Chapter 10 – Momentum and Conservation of Momentum

UM-SJTU Joint Institute
Physics I (Summer 2020)
Mateusz Krzyzosiak

Agenda

Momentum
Conservation of Momentum
Collisions
Center of Mass
Motion of Objects with Varying Mass., Rocket Propulsion

Definition. Momentum-Impulse Theorem
Momentum–Impulse Theorem
Kinetic Energy vs. Momentum

Momentum

Newton's idea $\rightarrow$ "quantity/ amount of motion"

2nd law of dynamics

$$\overline{F} = m\overline{a} = m\frac{d\overline{v}}{dt} = \frac{d}{dt}(m\overline{v}) = \frac{d\overline{p}}{dt}$$

$\overline{p} = m\overline{v}$ – **momentum (linear momentum)**; units $[\text{kg} \cdot \text{m/s}]$

Newton's 2nd law in terms of linear momentum $\overline{p} = m\overline{v}$

$$\boxed{\overline{F} = \frac{d\overline{p}}{dt}}$$ $\rightarrow$ the net force acting on a particle is equal to the time rate of change of particle's momentum
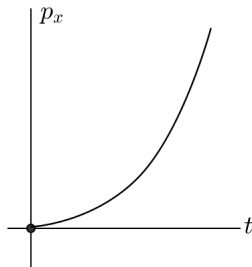
Illustration

$$\overline{F} = \frac{\mathrm{d}\overline{p}}{\mathrm{d}t}$$
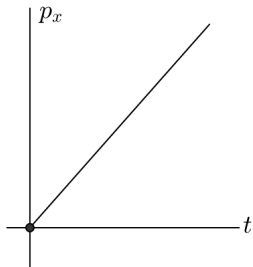
constant force $F_x = \text{const} > 0$

increasing force $F_x = \dfrac{\mathrm{d}p_x}{\mathrm{d}t} > 0$

$$F_x = \frac{\mathrm{d}p_x}{\mathrm{d}t} \qquad \implies \qquad F_x\,\mathrm{d}t = \mathrm{d}p_x \qquad \implies \qquad \int\limits_{t_1}^{t_2} F_x\,\mathrm{d}t = \int\limits_{p_1}^{p_2} \mathrm{d}p_x$$

Momentum-Impulse Theorem (1D)

$$p_2 - p_1 = \underbrace{\int\limits_{t_1}^{t_2} \overset{\text{net force}}{F_x}\,\mathrm{d}t}_{\text{impulse of the net force}}$$

Observation

the same net change in momentum can be achieved in various ways (different forces acting over different intervals of time), if the areas under both curves are the same, the resulting net changes of the momentum are equal.
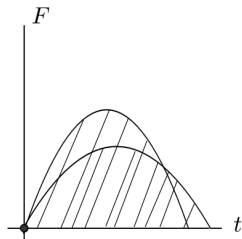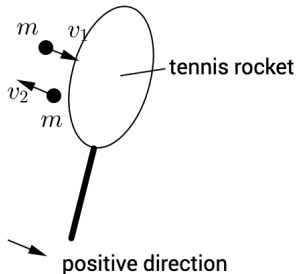
$$\frac{\Delta p}{\Delta t} = \frac{m(-v_2 - v_1)}{\Delta t} = F_{av}$$

where Δt is period of time when ball was in contact with rocket, F_{av} is average force exerted on ball.

$F_{av} = ?$

Both are measures of the "amount of motion".

Kinetic Energy

* scalar
* changes defined by work-kinetic energy theorem

$$dK = \overline{F} \circ d\overline{r}$$

force over distance

Momentum

* vector
* changes defined by momentum - impulse theorem

$$d\overline{p} = \overline{F} \cdot dt$$

force over period of time

Kinetic Energy vs. Momentum. Illustration

Suppose we want to bring two objects to a stop.

$$v_A = 10 \text{ [m/s]} \qquad\qquad v_B = 5 \text{ [m/s]}$$
$$m_A = 400 \text{ [kg]} \qquad\qquad m_B = 800 \text{ [kg]}$$
$$p_A = 4000 \text{ [kg m/s]} \quad = \quad p_B = 4000 \text{ [kg m/s]}$$
$$K_A = 20 \text{ [kJ]} \qquad\qquad K_B = 10 \text{ [kJ]}$$

What is the average stopping force if both A and B have stopped

* after travelling a distance $s = 100$ m? $\Delta K = -F \cdot s$
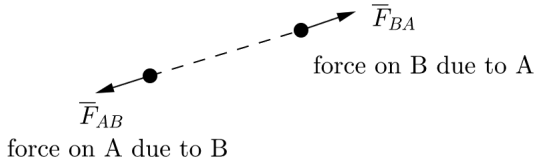
$$F_A = 200 \text{ [N]} \qquad F_B = 100 \text{ [N]}$$

* after travelling for $t = 10$ s? $\Delta p = -F \cdot t$

$$F_A = 400 \text{ [N]} \qquad F_B = 400 \text{ [N]}$$

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Motivation
Conservation of Momentum Law. Examples

Conservation of Momentum

Conservation of Momentum

$\overline{F}_{BA}$
force on B due to A

$\overline{F}_{AB}$
force on A due to B

Assume that there are no external forces acting on the two particles (or the sum of all these external force is zero), that is there are only interactions between the particles.

Newton's 3$^{\text{rd}}$ law

$$\overline{F}_{AB} = -\overline{F}_{BA}$$

Newton's 2$^{\text{nd}}$ law

$$\overline{F}_{AB} = \frac{\mathrm{d}\overline{p}_A}{\mathrm{d}t} \qquad F_{BA} = \frac{\mathrm{d}\overline{p}_B}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\overline{p}_A}{\mathrm{d}t} = -\frac{\mathrm{d}\overline{p}_B}{\mathrm{d}t} \qquad \Longrightarrow \qquad \frac{\mathrm{d}\overline{p}_A}{\mathrm{d}t} + \frac{\mathrm{d}\overline{p}_B}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overline{p}_A + \overline{p}_B) = 0$$

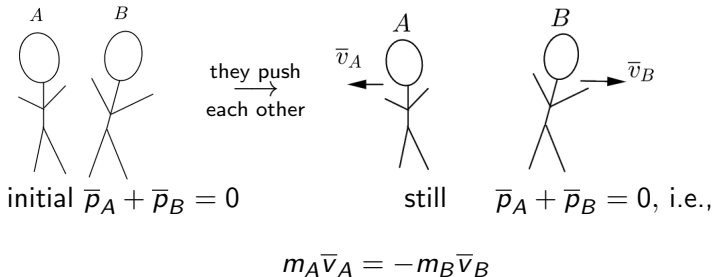Hence (recall the assumption: the net external force is zero)

$$\boxed{\overline{p}_A + \overline{p}_B = \text{const}}$$
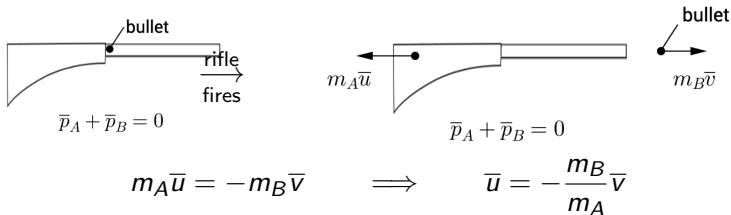
Conclusion

If the sum of all external forces on a system of two particles is equal to zero, then the total momentum of the system is constant.

Illustrations

(a) Two ice-skaters

initial $\overline{p}_A + \overline{p}_B = 0$ still $\overline{p}_A + \overline{p}_B = 0$, i.e.,

$$m_A \overline{v}_A = -m_B \overline{v}_B$$

(b) recoil of a rifle

$\overline{p}_A + \overline{p}_B = 0$ $\overline{p}_A + \overline{p}_B = 0$

$$m_A \overline{u} = -m_B \overline{v} \qquad \Longrightarrow \qquad \overline{u} = -\frac{m_B}{m_A} \overline{v}$$

Conservation of Momentum Law

The previous statement for two particles can be generalised to systems of many particles and formulated as a *conservation principle* for the momentum.
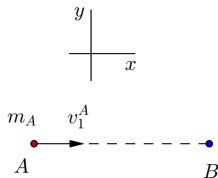
> **Conservation of Momentum Law**
>
> If the sum of external forces on a system is equal to zero, then the total momentum of the system is constant (i.e. is conserved).
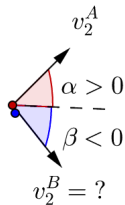
Illustration: exploding grenade

$\bar{p} = 0$

$\bar{p}_1$ $\bar{p}_2$ $\bar{p}_i$ $\bar{p}_N$

$$\overline{P} = \sum_{i=1}^{N} \bar{p}_i = 0$$
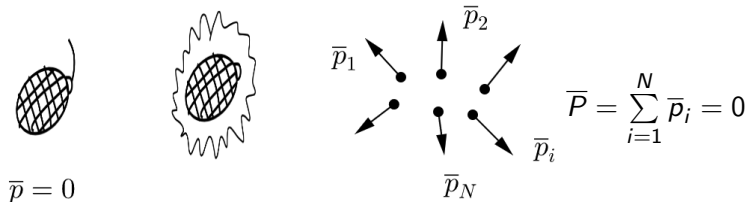
Initial (1)

Final (2)

$$\overline{p}_1 = (m_A v_1^A, 0)$$

$$
\begin{aligned}
\overline{p}_2 &= \overline{p}_2^A + \overline{p}_2^B \\
&= (m_A v_2^A \cos \alpha + m_B v_2^B \cos \beta, \\
&\qquad m_A v_2^A \sin \alpha + m_B v_2^B \sin \beta)
\end{aligned}
$$

No external forces $\implies$ $\overline{p}_1 = \overline{p}_2$

$$(m_A v_1^A,\, 0) = (m_A v_2^A \cos\alpha + m_B v_2^B \cos\beta,\ m_A v_2^A \sin\alpha + m_B v_2^B \sin\beta)$$

$$\begin{cases} m_A v_1^A = m_A v_2^A \cos\alpha + m_B v_2^B \cos\beta \\ 0 = m_A v_2^A \sin\alpha + m_B v_2^B \sin\beta \end{cases}$$

$$\begin{cases} m_A(v_1^A - v_2^A \cos\alpha) = m_B v_2^B \cos\beta \\ -m_A v_2^A \sin\alpha = m_B v_2^B \sin\beta \end{cases}$$

$$\implies \boxed{\begin{aligned} \tan\beta &= -\frac{v_2^A \sin\alpha}{v_1^A - v_2^A \cos\alpha} \\ v_2^B &= -\frac{m_A v_2^A \sin\alpha}{m_B \sin\beta} \end{aligned}}$$

If v_1^A, v_2^A, α and m_A, m_B are known, β and v_2^B may be found.

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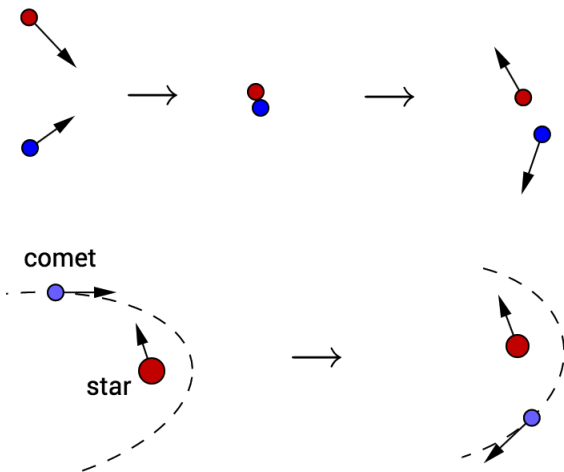
Classification
Completely Inelastic Collisions. Examples
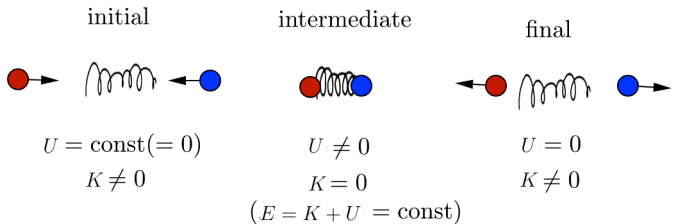Elastic Collision. Examples

Collisions

Collisions

Collision – two objects interact (directly or indirectly) over a finite time-interval.
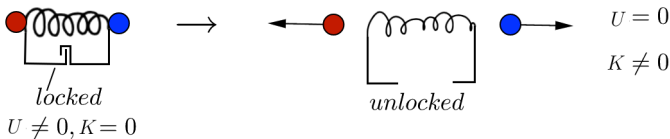
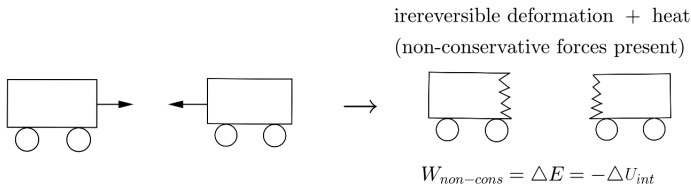# Classification of Collisions

- elastic collision

initial   intermediate   final

$U = \mathrm{const}(= 0)$ $U \neq 0$ $U = 0$
$K \neq 0$ $K = 0$ $K \neq 0$
$(E = K + U = \mathrm{const})$

$$K_{\mathsf{initial}} = K_{\mathsf{final}}$$

- superelastic collision

$U = 0$
$K \neq 0$

locked
$U \neq 0, K = 0$ unlocked

$$K_{\mathsf{final}} > K_{\mathsf{initial}}$$

Classification of Collisions

- inelastic collision

ireversible deformation + heat
(non-conservative forces present)

$$W_{non-cons} = \triangle E = -\triangle U_{int}$$

$$K_{\text{final}} < K_{\text{initial}}$$

In general

$$K_{\text{final}} = K_{\text{initial}} + Q \qquad \text{with } Q \begin{cases} = 0 & \text{(elastic)} \\ > 0 & \text{(superelastic)} \\ < 0 & \text{(inelastic)} \end{cases}$$

Elastic

Internal forces involved are potential (conservative), hence the mechanical energy is conserved $\Delta E = 0$ (usually in our examples $U = $ const, hence $K = $ const)
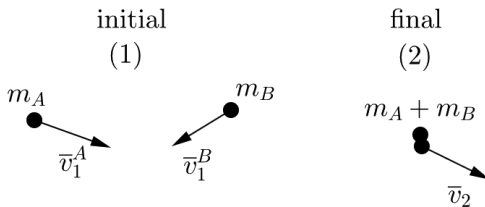
Inelastic

Internal forces are non-conservative, mechanical energy not conserved (total energy is conserved, but part of mechanical energy is transformed irreversibly into internal energy)

* **completely inelastic collisions** (colliding particles move as one object after the collision, stick to each other)

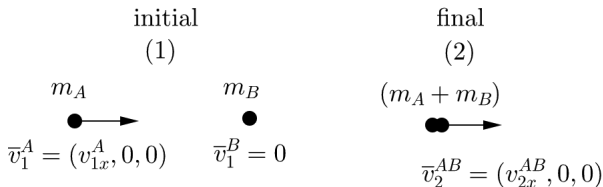In both cases, the total momentum of the system is conserved.

$$m_A \overline{v}_1^A + m_B \overline{v}_1^B \quad = \quad (m_A + m_B) \overline{v}_2$$

If we know: m_A, m_B and $\overline{v}_1^A$, $\overline{v}_2^B$, we can find

$$\overline{v}_2 = \frac{m_A \overline{v}_1^A + m_B \overline{v}_1^B}{m_A + m_B}$$

Completely Inelastic Collisions. Example I (head-on collision)

More specific example: head-on, completely inelastic collision.

$$\overline{p}_1 = \overline{p}_2 \qquad \Longrightarrow \qquad \boxed{v_{2x}^{AB} = \frac{m_A v_{1x}^A}{m_A + m_B}}$$

Compare the kinetic energy before and after the collision (we assume $U = \text{const}$)

$$
\begin{aligned}
K_1 &= \frac{1}{2} m_A (v_{1x}^A)^2 \\
K_2 &= \frac{1}{2}(m_A + m_B)(v_{2x}^{AB})^2 = \frac{1}{2}(m_A + m_B)\frac{m_A^2 (v_{1x}^A)^2}{(m_A + m_B)^2}
\end{aligned}
$$

$$= \frac{1}{2} \frac{m_A^2}{m_A + m_B}(v_{1x}^A)^2 = \frac{m_A}{m_A + m_B}K_1.$$

Hence

$$\Delta K = K_2 - K_1 = K_1 \left(\frac{m_A}{m_A + m_B} - 1 \right) < 0.$$

Discussion

- limit case $\quad m_A \gg m_B$

$$\overline{v}_2^{AB} = \left(\frac{m_A v_{1x}^A}{m_A + m_B}, 0, 0 \right) \approx (v_{1x}^A, 0, 0), \qquad \Delta K \approx 0-\text{negligible}$$

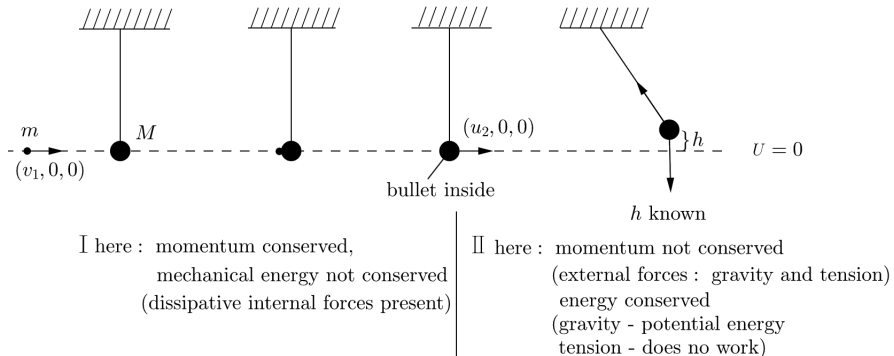- limit case $\quad m_A \ll m_B$

$$\overline{v}_2^{AB} - \text{negligible}, \qquad \Delta K \approx -K_1$$

Almost all the initial kinetic energy is dissipated.

Completely Inelastic Collisions. Example II (ballistic pendulum)

Ballistic pendulum – used to estimate the speed of a bullet.

m

$(v_1, 0, 0)$

M

$(u_2, 0, 0)$

bullet inside

$\} h$

$U = 0$

h known

I here : momentum conserved,
mechanical energy not conserved
(dissipative internal forces present)

II here : momentum not conserved
(external forces : gravity and tension)
energy conserved
(gravity - potential energy
tension - does no work)

Phase I. Use the conservation of momentum (its horizontal component) to find u_2

$$\begin{aligned}
\overline{p}_1 &= \overline{p}_2 \\
mv_1 &= (M + m)u_2 \\
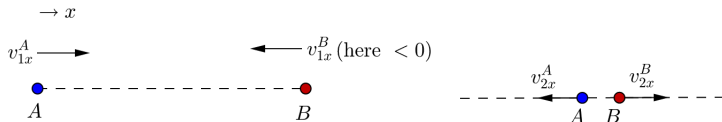\Rightarrow u_2 &= \frac{m}{M + m}v_1
\end{aligned}$$

Phase II. Use the conservation of energy to relate h with u_2 (hence with v_1)

$$\begin{aligned}
\frac{1}{2}(M + m)u_2^2 &= (M + m)gh \\
\frac{1}{2}(M + m)\frac{m^2}{(M + m)^2}v_1^2 &= (M + m)gh
\end{aligned}$$

$$\boxed{v_1 = \frac{M + m}{m}\sqrt{2gh}}$$

Elastic Collisions. Example (head-on collision, general case)

General 1D case (head-on collision) $\rightarrow$ before/after the collision the velocities of both particles are colinear

① Momentum conserved

$$m_A v_{1x}^A + m_B v_{1x}^B = m_A v_{2x}^A + m_B v_{2x}^B$$

② Energy conserved

$$\frac{1}{2} m_A (v_{1x}^A)^2 + \frac{1}{2} m_B (v_{1x}^B)^2 = \frac{1}{2} m_A (v_{2x}^A)^2 + \frac{1}{2} m_B (v_{2x}^B)^2$$

More specific example: B initially at rest ($v_{1x}^B = 0$). Then

$$\begin{cases} m_A v_{1x}^A = m_A v_{2x}^A + m_B v_{2x}^B \\ \frac{1}{2} m_A (v_{1x}^A)^2 = \frac{1}{2} m_A (v_{2x}^A)^2 + \frac{1}{2} m_B (v_{2x}^B)^2 \end{cases}$$

$$\begin{cases} m_A (v_{1x}^A - v_{2x}^A) = m_B v_{2x}^B \qquad (*) \\ m_A \underbrace{(v_{1x}^A - v_{2x}^A)(v_{1x}^A + v_{2x}^A)}_{(v_{1x}^A)^2 - (v_{2x}^A)^2} = m_B (v_{2x}^B)^2 \end{cases}$$

Divide side-by-side to get

$$v_{1x}^A + v_{2x}^A = v_{2x}^B \qquad (**)$$

Substitute v_{2x}^B from (**) into (*) and solve for v_{2x}^A, substitute back into (**) to get v_{2x}^B.

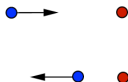$$v_{2x}^A = \frac{m_A - m_B}{m_A + m_B} v_{1x}^A$$

$$v_{2x}^B = \frac{2m_A}{m_A + m_B} v_{1x}^A$$

Discussion

a $m_A \ll m_B$

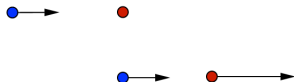$$v_{2x}^A \approx -v_{1x}^A, \qquad v_{2x}^B \ll v_{1x}^A$$
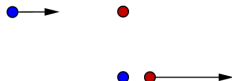
ping-pong ball

b $m_A \gg m_B$

$$v_{2x}^A \approx v_{1x}^A, \qquad v_{2x}^B \approx 2v_{1x}^A$$

ping-pong ball

c $m_A = m_B$

$$v_{2x}^A = 0, \qquad v_{2x}^B = v_{1x}^A$$

A ball collides in a non-head-on elastic collision with another identical ball which is initially at rest. Show that after the collision the balls move in perpendicular directions.

Solution

momentum: $\begin{cases} m\overline{v}_1 = m\overline{u}_1 + m\overline{u}_2 \\ \dfrac{1}{2}mv_1^2 = \dfrac{1}{2}mu_1^2 + \dfrac{1}{2}mu_2^2 \end{cases}$

mechanical energy ($U = $ const) :

$$\begin{cases} \overline{v}_1 \circ \overline{v}_1 = \overline{v}_1^2 = (\overline{u}_1 + \overline{u}_2) \circ (\overline{u}_1 + \overline{u}_2) \\ v_1^2 = u_1^2 + u_2^2 \end{cases}$$

$$\begin{cases} v_1^2 = u_1^2 + u_2^2 + \underbrace{\overline{u}_1 \circ \overline{u}_2 + \overline{u}_2 \circ \overline{u}_1}_{2\overline{u}_1 \circ \overline{u}_2} \\ v_1^2 = u_1^2 + u_2^2 \end{cases}$$

Subtracting side-by-side

$$\overline{u}_1 \circ \overline{u}_2 = 0 \Rightarrow \boxed{\overline{u}_1 \perp \overline{u}_2}$$
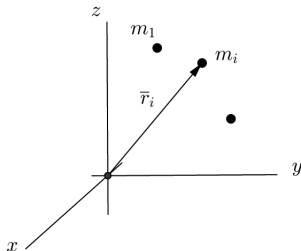
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Definition
Properties and Applications

Center of Mass

The position vector of the **center of mass** (system of particles)

$$\overline{r}_{cm} = \frac{\sum_{i=1}^{N} m_i \overline{r}_i}{\sum_{i=1}^{N} m_i} \rightarrow \text{discrete distribution of mass}$$

Note. The sum is replaced by an integral for a continuous distribution of mass.

Center of Mass. Properties

$$\underbrace{\sum_{i=1}^{N} m_i}_{\text{total mass}} \bar{r}_{cm} = \sum_{i=1}^{N} m_i \bar{r}_i \qquad \Big/ \frac{\mathrm{d}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\sum_{i=1}^{N} m_i)\bar{r}_{cm}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N} m_i \bar{r}_i$$

$$(\sum_{i=1}^{N} m_i)\frac{\mathrm{d}\bar{r}_{cm}}{\mathrm{d}t} = \sum_{i=1}^{N} m_i \underbrace{\underbrace{\frac{\mathrm{d}\bar{r}_i}{\mathrm{d}t}}_{\bar{v}_i}}_{\bar{p}_i}$$

Hence

$$\boxed{M\bar{v}_{cm} = \sum_{i=1}^{N} \bar{p}_i = \bar{p}} \qquad \text{where } M = \sum_{i=1}^{N} m_i$$

Conclusion

> *The total momentum of the system is equal to the momentum of a hypothetical particle of mass M moving with velocity $\overline{v}_{cm}$.*

Differentiate once again

$$\frac{d\overline{p}}{dt} = \frac{d}{dt}(M\overline{v}_{cm}) = M\frac{d\overline{v}_{cm}}{dt}$$

But $\frac{d\overline{p}}{dt} = \sum_{i=1}^{N} \overline{F}_i^{ext} = \overline{F}^{ext}$. Hence, if $\overline{F}^{ext} = 0$, then $\overline{v}_{cm} = \text{const}$.
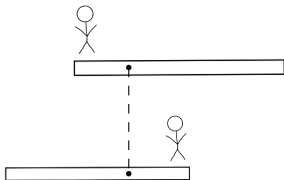
Conclusion

> *If the sum of all external forces acting on the system is equal to zero, the center of mass moves with a constant velocity.*

> *[In particular, if it was at rest initially, it remains at rest.]*

Center of Mass. Examples

ⓐ boat

$$\overline{F}^{ext} = 0$$
$$\overline{v}_{cm} = 0$$

ⓑ grenade

$$\overline{F}^{ext} \neq 0$$
$$\frac{d\overline{v}_{cm}}{dt} = \frac{\overline{F}^{ext}}{m}$$
$$\Downarrow$$
$$\overline{r}_{cm} = \overline{r}_{cm}(t) \to \text{parabola}$$

Motion of Objects with Varying Mass. Rocket Propulsion
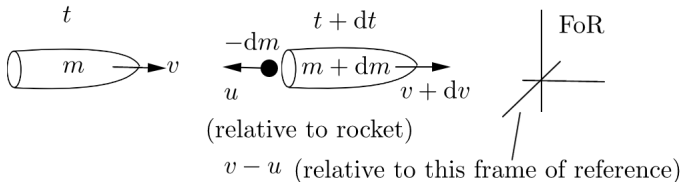
propeller
↓
needs air to work

rocket
↓
works in vacuum

Idea: burn fuel $\Rightarrow$ eject burned fuel products $\Rightarrow$
$\Rightarrow$ rocket moves forward

Momentum conservation
[all momenta in the same (fixed, inertial) frame of reference!]

$$mv = \overbrace{(m + dm)(v + dv)}^{\text{rocket}} \underbrace{\overbrace{-dm(v - u)}^{\text{burned fuel}}}_{>0}$$

$$mv = mv + m\,dv + v\,dm + \underbrace{(dm)(dv)}_{} - v\,dm + u\,dm$$

much smaller than
the other terms
(product of 2 infinitesimally
small quantities)

Eventually,

$$m\,\mathrm{d}v = -u\,\mathrm{d}m \quad (\#)$$

and

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -u\frac{\mathrm{d}m}{\mathrm{d}t}$$

$$\boxed{a} = \frac{\mathrm{d}v}{\mathrm{d}t} = \boxed{-\frac{u}{m(t)}\frac{\mathrm{d}m}{\mathrm{d}t}}$$

Discussion. An effective rocket

- burns fuel at a fast rate ($|\mathrm{d}m/\mathrm{d}t|$ is large),

- ejects burned fuel products with a great speed (u is large).

To find the velocity as a function of time, look at Eq. $(\#)$ again

$$dv = -u \frac{dm}{m}$$

$$\int\limits_{v_0}^{v(t)} dv = -u \int\limits_{m_0}^{m(t)} \frac{dm}{m}$$

$$\boxed{v(t)} = v_0 - u \ln \frac{m(t)}{m_0} = \boxed{v_0 + u \ln \frac{m_0}{m(t)}}$$

Conclusions

* The ratio of the initial mass of the rocket to the mass of the rocket after all fuel is burned should be as large as possible.
* The final velocity is greater in magnitude than u if $m_0/m(t_{final}) > e$ (for $v_0 = 0$).

Numerical example: If $m(t_{final}) = m_0/4$, $t_{final} = 30$ s, and $u = 2400$ m/s, $v_0 = 0$, then $v_{final} = 3327$ m/s ["final" means after all fuel has been used up].