Section Notes for PSTAT 213

Haosheng Zhou

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Contents

This note contains extra exercises, examples and materials for **PSTAT 213.** The notes may be subject to typos, and you are welcome to email me at **hzhou593@ucsb.edu for any possible advice.**

Week 1

Example for Indicator

Lemma 1 (Example)**.** *The indicator I_A of event A is a random variable defined as*

$$I_A(\omega) = \begin{cases} 1 & if\ \omega \in A \\ 0 & else \end{cases} \tag{1}$$

(a): Write PMF for $X = I_A$ and calculate $\mathbb{E}X, Var(X), G_X(s)$.

(b): For random variable $Y \geq 0$ and ϕ as any non-negative increasing function on $[0, +\infty)$, show that $\forall a > 0, \phi(a) \cdot \mathbb{P}\left(Y \geq a\right) \leq \mathbb{E}\phi(Y)$ so that $\forall \varepsilon > 0, \mathbb{P}\left(|Z| \geq \varepsilon\right) \leq \frac{\mathbb{E}Z^2}{\varepsilon^2}$ for any random variable Z.

(c): Assume Y is a random variable such that its MGF $M_Y(t) = \mathbb{E}e^{tY}$ is finite for all $t \in \mathbb{R}$, show that when $t \geq 0$, $\mathbb{P}\left(X \geq x\right) \leq e^{-tx}M_X(t)$ so that $\mathbb{P}\left(X \geq x\right) \leq \inf_{t \geq 0} e^{-tx}M_X(t)$.

Proof. (a): X has support $\{0, 1\}$ with $\mathbb{P}\left(X = 1\right) = \mathbb{P}\left(A\right), \mathbb{P}\left(X = 0\right) = \mathbb{P}\left(A^c\right) = 1 - \mathbb{P}\left(A\right)$ gives the PMF.

From the PMF, it's easy to calculate $\mathbb{E}X = \mathbb{P}\left(A\right), \mathbb{E}X^2 = \mathbb{P}\left(A\right)$ so $Var(X) = \mathbb{E}X^2 - \mathbb{E}^2X = \mathbb{P}\left(A\right) - [\mathbb{P}\left(A\right)]^2$.

$$G_X(s) = \mathbb{E}s^X = 1 \cdot \mathbb{P}\left(X = 0\right) + s \cdot \mathbb{P}\left(X = 1\right) = 1 - \mathbb{P}\left(A\right) + s \cdot \mathbb{P}\left(A\right) \tag{2}$$

(b): This is the classical trick on indicator

$$\forall a > 0, \phi(a) \cdot \mathbb{P}\left(Y \geq a\right) = \mathbb{E}[\phi(a)\mathbb{I}_{Y \geq a}] \leq \mathbb{E}[\phi(Y)\mathbb{I}_{Y \geq a}] \leq \mathbb{E}\phi(Y) \tag{3}$$

since ϕ is increasing and indicator is non-negative and takes value no larger than 1.

Consider $\phi(x) = x^2$ non-negative and increasing on $[0, +\infty)$ plugging in $Y = |Z| \geq 0, a = \varepsilon$ to conclude the proof.

(c):

Since for $t > 0$, e^{tx} is non-negative and increasing in x, resulting in

$$\mathbb{P}\left(X \geq x\right) = \mathbb{P}\left(e^{tX} \geq e^{tx}\right) \leq e^{-tx}\mathbb{E}e^{tX} = e^{-tx}M_X(t) \tag{4}$$

applying the conclusion in (b) for $Y = e^{tX}, a = e^{tx}, \phi(x) = x$. When $t = 0$, check $e^{-tx}M_X(0) = 1$ so $\mathbb{P}\left(X \geq x\right) \leq 1$ naturally holds. This proves that the inequality holds for $\forall t \geq 0$. Taking inf on both sides w.r.t. t concludes the proof.

$\square$

Remark. *Part (c) is a very important technique that will appear once again in 213BC to derive the Chernoff bound of concentration of measures. The basic idea is to* **introduce some unspecified parameter t, build a bound for the probability and optimize the bound to get the tightest bound by specifying an appropriate value of t.**

Example for Branching Process

Lemma 2 (Example). *There is an isolated island with the original stock of 100 family surnames, and the survival of family names is modelled by branching process, different surnames' survivals are independent. Each surname has extinction probability $\eta = \frac{9}{10}$.*

(a): After many generations how many surnames do you expect to be on the island?

(b): Do you expect the total population on the island to be increasing or decreasing?

Proof. (a): Each surname has η probability of disappearing independent of other surnames so the number of surname survived after a long enough time denoted X has binomial distribution $X \sim B(100, 1 - \eta)$. It's clear that $\mathbb{E}X = 100(1 - \eta) = 10$.

(b): Since $\eta > 0, \eta \neq 1$, the branching process $\{Z_n\}$ for each family surname is in the supercritical phase with offspring mean $\mu > 1$. It's clear that $\mathbb{E}Z_n = \mu^n \to +\infty \ (n \to \infty)$ so the expected total population is increasing.

$\square$

Lemma 3 (Example). *Branching process $\{Z_n\}$ originates from one individual, i.e. $Z_0 = 1$ has Poisson offspring distribution $Z_1 \sim P(\lambda) \ (\lambda > 1)$. If it's known that a branching process conditional on extinction is still a branching process, i.e. let A stands for the event that $\{Z_n\}$ extinct, $\{E_n\} = \{Z_n\}|_A$ is still a branching process. Can you derive the offspring distribution for $\{E_n\}$?*

Proof. Since $E_0 = Z_0|_A = 1$, the offspring distribution for $\{E_n\}$ is just the distribution of E_1. Let's denote η as the extinction probability of $\{Z_n\}$, i.e. $\eta = \mathbb{P}(A)$ and $p_k = \mathbb{P}(Z_1 = k)$ as the offspring distribution PMF of $\{Z_n\}$.

$$\mathbb{P}(E_1 = k) = \mathbb{P}(Z_1 = k|A) = \frac{\mathbb{P}(A|Z_1 = k)\mathbb{P}(Z_1 = k)}{\mathbb{P}(A)} \tag{5}$$

using Bayes formula. Notice that conditional on $Z_1 = k$, extinction happens if and only if all k subtrees generated in generation 1 are extinct. Since all k subtrees are independent and follow the same offspring distribution, they have exactly the same probability of being extinct, resulting in

$$\mathbb{P}(A|Z_1 = k) = [\mathbb{P}(A|Z_1 = 1)]^k = [\mathbb{P}(A)]^k = \eta^k \tag{6}$$

where the second equation comes from the fact that if $Z_1 = 1$, restarting the branching process at generation 1 makes no difference to the extinction probability (this is actually the Markov property of branching process). At this point, we see that

$$\mathbb{P}(E_1 = k) = \eta^{k-1}p_k = \eta^{k-1}\frac{\lambda^k}{k!}e^{-\lambda} \tag{7}$$

Since $\lambda > 1$, the offspring mean is larger than 1, the extinction probability η is thus the fixed point of $G(s)$ with

$$G(s) = \mathbb{E}s^{Z_1} = \sum_{k=0}^{\infty} s^k \frac{\lambda^k}{k!} e^{-\lambda} = e^{s\lambda - \lambda} \tag{8}$$

telling us

$$e^{\eta\lambda - \lambda} = \eta \tag{9}$$

turning it into $e^{-\lambda} = \eta e^{-\eta\lambda}$ and replace the $e^{-\lambda}$ term in the expression of $\mathbb{P}(E_1 = k)$ to get

$$\mathbb{P}(E_1 = k) = \frac{(\eta\lambda)^k}{k!} e^{-\eta\lambda}, E_1 \sim P(\eta\lambda) \tag{10}$$

the offspring distribution of $\{E_n\}$ is still Poisson but it's $P(\eta\lambda)$.

$\square$

Remark. *Actually **any branching process conditional on extinction is still a branching process**. Unfortunately, there is no easy approach to prove this conclusion since it's a statement for the whole process but not for pointwise evaluation of the process. Proving this conclusion requires the correspondence between branching process and random walk which we might have the chance to introduce in the future.*

However, we can do heuristic calculations as above to calculate the offspring distribution of the new branching process. From what we have shown above, the new branching process $\{E_n\}$ has offspring distribution with PMF

$$\mathbb{P}(E_1 = k) = p_k' = \eta^{k-1} p_k \tag{11}$$

*this is called **the duality principle of branching process**. In particular, Poisson branching process conditional on extinction still provides a Poisson branching process.*

Lemma 4 (Example)**.** *A branching process $\{Z_n\}$ is given such that $Z_0 = 8$ with offspring distribution PMF $p_0 = 0.2, p_1 = 0.5, p_2 = 0.3$.*

(a): Derive its extinction probability η.

(b): Derive the probability that the process is extinct in generation 3 but survives in generation 1 and generation 2.

Proof. (a): Such branching process is actually the sum of 8 branching process $\left\{Z_n^{(1)}\right\}, ..., \left\{Z_n^{(8)}\right\}$ with the same offspring distribution but with $Z_0^{(1)} = ... = Z_0^{(8)} = 1$. Moreover, those 8 branching processes are independent (by the definition of branching process).

Denote $E_n^{(i)}$ as the event that $\left\{Z_n^{(i)}\right\}$ is extinct in generation n and $S_n^{(i)}$ as the event that $\left\{Z_n^{(i)}\right\}$ survives in generation n, $E^{(i)}$ as the event that $\left\{Z_n^{(i)}\right\}$ is extinct. It's clear that $\{Z_n\}$ is extinct if and only if $\left\{Z_n^{(1)}\right\}, ..., \left\{Z_n^{(8)}\right\}$ are all extinct.

$$\eta = \mathbb{P}\left(E^{(1)}, E^{(2)}, ..., E^{(8)}\right) = \left[\mathbb{P}\left(E^{(1)}\right)\right]^8 \tag{12}$$

since offspring mean $\mu = 0.5 + 2 \times 0.3 = 1.1 > 1$, $\left\{Z_n^{(i)}\right\}$ is in supercritical phase, $\mathbb{P}\left(E^{(1)}\right)$ is the fixed point of $G(s)$. Let's first derive generating function

$$G(s) = 0.2 + 0.5s + 0.3s^2 \tag{13}$$

and solve $G(s) = s$ to get the solution $\mathbb{P}\left(E^{(1)}\right) = \frac{2}{3}$. We get the answer

$$\eta = \left(\frac{2}{3}\right)^8 \tag{14}$$

(b): $\{Z_n\}$ is extinct in generation 3 iff all $\left\{Z_n^{(i)}\right\}$ are extinct in generation 3. $\{Z_n\}$ survives in generation 2 iff there exists some $\left\{Z_n^{(i)}\right\}$ survive in generation 2. Notice that $\{Z_n\}$ survives in generation 2 implies $\{Z_n\}$ survives in generation 1 so the probability we want to find is the probability that $\{Z_n\}$ is extinct in generation 3 and survives in generation 2.

$$\mathbb{P}\left(\bigcap_{i=1}^{8} E_3^{(i)} \cap \bigcup_{i=1}^{8} S_2^{(i)}\right) = \mathbb{P}\left(\bigcap_{i=1}^{8} E_3^{(i)}\right) - \mathbb{P}\left(\bigcap_{i=1}^{8} E_3^{(i)} \cap \left[\bigcup_{i=1}^{8} S_2^{(i)}\right]^c\right) \tag{15}$$

$$= \mathbb{P}\left(\bigcap_{i=1}^{8} E_3^{(i)}\right) - \mathbb{P}\left(\bigcap_{i=1}^{8} E_3^{(i)} \cap \bigcap_{i=1}^{8} \left[S_2^{(i)}\right]^c\right) \tag{16}$$

$$= \mathbb{P}\left(\bigcap_{i=1}^{8} E_3^{(i)}\right) - \mathbb{P}\left(\bigcap_{i=1}^{8} E_3^{(i)} \cap \bigcap_{i=1}^{8} E_2^{(i)}\right) \tag{17}$$

is what we want to calculate by noticing $\forall n, i, \left[S_n^{(i)}\right]^c = E_n^{(i)}$. Use the fact that extinction in generation 2 implies extinction in generation 3, this tells us

$$\mathbb{P}\left(\bigcap_{i=1}^{8} E_3^{(i)} \cap \bigcap_{i=1}^{8} E_2^{(i)}\right) = \mathbb{P}\left(\bigcap_{i=1}^{8} (E_3^{(i)} \cap E_2^{(i)})\right) = \mathbb{P}\left(\bigcap_{i=1}^{8} E_2^{(i)}\right) \tag{18}$$

the structure of independence helps us again

$$\mathbb{P}\left(\bigcap_{i=1}^{8} E_3^{(i)} \cap \bigcup_{i=1}^{8} S_2^{(i)}\right) = \mathbb{P}\left(\bigcap_{i=1}^{8} E_3^{(i)}\right) - \mathbb{P}\left(\bigcap_{i=1}^{8} E_2^{(i)}\right) = \left[\mathbb{P}\left(E_3^{(1)}\right)\right]^8 - \left[\mathbb{P}\left(E_2^{(1)}\right)\right]^8 \tag{19}$$

the final step is to calculate those two probabilities. Recall the property of generating function that $G_X(0) = \mathbb{P}(X = 0)$. Now $\mathbb{P}\left(E_2^{(1)}\right) = \mathbb{P}\left(Z_2^{(1)} = 0\right) = G_{Z_2^{(1)}}(0)$ and we have proved in class that $Z_2^{(1)}$ has generating function

$G(G(s))$. This tells us

$$
\begin{cases}
\mathbb{P}\left(E_2^{(1)}\right) = G(G(0)) = G(0.2) = 0.312 \\
\mathbb{P}\left(E_3^{(1)}\right) = G(G(G(0))) = G(0.312) = 0.3852
\end{cases}
\tag{20}
$$

so the probability we want to find is

$$
0.3852^8 - 0.312^8
\tag{21}
$$

$\square$

Extra Materials: Total Progeny

For branching process $\{Z_n\}$ with $Z_0 = 1$, offspring distribution $\{p_k\}$ and generating function of offspring distribution $G(s)$, the **total progeny** is defined as

$$
T = \sum_{n=0}^{\infty} Z_n
\tag{22}
$$

the overall number of individuals in the branching process. It's easy to see that if extinction probability $\eta = 1$, then $T < \infty$ *a.s.*, otherwise T has positive probability taking value ∞. Due to this fact, the generating function of the total progeny is defined as

$$
G_T(s) = \mathbb{E}\left(s^T \cdot \mathbb{I}_{T<\infty}\right)
\tag{23}
$$

with the indicator added to make sure that $G_T(s)$ is well-defined. Deriving the generating function of T would provide us with a taste of how things work in branching process.

Theorem 1. *(Generating Function of Total Progeny)*

$$
\forall s \in [0, 1), G_T(s) = s \cdot G(G_T(s))
\tag{24}
$$

Proof. Tear apart the expectation w.r.t. the value of Z_1 to get

$$
G_T(s) = \sum_{k=0}^{\infty} \mathbb{P}(Z_1 = k) \cdot \mathbb{E}(s^T \cdot \mathbb{I}_{T<\infty}|Z_1 = k)
\tag{25}
$$

now under the condition that $Z_1 = k$, $T = 1 + T_1 + ... + T_k$ where T_j denotes the total progeny of the descendants of the j-th person in generation 1

$$
G_T(s) = \sum_{k=0}^{\infty} p_k \cdot s \cdot \mathbb{E}(s^{T_1}...s^{T_k} \cdot \mathbb{I}_{T_1<\infty}...\mathbb{I}_{T_k<\infty}|Z_1 = k)
\tag{26}
$$

notice that $T_1, ..., T_k, Z_1$ are independent and $T_1, ..., T_k$ are identically distributed, so

$$G_T(s) = \sum_{k=0}^{\infty} p_k \cdot s \cdot \mathbb{E}(s^{T_1}...s^{T_k} \cdot \mathbb{I}_{T_1 < \infty}...\mathbb{I}_{T_k < \infty}) \tag{27}$$

$$= s \cdot \sum_{k=0}^{\infty} p_k \cdot \mathbb{E}(s^{T_1} \cdot \mathbb{I}_{T_1 < \infty})...\mathbb{E}(s^{T_k} \cdot \mathbb{I}_{T_k < \infty}) \tag{28}$$

$$= s \cdot \sum_{k=0}^{\infty} p_k \cdot [G_{T_1}(s)]^k \tag{29}$$

$$= s \cdot G(G_{T_1}(s)) \tag{30}$$

at last notice that $T \stackrel{d}{=} T_1$ since the branching process starting from generation 0 with 1 individual is the same in distribution as the branching process starting from generation 1 with 1 individual, so the distribution of the total progeny in these two cases are the same. We conclude that

$$G_T(s) = s \cdot G(G_T(s)) \tag{31}$$

$\square$

Remark. *By noticing the continuity of G_T and taking $s \to 1^-$, one may find that*

$$G_T(1) = G(G_T(1)) \tag{32}$$

*when $\eta = 1$, it's obvious that $G_T(1) = \mathbb{P}(T < \infty) = 1$. When $\eta < 1$, however, $G_T(1) < 1$ and is the fixed point of the generating function $G(s)$. Since in supercritical phase, the fixed point of $G(s)$ in $[0, 1)$ exists and is uniquely the extinction probability η, we conclude that $G_T(1) = \mathbb{P}(T < \infty) = \eta$. This provides **another perspective understanding the extinction probability**.*

Week 2

Interpretation of Markov Property

From what we have learnt about discrete-state discrete-time Markov chain, it's a stochastic process $\{X_n\}$ satisfying the Markov property

$$\mathbb{P}\left(X_n = i_n | X_0 = i_0, ..., X_{n-1} = i_{n-1}\right) = \mathbb{P}\left(X_n = i_n | X_{n-1} = i_{n-1}\right) \tag{33}$$

many different interpretations can be made on the Markov property. The most intuitive one is saying that conditional on the value of X_{n-1}, X_n is independent of $X_0, ..., X_{n-2}$. In short, **conditional on present (observation at time $n-1$), past (observation prior to time $n-1$) is independent of future (observation after time $n-1$)**.

Another useful interpretation of Markov property is that Markov chain is a process that is **memoryless**. Since Markov property is saying that if one cares about the future behavior of Markov chain, only the most recent past matters, if we have already observed the event $\{X_n = 0\}$ happening, we can actually forget about $X_0, ..., X_{n-1}$ when investigating the behavior of the Markov chain after time n. This interpretation will be made clearer a little bit afterwards.

Due to the presence of Markov property, one can define the transition probability of Markov chain $p_{ij}^n(1) = \mathbb{P}\left(X_{n+1} = j | X_n = i\right)$ as the probability of transiting from state i to state j at time n. For simplification, we will only consider the time-homogeneous Markov chain, i.e. Markov chain such that $\mathbb{P}\left(X_{n+1} = j | X_n = i\right)$ does not depend on n so the same transition law applies at each time point. After knowing the transition probability, one last thing to know in order to fix the distribution of the whole Markov chain is just the information on where it starts, i.e. the initial distribution of X_0 denoted μ. As a result, **the distribution of a Markov chain is fixed iff the initial distribution and the transition probability are known**.

Remark. *At this point considering time-homogeneous Markov chain, one can always stop the Markov chain and restart it. For example, if we have already observed the event $\{X_n = 0\}$ happening, we can actually forget about $X_0, ..., X_{n-1}$ when we investigate the behavior of the Markov chain after time n. This is equivalent to **stopping the current Markov chain at time n and restarting it with the same transition rule but act as if it has initial value** 0. We will come back to this interpretation a lot of times in the future.*

Examples of Markov Chain

Let's look at some examples of Markov chain. The easiest one is the two-state Markov chain with state space $S = \{0, 1\}$ (state of the phone, 0 means free and 1 means busy). It's assumed that at each time point there's probability p that a call is coming in and if the phone was busy then there is q probability that the call will end at this time point. This results in the transition matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}, \ p, q \in [0, 1] \tag{34}$$

however, if the system can put one caller on hold, the state space is extended to $S = \{0, 1, 2\}$ where state 1 means that the phone is busy but no caller is on hold and state 2 means that the phone is busy and there's also one caller on hold. The transition matrix now becomes

$$P = \begin{bmatrix} 1-p & p & 0 \\ q(1-p) & 1 - p(1-q) - q(1-p) & p(1-q) \\ 0 & q & 1-q \end{bmatrix}, \ p, q \in [0, 1] \tag{35}$$

the second row comes from the fact that state 1 transits to state 2 with probability $p(1-q)$ (the old call has not ended and a new call comes in) while state 1 transits to state 0 with probability $q(1-p)$ (the old call has ended and no new call comes in). This provides a simple queueing model that we will be able to analyze later on.

On the other hand, Markov chain can also be formed in structures other than the real line $\mathbb{R}$. Consider the random walk on any undirected graph with $S = \{v_1, ..., v_n\}$ as the set of all vertices. Let $N(v_i) = \{v_j \in V : v_j \sim v_i\}$ be the neighborhood of vertex v_i so that $d_{v_i} = |N(v_i)|$ is called the degree of vertex v_i. When the state is at v_i, it has $\frac{1}{d_{v_i}}$ probability transiting to any one of the states in $N(v_i)$. It's called a **random walk on graph** and it also turns out to be a Markov chain. Another famous example would be the **random walk on infinite binary tree**, we will come back to this interesting example later. Different from the random walk on finite graph, this example is a random walk on infinite graph.

Another useful example to mention is that if $\{X_n\}$ is a Markov chain with state space S, the tuple $Z_n = (X_n, X_{n+1})$ that tracks the **two-step history** of $\{X_n\}$ is also a Markov chain with state space $S \times S$. An easy proof can be given below that

$$\mathbb{P}\left(Z_n = (i_n, i_{n+1}) | Z_0 = (i_0, i_1), ..., Z_{n-1} = (i_{n-1}, i_n)\right) \tag{36}$$

$$= \mathbb{P}\left(X_n = i_n, X_{n+1} = i_{n+1} | X_0 = i_0, X_1 = i_1, ..., X_n = i_n\right) \tag{37}$$

$$= \mathbb{P}\left(X_{n+1} = i_{n+1} | X_0 = i_0, X_1 = i_1, ..., X_n = i_n\right) \tag{38}$$

$$= \mathbb{P}\left(X_{n+1} = i_{n+1} | X_{n-1} = i_{n-1}, X_n = i_n\right) \tag{39}$$

$$= \mathbb{P}\left(X_n = i_n, X_{n+1} = i_{n+1} | X_{n-1} = i_{n-1}, X_n = i_n\right) \tag{40}$$

$$= \mathbb{P}\left(Z_n = (i_n, i_{n+1}) | Z_{n-1} = (i_{n-1}, i_n)\right) \tag{41}$$

using the Markov property of $\{X_n\}$. An example would be to set $\{X_n\}$ as the output sequence one is getting by tossing a coin independently. For this Markov chain, $S = \{0, 1\}$ so $\{Z_n\}$ is a Markov chain on $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$, with transition probability matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{42}$$

and initial distribution as that Z_0 has $\frac{1}{4}$ probability taking all four possible states. This Markov chain is useful if we

want to find, e.g. the expected time of tosses we have to make until we see two consecutive heads.

Remark. *Markov process theory is the core part of probability theory since in real life we rarely see a sequence of independent random variables. Markov process considers a sequence of dependent random variables by putting mild restrictions on the dependency.*

However, one might be curious about the way we deal with non-Markov processes. For example, consider process $\{X_n\}$ with state space $\{0,1\}$, the transition rule is that

- *If $X_{n-1} = 0, X_{n-2} = 0$, then $X_n \sim B(1, \frac{1}{2})$*

- *If $X_{n-1} = 0, X_{n-2} = 1$, then $X_n \sim B(1, \frac{3}{4})$*

- *If $X_{n-1} = 1, X_{n-2} = 0$, then $X_n \sim B(1, \frac{1}{4})$*

- *If $X_{n-1} = 1, X_{n-2} = 1$, then $X_n \sim B(1, \frac{2}{3})$*

it's quite obvious that $\{X_n\}$ is not a Markov chain since the transition rule at time n differs according to different values of X_{n-2}.

However, $Z_n = (X_n, X_{n+1})$ turns out to be a Markov chain with transition matrix (please check)

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \tag{43}$$

this is because for $\{X_n\}$ the transition rule is fixed based on its two-step history and the construction of Z_n as a tuple keeps track of the two-step history of $\{X_n\}$ at each time point so it's Markov. More generally, **when $\{X_n\}$ is not Markov but its transition rule depends on k-step history, setting $Z_n = (X_n, X_{n+1}, ..., X_{n+k})$ always creates a Markov chain $\{Z_n\}$ at the cost of enlarging the state space**.

Chapman-Kolmogorov Equation

It should be familiar that Markov property implies Cahpman-Kolmogorov equation, however, here we raise a counterexample to show that the converse is not true.

Consider $Y_1, Y_3, ...$ as *i.i.d.* random variables taking value ± 1 with probability $\frac{1}{2}$ and set $Y_{2k} = Y_{2k-1}Y_{2k+1}$ so $Y_2, Y_4, ...$ is also a sequence of *i.i.d.* random variables taking value ± 1 with probability $\frac{1}{2}$. Moreover, the sequence of random variables $Y_1, Y_2, Y_3, ...$ are pairwise independent. Those facts can be checked below

$$\mathbb{P}(Y_2 = 1) = \mathbb{P}(Y_1 Y_3 = 1) = \mathbb{P}(Y_1 = 1, Y_3 = 1) + \mathbb{P}(Y_1 = -1, Y_3 = -1) \tag{44}$$

$$= \mathbb{P}(Y_1 = 1)\mathbb{P}(Y_3 = 1) + \mathbb{P}(Y_1 = -1)\mathbb{P}(Y_3 = -1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \tag{45}$$

due to independence of Y_1 and Y_3

$$\mathbb{P}\left(Y_2 = 1, Y_4 = 1, ..., Y_{2k} = 1\right) = \mathbb{P}\left(Y_1 Y_3 = 1, Y_3 Y_5 = 1, ..., Y_{2k-1} Y_{2k+1} = 1\right) \tag{46}$$

$$= \mathbb{P}\left(Y_1 = 1, Y_3 = 1, ..., Y_{2k+1} = 1\right) + \mathbb{P}\left(Y_1 = -1, Y_3 = -1, ..., Y_{2k+1} = -1\right) \tag{47}$$

$$= \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^k} = \mathbb{P}\left(Y_2 = 1\right)\mathbb{P}\left(Y_4 = 1\right)...\mathbb{P}\left(Y_{2k} = 1\right) \tag{48}$$

due to the sequence $Y_1, Y_3, ...$ being *i.i.d.*, for the pairwise independence of $Y_1, Y_2, Y_3, ...$, we only have to check the independence of Y_1 and Y_{2k} without loss of generality

$$\mathbb{P}\left(Y_1 = 1, Y_{2k} = 1\right) = \mathbb{P}\left(Y_1 = 1, Y_{2k-1} Y_{2k+1} = 1\right) \tag{49}$$

$$= \mathbb{P}\left(Y_1 = 1, Y_{2k-1} = 1, Y_{2k+1} = 1\right) + \mathbb{P}\left(Y_1 = 1, Y_{2k-1} = -1, Y_{2k+1} = -1\right) \tag{50}$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4} = \mathbb{P}\left(Y_1 = 1\right)\mathbb{P}\left(Y_{2k} = 1\right) \tag{51}$$

Using the facts mentioned above, the transition probability is well-defined

$$\forall i, j \in \{-1, 1\}, \forall m, p_{ij}(m) = \mathbb{P}\left(Y_{n+m} = j | Y_n = i\right) = \mathbb{P}\left(Y_{n+m} = j\right) = \frac{1}{2} \tag{52}$$

and the m-step transition matrix is

$$P^{(m)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{53}$$

let's check Chapman-Kolmogorov equation

$$P^{(m)} P^{(1)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = P^{(m+1)} \tag{54}$$

proves that $P^{(m)} = [P^{(1)}]^m$. However, this process $\{Y_n\}$ is not a Markov chain since

$$\mathbb{P}\left(Y_3 = 1 | Y_2 = 1, Y_1 = 1\right) = \mathbb{P}\left(Y_3 = 1 | Y_1 Y_3 = 1, Y_1 = 1\right) = 1 \tag{55}$$

$$\mathbb{P}\left(Y_3 = 1 | Y_2 = 1\right) = \mathbb{P}\left(Y_3 = 1\right) = \frac{1}{2} \tag{56}$$

violates the Markov property.

Remark. *When one is asked to prove that a process is Markov, merely arguing the existence of the transition matrix or checking Chapman-Kolmogorov equation does not suffice.*