Section Notes for PSTAT 213C

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Contents

Week 1

Review: Conditional Expectation

Exercise 1. *Let $\mathscr{F}_2 \subset \mathscr{F}_1 \subset \mathscr{F}$ be σ-fields on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For $X \in L^1$, show that*

$$\mathbb{E}(X - \mathbb{E}(X|\mathscr{F}_1))^2 \leq \mathbb{E}(X - \mathbb{E}(X|\mathscr{F}_2))^2. \tag{1}$$

If $X \in L^2$, what is the geometric interpretation?

Hints. Expand squares, it suffices to show that $\mathbb{E}[\mathbb{E}(X|\mathscr{F}_1)]^2 \geq \mathbb{E}[\mathbb{E}(X|\mathscr{F}_2)]^2$. Apply Jensen's inequality for the r.v. $\mathbb{E}(X|\mathscr{F}_1)$ under the conditional expectation $\mathbb{E}(\cdot|\mathscr{F}_2)$ yields $\mathbb{E}([\mathbb{E}(X|\mathscr{F}_1)]^2|\mathscr{F}_2) \geq [\mathbb{E}(\mathbb{E}(X|\mathscr{F}_1)|\mathscr{F}_2)]^2$. Taking expectations on both sides and using the tower property conclude the proof.

Geometrically, the distance from a fixed point to a vector space V (e.g., a plane) is no larger than the distance from the same point to a linear subspace of V (e.g., any line within the plane). □

Exercise 2. *Define the conditional independence $\mathscr{F}_1 \perp \mathscr{F}_2|\mathscr{G}$ to hold if and only if*

$$\mathbb{E}(X_1X_2|\mathscr{G}) = \mathbb{E}(X_1|\mathscr{G}) \cdot \mathbb{E}(X_2|\mathscr{G}),\ \forall X_1 \in \mathscr{F}_1, X_2 \in \mathscr{F}_2. \tag{2}$$

Show that:

(i): $\mathscr{F}_1 \perp \mathscr{F}_2|\mathscr{G}$ if and only if $(\mathscr{F}_1 \vee \mathscr{G}) \perp \mathscr{F}_2|\mathscr{G}$, where $\mathscr{F}_1 \vee \mathscr{G} := \sigma(\mathscr{F}_1 \cup \mathscr{G})$.

(ii): If $\mathscr{G} \subset \mathscr{F}_1$, then $\mathscr{F}_1 \perp \mathscr{F}_2|\mathscr{G}$ if and only if $\mathbb{E}(X_2|\mathscr{F}_1) \in \mathscr{G}$ for $\forall X_2 \in \mathscr{F}_2$.

Hints. (i): Direction $\Leftarrow$ is trivial. Direction $\Rightarrow$ requires $\pi - \lambda$ theorem (first prove the statement for $X_1\mathbb{I}_G$, where $X_1 \in \mathscr{F}_1$ and $G \in \mathscr{G}$).

(ii): Direction $\Leftarrow$ is trivial (tower property). Direction $\Rightarrow$ requires proving $\mathbb{E}(X_2|\mathscr{F}_1) = \mathbb{E}(X_2|\mathscr{G})$, using the definition of conditional expectation. □

Martingale

Exercise 3. *$\{X_n\}$ is a discrete-state Markov chain (with countable state space S) and transition probability matrix P. If there exists a bounded function $\psi : S \to \mathbb{R}$ such that*

$$\sum_{j\in S} P_{ij}\psi(j) = \psi(i),\ \forall i \in S, \tag{3}$$

check that $\{X_n\}$ is a martingale under the natural filtration.

A discrete-state Markov chain induces a difference operator L such that

$$(L\psi)(i) := \sum_{j\in S} P_{ij}\psi(j) - \psi(i),\ \forall i \in S. \tag{4}$$

What about a continuous-state Markov chain (e.g., Brownian motion)? That induces a differential operator on $\mathbb{R}^d$, which is one half the Laplacian $\Delta := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$. With time evolution of the Markov chain introduced, this provides crucial connection between probability and PDE (generator, forward/backward equations).

Exercise 4. *$\{X_n\}$ is a sequence of i.i.d. r.v. with an unknown density f. The density f is known to be either p or q (both are known and strictly positive), which results in a likelihood ratio test*

$$\begin{cases} H_0 : f = q \\ H_1 : f = p \end{cases}, \tag{5}$$

with a test statistic

$$Y_n := \prod_{i=1}^n \frac{p(X_i)}{q(X_i)}. \tag{6}$$

Show that $\{Y_n\}$ is a martingale under the natural filtration of $\{X_n\}$ if H_0 is true.

The rejection region of this test: $Y_n \geq a$. The simplest example of sequential hypothesis testing (continue collecting evidence until a conclusion can be drawn).

Does $\{Y_n\}$ admit a limit? If one terminates the test only when a rejection is made, can we control the probability of rejecting H_0 if H_0 is true (type-I error)? Will be answered next week.

Exercise 5. *A village contains $N+1$ people, one of whom suffers from an infectious illness that cannot be cured. S_t is the number of susceptibles at time t, I_t is the number of infectives and D_t is the number of deads such that $S_t + I_t + D_t = N+1$, while $S_0 = N, I_0 = 1, D_0 = 0$.*

Once a person is infected, his remaining lifespan is a random time that follows $\mathcal{E}(\mu)$. Once a susceptible interacts with an infective, he gets infected after a random time that follows $\mathcal{E}(\lambda)$. It is assumed that at any fixed time, any two individuals within the system interact, and the random times are all independent.

(i): Specify the state space S_X and the dynamics of the continuous-time discrete-state Markov chain (S_t, I_t).

(ii): Specify the generator G of (S_t, I_t).

(iii): If $G\psi = 0$ for some $\psi : S_X \to \mathbb{R}$, show that $Y_t := \psi(S_t, I_t)$ is a martingale under the natural filtration of $\{(S_t, I_t)\}$.

(iv): Find one such $\psi(s,i) = \alpha(s)\beta(i)$ that reveals the martingale structure of the epidemic model.

This epidemic model must result in one of the two situations: either everyone dies due to illness or the illness dies out itself before infecting everybody. How to calculate the probability of those situations happening? Will also be answered next week.

Hints. (i): $S_X = \{(s,i) : s + i \leq N+1, s \geq 0, i \geq 0\}$. Given $(S_t, I_t) = (s,i)$, the next state transition is either a death $(s,i) \to (s, i-1)$ or an infection $(s,i) \to (s-1, i+1)$.

The time until the next death is the minimum of the lifespan of each infective, which is the minimum of i independent $\mathcal{E}(\mu)$ r.v. Consequently, death rate $DR_{(s,i)} = i\mu$.

The time until the next infection is the minimum of is independent $\mathcal{E}(\lambda)$ r.v. (since each of the s susceptible interacts with all i infectives). Consequently, infection rate $IR_{(s,i)} = is\lambda$.

Therefore, the holding rate of state (s,i) is $i(\mu+s\lambda)$. Whenever a state transition happens, there is $\frac{i\mu}{i\mu+is\lambda}$ probability transiting to $(s,i-1)$, there is $\frac{is\lambda}{i\mu+is\lambda}$ probability transiting to $(s-1,i+1)$.

(ii): By definition,

$$G_{(s,i),(s,i-1)} = i\mu, \quad G_{(s,i),(s-1,i+1)} = is\lambda, \quad G_{(s,i),(s,i)} = -i(\mu+s\lambda), \tag{7}$$

while all other entries are zero.

(iii): Prove the martingale property within an infinitesimal interval $[t, t+\Delta t]$.

(iv): Assume $\alpha(N) = 1$ and solve $G\psi = 0$ to get

$$\psi(s,i) = \prod_{k=s+1}^{N} \frac{kB\lambda - (1-B)\mu}{k\lambda B^2} B^i, \tag{8}$$

which is a valid solution for an arbitrary $B > 0$. □

Remark. *One obtains the freedom of choosing $B > 0$ in the example above that ensures*

$$Y_t = \prod_{k=S_t+1}^{N} \frac{kB\lambda - (1-B)\mu}{k\lambda B^2} B^{I_t} \tag{9}$$

being a martingale.

As we shall see later, martingales have very nice structures and are the easiest stochastic processes to investigate. Unfortunately, processes of general interests (e.g., branching processes, diffusions, asymmetric random walks) are typically not martingales. This example shows how one should discover the hidden martingale structure to be able to apply technical tools that are specifically designed for martingales (e.g., optional stopping theorem, maximal inequalities, etc.). ***The martingale structure is crucial but never for free!***