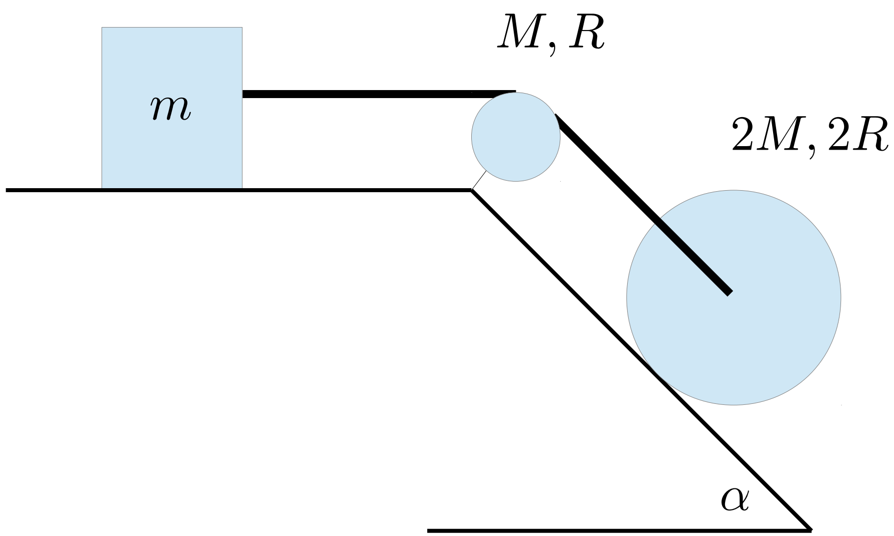
Rigid Body Dynamics:

Additional Examples

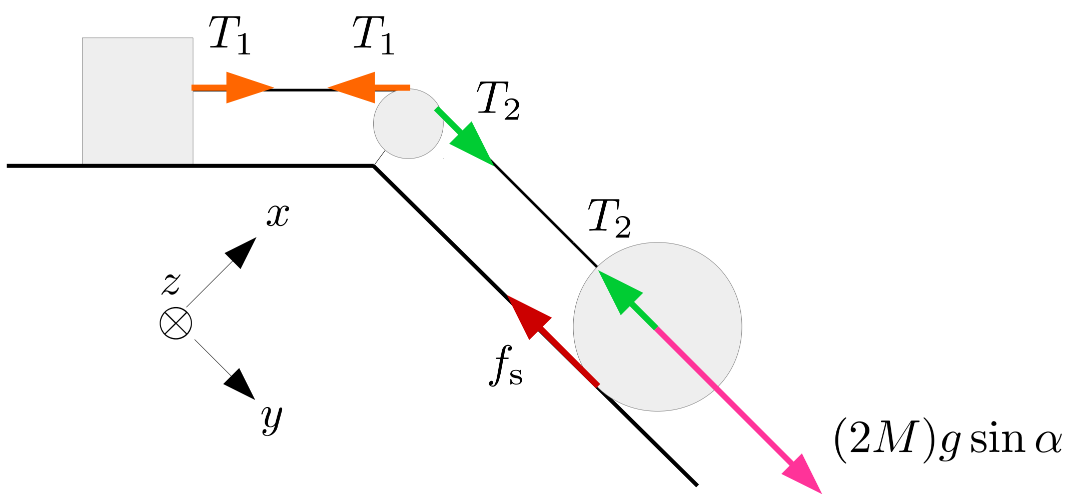
Example 1: Inclined plane

A light inextensible string is attached to an axle going through the center of mass of a ball with radius $2R$ and mass $2M$, placed on a plane inclined at an angle α. The string is parallel to the incline and goes over a cylindrical pulley with radius R and mass M, that can rotate freely about the axis of symmetry. The other end of the string is attached to a block with mass m placed on a smooth horizontal surface. There is no slipping anywhere in the system.

Find the linear acceleration of the center of mass of the ball.

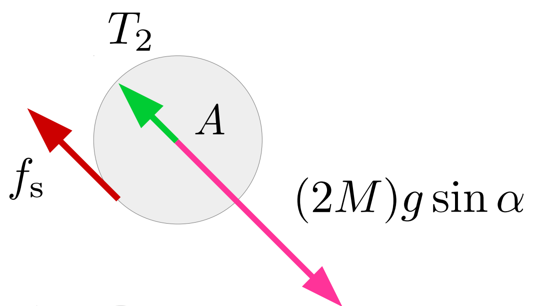


Example 1: Inclined plane



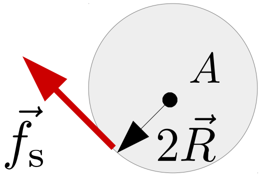
(only components in the direction of translational motion are shown)

Free body diagram and 2nd law of dynamics – ball



$$(2M)a_y = (2M)g\sin\alpha - f_s - T_2$$

(translational motion of the c.m.)



$$\vec{\tau}_A = 2\vec{R} \times \vec{f}_s$$

$\otimes$ (positive)

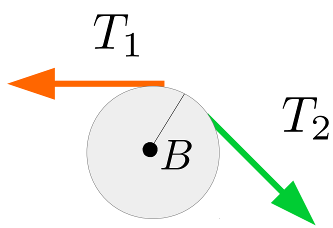
$$I_A \varepsilon_{A,z} = 2Rf_s$$

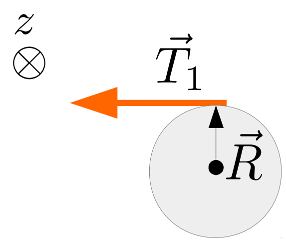
(rotation about the axis A through the c.m.)

$$I_A = \frac{2}{5}(2M)(2R)^2 = \frac{16}{5}MR^2$$

Example 1: Inclined plane

Free body diagram and 2nd law of dynamics – pulley

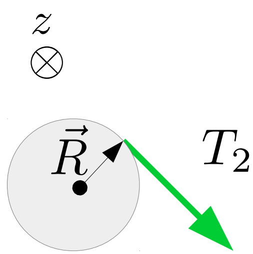




$\vec{R} \times \vec{T}_1$

$\odot$

(negative)



$\vec{R} \times \vec{T}_2$

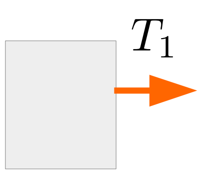
$\otimes$

(positive)

$$\boxed{I_B \varepsilon_{B,z} = (T_2 - T_1) R}$$

$$I_B = \frac{1}{2} M R^2$$

Free body diagram and 2nd law of dynamics – block



$$\boxed{m a_y = T_1}$$

Example 1: Inclined plane

System of equations to solve (six unknowns)

$$(2M)a_y = (2M)g\sin\alpha - f_\mathrm{s} - T_2$$

$$I_A\varepsilon_{A,z} = 2Rf_\mathrm{s}$$

$$I_B\varepsilon_{B,z} = (T_2 - T_1)R$$

$$ma_y = T_1$$

$$a_y = 2R\varepsilon_{A,z}$$ (ball does not slip while rolling)

$$a_y = R\varepsilon_{B,z}$$ (string inextensible and does not slip on the pulley)

Answer: $$a_y = \frac{2Mg\sin\alpha}{\frac{33}{10}M + m}$$

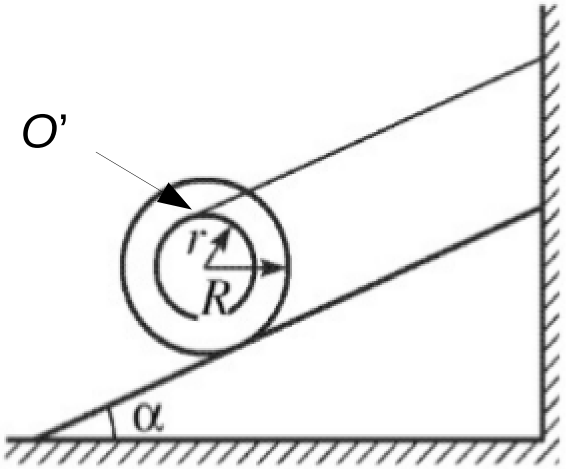
Of course, can also find other quantities, e.g. $$f_\mathrm{s} = \frac{\frac{8}{5}M^2 g\sin\alpha}{\frac{33}{10}M + m}$$

Example 2: Reel on an incline

A light inextensible string is winded on a reel with mass m, inner radius r, and outer radius R placed on a rough ramp (coefficient of kinetic friction is μ) inclined at an angle α to the horizontal. The other end of the string is attached to a wall, so that the string is parallel to the plane. The string does not slip on the reel.

What is the acceleration of the center of mass of the reel?

The moment of inertia of the reel about the axis of symmetry is I_0.

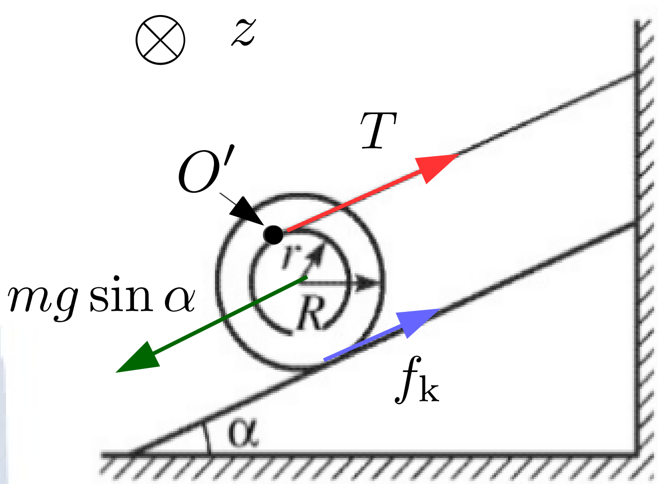


Solution:

Of course for the string to unwind and the reel move down the plane, the reel must be slipping on the incline (here friction is kinetic!).

Therefore we will consider the reel as rotating about the instantaneous axis of rotation through the point *O*' on the inner radius of the reel, where the string is still in contact with the reel.

Example 2: Reel on an incline



Translational motion of the c.m.

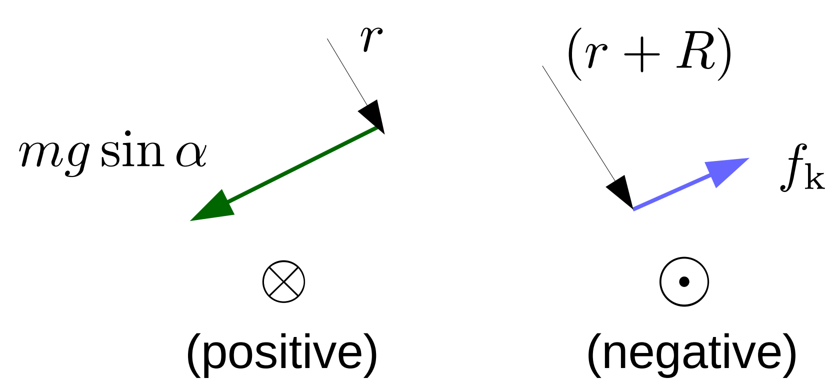
$$ma_\mathrm{cm} = mg\sin\alpha - T - f_\mathrm{k}$$

but the friction is kinetic, so $f_\mathrm{k} = \mu N = \mu mg\cos\alpha$

$$\boxed{ma_\mathrm{cm} = mg\sin\alpha - T - \mu mg\cos\alpha}$$

Rotation about the axis through point *O'* (instantaneously at rest)

$$\boxed{I_{O'}\varepsilon_z = rmg\sin\alpha - (R+r)\underbrace{\mu mg\cos\alpha}_{f_\mathrm{k}}}$$



Parallel axis (Steiner) theorem: $I_{O'} = I_O + mr^2$

Example 2: Reel on an incline

System of equations to solve (three unknowns)

$$ma_{\rm cm} = mg\sin\alpha - T - \mu mg\cos\alpha$$

$$(I_O + mr^2)\varepsilon_z = rmg\sin\alpha - (R+r)\mu mg\cos\alpha$$

$\varepsilon_z = a_{\rm cm}/r$ (pure rotation about O')

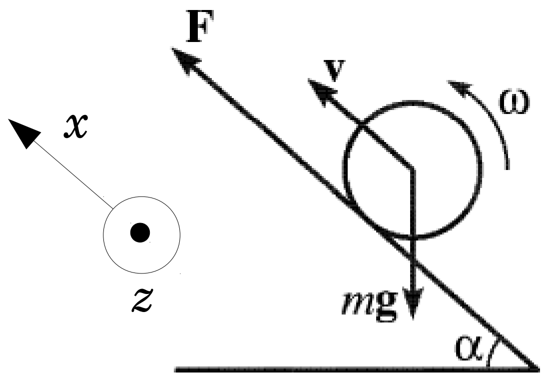
Answer:

$$a = \frac{r\sin\alpha - \mu(r+R)\cos\alpha}{I_O + mr^2}mgr$$

Example 3: Cylinder rolling up an incline

A uniform cylinder with radius *R*, rotating with an initial angular speed ω_0, is placed on a plane inclined at an angle α to the horizontal, so that the initial velocity of its translational motion is zero. The cylinder starts rolling upwards. Find the time needed for the cylinder to reach the highest point on the plane.

Initially, when the cylinder is slipping, ***F*** is the force of kinetic friction. After the cylinder starts rolling upwards without slipping it becomes the force of static friction.



Regardless of the character of the frictional force, the equation of motion for translational motion of the center of mass in the direction parallel to the plane is

$$m\frac{dv_{\mathrm{c.m.}}}{dt} = F - mg\sin\alpha$$

Example 3: Cylinder rolling up an incline

And the equation of motion for rotational motion about the axis *z* (through the center of mass) is

$$I\frac{d\omega}{dt} = -FR$$

Combining both equations, we get

$$mR\frac{dv_{\text{c.m.}}}{dt} = -I\frac{d\omega}{dt} - mgR\sin\alpha$$

Integration with respect to time from *0* to *t*, with the initial conditions

$$\omega(t=0) = \omega_0, \qquad v_{\text{c.m.}}(t=0) = 0$$

yields

$$mRv_{\text{c.m.}} = I(\omega_0 - \omega) - mgRt\sin\alpha$$

Example 3: Cylinder rolling up an incline

$$mRv_{\mathrm{c.m.}} = I(\omega_0 - \omega) - mgRt\sin\alpha$$

At the highest point, the cylinder stops instantaneously, that is both

$$\omega = 0, \qquad v_{\mathrm{c.m.}} = 0$$

Hence, the time needed for the cylinder to reach height *h* is

$$t_{\mathrm{h}} = \frac{I\omega_0}{mgR\sin\alpha} = \frac{R\omega_0}{2g\sin\alpha}$$

Comment

The result does not depend on the coefficient of friction, but friction needs to be large enough for the cylinder to roll upwards, as assumed.

Question

Does the maximum height depend on the coefficient of friction?

Rigid Body Dynamics:

The Gyroscopic Effect

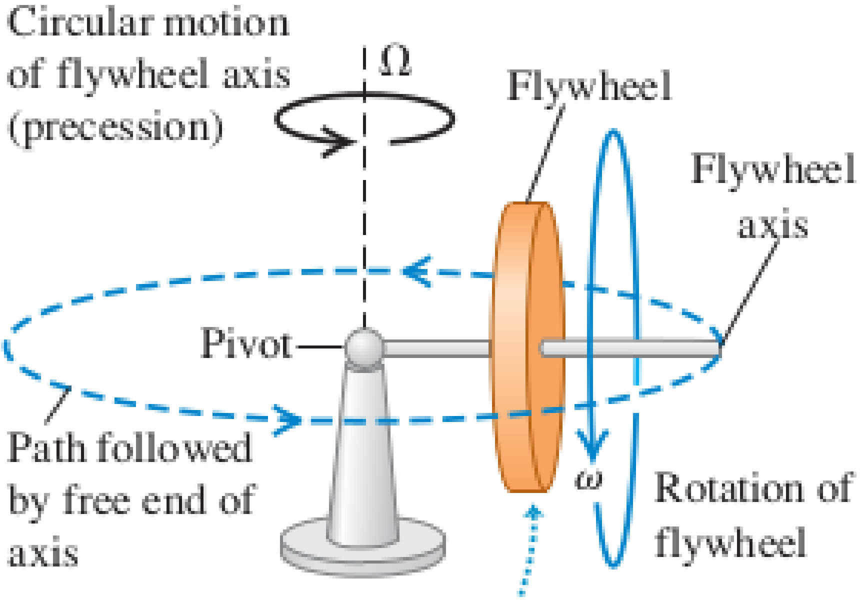
Introduction

So far: rotation axis fixed or moving, but without changing its direction.

Many interesting phenomena appear when the axis is free to change its direction.

Example (toy-gyroscope)

When the flywheel rotates, the axis keeps horizontal orientation and rotates about the pivot.



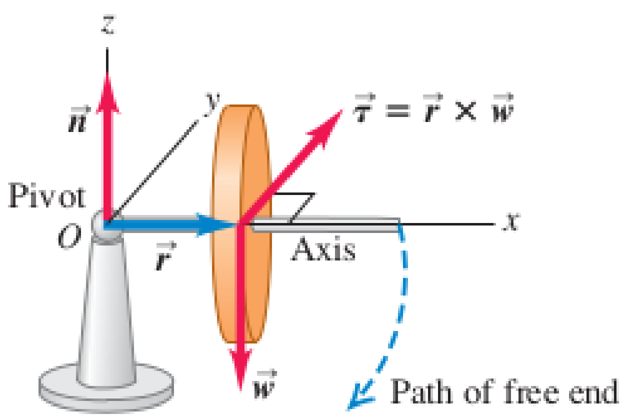
When the flywheel and its axis are stationary, they will fall to the table surface. When the flywheel spins, it and its axis "float" in the air while moving in a circle about the pivot.

Example: Bicycle Wheel

http://techtv.mit.edu/videos/717-mit-physics-demo----bicycle-wheel-gyroscope

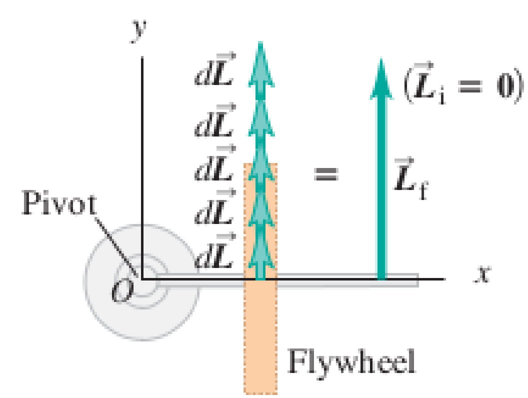
Explanation: non-rotating flywheel

(a) Nonrotating flywheel falls



When the flywheel is not rotating, its weight creates a torque around the pivot, causing it to fall along a circular path until its axis rests on the table surface.

(b) View from above as flywheel falls

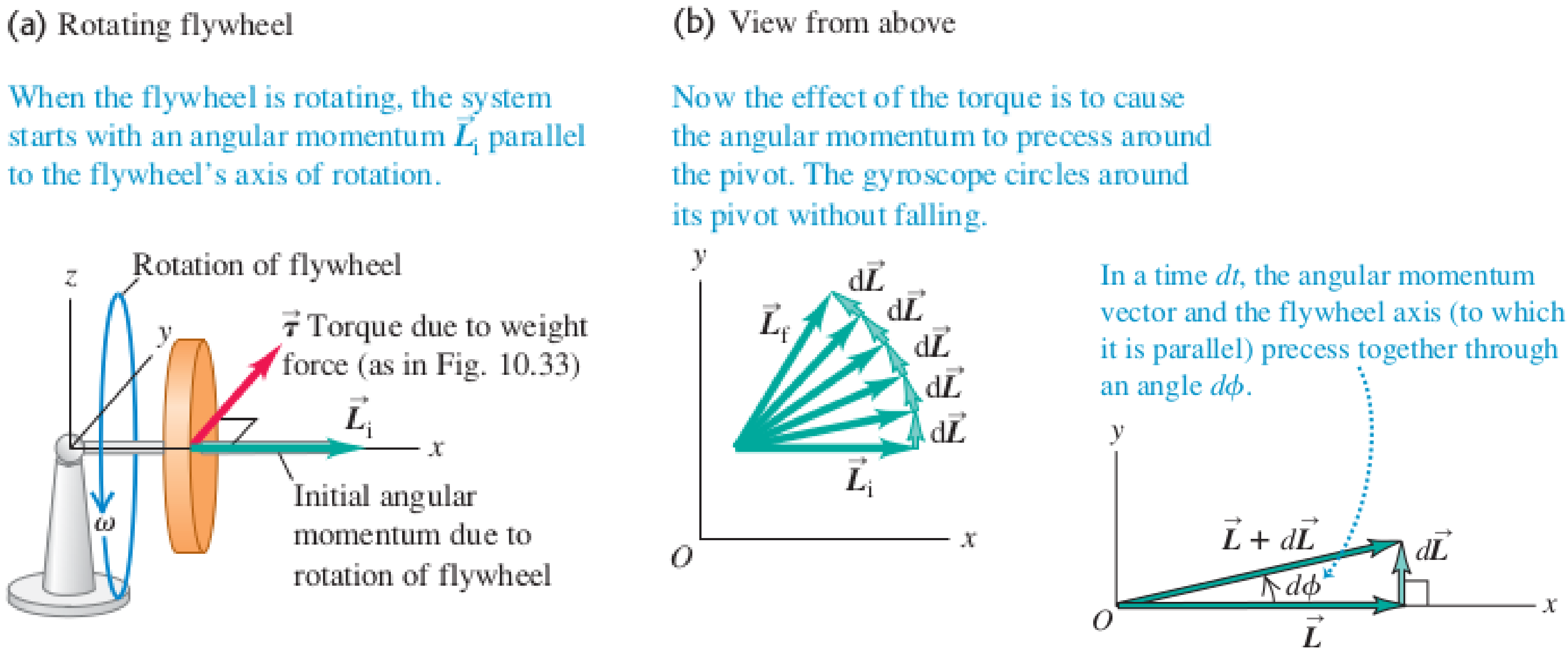


In falling, the flywheel rotates about the pivot and thus acquires an angular momentum $\vec{L}$. The *direction* of $\vec{L}$ stays constant.

$$\boldsymbol{\tau}^{\text{ext}} = \frac{d\mathbf{L}}{dt} \iff d\mathbf{L} = \boldsymbol{\tau}^{\text{ext}} dt \quad \text{(always valid)}$$

Observation: non-zero torque of the gravitational force about the pivot results in increase of the angular momentum along the *y* axis, i.e. the flywheel's axis rotates in the *x-z* plane about the pivot and eventually hits the table.

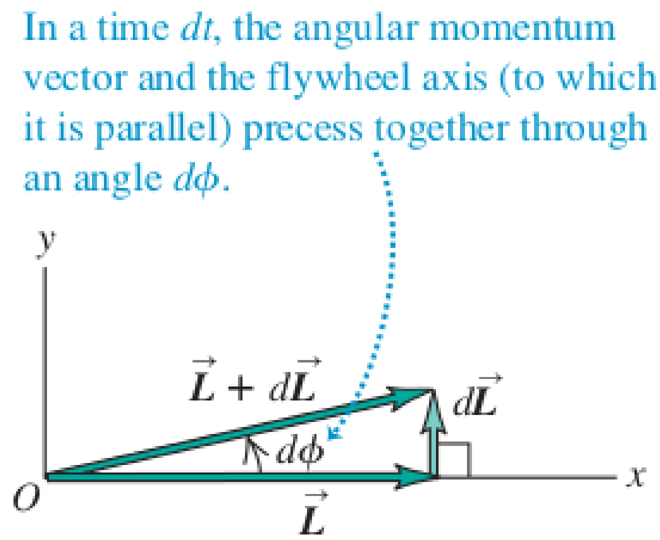
Explanation: rotating flywheel



Conclusion: a non-zero torque of the gravitational force about the pivot results in the change d***L*** of the angular momentum. d***L*** is perpendicular to the axis of rotation of the flywheel, i.e. changes the direction of ***L***, but not its magnitude (**the flywheel disk is rapidly spinning, |*L*| is almost constant**). Since d***L*** is always in the *x-y* plane, ***L*** (and the axis of the flywheel) stays in the *x-y* plane and rotates about the *z* axis.

Rotation of the flywheel's axis about the *z* axis is called ***precession***.

Angular Speed of Precession



$$d\mathbf{L} = \boldsymbol{\tau}^{\text{ext}} dt \perp \mathbf{L}$$

corresponding angular displacement

$$d\phi = \frac{|d\mathbf{L}|}{|\mathbf{L}|}$$

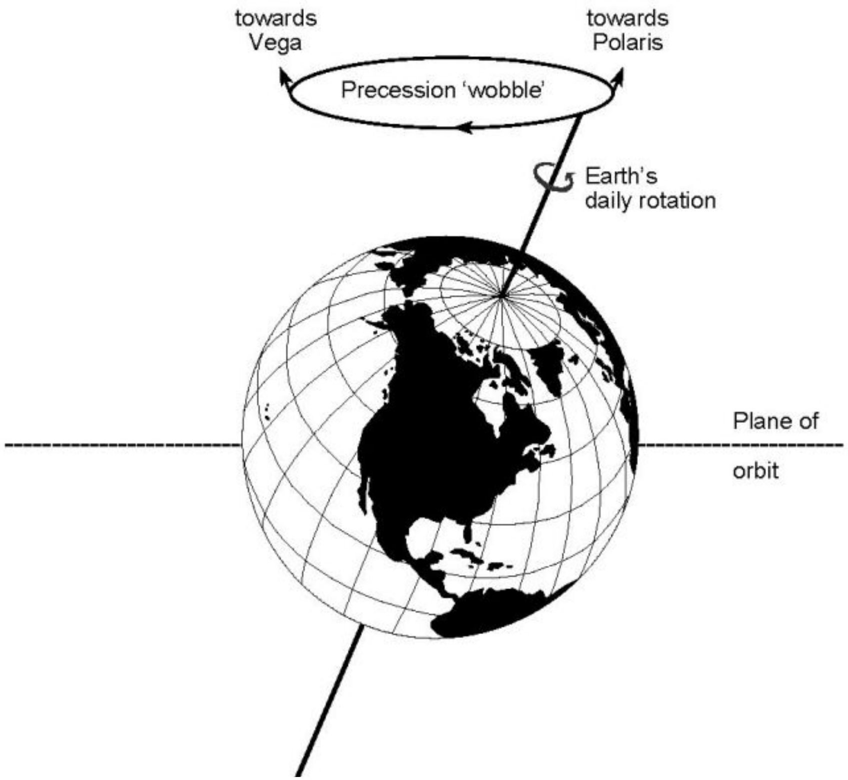
The rate at which the flywheel axis rotates about the *z* axis (*precession angular speed*)

$$\Omega = \frac{d\phi}{dt} = \frac{\frac{|d\mathbf{L}|}{|\mathbf{L}|}}{dt} = \frac{\tau^{\text{ext}}}{L} = \frac{mgr}{I\omega}$$

Observation: the angular speed of precession is inversely proportional to the angular speed of rotation (spinning) of the flywheel (in other words: a rapidly spinning gyroscope precesses slowly).

Note. We have ignored the contribution to the angular momentum due to precession (this component is along the *z* axis). Our simplified analysis is valid for a rapidly spinning flywheel.

Precession of the Earth



$$T_p = 2\pi/\Omega = 26000 \text{ years}$$

While the Pole Star in the northern hemisphere is now Polaris, in 3000 B.C., the north celestial pole coincided with Thuban, a star in the constellation of Draco. In 14,000 A.D. Vega, in Lyra, will be the northern pole star.

http://www.soes.soton.ac.uk/staff/ejr/DarkMed/ch5.html
http://www.astro.cornell.edu/academics/courses/astro201/earth_precess.htm