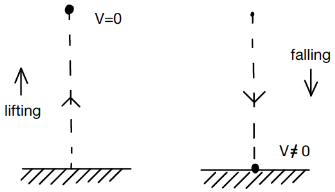
Chapter 9 – Potential Energy, Conservative Forces, and Conservation of Mechanical Energy

UM-SJTU Joint Institute
Physics I (Summer 2021)
Mateusz Krzyzosiak

Agenda

Introduction

Example. Work done on a particle close to the earth surface.



Lifting an object $\rightarrow$ external force does positive work, gravitational force does negative work.

Falling object $\rightarrow$ gravitational force (net force) does positive work.
W>0 $\Rightarrow$ K>0.

Energy associated with the position in the earth-particle system
Potential Energy

$$\updownarrow$$

Energy associated with motion
Kinetic Energy

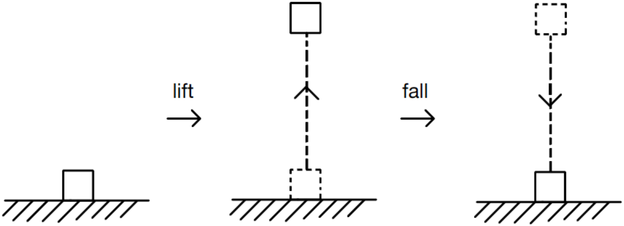
Whenever we can define the *potential energy* (not always possible), we can formulate the *law of conservation of mechanical energy*.

$$\text{Mechanical Energy} = \text{Kinetic Energy} + \text{Potential Energy}$$

Potential Energy and Energy Conservation

Gravitational Force. Gravitational Potential Energy

Qualitative Discussion



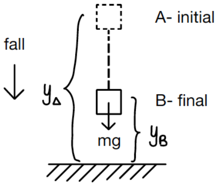
Lifting

External force does positive work; creates a "*potential*" for positive work to be done by gravitational force. *Potential energy* increases.

Falling

Gravitational force uses this "*potential*" to do positive work on the object. *Potential energy* decreases, kinetic energy increases.

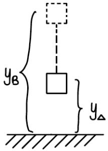
Work Done by Gravitational Force. Gravitational Potential Energy



Here only the gravitational force present (net force = gravitational force)

$$W_{grav} = mg(y_A - y_B) = mgy_A - mgy_B > 0$$

Note that this formula is also valid for the case when the object is being lifted (with the gravitational force opposite to displacement). In that case $W_{grav} = mgy_A - mgy_B < 0$



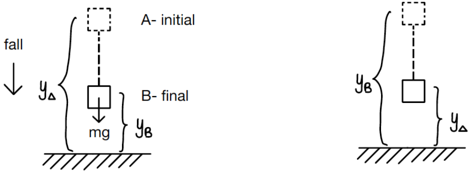
Define the function $\boxed{U_{grav}(y) = mgy}$. Then

$$W_{grav} = \underbrace{mgy_A}_{U_{grav,A}} - \underbrace{mgy_B}_{U_{grav,B}} = -(U_{grav,B} - U_{grav,A}) = -\Delta U_{grav}.$$

Comment. The minus sign is important.

Interpretation

$$W_{grav} = \underbrace{mgy_A}_{U_{grav,A}} - \underbrace{mgy_B}_{U_{grav,B}} = -\Delta U_{grav}.$$



- particle moves downwards (y decreases)

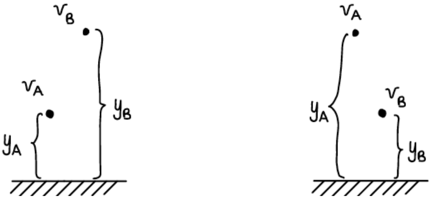
$$W_{grav} > 0 \quad \Longrightarrow \quad \Delta U_{grav} < 0$$

- particle moves upwards (y increases)

$$W_{grav} < 0 \quad \Longrightarrow \quad \Delta U_{grav} > 0$$

The function U_{grav} is called the gravitational potential energy.

Conservation of Mechanical Energy (I) (Only Gravitational Forces Acting)



A- initial

B- final

Work-kinetic energy theorem

$$W = K_B - K_A = \Delta K,$$

where W is the work done by the net force. Here *net force* = *gravitational force* (assumed $\overline{F}_{other} = 0$), so that

$$W = W_{grav} = K_B - K_A = \boxed{\Delta K}.$$

But

$$W_{grav} = U_{grav,A} - U_{grav,B} = \boxed{-\Delta U_{grav}}.$$

Hence $\Delta K = -\Delta U_{grav}$ or, equivalently,

$$\boxed{\Delta K + \Delta U_{grav} = 0.}$$

Conservation of Mechanical Energy (only gravitational forces present)

When only the force of gravity does work, the sum

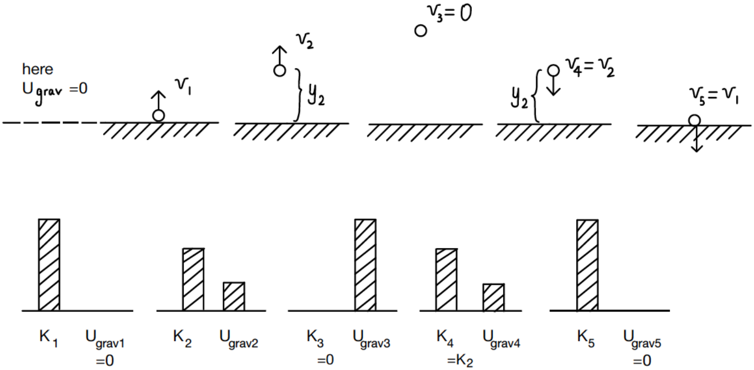
$$E = U_{grav} + K,$$

called the *total mechanical energy* of the system is constant (that is, it is *conserved*).

Comments

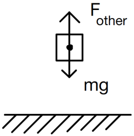
1. Explicitly, $mgy + \frac{1}{2}mv^2 = \mathrm{const.}$
2. U_{grav} is determined up to an additive constant, $U_{grav} = mgy + C$. Only the difference ΔU_{grav} is measurable (physical), hence the constant plays no role (we can choose the reference level for U_{grav} at will.)

Illustration



$$K_1 + U_{grav,1} = K_2 + U_{grav,2} = \dots = K_5 + U_{grav,5} = const$$

What If Other Forces (Other Than Gravity) Are Present?



Total work

$$W = W_{grav} + W_{other}$$

Use work-kinetic energy theorem

$$W = \Delta K \quad \Longrightarrow \quad W_{grav} + W_{other} = K_B - K_A,$$

where $W_{grav} = -\Delta U_{grav} = U_{grav,A} - U_{grav,B}$. Hence,

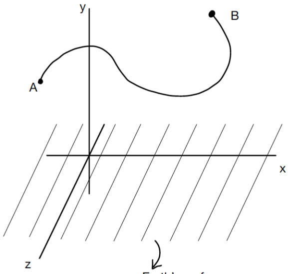
$$\underbrace{K_A + U_{grav,A}}_{E_A} + W_{other} = \underbrace{K_B + U_{grav,B}}_{E_B}.$$

So that $W_{other} = E_B - E_A = \Delta E$.

Conclusion

The work done by all forces other than the gravitational force is equal to the change in the total mechanical energy $E = U_{grav} + K$ of the system.

Work Done by Gravitational Force Along a Curved Path. Path Independence



$$W_{grav,A\to B} = \int\limits_{\Gamma_{AB}} \overline{F}_{grav} \circ \mathrm{d}\overline{r}$$

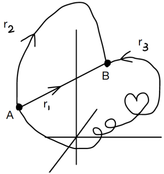
$$\overline{F}_{grav} = (0, -mg, 0)$$

$$\mathrm{d}\overline{r} = (dx, dy, dz)$$

$$\begin{aligned} W_{grav,A\to B} &= \int\limits_{\Gamma_{AB}} \overline{F}_{grav} \circ \mathrm{d}\overline{r} = \int\limits_{y_A}^{y_B} (-mg)\,\mathrm{d}y = mgy_A - mgy_B \\ &= U_{grav,A} - U_{grav,B} \end{aligned}$$

$$\boxed{W_{grav,A\to B} = U_{grav,A} - U_{grav,B}}$$

Work done by the gravitational force on a particle does not depend on path! It depends only on the initial and the final position of the particle.



$$W_{grav,\Gamma_1} = W_{grav,\Gamma_2} = W_{grav,\Gamma_3}$$

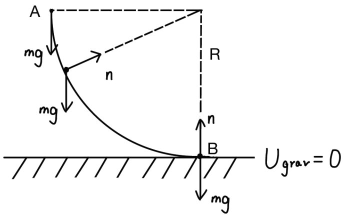
Note. This is true for any constant force.

A force with the property that work done by it on a particle does not depend on the path the particle is moved along, but only on the initial and the final position of the particle, is called a **conservative** (or **potential**) **force**.

Observation. $\oint_\Gamma \bar{F} \circ d\bar{r} = 0$ for any conservative force and any closed path Γ.

Example (a). Block Sliding down a Frictionless Ramp

What is the speed of the object as it leaves the ramp?



$$K_A = 0, \qquad U_{grav,A} = mgR$$
$$K_B = \frac{1}{2}mv_B^2, \qquad U_{grav,B} = 0$$

FoR — ramp (inertial)

Other forces (here, the normal force) present but do no work, because $\overline{n} \circ d\overline{r} = 0$. Can use conservation of mechanical energy $K_A + U_{grav,A} = K_B + U_{grav,B}$, to find

$$mgR = \frac{1}{2}mv_B^2 \qquad \Longrightarrow \qquad \boxed{v_B = \sqrt{2gR}.}$$

Comment. The result does not depend on the shape of the ramp.

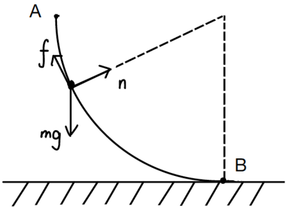
What is the magnitude of the normal force that the object exerts on the ground at point B?

$$F_{centripetal} = n - mg \qquad \Longrightarrow \qquad n = m \underbrace{a_{centripetal}}_{v^2/R} + mg\,.$$

As $v_B = \sqrt{2gR}$, hence at the point B, we have $ma_{centripetal} = 2mg$ and

$$\boxed{n} = 2mg + mg = \boxed{3mg}$$

Example (b). Block Sliding down a Rough Ramp



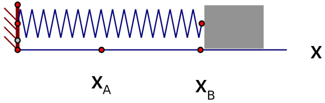
$$W_{other} = \Delta E < 0$$

$$v_B < \sqrt{2gR}$$

Here v_B depends on the shape of the ramp. Frictional forces are non-conservative and their work depends on path!

Elastic Force. Elastic Potential Energy

Elastic Potential Energy



Recall

- work done by an external force on the spring
  $W_{\text{ext}} = \frac{1}{2}kx_B^2 - \frac{1}{2}kx_A^2$,
- work done by the spring (the elastic restoring force), e.g. on a block
  $W_{\text{el}} = \frac{1}{2}kx_A^2 - \frac{1}{2}kx_B^2$.

Define the **elastic potential energy** $\boxed{U_{\text{el}} = \frac{1}{2}kx^2.}$

Now, work done on the block by the elastic force

$$W_{\text{el}} = \frac{1}{2}kx_A^2 - \frac{1}{2}kx_B^2 = U_{\text{el},A} - U_{\text{el},B} = -\Delta U_{\text{el}}$$

Examples. For a spring that is being stretched $W_{\text{el}} < 0$ and $\Delta U_{\text{el}} > 0$. For a spring that relaxes (from the stretched length) $W_{\text{el}} > 0$ and $\Delta U_{\text{el}} < 0$.

Comment

Note that U_{el} is determined up to an additive constant, that is any function

$$U_{\text{el}}(x) = \frac{1}{2}kx^2 + C,$$

where C is a constant, can be defined as the elastic potential energy. This is because only the potential energy difference is measurable (physical).

For convenience, we usually set $C = 0$.

Conservation of Mechanical Energy (II) (Only Elastic Forces Acting)

From the work-kinetic energy theorem $W = K_B - K_A = \Delta K$, where W is the work done by the net force. Here *net force* = *elastic force* (assumed $\overline{F}_{other} = 0$), so that

$$W = W_{\text{el}} = K_B - K_A = \boxed{\Delta K}.$$

But

$$W_{\text{el}} = U_{\text{el},A} - U_{\text{el},B} = \boxed{-\Delta U_{\text{el}}}.$$

Hence $\Delta K = -\Delta U_{\text{el}}$ or, equivalently,

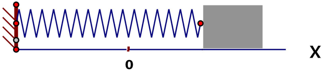
$$\boxed{\Delta K + \Delta U_{\text{el}} = 0.}$$

Conservation of Mechanical Energy (only elastic force present)

When only the elastic force does work, the sum $E = U_{\text{el}} + K$, called the *total mechanical energy* of the system is constant (that is, it is *conserved*).

Comment. Explicitly, $\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \text{const.}$

Example. Conservation of Mechanical Energy in SHM



$\omega_0^2 = k/m$

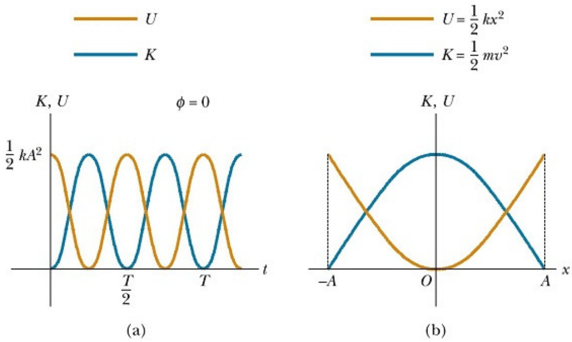
Recall

$$x(t) = A\cos(\omega_0 t + \phi), \qquad v(t) = \dot{x}(t) = -A\omega_0 \sin(\omega_0 t + \phi).$$

The total mechanical energy of the simple harmonic oscillator

$$\begin{aligned} E &= \overbrace{\frac{1}{2}kx^2(t)}^{U_{\rm el}} + \overbrace{\frac{1}{2}mv^2(t)}^{K} \\ &= \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi) + \frac{1}{2}A^2\underbrace{\omega_0^2 m}_{k}\sin^2(\omega_0 t + \phi) \\ &= \frac{1}{2}kA^2[\cos^2(\omega_0 t + \phi) + \sin^2(\omega_0 t + \phi)] = \frac{1}{2}kA^2 = \text{const} \end{aligned}$$

Illustration



www.socratic.org

Elastic Force and Gravitational Force

Conservation of Mechanical Energy (III) (Elastic and Gravitational Forces Acting)

If only gravitational and elastic forces are acting, then the work done by the net force

$$W = W_{\text{grav}} + W_{\text{el}} \overset{work-k.e.\ thm}{=} K_B - K_A = \Delta K.$$

But $W_{\text{grav}} = -\Delta U_{\text{grav}}$ and $W_{\text{el}} = -\Delta U_{\text{el}}$. Hence

$$\Delta U_{\text{grav}} + \Delta U_{\text{el}} + \Delta K = \Delta E = 0.$$

That is $\boxed{E = U_{\text{grav}} + U_{\text{el}} + K = \text{const.}}$

If other forces are acting, then the work done by the net force is $W = W_{\text{grav}} + W_{\text{el}} + W_{\text{other}} = \Delta K$ and, eventually,

$$\boxed{W_{\text{other}} = \Delta E.}$$

The work done by all forces other than the gravitational force or the elastic force equals the change in the total mechanical energy of the system ($U_{grav} + U_{el} + K$).

Conservative Force and Energy Conservation. General Remarks

Conservative Forces in 1D

In general, for any conservative (potential) force in 1D

$$F_x = -\frac{\mathrm{d}U}{\mathrm{d}x},$$

where U is the corresponding potential energy, and x — the position. Hence, the elementary work done by F_x is

$$\delta W = F_x \, \mathrm{d}x = -\frac{\mathrm{d}U}{\mathrm{d}x} \, \mathrm{d}x = -\mathrm{d}U \qquad \text{(total differential of } U\text{)}$$

and the total work ($A \to B$)

$$W_{A\to B} = \int_{A\to B} \delta W = \int_A^B F_x \, \mathrm{d}x = -\int_A^B \mathrm{d}U = U(A) - U(B)$$

$$\boxed{W_{A\to B} = U(A) - U(B)}$$

That is, the total work depends only on the initial and the final position.

Note that if the net force is potential

$$\left.\begin{array}{l}\delta W = -\mathrm{d}U \\ \delta W = \mathrm{d}K\end{array}\right\} \quad \Longleftrightarrow \quad -\mathrm{d}U = \mathrm{d}K \quad \Longleftrightarrow \quad \mathrm{d}(U + K) = 0$$

Hence, if only potential forces act in the system, the total mechanical energy is conserved

$$\boxed{E = U + K = \text{const.}}$$

Conservative Forces in 3D

The concepts can be generalised to 3D. For conservative forces in 3D

$$\overline{F} = \left(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}\right) = -\nabla U.$$

Then

$$\delta W = \overline{F} \circ \mathrm{d}\overline{r} = -\frac{\partial U}{\partial x}\,\mathrm{d}x - \frac{\partial U}{\partial y}\,\mathrm{d}y - \frac{\partial U}{\partial z}\,\mathrm{d}z = -\mathrm{d}U,$$

so that again

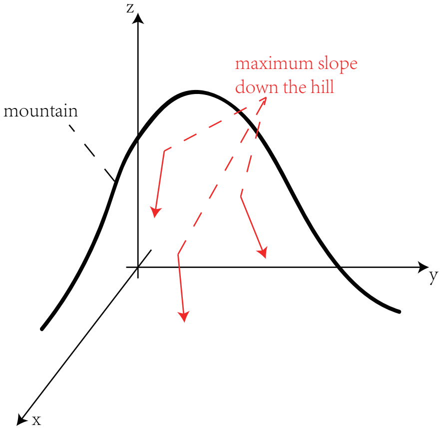
$$\boxed{W_{A\to B}} = \int_{\Gamma_{AB}} \overline{F} \circ \mathrm{d}\overline{r} = -\int_A^B \mathrm{d}U = \boxed{U(A) - U(B).}$$

Consequences

- Work $W_{A\to B}$ does not depend on the path Γ_{AB}, but only on the initial/final position.
- Work along any loop is zero ($\oint_\Gamma \overline{F} \circ \mathrm{d}\overline{r} = 0$ for any path Γ).

Force and Potential Energy

In 1D, the force always points in the direction of decreasing potential energy. In 3D, $\overline{F} = -\text{grad}\, U$, and this vector shows the direction at which at that point the potential energy decreases at the fastest rate.



What if Non-Conservative Forces Are Acting?

$$\boxed{\text{net force} = \text{conservative force} + \text{non-conservative force}}$$

$$\delta W = \underbrace{\delta W_{\text{cons}}}_{-\mathrm{d}U} + \delta W_{\text{non-cons}} \overset{work-k.e.\ thm}{=} \mathrm{d}K.$$

Hence $-\mathrm{d}U + \delta W_{\text{non-cons}} = \mathrm{d}K$ and, equivalently,

$$\boxed{\delta W_{\text{non-cons}} = \mathrm{d}(K+U) = \mathrm{d}E} \quad \text{or} \quad \boxed{\Delta W_{\text{non-cons}} = \Delta K + \Delta U = \Delta E}$$

Conclusion. Work done by non-conservative forces changes the mechanical energy of the system.

Experiments show that $\Delta W_{\text{non-cons}} = -\Delta U_{\text{int}}$, where U_{int} is the *internal energy* (all forms of energy other than the kinetic energy and the potential energy due to external forces)

Hence, the **law of conservation of the total energy**

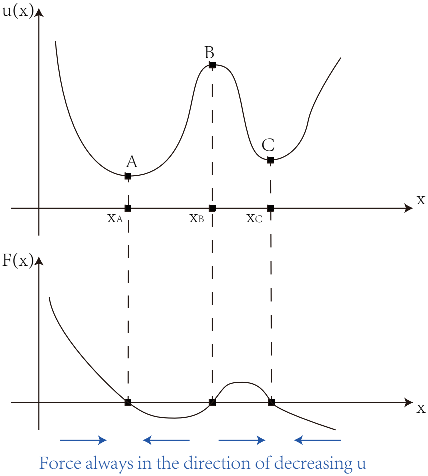
$$\boxed{\Delta K + \Delta U + \Delta U_{\text{int}} = 0}$$

Interpretation. Energy can be transformed between its different forms but the net change is always zero.

Energy Diagrams

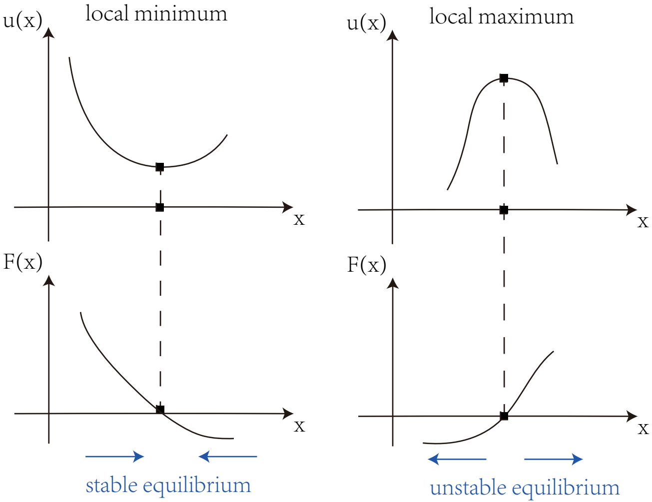
Energy Diagrams

Consider a particle moving along the x-axis, which is acted upon a conservative net force $\overline{F} = (F(x), 0, 0)$ with the corresponding potential energy $U = U(x)$, so that $F(x) = -\frac{\mathrm{d}U}{\mathrm{d}x}$. The energy diagram $U = U(x)$ can be used to analyze the motion.



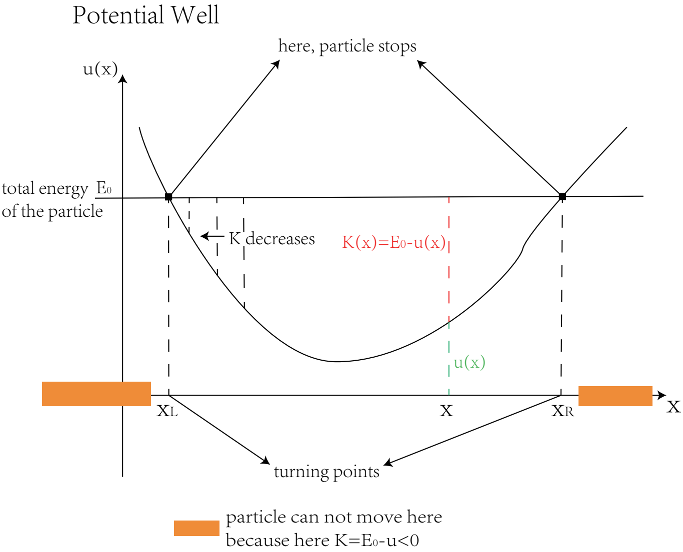
Equilibrium Points

At the positions x_A, x_B, x_C, the potential energy is minimum or maximum and $\frac{dU}{dx} = 0 \Rightarrow$, so that $F = -\frac{dU}{dx} = 0$. Consequently, if a particle is placed at such a point at rest, it will remain at rest. Therefore such points are called *equilibrium points*.



Motion in a Potential Well. Harmonic Approximation

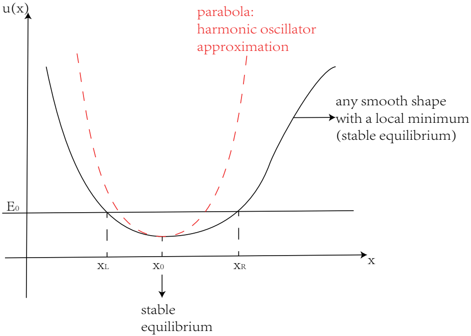
Potential Well



Observation. The particle moves back and forth between the turning points.

Harmonic Approximation

Consider motion of a particle close to the stable equilibrium at x_0 (small oscillations).



Assuming that the potential energy function is smooth, we can expand it into Taylor series around x_0

$$U(x) = U(x_0) + \frac{1}{1!}\left.\frac{dU}{dx}\right|_{x=x_0}(x-x_0) + \frac{1}{2!}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}(x-x_0)^2 + \frac{1}{3!}\left.\frac{d^3U}{dx^3}\right|_{x=x_0}(x-x_0)^3 + \dots$$

...and neglect all terms of the order higher than two in $(x-x_0)$. Recall we consider *small oscillations* and $|x-x_0| \ll (x_R - x_L)$.

Hence, within this *harmonic approximation*

$$U(x) \approx U(x_0) + \frac{1}{1!}U'(x_0)(x - x_0) + \frac{1}{2!}U''(x_0)(x - x_0)^2.$$

But x_0 is the equilibrium point, so that

$$F(x_0) = -\left.\frac{\mathrm{d}U}{\mathrm{d}x}\right|_{x=x_0} = 0 \Rightarrow U'(x_0) = 0.$$

Therefore, the harmonic approximation eventually yields

$$\boxed{U(x) \approx U(x_0) + \frac{1}{2!}U''(x_0)(x - x_0)^2.}$$

The corresponding force of a harmonic oscillator is

$$F(x) = -\frac{\mathrm{d}U}{\mathrm{d}x} = -k(x - x_0),$$

where $k = U''(x_0)$ (i.e. the curvature of $U = U(x)$ at the equilibrium position) plays the role of the *spring constant*.

Discussion

Without any loss of generality we may assume $x_0 = 0$ and $U(x_0) = 0$. Then

$$\boxed{U(x) = \frac{1}{2}kx^2 \qquad \text{and} \qquad F(x) = -kx.}$$

Conclusion. A particle moving in a potential well of *any smooth shape*, not too far away from a stable equilibrium, moves as if it was in harmonic motion with the natural frequency

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{U''(x_0)}{m}}$$

Consequently, $x(t) = A\cos(\omega_0 t + \phi)$.