Chapter 8 – Work and Kinetic Energy

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Physics I (Summer 2021)
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Agenda

1. Motivation
2. Work
   - How to Calculate Work in Various Situations?
   - Example: Work Done on/by a Spring
   - Examples: Work Along a Curved Path
3. Work and Motion
   - Kinetic Energy and Work-Kinetic Energy Theorem
   - Power

Motivation

Motivation

Mechanics

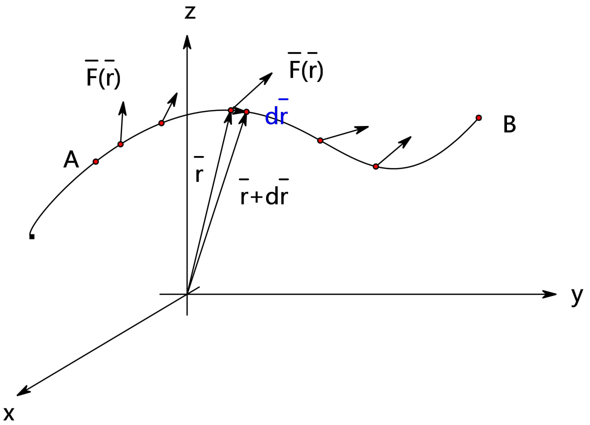
- Kinematics $(\bar{r}, \bar{v}, \bar{a})$ *HOW?*
- Dynamics $(\bar{F}, \bar{a} = \frac{\bar{F}}{m})$ *WHY?*

New scalar quantities

- Concept of *work*: How to measure the "effort" put into moving the particle from one place to another?
- *Kinetic energy* and *work-kinetic energy theorem*: How does this "effort" change the state of the particle (e.g. its speed)?

Work

Work



In general,

$$\bar{F} = \bar{F}(\bar{r})$$

(position-dependent force; vector field)

Elementary work done by $\bar{F}$ when the particle moves from $\bar{r}$ to $\bar{r} + d\bar{r}$

$$\boxed{\delta W = \bar{F} \circ d\bar{r}}$$

Work

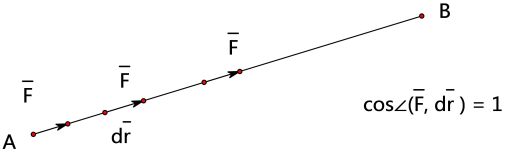
Total work done by $\bar{F}$ when the particle moves from A to B along the curve Γ_{AB} — add all infinitesimal contributions

$$\boxed{W_{AB} = \int_{\Gamma_{AB}} \bar{F} \circ d\bar{r}}$$

How to calculate work in various situations?

(A) Constant force in the direction of the displacement; displacement along a straight line

$$\delta W = \bar{F} \circ d\bar{r} = |\bar{F}| \cdot |d\bar{r}|$$



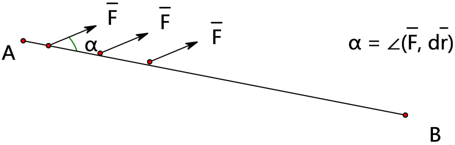
$$W_{AB} = \int_{\Gamma_{AB}} \bar{F} \circ d\bar{r} = \int_{\Gamma_{AB}} |\bar{F}||d\bar{r}| = |\bar{F}| \underbrace{\int_{\Gamma_{AB}} |d\bar{r}|}_{S_{AB}} = |\bar{F}|\, S_{AB},$$

where S_{AB} is the length of AB. Hence, in this case

$$\boxed{W_{AB} = |\bar{F}|\, S_{AB}}$$

(B) Constant force acting at an angle to the direction of straight-line displacement

$$\delta W = \bar{F} \circ d\bar{r} = |\bar{F}| \cdot |d\bar{r}| \cos\alpha$$



α = ∠(F, dr) smaller angle

$$W_{AB} = \int_{\Gamma_{AB}} \bar{F} \circ d\bar{r} = \int_{\Gamma_{AB}} |\bar{F}| \cdot |d\bar{r}| \cos\alpha = |\bar{F}| \cos\alpha \underbrace{\int_{\Gamma_{AB}} |d\bar{r}|}_{s_{AB}}$$

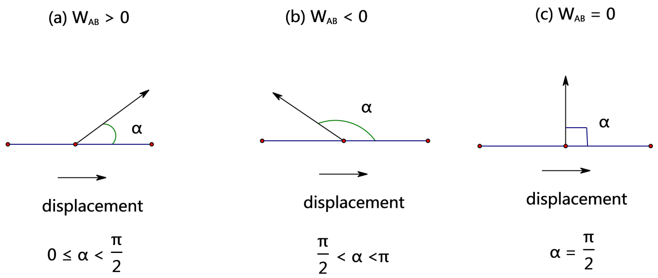
$$= |\bar{F}| S_{AB} \cos\alpha$$

Hence, in this case,

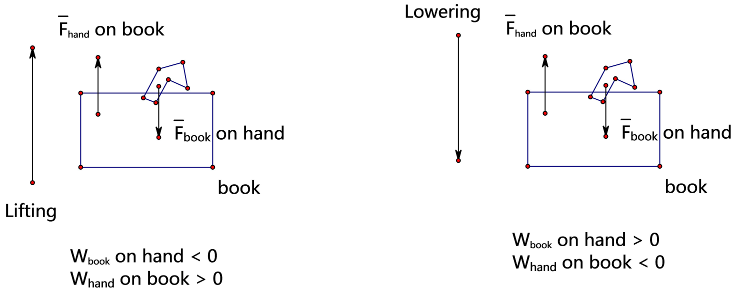
$$\boxed{W_{AB} = |\bar{F}|\, S_{AB} \cos\alpha}$$

Conclusion: Work may either positive or negative or zero.

Illustration. Positive/negative work.



Remember what does work on what!



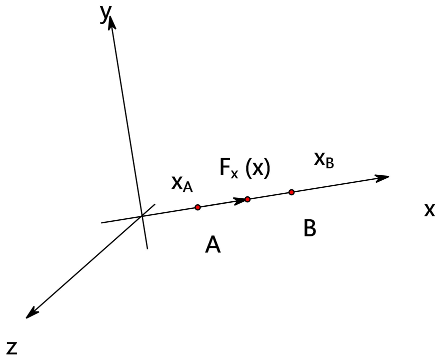
Remark. Elementary work done in the case when multiple forces act on a particle

$$\delta W = (\bar{F}_1 + \bar{F}_2 + ... + \bar{F}_N) \circ d\bar{r} = \bar{F}_1 \circ d\bar{r} + \bar{F}_2 \circ d\bar{r} + ... + \bar{F}_N \circ d\bar{r}$$
$$= \delta W_1 + \delta W_2 + ... + \delta W_N$$

or, equivalently,

$$\delta W = \bar{F}_{net} \circ d\bar{r}$$

(C) Force of varying magnitude acting along a straight-line path



$$\bar{F} = (\underbrace{F_x(x)}_{\text{function of x only}}, 0, 0)$$

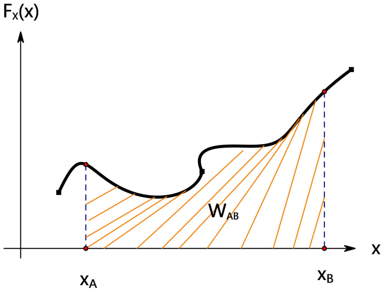
and

$$d\bar{r} = (dx, 0, 0)$$

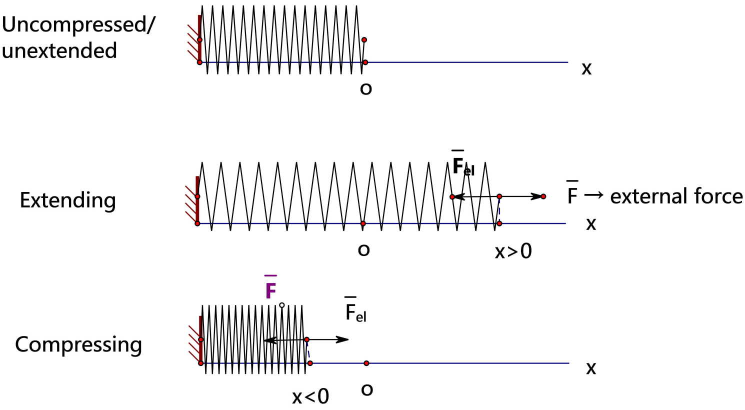
$$\delta W = \bar{F} \circ d\bar{r} = F_x(x)\, dx \quad \text{(takes care of the sign)}$$

$$W_{AB} = \int_{\Gamma_{AB}} \delta W = \int_{\Gamma_{AB}} F_x(x)\, dx = \int_{x_A}^{x_B} F_x(x)\, dx$$

Interpretation (area under the curve)



Example: Work Done on/by a Spring

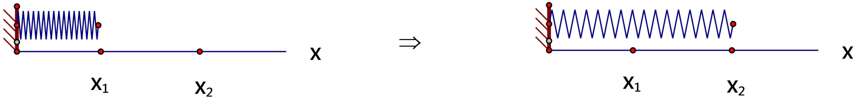


$$\underbrace{\bar{F}_{el}}_{\text{elastic (restoring) force}} = (-kx, 0, 0)$$

$$\bar{F} = (kx, 0, 0)$$

$$d\bar{r} = (dx, 0, 0)$$

If the position of the right (free) end of the spring changes from x_1 to x_2



Work done by

- the external force (e.g. hand) on the spring

$$W^{\text{ext}}_{1\to 2} = \int_{x_1}^{x_2} kx\, dx = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

- the elastic force (spring) on the hand

$$W^{\text{el}}_{1\to 2} = \int_{x_1}^{x_2} -kx\, dx = -\left(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2\right) = -W^{\text{ext}}_{1\to 2}$$

Illustration

(*)

$W_{1\to2}^{ext} > 0$
$W_{1\to2}^{el} < 0$

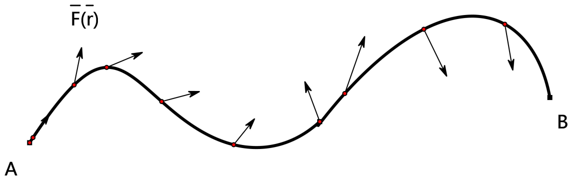
(**)

$W_{1\to2}^{ext} > 0$
$W_{1\to2}^{el} < 0$

(***)

$W_{1\to2}^{ext} < 0$
$W_{1\to2}^{el} > 0$

(D) General case – varying force, curved path



$$\bar{F} = \bar{F}(\bar{r}) \neq \text{const}$$

$$\delta W = \bar{F} \circ d\bar{r}$$

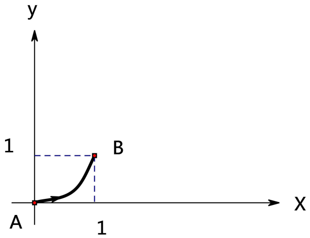
and

$$\boxed{W_{AB} = \int_{\Gamma_{AB}} \bar{F} \circ d\bar{r}}$$

Example (work along a parabola)

Calculate work done by the force $\bar{F}(\bar{r}) = (x^2 + y^2)\hat{n}_x + x\hat{n}_y$ [N], acting on a particle moving from (0,0) to (1,1) along a segment of parabola $y = x^2$.

Example 1 (work along a parabola)



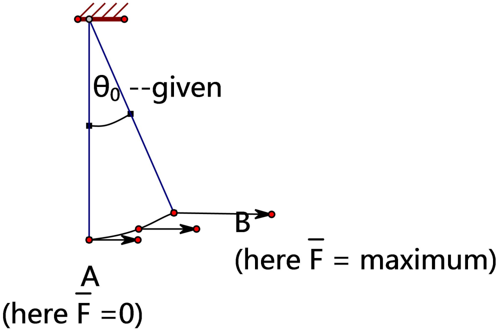
$$\Gamma_{AB} : \begin{cases} 0 \leq x \leq 1 \;\Rightarrow & dx \\ y = x^2 \;\Rightarrow & dy = 2x\,dx \end{cases}$$

Note that for all the points on the parabola $dy = 2x\,dx$.

$$\begin{aligned} W_{AB} &= \int\limits_{\Gamma_{AB}} \bar{F} \circ d\bar{r} = \int\limits_{\Gamma_{AB}} F_x\,dx + F_y\,dy = \int\limits_{\Gamma_{AB}} (x^2 + y^2)\,dx + x\,dy \\ &= \int_0^1 [(x^2 + x^4) + (x \cdot 2x)]\,dx = \frac{1}{3} + \frac{1}{5} + \frac{2}{3} = \frac{18}{15} \quad [\mathrm{J}] \end{aligned}$$

Example 2 (less abstract one; work along a curve)

Find the minimum work needed to move an object of weight Q, suspended on a light rod with length R, from A to B, acting with a horizontal force.



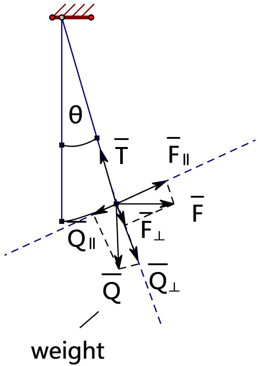
$$W_{A\to B} = \int_{\Gamma_{AB}} \bar{F} \circ d\bar{r}$$

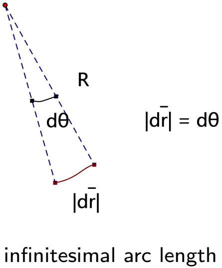
First Method

Idea: Decompose the force

$$\delta W = \bar{F} \circ d\bar{r} = (\bar{F}_{\|} + \bar{F}_{\perp}) \circ d\bar{r} = \underbrace{\bar{F}_{\|}}_{\text{varies}} \circ d\bar{r}$$

and express everything in terms of θ





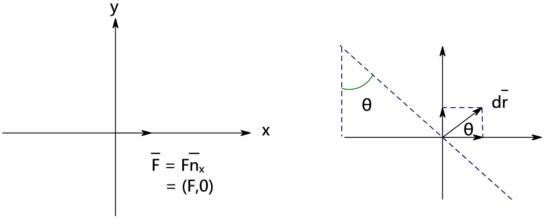
$$\delta W = F_{\|} |d\bar{r}| = Q \sin\theta R \, d\theta$$

Work done by $\bar{F}$ on the path $A \to B$ (arc of a circle with radius R)

$$\begin{aligned}
\boxed{W_{A\to B}} &= \int\limits_{\Gamma_{AB}} \bar{F} \circ d\bar{r} = \int\limits_{0}^{\theta_0} \underbrace{Q \sin\theta}_{F_{\parallel}} R\, d\theta = QR \int\limits_{0}^{\theta_0} \sin\theta\, d\theta \\
&= \boxed{QR\,[1 - \cos\theta_0]}
\end{aligned}$$

Second Method

Idea: Rewrite both the force and the elementary displacement in terms of their Cartesian coordinates.



$$d\bar{r} = |d\bar{r}| \cos\theta \, \hat{n}_x + |d\bar{r}| \sin\theta \, \hat{n}_y = \left(|d\bar{r}| \cos\theta, \; |d\bar{r}| \sin\theta\right)$$

$$\delta W = \bar{F} \circ d\bar{r} = F \cos\theta \underbrace{|d\bar{r}|}_{R\,d\theta} = FR \cos\theta \, d\theta$$

and $F \cos\theta = Q \sin\theta$, so that $\delta W = Q \sin\theta R \, d\theta$. Eventually,

$$W_{A \to B} = \int_0^{\theta_0} Q \sin\theta R \, d\theta = QR \left[1 - \cos\theta_0\right]$$

Work and Motion

Motivation

Observations

- $W > 0 \quad \Longrightarrow \quad$ particle speeds up
- $W < 0 \quad \Longrightarrow \quad$ particle slows down
- $W = 0 \quad \Longrightarrow \quad$ no change in speed

Conclusion. Work done by the net force implies a change in the particle's speed (or another quantity that is a function of speed).

Kinetic Energy and Work-Kinetic Energy Theorem

Recall that $\delta W = \bar{F} \circ d\bar{r}$, so the rate of work being done by the net force on a particle

$$\underbrace{\frac{\delta W}{dt}} = \bar{F} \circ \frac{d\bar{r}}{dt} = \bar{F} \circ \dot{\bar{r}} = m\,\bar{a} \circ \bar{v} = m\,\dot{\bar{v}} \circ \bar{v}$$

But $v^2 = \bar{v} \circ \bar{v}$, so $\frac{d}{dt}(v^2) = \frac{d}{dt}(\bar{v} \circ \bar{v}) = \dot{\bar{v}} \circ \bar{v} + \bar{v} \circ \dot{\bar{v}} = 2\bar{v} \circ \dot{\bar{v}}$

$$\frac{\delta W}{dt} = \frac{d}{dt}\underbrace{\left(\frac{1}{2}mv^2\right)}_{K}$$

where $\boxed{K = \frac{1}{2}mv^2}$ is the **kinetic energy**. Hence,

$$\frac{\delta W}{dt} = \frac{dK}{dt} \qquad \text{or} \qquad \boxed{\delta W = dK}$$

Work–Kinetic Energy Theorem

Work–Kinetic Energy Theorem

Work done by the net force on a particle is equal to the change in the particle's kinetic energy.

For finite changes

$$W = \Delta K$$

$$\begin{aligned} W > 0 \quad &\Longrightarrow \quad \Delta K > 0 \quad \Longrightarrow \Delta v > 0 \\ W < 0 \quad &\Longrightarrow \quad \Delta K < 0 \quad \Longrightarrow \Delta v < 0 \\ W = 0 \quad &\Longrightarrow \quad K = \text{const} \quad \Longrightarrow v = \text{const} \end{aligned}$$

1. Derived in a general case; no assumptions about the nature of the force or the particle's trajectory made
2. Used Newton's second law; can use only in inertial FoRs.
3. Kinetic energy has the units of work $[\mathrm{J} = \mathrm{N} \cdot \mathrm{M} = \mathrm{kg}\frac{\mathrm{m}^2}{\mathrm{s}^2}]$

Power

Power – characterizes "how fast" work is being done

$$\underbrace{\frac{\delta W}{dt}}_{\text{rate of work done}} = \bar{F} \circ \bar{v} = \underbrace{P}_{\text{instantaneous power}}$$

$$\frac{W}{\Delta t} = P_{\text{av}}$$

Units: [W] (Watt)

Alternative units: $\underbrace{\text{hp}}_{\text{horse power}} = 746W = 0.746kW$

Example (*Power of engines in vehicles*): Abrams battle tank 1500 hp, Volvo C30 108hp, Harley-Davidson 1600ccm 63 hp, Porsche 911 GT2 523 hp, Boeing 747-300 (@ 0.84 Mach, 35k feet) 330 000 hp, Diesel locomotive 5000 hp