)$ so $f = 0$ on $\partial\Omega$. We have argued that $\forall y \in \overline{\Omega}, y \neq x, f \geq 0$.

Remark. *By the interpretation of Green's function in the context above, one may see that notationally Green's function is the solution to the Dirichlet problem*

$$\begin{cases} \Delta_y G(x,y) = -\delta_x \leq 0 \\ G(x,y)|_{y\in\partial\Omega} = 0 \end{cases} \tag{511}$$

As a result, Green's function can be seen as a super-harmonic function and the minimum principle holds. That's why Green's function is always non-negative.

Remark. *One can also mimic the proof for the non-negativity of Poisson kernel to construct $f \in [0,1], g = 0$ such that $u(x) = \int_\Omega G(x,y)f(y)\,dy < 0$.*

Week 7

Actually, if the Poisson integration formula is already known, we can derive the Green's function easily by noticing that the corrector function $h^x(y)$ is actually a solution to the Dirichlet problem with boundary value $\forall y \in \partial\Omega, h^x(y) = \Gamma(y-x)$ and it's also harmonic inside Ω.

$$h^x(y) = \int_{\partial\Omega} k(x,y)\Gamma(y-x)\, dS(y) \tag{512}$$

recovers the corrector function and

$$G(x,y) = \Gamma(y-x) - h^x(y) \tag{513}$$

gives the Green's function.

Remark. *Poisson integration formula also indicates the mean-value property of harmonic function. Set $\Omega = B_r(x_0)$ as the ball and*

$$\forall x \in B_r(x_0), u(x) = \frac{r^2 - ||x-x_0||^2}{rnV_n(1)} \int_{\partial B_r(x_0)} \frac{u(y)}{||y-x||^n}\, dS(y) \tag{514}$$

set $x = x_0$ to get

$$u(x_0) = \frac{r^2}{rnV_n(1)} \int_{\partial B_r(x_0)} \frac{u(y)}{||y-x_0||^n}\, dS(y) \tag{515}$$

$$= \frac{1}{r^{n-1}nV_n(1)} \int_{\partial B_r(x_0)} u(y)\, dS(y) \tag{516}$$

and $|\partial B_r(x_0)| = r^{n-1}nV_n(1)$ recovers the mean-value property on sphere.

Remark. *Poisson Integration formula also provides detailed bounds for Harnack's inequality. Consider the region to be the open ball, then*

$$\forall x \in B_R(x_0), |u(x)| = \frac{R^2 - ||x-x_0||^2}{RnV_n(1)} \left| \int_{\partial B_R(x_0)} \frac{u(y)}{||y-x||^n}\, dS(y) \right| \tag{517}$$

denote $r = ||x-x_0||$ and notice that $R - r \leq ||y-x|| \leq R + r$ with the mean-value property to get

$$|u(x)| \leq \frac{R^2 - r^2}{(R-r)^n RnV_n(1)} \left| \int_{\partial B_R(x_0)} u(y)\, dS(y) \right| \tag{518}$$

$$= \frac{R^{n-2}(R^2-r^2)}{(R-r)^n} |u(x_0)| = \frac{1 + \frac{r}{R}}{(1 - \frac{r}{R})^{n-1}} |u(x_0)| \tag{519}$$

and on the other direction

$$|u(x)| \geq \frac{R^2 - r^2}{(R+r)^n RnV_n(1)} \left| \int_{\partial B_R(x_0)} u(y)\, dS(y) \right| \tag{520}$$

$$= \frac{R^{n-2}(R^2 - r^2)}{(R+r)^n} |u(x_0)| = \frac{1 - \frac{r}{R}}{\left(1 + \frac{r}{R}\right)^{n-1}} |u(x_0)| \tag{521}$$

As a result, we conclude that for harmonic function u on Ω and $\forall x_0 \in \Omega, \forall R > 0$ such that $B_R(x_0) \subset\subset \Omega, \forall x \in B_R(x_0)$ denote $r = ||x - x_0||$, then

$$\frac{1 - \frac{r}{R}}{\left(1 + \frac{r}{R}\right)^{n-1}} |u(x_0)| \leq |u(x)| \leq \frac{1 + \frac{r}{R}}{\left(1 - \frac{r}{R}\right)^{n-1}} |u(x_0)| \tag{522}$$

and this inequality holds without absolute values for non-negative harmonic u.

For non-negative harmonic function u on $\mathbb{R}^2$, this reduces to

$$\frac{R-r}{R+r} u(x_0) \leq u(x) \leq \frac{R+r}{R-r} u(x_0) \tag{523}$$

and if u is non-negative harmonic in unit open disk centered at origin in $\mathbb{R}^2$ with $u(0) = 1$, we can know that

$$\frac{1}{3} \leq u\left(\frac{1}{2}\right) \leq 3 \tag{524}$$

*which is a very strong statement, showing the **global rigidity** of harmonic functions.*

Weak Derivative

The appearance of weak derivative is to solve the regularity problems in solving PDEs. For example, when applying the method of characteristic for first-order PDE, we are actually assuming that the solution has C^2 regularity (necessary to get the equation w.r.t. $p(s)$) although the PDE itself only requires C^1 regularity. If regularity conditions can be relaxed to some extent, then we might be able to generalize those methods and apply them for more general PDEs.

Different from traditional derivatives, the weak derivative is motivated by the **integration by parts**, one of the crucial properties of derivatives. Consider any **test function**, i.e. $\forall \phi \in C_c^\infty(\Omega)$, if the integration by parts hold for $f \in L^1_{loc}(\Omega)$, we would get in 1-dimension that

$$\int_\Omega f(x) \cdot \phi'(x)\, dx = f(x)\phi(x)|_{\partial\Omega} - \int_\Omega f'(x) \cdot \phi(x)\, dx \tag{525}$$

$$= -\int_\Omega f'(x) \cdot \phi(x)\, dx \tag{526}$$

since here ϕ has compact support and Ω is an open area. Here we simply **define f' as the weak derivative of f on Ω**. By definition, integration by parts naturally holds for weak derivatives.

Similarly, we can define **weak i-th partial derivatives g_i of f on Ω** if

$$\forall \phi \in C_c^\infty(\Omega), \int_\Omega f(x) \cdot \partial_i \phi(x)\, dx = -\int_\Omega \partial_i g(x) \cdot \phi(x)\, dx \tag{527}$$

and for **higher order** weak partial derivatives, suppose α is a multi-index, function $f \in L^1_{loc}$ has weak derivative $\partial^\alpha f \in L^1_{loc}(\Omega)$ if

$$\forall \phi \in C_c^\infty(\Omega), \int_\Omega \partial^\alpha f(x) \cdot \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega f(x) \cdot \partial^\alpha \phi(x)\, dx \tag{528}$$

each application of integration by parts gives one -1, that's why $(-1)^{|\alpha|}$ appears.

Remark. *It's clear that the weak derivative has to be at least $L^1_{loc}(\Omega)$ such that its integral on Ω is well-defined. In the following context, still use f' for the weak derivatives.*

The weak derivative is unique if it exists. Notice that since the definition depends on integral, the uniqueness is in the almost everywhere sense.

Theorem 32. ***(Uniqueness of Weak Derivative)*** *If weak derivative exists, it must be unique in the almost everywhere sense.*

Proof. If u, v are both weak derivatives of f on Ω,

$$\forall \phi \in C_c^\infty(\Omega), \int_\Omega \phi(x) \cdot (u(x) - v(x))\, dx = 0 \tag{529}$$

we can conclude that $u = v$ *a.e.*. $\square$

Remark. *There are some hidden details for the last theorem.*

$$\exists f \in L^1_{loc}(\Omega), \forall \phi \in C_c^\infty(\Omega), \int_\Omega f(x) \cdot \phi(x)\, dx = 0 \Longrightarrow f(x) = 0 \ a.e. \tag{530}$$

in brief, C_c^∞ ***is sufficiently large to work as test functions for the space*** L^1_{loc}.

To prove this, for any compact set $K \subset \Omega$*, let's prove that* $f = 0$ *a.e. on* K*, which is sufficient to prove that* $f = 0$ *a.e..*

Firstly, take $\psi \in C_c^\infty(\Omega)$ *such that* $\psi|_K = 1$ *and define*

$$f_\psi(x) = f(x) \cdot \psi(x) \cdot \mathbb{I}_\Omega(x) \tag{531}$$

so f_ψ *is the version of* f *supported within* Ω *and is equal to* f *on* K*. Since* f *is locally integrable,* f_ψ *is integrable on the whole* $\mathbb{R}^n$*. The problem now is that* $\psi \cdot \mathbb{I}_\Omega$ *is not even continuous, so how shall we turn it into a smooth function? Recall that standard mollifiers* η^ε *provides us with an approach to get better regularity. By the approximation identity,*

$$\eta^\varepsilon * f_\psi \overset{a.e.}{\to} f_\psi \ (\varepsilon \to 0) \tag{532}$$

and

$$\eta^\varepsilon * f_\psi(x) = \int f_\psi(y) \cdot \eta^\varepsilon(x-y)\, dy \tag{533}$$

$$= \int_\Omega f(y) \cdot \psi(y) \cdot \eta^\varepsilon(x-y)\, dy \tag{534}$$

with $\psi(y) \cdot \eta^\varepsilon(x-y) \in C_c^\infty(\Omega)$*, so it's easy to see that by taking* $\varepsilon \to 0$*, we can conclude that* $f_\psi = 0$ *a.e., which proves that* $f = 0$ *a.e. on* K*.*

In $C^1(\Omega)$ function space where the true derivative exists, weak derivative exists and is consistent with the true derivative as expected.

Theorem 33. ***(Consistency of Weak Derivative with True Derivative)*** *If* $f \in C^1(\Omega)$*, weak derivative exists and is equal to the true derivative almost everywhere.*

Proof. By noticing that the true derivative has integration by parts to hold, so the true derivative must be weak derivative. By the uniqueness argument above, they are equal almost everywhere. $\square$

The existence of weak derivative is interesting to look into since the existence of weak derivative is not equivalent to the almost everywhere existence of the true derivative.

There are actually some results on the existence of the weak derivatives. Here we just list them without proving. These results show that the existence has some connections with concepts like absolute continuity, Lipschitz and bounded variation.

Theorem 34. *If $f \in L^1_{loc}(a,b)$, then f is Lipschitz on $[a,b]$ if and only if f is weakly differentiable on (a,b) and $f' \in L^\infty(a,b)$.*

Theorem 35. *If $f \in L^1_{loc}(a,b)$, then f is absolutely continuous on $[a,b]$ if and only if f is weakly differentiable on (a,b) and $f' \in L^1(a,b)$.*

The following theorem ensures the uniqueness of the weak solution defined later.

Theorem 36. ***(Uniqueness of Weak Solution)*** *If $f : (a,b) \to \mathbb{R}$ is weakly differentiable and $f' = 0$ a.e., then f is constant.*

Proof. To prove that f is constant, we would think of proving that for some fixed constant C,

$$\forall \phi \in C_c^\infty(\mathbb{R}), \int_a^b (f(x) - C) \cdot \phi(x)\, dx = 0 \tag{535}$$

$$\forall \phi \in C_c^\infty(\mathbb{R}), \int_a^b f(x) \cdot \phi(x)\, dx = C \int_a^b \phi(x)\, dx \tag{536}$$

now take $\eta \in C_c^\infty(a,b), \int_a^b \eta(x)\, dx = 1$ as a smooth density function and consider the decomposition

$$\forall \phi \in C_c^\infty(\mathbb{R}), \phi(x) = A\eta(x) + \phi(x) - A\eta(x) \tag{537}$$

$$= A\eta(x) + \psi'(x) \tag{538}$$

where $\psi(x) = \int_a^x \phi(t) - A\eta(t)\, dt \in C_c^\infty(a,b)$. As a result,

$$\int_a^b f(x) \cdot \phi(x)\, dx = A \int_a^b f(x) \cdot \eta(x)\, dx + \int_a^b f(x) \cdot \psi'(x)\, dx \tag{539}$$

$$= A \int_a^b f(x) \cdot \eta(x)\, dx \tag{540}$$

setting $A = \int_a^b \phi(x)\, dx, C = \int_a^b f(x) \cdot \eta(x)\, dx$ would perfectly match our objective. That's why we conclude that $f(x)$ is constant and $f(x) = C = \int_a^b f(x) \cdot \eta(x)\, dx$. $\square$

Example

Consider $f(x) = x \vee 0$, it's not C^1 on $\mathbb{R}$ but **has weak derivative**. It's easy to guess that $f' = \mathbb{I}_{(0,\infty)}$. To prove it, notice that

$$\forall \phi \in C_c(\mathbb{R}), \int f(x) \cdot \phi'(x)\, dx = \int_0^\infty x \cdot \phi'(x)\, dx \tag{541}$$

$$= -\int_0^\infty \phi(x)\, dx \tag{542}$$

$$= -\int \mathbb{I}_{(0,\infty)}(x) \cdot \phi(x)\, dx \tag{543}$$

However, if we consider $f(x) = \mathbb{I}_{(0,\infty)}(x)$, this function has **no weak derivative**. Assume that it's weak derivative is $g(x)$.

$$\forall \phi \in C_c^\infty(\mathbb{R}), \int g(x) \cdot \phi(x)\, dx = -\int f(x) \cdot \phi'(x)\, dx \tag{544}$$

$$= -\int_0^\infty \phi'(x)\, dx \tag{545}$$

$$= \phi(0) \tag{546}$$

take $\phi^y(x) = \eta^\varepsilon(y-x)$ as the translated standard mollifier to see that

$$\int g(x) \cdot \phi^y(x)\, dx = g * \eta^\varepsilon(y) \stackrel{a.e.}{\to} g(y)\ (\varepsilon \to 0) \tag{547}$$

so by the conclusion above

$$\forall y \in \mathbb{R}, g(y) = \phi(0)\ a.e. \tag{548}$$

the weak derivative is constant, and this leads to the result that

$$\forall \phi \in C_c^\infty(\mathbb{R}), \int g(x) \cdot \phi(x)\, dx = \phi(0) \int \phi(x)\, dx = \phi(0) \tag{549}$$

which is a contradiction since $\int \phi(x)\, dx$ is not necessarily 1.

There are also function **continuous with derivative existing almost everywhere but has no weak derivative**. The example is the Cantor function $C(x)$ on $[0,1]$. It's derivative is 0 almost everywhere since Cantor set has measure 0. If it has weak derivative g, then

$$\forall \phi \in C_c^\infty(0,1), \int_0^1 C(x) \cdot \phi'(x)\, dx = -\int_0^1 g(x) \cdot \phi(x)\, dx \tag{550}$$

notice that $\int_0^1 C(x) \cdot \phi'(x)\, dx = -\int_0^1 \phi(x)\, dC(x)$, so

$$\forall \phi \in C_c^\infty(0,1), \int_0^1 \phi(x)\, dC(x) = \int_0^1 g(x) \cdot \phi(x)\, dx \tag{551}$$

shrink the support of $\phi(x)$ to the subsets of the complement of Cantor set to see that if such equation holds, $g = 0\ a.e.$

$$\forall \phi \in C_c^\infty(0,1), \int_0^1 \phi(x)\, dC(x) = 0 \tag{552}$$

$$\forall \phi \in C_c^\infty(0,1), X \sim Cantor, \mathbb{E}\phi(X) = 0 \tag{553}$$

so $X \stackrel{d}{=} 0$, a contradiction! (equality in distribution has continuous bounded functions as test functions)

Conversely, **a weakly differentiable function is not necessarily almost everywhere differentiable**. The counterexample is $\mathbb{I}_{\mathbb{Q}}(x)$ on $\mathbb{R}$. It's obvious that the weak derivative of this function is 0 since

$$\forall \phi \in C_c^\infty(\mathbb{R}), \int \mathbb{I}_{\mathbb{Q}}(x) \cdot \phi(x)\, dx = 0 \tag{554}$$

however, this function is not continuous everywhere and is not differentiable everywhere.

Remark. *The equivalence fails because differentiability is a local concept, i.e. it only has something to do with the behavior of the function in a neighborhood of a point. However, the weak differentiability is a global concept since it's defined by integrals.*

Actually, one might be able to see that continuous functions are "almost" weakly differentiable. The function is truly weakly differentiable if absolute continuity condition also holds. Another example showing the gap between continuity and weakly differentiable is the path of Brownian motion.

Let's look at the last example which provides some intuition on weak derivatives. Consider $a \in \mathbb{R}, f : B_1(0) \to \mathbb{R}, f(x) = ||x||^{-a}$ and discuss for what values of a will weak derivatives exist. We see that except the origin, the function is C^1 with partial derivative

$$\partial_i f(x) = -\frac{ax_i}{||x||^{a+2}} \tag{555}$$

heuristically, the existence of weak derivative on $\mathbb{R}^n$ depends on whether $\partial_i f$ is locally integrable at 0. See the partial derivative as $\partial_i f(x) = -a\frac{x_i}{||x||}\frac{1}{||x||^{a+1}}$, then the locally integrability at 0 depends on the term $\frac{1}{||x||^{a+1}}$. Since in $\mathbb{R}^n$, $\frac{1}{||x||^\beta}$ is integrable at 0 if and only if $\beta < n$, guess that **the weak derivative of such function exists and is integrable if and only if** $a < n - 1$.

Let's only prove the non-trivial direction from $a < n-1$ to the existence of weak derivative. Apply the divergence theorem for $(0, ..., 0, f(x) \cdot \phi(x), 0, ..., 0)$ for which only the i-th component is nonzero on the area $B_1(0) - B_r(0)$ for $\forall r > 0, \forall \phi \in C_c^\infty(B_1(0))$ to get

$$\int_{B_1(0)-B_r(0)} f_i(x) \cdot \phi(x)\, dx + \int_{B_1(0)-B_r(0)} \phi_i(x) \cdot f(x)\, dx = -\int_{\partial B_r(0)} f(x) \cdot \phi(x) \cdot \nu_i\, dS(x) \tag{556}$$

where ν is the outward unit normal vector of $\partial B_r(0)$. Now we conclude that the boundary term $\int_{\partial B_r(0)} f(x) \cdot \phi(x) \cdot \nu_i\, dS(x)$ vanishes when $r \to 0$.

$$\left|\int_{\partial B_r(0)} f(x) \cdot \phi(x) \cdot \nu_i\, dS(x)\right| \le C \left|\int_{\partial B_r(0)} r^{-a}\, dS(y)\right| = C' \cdot r^{n-1-a} \to 0 \ (r \to 0) \tag{557}$$

since ϕ is compactly supported and $n - 1 - a > 0$. That proves

$$\int_{B_1(0)} f_i(x) \cdot \phi(x)\, dx = -\int_{B_1(0)} \phi_i(x) \cdot f(x)\, dx \tag{558}$$

since f_i, f are both locally integrable at 0 when taking the limit $r \to 0$. This is exactly the definition of weak derivative.

Property of Weak Derivative

Theorem 37. ***(Weak Derivative of the Product)*** *If $f \in L^1_{loc}(\Omega)$ has weak partial derivative $\partial_i f \in L^1_{loc}(\Omega)$ and $\psi \in C^\infty(\Omega)$, then $\psi \cdot f$ is weakly differentiable w.r.t. x_i and $\partial_i(\psi \cdot f) = (\partial_i \psi) \cdot f + \psi \cdot (\partial_i f)$.*

Proof.

$$\forall \phi \in C_c^\infty(\Omega), \int_\Omega \partial_i f \cdot (\phi \cdot \psi)\, dx = -\int_\Omega f \cdot \partial_i(\phi \cdot \psi)\, dx \tag{559}$$

$$= -\int_\Omega f \cdot \phi \cdot \partial_i \psi\, dx - \int_\Omega f \cdot \partial_i \phi \cdot \psi\, dx \tag{560}$$

$$\int_\Omega f \cdot \partial_i \phi \cdot \psi\, dx = -\int_\Omega \phi \cdot (\partial_i f \cdot \psi + \partial_i \psi \cdot f)\, dx \tag{561}$$

telling us that $\partial_i(f \cdot \psi) = (\partial_i \psi) \cdot f + \psi \cdot (\partial_i f)$ □

Remark. *Here ψ is assumed to be C^∞ such that $\phi \cdot \psi \in C^\infty$, which is necessary to use the definition of the weak derivative of f. However, the condition for such psi can be weakened to $C^1(\Omega)$ and the result will become stronger. That proof can't be derived directly from the definition of the weak derivative.*

Theorem 38. ***(Commutativity of Weak Derivatives)*** *$f \in L^1_{loc}(\Omega)$ has weak partial derivatives $\partial^\alpha f, \partial^\beta f$ exist for multi-index α, β. If one of the weak derivative $\partial^{\alpha+\beta} f, \partial^\alpha \partial^\beta f, \partial^\beta \partial^\alpha f$ exists, then all three weak derivatives exist and are equal.*

Proof. Still by definition,

$$\forall \phi \in C_c^\infty(\Omega), \int_\Omega \partial^\alpha f \cdot \partial^\beta \phi\, dx = (-1)^{|\alpha|} \int_\Omega f \cdot \partial^{\alpha+\beta} \phi\, dx \tag{562}$$

if $\partial^{\alpha+\beta} f$ exists, then

$$\forall \phi \in C_c^\infty(\Omega), \int_\Omega \partial^\alpha f \cdot \partial^\beta \phi\, dx = (-1)^{|\beta|} \int_\Omega \partial^{\alpha+\beta} f \cdot \phi\, dx \tag{563}$$

so $\partial^{\alpha+\beta} f$ is actually working as the ∂^β of $\partial^\alpha f$, so $\partial^\beta \partial^\alpha f$ exists. By the uniqueness of weak derivative, they are equal almost everywhere.

Conversely, if $\partial^\beta \partial^\alpha f$ exists, then

$$\forall \phi \in C_c^\infty(\Omega), (-1)^{|\beta|} \int_\Omega \partial^\beta \partial^\alpha f \cdot \phi\, dx = (-1)^{|\alpha|} \int_\Omega f \cdot \partial^{\alpha+\beta} \phi\, dx \tag{564}$$

so $\partial^\beta \partial^\alpha f$ is actually working as the $\partial^{\alpha+\beta}$ of f, so $\partial^{\alpha+\beta} f$ exists. By the uniqueness of weak derivative, they are equal almost everywhere. □

Example

Consider

$$u(x,y) = f(x) + g(y) \in L^1_{loc}(\mathbb{R}) \tag{565}$$

it's clear that $u \in L^1_{loc}(\mathbb{R})$ if and only if $f, g \in L^1_{loc}(\mathbb{R})$. It's not at all surprising to state that $\partial_x u$ **exists if and only if the weak derivative of f exists, and $\partial_x u = f'$.**

$$\forall \phi \in C_c^\infty(\mathbb{R}^2), \int u(x,y) \cdot \partial_x \phi(x,y)\, dx\, dy \tag{566}$$

$$= \int f(x) \int \partial_x \phi(x,y)\, dy\, dx + \int g(y) \int \partial_x \phi(x,y)\, dx\, dy \tag{567}$$

$$= \int f(x) \int \partial_x \phi(x,y)\, dy\, dx \tag{568}$$

by Fubini's theorem. Now set $\xi(x) = \int \phi(x,y)\, dy \in C_c^\infty(\mathbb{R})$, and notice that

$$\forall \phi \in C_c^\infty(\mathbb{R}^2), \int u(x,y) \cdot \partial_x \phi(x,y)\, dx\, dy = \int f(x) \cdot \xi'(x)\, dx \tag{569}$$

Now if $\partial_x u$ exists, then

$$\forall \phi \in C_c^\infty(\mathbb{R}^2), -\int \partial_x u(x,y) \cdot \phi(x,y)\, dx\, dy = \int f(x) \cdot \xi'(x)\, dx \tag{570}$$

$$= \int f(x) \cdot \int \partial_x \phi(x,y)\, dy\, dx \tag{571}$$

meaning that $\partial_x u$ is working as f', weak derivative exists and by uniqueness they are equal.

Now if f' exists, then

$$\forall \phi \in C_c^\infty(\mathbb{R}^2), \int u(x,y) \cdot \partial_x \phi(x,y)\, dx\, dy = -\int f'(x) \cdot \xi(x)\, dx \tag{572}$$

$$= -\int f'(x) \cdot \int \phi(x,y)\, dy\, dx \tag{573}$$

meaning that f' is working as $\partial_x u$, weak derivative exists and by uniqueness they are equal.

Now let's consider second-order weak derivatives by their definitions, e.g. $\partial_x \partial_y u$.

$$\forall \phi \in C_c^\infty(\mathbb{R}^2), \int \partial_y u(x,y) \cdot \partial_x \phi(x,y)\, dx\, dy \tag{574}$$

$$= \int \partial_y f(x) \int \partial_x \phi(x,y)\, dy\, dx + \int \partial_y g(y) \int \partial_x \phi(x,y)\, dx\, dy \tag{575}$$

$$= \int \partial_y g(y) \int \partial_x \phi(x,y)\, dx\, dy = 0 \tag{576}$$

since ϕ has compact support, telling us that $\partial_x \partial_y u$ exists and is 0. Then by the commutativity of weak derivatives, $\partial_{xy} u = \partial_y \partial_x = 0$. This can be seen both through the definition of weak derivatives and the fact that $\partial_y \partial_x u(x, y) = \partial_y f'(x) = 0$.

Week 8

Properties of Weak Derivative

The following theorem shows that the real derivative of a mollified locally integrable function is just the mollified weak derivative of such function. This result directly leads to the generalization of the approximation identity mentioned above for the mollifier.

Theorem 39. ***(Approximation Identity for Weak Derivative)*** *Suppose $f \in L^1_{loc}(\Omega)$ has weak derivative $\partial^\alpha f \in L^1_{loc}(\Omega)$, then $\eta^\varepsilon * f \in C^\infty(\Omega)$ and*

$$\partial^\alpha(\eta^\varepsilon * f) = \eta^\varepsilon * \partial^\alpha f \tag{577}$$

Moreover,

$$\partial^\alpha(\eta^\varepsilon * f) \overset{L^1_{loc}(\Omega)}{\to} \partial^\alpha f \ (\varepsilon \to 0) \tag{578}$$

Proof. Notice the fact that

$$\partial^\alpha(\eta^\varepsilon * f) = (\partial^\alpha \eta^\varepsilon) * f \tag{579}$$

this is due to the dominated convergence theorem since $\partial^\alpha \eta^\varepsilon$ is always bounded for each fixed ε. By the definition of the weak derivative,

$$\begin{aligned}
(\partial^\alpha \eta^\varepsilon) * f(x) &= \int \partial_x^\alpha \eta^\varepsilon(x-y) f(y)\, dy && (580)\\
&= (-1)^{|\alpha|} \int \partial_y^\alpha \eta^\varepsilon(x-y) f(y)\, dy && (581)\\
&= \int \eta^\varepsilon(x-y) \cdot \partial^\alpha f(y)\, dy && (582)\\
&= \eta^\varepsilon * \partial^\alpha f(x) && (583)
\end{aligned}$$

Due to the approximation identity of mollifier, for $\partial^\alpha f \in L^1_{loc}(\Omega)$,

$$\partial^\alpha(\eta^\varepsilon * f) = \eta^\varepsilon * \partial^\alpha f \overset{L^1_{loc}(\Omega)}{\to} \partial^\alpha f \ (\varepsilon \to 0) \tag{584}$$

□

Such approximation identity leads to the characterization of weak derivative with the approximation using smooth functions.

Theorem 40. ***(Characterization of Weak Derivative with Smooth Function Approximation)***

$f \in L^1_{loc}(\Omega)$ is weakly differentiable if and only if there exists a sequence of smooth function $f_n \in C^\infty(\Omega)$ and $g \in L^1_{loc}(\Omega)$ such that

$$f_n \overset{L^1_{loc}(\Omega)}{\to} f \ (n \to \infty) \tag{585}$$

$$\partial^\alpha f_n \overset{L^1_{loc}(\Omega)}{\to} g \ (n \to \infty) \tag{586}$$

If either condition above holds, one may conclude that $g = \partial^\alpha f$ is just the weak derivative.

Proof. If f is weakly differentiable, by the approximation identity, $f_n = \eta^\varepsilon * f, \varepsilon = \frac{1}{n}$ just provides the construction and $\partial^\alpha f_n \overset{L^1_{loc}(\Omega)}{\to} \partial^\alpha f = g \ (n \to \infty)$.

Conversely, if f has such smooth function approximation, then

$$\forall \phi \in C_c^\infty(\Omega), \partial^\alpha \phi \in C_c^\infty(\Omega) \tag{587}$$

$$\int f_n \cdot \partial^\alpha \phi \to \int f \cdot \partial^\alpha \phi \ (n \to \infty) \tag{588}$$

apply the definition of weak derivative to find

$$\int f_n \cdot \partial^\alpha \phi = (-1)^{|\alpha|} \int \partial^\alpha f_n \cdot \phi \tag{589}$$

$$\lim_{n\to\infty} (-1)^{|\alpha|} \int \partial^\alpha f_n \cdot \phi = \int f \cdot \partial^\alpha \phi = (-1)^{|\alpha|} \int \partial^\alpha g \cdot \phi \tag{590}$$

that's why f is weakly differentiable and $g = \partial^\alpha f$ by the uniqueness of weak derivative.

□

From the perspective that views weak differentiable function as the limit of smooth functions, one can prove the properties of weak derivative for a wider class of functions.

Theorem 41. ***(Product Law and Chain Rule)*** *Suppose $a \in C^1(\Omega), f \in L^1_{loc}(\Omega)$ and f is weakly differentiable, then au is weakly differentiable and*

$$\partial_i(af) = a \cdot \partial_i f + f \cdot \partial_i a \tag{591}$$

If $g : \mathbb{R} \to \mathbb{R} \in C^1(\mathbb{R})$ and g' is bounded, then $h = g \circ f$ is weakly differentiable and

$$\partial_i h = g'(f) \cdot \partial_i f \tag{592}$$

Proof. There exists f_n to be a series of smooth functions on Ω such that

$$f_n \overset{L^1_{loc}(\Omega)}{\to} f \ (n \to \infty) \tag{593}$$

$$\partial_i f_n \overset{L^1_{loc}(\Omega)}{\to} \partial_i f \ (n \to \infty) \tag{594}$$

then

$$af_n \overset{L^1_{loc}(\Omega)}{\to} af \ (n \to \infty) \tag{595}$$

since C^1 functions are bounded on compact sets and by the convergence of weak derivative

$$\partial_i(af_n) = a \cdot \partial_i f_n + f_n \cdot \partial_i a \overset{L^1_{loc}(\Omega)}{\to} a \cdot \partial_i f + f \cdot \partial_i a \ (n \to \infty) \tag{596}$$

the characterization weak derivative enables us to conclude that af is differentiable and $\partial_i(af) = a \cdot \partial_i f + f \cdot \partial_i a$.

For the chain rule, consider $h_n = g \circ f_n$ to be a series of C^1 functions with $\partial_i h_n = g'(f_n) \cdot \partial_i f_n$. Since g' is bounded and C^1, it's obvious that g has to be globally Lipschitz with $\exists M > 0, \forall x, y \in \Omega, |g(x) - g(y)| \leq M|x - y|$. For $\forall K \subset\subset \Omega$,

$$\int_K |h_n - h| = \int_K |g(f_n(x)) - g(f(x))| \, dx \tag{597}$$

$$\leq M \int_K |f_n(x) - f(x)| \, dx \to 0 \ (n \to \infty) \tag{598}$$

$$h_n \overset{L^1_{loc}(\Omega)}{\to} h \ (n \to \infty) \tag{599}$$

now consider the convergence of the derivative of h_n

$$\int_K |g'(f_n) \cdot \partial_i f_n - g'(f) \cdot \partial_i f| \leq \int_K |g'(f_n)| \cdot |\partial_i f_n - \partial_i f| + \int_K |g'(f_n) - g'(f)| \cdot |\partial_i f| \tag{600}$$

$$\leq M \int_K |\partial_i f_n - \partial_i f| + \int_K |g'(f_n) - g'(f)| \cdot |\partial_i f| \tag{601}$$

the first term goes to 0 obviously. The second term actually also goes to 0 because by our construction of f_n using the mollifier, we can add a property to f_n such that

$$f_n \overset{a.e.}{\to} f \ (n \to \infty) \tag{602}$$

by the approximation identity of mollifier. As a result, $|g'(f_n) - g'(f)| \cdot |\partial_i f| \leq 2M|\partial_i f|$ which is integrable on K, so by dominated convergence theorem,

$$\int_K |g'(f_n) \cdot \partial_i f_n - g'(f) \cdot \partial_i f| \to 0 \ (n \to \infty) \tag{603}$$

this is proving that

$$h_n \overset{L^1_{loc}(\Omega)}{\to} g'(f) \cdot \partial_i f \ (n \to \infty) \tag{604}$$

by the characterization of weak derivative, we have proved that h is weakly differentiable and $\partial_i h = g'(f) \cdot \partial_i f$. $\square$

Remark. *Compared to the product rule proved above which requires such $a \in C^\infty(\Omega)$, the condition now is requiring much less regularity that $a \in C^1(\Omega)$. This is because we are shifting from the definition of the weak derivative to the application of mollifier and notice the characterization of weak derivative with smooth function approximation.*

In fact, the regularity can be further weakened into $f \in W^{1,\infty}(\mathbb{R})$ instead of $C^1(\Omega)$ since the global Lipschitz condition is playing the crucial role.

Example

Consider $u \in L^1_{loc}(\Omega)$ with weak derivative $\partial_i u$, then $|u|$ is always weakly differentiable. This example is telling us that the C^1 condition in the last theorem is actually not necessary. To find out the weak derivative of $|u|$, one might naturally guess that

$$\partial_i|u| = \begin{cases} \partial_i u & u > 0 \\ 0 & u = 0 \\ -\partial_i u & u < 0 \end{cases} \tag{605}$$

Let's prove this by considering $f^\varepsilon(x) = \sqrt{x^2+\varepsilon^2}$, a C^1 approximation of $|u|$ when $\varepsilon \to 0^+$. Let's first consider

$$\partial_i(f^\varepsilon \circ u) = \frac{u}{\sqrt{u^2+\varepsilon^2}} \cdot \partial_i u \tag{606}$$

by the definition of weak derivative

$$\forall \phi \in C_c^\infty(\Omega), \int_\Omega \sqrt{u^2+\varepsilon^2} \cdot \partial_i \phi = \int_\Omega \frac{u}{\sqrt{u^2+\varepsilon^2}} \cdot \partial_i u \cdot \phi \tag{607}$$

now set $\varepsilon \to 0$ to find that

$$\int_\Omega |u| \cdot \partial_i \phi = \int_\Omega \partial_i |u| \cdot \phi \tag{608}$$

due to the dominated convergence theorem ($u, \partial_i u$ are locally integrable and ϕ has compact support). By definition, we proved the piecewise expression above for $\partial_i|u|$.

L^p Spaces

Define for $1 \le p < \infty$ the space $L^p(\Omega)$ consisting of Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ such that $\int_\Omega |f|^p < \infty$. $L^\infty(\Omega)$ consists of all essentially bounded functions $f : \Omega \to \mathbb{R}$. The norm on those spaces are defined as

$$||f||_p = \left(\int_\Omega |f|^p\right)^{\frac{1}{p}} \tag{609}$$

$$||f||_\infty = \sup_\Omega |f| \tag{610}$$

The following theorem allows us to bound the norm of different L^p spaces.

Theorem 42. ***(Interpolation Inequality)*** *For $1 \le p \le q \le r$, $L^p(\Omega) \cap L^r(\Omega) \subset L^q(\Omega)$ and there exists $\theta \in [0,1], \frac{\theta}{p} + \frac{1-\theta}{r} = \frac{1}{q}$ such that $||f||_q \le ||f||_p^\theta \cdot ||f||_r^{1-\theta}$.*

Proof. Apply Holder's inequality for the conjugate pair $\frac{p}{q\theta}, \frac{r}{q(1-\theta)}$ by the definition of θ and split the power q to get

$$||f||_q^q = \int |f|^q = \int |f|^{q\theta}|f|^{q(1-\theta)} \tag{611}$$

$$\le \left[\int (|f|^{q\theta})^{\frac{p}{q\theta}}\right]^{\frac{q\theta}{p}} \cdot \left[\int (|f|^{q(1-\theta)})^{\frac{r}{q(1-\theta)}}\right]^{\frac{q(1-\theta)}{r}} \tag{612}$$

$$= \left[\int |f|^p\right]^{\frac{q\theta}{p}} \cdot \left[\int |f|^r\right]^{\frac{q(1-\theta)}{r}} \tag{613}$$

taking power $\frac{1}{q}$ on both sides to conclude

$$||f||_q \le ||f||_p^\theta \cdot ||f||_r^{1-\theta} \tag{614}$$

this proves that if a function is in L^p, L^r in the same time, it must be in L^q for any q in between. $\square$

Week 9

Sobolev Spaces

The definition of the Sobolev space is for $k \in \mathbb{N}, 1 \leq p \leq \infty$ that

$$W^{k,p}(\Omega) = \left\{ f \in L^1_{loc}(\Omega) : \forall |\alpha| \leq k, \partial^\alpha f \in L^p(\Omega) \right\} \tag{615}$$

contains all locally integrable function such that all their k-th order weak partial derivatives are in the L^p space. When $p = 2$, denote $W^{k,2}(\Omega) = H^k(\Omega)$ since such space will be a Hilbert space.

Sobolev space is equipped with norm

$$||f||_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha f|^p \, dx \right)^{\frac{1}{p}} \tag{616}$$

for $1 \leq p < \infty$ and for $p = \infty$

$$||f||_{W^{k,\infty}} = \max_{|\alpha| \leq k} \sup_\Omega |\partial^\alpha f| \tag{617}$$

for $p = 2$, the inner product is defined as

$$\langle f, g \rangle_{H^k} = \sum_{|\alpha| \leq k} \int_\Omega \partial^\alpha f \cdot \partial^\alpha g \, dx \tag{618}$$

Remark. *Similar to L^p space, the element in Sobolev space is actually a representative among the equivalence relationship of two functions having all weak partial derivatives with order no larger than k being equal almost everywhere. To see this, if $||f||_{W^{k,p}} = 0$, then*

$$\sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha f|^p \, dx = 0 \tag{619}$$

$$\forall |\alpha| \leq k, \int_\Omega |\partial^\alpha f|^p \, dx = 0 \tag{620}$$

$$\forall |\alpha| \leq k, \partial^\alpha f = 0 \text{ a.e.} \tag{621}$$

*Although not proved here, all Sobolev spaces are complete so they are all **Banach spaces**.*

Approximation of Sobolev Functions

For Sobolev functions in $W^{k,p}(\mathbb{R}^n)$, the whole space, one can always find out its dense subset easily as the set of compactly supported smooth functions.

Theorem 43. ***(Approximation of Sobolev Functions on the Whole Space for $p < \infty$)***

For $1 \leq p < \infty$, $C_c^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is the dense subset of $W^{k,p}(\mathbb{R}^n)$.

Proof. To get smoothness, think about using mollifier. $\forall f \in W^{k,p}(\mathbb{R}^n)$, consider $\eta^\varepsilon * f \in W^{k,p}(\mathbb{R}^n)$ where η^ε is the standard mollifier. By the approximation identity of weak derivative,

$$\forall |\alpha| \leq k, \partial^\alpha(\eta^\varepsilon * f) \overset{L^p_{loc}(\mathbb{R}^n)}{\to} \partial^\alpha f \ (\varepsilon \to 0) \tag{622}$$

where the $L^p_{loc}(\mathbb{R}^n)$ convergence is due to the *a.e.* convergence in the approximation identity of the mollifier and the dominated convergence theorem. As a result, this is implying

$$\eta^\varepsilon * f \overset{W^{k,p}(\mathbb{R}^n)}{\to} f \ (\varepsilon \to 0) \tag{623}$$

and we have proved that smooth functions in $W^{k,p}(\mathbb{R}^n)$ are dense.

Next, let's prove that compactly supported smooth functions in $W^{k,p}(\mathbb{R}^n)$ are dense w.r.t. the smooth functions in $W^{k,p}(\mathbb{R}^n)$. Take $g \in C^\infty \cap W^{k,p}$, consider $\phi \in C_c^\infty$ such that $\forall ||x|| \leq 1, \phi(x) = 1, \forall ||x|| \geq 2, \phi(x) = 0, \forall x \in \mathbb{R}^n, 0 \leq \phi(x) \leq 1$. Set

$$\phi_R(x) = \phi\left(\frac{x}{R}\right) \tag{624}$$

such that the support of ϕ_R is a subset of $||x|| \leq 2R$. By setting $g_R = g \cdot \phi_R \in C_c^\infty(\mathbb{R}^n)$, one can prove that such function can approximate g well enough.

$$\begin{aligned} \forall 0 < |\alpha| \leq k, \partial^\alpha g_R &= \partial^\alpha g \cdot \phi_R + g \cdot \partial^\alpha \phi_R && (625) \\ &= \partial^\alpha g \cdot \phi_R + \frac{1}{R} h_R && (626) \end{aligned}$$

where h_R is a function in R with $||h_R||_{L^p} \leq M$ uniform w.r.t. R, set $R \to \infty$ to find

$$\forall |\alpha| \leq k, \partial^\alpha g_R \overset{L^p(\mathbb{R}^n)}{\to} \partial^\alpha g \ (R \to \infty) \tag{627}$$

this is equivalent to saying that

$$\forall |\alpha| \leq k, g_R \overset{W^{k,p}(\mathbb{R}^n)}{\to} g \ (R \to \infty) \tag{628}$$

so it's proved. □

Remark. ***The approximation of Sobolev functions does not hold for*** $p = \infty$. *To see the counterexample, consider $f = \mathbb{I}_{(0,\infty)}$ and for any $g \in C(\mathbb{R})$, we can prove that $||f - g||_{L^\infty} \geq \frac{1}{3}$ since otherwise exists $g \in C(\mathbb{R})$ such* *that* $||f-g||_{L^\infty} < \frac{1}{3}$

$$\forall \varepsilon > 0, |g(\varepsilon)-g(-\varepsilon)| \geq \Big| |g(\varepsilon)-f(\varepsilon)| - |g(-\varepsilon)-f(-\varepsilon)-1| \Big| \tag{629}$$

since $f(\varepsilon)=1, f(-\varepsilon)=0$. *However, notice that* $|g(\varepsilon)-f(\varepsilon)| < \frac{1}{3}$ *and* $|g(-\varepsilon)-f(-\varepsilon)-1| \in (\frac{2}{3}, \frac{4}{3})$ *so*

$$|g(\varepsilon)-g(-\varepsilon)| \geq \frac{1}{3} \tag{630}$$

contradiction with the fact that g *is continuous if set* $\varepsilon \to 0$.

This counterexample is telling us that the set of continuous functions are even not dense in $L^\infty = W^{0,\infty}$, *the set of compactly supported smooth functions cannot be dense.*

Remark. ***This theorem holds if and only if*** $\Omega = \mathbb{R}^n$ ***is the whole space***. *To see this, if* Ω *has any boundary, compactly supported smooth functions on* Ω *will definitely be 0 near the boundary. However, functions in* $W^{k,p}(\Omega)$ *can have arbitrary boundary behavior near* $\partial\Omega$. *That's the intuition of why the theorem fails. To see the detailed arguments,*

$$\forall K \subset\subset \Omega, K \cap \partial\Omega = \emptyset \tag{631}$$

$$\forall \phi \in C_c^\infty(\Omega), \forall x \in \partial\Omega, \phi(x) = 0 \tag{632}$$

so the pointwise limit of a series of compactly supported smooth functions on Ω *vanishes on* $\partial\Omega$. *If the approximation of Sobolev function if true on such* Ω, *then* $\forall f \in W^{k,p}(\Omega), \forall x \in \partial\Omega, \lim_{x_n \to x, x_n \in \Omega} f(x_n) = 0$, *which is obviously not the case.*

This failure motivates us to define $W_0^{k,p}$, *the space of Sobolev functions that vanishes on the boundary to achieve better properties.*

Sobolev Embedding

The Sobolev embedding aims at establishing an estimate connecting the norm of f and the norm of Df for compactly supported smooth function f on $\mathbb{R}^n$.

Let's assume that the conclusion we want to derive is

$$\exists C, \forall f \in C_c^\infty(\mathbb{R}^n), ||f||_{L^q} \leq C||Df||_{L^p} \tag{633}$$

for some p, q, n where the constant C only depends on p, q, n. To make this estimation useful, we hope that this inequality is **scaling invariant**, i.e. for $\forall \lambda > 0, f_\lambda(x) = f\left(\frac{x}{\lambda}\right)$, we would expect the inequality to hold for the same scaling on both sides.

It's easy to see that $D(f_\lambda) = \frac{1}{\lambda}(Df)_\lambda$ and plugging in the inequality to see

$$||f_\lambda||_{L^q} \leq C||D(f_\lambda)||_{L^p} \tag{634}$$

with change of variables one can see

$$||f_\lambda||_{L^q} = \left(\lambda^n \int_{\mathbb{R}^n} f^q(u)\,du\right)^{\frac{1}{q}} = \lambda^{\frac{n}{q}}||f||_{L^q} \tag{635}$$

$$||D(f_\lambda)||_{L^p} = \frac{1}{\lambda}\left(\lambda^n \int_{\mathbb{R}^n} [Df(u)]^p\,du\right)^{\frac{1}{p}} = \lambda^{\frac{n}{p}-1}||Df||_{L^p} \tag{636}$$

where n is the dimension of the space. So if this property shall hold for f_λ, we get

$$\lambda^{\frac{n}{q}}||f||_{L^q} \leq C\lambda^{\frac{n}{p}-1}||Df||_{L^p} \tag{637}$$

comparing with original inequality to set

$$\frac{n}{q} = \frac{n}{p} - 1 \tag{638}$$

such that for any λ the inequality is scaling invariant (we cannot put λ in C since there will still be scaling problems if $\lambda \to 0$ or $\lambda \to \infty$). This gives the definition of **the Sobolev conjugate of** p which is p^* such that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \tag{639}$$

Remark. *The scaling invariant property is ensuring that for the same function f, the inequality holds regardless of the speed the variable is going through all the values. This is an ideal condition for a useful property since otherwise the constant C will also contain λ, meaning that the constant has something to do this the selection of the function f.*

*One might also notice that the Sobolev conjugate is **only well defined for** $p < n$ since otherwise the Sobolev conjugate will be ∞ or a negative number, which is not acceptable since we want the space L^{p^*} to be well-defined. If the Sobolev conjugate p^* exists, then $1 \leq p < p^* < \infty$. That's why the main discussion in the following context is based on the assumption that $p < n$.*

Now we are clear that we have to prove the inequality above for $q = p^*$ to be the Sobolev conjugate of p. The proof of the result is based on the following theorem to bound the integral of the product of the leave-one-out functions from above by the product of the norm of these functions. To build up the tools needed, the notations are defined as: for $x \in \mathbb{R}^n$, $x_i' = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in \mathbb{R}^{n-1}$ be the vector leaving component x_i out and $x = (x_i, x_i')$ denotes the original vector and we denote

$$f(x) = f(x_i, x_i') \tag{640}$$

The following easy lemma enables us to bound the upper integral of a function by the L^1 norm for integrable function with 0 integral on the real line.

Lemma 3. *For $g : \mathbb{R} \to \mathbb{R}$ integrable with compact support and $\int_{\mathbb{R}} g(x)\,dx = 0$, then if*

$$f(x) = \int_{-\infty}^{x} g(t)\,dt \tag{641}$$

then

$$|f(x)| \le \frac{1}{2}\int_{\mathbb{R}} |g(t)|\,dt \tag{642}$$

Proof. Tear g into the positive and negative part to see $\int g_-(x)\,dx = \int g_+(x)\,dx$ and both have compact support.

Notice that

$$f(x) \le \int_{-\infty}^{x} g_+(t)\,dt \le \int_{\mathbb{R}} g_+(t)\,dt = \frac{1}{2}\int |g(t)|\,dt \tag{643}$$

by same reasoning, $f(x) \ge -\frac{1}{2}\int |g(t)|\,dt$ so $|f(x)| \le \frac{1}{2}\int |g(t)|\,dt$.

□

Applying the lemma above for f, one would get the following estimate to bound f by the integral w.r.t. the leave-one-out vector.

Lemma 4. *f is compactly supported smooth function on $\mathbb{R}^n$, then*

$$\forall 1 \le i \le n, |f(x)| \le \frac{1}{2}\int_{\mathbb{R}} |\partial_i f(t, x_i')|\,dt \tag{644}$$

Proof. Since $f(x) = \int_{-\infty}^{x_i} \partial_i f(t, x_i')\,dt$ and $\int_{\mathbb{R}} \partial_i f(t, x_i')\,dt = 0$, apply the lemma to find $|f(x)| \le \frac{1}{2}\int_{\mathbb{R}} |\partial_i f(t, x_i')|\,dt$.

□

Theorem 44. ***(Bound the Integral of Product by the Product of Norm)*** *For $n \ge 2$ and $g_1, ..., g_n \in C_c^\infty(\mathbb{R}^{n-1})$ are non-negative functions, define*

$$g(x) = \prod_{i=1}^{n} g_i(x_i') \in C_c^\infty(\mathbb{R}^n) \tag{645}$$

then

$$\int_{\mathbb{R}^n} g(x)\,dx \le \prod_{i=1}^{n} ||g_i||_{L^{n-1}} \tag{646}$$

Proof. When $n = 2$, by Fubini theorem,

$$\int_{\mathbb{R}^2} g(x)\,dx = \int_{\mathbb{R}^2} g_1(x_2)g_2(x_1)\,dx_1\,dx_2 = \int_{\mathbb{R}} g_1(x_2)\,dx_2 \cdot \int_{\mathbb{R}} g_2(x_1)\,dx_1 \le ||g_1||_{L^1} \cdot ||g_2||_{L^1} \tag{647}$$

so it holds.

By induction, assume that the conclusion holds for dimension $n-1$. When the dimension is n, fix one more coordinate x_1 to set $g_{x_1}(x_1') = g(x_1, x_1')$. Now denote $x_i' = (x_1, x_{1,i}')$ for $2 \leq i \leq n$ where

$$x_{1,i}' = (x_2, ..., x_{i-1}, x_{i+1}, ..., x_n) \in \mathbb{R}^{n-2} \tag{648}$$

be the vector leaving out x_1, x_i. Denote

$$g_{i,x_1}(x_{1,i}') = g_i(x_1, x_{1,i}') \tag{649}$$

so now $g_{x_1}(x_1') = g(x_1, x_1') = \prod_{i=1}^n g_i(x_i') = g_1(x_1') \cdot \prod_{i=2}^n g_{i,x_1}(x_{1,i}')$

Now we only have to bound the part of $g_1(x_1')$ since the other parts can be bounded by the induction hypothesis because the x_1 in them is fixed. In order to let the norm in L^{n-1} appear to bound the L^1 norm, apply Holder's inequality with conjugate numbers $n-1$ and $\frac{n-1}{n-2}$ to conclude that

$$\int g_{x_1}(x_1')\, dx_1' = \int g_1(x_1') \cdot \prod_{i=2}^n g_{i,x_1}(x_{1,i}')\, dx_1' \tag{650}$$

$$\leq ||g_1||_{L^{n-1}} \cdot \left[\int \left(\prod_{i=2}^n g_{i,x_1}(x_{1,i}')\right)^{\frac{n-1}{n-2}} dx_1'\right]^{\frac{n-2}{n-1}} \tag{651}$$

apply the induction hypothesis for $\prod_{i=2}^n g_{i,x_1}(x_{1,i}')$ to conclude that

$$\int \left(\prod_{i=2}^n g_{i,x_1}(x_{1,i}')\right)^{\frac{n-1}{n-2}} dx_1' \leq \prod_{i=2}^n ||g_{i,x_1}^{\frac{n-1}{n-2}}||_{L^{n-2}} \tag{652}$$

$$= \prod_{i=2}^n ||g_{i,x_1}||_{L^{n-1}}^{\frac{n-1}{n-2}} \tag{653}$$

now we conclude that $\int g_{x_1}(x_1')\, dx_1' \leq ||g_1||_{L^{n-1}} \cdot \prod_{i=2}^n ||g_{i,x_1}||_{L^{n-1}}$.

By Fubini theorem, $\int g(x)\, dx = \int (\int g_{x_1}(x_1')\, dx_1')\, dx_1$ so integrate on both sides w.r.t. x_1 to recover the integral

$$\int g(x)\, dx \leq ||g_1||_{L^{n-1}} \cdot \int \prod_{i=2}^n ||g_{i,x_1}||_{L^{n-1}}\, dx_1 \tag{654}$$

now we have an integral of the product. To split it w.r.t. g_{i,x_1}, we have to apply Holder's inequality again for the product of $n-1$ terms with conjugate numbers $n-1, ..., n-1$ to conclude that

$$\int \prod_{i=2}^n ||g_{i,x_1}||_{L^{n-1}}\, dx_1 \leq \prod_{i=2}^n \left(\int ||g_{i,x_1}||_{L^{n-1}}^{n-1}\, dx_1\right)^{\frac{1}{n-1}} = \prod_{i=2}^n \left(\int\int |g_{i,x_1}(x_{1,i}')|^{n-1}\, dx_{1,i}'\, dx_1\right)^{\frac{1}{n-1}} \tag{655}$$

$$= \prod_{i=2}^n \left(\int |g_i(x_i')|^{n-1}\, dx_i'\right)^{\frac{1}{n-1}} = \prod_{i=2}^n ||g_i||_{L^{n-1}} \tag{656}$$

As a result, we have proved that

$$\int g(x)\,dx \leq ||g_1||_{L^{n-1}} \cdot \prod_{i=2}^{n} ||g_i||_{L^{n-1}} = \prod_{i=1}^{n} ||g_i||_{L^{n-1}} \tag{657}$$

finishes the induction. □

Remark. *For $n = 2$, the result is trivial and direct from the Fubini theorem. For $n \geq 3$, the induction is based on first fixing the x_1 coordinate and integrate w.r.t. it later, with a lot of calculations and inequalities. To get an intuitive impression of what the conclusion looks like, for $n = 3$, the conclusion looks like if*

$$g(x_1, x_2, x_3) = g_1(x_2, x_3) \cdot g_2(x_1, x_3) \cdot g_3(x_1, x_2) \tag{658}$$

then

$$\int_{\mathbb{R}^3} g(x)\,dx \leq ||g_1||_{L^2} \cdot ||g_2||_{L^2} \cdot ||g_3||_{L^2} \tag{659}$$

$$= \left(\int_{\mathbb{R}^2} g_1^2(x_2, x_3)\,dx_2\,dx_3\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^2} g_2^2(x_1, x_3)\,dx_1\,dx_3\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^2} g_3^2(x_1, x_2)\,dx_1\,dx_2\right)^{\frac{1}{2}} \tag{660}$$

so the integral of the product consisting of leave-one-variable-out functions can be bounded above by the product of the norm of each leave-one-variable-out functions.

By using those technical results, we are able to prove the functional inequality mentioned above to bound the norm of the Sobolev functions on the whole space by the norm of its gradient.

Theorem 45. ***(Sobolev Inequality)*** *For $1 \leq p < n, n \geq 2$, set p^* as the Sobolev conjugate of p, then*

$$\exists C > 0, \forall f \in C_c^\infty(\mathbb{R}^n), ||f||_{L^{p^*}} \leq C \cdot ||Df||_{L^p} \tag{661}$$

where the constant C only depends on n and p.

Proof. We first show it for $p = 1$. Use the lemma that $|f(x)| \leq \frac{1}{2}\int |\partial_i(t, x_i')|\,dt$. Set $g_i(x_i') = \left(\int |\partial_i f(t, x_i')|\,dt\right)^{\frac{1}{n-1}}$ and $g(x) = \prod_{i=1}^n g_i(x_i')$ to get

$$g(x) \geq 2^{\frac{n}{n-1}} |f(x)|^{\frac{n}{n-1}} \tag{662}$$

the theorem we have just proved tells us that

$$\int g(x)\,dx \leq \prod_{i=1}^{n} ||g_i||_{L^{n-1}} = \prod_{i=1}^{n} \left(\int_{\mathbb{R}} \left(\int |\partial_i f(t, x_i')|\,dt\right)\,dx_i'\right)^{\frac{1}{n-1}} = \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |\partial_i f(x)|\,dx\right)^{\frac{1}{n-1}} \tag{663}$$

as a result, we get

$$2^{\frac{n}{n-1}}\int_{\mathbb{R}^n}|f(x)|^{\frac{n}{n-1}}\,dx\leq\prod_{i=1}^n\left(\int_{\mathbb{R}^n}|\partial_i f(x)|\,dx\right)^{\frac{1}{n-1}} \tag{664}$$

notice that the Sobolev conjugate of 1 is $\frac{n}{n-1}$, so

$$\left(\int_{\mathbb{R}^n}|f(x)|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}}\leq\frac{1}{2}\prod_{i=1}^n\left(\int_{\mathbb{R}^n}|\partial_i f(x)|\,dx\right)^{\frac{1}{n}} \tag{665}$$

$$\left(\int_{\mathbb{R}^n}|f(x)|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}}\leq\frac{1}{2n}\sum_{i=1}^n\int_{\mathbb{R}^n}|\partial_i f(x)|\,dx \tag{666}$$

by the arithmetic-geometric mean inequality, and we conclude that (if one does not understand the reasoning here, refer to the remark after the theorem)

$$||f||_{L^{\frac{n}{n-1}}}\leq C\cdot||Df||_{L^1} \tag{667}$$

For general $p>1$, we can actually use the proved version for $p=1$ to construct a proof. The trick is to add a fixed number $s>1$ which will be specified later and apply the conclusion for $|f|^s$ to get

$$\left(\int_{\mathbb{R}^n}|f(x)|^{\frac{ns}{n-1}}\,dx\right)^{\frac{n-1}{n}}\leq C\cdot||D(|f|^s)||_{L^1} \tag{668}$$

notice that $D(|f|^s)=s|f|^{s-1}\cdot D|f|$ and by Holder's inequality, in order to let $||Dp||_{L^p}$ appear, pick conjugate numbers $p,\frac{p}{p-1}$ to get

$$\int|f|^{s-1}\cdot D|f|\,dx\leq\left(\int|f|^{\frac{(s-1)p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\cdot\left(\int(D|f|)^p\,dx\right)^{\frac{1}{p}} \tag{669}$$

As a result, now we have proved

$$\left(\int_{\mathbb{R}^n}|f(x)|^{\frac{ns}{n-1}}\,dx\right)^{\frac{n-1}{n}}\leq C\cdot\left(\int|f|^{\frac{(s-1)p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\cdot||Df||_{L^p} \tag{670}$$

the only work left is to merge the LHS and the first term on the RHS. Naturally, we set

$$\frac{ns}{n-1}=\frac{(s-1)p}{p-1} \tag{671}$$

$$p^*=\frac{np}{n-p}=\frac{ns}{n-1} \tag{672}$$

and this specification of s tells us

$$||f||_{L^{\frac{ns}{n-1}}} \leq C \cdot ||Df||_{L^p} \tag{673}$$

and the theorem is finally proved! □

Remark. *In the proof of the theorem for $p = 1$, we conclude that $||f||_{L^{\frac{n}{n-1}}} \leq C \cdot ||Df||_{L^1}$ after getting $||f||_{L^{\frac{n}{n-1}}} \leq \frac{1}{2n} \sum_{i=1}^n \int_{\mathbb{R}^n} |\partial_i f(x)|\, dx$. Here one might be confused with the definition of $||Df||_{L^1}$.*

Actually here we are using the definition

$$||Df||_{L^1} = \int ||Df||_1 \, dx = \int_{\mathbb{R}^n} \sum_{i=1}^n |\partial_i f(x)| \, dx \tag{674}$$

*with the norm of Df taken as the l_1 vector norm. Note that here one can choose whatever norm one prefer since $Df \in \mathbb{R}^n$ and **any two norms on the finite-dimensional space are equivalent**.*

Remark. *For the case where $p = 1$, the optimal constant in the Sobolev inequality is*

$$C(n, 1) = \frac{1}{n(V_n)^{\frac{1}{n}}} \tag{675}$$

where V_n is the volume of n-dimensional unit ball.

For the general case $1 < p < n$, the optimal constant is

$$C(n, p) = \frac{1}{n^{\frac{1}{p}}\sqrt{\pi}} \left(\frac{p-1}{n-p}\right)^{1-\frac{1}{p}} \left[\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right]^{\frac{1}{n}} \tag{676}$$

achieved when $f(x) = \left(a + b|x|^{\frac{p}{p-1}}\right)^{1-\frac{n}{p}}$ $(a, b > 0)$.

Remark. *The Sobolev inequality only holds for $p < n$ and **does not hold when** $p = n$. To see this, when $p = n$, the Sobolev conjugate $p^* = \infty$. Consider $\phi(x) \in C_c^\infty(\mathbb{R}^n)$ taken such that $\phi \in [0, 1], \forall x, ||x|| \leq 1, \phi(x) = 1$ and $\forall x, ||x|| \geq 2, \phi(x) = 0$, set*

$$f(x) = \phi(x) \cdot \log\log\left(1 + \frac{1}{||x||}\right) \tag{677}$$

then it's easy to see that

$$\int |f(x)|^n \, dx \leq C + \int_{B_1(0)} \left|\log\log\left(1 + \frac{1}{||x||}\right)\right|^n dx < \infty \tag{678}$$

since $\left|\log\log\left(1+\frac{1}{||x||}\right)\right|^n \leq \left|\log\frac{1}{||x||}\right|^n = (-\log||x||)^n$ *and by changing variables*

$$\int_{B_1(0)} \left|\log\log\left(1+\frac{1}{||x||}\right)\right|^n dx \leq |\partial B_1(0)| \cdot \int_0^1 r^{n-1}\log^n\frac{1}{r}\,dr < \infty \tag{679}$$

so $f \in L^n(\mathbb{R}^n)$*, consider the norm of its gradient*

$$||Df(x)||_{L^n}^n = \int_{\mathbb{R}^n}\left(\sum_{i=1}^n |\partial_i f(x)|\right)^n dx < \infty \tag{680}$$

$$\partial_i f(x) = \partial_i\phi\cdot\log\log\left(1+\frac{1}{||x||}\right) - \frac{\phi\cdot x_i}{||x||^2(||x||+1)\log\left(1+\frac{1}{||x||}\right)} \tag{681}$$

and it's easy to verify the convergence of the integral. The first term in $\partial_i f(x)$ *has only finite contribution with the proof being same as that for* $f\in L^n(\mathbb{R}^n)$*. The second term has contribution*

$$\leq C\cdot\int_{B_1(0)}\frac{1}{||x||^n\log^n\left(1+\frac{1}{||x||}\right)}\,dx \tag{682}$$

$$\leq C'\cdot\int_0^1\frac{1}{r^n\log^n\left(1+\frac{1}{r}\right)}r^{n-1}\,dr \tag{683}$$

$$\leq C'\cdot\int_0^1\frac{1}{r\log^n\left(1+\frac{1}{r}\right)}\,dr \tag{684}$$

$$= C'\cdot\int_{\log 2}^{\infty}\frac{e^u}{e^u-1}u^{-n}\,du < \infty \tag{685}$$

where r^{n-1} *is the Jacobian of the polar coordinate transformation in* $\mathbb{R}^n$*. So we have argued that* $f\in W^{1,n}(\mathbb{R}^n)$*. However, it's very clear that* $|f(x)|\to+\infty$ $(||x||\to 0)$*, so* $f\notin L^\infty(\mathbb{R}^n)$*.*

Remark. *The Sobolev inequality provides some very interesting insights since it's telling us that if a compactly supported smooth function* f *is in the space* $W^{1,p}(\mathbb{R}^n)$ *with* $1\leq p<n$*, then* $f\in L^{p^*}$ *where* $p<p^*$*. The natural interpretation of Sobolev inequality is:* ***when the weak derivative of a function is not bad, there is some extra information known about the original function that results in higher regularity*** *since taking derivative always makes the property of the function "worse". Although this inequality only holds for compactly supported smooth functions, by noticing that the set of compactly supported smooth functions is dense in* $W^{k,p}(\mathbb{R}^n)$ *and that weak derivative can be characterized using approximation of smooth functions, we expect a similar property to hold for general Sobolev functions.*

To investigate Sobolev embeddings, let's define the meaning of embedding. For two Banach spaces $X<Y$, say X can be **embedded into** Y if there exists injective linear bounded functional $i: X\to Y$. We denote the embedding as $X\hookrightarrow Y$.

Although the Sobolev inequality only holds for compactly supported smooth functions, but noticing the characterization of weak derivative by the approximation of smooth functions, one can turn the conclusion into the following embedding theorem for general functions in the Sobolev space $W^{1,p}$.

Theorem 46. ***(Sobolev Embedding for*** $W^{1,p}(\mathbb{R}^n)$***)*** *Suppose* $1 \leq p < n, p \leq q \leq p^*$*, then* $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ *and*

$$\exists C > 0, \forall f \in W^{1,p}(\mathbb{R}^n), ||f||_{L^q} \leq C \cdot ||f||_{W^{1,p}} \tag{686}$$

where the constant $C = C(n, p, q)$.

Proof. For $\forall f \in W^{1,p}(\mathbb{R}^n)$, according to the Sobolev approximation theorem, there exists $f_n \in C_c^\infty(\mathbb{R}^n)$ such that

$$f_n \overset{W^{1,p}(\mathbb{R}^n)}{\rightarrow} f \ (n \rightarrow \infty) \tag{687}$$

apply the Sobolev inequality for f_n to know

$$\exists C > 0, \forall n, ||f_n||_{L^{p^*}} \leq C \cdot ||Df_n||_{L^p} \tag{688}$$

so f_n is a Cauchy sequence in L^{p^*} with the same limit f (otherwise there exists a subsequence of f_n converging to f and the new limit in the almost everywhrere sense at the same time) and set $n \rightarrow \infty$ to find

$$\exists C > 0, ||f||_{L^{p^*}} \leq C \cdot ||Df||_{L^p} \tag{689}$$

now for $p \leq q \leq p^*$, apply the interpolation inequality to conclude

$$\exists \theta \in (0, 1), ||f||_{L^q} \leq ||f||_{L^p}^\theta \cdot ||f||_{L^{p^*}}^{1-\theta} \tag{690}$$

now $||f||_{L^{p^*}} < \infty$ and $f \in L^p$ so we conclude that $f \in L^q$. As a result, the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is obvious. ☐

Remark. *The condition that* $Df \in L^p(\mathbb{R}^n)$ *is not enough to ensure that* $f \in L^q(\mathbb{R}^n)$ *and* $f \in L^p(\mathbb{R}^n)$ *is necessary. An easy counterexample is the nonzero constant function.*

Remark. *When the area in the Sobolev embedding theorem is not the whole space* $\mathbb{R}^n$ *but an* ***open set*** Ω ***with finite Lebesgue measure****, one immediately knows that*

$$... \subset L^n(\Omega) \subset ... \subset L^2(\Omega) \subset L^1(\Omega) \tag{691}$$

so one may get the stronger conclusion that $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for* $1 \leq q \leq p^*$*, a wider range of* q.

One might be confused with the appearance of the $W^{1,p}(\Omega)$ *space here and assert that such conclusion shall only hold for the space* $W_0^{1,p}(\Omega)$*, this is a good observation! For the Sobolev approximation using compactly supported smooth functions to hold, one does need to add the condition that Sobolev functions have* ***vanishing boundary values*** *as we have mentioned in the remarks for Sobolev approximation.*

However, there's actually a way for us to circumvent this issue by using the ***extension of Sobolev functions*** *stated below.*

Theorem 47. ***(Sobolev Extension Theorem)*** *For $1 \le p \le \infty$, if Ω is bounded with C^1 boundary, and there exists bounded open set Ω' such that $\Omega \subset\subset \Omega'$, then there exists a linear bounded operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ having support within Ω' such that*

$$\forall f \in W^{1,p}(\Omega), \forall x \in \Omega, Ef(x) = f(x) \ a.e. \tag{692}$$

and the extension is bounded in norm

$$\exists C, \forall f \in W^{1,p}(\Omega), ||Ef||_{W^{1,p}(\mathbb{R}^n)} \le C \cdot ||f||_{W^{1,p}(\Omega)} \tag{693}$$

where the constant $C = C(p, \Omega, \Omega')$.

Theorem 48. ***(Sobolev Embedding Theorem in General Bounded Areas)*** *Suppose Ω is bounded with C^1 boundary, and $1 \le p < n, 1 \le q \le p^*$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ and*

$$\exists C > 0, \forall f \in W^{1,p}(\Omega), ||f||_{L^q} \le C \cdot ||f||_{W^{1,p}} \tag{694}$$

where the constant $C = C(n, p, \Omega)$.

In the context above, we are discussing the case where $p < n$. The motivation is from the scaling invariant property and the existence of Sobolev conjugate. In the following context, we focus on the case where $p > n$. Actually in this high regularity case, the Sobolev functions will behave better.

Firstly define Holder spaces. For $0 < \alpha \le 1$, a function $f : \Omega \to \mathbb{R}$ is called **α-Holder continuous on Ω** if

$$[f]_{\alpha,\Omega} \stackrel{def}{=} \sup_{x,y\in\Omega, x\neq y} \frac{|f(x) - f(y)|}{||x - y||^\alpha} < \infty \tag{695}$$

and it's called **locally α-Holder continuous on Ω** if

$$\forall \Omega' \subset\subset \Omega, [f]_{\alpha,\Omega'} < \infty \tag{696}$$

denote $C^{0,\alpha}(\Omega)$ as the space of locally α-Holder continuous functions on Ω. For bounded Ω, $C^{0,\alpha}(\overline{\Omega})$ denotes the space of α-Holder continuous functions on Ω.

Remark. *It's easy to see that if a function is α-Holder continuous on Ω, it must be continuous. When $\alpha = 1$, the α-Holder continuity is just the Lipschitz and locally Lipschitz continuity. Note that α-Holder continuity generally has nothing to do with regularity and smoothness, e.g. $f(x) = |x|$. When $\alpha > 1$, however, the case is trivial since*

$$\forall \Omega' \subset\subset \Omega, \exists C, \forall x, y \in \Omega', x \neq y, \frac{|f(x) - f(y)|}{||x - y||} \le C||x - y||^{\alpha-1} \to 0 \ (x \to y) \tag{697}$$

so the derivative at any point exists and is always 0, f has to be constant.

Remark. $[f]_{\alpha,\Omega}$ *is generally not a norm since all constant f has $[f]_{\alpha,\Omega} = 0$, but it's a semi-norm.*

To construct a norm on the Holder space $C^{0,\alpha}(\overline{\Omega})$ for bounded opens et Ω, one can notice that

$$||f||_{C^{0,\alpha}(\overline{\Omega})} = \sup_{\overline{\Omega}} |f| + [f]_{\alpha,\overline{\Omega}} \tag{698}$$

is a norm under the equality of *a.e.* sense and is making $C^{0,\alpha}(\overline{\Omega})$ a **Banach space**.

Similarly, one might generalize those concepts to that for higher order derivatives of f. Define

$$C^{k,\alpha}(\Omega) = \left\{ f \in C^k(\Omega) : \forall |\beta| = k, \partial^\beta f \in C^{0,\alpha}(\Omega) \right\} \tag{699}$$

and for bounded Ω define

$$C^{k,\alpha}(\overline{\Omega}) = \left\{ f \in C^k(\overline{\Omega}) : \forall |\beta| = k, \partial^\beta f \in C^{0,\alpha}(\overline{\Omega}) \right\} \tag{700}$$

a natural norm on such space is

$$||f||_{C^{k,\alpha}(\overline{\Omega})} = \sum_{|\beta| \le k} \sup_{\overline{\Omega}} |\partial^\beta f| + \sum_{|\beta| = k} [\partial^\beta f]_{\alpha,\overline{\Omega}} \tag{701}$$

the space $C^{k,\alpha}(\overline{\Omega})$ equipped with such norm is a **Banach space**.

One can also prove the interpolation inequality of the Holder norm.

Theorem 49. ***(Interpolation Inequality for Holder Norm)*** *For open set $\Omega \subset \mathbb{R}^n$, assume $0 < \beta < \gamma \le 1$, then*

$$||f||_{C^{0,\gamma}(\Omega)} \le ||f||_{C^{0,\beta}(\Omega)}^{\frac{1-\gamma}{1-\beta}} \cdot ||f||_{C^{0,1}(\Omega)}^{\frac{\gamma-\beta}{1-\beta}} \tag{702}$$

Proof. Consider how the fraction on the power appears, one may find that by setting $(1-t)\beta + t = \gamma$, one would get $t = \frac{\gamma-\beta}{1-\beta} \in (0, 1]$.

First let's consider the Holder semi-norm to find

$$\frac{|f(x) - f(y)|}{||x-y||^\gamma} = \frac{|f(x) - f(y)|}{||x-y||^{(1-t)\beta}} \cdot \frac{|f(x) - f(y)|}{||x-y||^t} \tag{703}$$

$$= \left(\frac{|f(x) - f(y)|}{||x-y||^\beta} \right)^{1-t} \cdot \left(\frac{|f(x) - f(y)|}{||x-y||} \right)^t \tag{704}$$

take supreme w.r.t. $\forall x, y \in \Omega, x \ne y$ to get

$$[f]_\gamma \le [f]_\beta^{1-t} \cdot [f]_1^t \tag{705}$$

Now based on this fact, we only have to prove that

$$||f||_{C^{0,\gamma}} = ||f||_\infty + [f]_\gamma \leq ||f||_\infty + [f]_\beta^{1-t} \cdot [f]_1^t \tag{706}$$

to prove the interpolation inequality, one only needs to show that

$$||f||_\infty + [f]_\beta^{1-t} \cdot [f]_1^t \leq (||f||_\infty + [f]_\beta)^{1-t} \cdot (||f||_\infty + [f]_1)^t \tag{707}$$

to see clearly the structure of this inequality, let's try to do some equivalent transformations

$$||f||_\infty + [f]_\beta \left(\frac{[f]_1}{[f]_\beta}\right)^t \leq (||f||_\infty + [f]_\beta) \left(\frac{||f||_\infty + [f]_1}{||f||_\infty + [f]_\beta}\right)^t \tag{708}$$

$$\frac{||f||_\infty}{||f||_\infty + [f]_\beta} 1^t + \frac{[f]_\beta}{||f||_\infty + [f]_\beta} \left(\frac{[f]_1}{[f]_\beta}\right)^t \leq \left(\frac{||f||_\infty + [f]_1}{||f||_\infty + [f]_\beta}\right)^t \tag{709}$$

we immediately see that this is saying that for function $h(x) = x^t$ where $t \in (0, 1]$,

$$\frac{||f||_\infty}{||f||_\infty + [f]_\beta} h(1) + \frac{[f]_\beta}{||f||_\infty + [f]_\beta} h\left(\frac{[f]_1}{[f]_\beta}\right) \leq h\left(\frac{||f||_\infty + [f]_1}{||f||_\infty + [f]_\beta}\right) \tag{710}$$

and $\frac{||f||_\infty}{||f||_\infty + [f]_\beta} + \frac{[f]_\beta}{||f||_\infty + [f]_\beta} = 1$ with those two terms being positive and

$$\frac{||f||_\infty}{||f||_\infty + [f]_\beta} \cdot 1 + \frac{[f]_\beta}{||f||_\infty + [f]_\beta} \frac{[f]_1}{[f]_\beta} = \frac{||f||_\infty + [f]_1}{||f||_\infty + [f]_\beta} \tag{711}$$

and one might be able to tell that this is just the Jensen's inequality applied for h. Notice that

$$h'(x) = tx^{t-1}, h''(x) = t(t-1)x^{t-2} \leq 0 \tag{712}$$

one can conclude that h is concave, thus Jensen's inequality directly provides the proof. $\square$

The following estimation tells us the bound of the $C^{k,\alpha}$ norm by Sobolev norm in the case where $p > n$.

Theorem 50. ***(Morrey's Inequality)*** *Let $n < p \leq \infty$ and $\alpha = 1 - \frac{n}{p}$, then there exists constant $C = C(n, p)$ such that*

$$\forall f \in C_c^\infty(\mathbb{R}^n), [f]_{\alpha,\mathbb{R}^n} \leq C \cdot ||Df||_{L^p}, \ \sup_{\mathbb{R}^n} |f| \leq C \cdot ||f||_{W^{1,p}} \tag{713}$$

this will be able to provide us the Sobolev embedding in the case where $p > n$.

Week 10

Theorem 51. ***(Morrey's Inequality)*** *Let $n < p \leq \infty$ and $\alpha = 1 - \frac{n}{p}$, then there exists constant $C = C(n, p)$ such that*

$$\forall f \in C_c^\infty(\mathbb{R}^n), [f]_{\alpha, \mathbb{R}^n} \leq C \cdot ||Df||_{L^p}, \ \sup_{\mathbb{R}^n} |f| \leq C \cdot ||f||_{W^{1,p}} \tag{714}$$

Proof. **First step:** prove that there exists constant $C = C(n)$ such that for any ball $B_r(x)$,

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x) - f(y)| \, dy \leq C \cdot \int_{B_r(x)} \frac{||Df(y)||}{||x - y||^{n-1}} \, dy \tag{715}$$

To prove this, first take $\forall w \in \partial B_1(0)$ and consider $f(x + sw) - f(x) = \int_0^s \frac{d}{dt} f(x + tw) \, dt = \int_0^s Df(x + tw) \cdot w \, dt$ for $s > 0$. Take absolute value on both sides to find

$$\forall s > 0, |f(x + sw) - f(x)| \leq \int_0^s |Df(x + tw) \cdot w| \, dt \leq \int_0^s ||Df(x + tw)|| \, dt \tag{716}$$

since $||w|| = 1$ by Cauchy-Schwarz. To connect RHS with the integral on the ball, note that t is actually serving as radial variable. As a result, we can take $w \in \partial B_1(0)$ as the angular variable and integrate w.r.t. it

$$\forall s > 0, \int_{\partial B_1(0)} |f(x + sw) - f(x)| \, dS(w) \leq \int_{\partial B_1(0)} \int_0^s ||Df(x + tw)|| \, dt \, dS(w) \tag{717}$$

$$= \int_{B_s(x)} \frac{||Df(y)||}{||x - y||^{n-1}} \, dy \ (y = x + tw) \tag{718}$$

with $||x - y||^{n-1} = t^{n-1}$ as the Jacobian of the polar coordinate transformation.

Now deal with the LHS by splitting it under polar coordinates

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x) - f(y)| \, dy = \frac{1}{|B_r(x)|} \int_0^r s^{n-1} \int_{\partial B_1(0)} |f(x) - f(x + sw)| \, dS(w) \, ds \tag{719}$$

and apply the estimate to get

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x) - f(y)| \, dy \leq \frac{1}{|B_r(x)|} \int_0^r s^{n-1} \int_{B_s(x)} \frac{||Df(y)||}{||x - y||^{n-1}} \, dy \, ds \tag{720}$$

$$\leq \frac{1}{|B_r(x)|} \int_0^r s^{n-1} \, ds \cdot \int_{B_r(x)} \frac{||Df(y)||}{||x - y||^{n-1}} \, dy \tag{721}$$

$$= \frac{r^n}{n|B_r(x)|} \int_{B_r(x)} \frac{||Df(y)||}{||x - y||^{n-1}} \, dy \tag{722}$$

note that $C = \frac{r^n}{n|B_r(x)|} = \frac{1}{nV_n(1)}$, this proved that the constant C only depends on n and has nothing to do with r.

Second step: Set $r = ||x-y||, \Omega = B_r(x) \cap B_r(y)$, so now

$$|f(x) - f(y)| \leq \frac{1}{|\Omega|} \int_\Omega |f(x) - f(z)|\, dz + \frac{1}{|\Omega|} \int_\Omega |f(z) - f(y)|\, dz \tag{723}$$

is averaging w.r.t. z. Notice that $\Omega \subset B_r(x)$

$$\begin{aligned}
\frac{1}{|\Omega|} \int_\Omega |f(x) - f(z)|\, dz &\leq \frac{|B_r(x)|}{|\Omega|} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x) - f(z)|\, dz && (724)\\
&\leq \frac{|B_r(x)|}{|\Omega|} \cdot C \cdot \int_{B_r(x)} \frac{||Df(z)||}{||x-z||^{n-1}}\, dz && (725)\\
&\leq C' \cdot \int_{B_r(x)} \frac{||Df(z)||}{||x-z||^{n-1}}\, dz && (726)
\end{aligned}$$

where C' only depends on n. By Holder's inequality applied for conjugate p, q, the norm of Df finally appears

$$\begin{aligned}
\frac{1}{|\Omega|} \int_\Omega |f(x) - f(z)|\, dz &\leq C' \left(\int_{B_r(x)} ||Df||^p\, dz \right)^{\frac{1}{p}} \cdot \left(\int_{B_r(x)} ||x-z||^{(1-n)q}\, dz \right)^{\frac{1}{q}} && (727)\\
&\leq C' \cdot ||Df||_{L^p} \cdot r^{1-n} \left(|B_r(x)| \right)^{\frac{1}{q}} && (728)\\
&\leq C'' \cdot ||Df||_{L^p} \cdot r^{1-n+\frac{n}{q}} && (729)\\
&= C'' \cdot ||Df||_{L^p} \cdot r^{1-\frac{n}{p}} && (730)
\end{aligned}$$

where constant $C = C(n,p)$ and $q = \frac{p}{p-1}$.

Return to the estimate on $|f(x) - f(y)|$ to see that

$$\begin{aligned}
|f(x) - f(y)| &\leq C \cdot ||Df||_{L^p} \cdot r^{1-\frac{n}{p}} && (731)\\
&= C \cdot ||Df||_{L^p} \cdot ||x-y||^{1-\frac{n}{p}} && (732)\\
\frac{|f(x) - f(y)|}{||x-y||^{1-\frac{n}{p}}} &\leq C \cdot ||Df||_{L^p} && (733)
\end{aligned}$$

so the fact that $[f]_{\alpha, \mathbb{R}^n} \leq C \cdot ||Df||_{L^p}$ is proved.

Third step: Consider $|f(x)| \leq |f(x) - f(y)| + |f(y)|$, integrate both sides over the unit ball w.r.t. y to get

$$\begin{aligned}
|f(x)| &\leq \frac{1}{|B_1(x)|} \int_{B_1(x)} |f(x) - f(y)|\, dy + \frac{1}{|B_1(x)|} \int_{B_1(x)} |f(y)|\, dy && (734)\\
&\leq C \cdot ||Df||_{L^p} + C \cdot ||f||_{L^p} && (735)\\
&\leq C \cdot ||f||_{W^{1,p}} && (736)
\end{aligned}$$

this proved the second inequality.

$\square$

Remark. *One might feel confused with the proof of Morrey's inequality since the steps seem to be messy and tricky. However, here we are actually estimating the difference* $|f(x) - f(y)|$ *w.r.t.* Df *and average the result over a ball* $B_r(x)$ *and apply Holder's inequality to make sure that* $||Df||_{L^p}$ *appears.*

Actually, the Morrey's inequality contains the information of the boundary behavior. Let's define

$$C_0^{0,\alpha}(\mathbb{R}^n) = \left\{f \in C^{0,\alpha}(\mathbb{R}^n) : f(x) \to 0 \ (x \to \infty)\right\} \tag{737}$$

as the space of α-Holder continuous functions with vanishing boundary values. This is equivalent to saying that $\forall \varepsilon > 0$, there exists compact K such that $\forall x \notin K, |f(x)| < \varepsilon$.

The following theorem shows the Sobolev embedding results in the case $p > n$.

Theorem 52. ***(Sobolev Embedding Theorem for*** $p > n$***)*** *Let* $n < p < \infty, \alpha = 1 - \frac{n}{p}$*, then* $W^{1,p}(\mathbb{R}^n) \hookrightarrow C_0^{0,\alpha}(\mathbb{R}^n)$ *and there is a constant* $C = C(n,p)$ *such that* $\forall f \in W^{1,p}(\mathbb{R}^n), ||f||_{C^{0,\alpha}} \leq C||f||_{W^{1,p}}$.

Proof. The proof of the theorem is the same as that for $p < n$. First approximate $\forall f \in W^{1,p}(\mathbb{R}^n)$ by a series of compactly supported smooth functions $f_m \in C_c^\infty(\mathbb{R}^n)$ with $f_m \overset{W^{1,p}(\mathbb{R}^n)}{\to} f \ (m \to \infty)$. Now by Morrey's inequality,

$$[f_m]_{\alpha,\mathbb{R}^n} \leq C \cdot ||Df_m||_{L^p} \tag{738}$$

$$\sup_{\mathbb{R}^n} |f_m| \leq C \cdot ||f_m||_{W^{1,p}} \tag{739}$$

since f_m is Cauchy in $W^{1,p}(\mathbb{R}^n)$, it's also Cauchy in $C^{0,\alpha}$ and L^∞ and the limit must also be f. So

$$[f]_{\alpha,\mathbb{R}^n} \leq C \cdot ||Df||_{L^p} \tag{740}$$

$$\sup_{\mathbb{R}^n} |f| \leq C \cdot ||f||_{W^{1,p}} \tag{741}$$

and this is concluding that $f \in C^{0,\alpha}(\mathbb{R}^n)$.

However, one might notice that here we have an extra convergence in L^∞ holding. As a result,

$$\sup_{\mathbb{R}^n} |f_m - f| \to 0 \ (m \to \infty) \tag{742}$$

and since $f_m \in C_c^\infty(\mathbb{R}^n)$, the f has to vanish at ∞, so $f \in C_0^{0,\alpha}(\mathbb{R}^n)$. ☐

Remark. *The interesting fact here is that Sobolev functions can actually be embedded into a smaller space* $C_0^{0,\alpha}(\mathbb{R}^n) \subset C^{0,\alpha}(\mathbb{R}^n)$ *than the space of locally* α*-Holder continuous functions with the same norm defined.*

Note that for $p = \infty$*,* $f \in W^{1,\infty}(\mathbb{R}^n)$ ***is globally Lipschitz with the inequality*** $[f]_{1,\mathbb{R}^n} \leq C \cdot ||Df||_{L^\infty}$ ***to hold****. In this case,* f *does not need to vanish at* ∞ *since the Holder norm cannot distinguish between constants. A counterexample is to set* f *as any constant function. That's why* ***when*** $p = \infty$***, the embedding*** $W^{1,p}(\mathbb{R}^n) \hookrightarrow C_0^{0,\alpha}(\mathbb{R}^n)$ ***no longer holds****. We also remind the reader here that the Sobolev approximation also does not work for* $W^{1,\infty}(\mathbb{R}^n)$ *and the set of all continuous functions is even not enough to be dense.*

Despite the fact that $W^{1,\infty}(\mathbb{R}^n)$ is not a nice enough space for us to work on, it provides characterizations of the globally Lipschitz functions.

Theorem 53. ***(Characterization of Lipschitz Functions)*** *$f \in L^1_{loc}(\mathbb{R}^n)$ is globally Lipschitz if and only if f is weakly differentiable and $Df \in L^\infty(\mathbb{R}^n)$.*

Despite the fact that the case where $p = \infty$ is eliminated from the Sobolev embedding theorem, the following property holds for $n < p \leq \infty$.

Theorem 54. ***(Consistency of Pointwise Derivative and Weak Derivative of Sobolev Functions)*** *For $n < p \leq \infty$, if $f \in W^{1,p}(\mathbb{R}^n)$, then f is differentiable almost everywhere and the pointwise derivative is just the weak derivative.*

More Inequality on the Norm of Derivatives

We also present another inequality here that displays the relationship of the L^p norm of $f, Df, D^2 f$. Similar to what we have done before, we first build this estimate for $f \in C_c^\infty(\Omega)$. The inequality can also be proved for Sobolev functions by applying approximations using compactly supported smooth functions.

Theorem 55. *For $\Omega \subset \mathbb{R}^n, 2 \leq p < \infty, \forall f \in C_c^\infty(\Omega)$, $||Df||_{L^p} \leq C \cdot ||f||_{L^p}^{\frac{1}{2}} \cdot ||D^2 f||_{L^p}^{\frac{1}{2}}$ for constant $C = C(n,p)$ where $D^2 f = vec(H)$ is the vectorized Hessian matrix.*

Proof. Let's denote $x' = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. Notice that

$$\int ||Df||_2^p \, dx = \sum_{i=1}^n \int (\partial_i f)^2 ||Df||_2^{p-2} \, dx \tag{743}$$

$$= \sum_{i=1}^n \int \int (\partial_i f)^2 ||Df||_2^{p-2} \, dx_i \, dx' \tag{744}$$

$$= \sum_{i=1}^n \int \int \partial_i f \cdot ||Df||_2^{p-2} \, df \, dx' \tag{745}$$

$$= -\sum_{i=1}^n \int \int f \, \partial_i [\partial_i f \cdot ||Df||_2^{p-2}] \, dx' \tag{746}$$

where $||\cdot||_2$ denotes the vector l_2 norm and the boundary term vanishes since f has compact support. Let's calculate the partial derivative $\partial_i[\partial_i f ||Df||_2^{p-2}]$ to find that

$$\frac{\partial}{\partial x_i}[\partial_i f \cdot ||Df||_2^{p-2}] = \partial_{ii} f \cdot ||Df||_2^{p-2} + \partial_i f \cdot \frac{\partial ||Df||_2^{p-2}}{\partial x_i} \tag{747}$$

$$= \partial_{ii} f \cdot ||Df||_2^{p-2} + (p-2)\partial_i f \cdot ||Df||_2^{p-4} \sum_{j=1}^n \partial_j f \cdot \partial_{ij} f \tag{748}$$

As a result, let's continue with previous calculations to find

$$\int ||Df||_2^p\,dx = -\sum_{i=1}^n \int\int f\left[\partial_{ii}f\cdot||Df||_2^{p-2} + (p-2)\partial_i f\cdot||Df||_2^{p-4}\sum_{j=1}^n \partial_j f\cdot\partial_{ij}f\right]dx_i\,dx' \tag{749}$$

$$= -\int f\cdot\Delta f\cdot||Df||_2^{p-2}\,dx - (p-2)\int f\cdot||Df||_2^{p-4}\cdot\sum_{i,j=1}^n \partial_i f\cdot\partial_j f\cdot\partial_{ij}f\,dx \tag{750}$$

$$\leq \int |f|\cdot|\Delta f|\cdot||Df||_2^{p-2}\,dx + (p-2)\int |f|\cdot||Df||_2^{p-4}\cdot\sum_{i,j=1}^n |\partial_i f|\cdot|\partial_j f|\cdot|\partial_{ij}f|\,dx \tag{751}$$

$$\leq \int |f|\cdot||D^2f||_1\cdot||Df||_2^{p-2}\,dx + (p-2)\int |f|\cdot||Df||_2^{p-2}\cdot||D^2f||_1\,dx \tag{752}$$

$$\leq C\int |f|\cdot||D^2f||_1\cdot||Df||_2^{p-2}\,dx \tag{753}$$

$$\leq C\int |f|\cdot||D^2f||_2\cdot||Df||_2^{p-2}\,dx \tag{754}$$

because any two norms on finite -dimensional space are equivalent. Now apply Holder's inequality for conjugate numbers $p, p, \frac{p}{p-2}$ to get

$$||Df||_{L^p}^p = \int ||Df||_2^p\,dx \leq C\cdot||f||_{L^p}\cdot||D^2f||_{L^p}\cdot||Df||_{L^p}^{p-2} \tag{755}$$

and the theorem is proved. ☐

Theorem 56. ***(Bound the Norm of Gradient by Function Norm and the Norm of Hessian)*** *For $\Omega\subset\mathbb{R}^n, 1\leq p<\infty, \forall f\in C_c^\infty(\Omega)$, $||Df||_{L^{2p}}\leq C\cdot||f||_{L^\infty}^{\frac{1}{2}}\cdot||D^2f||_{L^p}^{\frac{1}{2}}$ for constant $C=C(n,p)$ where $D^2f=vec(H)$ is the vectorized Hessian matrix.*

Proof. The proof of this theorem is basically the same as the last one. Just do the same calculation for $2p$ to get

$$\int ||Df||_2^{2p}\,dx \leq C\int |f|\cdot||D^2f||_2\cdot||Df||_2^{2p-2}\,dx \tag{756}$$

to proceed, apply Holder's inequality for conjugate numbers $p, \frac{p}{p-1}$ to get

$$||Df||_{L^{2p}}^{2p} = \int ||Df||_2^{2p}\,dx \leq C\cdot||f||_\infty\cdot\int ||D^2f||_2\cdot||Df||_2^{2p-2}\,dx \tag{757}$$

$$\leq C\cdot||f||_\infty\cdot||D^2f||_{L^p}\cdot||Df||_{L^{2p}}^{2p-2} \tag{758}$$

and it's proved. ☐

Boundary Values of Sobolev Functions

For function $f \in C(\overline{\Omega})$, one can define the boundary value of such continuous function in the limiting sense as

$$\forall x \in \partial\Omega, f(x) = \lim_{x_n \to x, x_n \in \Omega} f(x_n) \tag{759}$$

however, this does not hold for Sobolev functions since Sobolev functions are not necessarily different from continuous functions on a zero measure set. That's why it's impossible to extend Sobolev functions as what we can do for continuous functions as what the following counterexample shows.

An example is that let operator $T : C^\infty([0,1]) \to \mathbb{R}$ be defined such that

$$T(\phi) = \phi(0) \tag{760}$$

consider $\phi^\varepsilon(x) = e^{-\frac{x^2}{\varepsilon}}$ $(\varepsilon > 0)$ to find

$$||\phi^\varepsilon||_{L^1} \leq 1 \tag{761}$$

and that's why

$$\lim_{\varepsilon \to 0^+} ||\phi^\varepsilon||_{L^1} = \int_0^1 \lim_{\varepsilon \to 0^+} e^{-\frac{x^2}{\varepsilon}}\, dx = 0 \tag{762}$$

notice that $\phi^\varepsilon(0) = 1$ always holds, so T cannot be a bounded operator on the L^1 space (otherwise $|\phi^\varepsilon(0)| \leq ||T|| \cdot ||\phi^\varepsilon||_{L^1}$ and $||T|| = \infty$) which is telling us that it's impossible to extend T by continuity to $L^1([0,1])$, which is a larger space.

However, one might have already seen the necessity of the extension theorem of Sobolev functions in the context above. The extension allows us to prove the conclusions for Sobolev functions on general areas by proving the conclusion only for the whole space $\mathbb{R}^n$. The theorem is stated again below

Theorem 57. ***(Sobolev Extension Theorem)*** *For $1 \leq p \leq \infty$, if Ω is bounded with C^1 boundary, and there exists bounded open set Ω' such that $\Omega \subset\subset \Omega'$, then there exists a linear bounded operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ having support within Ω' such that*

$$\forall f \in W^{1,p}(\Omega), \forall x \in \Omega, Ef(x) = f(x) \text{ a.e.} \tag{763}$$

and the extension is bounded in norm

$$\exists C, \forall f \in W^{1,p}(\Omega), ||Ef||_{W^{1,p}(\mathbb{R}^n)} \leq C \cdot ||f||_{W^{1,p}(\Omega)} \tag{764}$$

where the constant $C = C(p, \Omega, \Omega')$.

To prove such a theorem, one actually only needs to deal with the half space as a special case and all bounded area with C^1 boundary can be dealt similarly with the help of diffeomorphism. Denote $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n = 0\}$ as the half space and $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} = \{x \in \mathbb{R}^n : x_n = 0\}$, so for $x \in \partial\mathbb{R}^n_+$, we denote $x = (x', 0), x' \in \mathbb{R}^{n-1}$. Let's first consider the extension from the upper half plane to the whole space.

Theorem 58. ***(Sobolev Extension Theorem for Half Space)*** *There exists linear bounded operator $E : W^{1,p}(\mathbb{R}^n_+) \to W^{1,p}(\mathbb{R}^n)$ such that $\forall f \in W^{1,p}(\mathbb{R}^n_+), Ef = f$ a.s. on $\mathbb{R}^n_+$ and $||Ef||_{W^{1,p}(\mathbb{R}^n)} \leq C \cdot ||f||_{W^{1,p}(\mathbb{R}^n_+)}$ where $C = C(n,p)$.*

Proof. The sketch of the proof is to first extend f as a function $\tilde{f}$ on $\mathbb{R}^n$ by reflecting the function value w.r.t. $\partial\mathbb{R}^n_+$. One can ensure the regularity on the boundary $\partial\mathbb{R}^n_+$ (it's still a Sobolev function across the boundary) such that $\tilde{f} \in W^{1,p}(\mathbb{R}^n)$. □

We have presented the dense argument that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$. However, this does not hold for general area and the price one has to pay such that similar dense argument still holds is to make sure that the function in the dense subset is compactly supported and smooth on the closure of the open area.

Theorem 59. ***(Sobolev Approximation for Upper Half Plane)*** *$C_c^\infty(\overline{\mathbb{R}^n_+})$ is dense in $W^{k,p}(\mathbb{R}^n_+)$ since the closure allows the function to have arbitrary boundary value.*

Remark. *This theorem makes sense because for compactly supported smooth functions on a closed set, the boundary value is not necessarily 0 any longer. On the other hand, our previous proof for the Sobolev Approximation on the whole space still holds to prove that the set $C^\infty(\mathbb{R}^n_+)$ consisting of all smooth functions on $\mathbb{R}^n_+$ is dense in $W^{k,p}(\mathbb{R}^n_+)$.*

Define the space $W_0^{k,p}(\mathbb{R}^n_+)$ as the closure of $C_c^\infty(\mathbb{R}^n_+)$ under the norm $||\cdot||_{W^{k,p}}$ as the space of $W^{k,p}$ Sobolev functions that have vanishing boundary values. The following theorem characterizes the Sobolev space with vanishing boundary values with trace maps.

Theorem 60. ***(Characterization of $W_0^{1,p}(\mathbb{R}^n_+)$ using Trace Maps)*** *For $1 \leq p < \infty$, there exists a bounded linear operator $T : W^{1,p}(\mathbb{R}^n_+) \to L^p(\partial\mathbb{R}^n_+)$ such that for any $f \in C_c^\infty(\overline{\mathbb{R}^n_+})$, $Tf(x') = f(x', 0)$ and $||Tf||_{L^p(\partial\mathbb{R}^n_+)} \leq C \cdot ||f||_{W^{1,p}(\mathbb{R}^n_+)}$ for $C = C(p)$. Moreover, $f \in W_0^{1,p}(\mathbb{R}^n_+)$ if and only if $Tf = 0$.*

Proof. First notice that for $f \in C_c^\infty(\overline{\mathbb{R}^n_+})$ and $x' \in \mathbb{R}^{n-1}, p \geq 1$, we can always represent $|f(x',0)|^p$ as an integral that

$$|f(x', 0)|^p \leq p \int_0^\infty |f(x', t)|^{p-1} |\partial_{x_n} f(x', t)| \, dt \tag{765}$$

since $\partial_{x_n}[f^p(x',t)] = p[f^{p-1}(x',t)] \cdot \partial_{x_n} f(x',t)$. Take this as a starting point, we want to see the $W^{1,p}$ norm on the RHS and the L^p norm on the LHS. As a result, it's natural to integrate both sides w.r.t. x' and apply Holder's inequality for conjugate $p, q = \frac{p}{p-1}$

$$\int_{\mathbb{R}^{n-1}} |f(x', 0)|^p \, dx' \leq p \int_{\mathbb{R}^{n-1}} \int_0^\infty |f(x', t)|^{p-1} |\partial_{x_n} f(x', t)| \, dt \, dx' \tag{766}$$

$$\leq p \left(\int_{\mathbb{R}^{n-1}} \int_0^\infty |f(x', t)|^p \, dt \, dx' \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^{n-1}} \int_0^\infty |\partial_{x_n} f(x', t)|^p \, dt \, dx' \right)^{\frac{1}{p}} \tag{767}$$

$$= p||f||_{L^p}^{p-1} \cdot ||\partial_{x_n} f||_{L^p} \tag{768}$$

$$\leq p||f||_{W^{1,p}}^p \tag{769}$$

we have proved that

$$||Tf||_{L^p(\partial\mathbb{R}^n_+)} \leq p^{\frac{1}{p}}||f||_{W^{1,p}(\mathbb{R}^n_+)} \tag{770}$$

with the constant $C = p^{\frac{1}{p}}$ only depends on p. Note that this estimation also tells us that T is a linear bounded operator both in L^p and $W^{1,p}$ norm. Since $C_c^\infty(\overline{\mathbb{R}^n_+})$ is dense in $W^{1,p}(\mathbb{R}^n_+)$, one is able to extend the operator T onto $W^{1,p}(\mathbb{R}^n_+)$. For $\forall f \in W^{1,p}(\mathbb{R}^n_+)$,

$$\exists f_n \in C_c^\infty(\overline{\mathbb{R}^n_+}), f_n \overset{W^{1,p}(\mathbb{R}^n_+)}{\to} f \ (n \to \infty) \tag{771}$$

$$T(f) \overset{def}{=} \lim_{n\to\infty} T(f_n) \tag{772}$$

note that $T(f_n)$ is guaranteed to be Cauchy so the limit exists under norm $||\cdot||_{L^p(\partial\mathbb{R}^n_+)}$.

For the last fact, if $f \in W_0^{1,p}(\mathbb{R}^n_+)$, by definition, this space is the closure of $C_c^\infty(\mathbb{R}^n_+)$ and

$$\exists f_n \in C_c^\infty(\mathbb{R}^n_+), f_n \overset{W^{1,p}(\mathbb{R}^n_+)}{\to} f \ (n \to \infty) \tag{773}$$

$$T(f) = \lim_{n\to\infty} T(f_n) = 0 \tag{774}$$

since $\forall n, T(f_n) = 0$.

The other direction is not that trivial and a sketch of the proof is that if $T(f) = 0$ we can first extend f to lower half plane and shift it upward a little bit such that it's 0 near the boundary. Use the mollifier to smooth the shifted function and apply the approximation identity of the mollifier to conclude.

$\square$

Remark. *Such T is called the **trace map**, which maps $W^{1,p}$ functions on the upper half plane to L^p functions on the boundary by filling in the last component as 0. The trace map is **a contraction in norm up to a constant multiple that only depends on** p and the characterization of $W_0^{1,p}(\mathbb{R}^n_+)$ is that*

$$KerT = W_0^{1,p}(\mathbb{R}^n_+) \tag{775}$$

*it's **the kernel of the trace map**.*

Remark. *The last part of this theorem does not hold for $W^{k,p}(\mathbb{R}^n_+)$ functions. In particular, $T(f) = 0$ is not enough to ensure that $f \in W_0^{k,p}(\mathbb{R}^n_+)$ and additional regularity is required. To see this, $\partial_i f \in W_0^{k-1,p}(\mathbb{R}^n_+)$ so one need the images of all partial derivatives up to order $k-1$ under T to be 0, i.e.*

$$\forall |\alpha| \leq k-1, T(\partial^\alpha f) = 0 \Longleftrightarrow f \in W^{k,p}(\mathbb{R}^n_+) \tag{776}$$

Remark. *For $p = 1$, the trace map $T : W^{1,1}(\mathbb{R}^n_+) \to L^1(\partial\mathbb{R}^n_+)$ is **surjective**. However, for $1 < p < \infty$, the trace map is not surjective and the image actually forms the Besov space $B^{1-\frac{1}{p},p}(\partial\mathbb{R}^n_+)$.*

To understand what the trace operator is doing intuitively, it's actually tracing the function values on the boundary of the area in order to extend the restriction of a function to its boundary to a generalized function in the Sobolev space. Unfortunately, although we are able to construct the trace operator from $W^{1,p}(\mathbb{R}^n_+)$ to $L^p(\partial\mathbb{R}^n_+)$ in the context above, the trace operator on general bounded area does not exist if we extend the function continuously to the boundary of the area. (a more general result than the counterexample we have raised above)

Theorem 61. ***(Non-existence of Trace Operator on Bounded Area for Continuous Extension of*** L^p ***Functions)*** *For* Ω *as bounded open area with* C^1 *boundary, there does not exist a bounded linear operator* $T : L^p(\Omega) \to L^p(\partial\Omega)$ *such that* $Tf = f$ *on* $\partial\Omega$ *for* $\forall f \in C(\overline{\Omega}) \cap L^p(\Omega)$.

Proof. Consider

$$f_n(x) = \max\{0, 1 - n \cdot dist(x, \partial\Omega)\} \tag{777}$$

then f_n is continuous since $dist$ is always continuous and $0 \leq f_n \leq 1$ so it's L^p on a bounded area. Notice that $\forall n, \forall x \in \partial\Omega, f_n(x) = 1$ since we want to satisfy the property that $f_n \in C(\overline{\Omega})$ to extend this series of functions continuously to the boundary. However, one might notice that

$$\forall x \in \Omega, f_n(x) \to 0 \ (n \to \infty) \tag{778}$$

so the pointwise limit f of f_n is constantly 0 inside Ω and thus

$$\forall x \in \partial\Omega, f(x) = 0 \tag{779}$$

is the continuous extension to the boundary.

However, since the trace operator is linear bounded operator (of course, continuous), it keeps the value of the function on the boundary, so

$$\forall x \in \partial\Omega, T(f_n)(x) = 1 \tag{780}$$

$$\forall x \in \partial\Omega, T(f)(x) = 0 \tag{781}$$

as a result, $\forall x \in \partial\Omega, \lim_{n\to\infty} T(f_n)(x) \neq T(f)(x)$, a contradiction with the continuity! □

Remark. *The spirit of the construction above is to find a series of function that stays flat inside the area but increases suddenly near the boundary.*

Now to prove the Sobolev extension theorem that works for general bounded domain Ω with $\overline{\Omega}$ compact ($\partial\Omega$ is C^1), we may cover $\overline{\Omega}$ by a collection of open balls that are contained in Ω or are centered on $\partial\Omega$. There exists finite sub-cover $B_1, ..., B_N$ so there exists a partition of the unity

$$\psi_i \in C_c^\infty(B_i), \psi_i \in [0,1], \sum_{i=1}^N \psi_i = 1 \ on \ \overline{\Omega} \tag{782}$$

For any given $f \in L^1_{loc}(\Omega)$, write

$$f = \sum_{i=1}^{N} f_i, f_i = \psi_i f \tag{783}$$

so either $supp(f_i) \subset \Omega$ which corresponds to $f_i \in L^1_{loc}(\mathbb{R}^n)$ or $supp(f_i) \cap \Omega \neq \emptyset$ which corresponds to $f_i \in L^1_{loc}(\mathbb{R}^n_+)$. The correspondence can be realized locally by applying the diffeomorphism.

The value of the Sobolev extension theorem has been stated in the previous context. It enables us to extend the Sobolev embedding theorem to general bounded area with C^1 boundary. For higher order derivatives, one can prove the similar embedding conclusion that

$$W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \tag{784}$$

for q to be the Sobolev conjugate of p defined as

$$\frac{1}{q} + \frac{k}{n} = \frac{1}{p} \tag{785}$$

when $p < n$. In particular, those conclusions also hold for general bounded area with C^1 boundary.