Chapter 18 – Mechanical Waves

UM-SJTU Joint Institute
Physics I (Summer 2021)
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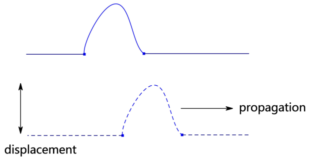
Agenda

Introduction

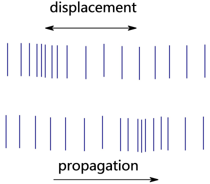
What is a Mechanical Wave?

Wave — "disturbance" of a medium propagating through space. Consequently, mechanical waves need a medium to propagate.

- **Transverse waves** — the direction of displacement of medium particles is perpendicular to the direction of wave propagation. Example: wave on a rope [animation].



- **Longitudinal waves** — the direction of displacement is parallel to the direction of propagation. Example: sound.



Sinusoidal (Harmonic) Waves

Harmonic Waves

A propagating mechanical wave in the shape of a cosine (or sine) function is called a *harmonic* wave. The disturbance of the medium, denoted by ξ, is

$$\boxed{\xi(x,t) = \xi_0 \cos\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right)}$$

where

- λ is the *wavelength*,
- and T is the *period*.

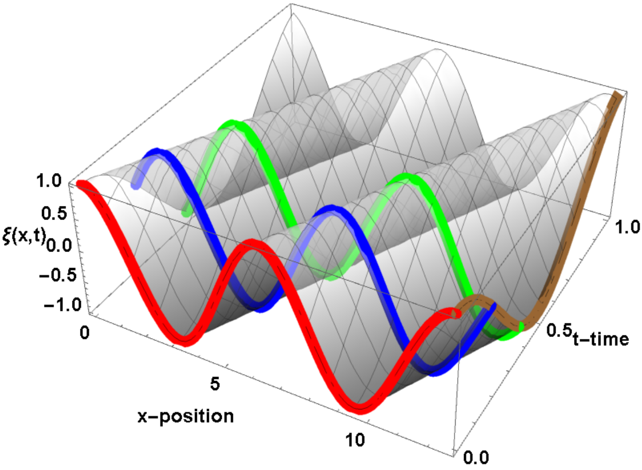
Or, equivalently,

$$\boxed{\xi(x,t) = \xi_0 \cos(kx - \omega t),}$$

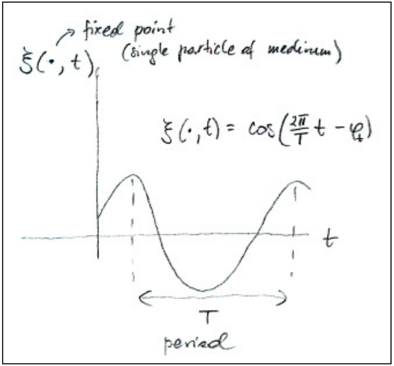
with

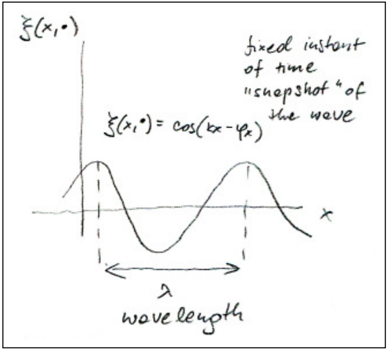
- $k = \frac{2\pi}{\lambda}$ — *wave number*,
- $\omega = \frac{2\pi}{T}$ — *angular frequency*.

$$\boxed{\xi(x,t) = \xi_0 \cos(kx - \omega t)}$$



Interpretation of Period T and Wavelength λ





Particles of the medium move in simple harmonic motion about the equilibrium position $\xi = 0$.

Phase Speed

Consider a propagating harmonic wave

$$\xi(x,t) = \xi_0 \cos(kx - \omega t).$$

Question: How fast does the wave propagate?

Look at a point with a fixed phase (the *wave front*)

$$kx - \omega t = \theta_0,$$

where θ_0 is a constant. Differentiating both sides with respect to time yields

$$k\dot{x} - \omega = 0 \implies \dot{x} = \frac{\omega}{k}$$

Hence the speed the wave propagates with (the *phase speed*) is

$$\boxed{v_{ph} = \frac{\omega}{k}.}$$

The Classical Wave Equation in 1D*

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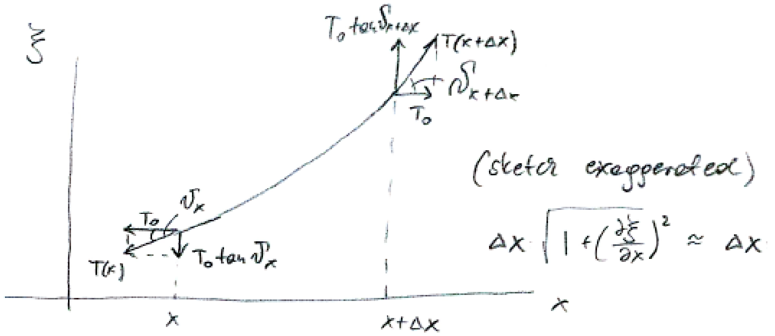


Consider a wave on a string with linear density of mass ϱ. The equation of motion in the vertical direction for the element of mass $\varrho\,\Delta x$ is

$$\varrho\,\Delta x\,\frac{\partial^2\xi}{\partial t^2} = T_0 \tan\vartheta_{x+\Delta x} - T_0 \tan\vartheta_x .$$

Note that $\tan\vartheta = \frac{\partial\xi}{\partial x}$.

Hence

$$\varrho\,\Delta x\,\frac{\partial^2\xi}{\partial t^2} = T_0\left(\left.\frac{\partial\xi}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial\xi}{\partial x}\right|_{x}\right) = T_0\,\frac{\partial^2\xi}{\partial x^2}\,\Delta x,$$

and, eventually,

$$\boxed{\frac{\partial^2\xi}{\partial x^2} - \frac{\varrho}{T_0}\frac{\partial^2\xi}{\partial t^2} = 0}$$

where $\dfrac{\varrho}{T_0} = \dfrac{1}{v_{ph}^2}$. This is the **classical wave equation** (in 1D).

Comments

- Harmonic waves $\xi(x,t) = \xi_0 \cos(kx - \omega t)$ satisfy the wave equation.

$$\frac{\partial \xi}{\partial x} = -k\xi_0 \sin(kx - \omega t); \quad \frac{\partial^2 \xi}{\partial x^2} = -k^2 \xi_0 \cos(kx - \omega t) = -k^2 \xi$$
$$\frac{\partial \xi}{\partial t} = \omega\xi_0 \sin(kx - \omega t); \quad \frac{\partial^2 \xi}{\partial t^2} = -k^2 \xi_0 \cos(kx - \omega t) = -\omega^2 \xi$$

  Check: $\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{v_{ph}^2}\frac{\partial^2 \xi}{\partial t^2} = -k^2\xi - \frac{1}{(\omega/k)^2}(-\omega)\xi \equiv 0$

- The wave equation is linear $\implies$ the superposition principle holds:
  If ξ_1 and ξ_2 are solutions of the same wave equation, then any linear combination $\alpha\xi_1 + \beta\xi_2$ is also a solution.

Note that linearity of the derivatives implies:

$$\begin{aligned}\frac{\partial^2(\alpha\xi_1+\beta\xi_2)}{\partial x^2} \quad - \quad & \frac{1}{v_{vp}^2}\frac{\partial^2(\alpha\xi_1+\beta\xi_2)}{\partial t^2} = \\ = \quad & \alpha\left(\frac{\partial^2\xi_1}{\partial x^2}-\frac{1}{v_{ph}^2}\frac{\partial^2\xi_1}{\partial t^2}\right)+\beta\left(\frac{\partial^2\xi_2}{\partial x^2}-\frac{1}{v_{ph}^2}\frac{\partial^2\xi_2}{\partial t^2}\right)\end{aligned}$$

- The shape of the wave impulse needs not to be sinusoidal. Suppose that $\xi(x,t) = f(x - vt)$, where $f(u)$ is any twice-differentiable function. We have

$$\frac{\partial\xi}{\partial x} = f'; \qquad \frac{\partial^2\xi}{\partial x^2} = f''$$
$$\frac{\partial\xi}{\partial t} = -vf'; \qquad \frac{\partial^2\xi}{\partial t^2} = v^2 f''$$

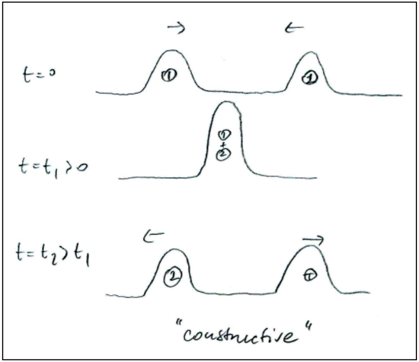
And

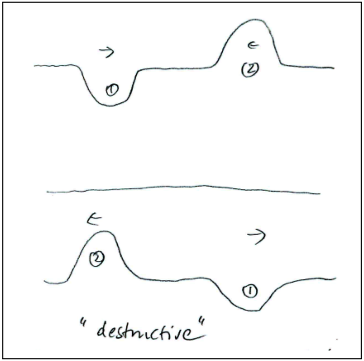
$$\frac{\partial^2\xi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2} = 0.$$

Interference of Waves. Standing Waves

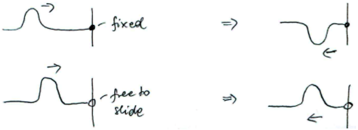
Interference

Idea: Two wave impulses propagate in space. The resultant wave $\xi(x,t) = \xi_1(x,t) + \xi_2(x,t)$ The interference may be *constructive* or *destructive*





Digression: Reflection of a pulse wave on a rope.



Different boundary conditions!

Animation (click)

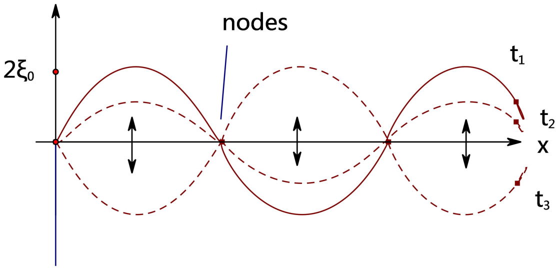
Example: Standing Waves

Suppose: two sinusoidal waves with the same wavelength propagating in opposite directions with the same speed.

$$\xi_1(x,t) = -\xi_0 \cos(kx + \omega t), \qquad \xi_2(x,t) = \xi_0 \cos(kx - \omega t).$$

Superposition

$$\begin{aligned}
\xi(x,t) = \xi_1(x,t) + \xi_2(x,t) &= \xi_0[-\cos(kx+\omega t) + \cos(kx - \omega t)] \\
&= -2\xi_0 \sin\frac{kx - \omega t + kx + \omega t}{2} \sin\frac{kx - \omega t - kx - \omega t}{2} \\
&= 2\xi_0 \sin(kx)\sin(\omega t)
\end{aligned}$$

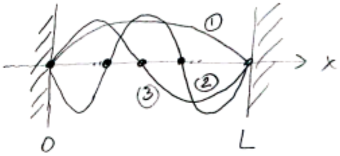


$$\boxed{\xi(x,t) = 2\xi_0 \sin(kx)\sin(\omega t)}$$

Positions of the nodes are fixed at $x_{\text{node}} = \frac{n\pi}{k} = \lambda\frac{n}{2}$, with $n = 0, 1, 2, \ldots$

Example. Standing wave on a string of length L clamped at both ends. What are the possible wavelengths?

Boundary conditions (for all t): $\xi(0,t) = \xi(L,t) = 0$



Possible wavelengths: $L = n\,\frac{\lambda}{2}$ (length of the string accommodates multiples of $\lambda/2$).

$$\boxed{\lambda = \frac{2L}{n} = \lambda_n}$$

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- 1st harmonic: $\lambda_1 = 2L$
- 2nd harmonic: $\lambda_2 = L$
- 3rd harmonic: $\lambda_3 = \frac{2}{3}L$
- ...