Chapter 6 – Harmonic Oscillator and Mechanical Resonance

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Agenda

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   - Uniform Circular Motion
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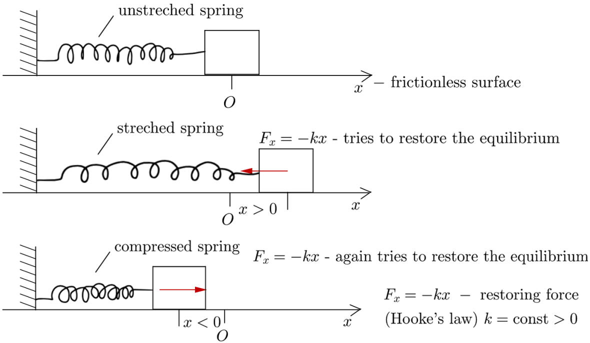
   - Steady-state Oscillations: Amplitude and Phase-Lag
   - Mechanical Resonance in Practice. Demonstrations

Motivation

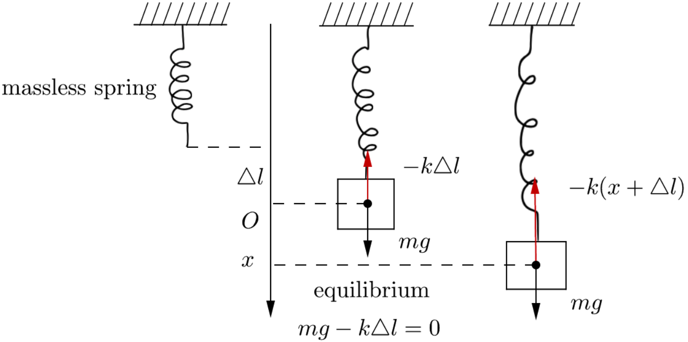
Example I: Horizontal Mass–Spring System



Equation of motion (net force = elastic force)

$$ma_x = F_x \qquad \Longrightarrow \qquad \boxed{\ddot{x} + \frac{k}{m}x = 0}$$

Example II. Vertical Mass–Spring System

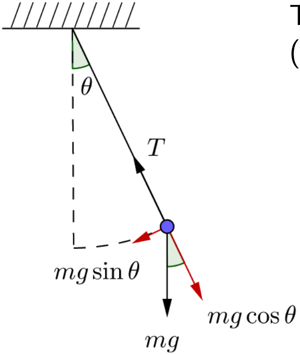


Equation of motion (net force = weight + elastic force)

$$ma_x = mg - k(x + \triangle l) \quad \Longrightarrow \quad ma_x = \underbrace{mg - k\triangle l}_{=0} - kx$$

$$ma_x = -kx \quad \Longrightarrow \quad \boxed{\ddot{x} + \frac{k}{m}x = 0}$$

Example III. Simple Pendulum



Tangential component on the net force (non-zero contribution only due to weight)

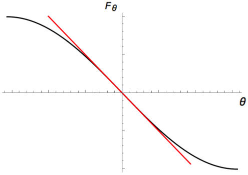
$$F_\theta = -mg\sin\theta \overset{\text{small angles only}}{\approx} -mg\theta$$

Because

$$\sin\theta = \underbrace{\theta}_{\text{keep the 1}^{\text{st}}\text{term}} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots$$

When is this approximation valid?

E.g., for $\theta = 0.1 \quad (6^\circ)$, $\quad \sin\theta = 0.0998$



Example III. Simple Pendulum

Motion along the arc (tangential components)

$$ma_\theta = F_\theta \qquad \stackrel{a_\theta = l\ddot{\theta}}{\Longleftrightarrow} \qquad ml\ddot{\theta} \approx -mg\theta$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$$

Observation:

In all three cases, the equation of motion is of the same form

$$\boxed{\ddot{x} + \omega_0^2 x = 0}$$

[with $\omega_0^2 = k/m$ for the mass–spring systems and $\omega_0^2 = g/l$ for the simple pendulum]

Any particle (system) with the equation of motion of the above form is called a **simple harmonic oscillator** (SHO).

Simple Harmonic Oscillator

Simple Harmonic Oscillator

Simple Harmonic Oscillator – only the restoring force acts.

$$\ddot{x} + \omega_0^2 x = 0 \qquad \Longrightarrow \qquad \boxed{x(t) = ?}$$

How to solve? Guess a solution and check...

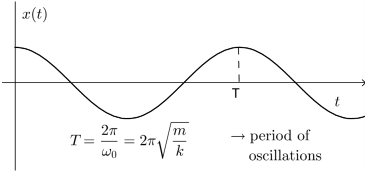
Our guess

$$x(t) = \cos(\omega_0 t)$$

Check

$$\dot{x} = -\omega_0 \sin(\omega_0 t) \qquad \text{and} \qquad \ddot{x} = -\omega_0^2 \cos(\omega_0 t) = -\omega_0^2 x$$

Periodic behavior (oscillations)



Have we found the most general form of the solution?

1.

$$\begin{aligned} x(t) &= A\cos\omega_0 t \\ \dot{x}(t) &= -\omega_0 A\sin\omega_0 t \\ \ddot{x}(t) &= -\cos^2 A\cos\omega_0 t = -\omega_0^2 x(t) \quad \to \text{also solves the equation} \end{aligned}$$

2.

$$\begin{aligned} x(t) &= A\cos(\omega_0 t + \phi) \\ \dot{x}(t) &= -\omega_0 A\sin(\omega_0 t + \phi) \\ \ddot{x}(t) &= -\cos^2 A\cos(\omega_0 t + \phi) = -\omega_0^2 x(t) \quad \to \text{solves it, too!} \end{aligned}$$

The most general solution

$$\boxed{x(t) = \underset{\text{amplitude}}{A}\cos(\,\omega_0 t + \underset{\text{initial phase}}{\phi})}$$

Equivalently (see Problem Set 4) the most general solution can be written as

$$x(t) = B \cos \omega_0 t + C \sin \omega_0 t$$

Note: We have two constants (now, B and C) again.

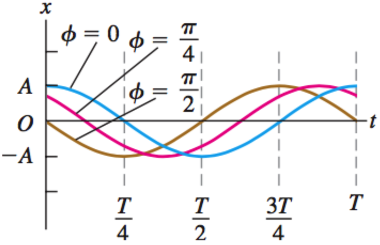
The constants A and ϕ (or B and C) are found by applying the initial conditions:

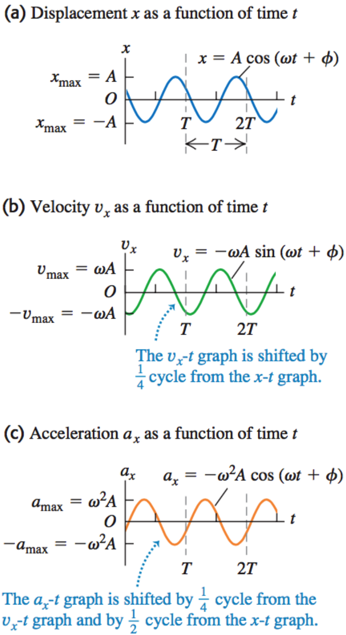
$$\begin{cases} x(0) = x_0 \\ v_x(0) = v_{0x} \end{cases}$$

then the problem has a unique solution.

Position, Velocity, and Acceleration in SHM

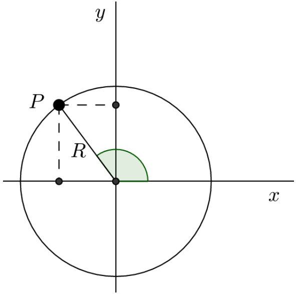
These three curves show SHM with the same period T and amplitude A but with different phase angles ϕ.





Uniform Circular Motion and Simple Harmonic Motion

Uniform Circular Motion and Simple Harmonic Motion



$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega_0 = \frac{v}{R} = \text{const}$$

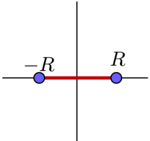
$$\Longrightarrow \quad \varphi = \omega_0 t \quad [\text{assume } \varphi(0) = 0]$$

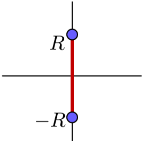
$$x = R\cos \overbrace{\omega_0 t}^{\varphi}, \qquad y = R\sin \overbrace{\omega_0 t}^{\varphi}$$

Differentiate twice w.r.t. time

$$\begin{cases} a_x = -R\omega^2 \cos\omega_0 t = -\omega_0^2 x \\ a_y = -\omega_0^2 y \end{cases}$$

Conclusion: The projection of P onto the x axis (or the y axis) moves as if it was in a simple harmonic motion.





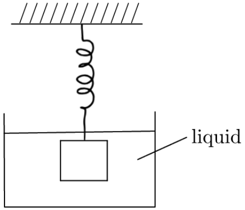
Damped Oscillations

Damped Oscillations

Oscillating object placed in a liquid – add a linear drag to the model ($b = const > 0$)

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -b\frac{\mathrm{d}x}{\mathrm{d}t} - kx$$

$$\boxed{\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 x = 0}$$



How to solve? *(not easy to guess directly...)*

Try $x(t) = e^{\lambda t}$ (will need to find λ). Then

$$\begin{aligned} \dot{x} &= \lambda e^{\lambda t} = \lambda x \\ \ddot{x} &= \lambda^2 e^{\lambda t} = \lambda^2 x \end{aligned}$$

Plugging back into the equation of motion

$$\lambda^2 x + \frac{b}{m}\lambda x + \omega_0^2 x = 0 \qquad \Longrightarrow \qquad \lambda^2 + \frac{b}{m}\lambda + \omega_0^2 = 0$$

Observation: A differential equation turned into an algebraic (quadratic) one. Easy to solve!

Solution (roots) of the algebraic equation depends on the sign of

$$\Delta = \left(\frac{b}{m}\right)^2 - 4\omega_0^2$$

1. $\Delta < 0$; complex roots $\lambda_{1,2} = -\frac{b}{2m} \pm i\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$
2. $\Delta > 0$; real & different roots $\lambda_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}$
3. $\Delta = 0$; real root (repeated) $\lambda = -\frac{b}{2m}$

Need to analyze these three cases.

Damped Oscillations: Case I (Underdamped Regime)

$$\Delta < 0 \quad \Longrightarrow \quad \left(\frac{b}{m}\right)^2 - 4\omega_0^2 < 0 \quad \Longrightarrow \quad \boxed{\left(\frac{b}{m}\right)^2 < 4\omega_0^2}$$

It is the case of weak damping (**underdamped regime**). Then, the general solution

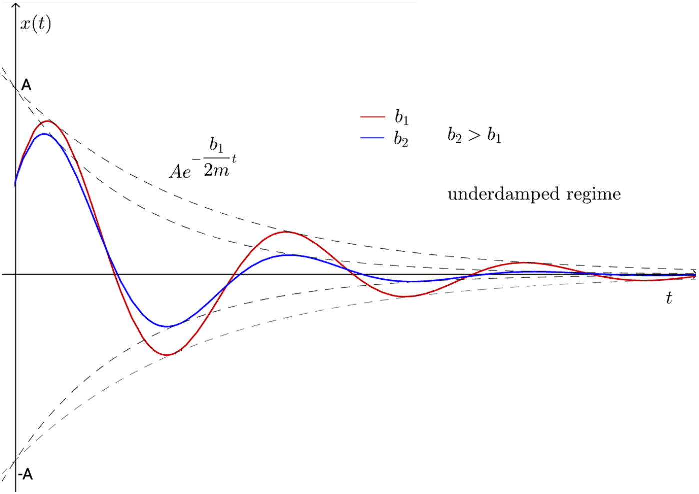
$$\begin{aligned} x(t) &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = \\ &= C_1 e^{-\frac{b}{2m}t} e^{i\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\,t} + C_2 e^{-\frac{b}{2m}t} e^{-i\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\,t} \end{aligned}$$

But $x(t)$ is a physical quantity (displacement from the equilibrium position); must be real, so

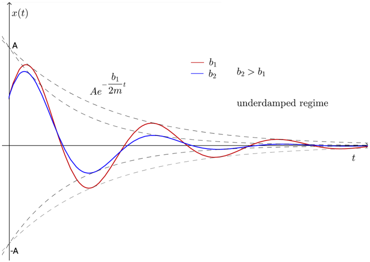
$$C_1 = \frac{1}{2} A e^{i\phi_0} = C_2^* \quad \Longrightarrow \quad C_2 = \frac{1}{2} A e^{-i\phi_0}$$

Hence (recall the formula: $e^{iu} = \cos u + i \sin u$)...

$$x(t) = Ae^{-\frac{b}{2m}t}\cos\left(\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\,t + \phi_0\right)$$



Effects of weak damping (underdamped regime)



* Motion still periodic, but the amplitude of oscillations decreases exponentially with time.
* The angular frequency of oscillations $\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} < \omega_0$, i.e. it is smaller than that for the undamped case (the natural angular frequency ω_0). Consequently, the period $T = 2\pi/\omega$ increases.

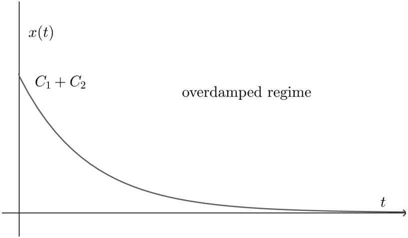
Case II. Overdamped Regime

Now, $\Delta > 0$ so that $\boxed{\left(\frac{b}{m}\right)^2 > 4\omega_0^2}$. It is the case of strong damping (**overdamped regime**)and the general solution

$$
\begin{aligned}
x(t) &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = \\
&= \boxed{C_1 e^{-\left(\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}\right) t} + C_2 e^{-\left(\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}\right) t}}
\end{aligned}
$$

Effects of strong damping (overdamped regime)

- No periodic behavior.
- Strong damping results in aperiodic motion: the particle returns aperiodically to the equilibrium position.



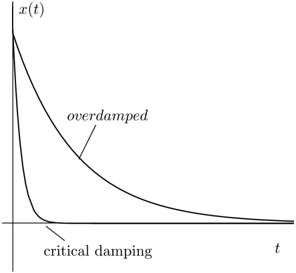
Case III. Critical Damping

Finally, is $\Delta = 0$ so that $\boxed{\left(\dfrac{b}{m}\right)^2 = 4\omega_0^2}$. In this case damping is called **critical**and the general solution[1]

$$\boxed{x(t) = D_1 e^{-\dfrac{b}{2m}t} + D_2 t e^{-\dfrac{b}{2m}t}}$$

Effects of critical damping

* No periodic behavior.
* The system may pass through the equilibrium position at most once (see Problem Set).



[1]** The second term includes the factor t in order to make it linearly independent from $e^{-\frac{b}{2m}t}$.

Forced (or Driven) Oscillations. Mechanical Resonance

Forced (or Driven) Oscillations

Now: restoring force + linear drag + **driving force** $F_{\rm dr}$

Simplest case to analyze ($\omega_{\rm dr}$ – driving frequency):

$$F_{\rm dr} = F_0 \cos \omega_{\rm dr} t \qquad \text{(sinusoidal time-dependence)}$$

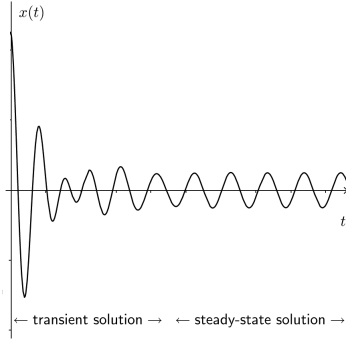
Equation of motion

$$ma_x = F_x = -kx - bv_x + \boxed{F_0 \cos \omega_{\rm dr} t}$$

$$\boxed{\ddot{x} + \frac{b}{m}\dot{x} + \underbrace{\frac{k}{m}}_{\omega_0^2} x = \frac{F_0}{m} \cos \omega_{\rm dr} t}$$

Observation

After some time the oscillations stabilize and the particle oscillates with the angular frequency of the driving force (there may be a shift in phase between the drive and the response though).



In general, the solution to the equation of motion in this case is of the form

$$x(t) \;=\; \underbrace{\boxed{\text{solution to the equation of motion for damped oscillator}}}_{\text{(vanishes as } t \to \infty)}$$

$$+\underbrace{\boxed{\text{periodic steady-state oscillations with angular frequency } \omega_{\mathsf{dr}}}}_{x_{\mathsf{s}}(t)}$$

Steady-State Solution

The steady-state solution

$$\boxed{x_s(t) = A\cos\left(\omega_{dr} t + \phi\right)}$$

ϕ is assumed to be negative, so that it has the interpretation of a *phase-lag*.

Detailed calculations (omitted here[2]) show that

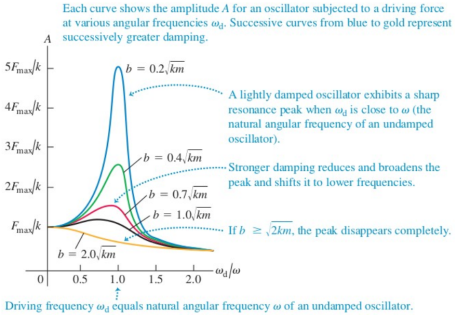
$$\boxed{\begin{aligned} A(\omega_{dr}) &= \frac{F_0}{m\sqrt{(\omega_0^2 - \omega_{dr}^2)^2 + \left(\frac{b\omega_{dr}}{m}\right)^2}} \\ \tan\phi &= \frac{b\omega_{dr}}{m(\omega_{dr}^2 - \omega_0^2)} \end{aligned}}$$

Observations. (1) A is a function of ω_{dr}. (2) In general, $\phi \neq 0$, hence F_{dr} and x_s **are not** in phase.

[2] You are welcome to take Honors Physics next time!

Discussion: Amplitude. Mechanical Resonance

$$A = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega_{dr}^2)^2 + \left(\frac{b\omega_{dr}}{m}\right)^2}}$$

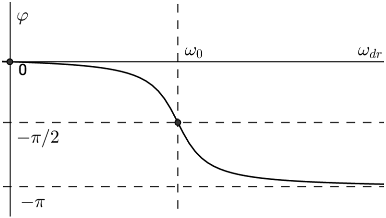


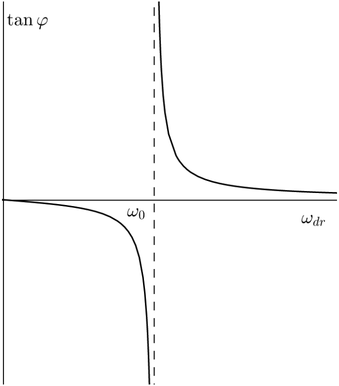
Features

- A peak in the curve $A = A(\omega_{dr})$ at the **resonance frequency**, i.e. for $\omega_{dr} = \omega_{res} = \sqrt{\omega_0^2 - b^2/2m^2}$ [see Problem Set]. The sharp increase in the amplitude of oscillations when $\omega_{dr} \to \omega_{res}$ is called the **mechanical resonance**.
- Increasing damping shifts the resonance frequency downwards.
- For $\omega_{dr} \to 0$ (i.e., $T_{dr} \to \infty$; constant force), then $A \to \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$.

Discussion: Phase-Shift

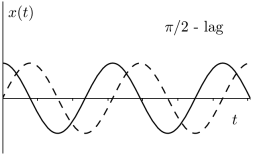
$$\boxed{\tan\phi = \frac{b\omega_{\mathrm{dr}}}{m(\omega_{\mathrm{dr}}^2 - \omega_0^2)}}$$





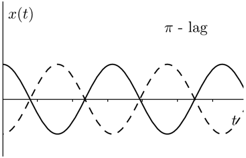
Phase-Shift

- If $\omega_{\mathsf{dr}} \to \omega_0$ (close to resonance[3]), then $\phi \to -\pi/2$.



The response ($x_s(t)$) lags the drive ($F_{dr}(t)$) by 1/4 of the cycle.

- If $\omega_{\mathsf{dr}} \to \infty$ (high frequencies), then $\phi \to -\pi$ the response lags the drive by 1/2 of the cycle (displacement and drive are in antiphase)



The response ($x_s(t)$) lags the drive ($F_{dr}(t)$) by 1/2 of the cycle.

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[3] Small damping is assumed here, so that $\omega_{\mathsf{res}} \approx \omega_0$.

Mechanical Resonance in Practice. Demonstrations

[videos]