Equilibrium

and

Elasticity

In this part we will...

- study the conditions for equilibrium of a rigid body
- define the center of gravity and understand how it relates to a body's stability
- learn how to solve problems for rigid bodies in equilibrium
- analyze situations involving tension, compression, pressure, and shear
- investigate what happens when a body is stretched so much that it deforms or breaks

Equilibrium

Introduction

Many bodies, such as bridges, aqueducts, and ladders, are designed so they do not accelerate.

Real materials are not truly rigid. They are elastic and do deform to some extent.

We shall introduce concepts such as stress and strain to understand the deformation of real bodies.

Conditions for Equilibrium

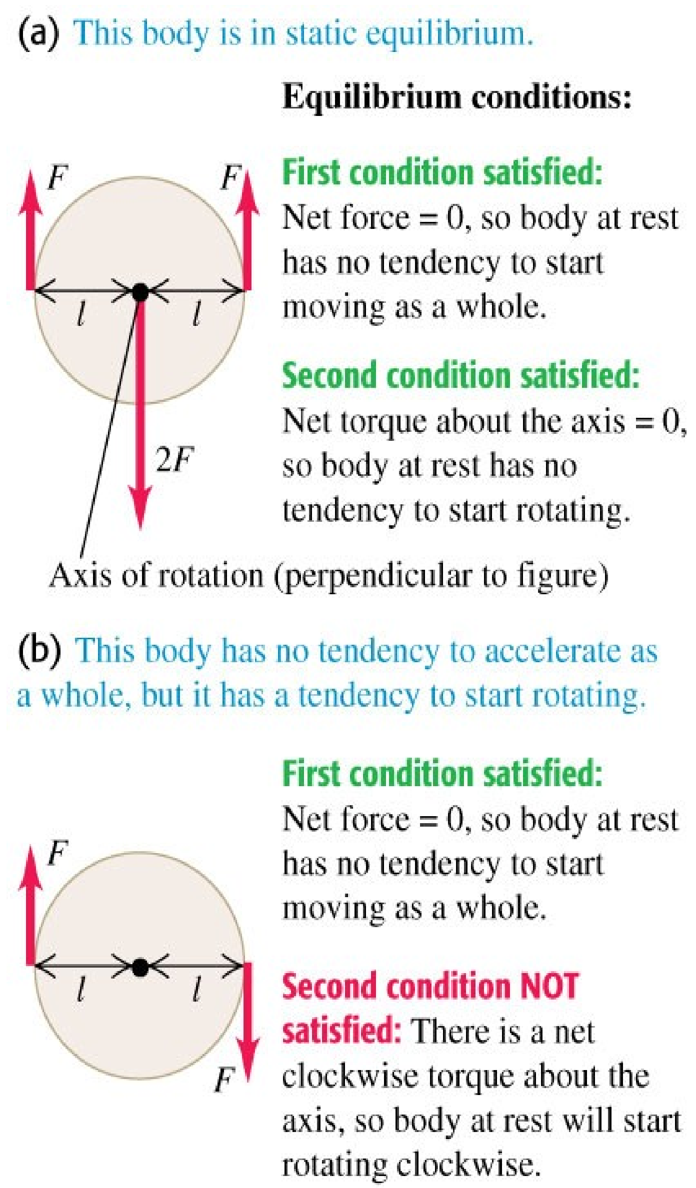
1. The sum of all the external forces is equal to zero.

$$\mathbf{F}^{\text{ext}} = 0$$

2. The sum of all torques of external forces about any point is equal to zero.

$$\boldsymbol{\tau}^{\text{ext}} = 0$$

If <u>both</u> conditions are satisfied, then the body is in equilibrium:
if it was initially at rest, then it will remain at rest.

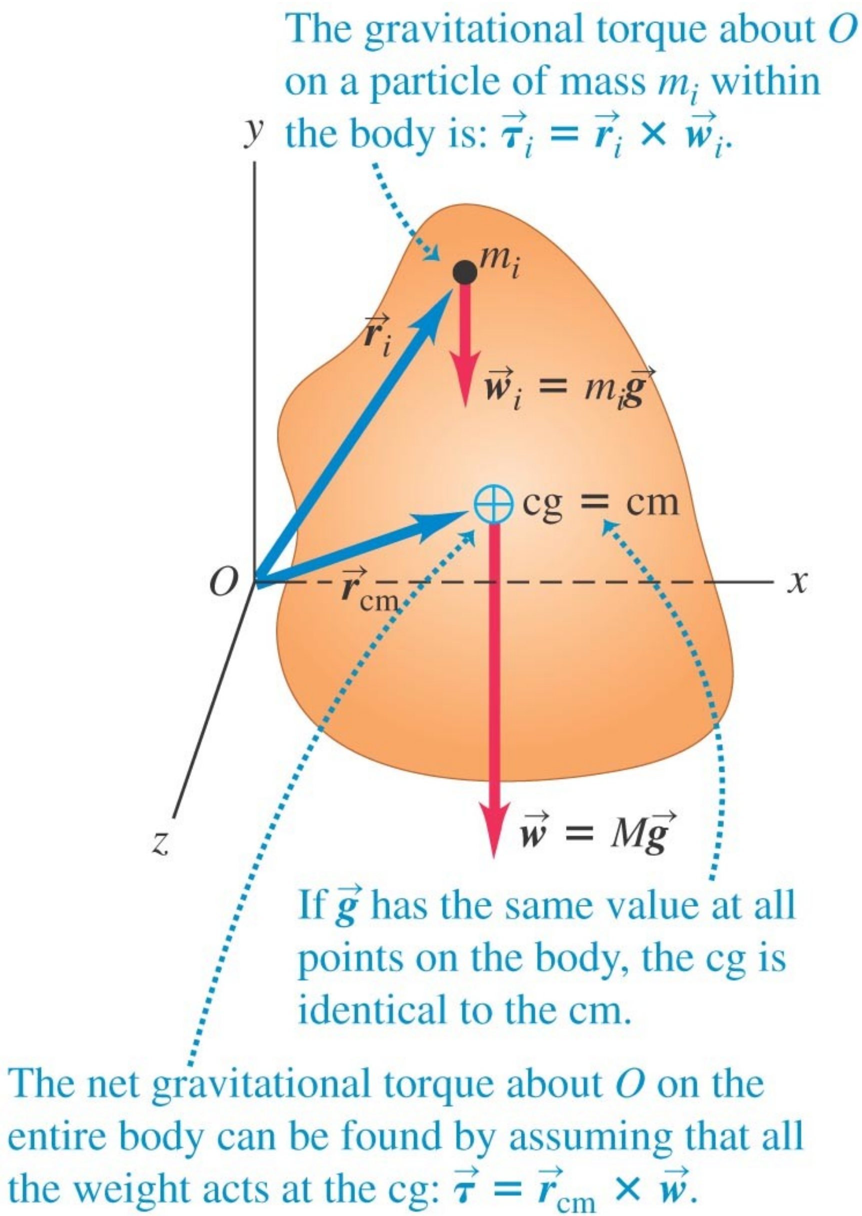


Center of Gravity

- We can treat a body’s weight as though it all acts at a single point—the *center of gravity*.

- If we can ignore the variation of gravity with altitude, the center of gravity is the same as the center of mass.

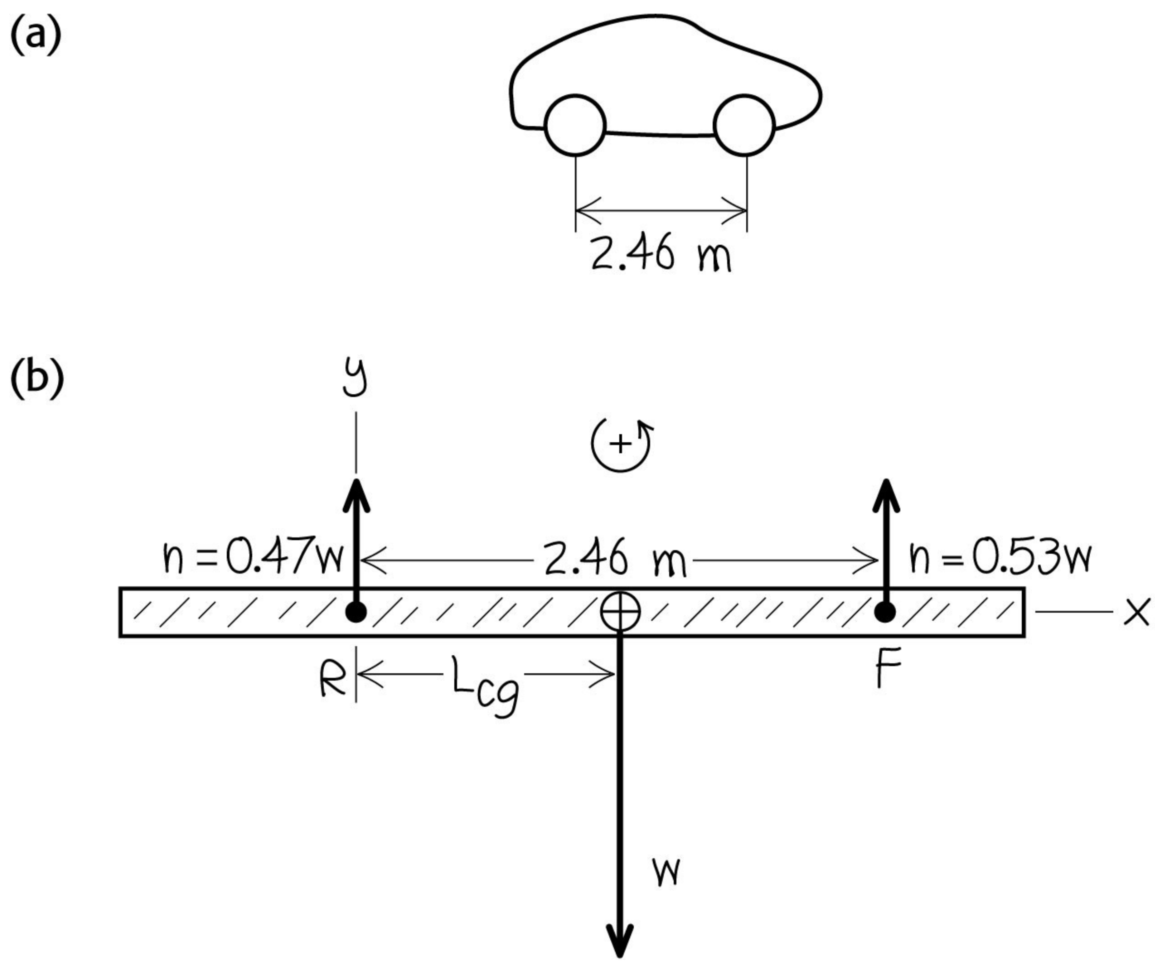
- How to find practically?



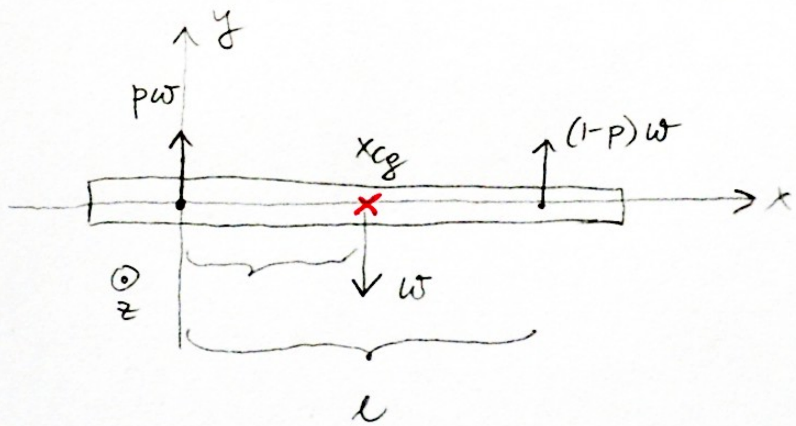
Solving Rigid-body Equilibrium Problems

- Sketch the situation; identify the object (do not treat it as a point!) in equilibrium.
- Draw a free-body diagram with forces attached to the points they act onto.
- Choose an appropriately placed coordinate system (can save calculations by eliminating torques of certain forces!).
- Write down equilibrium conditions for forces and torques.
- Solve for unknowns.

Example: Center of Gravity of a Car



Example: Center of Gravity of a Car



- origin at the rear axle

Equilibrium:

forces

$$pw + (1-p)w + w = 0$$

torques (about the origin)

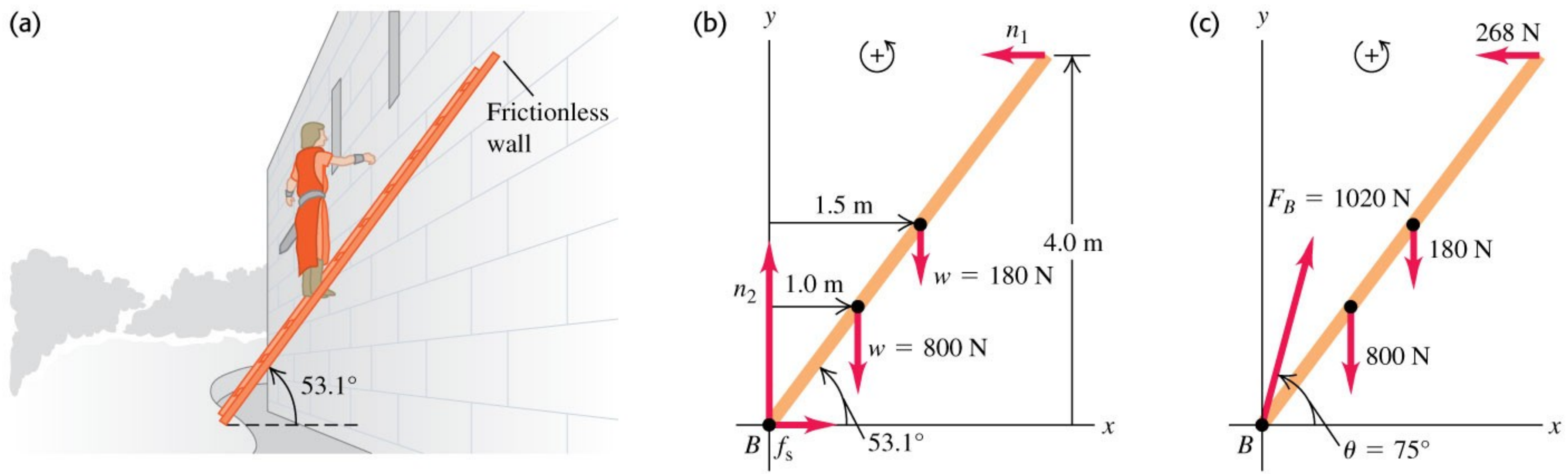
$$0 \qquad \otimes \qquad \odot$$

$$0 - x_{cg}w + \ell \cdot (1-p)w = 0 \Rightarrow \boxed{x_{cg} = \ell(1-p)}$$

if $0 < p < \frac{1}{2}$ then $\frac{\ell}{2} < x_{cg} < \ell$

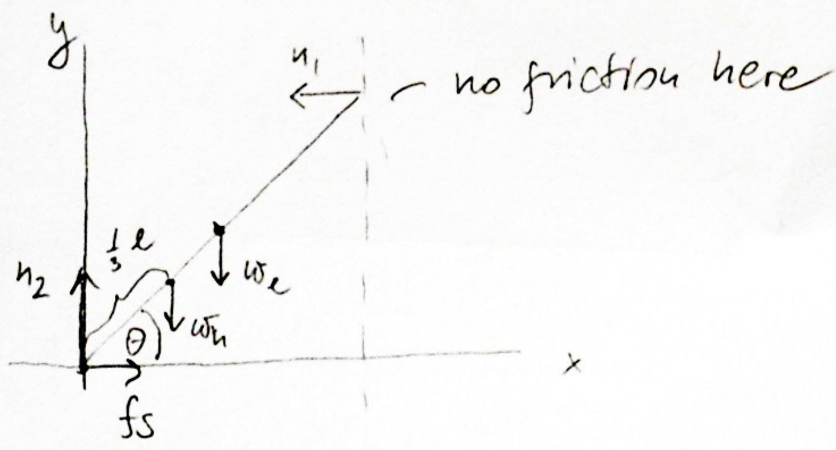
(towards front of the car)

Example: Will the Ladder Slip?



What is the minimum coefficient of static friction, so that the ladder does not slip at the base?

Example: Will the Ladder Slip?



Origin at the point where the ladder touches the floor

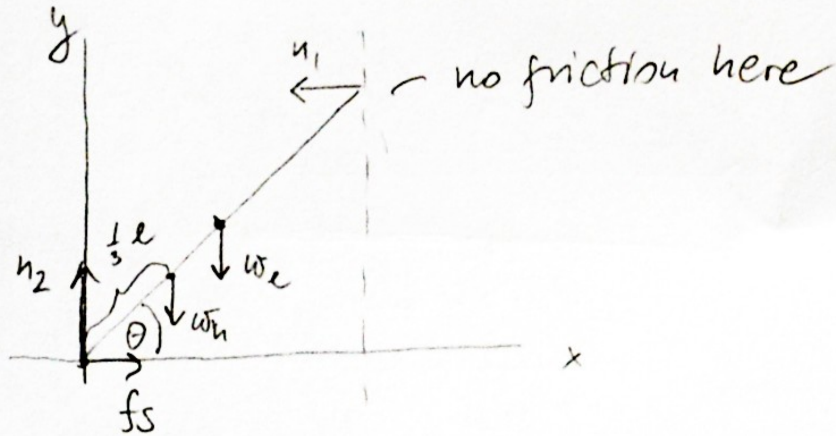
Equilibrium

forces

"x" component $\quad f_s - n_1 = 0$

"y" component $\quad n_2 - w_h - w_e = 0 \quad \Rightarrow \quad n_2 = w_h + w_e$

Example: Will the Ladder Slip?



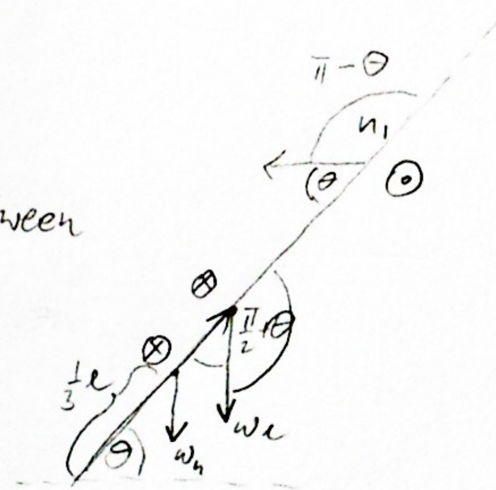
Origin at the point where the ladder touches the floor

Equilibrium

torques

(about the origin)

angles between $\bar{r}$ and $\bar{F}$



$$-\frac{1}{3} l w_h \sin\left(\frac{\pi}{2}+\theta\right) - \frac{1}{2} l w_l \sin\left(\frac{\pi}{2}+\theta\right) + l n_1 \sin(\pi-\theta) = 0$$

$$-\frac{1}{3} l w_h \cos\theta - \frac{1}{2} l w_l \cos\theta + l n_1 \sin\theta = 0$$

Example (contd): Will the Ladder Slip?

Hence, the equilibrium conditions yield a system of three equations

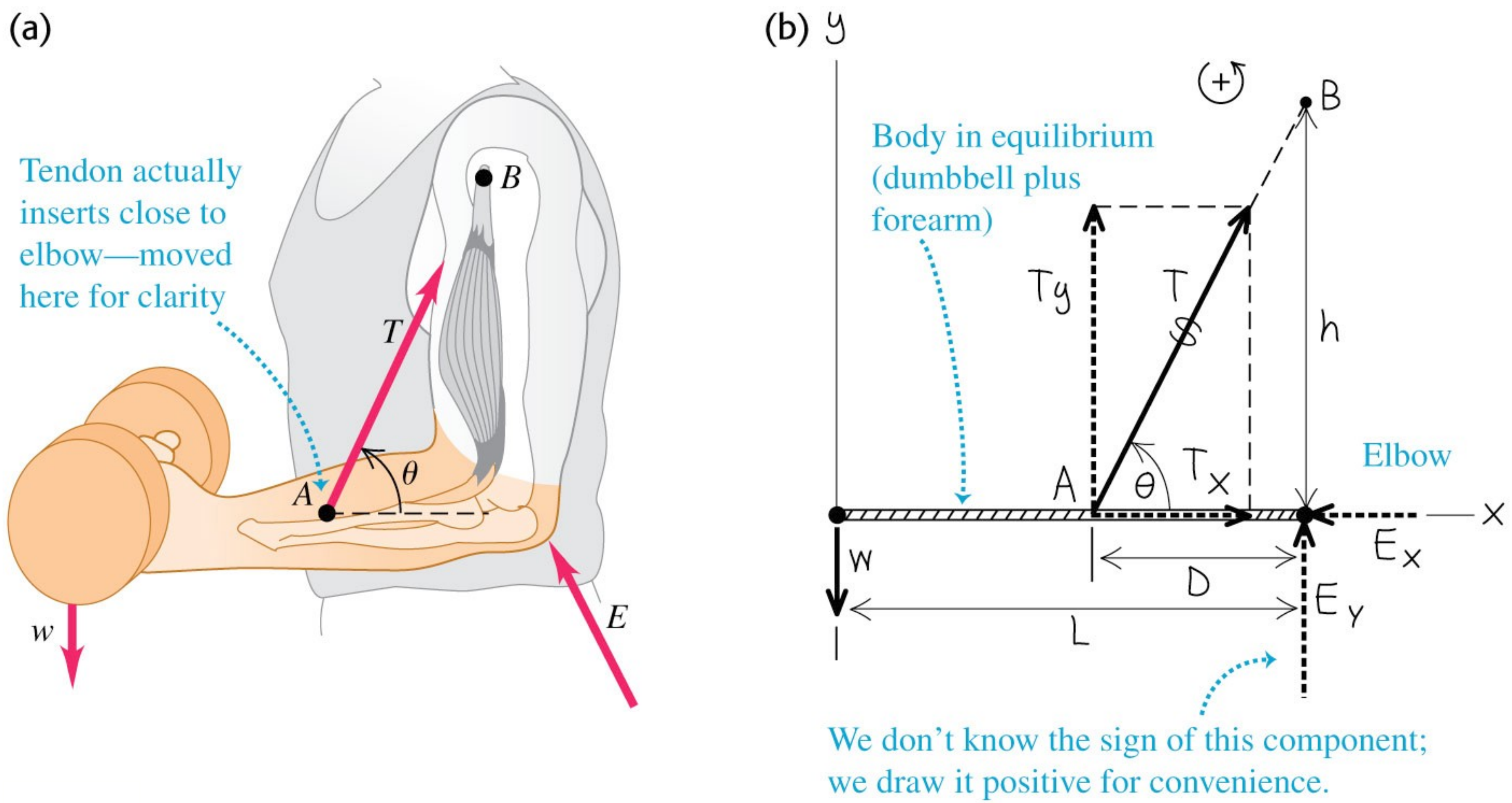
$$\begin{cases} f_s - n_1 = 0 \\ n_2 - w_h - w_e = 0 \\ \frac{1}{3} w_h \cos\theta + \frac{1}{2} w_e \cos\theta - n_1 \sin\theta = 0 \end{cases}$$

3 unknowns f_s (or μ_s), n_1, n_2

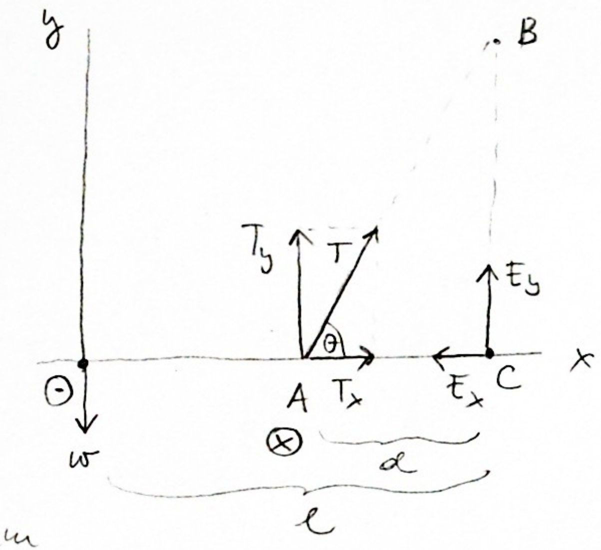
The magnitude of the static friction force is between 0 and $\mu_s n_2$, hence the minimum coefficient of friction is

$$\mu_{\mathrm{s,min}} = \frac{f_{\mathrm{s}}}{n_2}$$

Example: Equilibrium and "Pumping Iron"



Example: Equilibrium and "Pumping Iron"



θ, l, d, w known

$$w = 200 \text{ N},\ d = 5 \text{ cm},\ l = 30 \text{ cm},\ \theta = 80^o$$

Equilibrium

torques (w.r.t. C) only due to w ($\odot$) and T_y ($\otimes$)

$$l w - d T_y = 0 \Rightarrow T_y = \frac{l w}{d} \overset{\frac{T_y}{T} = \sin\theta}{\Rightarrow} \boxed{T = \frac{l w}{d \sin\theta}}$$

forces

"x" component $\quad T_x - E_x = 0 \Rightarrow \boxed{E_x = T_x \overset{\frac{T_x}{T} = \cos\theta}{=} \frac{l w}{d} \cot\theta}$

"y" component $\quad T_y + E_y - w = 0 \Rightarrow \boxed{E_y = w - T_y = w\left(1 - \frac{l}{d}\right) < 0}$

points down

Plug the numbers: *T* = 1220 N, *E* = 1020 N (significant foces!)

Beyond the Rigid Body Model:

Elasticity

Strain, Stress, and Elastic Modulus

- Stretching, squeezing, and twisting a real body causes it to deform. We shall study the relationship between forces and the deformations they cause.

- *Stress* is the force per unit area and *strain* is the fractional deformation due to the stress. *Elastic modulus* is stress divided by strain.

- When the strain and stress are small enough, we often find that the two are directly proportional. The proportionality of stress and strain is called *Hooke's law*.

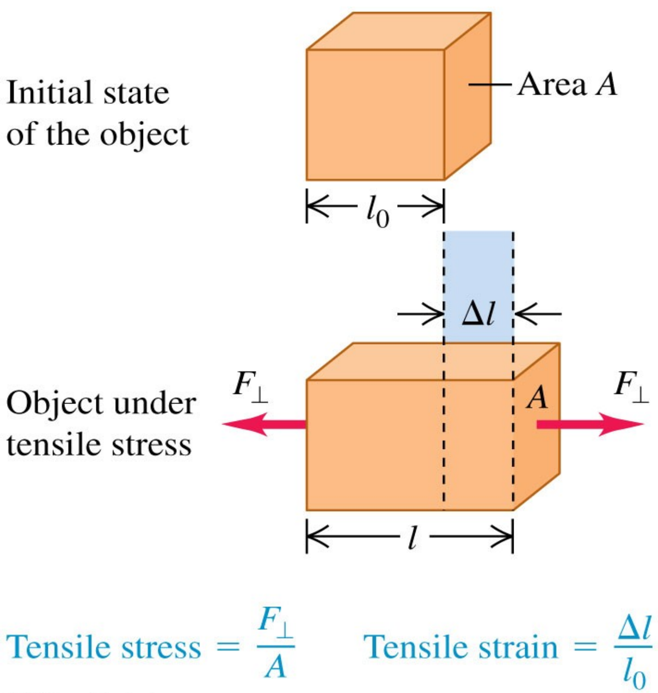
$$\frac{\text{stress}}{\text{strain}} = \text{elastic modulus}$$

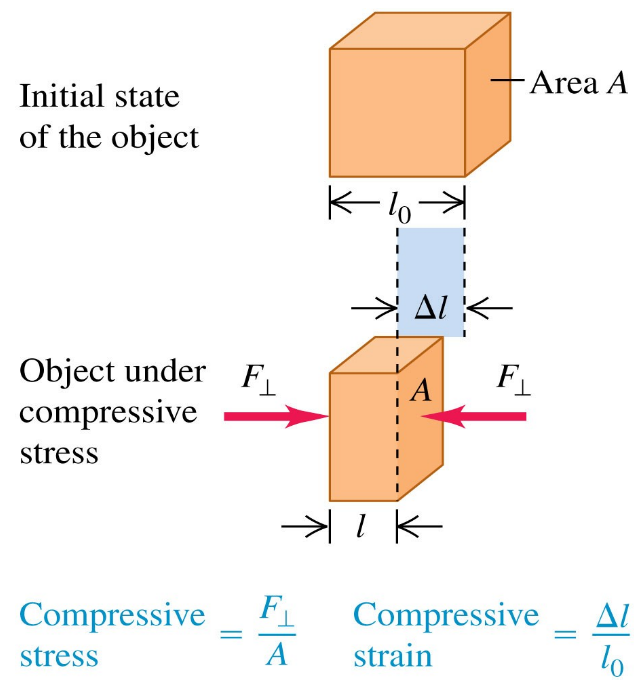
Tensile and Compressive Stress and Strain

- *Tensile stress* = $F_\perp / A$ and *tensile strain* = $\Delta l / l_0$.

  Compressive stress and *compressive strain* are defined in a similar way.
- *Young's modulus* is tensile stress divided by tensile strain, and is given by

$$Y = \frac{\frac{F_\perp}{A}}{\frac{\Delta l}{l_0}}$$





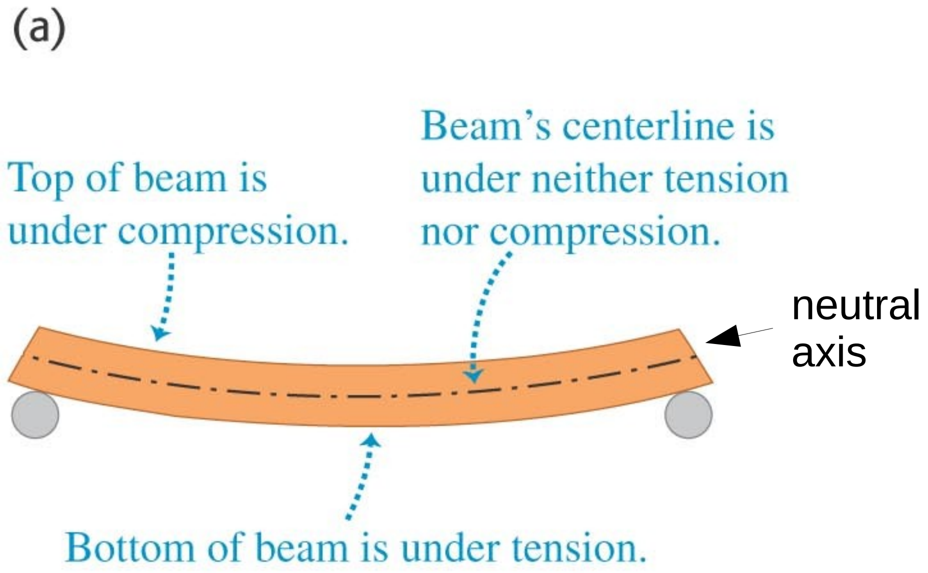
Values of Young's Modulus

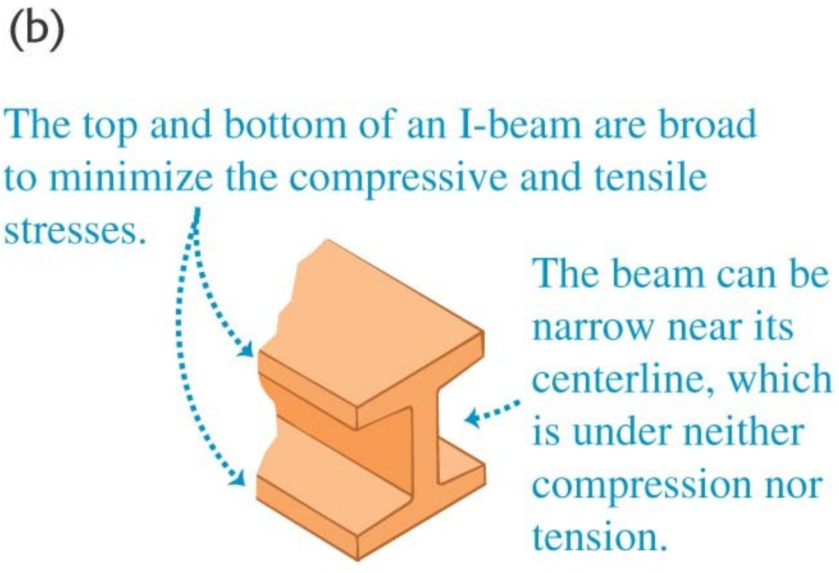
Table 11.1 Approximate Elastic Moduli

Material	Young's Modulus, Y (Pa)
Aluminum	7.0×10^{10}
Brass	9.0×10^{10}
Copper	11×10^{10}
Crown glass	6.0×10^{10}
Iron	21×10^{10}
Lead	1.6×10^{10}
Nickel	21×10^{10}
Steel	20×10^{10}

Tensile Stress and Strain

In many cases, a body can experience both tensile and compressive stress at the same time, as shown in figure below.

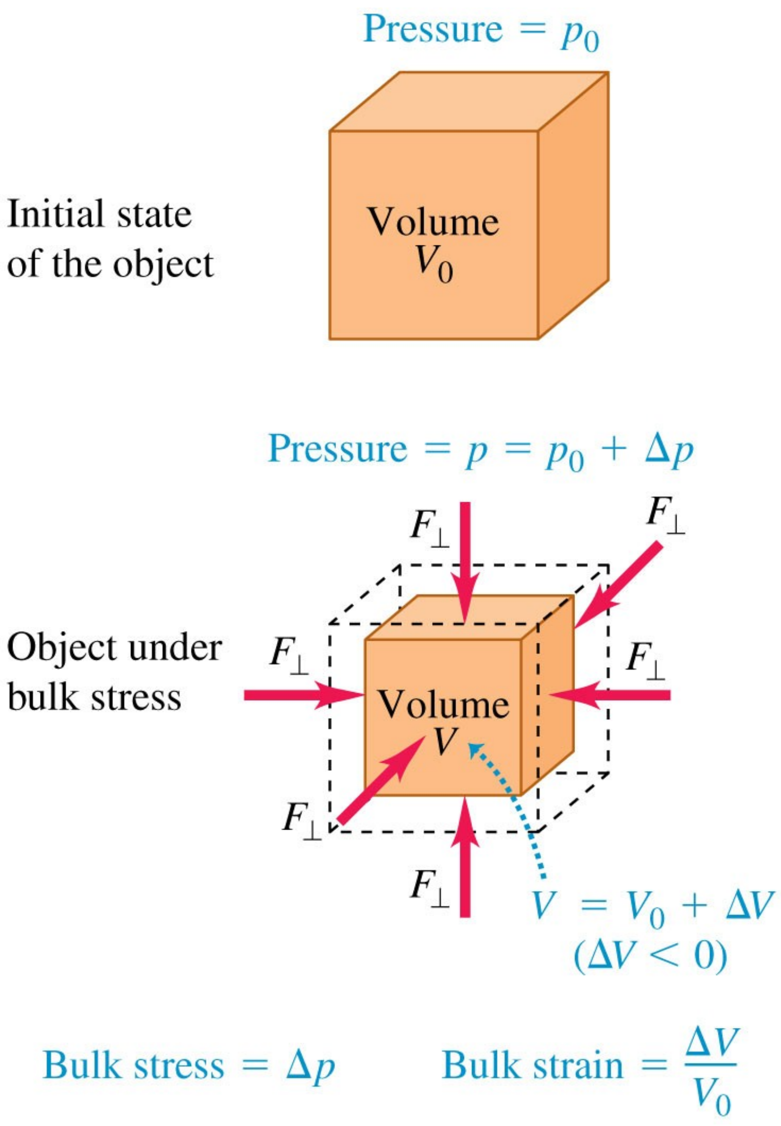




Bulk Stress and Strain

- Pressure in a fluid is force per unit area: $p = F_\perp / A$.
- *Bulk stress* is pressure change Δp and *bulk strain* is fractional volume change $\Delta V / V_0$ (see figure)
- *Bulk modulus* is bulk stress divided by bulk strain and is given by

$$B = -\frac{\Delta p}{\frac{\Delta V}{V_0}}$$



Values of elastic moduli

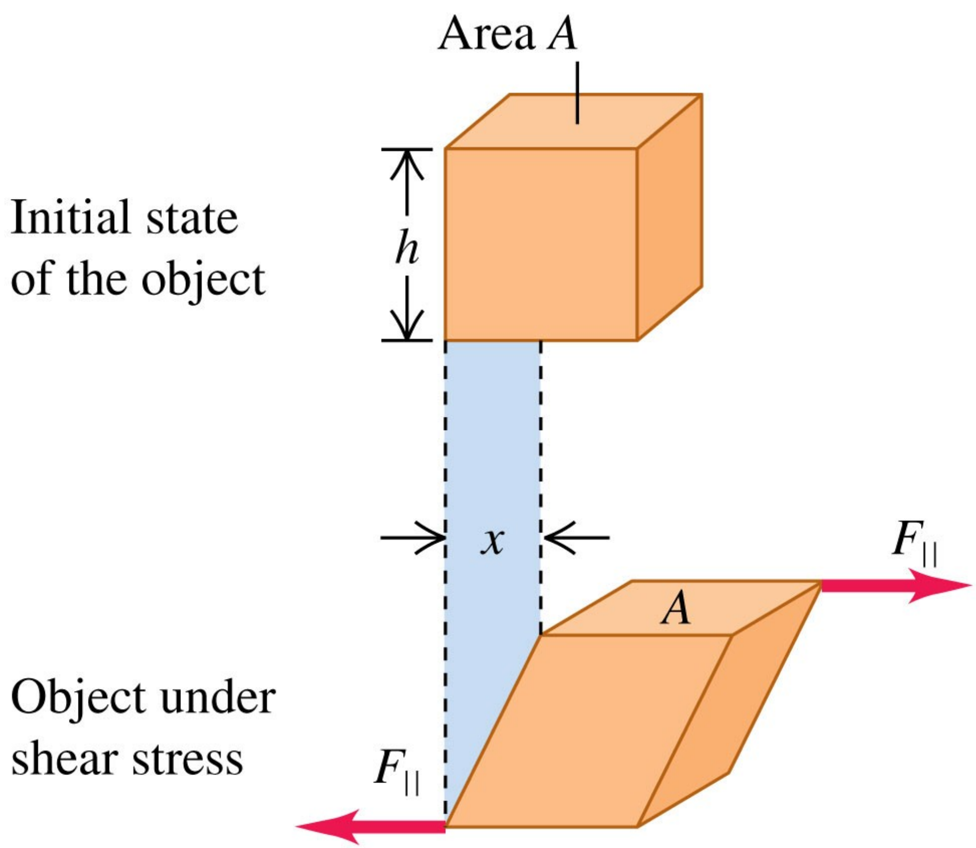
Table 11.1 Approximate Elastic Moduli

Material	Young's Modulus, Y (Pa)	Bulk Modulus, B (Pa)
Aluminum	7.0×10^{10}	7.5×10^{10}
Brass	9.0×10^{10}	6.0×10^{10}
Copper	11×10^{10}	14×10^{10}
Crown glass	6.0×10^{10}	5.0×10^{10}
Iron	21×10^{10}	16×10^{10}
Lead	1.6×10^{10}	4.1×10^{10}
Nickel	21×10^{10}	17×10^{10}
Steel	20×10^{10}	16×10^{10}

Shear stress and strain

- *Shear stress* is $F_{||}/A$ and *shear strain* is x/h, as shown in figure
- *Shear modulus* is shear stress divided by shear strain, and is given by

$$S = \frac{\frac{F_{||}}{A}}{\frac{x}{h}}$$



$$\text{Shear stress} = \frac{F_{||}}{A} \qquad \text{Shear strain} = \frac{x}{h}$$

Values of elastic moduli

Table 11.1 Approximate Elastic Moduli

Material	Young's Modulus, Y (Pa)	Bulk Modulus, B (Pa)	Shear Modulus, S (Pa)
Aluminum	7.0×10^{10}	7.5×10^{10}	2.5×10^{10}
Brass	9.0×10^{10}	6.0×10^{10}	3.5×10^{10}
Copper	11×10^{10}	14×10^{10}	4.4×10^{10}
Crown glass	6.0×10^{10}	5.0×10^{10}	2.5×10^{10}
Iron	21×10^{10}	16×10^{10}	7.7×10^{10}
Lead	1.6×10^{10}	4.1×10^{10}	0.6×10^{10}
Nickel	21×10^{10}	17×10^{10}	7.8×10^{10}
Steel	20×10^{10}	16×10^{10}	7.5×10^{10}

Elasticity and Plasticity

- Hooke's law applies up to point *a* in the figure below.
- The table shows some approximate breaking stresses.

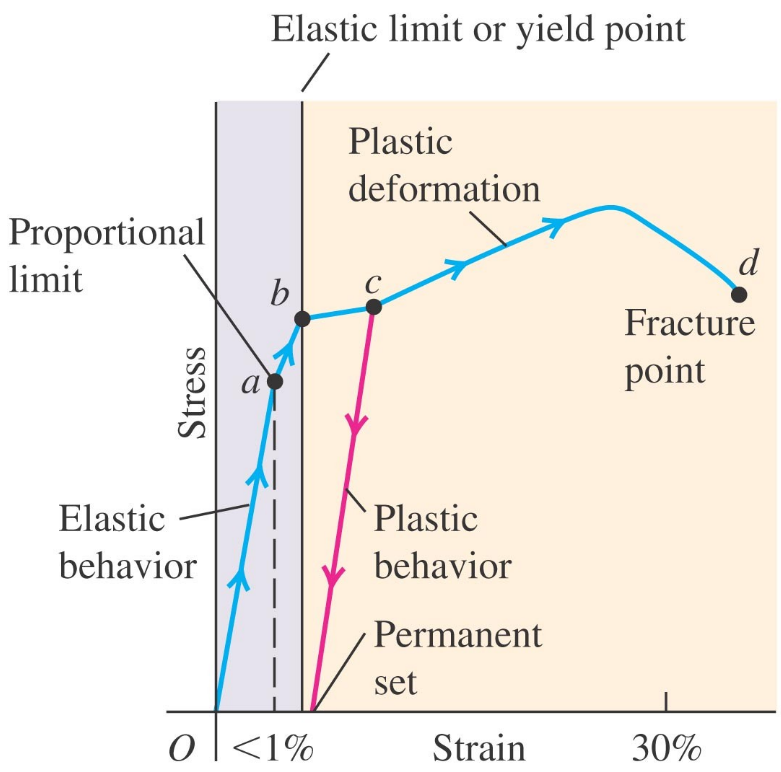


Table 11.3 Approximate Breaking Stresses

Material	Breaking Stress (Pa or N/m^2)
Aluminum	2.2×10^8
Brass	4.7×10^8
Glass	10×10^8
Iron	3.0×10^8
Phosphor bronze	5.6×10^8
Steel	$5 - 20 \times 10^8$