Basic Kinematic Quantities in Motion Along a Straight Line
Analysis of $v_x(t)$ and $x(t)$ Graphs
Velocity vs Speed
Example: Free Fall
Example: Motion with Varying Acceleration
Galilean Transformation and Velocity Addition Rule

Chapter 2 – Kinematics in 1D

UM-SJTU Joint Institute
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Agenda

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Average and Instantaneous Velocity
Average and Instantaneous Acceleration

Basic Kinematic Quantities in Motion Along a Straight Line

Kinematics – describes motion quantitatively, does not discuss the cause of motion.

In 1D all vectors have just a single component; choice of the positive direction is arbitrary. Hence, the **position** in 1D is uniquely determined by a single number x (can use scalars).
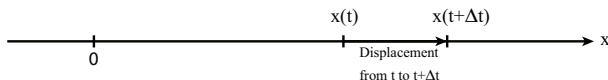
In general, the position changes with time $x = x(t)$, and we can characterize these changes by, e.g., the **average velocity** over a time interval $(t, t + \Delta t)$
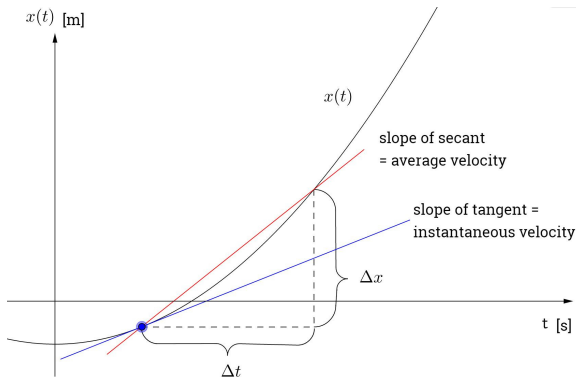
$$v_{\mathrm{av},x} \overset{\text{def}}{=} \frac{x(t + \Delta t) - x(t)}{\Delta t} \qquad (\text{units } [m/s])$$

Instantaneous Velocity

If we let $\Delta t \to 0$, then

$$\lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \boxed{\dot{x}(t) \stackrel{\text{def}}{=} v_x(t)} \qquad \text{(units } [m/s])$$

which is called the **instantaneous velocity**.
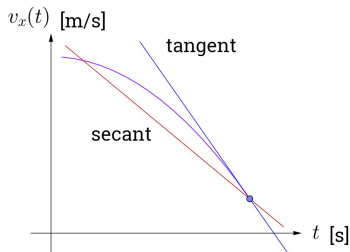
Average and Instantaneous Acceleration

Analogously, we can define the average and the instantaneous acceleration, quantifying changes of the velocity with time.

The **average acceleration**, over a time interval $(t, t + \Delta t)$, is defined as

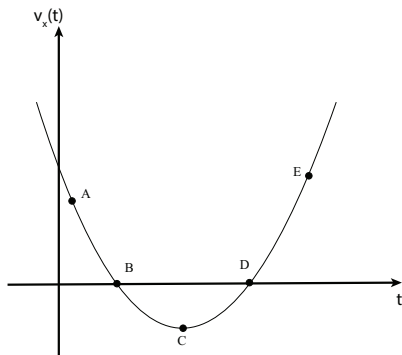$$a_{\text{av,x}} \stackrel{\text{def}}{=} \frac{v_x(t + \Delta t) - v_x(t)}{\Delta t} \qquad (\text{units } [m/s^2]),$$

while the **instantaneous acceleration** (having the same units) is defined as

$$\boxed{a_x(t)} \stackrel{\text{def}}{=} \lim_{\Delta t \to 0} \frac{v_x(t + \Delta t) - v_x(t)}{\Delta t} = \frac{dv_x(t)}{dt} = \boxed{\dot{v}_x(t)} = \frac{d^2 x(t)}{dt^2} = \ddot{x}(t).$$

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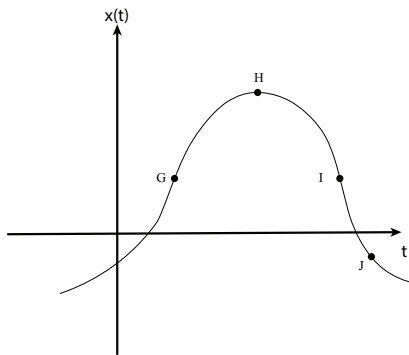# Analysis of $v_x(t)$ and $x(t)$ Graphs

Example. Analysis of a $v_x(t)$ Graph

- A : $v_x > 0$, $a_x < 0$ moves to the right, slowing down
- B : $v_x = 0$, $a_x < 0$ stops; about to start moving to the left
- C : $v_x < 0$, $a_x = 0$ moves to the left; no instant. acceleration
- D : $v_x = 0$, $a_x > 0$ stops; about to start moving to the right
- E : $v_x > 0$, $a_x > 0$ moves to the right; speeding up

Example. Analysis of a $x(t)$ Graph

- G : $v_x > 0$, $a_x = 0$ (inflection point)
- H : $v_x = 0$, $a_x < 0$
- I : $v_x < 0$, $a_x = 0$ (inflection point)
- J : $v_x < 0$, $a_x > 0$

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Velocity vs Speed

Note that, the instantaneous speed is just the magnitude of instantaneous velocity.

However, the average speed <u>is not</u> equal to the magnitude of the average velocity.

These two quantities differ in that

$$\text{average speed} \; = \; \frac{\text{distance traveled}}{\text{time interval}}$$
$$\text{average velocity} \; = \; \frac{\text{displacement}}{\text{time interval}}$$

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General Problem and Solution Strategy
Galileo's Experiment: Free Fall Close to Earth's Surface
Final Remarks

Example: Free Fall

General remarks on mutual relations between x, v_x and a_x

$$
\begin{array}{rcl}
\text{position} & & x(t) \\
\text{instantaneous velocity} & v_x(t) = & \dot{x}(t) \\
\text{instantaneous acceleration} \quad a_x(t) = & \dot{v}_x(t) = & \ddot{x}(t)
\end{array}
$$

$$
\begin{array}{ll}
x \to v_x \to a_x & \text{differentiate (easy)} \\
a_x \to v_x \to x & ?
\end{array}
$$

<u>Question</u>: Given acceleration, how to find velocity and position?

...integrate!

$$a_x(t) = \frac{\mathrm{d}v_x}{\mathrm{d}t} \quad \Rightarrow \quad \mathrm{d}v_x = a_x(t)\,\mathrm{d}t \quad \Rightarrow \quad \int \mathrm{d}v_x = \int a_x(t)\,\mathrm{d}t$$

Finally we have

$$v_x(t) = \int a_x(t)\,\mathrm{d}t \qquad \text{(determined up to an additive constant)}$$

Similarly,

$$x(t) = \int v_x(t)\,\mathrm{d}t \qquad \text{(determined up to an additive constant)}$$

The additive constants in the two formulas above are found from *initial conditions*.

Free Fall Close to Earth's Surface

Let us analyze Galileo's experiment, that is motion of a free falling object close to the Earth's surface, neglecting air drag.

The acceleration is constant

$$a_y(t) = g = \text{const}$$

Earth

Hence, the velocity,

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = g \quad \Rightarrow \quad \mathrm{d}v_y = g\,\mathrm{d}t \quad \Rightarrow \quad \int \mathrm{d}v_y = \int g\,\mathrm{d}t$$

Eventually

$$v_y(t) = gt + C_1 \qquad (C_1 \text{ is found from an initial condition})$$

Free Fall Close to Earth's Surface

Combining the result with the initial condition $v_y(t_0) = v_{0y}$, we have

$$v_y(t_0) = gt_0 + C_1 \quad \Rightarrow \quad C_1 = v_{0y} - gt_0$$

and, eventually,

$$v_y(t) = g(t - t_0) + v_{0y}.$$

Similarly, for the position,

$$
\begin{aligned}
\frac{dy}{dt} &= g(t - t_0) + v_{0y} \\
\int dy &= \int [g(t - t_0) + v_{0y}] \, dt \\
y(t) &= \frac{1}{2}gt^2 - gt_0 t + v_{0y}t + C_2 = \frac{1}{2}gt^2 + (v_{0y} - gt_0)t + C_2
\end{aligned}
$$

To determine the constant C_2, another initial condition is needed: $y(t_0) = y_0$. Then

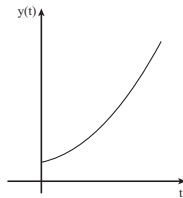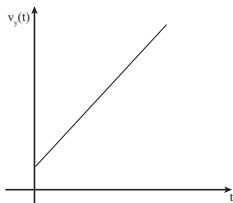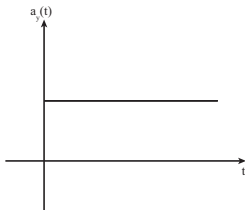$$y(t_0) = \frac{1}{2}gt_0^2 + (v_{0y} - gt_0)t_0 + C_2 \quad \Rightarrow C_2 = \frac{1}{2}gt_0^2 - v_{0y}t_0 + y_0$$

Eventually

$$y(t) = \frac{1}{2}g(t^2 + t_0^2) + v_{0y}(t - t_0) - gt_0 t + y_0$$

Free Fall Close to Earth's Surface

If we choose $t_0 = 0$ (the usual choice)

$$v_y(t) = gt + v_{0y}, \qquad\qquad y(t) = \frac{1}{2}gt^2 + v_{0y}t + y_0$$

Summary

$$
\begin{aligned}
a_y(t) &= g = \text{const} \\
v_y(t) &= gt + v_{0y} \\
y(t) &= \frac{1}{2}gt^2 + v_{0y}t + y_0
\end{aligned}
$$

(1) Using definite integrals takes care of initial conditions "automatically"

$$\int_{v_{0y}}^{v_y(t)} \mathrm{d}v_y = \int_{t_0}^{t} a_y(t)\,\mathrm{d}t \quad \Rightarrow \quad v_y(t) - v_{0y} = \int_{t_0}^{t} a_y(t)\,\mathrm{d}t$$

$$\int_{y_0}^{y(t)} \mathrm{d}y = \int_{t_0}^{t} v_y(t)\mathrm{d}t \quad \Rightarrow \quad y(t) - y_0 = \int_{t_0}^{t} v_y(t)\,\mathrm{d}t$$
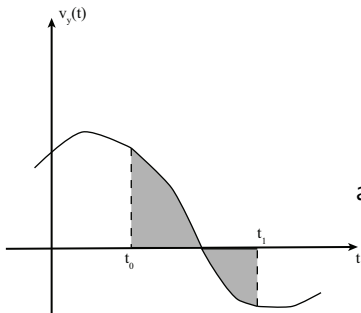
Final Remarks

(2) The net displacement between the instants t_0 and t_1 is

$$\int_{t_0}^{t_1} v_y(t)\, \mathrm{d}t = y(t_1) - y(t_0)$$

and the corresponding distance travelled between t_0 and t_1 is the total area of the shaded region

$$\text{distance} = \int_{t_0}^{t_1} |v_y(t)|\, \mathrm{d}t$$

$$\text{average speed} \;\; = \;\; \frac{\int_{t_0}^{t_1} |v_y(t)|\, \mathrm{d}t}{t_1 - t_0}$$

$$\text{average velocity} \;\; = \;\; \frac{\int_{t_0}^{t_1} v_y(t)\, \mathrm{d}t}{t_1 - t_0}$$

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Example: Motion with Varying Acceleration

Example. Motion with Varying Acceleration

The acceleration of an iron object moving in a straight line close to a magnet turns out to be related to its position with respect to the magnet, $x > 0$, as $a_x = -k/x$, where k is a positive constant. It has been experimentally determined that the velocity of the object is $\sqrt{2}v_0 > 0$, when the object is at $x = x_0 > 0$ and it is v_0 when $x = 2x_0$.

What is the velocity of this particle when $x = 3x_0$?

Note that $a_x = \frac{dv_x}{dt} \overset{\text{chain rule}}{=} \frac{dv_x}{dx} \underbrace{\frac{dx}{dt}}_{v_x} = \frac{dv_x}{dx} v_x$.

Hence $-\frac{k}{x} = v_x \frac{dv_x}{dx}$. Integrating and using the information provided

$$\int\limits_{\sqrt{2}v_0}^{v_0} v_x \, dv_x = -\int\limits_{x_0}^{2x_0} \frac{k}{x} \, dx \qquad \implies \qquad \frac{1}{2}v_0^2 - v_0^2 = -k \ln 2,$$

so that $\boxed{k = \dfrac{v_0^2}{2\ln 2}}$.

To find out the answer, integrate with the corresponding limits

$$\int\limits_{\sqrt{2}v_0}^{v_x(3x_0)} v_x \, \mathrm{d}v_x = -\int\limits_{x_0}^{3x_0} \frac{k}{x} \, \mathrm{d}x$$

to get

$$\frac{1}{2} v_x^2(3x_0) - v_0^2 = -k \ln 3 = -\frac{v_0^2}{2} \frac{\ln 3}{\ln 2}.$$

Solving for $v_x(3x_0)$ yields

$$\boxed{v_x(3x_0) = v_0 \sqrt{2 - \frac{\ln 3}{\ln 2}}.}$$

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Galilean Transformation and Velocity Addition Rule

In 1D kinematics, the same particle may have different positions, velocities or accelerations, depending on the choice of the coordinate axis. An axis with respect to which we describe motion is called a <u>frame of reference</u> (abbreviated as *FoR*).

Example. Student in a moving train car: A is the *FoR* associated with the train station's platform ground; A' is the FoR of the train car itself.
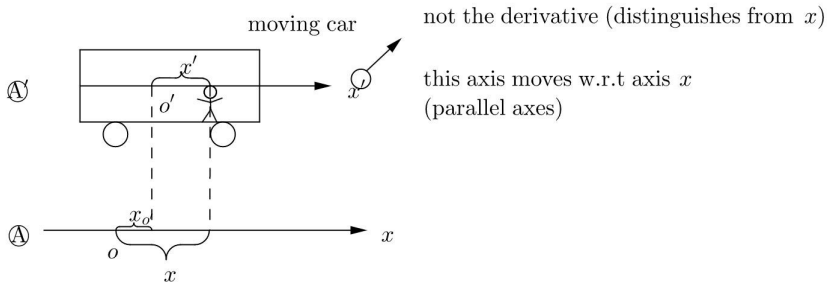
The student's position in *FoR A* is

$$x = x_{O'} + x'$$

where $x_{O'}$ is the position of the origin of *A'* in *FoR A*, and x' is the position of the student in FoR *A'*.

Differentiating the equation above with respect to time,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x_{O'}}{\mathrm{d}t} + \frac{\mathrm{d}x'}{\mathrm{d}t} \qquad \Longrightarrow \qquad \boxed{v_x = v_{O'x} + v'_x}$$

which is the well-known velocity addition rule.

Consequently, for accelerations, $a_x = a_{O'x} + a'_x$.

In the special case when $O' = O$ at $t = 0$, and $v_{O'x} = $ const, we have $a_x = a_x$ and the relation between student's positions in A and A' is

$$x = v_{O'x}t + x'.$$

The boxed formula is known as the **Galilean transformation**.

There is an implicit assumption in the derivation of the Galilean transformation.

What did we assume?