Chapter 17 – Gravitation

UM-SJTU Joint Institute
Physics I (Summer 2021)
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Agenda

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3. Motion of Satellites in Circular Orbits
4. Motion of Planets in Solar System. Kepler's Laws

Newton's Law of Gravitation

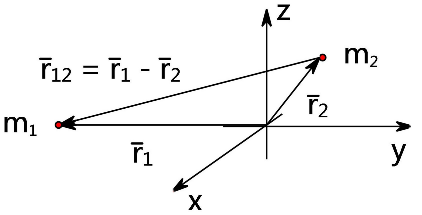
Introduction

Features of the gravitational interaction

- $\rightarrow$ source: any mass (matter)
- $\rightarrow$ one of the four fundamental interactions
- $\rightarrow$ relative strength: much weaker than the other three interactions (example: two electrons)
- $\rightarrow$ long-range; important for the evolution of the Universe
- $\rightarrow$ always attractive

Newton's Law of Gravitation

Force of gravitational attraction between two particles
(particle = mass concentrated at a single point of space)

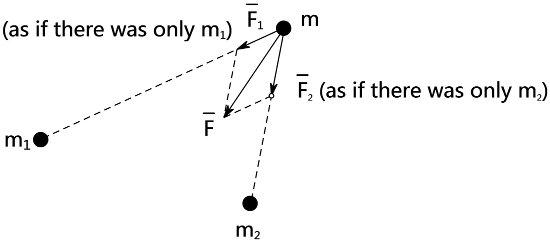


$$\boxed{\underbrace{\bar{F}_{12}}_{\text{on "1" due to "2"}} = -G\,\frac{m_1 m_2}{r_{12}^2}\,\frac{\bar{r}_{12}}{|\bar{r}_{12}|}}$$

$G = 6.67250(85) \cdot 10^{-11} \left[\mathrm{N} \cdot \frac{\mathrm{m}^2}{\mathrm{kg}^2}\right]$
Gravitational constant

Note. $\bar{F}_{12} = -\bar{F}_{21}$

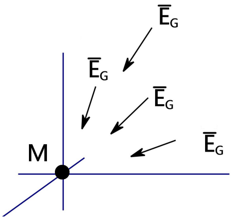
The superposition principle



Gravitational Field

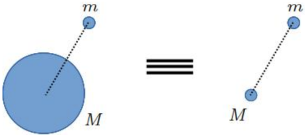
The gravitational interaction defines a vector field in space

$$\bar{E}_G = \frac{\bar{F}_G}{m} = -G\,\frac{M}{r^2}\frac{\bar{r}}{r}$$



Fact (will be justified soon)

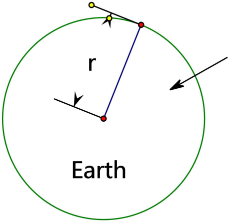
> *For a spherically symmetric distribution of mass in a region Ω, the gravitational field outside of Ω is as if the whole mass was concentrated at the center of Ω.*



Weight

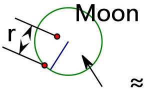
The weight of a body is the total gravitational force exerted on a body by all other objects in the Universe.

In practice,



≈ only due to earth

$$F \approx G\frac{M_{earth}m}{r^2}$$



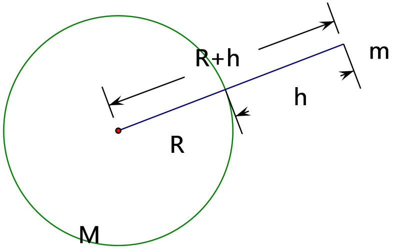
≈ only due to moon

$$F \approx G\frac{M_{moon}m}{r^2}$$

At the Earth's surface, $F = G\frac{M_{\text{Earth}}m}{R^2} = mg$, where R is the radius of the Earth.

Note. By measuring g, it is possible to estimate the mass of the Earth as $M_{\text{Earth}} = gR^2/G \approx 5.98 \times 10^{24}$ kg. Hence, the average density of the Earth's mass $\rho_{av} = M_{Earth}/\frac{4}{3}\pi R^3 \approx 5.5 \times 10^3$ kg/m^3

Variation of the weight with altitude (close to the Earth)



$$\text{weight} = G\frac{Mm}{(R+h)^2} = G\frac{Mm}{R^2(1+\frac{h}{R})^2} = G\frac{M}{R^2}m\left(1+\frac{h}{R}\right)^{-2}$$

$$= G\frac{M}{R^2}m\left(1-2\frac{h}{R}+3\left(\frac{h}{R}\right)^2-\ldots\right) \quad [and, if\ \ h \ll R]$$

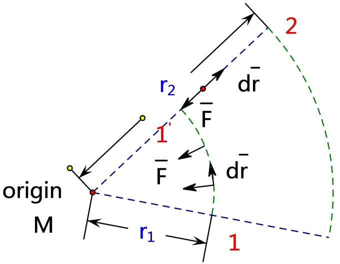
$$\approx \underbrace{G\frac{M}{R^2}}_{=g}m = mg \quad \text{(close the Earth's surface)}$$

Gravitational Potential Energy

Gravitational Potential Energy

A force $\bar{F}$ is called *central*, if $\bar{F} = f(r)\bar{r}$. Example: the gravitational force due to a point mass placed at the origin $\bar{F} = -G\frac{Mm}{r^2}\frac{\bar{r}}{|r|} = f(r)\bar{r}$, with $f(r) = -G\frac{Mm}{r^3}$.

Fact. Central forces are conservative $\implies$ their work does not depend on the path.



Choose the path:
$1 \to$ *arc of a circle* $\to$
$1' \to$ *radius* $\to 2'$

$$W_{1\to 2} = \int\limits_{1\to 2} \bar{F}\circ d\bar{r} = \underbrace{\int\limits_{1\to 1'} \bar{F}\circ d\bar{r}}_{=0 \quad (\bar{F}\perp d\bar{r})} + \underbrace{\int\limits_{1'\to 2'} \bar{F}\circ d\bar{r}}_{\angle(\bar{F},d\bar{r})=\pi} = -\int_{r_1}^{r_2} |\bar{F}|\cdot|d\bar{r}| =$$

$$= -\int_{r_1}^{r_2} G\frac{Mm}{r^2}dr = G\frac{Mm}{r}\Big|_{r_1}^{r_2} = G\frac{Mm}{r_2} - G\frac{Mm}{r_1}$$

On the other hand, for potential forces, $W_{1\to 2} = U_1 - U_2$.Hence, the gravitational potential energy

$$\boxed{U(r) = -G\frac{Mm}{r} + C}$$

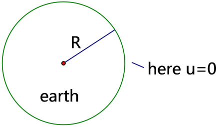
the additive constant C can be chosen arbitrarily (no physical meaning) as only ΔU is measurable/physical.

Note. $\bar{F} = -\text{grad}\, U = -\nabla U$.

Choice of C ("choice of the gauge") — infinitely many possibilities. Two examples:

1. $U(\infty) = 0 \implies C = 0$ *and* $\boxed{U(r) = -G\dfrac{Mm}{r}}$
2. $U(R) = 0 \implies -G\frac{Mm}{R} + C = 0 \implies C = G\frac{Mm}{R}$ and

$$\boxed{U(r) = GMm\left(\frac{1}{R} - \frac{1}{r}\right)}$$



Let $r = R + h$ (with $h \ll R$), then

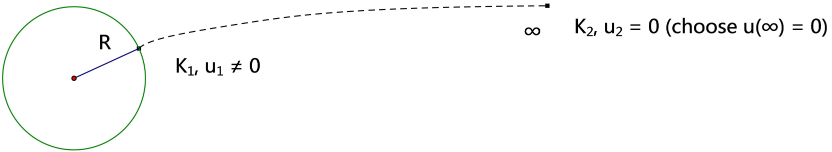
$$U(r) = GMm\left(\frac{1}{R} - \frac{1}{R+h}\right) = G\frac{Mm}{R}\left(1 - \frac{1}{1+\frac{h}{R}}\right)$$

$$= G\frac{Mm}{R}\left(1 - 1 + \frac{h}{R} - \left(\frac{h}{R}\right)^2 + \ldots\right) \quad [\textit{for } h \ll R]$$

$$\approx G\frac{Mm}{R}\left(1 - 1 + \frac{h}{R}\right) = G\frac{Mm}{R^2}h = mgh$$

Example. Escape Speed

Escape speed — the minimum speed a particle should have to be able to move away from a planet to ∞



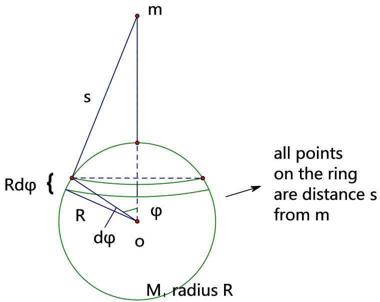
Conservation of energy

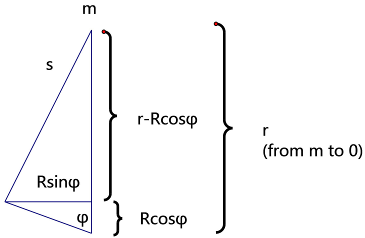
$$K_1 + U_1 = K_2 + U_2$$

Hence, e.g. for the Earth,

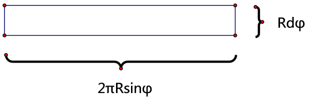
$$\frac{1}{2} m v_{esc}^2 - G \frac{M_{\text{Earth}} m}{R} = 0 \implies \boxed{v_{esc} = \sqrt{\frac{2GM_{\text{Earth}}}{R}} \approx 11.2 km/s}$$

Example. Potential Energy Due to a Uniform Spherical Shell





Area of the infinitesimal ring $dA = 2\pi R \sin\varphi R\, d\varphi$



Mass on the ring $dM = \frac{M}{4\pi R^2}\, dA = \frac{1}{2} M \sin\varphi\, d\varphi$. Contribution of the ring to the potential energy (choose gauge with $U(\infty) = 0$)

$$dU = -G\frac{m\, dM}{s} = -G\frac{mM}{2s}\sin\varphi\, d\varphi$$

Now, express everything in terms of s; then integrate w.r.t. s from $r - R$ ("north pole") to $r + R$ ("south pole")

$$s^2 = (r - R\cos\varphi)^2 + (R\sin\varphi)^2 = r^2 - 2rR\cos\varphi + R^2$$

$$2s\,ds = 2rR\sin\varphi\,d\varphi \implies \sin\varphi\,d\varphi = \frac{s}{Rr}\,ds$$

Hence,

$$dU = -G\frac{Mm}{2s}\frac{s}{Rr}\,ds = -G\frac{Mm}{2Rr}\,ds$$

and,

$$U(r) = \int_{r-R}^{r+R} (-G\frac{Mm}{2Rr})\,ds = -G\frac{Mm}{2Rr}[r + R - r + R]$$

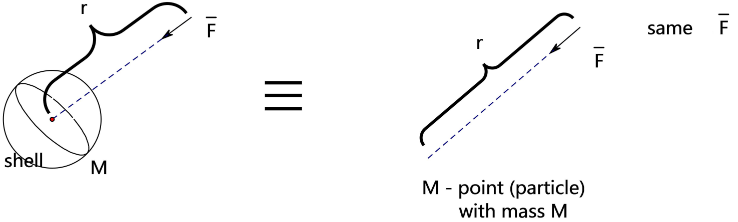
$$= -G\frac{Mm}{r} \qquad \implies \boxed{U(r) = -G\frac{Mm}{r}}$$

The corresponding force

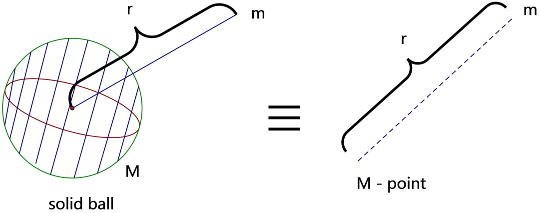
$$\bar{F} = -\text{grad}\,U = -G\frac{Mm}{r^2}\,\frac{\bar{r}}{r}$$

Discussion of the result

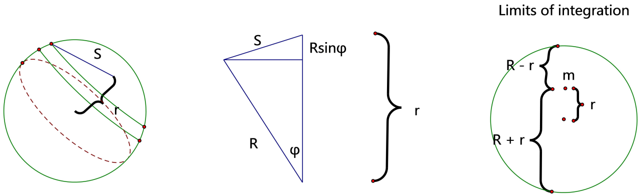
(*)



(*) For a solid ball with a spherically symmetric distribution of mass, divide it into shells and use the above fact



(*) If m is inside the sphere

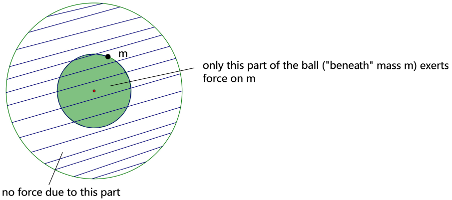


$$\boxed{U(r)} = -G\frac{mM}{2rR}\int_{R-r}^{R+r} ds = \boxed{-G\frac{Mm}{R} = \text{const}}$$

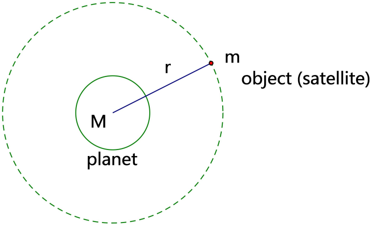
$$\Longrightarrow \bar{F} = -\text{grad}\, U \equiv 0$$

Gravitational force on mass m inside the sphere is zero!

(*) Consequently



Motion of Satellites in Circular Orbits



Gravitational force plays the role of the centripetal force

$$\bar{F}_{grav} = m\bar{a}_{centripetal}$$

Magnitude

$$G\frac{Mm}{r^2} = \frac{mv^2}{r} \quad (or \quad m\omega^2 r)$$

Satellite's linear speed in a circular orbit $\boxed{v = \sqrt{\frac{GM}{r}}}$.

Note. This value (if $r = R$, i.e. for low orbits) is exactly $\sqrt{2}$ times smaller than v_{esc}.

Period of motion in a circular orbit

$$\boxed{T} = \frac{2\pi r}{v} = 2\pi \frac{r^{3/2}}{\sqrt{GM}} \boxed{\sim r^{3/2}}.$$

Given the orbit's radius r, the speed is determined. Note that further orbits mean slower speeds ($v \sim 1/\sqrt{r}$) and longer periods.

Examples

(a) geostationary satellite ($T_{geost} = 24h$)

$$r_{geost} \simeq 35786 \text{ km}$$

$$v_{geost} \simeq 3.1 \text{ km/s}$$

(b) ISS ($r = 6800$ km)

$$v = 7.7 \text{ km/s}$$

$$T = 93 \text{ min}$$

(c) moon ($r \simeq 380000$ km)

$$v = 1 \text{ km/s}$$

$$T \simeq 28 \text{ days}$$

Motion of Planets in Solar System. Kepler's Laws

Motion in a Central Field

Recall: a force $\bar{F}$ is called **central** if $\bar{F}(\bar{r}) = f(r)\,\bar{r}$. Hence, for a central force

$$\bar{r} \times \bar{F} = \bar{r} \times f(r)\bar{r} = 0.$$

On the other hand,

$$\bar{r} \times \bar{F} = \bar{\tau} = \frac{d\bar{L}}{dt}.$$

Eventually, for motion in the field of a central force,

$$\boxed{\frac{d\bar{L}}{dt} = 0}.$$

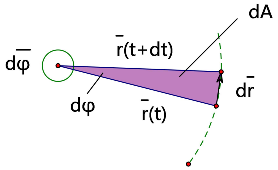
Conclusion

> *For motion in the field of a central force, $\bar{L} = const$ (both magnitude and direction are constant).*

Note. Constant direction of $\bar{L} \implies$ motion in a plane.

Aerial Velocity

For planer motion, the **aerial velocity** may be defined



The surface area swept by $\bar{r}$ over the time dt is $dA = \left|\frac{1}{2}\bar{r} \times d\bar{r}\right|$ and the rate of change of that area

$$\frac{dA}{dt} = \frac{1}{2}\left|\bar{r} \times \frac{d\bar{r}}{dt}\right| = \frac{1}{2}\left|\bar{r} \times \bar{v}\right|.$$

Aerial velocity vector (direction — right-hand rule)

$$\boxed{\bar{\sigma} = \frac{1}{2}(\bar{r} \times \bar{v})} \qquad \text{(direction same as } d\bar{\varphi}\text{)}$$

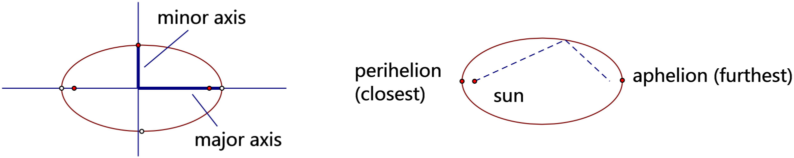
Recall: $\bar{L} = \bar{r} \times \bar{p} = \underbrace{\bar{r} \times m\bar{v}}_{=\bar{\sigma}\cdot 2m}$. Hence $\bar{L} = const \Leftrightarrow \bar{\sigma} = const$.

Consequently, for motion in a central force field $\bar{\sigma} = \text{const}$.

Kepler's Laws of Planetary Motion

1. Each planet moves in an elliptical orbit with the Sun at one of the focal points.
2. The line from the Sun to a given planet sweeps out equal areas in equal times.
3. The period of motion of a planet is proportional to the 3/2-power of the major axis length of its orbit

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM_{\text{Sun}}}}.$$



Note. Precisely, motion of both the Sun and the planet is about their center of mass (in practice, the center of mass is very close to the center of the Sun).