PHYSICS I Problem Set 3

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Problem 1 **Solution**

(a) Since the tangential component of the particle's acceleration is a constant, we may as well suppose this constant as a_τ. Then $v_{(t)} = a_\tau t$.

For polar coordinate system, $\bar{a} = (\ddot{r} - r(\dot{\varphi})^2)\hat{n}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{n}_\varphi$

In circular motion we know

$$r = \text{R} = \text{constant} \quad \text{Hence } \dot{r} = \ddot{r} = 0$$
$$\text{Besides, radial} = \text{normal}$$

Therefore, $|\bar{a}_n| = r(\dot{\varphi})^2$. Note that $\bar{v} = \dot{r}\,\hat{n}_r + r\,\dot{\varphi}\,\hat{n}_\varphi$, namely, $v = r\,\dot{\varphi}$. We rewrite

$$|\bar{a}_n|_{(t)} = \frac{v^2}{R} = \frac{a_\tau^2 t^2}{R}$$

(b) According to (a), we have $|\bar{a}_n|_{(t)} = \frac{a_\tau^2 t^2}{R}$. We then know

$$|\bar{a}_{(t)}| = \sqrt{a_\tau^2 + a_n^2} = \sqrt{a_\tau^2 + (\frac{a_\tau^2 t^2}{R})^2} = \frac{a_\tau}{R}\sqrt{R^2 + a_\tau^2 t^4}$$

Based on what we have calculated so far, we can easily calculate the angle θ that the vector $\mathbf{a}$ forms with the position vector $\mathbf{r}$. Since $\mathbf{a} = \bar{a}_n + \bar{a}_\tau$ and $\bar{r}$ is negative direction to $\bar{a}_n$, we have

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{r}}{|\mathbf{a}|\cdot|\mathbf{r}|} = \frac{\bar{a_\tau}\bar{r} + \bar{a_n}\bar{r}}{\sqrt{a_\tau^2 + a_n^2}\cdot R} = \frac{-|\bar{a}_n|_{(t)}}{\sqrt{a_\tau^2 + a_n^2}} = \frac{-a_\tau t^2}{\sqrt{R^2 + a_\tau^2 t^4}}$$
$$\Rightarrow \quad \theta = \arccos(\frac{-a_\tau t^2}{\sqrt{R^2 + a_\tau^2 t^4}})$$

Problem 2 **Solution**

(a) Set the ground as our frame of reference.

According to Newton's second law, since the magnitude and direction of the ball's velocity remains constant, the net force the ball exerted is zero. Therefore, angle that the string forms with the vertical direction is $0°$.

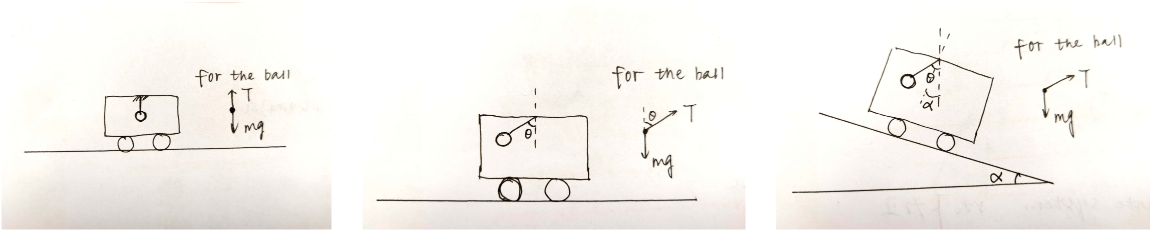
(b) Suppose the magnitude of acceleration is a. According to free body diagram(b), we have

$$\begin{cases} T\cos\theta = mg \\ T\sin\theta = ma \end{cases} \quad \Rightarrow \quad \theta = \arctan(\frac{a}{g})$$

Therefore, the angle that the string forms with the vertical direction is $\arctan(\frac{a}{g})°$.

(c) Since the ball is attached to the roof of the car, it shares the same acceleration of that car. Then according to free body diagram(c), for the system we have

$$\begin{cases} T\cos(\theta-\alpha) = mg\cos\alpha \\ m_{\text{ball}}a = m_{\text{ball}}g\sin\alpha + T\sin(\theta-\alpha) \\ m_{\text{car}}a = m_{\text{car}}g\sin\alpha \end{cases} \Rightarrow \begin{cases} \theta = \alpha \\ T = mg \end{cases}$$



(a) Constant speed along a straight line

(b) Constant acceleration along a straight line

(c) Sliding down a slide

Figure 1: Free body diagrams in Problem 2

Problem 3 Solution

(a) We first consider m_1 and m_2 as a whole, and then draw the free body diagram of m_3 and the system of m_1 and m_2. Then we have

$$m_3 a = m_3 g\sin 2\alpha - T_3$$
$$(m_1+m_2)a = T_3 - (m_1+m_2)g\sin\alpha - \mu_1 m_1 g\cos\alpha - \mu_2 m_2 g\cos\alpha$$

$\Rightarrow$

$$a = \frac{m_3 g\sin 2\alpha - (m_1+m_2)g\sin\alpha - \mu_1 m_1 g\cos\alpha - \mu_2 m_2\cos\alpha}{m_1+m_2+m_3}$$
$$T_3 = \frac{m_3}{m_1+m_2+m_3}\cdot((m_1+m_2)(\sin 2\alpha+\sin\alpha)g + \mu_1 m_1 g\cos\alpha + \mu_2 m_2 g\cos\alpha)$$

Then we analyze m_1 and m_2 separately

$$m_2 a = T_2 - T_1 - m_2 g\sin\alpha - \mu_2 m_2 g\cos\alpha$$
$$m_1 a = T_1 - m_1 g\sin\alpha - \mu_1 m_1 g\cos\alpha$$

$\Rightarrow$

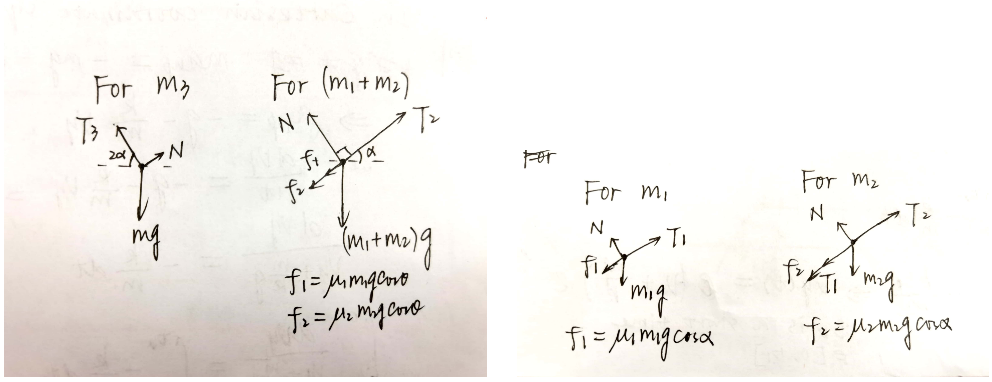
$$T_1 = \frac{m_1}{m_1+m_2+m_3}\cdot(m_3 g\sin 2\alpha + m_3 g\sin\alpha + \mu_1 m_3 g\cos\alpha + \mu_1 m_2 g\cos\alpha - \mu_2 m_2 g\cos\alpha)$$
$$T_2 = \frac{m_3 g}{m_1+m_2+m_3}\cdot((m_1+m_2)(\sin 2\alpha+\sin\alpha) + (\mu_1 m_1+\mu_2 m_2)\cos\alpha)$$

(b) Since acceleration and tensions cannot be negative and denominator cannot be zero, we have

$$m_3\sin 2\alpha > (m_1+m_2)\sin\alpha + \mu_1 m_1\cos\alpha + \mu_2 m_2\cos\alpha$$
$$m_3 - m_1 - m_2 > 0$$
$$\mu_1 m_1 g\cos\alpha \le m_1 g\sin\alpha$$
$$\mu_2 m_2 g\cos\alpha \le m_2 g\sin\alpha$$

$\Rightarrow$

$$m_3 > \frac{m_1 + m_2}{2\cos\alpha} + \frac{\mu_1 m_1 + \mu_2 m_2}{2\sin\alpha} \quad \text{and} \quad \mu_1 \le \tan\alpha \quad \text{and} \quad \mu_2 \le \tan\alpha$$



(a) Free body diagram of m_3 and the system of m_1 and m_2

(b) Free body diagram of m_1 and m_2

Figure 2: Free body diagrams in Problem 3

Problem 4 **Solution**

Set the upwards as the positive direction.

(a) According to Newton's third law, the rope gives the student a reactive force when the student exerts an acting force on the rope, namely, $F' = F$. We draw a free body diagram and analyze the student and the bar as a whole system.

According to the diagram, suppose the rope around pulley exerts T on the bar, the mass of the student is m_1, the mass of the bar is m_2, we have

$$\begin{cases} T = F' \\ (m_1 + m_2)a = T + F' - (m_1 + m_2)g \end{cases} \quad \Rightarrow \quad a = \frac{5}{12}\ m/s^2 = 0.417\ m/s^2$$

(b) We then analyze the bar only

Suppose the force the student exerts on the bar is F_T. According to the free body diagram, we have

$$m_2 a = F_T - m_2 g + T$$
$$\Rightarrow F_T = \frac{250}{3} N = -83.3N$$

Namely, the magnitude of F_T is 83.3 N, with pointing downwards.

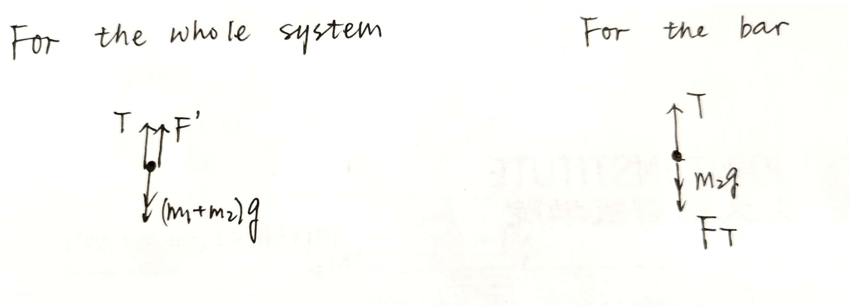


Figure 3: Free body diagrams in Problem 4

Problem 5 **Solution**

For the whole system

$$F - \mu_k Mg = (m+M)a \quad \Rightarrow \quad a = \frac{F - \mu_k Mg}{m+M}$$

Denote the tension as T and the distance from the block is x. For the rope

$$T - \mu_k Mg = (\frac{m}{d} \cdot x + M)a$$
$$\Rightarrow \quad T = \frac{mF - \mu_k mMg}{md + Md} \cdot x + \frac{\mu_k mMg + MF}{m+M}$$

Problem 6 **Solution**

According Newton's second law

$$ma = F$$
$$\Rightarrow \quad \bar{a} = (2\,\sin 2t, 3t - 6, -3e^{-3t})$$

The integrate!

$$\begin{cases} \int_{v(0)}^{v(t)} dv = \int_0^t \bar{a} dt \\ \int_{r(0)}^{r(t)} dr = \int_0^t \bar{v} dt \end{cases} \quad \Rightarrow \quad \begin{array}{c} v(t) = (3 - \cos 2t, \frac{3}{2}t^2 - 6t, e^{-3t}) \\ r(t) = (-\frac{1}{2}\sin 2t + 3t + 5, \frac{1}{2}t^3 - 3t^2 + 2, -\frac{1}{3}e^{-3t} - \frac{8}{3}) \end{array}$$

Problem 7 **Solution**

We set upwards as the positive direction.

1) We first calculate the uprising procedure of the ball. According to Newton's second law

$$ma_u = -mg - kv_y$$
$$\Rightarrow \quad a_u = -(g + \frac{k}{m}v_y)$$

Notice that $a_u = \frac{dv_y}{dt}$, we then get

$$a_u = \frac{dv_y}{dt} = -g - \frac{k}{m}v_y = -\frac{k}{m}\left(v_y + \frac{m}{k}g\right)$$
$$\frac{dv_y}{v_y + \frac{m}{k}g} = -\frac{k}{m}dt$$
$$\int_{v_0}^{v_{y(t)}} \frac{dv_y}{v_y + \frac{m}{k}g} = \int_0^t \frac{k}{m}dt$$
$$ln\left(\frac{v_{y(t)} + \frac{m}{k}g}{v_0 + \frac{m}{k}g}\right) = \frac{-kt}{m}$$
$$\Rightarrow \quad v_{y_{(t)}} = \left(v_0 + \frac{mg}{k}\right)e^{\frac{-kt}{m}} - \frac{mg}{k}$$

Analougously, we can calculate the $y_{(t)}$ as follows

$$
\begin{aligned}
v_{y(t)} &= \frac{dy_{(t)}}{dt} = \left(v_0 + \frac{mg}{k}\right) e^{-\frac{kt}{m}} - \frac{mg}{k} \\
\int_0^{y_{(t)}} dy &= \int_0^t \left(\left(v_0 + \frac{mg}{k}\right) e^{-\frac{kt}{m}} - \frac{mg}{k}\right) dt \\
y_{(t)} &= \left(v_0 + \frac{mg}{k}\right) \int_0^t e^{-\frac{kt}{m}} - \int_0^t \frac{mg}{k} dt \\
y_{(t)} &= -\left(\frac{mv_0}{k} + \frac{m^2 g}{k^2}\right) e^{-\frac{kt}{m}} - \frac{mg}{k} t + \frac{m^2 g}{k^2} + \frac{mv_0}{k}
\end{aligned}
$$

Suppose the time when the ball reaches its highest point is t_s, then $v_{y(t_s)} = 0$. Namely

$$
\begin{aligned}
& \left(v_0 + \frac{mg}{k}\right) e^{-\frac{kt}{m}} - \frac{mg}{k} = 0 \\
\Rightarrow \qquad & t_s = \frac{m}{k}\, ln\left(\frac{mg/k + v_0}{mg/k}\right)
\end{aligned}
$$

Therefore, the highest point of the ball's trajectory is

$$
\begin{aligned}
h_{max} = |y_{(t_s)}| &= -\left(\frac{m^2 g}{k^2} + \frac{mv_0}{k}\right) \cdot \frac{mg/k}{mg/k + v_0} - \frac{m^2 g}{k^2} ln\left(\frac{mg/k + v_0}{mg/k}\right) + \frac{m^2 g}{k^2} + \frac{mv_0}{k} \\
&= \frac{mv_0}{k} - \frac{m^2 g}{k^2} ln\left(\frac{mg/k + v_0}{mg/k}\right)
\end{aligned}
$$

2) We then calculate the falling procedure of the ball. Set downwards the positive disrection. According to Newton's second law

$$
\begin{aligned}
& ma_d = mg - kv_y \\
\Rightarrow \qquad & a_d = g - \frac{k}{m} v_y
\end{aligned}
$$

Analougously, we get

$$
\begin{aligned}
v_{y(t)} &= \frac{mg}{k} - \frac{mg}{k} e^{-\frac{k}{m}\left(t - \frac{m}{k}\, ln\left(\frac{mg/k+v_0}{mg/k}\right)\right)} = \frac{mg}{k} - \left(\frac{mg}{k} + v_0\right) e^{-\frac{kt}{m}} \\
y_{(t)} &= \frac{m^2 g}{k^2} e^{-\frac{k}{m}\left(t - \frac{m}{k}\, ln\left(\frac{mg/k+v_0}{mg/k}\right)\right)} + \frac{mg}{k}\left(t - \frac{m}{k}\, ln\left(\frac{mg/k + v_0}{mg/k}\right)\right) - \frac{m^2 g}{k^2} + h_{max} \\
&= \frac{m^2 g}{k^2} \cdot \frac{mg + kv_0}{mg} e^{-\frac{kt}{m}} + \frac{mg}{k} \cdot t - \frac{2m^2 g}{k^2} ln\left(\frac{mg/k + v_0}{mg/k}\right) - \frac{m^2 g}{k^2} + \frac{mv_0}{k}
\end{aligned}
$$

Therefore, we get

$$
y_{(t)} = \begin{cases} -\left(\frac{mv_0}{k} + \frac{m^2 g}{k^2}\right) e^{-\frac{kt}{m}} - \frac{mg}{k} t + \frac{m^2 g}{k^2} + \frac{mv_0}{k} & \left(0 \le t \le \frac{m}{k}\, ln\left(\frac{mg/k+v_0}{mg/k}\right)\right) \\ \frac{m^2 g}{k^2} \cdot \frac{mg+kv_0}{mg} e^{-\frac{kt}{m}} + \frac{mg}{k} \cdot t - \frac{2m^2 g}{k^2} ln\left(\frac{mg/k+v_0}{mg/k}\right) - \frac{m^2 g}{k^2} + \frac{mv_0}{k} & \left(t \ge \frac{m}{k}\, ln\left(\frac{mg/k+v_0}{mg/k}\right)\right) \end{cases}
$$

Problem 8 **Solution**

(a) We notice that $a_x = \frac{dv_x}{dt}$, then we get

$$\begin{aligned} a_x = \frac{dv_x}{dt} &= -kv_x \\ \int_{v_0}^{v_{x(t)}} \frac{dv_x}{v_x} &= \int_0^t -kdt \\ ln\left(\frac{v_{x(t)}}{v_0}\right) &= -kt \\ \Rightarrow \qquad v_{x(t)} &= v_0 e^{-kt} \end{aligned}$$

Notice that the particle will never stop in theory, so when $t \to \infty$ the particle stops. Suppose the distance the particle travels is x, we now integrate!

$$x = \int_0^\infty v_{x(t)} dt = \int_0^\infty v_0 e^{-kt} dt = \frac{v_0}{k}$$

(b) Suppose when $t = t_s$ the particle travels s. We just list the equation as follows

$$\begin{aligned} s &= \int_0^{t_s} v_0 e^{-kt} dt \\ -\frac{v_0}{k} e^{-kt}|_0^{t_s} &= s \\ \frac{v_0}{k}\left(1 - e^{-kt_s}\right) &= s \\ \Rightarrow \qquad t_s &= -\frac{1}{k} ln\left(1 - \frac{ks}{v_0}\right) \end{aligned}$$