Elements of Fluid Dynamics

Outline

Introduction

Fluids at rest
- Density and pressure in a liquid
- Pascal's law
- Buoyancy and Archimedes principle

Fluids in motion
- Basic concepts; ideal liquid and continuity equation
- Bernoulli's equation
- Beyond the ideal liquid and laminar flow picture: viscosity and turbulence

Introduction
and
Fluids at Rest

Density of Mass

Density of mass (bulk)

$$\varrho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}$$

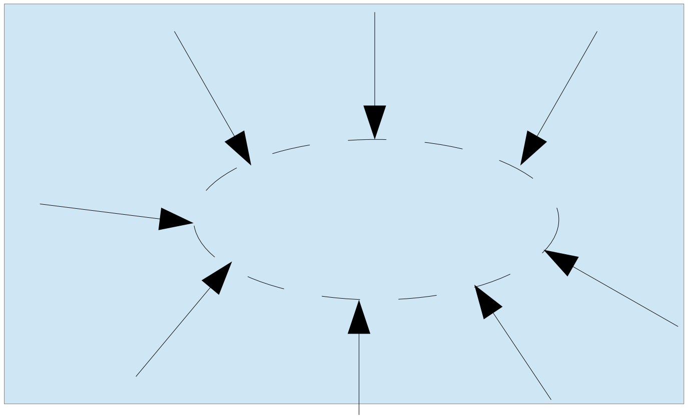
In general, for *inhomogeneous* objects, the density of mass depends on position. If the density is constant throughout an object, then the object is said to be *homogeneous* (mass is distributed uniformly).

Example. The density of air at 20° C is 1.2 kg/m³. What is the mass of the air in our classroom?

Relative density ("specific gravity")

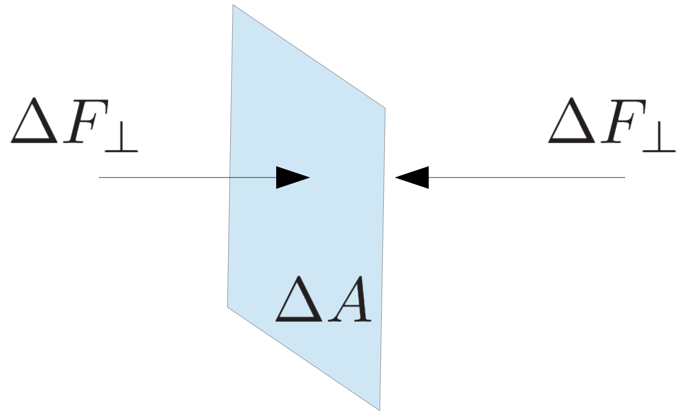
$$r = \frac{\varrho}{\varrho_{\mathrm{H_2O},4^\circ\mathrm{C}}} = \frac{\varrho}{1000\ \mathrm{kg/m^3}}$$

Pressure in a Fluid



force exerted by molecules of the liquid; molecules in motion, constantly colliding with each other and the walls of the container

Assume that the fluid is at rest



both forces equal in magnitude; otherwise non-zero net force → motion

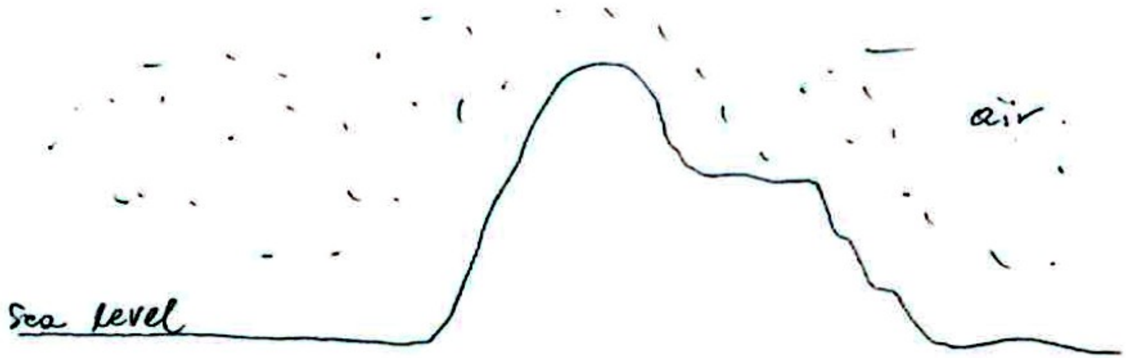
$$p \stackrel{\text{def}}{=} \frac{\Delta F_\perp}{\Delta A} \qquad [\mathrm{Pa} = \mathrm{N/m^2}]$$

→ in general position-dependent $p = p(\vec{r})$

→ if constant at all points of surface *A*, then $p = \dfrac{F_\perp}{A}$

Alternative units: 1 bar = 10^5 Pa
1 psi = 1 lb/in^2 = 6 895 Pa

Atmospheric Pressure

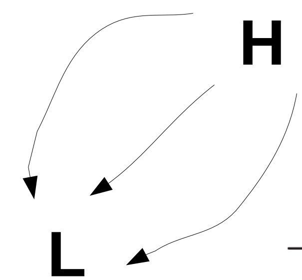


Atmospheric pressure – pressure due to the Earth's atmosphere

altitude

changes of atmospheric pressure

weather

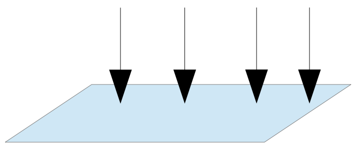


wind

$-\nabla p(\vec{r})$

Normal atmospheric pressure a.s.l. (at the sea level): $p_{atm} = 1.013 \times 10^5$ Pa =
= 14.70 lb/in^2 = 1 atm

Example. Force on classroom's floor due to air at p_{atm}.

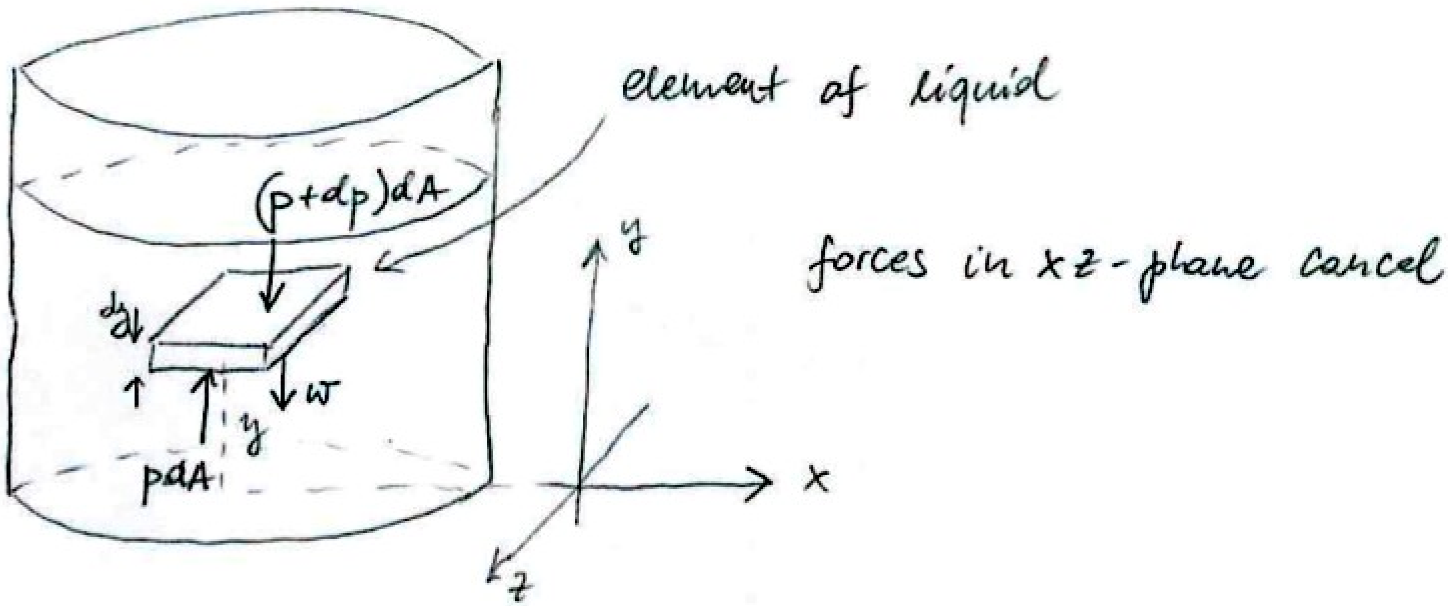


$$F_{\perp} = p \cdot A = 1.013 \times 10^5 \cdot 200 = 2.026 \times 10^7 \ \ \mathrm{N}$$

corresponding to 2000 tonnes!

Why doesn't it collapse? Upward force from the room downstairs (has the same magnitude if floor thickness is small.)

Pressure at a Depth



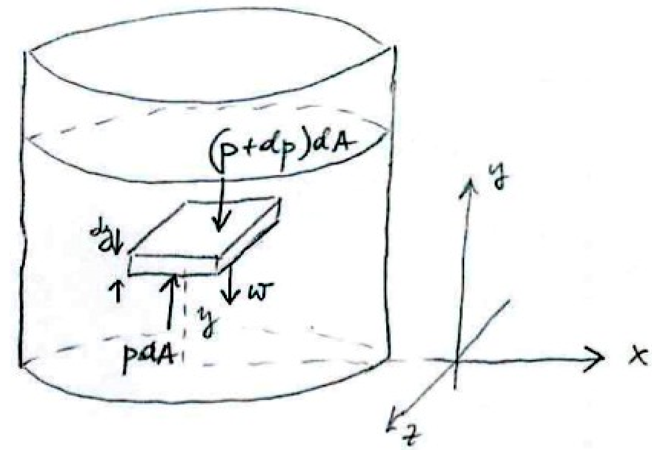
If weight of the liquid could be neglected, then the pressure would be the same throughout the volume of the liquid; otherwise – pressure varies with depth.

Everyday experience: Force exerted by water on eardrums varies when we dive.

Pressure at a Depth

In equilibrium, the net force on the element of the liquid is zero

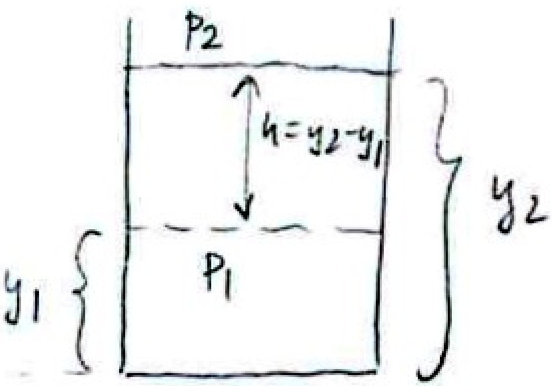
$$F_x = 0, \;\; F_z = 0$$



$$F_y = p\,\mathrm{d}A - (p + \mathrm{d}p)\,\mathrm{d}A - \underbrace{\varrho g\,\mathrm{d}A\,\mathrm{d}y}_{w} = 0$$

$$-\mathrm{d}p\,\mathrm{d}A = \varrho g\,\mathrm{d}A\,\mathrm{d}y \qquad \Longrightarrow \qquad \boxed{\mathrm{d}p = -\varrho g\,\mathrm{d}y}$$

pressure decreases as we move towards the liquid's surface



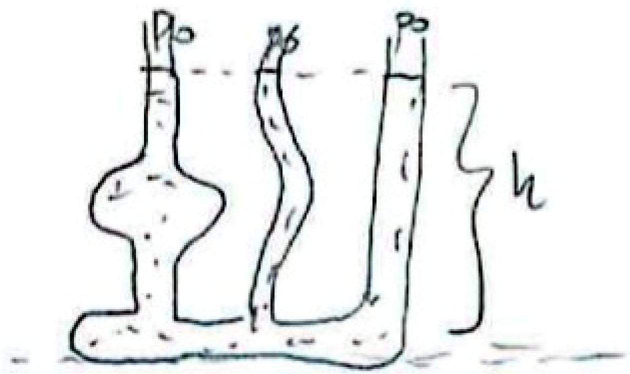
$$\int_{p_1}^{p_2} \mathrm{d}p = -\int_{y_1}^{y_2} \varrho g \mathrm{d}y \Longrightarrow \boxed{p_2 - p_1 = -\varrho g (y_2 - y_1)}$$

Or, introducing the symbols $p_1 = p, \;\; p_2 = p_0, \;\; y_2 - y_1 = h$

$$\boxed{p = p_0 + \varrho g h}$$

Pressure at a Depth

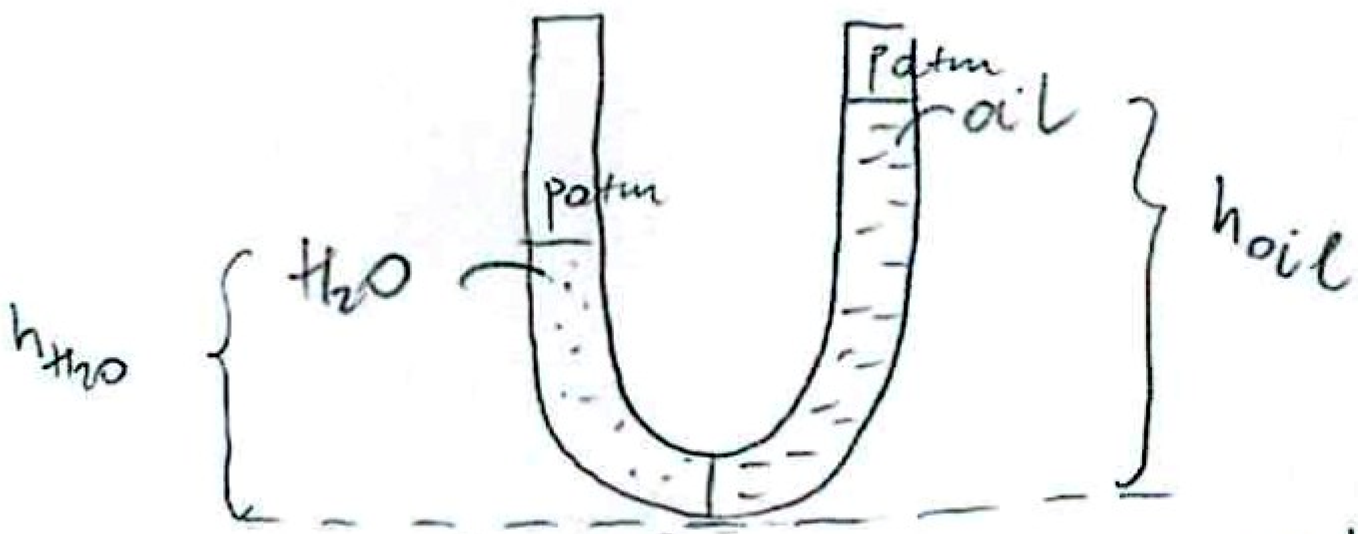
Note. Pressure does not depend on the shape of the container (cross-sectional area does not enter the formula).



$$p_A = p_B = p_C = p_0 + \rho g h$$

Pressure at Depth. Example

Consider two non-mixing fluids (e.g. water and oil) in a U-shaped tube.
If the boundary between the fluids is exactly at the bottom of the tube,
what is the ratio of the heights of both liquids in the arms of the tube.



$$p_{atm} + \varrho_{H_2O}\, g\, h_{H_2O} = p_{atm} + \varrho_{oil}\, g\, h_{oil}$$

$$h_{\rm oil} = \frac{\varrho_{H_2O}}{\varrho_{\rm oil}} h_{\rm H_2O} > h_{\rm H_2O}$$

because the density of water
is greater than that of oil

Pascal's Law

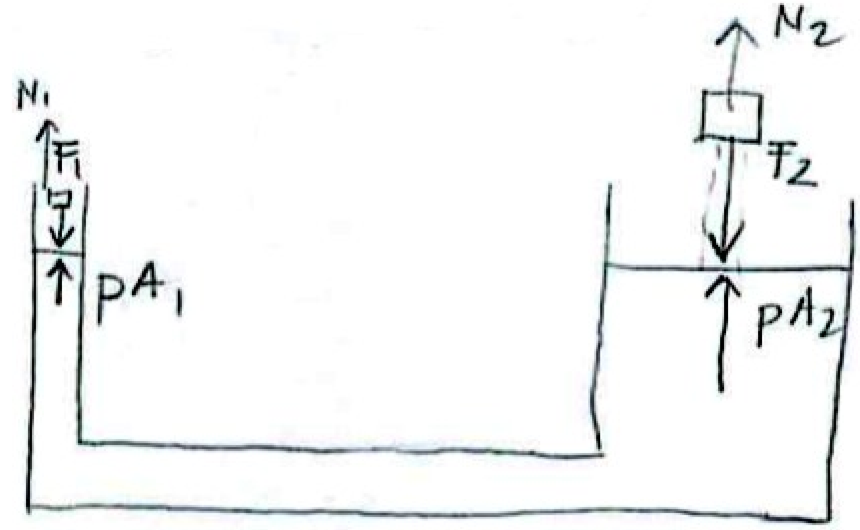
Pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the liquid and the walls of the container.

Application: the hydraulic lift

Statics: (magnitudes)

$$N_1 = F_1 = pA_1$$

$$N_2 = F_2 = pA_2$$



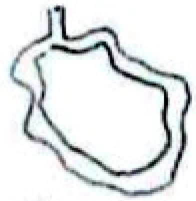
$$\frac{F_2}{F_1} = \frac{A_2}{A_1} \quad \Longrightarrow \quad F_2 = F_1 \frac{A_2}{A_1}$$

If the ratio of cross-sectional areas A_2/A_1 is large, a small force $\mathbf{F}_1$ applied to the surface 1 results in a large lifting force at surface 2.

Of course, the same work is done at both ends of the lift (surface A_1 needs to be moved over a longer distance).

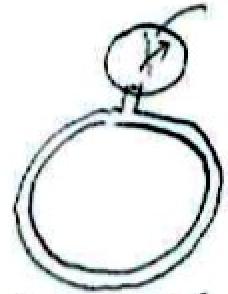
Note. If the pistons are not at the same height, need to take into account the term ρgh.

Absolute Pressure vs. Gauge Pressure



flat tire

$$p = p_{\mathrm{atm}}$$



pumped tire

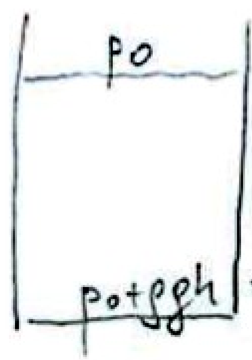
total (absolute) pressure

$$p = p_{\mathrm{atm}} + p_{\mathrm{gauge}}$$

gauge pressure

$$p_{\mathrm{gauge}} = \underbrace{p - p_{\mathrm{atm}}}_{>0}$$

Illustration. Pressure at depth

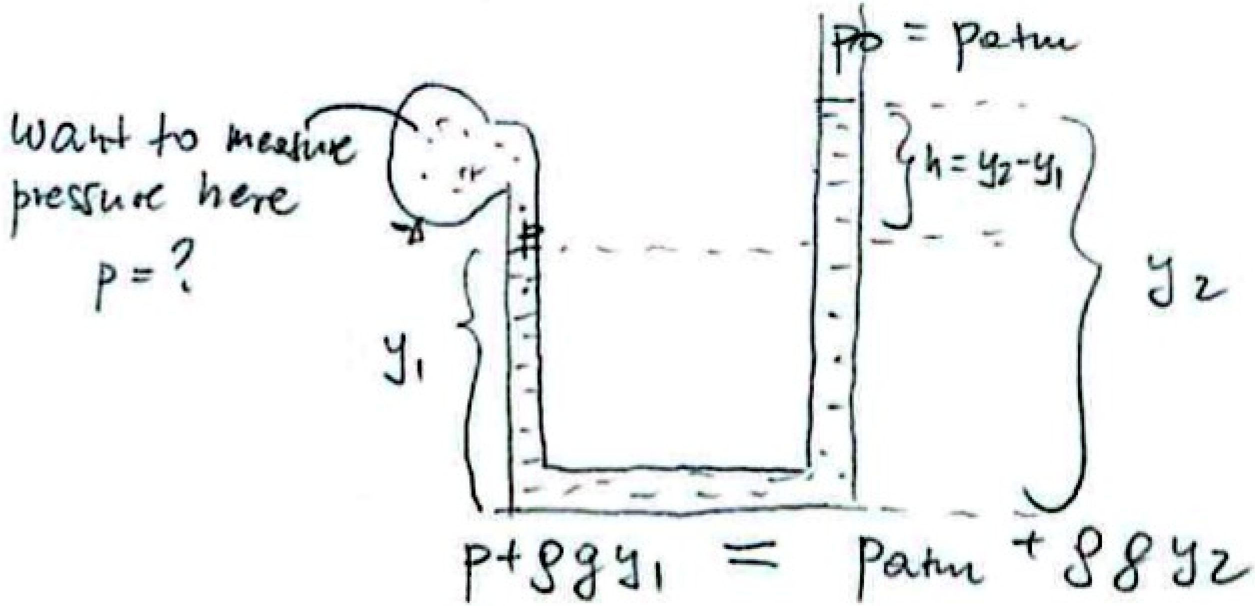


$$p - p_0 = \rho g h$$

gauge pressure
at the bottom of the container

Pressure Measurement – Pressure Gauges

(1) Open-tube Manometer

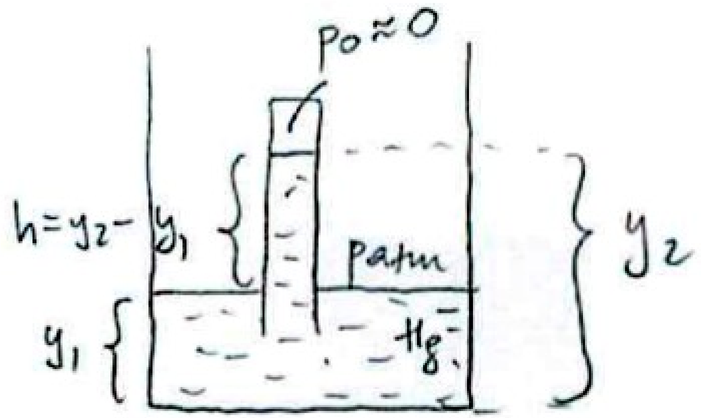


$$\boxed{p_{\mathrm{gauge}}} = p - p_{\mathrm{atm}} = \varrho g(y_2 - y_1) = \boxed{\varrho g h}$$

excess height of the liquid
in the right column

Pressure Measurement – Pressure Gauges

(2) Mercury Manometer



$$\boxed{p_{\mathrm{atm}}} = p_0 + \varrho g h = \boxed{\varrho g h}$$

Alternative unit of pressure: 1 mmHg = 1 torr = 133.3 Pa

Disadvantage: The density of mercury varies with temperature.

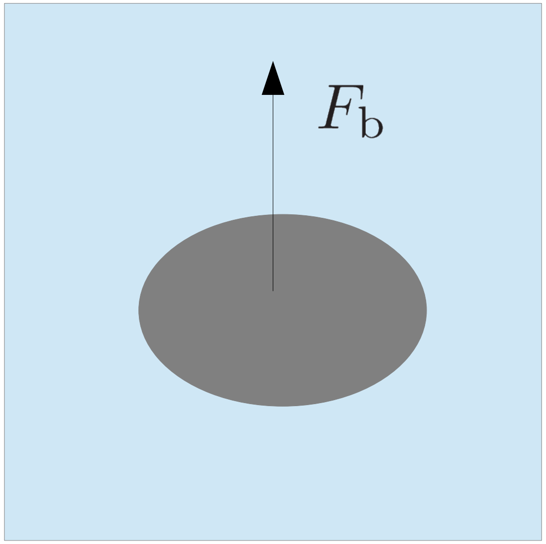
Buoyancy. Archimedes Principle

Buoyancy results from the fact that because of the gravitational force, fluid pressure increases with depth and from the fact that the pressure is exerted in all directions (Pascal's law), so that there is a net upward force on the bottom of a submerged object.

Archimedes' Principle

When a body is immersed in a fluid, the fluid exerts an upward force on the body equal to the weight of the fluid displaced by the body.

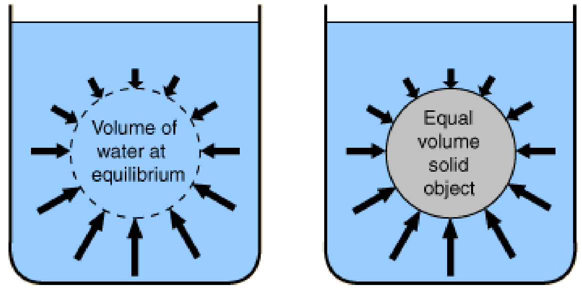
$$F_\mathrm{b} = \underbrace{\varrho_\mathrm{liquid} V_\mathrm{immersed}}_{m_\mathrm{liquid}} g$$



Note. All objects of *equal volume* experience *equal buoyant forces*.

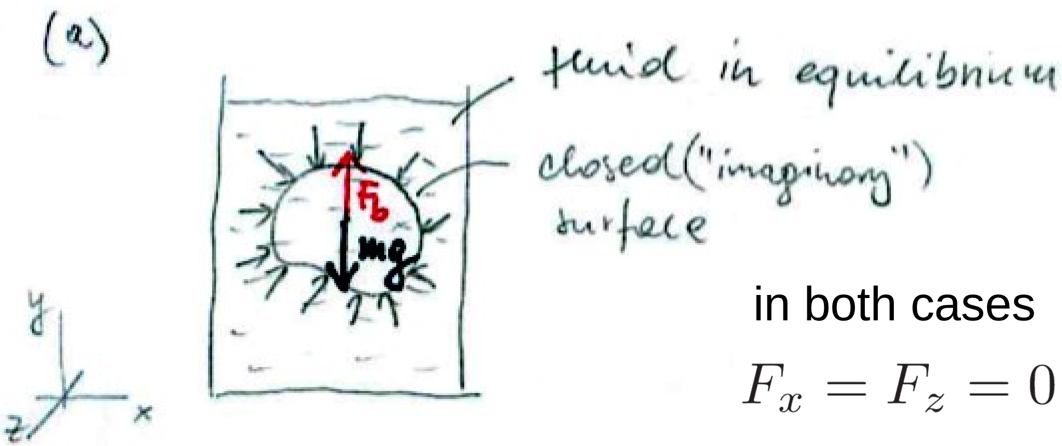
Archimedes Principle

Justification – idea



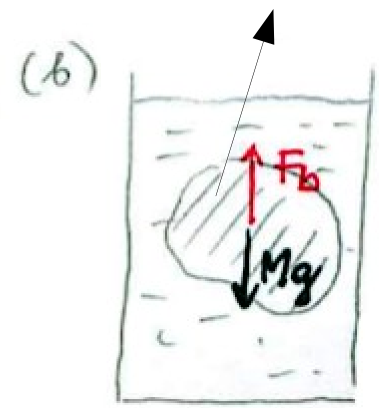
All objects of *equal volume* experience *equal buoyant forces.*

fill the region with a body



in both cases

$$F_x = F_z = 0$$



at any point of the surface pressure is the same as in (a); depends on the shape only

mass of liquid

$$F_y = F_{\rm b} - mg \quad \Longrightarrow$$

$$\Longrightarrow \quad F_{\rm b} = mg = \varrho_{\rm liquid} V g$$

mass of the body

$$F_y = F_{\rm b} - Mg = (m - M)g =$$

$$= (\varrho_{\rm liquid} - \varrho_{\rm body}) V g$$

Archimedes Principle. Examples

(1) Floating fish

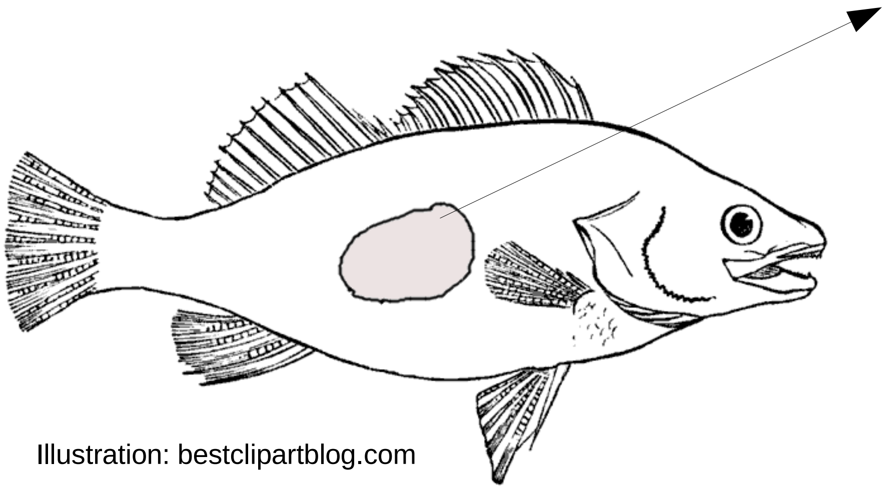


Illustration: bestclipartblog.com

Inflatable swim bladder controls the density of the body; can adjust the net force on the fish in the vertical direction

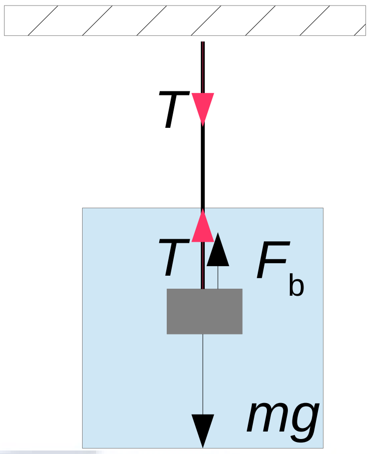
Although $\varrho_{\mathrm{tissue}} > \varrho_{\mathrm{H_2O}}$

filling the swim bladder with air can make $\varrho_{\mathrm{fish}} \leq \varrho_{\mathrm{H_2O}}$

and there will be a net upward force on the fish.

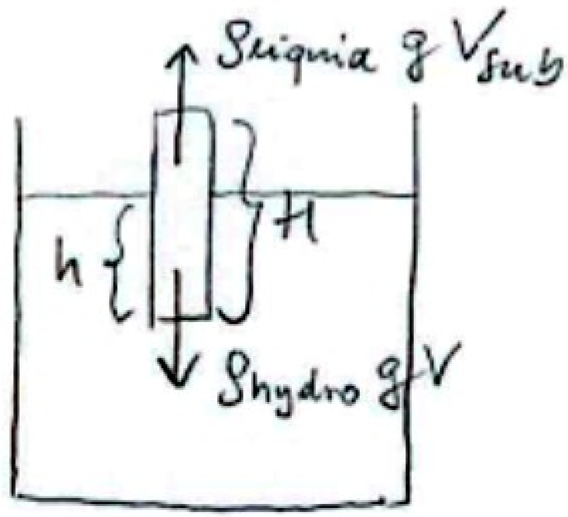
(2) Tension in a cord supporting a submerged object with density greater than that of the liquid

$$T + F_{\mathrm{b}} - mg = 0 \quad \Longrightarrow \quad T = mg - F_{\mathrm{b}}$$



Archimedes Principle. Examples

(3) Practical application: the hydrometer



The hydrometer is in equilibrium when

$$\varrho_{\rm liquid} V_{\rm sub} g = \varrho_{\rm hydro} V g$$

Hence

$$\varrho_{\rm liquid} = \varrho_{\rm hydro} \frac{V}{V_{\rm sub}}$$

If the hydrometer has a cylindrical shape with cross sectional area *A*, then

$$V = A \cdot H, \qquad V_{\rm sub} = A \cdot h$$

$$\varrho_{\rm liquid} = \varrho_{\rm hydro} \frac{H}{h}$$

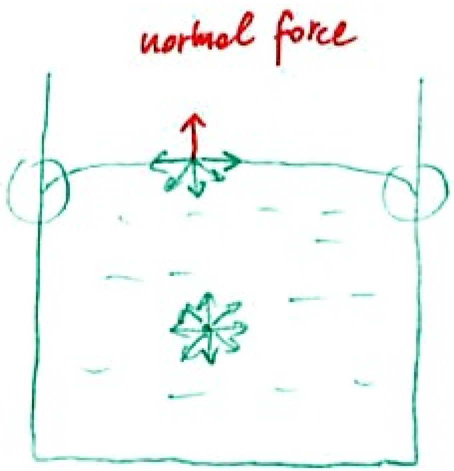
Marking the side of the cylinder allows us to directly read out the density of the liquid.

Surface Tension

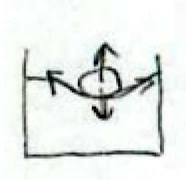
Demonstration: A paper clip floating on the surface of the liquid.

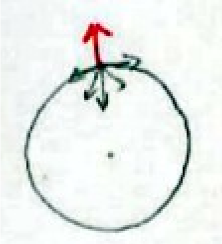
edtech.boisestate.edu

Origin: attractive interaction between molecules of the liquid.



interaction between neighboring molecules (cohesion force)

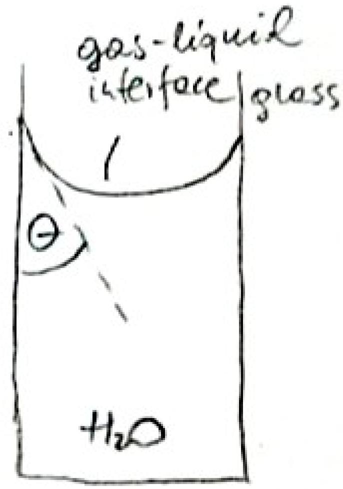


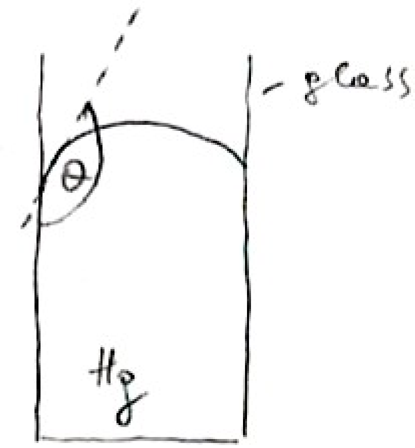


Note. In the weightless state, the liquid will form a ball – a solid that, with a given volume, has minimum surface area, hence minimizes the energy due to surface tension.

Meniscus and capillarity

As a result of competition between cohesion forces (forces between the molecules of the liquid) and adhesion forces (forces between the liquid and the container), the surface of the liquid may take up various shapes.





wetting liquid $0 < \theta < \dfrac{\pi}{2}$

adhesion forces dominate

non-wetting liquid $\dfrac{\pi}{2} < \theta < \pi$

cohesion forces dominate

Capillarity (capilar action) – ability of a liquid to flow in narrow tubes in opposition external forces (e.g. gravity).

Fluids in Motion

Vocabulary

Ideal Fluid

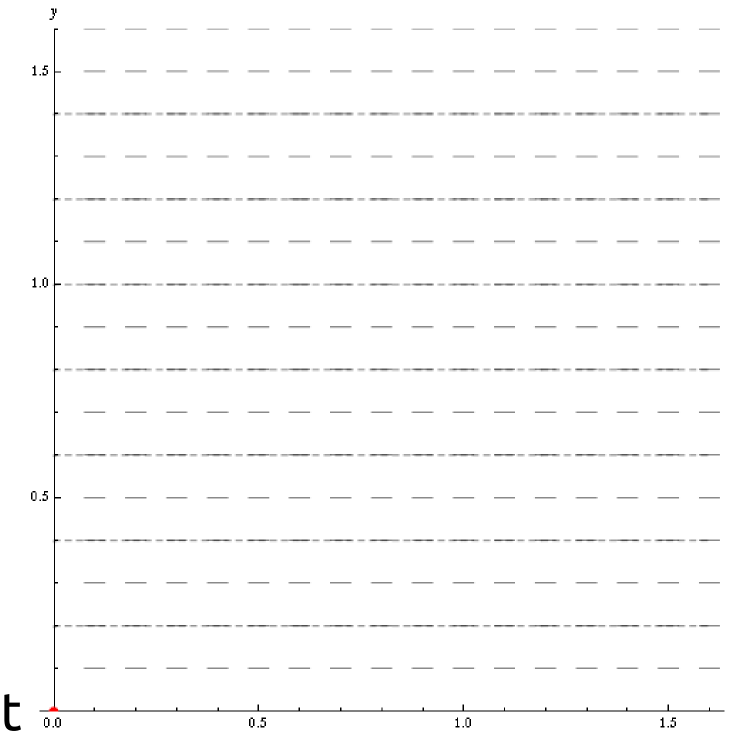
fluid density does not change (the fluid is incompressible);
experiences no internal friction (no viscosity)

Flow Lines (Path Lines)

trajectories of individual particles in a fluid

Stream Lines

family of curves such that at each point, the
tangent line to the curve coincides with
the direction of the fluid velocity at that point

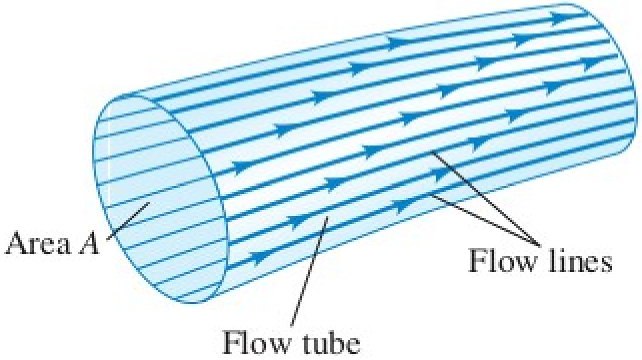


Note. When the flow pattern changes, flow lines do not coincide with stream lines.
We will discuss only steady-flow situation, when the two do coincide.

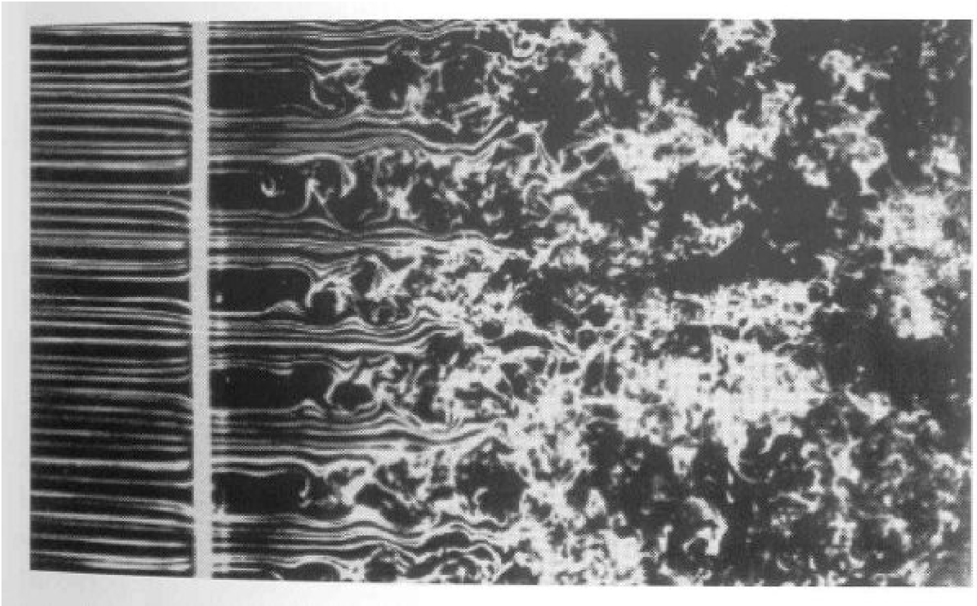
Vocabulary

Flow Tube

a tube formed by flow lines passing through the edge of an imaginary element of area. Note that in steady (laminar) flow no fluid can cross the side walls of a flow tube; fluids in different flow tubes cannot mix.



Laminar vs. Turbulent Flow

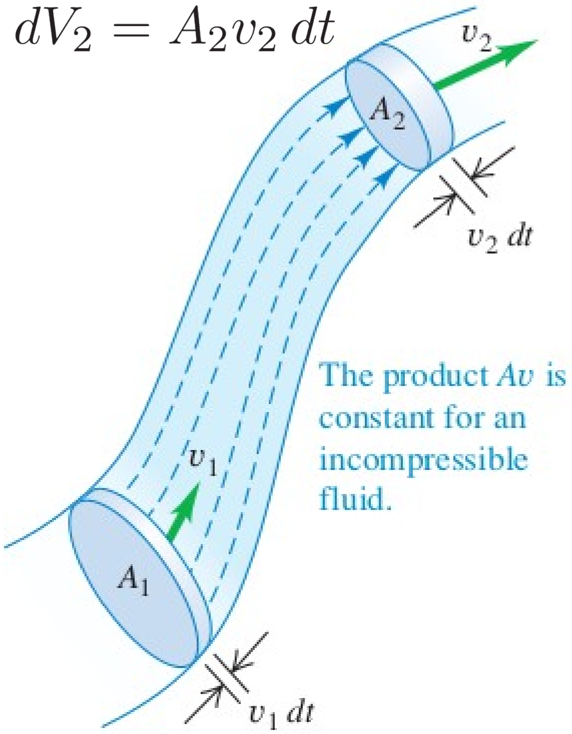


http://num3sis.inria.fr

Continuity Equation

Follows from the fact that mass is conserved (mass of a fluid element does not change as it flows)

$$dV_2 = A_2 v_2 \, dt$$



$$dV_1 = A_1 v_1 \, dt$$

$$\varrho A_1 v_1 \, dt = \varrho A_2 v_2 \, dt$$

For a homogeneous, incompressible fluid (steady flow)

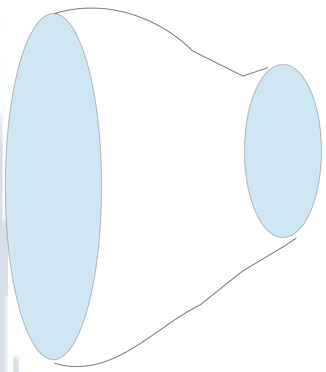
$$\varrho = \text{const}$$

$$\boxed{A_1 v_1 = A_2 v_2}$$

$$\frac{dV}{dt} \longrightarrow \text{volume flow rate}$$

Note. In case of a fluid that is not incompressible $\varrho_1 A_1 v_1 = \varrho_2 A_2 v_2$

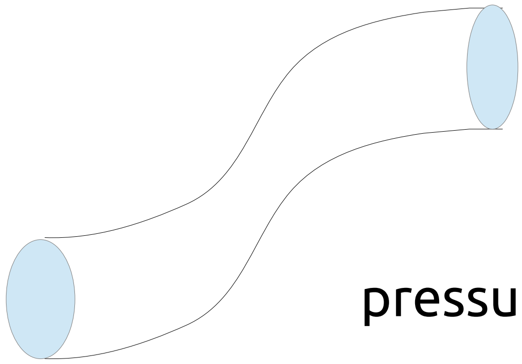
Bernoulli's Equation. Motivation



when an incompressible fluid flows through a horizontal tube of variable cross-section, its speed must change (continuity equation)

speed change means acceleration (force due to surrounding liquid)

pressure is different in regions of different cross-section



pressure difference due to difference in height

Note

In the following discussion we assume: incompressible fluid, steady flow, no viscosity.

Bernoulli's Equation.

Idea: Find the change in the (gravitational) potential and the kinetic energy of an element of liquid's volume. It is equal to the work of the force due to pressure in the liquid.

element of volume that is displaced from a-c to b-d

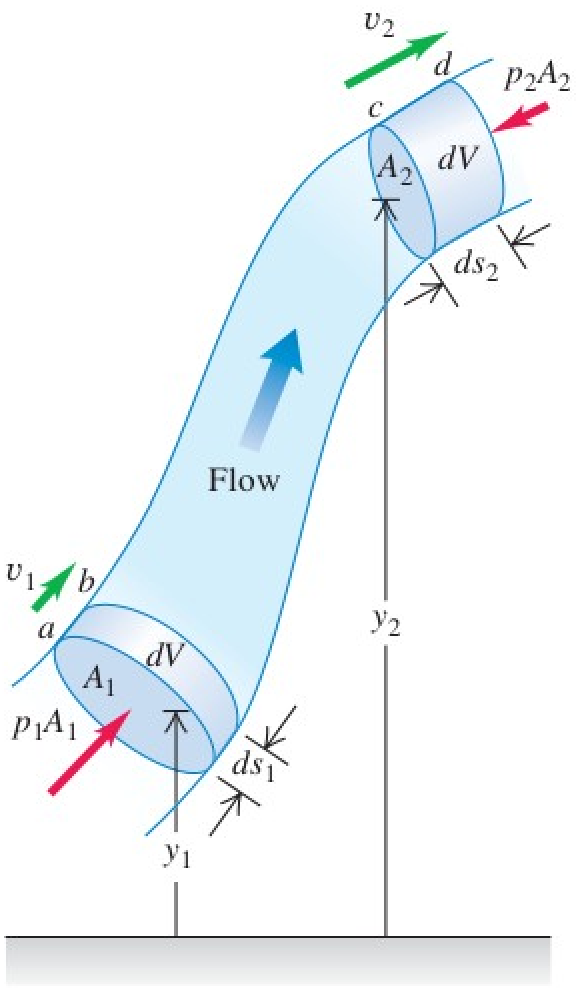
$$dV = A_1\, ds_1 = A_2\, ds_2$$

work done on this element by the surrounding liquid

$$\delta W = p_1 \underbrace{A_1\, ds_1}_{dV} - p_2 \underbrace{A_2\, ds_2}_{dV}$$

on the other hand

$$\delta W = dE = d(K + U_{\text{grav}}) = dK + dU_{\text{grav}}$$



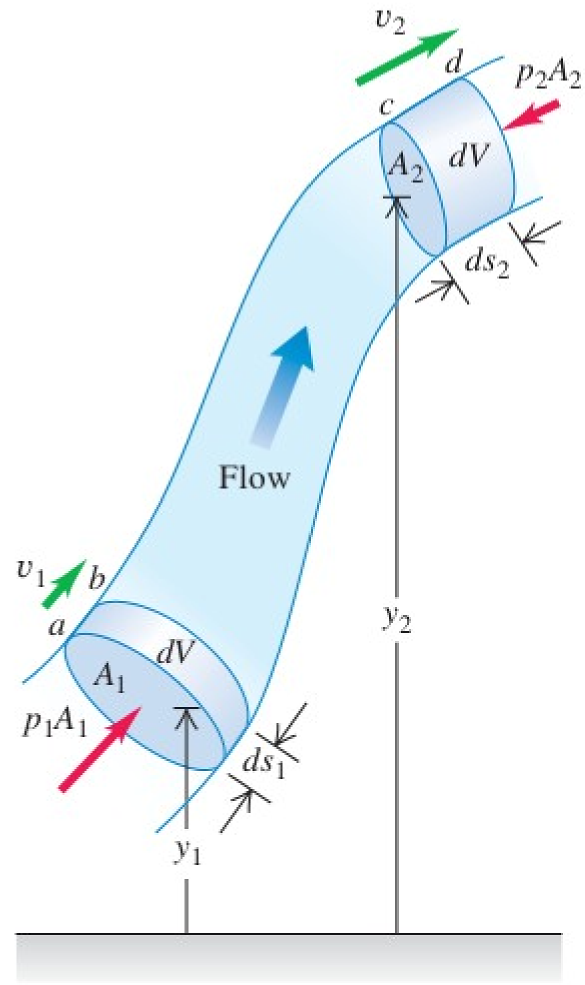
Bernoulli's Equation. Derivation

change in the kinetic energy

[contribution due to the volume between *b* and *c* cancels out;
effectively only the bottom and the top slice need to be taken into account]

(bottom slice in motion)		(top slice in motion)
$\varrho A_1\, ds_1$	mass	$\varrho A_2\, ds_2$
$\frac{1}{2}\varrho\,(A_1\, ds_1)\, v_1^2$	kinetic energy	$\frac{1}{2}\varrho\,(A_2\, ds_2)\, v_2^2$

$$dK = \frac{1}{2}\varrho\left(v_2^2 - v_1^2\right)\, dV$$



Bernoulli's Equation. Derivation

change of the (gravitational) potential energy
[again, contribution due to the volume between *b* and *c* cancels out;
effectively only the bottom and the top slice need to be taken into account]

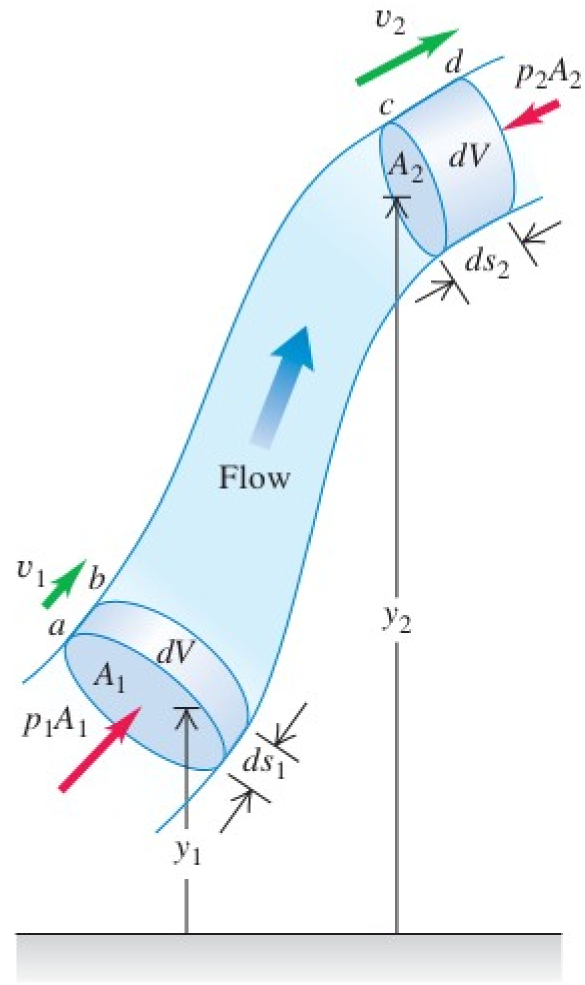
(bottom slice)

$$g y_1 \, dm = \varrho g y_1 \, dV$$

(top slice)

$$g y_2 \, dm = \varrho g y_2 \, dV$$

$$dU_{\mathrm{grav}} = \varrho g \left(y_2 - y_1 \right) \, dV$$



Bernoulli's Equation

combining both

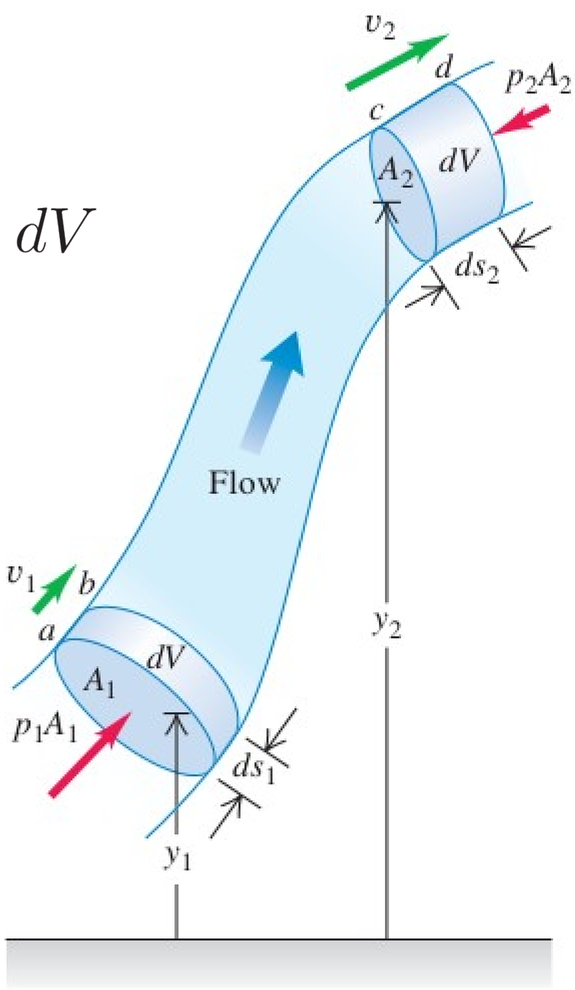
$$\delta W = dK + dU_{\text{grav}}$$

$$(p_1 - p_2)\ dV = \frac{1}{2}\varrho\left(v_2^2 - v_1^2\right)\ dV + \varrho g\left(y_2 - y_1\right)\ dV$$

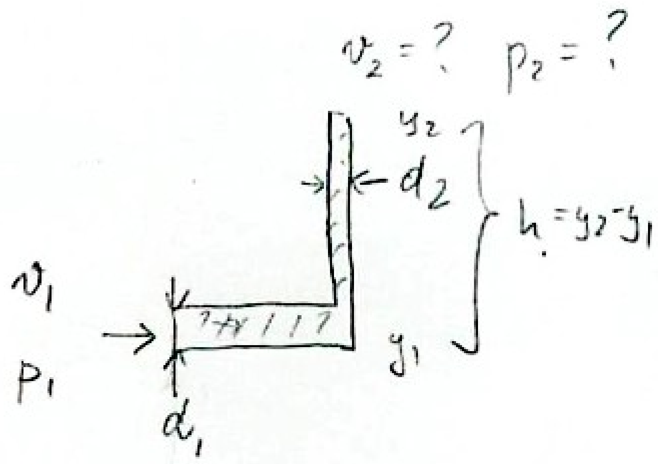
$$(p_1 - p_2) = \frac{1}{2}\varrho\left(v_2^2 - v_1^2\right) + \varrho g\left(y_2 - y_1\right)$$

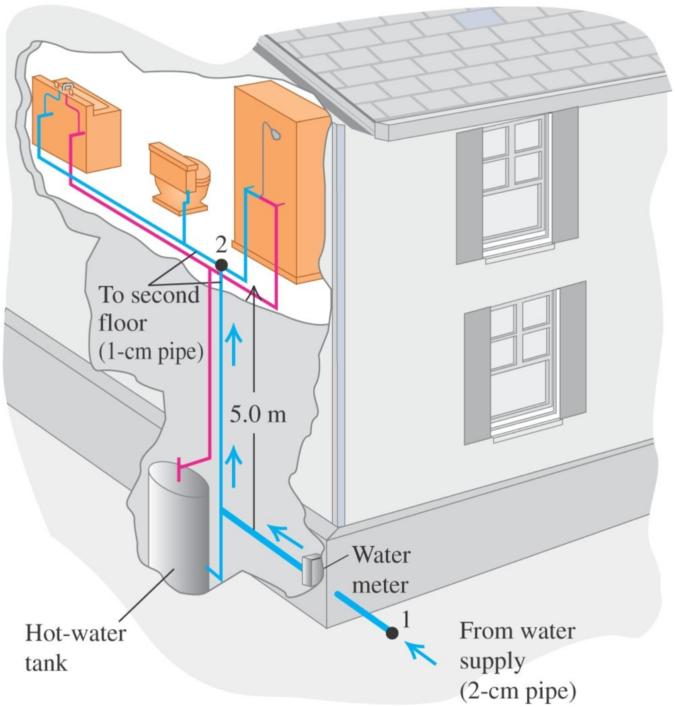
$$p_1 + \frac{1}{2}\varrho v_1^2 + \varrho g y_1 = p_2 + \frac{1}{2}\varrho v_2^2 + \varrho g y_2$$

$$\boxed{p + \frac{1}{2}\varrho v^2 + \varrho g y = \text{const}}$$



Example. Water Pressure in a Building





continuity equation $A_1 v_1 = A_2 v_2$

$$\pi \frac{d_1^2}{4} v_1 = \pi \frac{d_2^2}{4} v_2 \quad \Longrightarrow \quad v_2 = \left(\frac{d_1}{d_2}\right)^2 v_1$$

Bernoulli's equation

$$p_1 + \frac{1}{2}\varrho v_1^2 = p_2 + \varrho g h + \frac{1}{2}\varrho v_2^2$$

$$p_2 = p_1 - \frac{1}{2}\varrho v_1^2 \left[\left(\frac{d_1}{d_2}\right)^4 - 1\right] - \varrho g h$$

Example. Speed of Efflux

Bernoulli's equation

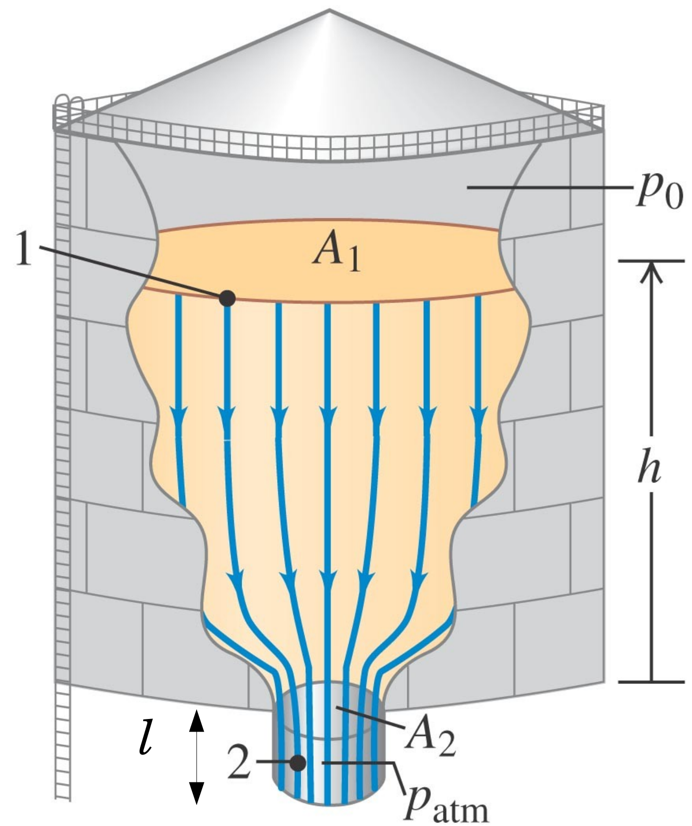
$$p_0 + \frac{1}{2}\varrho v_1^2 + \varrho g(h+l) = p_{\mathrm{atm}} + \frac{1}{2}\varrho v_2^2$$

taking into account

$l \ll h$ and $A_1 \gg A_2 \Longrightarrow v_1 = \dfrac{A_2}{A_1} v_2 \approx 0$

$$p_0 + \varrho g h = p_{\mathrm{atm}} + \frac{1}{2}\varrho v_2^2$$

$$v_2 = \sqrt{2\left(\frac{p_0 - p_{\mathrm{atm}}}{\varrho} + gh\right)}$$



Note. If the lid is open $p_0 = p_{\mathrm{atm}}$ and $v_2 = \sqrt{2gh}$

Example: Venturi Meter

How to measure flow speed in a tube?

continuity equation

$$v_1 A_1 = v_2 A_2 \Longrightarrow v_2 = \frac{A_1}{A_2} v_1$$

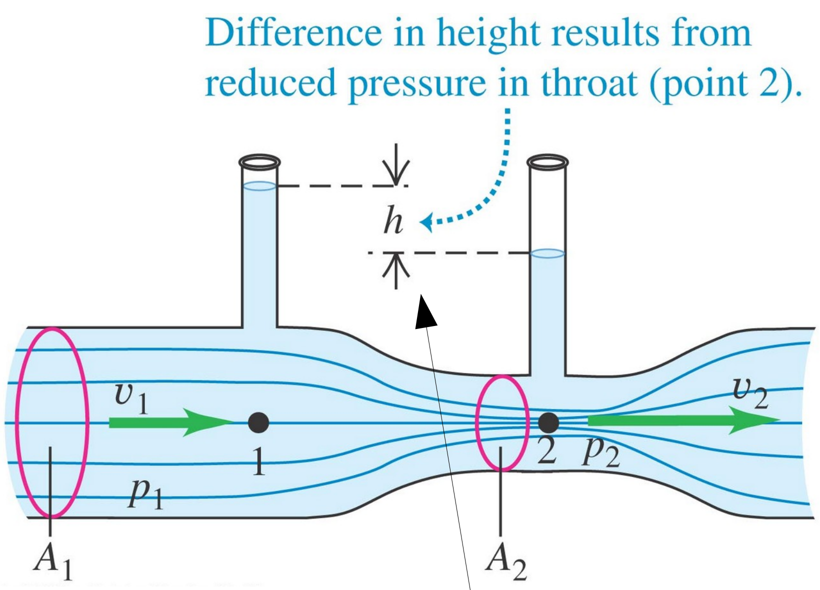
Bernoulli's equation

$$p_1 + \frac{1}{2}\varrho v_1^2 = p_2 + \frac{1}{2}\varrho v_2^2$$

$$p_1 - p_2 = \frac{1}{2}\varrho v_1^2 \left[\left(\frac{A_1}{A_2} \right)^2 - 1 \right]$$

on the other hand

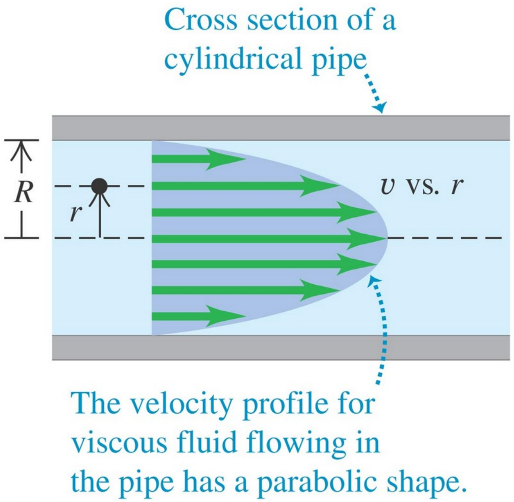
$$p_1 - p_2 = \varrho g h$$



$$v_1 = \sqrt{\frac{2gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}}$$

Viscosity and Turbulence

- *Viscosity* is internal friction in a fluid.
- *Turbulence* is irregular chaotic flow that is no longer laminar.



Example: A Curve Ball