Chapter 5a – Motion with Fluid/Air Resistance

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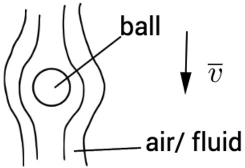
Introduction. Models for Air Resistance

Stokes (or linear) drag [small objects/low speeds]

$$\boxed{\overline{F}_{\text{drag}} = -k\overline{v} = -kv\left(\frac{\overline{v}}{v}\right)}$$



where $k = 6\pi\eta R_s = \mathrm{const}$, η is the fluid viscosity, R_s is the Stokes' radius (property of the object)

Quadratic drag [large objects/high speeds]

$$\boxed{\overline{F}_{\text{drag}} = -bv^2\left(\frac{\overline{v}}{v}\right)}$$

where $b = \frac{1}{2}\varrho C_d A = \mathrm{const}$, ϱ is the fluid density, C_d is a drag coefficient (e.g. for cars 0.25-0.5), A is the cross-sectional area of the object perpendicular to the direction of motion

Qualitative Analysis [fall with linear air drag, no initial velocity]

<u>initial phase</u>

$$t \approx 0 \quad \Rightarrow \quad v \approx 0$$

$$F_{\text{drag}} \quad \propto \quad v = 0$$

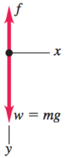
$$\Downarrow$$

$$a \approx g$$

In the initial phase
the particle moves as it was
free-falling

<u>final phase</u>

speed increases $\Rightarrow$ air drag increases



net force: $\quad mg - kv_\infty = 0$

$\rightarrow$ in the final phase, the drag balances the weight

$\Rightarrow$ particle moves with constant speed

terminal speed: $v_\infty = \dfrac{mg}{k}$

Observation: Heavier objects reach greater terminal speeds (that is, fall down faster).

General Problem: Strategy

General problem

$$a_y = \frac{F(v_y)}{m} \qquad \rightarrow \text{Newton's } 2^{\text{nd}} \text{ law (equation of motion)}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{F\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)}{m} \qquad \rightarrow$$

Problem: the second derivative...

Idea: Try the substitution $v_y = \dfrac{\mathrm{d}y}{\mathrm{d}t}$

Separate the variables...

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{F(v_y)}{m} \qquad \Longrightarrow \qquad m\frac{\mathrm{d}v_y}{F(v_y)} = \mathrm{d}t$$

...and integrate

$$\int_{v_{y_0}}^{v_y(t)} \frac{m}{F(v_y)}\mathrm{d}v_y = \int_0^t \mathrm{d}t$$

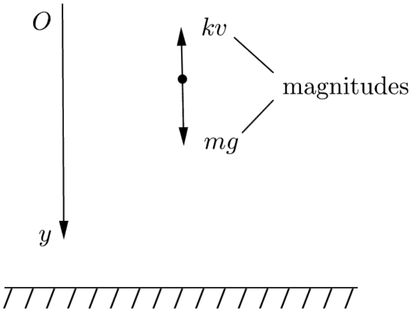
$$\boxed{\int\limits_{v_{y0}}^{v_y(t)} \frac{m}{F(v_y)} \mathrm{d}v_y = t} \quad \stackrel{\text{yields}}{\Longrightarrow} \quad v_y(t) = \ldots$$

Once the velocity as a function of time is known, both the acceleration and the position as functions of time can be found as well

$$\text{acceleration} \quad a_y = \frac{\mathrm{d}v_y}{\mathrm{d}t}$$

$$\text{position} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = v_y(t) \qquad \Rightarrow \qquad y(t) = \int\limits_0^t v_y(t)\mathrm{d}t + y_0$$

Example: Fall with Linear Drag



Initial conditions

$$\begin{cases} y(0) = 0 \\ v_y(0) = 0 \end{cases}$$

Newton's second law (equation of motion)

$$ma_y = mg - kv_y \quad \text{(net force)} \qquad \Longrightarrow \qquad a_y = g - \frac{k}{m} v_y$$

But $a_y = \dfrac{\mathrm{d}v_y}{\mathrm{d}t}$, so that

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = g - \frac{k}{m} v_y \qquad \Longrightarrow \qquad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -\frac{k}{m}\left(v_y - \frac{mg}{k}\right).$$

Separating the variables v_y and t,

$$\frac{\mathrm{d}v_y}{v_y - \frac{mg}{k}} = -\frac{k}{m}\mathrm{d}t$$

and integrating with the given initial conditions

$$\int_0^{v_y(t)} \frac{\mathrm{d}v_y}{v_y - \frac{mg}{k}} = -\frac{k}{m}\int_0^t \mathrm{d}t \qquad \Longrightarrow \qquad \ln\left|\frac{v_y(t) - \frac{mg}{k}}{-\frac{mg}{k}}\right| = -\frac{k}{m}t.$$

But $v_y(t) < \frac{mg}{k}$ (terminal speed), so that $\ln \frac{\frac{mg}{k} - v_y(t)}{\frac{mg}{k}} = -\frac{k}{m}t$

or, equivalently,

$$\frac{mg}{k} - v_y(t) = \frac{mg}{k}e^{-\frac{k}{m}t}.$$

Solving for $v_y(t)$

$$\boxed{v_y(t) = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right).}$$

Discussion

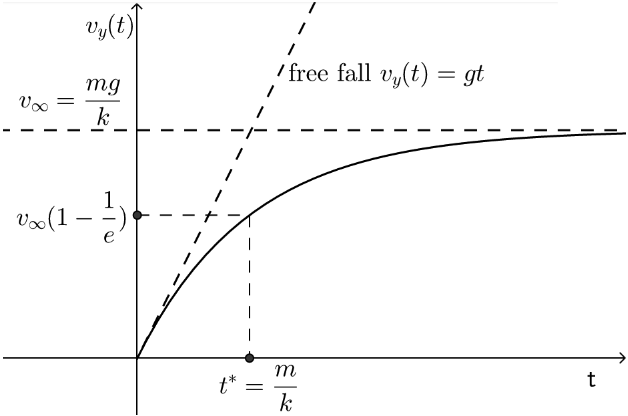
1° For short times, i.e. , $t \ll \frac{m}{k}$, then $\frac{k}{m}t \ll 1$, we can expand the exponential function in the Taylor (Maclaurin) series $e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \ldots$ and approximate by keeping the first two terms in the expansion

$$v_y(t) = \frac{mg}{k}(1 - e^{-\frac{k}{m}t}) \approx \frac{mg}{k}(1 - 1 + \frac{k}{m}t) = gt$$

$$v_y(t) \approx gt \qquad \text{constant acceleration}$$

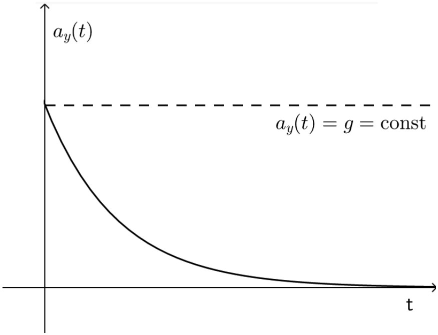
2° Long times, $t \gg \frac{m}{k}$, i.e. $t \to \infty$

$$v_\infty = \lim_{t\to\infty} \frac{mg}{k}(1 - e^{-\frac{k}{m}t}) = \frac{mg}{k} = \text{ const} \qquad \text{(constant velocity)}$$


$v_y(t)$
free fall $v_y(t) = gt$
$v_\infty = \frac{mg}{k}$
$v_\infty(1 - \frac{1}{e})$
$t^* = \frac{m}{k}$
t

Acceleration

$$\boxed{a_y(t)} = \frac{\mathrm{d}v_y(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)\right] = \boxed{ge^{-\frac{k}{m}t}}$$



Position

$$v_y(t) = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

$$\int_0^{y(t)} \mathrm{d}y = \frac{mg}{k}\int_0^t \left(1 - e^{-\frac{k}{m}t}\right)\mathrm{d}t$$

$$y(t) = \frac{mg}{k}\left(t - \left(-\frac{m}{k}\right)e^{-\frac{k}{m}t}\right)\Bigg|_0^t = \frac{mg}{k}\left(t + \frac{m}{k}e^{-\frac{k}{m}t} - \frac{m}{k}\right)$$

$$\Longrightarrow \quad \boxed{y(t) = \frac{mg}{k}\left[t + \frac{m}{k}\left(e^{-\frac{k}{m}t} - 1\right)\right]}$$

Discussion

1° Short times (now, add one more term in the Taylor polynomial)

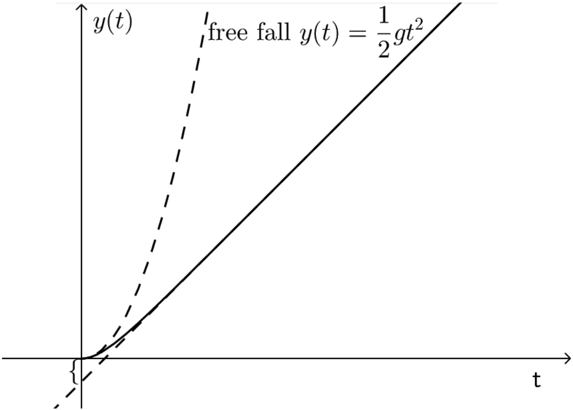
$$y(t) \approx \frac{mg}{k}\left[t + \frac{m}{k}\left(1 - \frac{k}{m}t + \frac{1}{2}\left(\frac{k}{m}t\right)^2 - 1\right)\right]$$

$$= \frac{mg}{k}\left[t - t + \frac{m}{k}\frac{1}{2}\left(\frac{k}{m}t\right)^2\right] = \frac{1}{2}gt^2$$

$$\boxed{y(t) \approx \frac{1}{2}gt^2}$$

2° Long times ($t \gg \frac{m}{k}$)

$$e^{-\frac{k}{m}t} \approx 0 \qquad \text{and} \qquad \boxed{y(t) \approx \frac{mg}{k}\left(t - \frac{m}{k}\right)}$$



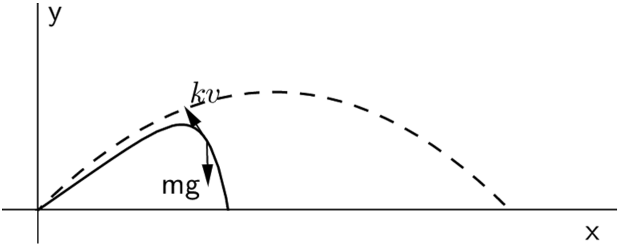
$y(t)$
free fall $y(t) = \frac{1}{2}gt^2$
t

Example: Projectile motion with linear air drag

Drag force $\overline{F}_{\text{drag}} = -k\overline{v} = -kv_x\hat{n}_x - kv_y\hat{n}_y$
Equation of motion $m\overline{a} = \overline{F}_{\text{grav}} + \overline{F}_{\text{drag}}$ and initial conditions

$$\begin{cases} ma_x = -kv_x \\ ma_y = -mg - kv_y \end{cases} \qquad \begin{cases} \overline{r}(0) = 0 \\ \overline{v}(0) = \overline{v}_0 \end{cases}$$

Solution strategy: same as before (equations for both components can be solved independently)



Effects of air drag

* reduces the maximum height
* shortens the range