Chapter 13 – Rigid Body Mechanics (III) Work and Conservation of Angular Momentum

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Physics I (Summer 2021)
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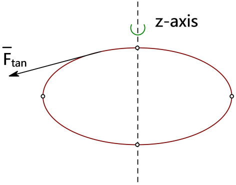
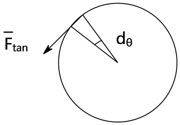
Agenda

1. Work and Power in Rotational Motion
2. Rotational Analogue of the Work-Kinetic Energy Theorem
3. Angular momentum
   - Single Particle
   - Rigid Body
4. Conservation of Angular Momentum

Work and Power in Rotational Motion

Idea





The axial (parallel to the axis of rotation) and the radial components do no work.

How much work is done by $\overline{F}_{tan}$ over a distance ds?

$$\delta W = F\,ds = \underbrace{F_{tan} R}_{\tau_z}\,d\theta = \tau_z\,d\theta$$

[Note that, $\overline{F}_{tan} \parallel d\overline{r}$, and $ds = |d\overline{r}| = R\,d\theta$]. Hence the total work $(\theta_1 \to \theta_2)$

$$\boxed{W_{\theta_1 \to \theta_2} = \int_{\theta_1}^{\theta_2} \tau_z\,d\theta}$$

E.g., if $\tau_z = \text{const} \implies W_{\theta_1 \to \theta_2} = \tau_z(\theta_2 - \theta_1)$.
Units of work [N·m = J]

Rotational Analogue of the Work-Kinetic Energy Theorem

Work–Kinetic Energy Theorem for Rotational Motion

Similarly, use the 2nd law of dynamics for rotational motion of a rigid body acted upon a net torque τ_z

$$\delta W = \tau_z \, \mathrm{d}\theta = (I\varepsilon_z)\, \mathrm{d}\theta = I \frac{\mathrm{d}\omega_z}{\mathrm{d}t}\, \mathrm{d}\theta = I \frac{\mathrm{d}\theta}{\mathrm{d}t}\, \mathrm{d}\omega_z = I\omega_z \, \mathrm{d}\omega_z = \mathrm{d}\left(\frac{1}{2} I\omega_z^2\right)$$

$$\boxed{\delta W} = \mathrm{d}\left(\frac{1}{2} I\omega_z^2\right) = \boxed{\mathrm{d}K_{rot}}$$

Where $K_{rot} = \frac{1}{2} I\omega_z^2$ is the kinetic energy in rotational motion. Hence, for finite changes: $W = \underbrace{K_2}_{\text{final}} - \underbrace{K_1}_{\text{initial}}$.

Power (rate of work being done)

$$\delta W = \tau_z \mathrm{d}\theta \qquad \Longrightarrow \qquad \frac{\delta W}{\mathrm{d}t} = \tau_z \frac{\mathrm{d}\theta}{\mathrm{d}t} = \tau_z \omega_z$$

$$\boxed{P = \tau_z \omega_z}$$

Angular momentum

Angular Momentum of a Single Particle

Start with 2nd law of dynamics for a particle

$$\underbrace{\bar{F}}_{\text{net force}} = \frac{\mathrm{d}\bar{p}}{\mathrm{d}t} \qquad / \; \bar{r} \times \ldots$$

$$\underbrace{\bar{r} \times \bar{F}}_{\text{torque of net force}} = \bar{r} \times \frac{\mathrm{d}\bar{p}}{\mathrm{d}t}$$

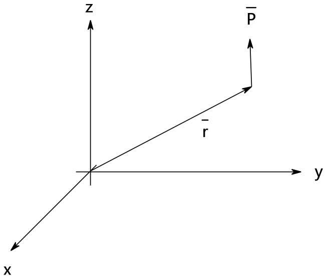
Note. $\frac{\mathrm{d}}{\mathrm{d}t}(\bar{r} \times \bar{p}) = \underbrace{\frac{\mathrm{d}\bar{r}}{\mathrm{d}t}}_{=\bar{v}} \times \underbrace{\bar{p}}_{=m\bar{v}} + \bar{r} \times \frac{\mathrm{d}\bar{p}}{\mathrm{d}t} = \bar{r} \times \frac{\mathrm{d}\bar{p}}{\mathrm{d}t}$. Hence

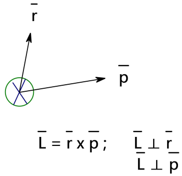
$$\bar{r} \times \bar{F} = \frac{\mathrm{d}}{\mathrm{d}t}(\bar{r} \times \bar{p}).$$

Define $\boxed{\bar{L} = \bar{r} \times \bar{p}}$, angular momentum with respect to the origin.
SI Unit: $[\mathrm{kg} \cdot \frac{\mathrm{m}^2}{\mathrm{s}}]$

Angular Momentum of a Single Particle

Interpretation: A measure of the "amount" of orbital motion (about a reference point).





Rewrite $\bar{r} \times \bar{F} = \frac{\mathrm{d}}{\mathrm{d}t}(\bar{r} \times \bar{p})$

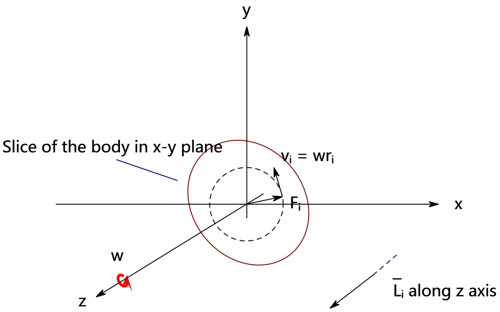
$$\boxed{\bar{\tau} = \frac{\mathrm{d}\bar{L}}{\mathrm{d}t}} \quad \text{(single particle)}$$

Conclusion. The rate of change of the angular momentum of a particle equals the torque of the net force acting on it (both calculated about the same reference point!).

Angular Momentum of a Rigid Body

How to find the total angular momentum of a rigid body rotating about z axis with angular velocity $\bar{\omega}$?

First, consider a single slice of the rigid body contained in the x–y plane.



For any element of mass m_i in the slice

$$\bar{r}_i \perp \bar{v}_i \qquad \Longrightarrow \qquad \bar{r}_i \perp \underbrace{\bar{p}_i}_{m_i \bar{v}_i}$$

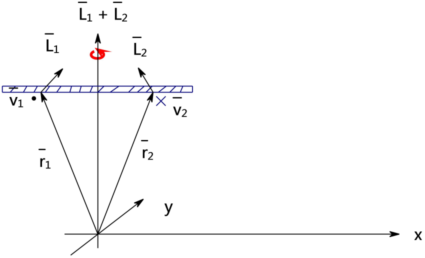
Hence, the contribution to the angular momentum of the slice (magnitude)

$$L_i = r_i p_i = r_i m_i v_i = r_i m_i \omega r_i = m_i r_i^2 \omega.$$

The total angular momentum due to that slice ($\bar{L} \parallel$ z-axis)

$$\underbrace{L}_{\text{z-component}} = \sum_{i=1}^{N_{slice}} L_i = \left(\sum_{i=1}^{N_{slice}} m_i \underbrace{r_i^2}_{=r_{i\perp}^2} \right) \omega = I_{slice} \omega$$

This procedure can be repeated for all other slices. However, for other slices we may, in general, have a non-zero component of $\bar{L}$ perpendicular to z-axis.

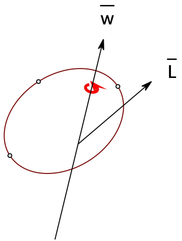


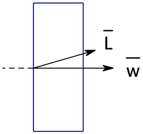
$\bar{r}_1, \bar{r}_2$ position vectors of elements of mass located symmetrically with respect to the z axis

Only if the z axis is *the axis of symmetry* of the rigid body, the perpendicular components of angular momentum due to the slices cancel out and the total angular momentum is still parallel to the z axis.

Hence, for rotation about an axis of symmetry[1], $\boxed{\bar{L} = I\bar{\omega}}$.

Note. In general (rotation about an arbitrary axis), $\bar{L} \nparallel \bar{\omega}$.





Car tire wobbles

Small weights are placed on the rim of the wheel to adjust the mass distribution, so that the wheel's axis of symmetry coincides with the axle. Then the wobbling is eliminated.

[1] Or for a planar object contained in the x–y plane.

Conservation of Angular Momentum

Conservation of Angular Momentum

In general (valid for any axis of rotation)

$$\boxed{\frac{\mathrm{d}\overline{L}}{\mathrm{d}t}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N}\overline{L}_i\right) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N}\overline{r}_i \times \overline{p}_i = \sum_{i=1}^{N}\overbrace{\underbrace{\frac{\mathrm{d}\overline{r}_i}{\mathrm{d}t}}_{=\overline{v}_i}\times\overline{p}_i}^{=0} + \sum_{i=1}^{N}\overline{r}_i \times \frac{\mathrm{d}\overline{p}_i}{\mathrm{d}t}$$

$$\overset{2^{\text{nd}}}{\underset{\text{law}}{=\!=}} \sum_{i=1}^{N}\overline{r}_i \times \overline{F}_i = \sum_{i=1}^{N}\overline{\tau}_i = \boxed{\overline{\tau}^{\text{ext}}}, \qquad \text{where } \overline{\tau}^{\text{ext}} = \sum_{i=1}^{N}\overline{r}_i \times \overline{F}_i^{\text{ext}}.$$

Conclusion

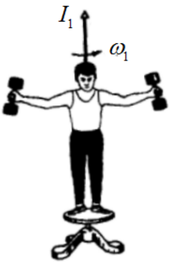
When the net external torque on a system is zero, then the total angular momentum of the system is conserved.

Recall that tor rotation about an axis of symmetry[2] (assume it is the z-axis) $L_z = I_z\omega_z$. Hence

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = \tau_z^{\text{ext}} \implies \qquad \boxed{\text{If } \tau_z^{\text{ext}} = 0, \text{ then } L_z = I_z\omega_z = \text{const}}$$

[2]Or for a planar object contained in the x–y plane.

Example (a). Spinning Ice-Skater



No external torque
(frictionless spinning)

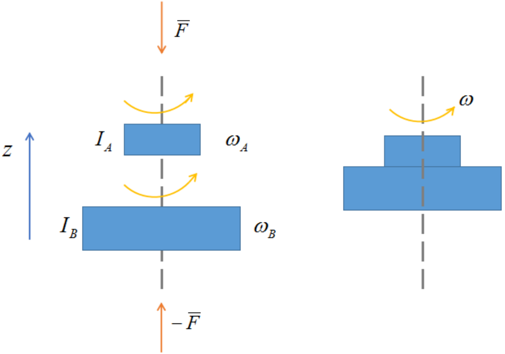
$$I\omega = \text{const}$$

$$I_1\omega_1 = I_2\omega_2 \implies \boxed{\omega_2 = \frac{I_1}{I_2}\omega_1 > \omega_1}$$

Increase of the angular speed by the factor of I_1/I_2.

Note. The kinetic energy increases here, because the skater does work when pulling his arms inwards.

Example (b). Completely Inelastic Rotational Collision



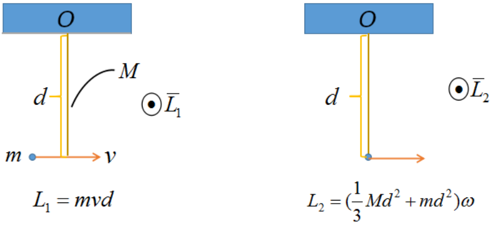
no torque due to $\overline{F}, -\overline{F} \Rightarrow \overline{L} =$ constant

Conservation of angular momentum

$$I_A\omega_A + I_B\omega_B = (I_A + I_B)\omega$$

$$\boxed{\omega = \frac{I_A\omega_A + I_B\omega_B}{I_A + I_B}}$$

Example (c). Ballistic Pendulum with Massive Rod



There are no external torques with respect to the suspension point O, hence the conservation of the total angular momentum implies

$$L_1 = L_2 \implies \boxed{\omega = \frac{mvd}{\frac{1}{3}Md^2 + md^2}}$$

Note. Here the rotation is not about a symmetry axis, but the rigid body (rod+bullet) is planar.