Chapter 7 – Dynamics of Circular Motion
Dynamics in Non-Inertial Frames of Reference

UM-SJTU Joint Institute
Physics I (Summer 2021)
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Agenda

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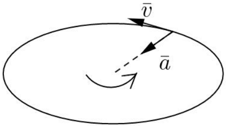
Dynamics of Circular Motion

Dynamics of Uniform Circular Motion



$|\bar{v}| = \text{const, but}$
$\bar{v} \neq \text{const}$

centripetal acceleration $\overset{2^{\text{nd}}\ \text{law}}{\Longleftrightarrow}$ **force**

(towards the center, always)

here radial $\parallel$ *normal*

$$\bar{a}_r = -\frac{v^2}{R}\hat{n}_r = -\omega^2 R\hat{n}_r$$

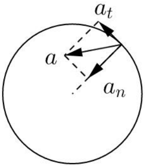
(called **centripetal force**)

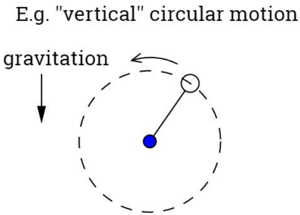
$$\bar{F}_r = m\bar{a}_r = -m\frac{v^2}{R}\hat{n}_r = -m\omega^2 R\hat{n}_r$$

Various forces may play the role of the centripetal force: tension in a cord (contact force); gravitational force; electromagnetic force (field forces),. . .

Dynamics of Non-Uniform Circular Motion

$$|\bar{v}| \neq \text{const} \quad \Rightarrow \quad a_t \neq 0 \quad \Rightarrow \quad \text{force in the tangential direction}$$



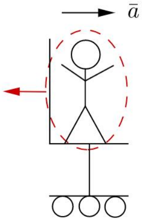


Dynamics in Non-Inertial Frames of Reference. Forces of Inertia

Motivation

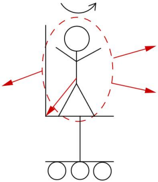
Examples/ demonstrations

(1) accelerating chair: moving along a straight line



feels a "force" pushing him in the direction opposite to the direction of acceleration

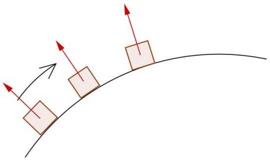
(2) rotating chair



feels a centrifugal "force"

Motivation

(3) chair moving along a curved path



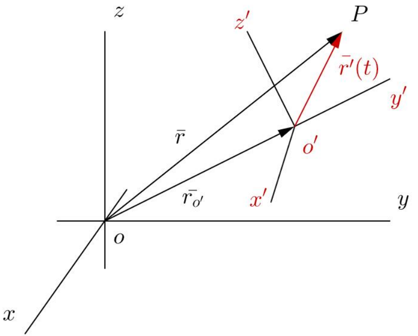
(view from the top)

feels a centrifugal "force" (directed outwards, along the instantaneous radius of curvature)

These "forces" cannot be regular forces: real forces are always of a material origin and always appear in pairs (Newton's third law).

What is the nature and the origin of these "forces"?

Equation of Motion in a Non-Inertial FoR



xyz — inertial FoR

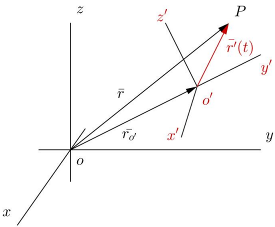
$x'y'z'$ — another FoR that moves arbitrarily w.r.t. *xyz* (may be accelerating, rotating with varying angular velocity, . . .)

Previously, we have considered a situation when $x'y'z'$ was moving with a constant velocity w.r.t *xyz* (then $x'y'z'$ is inertial as well). Now, it moves arbitrarily, hence is not inertial.

Goal: Derive a (kinematic) relationship between accelerations of a particle P in both frames of reference.

Relationship between position vectors in both FoR

$$\bar{r} = \bar{r}_{O'} + \bar{r}'$$



where

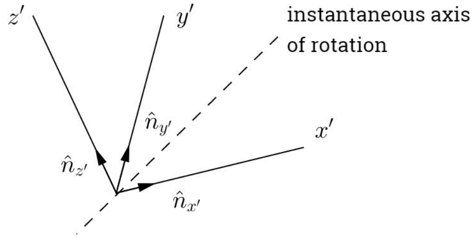
- $\bar{r}$ – position of the particle in xyz (we use fixed unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$ here)
- $\bar{r}_{O'}$ — position of the origin of $x'y'z'$ as seen from xyz;
- $\bar{r}'$ — position of the particle in $x'y'z'$ (here, we have to use $\hat{n}_{x'}, \hat{n}_{y'}, \hat{n}_{z'}$, that are not fixed; in particular may rotate)

Complication: Motion of $x'y'z'$ is arbitrary, will need to take into account that $\hat{n}_{x'}, \hat{n}_{y'}, \hat{n}_{z'}$ are not fixed (i.e. will need to know how to calculate their derivatives w.r.t. time).

Relation between velocities (mathematical details skipped here)

$$\boxed{\bar{v} = \bar{v}_{O'} + \bar{v}' + (\bar{\omega} \times \bar{r}')}$$

Comment: The arbitrary motion of $x'y'z'$ can be decomposed into a translational motion and a rotational motion about an instantaneous axis of rotation; the last term is due to the latter.



Eventually, the **relation between accelerations**

$$\boxed{\bar{a} = \bar{a}_{O'} + \bar{a}' + 2\bar{\omega} \times \bar{v}' + \frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}' + \bar{\omega} \times (\bar{\omega} \times \bar{r}')}$$

Multiply by m and rearrange (leave $m\bar{a'}$ on one side; move all other terms move to the other side)

$$m\bar{a}' = m\bar{a} - m\bar{a}_{O'} - m\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}' - 2m(\bar{\omega} \times \bar{v}') - m\bar{\omega} \times (\bar{\omega} \times \bar{r}')$$

But $m\bar{a} = \bar{F}$ because xyz is an inertial FoR (can use 2^{nd} law)

$\downarrow$

real force (i.e. of material origin)

Equation of Motion in a Non-Inertial FoR

$$m\bar{a}' = \underbrace{\bar{F}}_{} \underbrace{-m\bar{a}_{O'} - m\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}' - 2m(\bar{\omega} \times \bar{v}') - m\bar{\omega} \times (\bar{\omega} \times \bar{r}')}_{}$$

real force
(of material origin)

pseudo-forces (also called *fictitious forces* or forces of inertia); these are kinematic corrections (which have units of [N]) that **must** be included due to the fact that we describe dynamics in a non-inertial FoR (*force* = *mass* × *acceleration* is valid only in inertial FoRs). These "forces" **never** appear in inertial FoRs!

Each of the fictitious forces has its own name

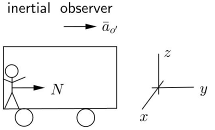
$-m\bar{a}_{O'}$	d'Alembert (or drift) "force"
$-m\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}'$	Euler "force"
$-2m(\bar{\omega} \times \bar{v}')$	Coriolis "force"
$-m\bar{\omega} \times (\bar{\omega} \times \bar{r}')$	centrifugal "force"

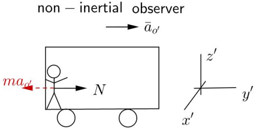
When discussing dynamics in a non-inertial FoR, do remember to distinguish these pseudo-forces (kinematic corrections) from real forces (due to interactions)!

Example I. Accelerating Car Moving in a Straight Line

$$\boxed{m\overline{a'} = \overline{F} - m\bar{a}_{O'} - m\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} \times \overline{r}' - 2m(\overline{\omega} \times \overline{v}') - m\overline{\omega} \times (\overline{\omega} \times \overline{r}')}$$

Assume $\bar{a}_{O'} \neq 0$, $\overline{\omega} = 0, \overline{v}' = 0$; e.g. accelerating car moving along a straight line.





$$ma_{O'} = N$$

"He moves with acceleration $\bar{a}_{O'}$, because there is a net force (normal force $\overline{N}$ due to the wall) acting upon him."

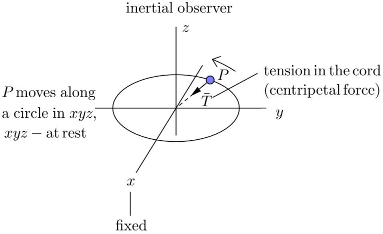
$$0 = N - ma_{O'}$$

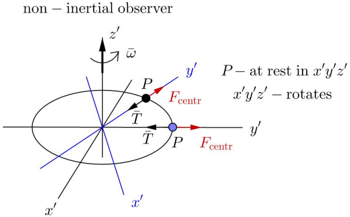
"He is at rest in my FoR because the normal force is balanced by the d'Alembert 'force'."

Mathematically, both equations are **identical**, but their physical interpretation is **different**!

Example II. Uniform Circular Motion

(2) $\overline{a}_{O'} = 0$; $\overline{\omega} = \text{const}$; $\overline{v}' = 0$; e.g. uniform circular motion





$$m\overline{a} = \overline{T}$$

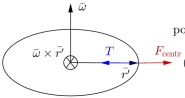
$$-m\omega^2 R\hat{n}_r = \overline{T}$$

$$0 = \overline{F}_{\text{centr}} + \boxed{\overline{T}}$$

$\downarrow$

cannot call it centripetal

"He moves in a circle with acceleration $-\omega^2 R\hat{n}_r$, because there is tension in the cord that plays a role of the centripetal force."



Since $\overline{r'} \perp \overline{\omega}$ and $\overline{\omega} \perp (\overline{\omega} \times \overline{r'})$ then

$$|\overline{F}_{\text{centr}}| = |-m\overline{\omega} \times (\overline{\omega} \times \overline{r'})| = m\omega^2 r' = m\omega^2 R.$$

Alternatively, use the formula for the double cross product

$$\overline{\omega}\times(\overline{\omega}\times\overline{r'}) = \overline{\omega}(\underbrace{\overline{\omega} \circ \overline{r'}}_{=0}) - \overline{r'}(\underbrace{\omega \circ \overline{\omega}}_{=\omega^2})$$

$$\overline{F}_{\text{centr}} = m\omega^2\overline{r'} = m\omega^2 R\hat{n}_r$$

So eventually

$$0 = m\omega^2 R\hat{n}_r + \overline{T}$$

"He rests in my frame of reference, because the tension in the cord is balanced by the 'centrifugal force'."

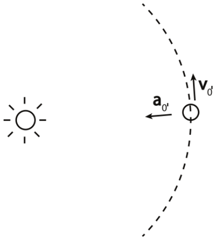
Again, both equations are algebraically identical, but they are formulated by the two observers using different vocabulary.

Earth as a Frame of Reference

Motion of the Earth

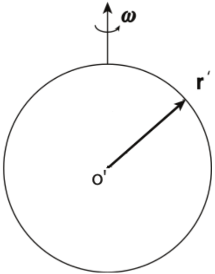
The Earth moves both in an orbital motion around the Sun and a rotational motion about its axis (with almost exactly constant angular velocity).

orbital motion



$$\overline{v}_{O'} \approx 30 \text{ km/s}$$

rotational motion



$$|\overline{\omega}| = \frac{2\pi}{T} = \frac{2\pi}{24 \text{ h}} \approx 7\times10^{-5} \, [\text{s}^{-1}]$$

The equation of motion of an object on the rotating Earth, in the FoR associated with the Earth and fixed at its center

$$m\bar{a}' = \overline{F} - m\bar{a}_{O'} - 2m(\overline{\omega} \times \overline{v}') - m\overline{\omega} \times (\overline{\omega} \times \overline{r}'),$$

where the net material force $\overline{F}_{\text{Sun}} + \overline{F}_{\text{Earth}} + \overline{(\text{other forces})}$.

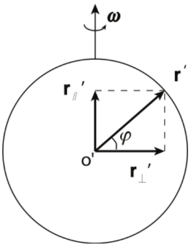
For the non-inertial observer sitting at the center of the Earth $\overline{F}_{\text{Sun}} = m\bar{a}_{O'}$, and eventually (assuming that "other forces" do not act), the equation of motion is

$$\boxed{m\bar{a}' = \overline{F}_{\text{Earth}} \underbrace{- m\overline{\omega} \times (\overline{\omega} \times \overline{r}')}_{\text{centrifugal force}} \underbrace{- 2m(\overline{\omega} \times \overline{v}')}_{\text{Coriolis force}}}$$

Task. *Investigate the effect of the centrifugal force and the Coriolis force to see whether their contributions can be ignored (that is whether the FoR of the Earth can be treated as inertial?*

Centrifugal Force Contribution

The position vector $\overline{r}'$ can de decomposed into $\overline{r}'_\perp$ and $\overline{r}'_\parallel$.



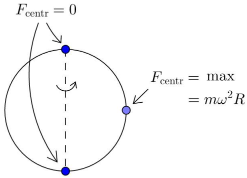
$$
\begin{aligned}
\overline{F}_{\text{centr}} &= -m\overline{\omega} \times (\overline{\omega} \times (\overline{r}'_\perp + \overline{r}'_\parallel)) \\
&= -m\overline{\omega} \times (\overline{\omega} \times \overline{r}'_\perp + \underbrace{\overline{\omega} \times \overline{r}'_\parallel}_{=0}) \\
&= -m\overline{\omega} \times (\overline{\omega} \times \overline{r}'_\perp).
\end{aligned}
$$

Hence

$$\boxed{F_{\text{centr}}} = m\omega^2 r'_\perp = \boxed{m\omega^2 R \cos\varphi}$$

where R — Earth's radius, and φ — latitude.

Note. The magnitude of the centrifugal force is different at different positions on the Earth: maximum on the equator and minimum (zero) at the poles.



Is the effect of the centrifugal force significant?

Estimate

$$\frac{(F_{\text{centr}})_{\max}}{mg} = \frac{\omega^2 R}{g} \approx 0.003$$

Conclusion. The effect of the centrifugal force is very small. To make this ratio ≈ 1, the Earth would need to spin 18 times faster. Then one day wold last only ca. 1h 20min.

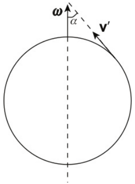
Digression. Because of the variation of the centrifugal force with latitude and the fact that the Earth's surface is not rigid, the real shape of the Earth is not exactly a sphere, but rather closer to an ellipsoid with $R_{\text{pole}} \approx 6357$ km and $R_{\text{equator}} \approx 6378$ km.

(Scale is greatly exaggerated); Source: `www.fws.gov`

Coriolis Force Contribution

For the Coriolis force to appear, the object must be moving in the Earth's FoR ($\overline{v}' \neq 0$)

$$\overline{F}_C = -2m(\overline{\omega} \times \overline{v}').$$



The magnitude $F_C = 2m|\overline{\omega}||\overline{v}'|\sin\alpha$, where $\alpha = \angle(\overline{\omega}, \overline{v}')$. It is maximum for $\alpha = \pi/2$ (free fall above the equator), and minimum (zero) for $\alpha = 0$ (free fall above poles).

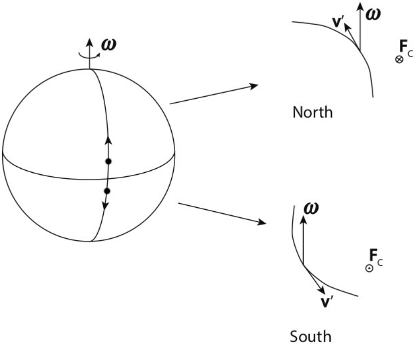
Is the effect of the Coriolis force significant?

$$\frac{(F_C)_{\max}}{mg} = \frac{2\omega}{g} v' \approx 1.5 \times 10^{-5} \left[\frac{\mathrm{s}}{\mathrm{m}}\right] v'.$$

Note. This ratio would equal to one if the object were moving at a speed of 66 km/s, i.e. 2.2 times faster than the Earth moves in its orbital motion!Therefore the effects due to the Coriolis force can *usually* be ignored (unless objects move very fast or are very massive).

Digression. Coriolis Force in Nature

Observation. On the northern hemisphere, objects are are deflected to the right, while on the southern hemisphere they are deflected to the left with respect to the original direction of motion.



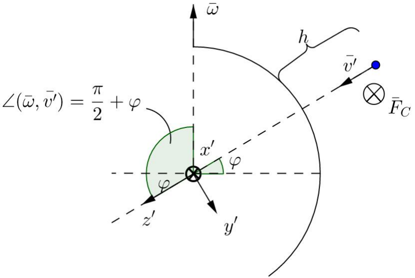
Implications

- Erosion of river banks;
- Foucault's pendulum;
- Wind patterns: cyclones, anticyclones, prevailing westerlies,. . . [see recitation class]

Example. Free Fall on a Rotating Earth

Free Fall on a Rotating Earth

Observation. When an object falls from a point above the surface of the earth, it will not fall directly on the position which is exactly vertically below on the surface. *Why?*



(the sketch is exaggerated; we still assume constant gravitational force; object close to the Earth's surface)

φ – latitude

Coriolis force

$$\bar{F}_C = -2m(\bar{\omega} \times \bar{v}'),$$

$$F_C = |\bar{F}_C| = 2m\omega v' \sin \angle(\bar{\omega}, \bar{v}').$$

But $\angle(\bar{\omega}, \bar{v}') = \frac{\pi}{2} + \varphi$, hence

$$F_C = 2m\omega v' \cos \varphi. \qquad (!)$$

In free fall $v' = gt$, so $F_C = 2mgt\omega\cos\varphi$.The acceleration due to this force (directed towards the east)

$$a_{x'} = \frac{F_C}{m} = 2g\omega t\cos\varphi.$$

So the velocity towards the east $v_{x'} = \int_0^t a_C\,\mathrm{d}t = g\omega t^2\cos\varphi$ and the deflection

$$x' = \int\limits_0^t v_{x'}\,\mathrm{d}t = \frac{1}{3}g\omega t^3\cos\varphi.$$

The time after the object reaches the ground is found from $h = \frac{gt_{\text{fall}}^2}{2}$ as $t_{\text{fall}} = \sqrt{\frac{2h}{g}}$. Therefore, the deflection after the object hits the ground is

$$x_{\text{defl}} = x'(t_{\text{fall}}) = \boxed{\frac{1}{3}g\omega\left(\frac{2h}{g}\right)^{\frac{3}{2}}\cos\varphi}.$$

Discussion

- This deflection is relatively small, but still measurable: for $h = 642$ m and $\varphi = 52^o$ N, we find $x_{\mathrm{defl}} = 22$ cm.
- On the poles (N or S), we have $\varphi = \frac{\pi}{2}$, and hence no deflection is observed.
- On the equator ($\varphi = 0$) the deflection is maximum
- **Caution!** This is an approximate analysis. In fact $\bar{v} = \bar{v}_{z'} + \bar{v}_{x'}$ and this total velocity should be used in the formula for the Coriolis' force.We have only included the z'-component (see the step marked with (!)). A complete analysis of the problem is complicated.