Problem Set 9

Due: 9 July 2021, 2.30 p.m.

Problem 1. Based on measurements of rotational energy levels of a HF molecule, its moment of inertia about the axis perpendicular to the line connecting both atoms, passing through the center of mass of the system, has been estimated at $1,37 \cdot 10^{-47}$ kg$\cdot$m^2. Find the distance between the H and the F atoms if their masses are $m_{\mathrm{H}} = 1,67 \cdot 10^{-27}$ kg, $m_{\mathrm{F}} = 3,17 \cdot 10^{-26}$ kg, respectively.

(*2 points*)

Problem 2. Justify the following statement that we mentioned in class when we were discussing the moment of inertia of a disk: *For a planar object, the moment of inertia about an axis perpendicular to the plane is the sum of the moments of inertia about two perpendicular axes through the same point in the plane of the object.*

It is known as *the perpendicular axis theorem.*

(*3 points*)

Problem 3. Find the moment of inertia of a hollow cylinder with inner radius R_1, outer radius R_2, and height H, about the axis of symmetry. The cylinder is not uniform and the bulk density of mass depends on the distance r from the axis of symmetry as αr with $\alpha > 0$.

(*3 points*)

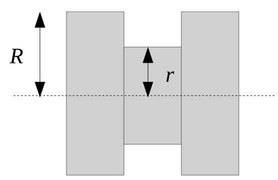
Problem 4. What is the moment of inertia of the cylinder from Problem 3 about the axis parallel to the axis of symmetry, and tangential to the outer side wall.

(*1 point*)

Problem 5. A yo-yo is made of two identical cylinders of radius R attached at their centers to a cylindrical axle with radius $r < R$. All three cylinders have the same height and are made of the same material (with constant density of mass). A string is attached to and wrapped around the axle. Holding the free end of the string firm, we release the yo-yo.

Find the acceleration of the center of mass of the yo-yo. Assume that the string does not slip while unwinding.

(*3 points*)



Problem 1. Based on measurements of rotational energy levels of a HF molecule, its moment of inertia about the axis perpendicular to the line connecting both atoms, passing through the center of mass of the system, has been estimated at $1,37 \cdot 10^{-47}$ kg$\cdot$m^2. Find the distance between the H and the F atoms if their masses are $m_{\mathrm{H}} = 1,67 \cdot 10^{-27}$ kg, $m_{\mathrm{F}} = 3,17 \cdot 10^{-26}$ kg, respectively.

Solution

Set the position of H atom as the origin and the distance between H and F atoms as r.

Then we calculate the position of the center of mass r_{cm}

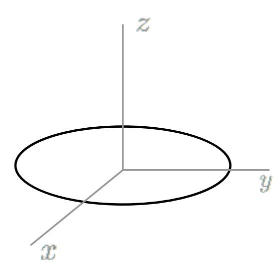
$$r_{cm} = \frac{m_F\, r}{m_H + m_F} = 0.95r$$

Therefore, we can express the moment of inertia as follows.

$$I = m_H r_{cm}^2 + m_F (r - r_{cm})^2 = 1.37 \times 10^{-47}$$
$$\Rightarrow r = 9.29 \times 10^{-11} m$$

Problem 2. Justify the following statement that we mentioned in class when we were discussing the moment of inertia of a disk: *For a planar object, the moment of inertia about an axis perpendicular to the plane is the sum of the moments of inertia about two perpendicular axes through the same point in the plane of the object.*

It is known as *the perpendicular axis theorem.*



Solution

According to the definition of the moment of inertia, we have

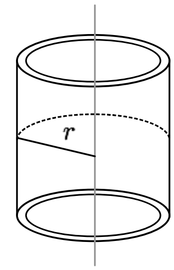
$$\begin{cases} I_x = \int\limits_{disk} (r_y^2 + r_z^2)\, dm \\ I_y = \int\limits_{disk} (r_x^2 + r_z^2)\, dm \\ I_z = \int\limits_{disk} (r_x^2 + r_y^2)\, dm \end{cases}$$

Since the disk is “planar object” as the question described, the component along z axis doesn’t exist. Therefore

$$I_x + I_y = \int\limits_{disk} r_y^2 dm + \int\limits_{disk} r_x^2 dm = \int\limits_{disk} (r_x^2 + r_y^2) dm = I_z$$

Problem 3. Find the moment of inertia of a hollow cylinder with inner radius R_1, outer radius R_2, and height H, about the axis of symmetry. The cylinder is not uniform and the bulk density of mass depends on the distance r from the axis of symmetry as αr with $\alpha > 0$.

Solution



For a tiny piece dm we can express is as

$$dm = \rho dV = \alpha r H(2\pi r)dr = 2\alpha\pi H r^2 dr$$

Then we calculate the moment of inertia

$$I = \int_{R_1}^{R_2} r^2 \cdot \left(2\alpha\pi H r^2\right) dr = \int_{R_1}^{R_2} 2\alpha\pi H r^4 dr = \frac{2}{5}\alpha\pi H(R_2^5 - R_1^5)$$

Problem 4. What is the moment of inertia of the cylinder from Problem 3 about the axis parallel to the axis of symmetry, and tangential to the outer side wall.

Solution

We here may apply the parallel axis theorem for a clearer calculation

$$\begin{cases} I_0 = \int_{R_1}^{R_2} r^2 \cdot \left(2\alpha\pi H r^2\right) dr = \int_{R_1}^{R_2} 2\alpha\pi H r^4 dr = \frac{2}{5}\alpha\pi H(R_2^5 - R_1^5) \\ I = I_0 + m{R_2}^2 \\ m = \int \rho dV = \int_{R_1}^{R_2} 2\alpha\pi H r^2 dr = \frac{2}{3}\alpha\pi H({R_2}^3 - {R_1}^3) \end{cases}$$

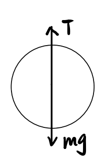
$$\Rightarrow I = \frac{16}{15}\alpha\pi H{R_2}^5 - \frac{2}{5}\alpha\pi H{R_1}^5 - \frac{2}{3}\alpha\pi H{R_1}^3{R_2}^2$$

Problem 5. A yo-yo is made of two identical cylinders of radius R attached at their centers to a cylindrical axle with radius $r < R$. All three cylinders have the same height and are made of the same material (with constant density of mass). A string is attached to and wrapped around the axle. Holding the free end of the string firm, we release the yo-yo.

Find the acceleration of the center of mass of the yo-yo. Assume that the string does not slip while unwinding.

Solution

For the center of mass, we draw the FBD



Linear acceleration of the whole yoyo

$$ma = mg - T$$

Rotation motion of the center of mass

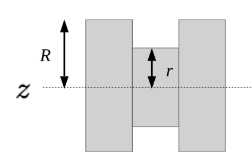
$$a = r\varepsilon_z$$

Second law of dynamics for rotation motion

$$\tau_z = rT = I_z\varepsilon_z$$

The moment of inertia

$$I_z = 2\int_0^R \rho h 2\pi x^3 dx + \int_0^r \rho h 2\pi x^3 dx$$



Mass of the yoyo

$$m = 2\pi R^2 h\rho + \pi r^2 h\rho$$

Solving all the equations

$$a = \frac{4R^2r^2 + 2r^4}{2R^4 + 4R^2r^2 + 3r^4}\, g$$