Chapter 3 – Kinematics in 3D

UM-SJTU Joint Institute
Physics I (Summer 2021)
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Agenda

1. Basic Kinematic Quantities in 3D Cartesian Coordinates
   - Position, Displacement, and Trajectory
   - Velocity
   - Acceleration
   - Tangential and Normal Components of Acceleration
   - Illustration
2. Example: Projectile Motion
   - Acceleration, Velocity, and Position
   - Trajectory
   - Tangential and Normal Components of Acceleration
   - Examples
3. Kinematics in Polar Coordinates (2D)
   - Position and Trajectory
   - How to Handle Time-dependent Unit Vectors?
   - Velocity and Acceleration. Radial and Transverse Components
   - Examples

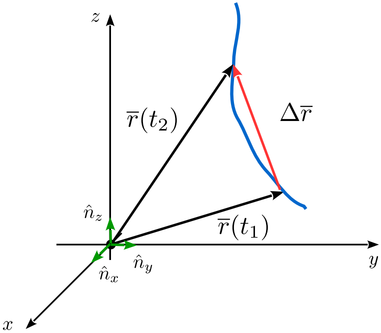
Agenda

4. Natural (or Kinematic) Coordinate System
   - Unit Vectors. Velocity
   - Acceleration. Tangential and Normal Components
   - Instantaneous Radius of Curvature
   - Examples: Circular Motion and Projectile Motion Revisited

5. Final Remarks
   - Average Speed vs Average Velocity
   - Relative Motion and Galilean Transformation

Basic Kinematic Quantities in 3D Cartesian Coordinates

Position, Displacement, and Trajectory



$\Delta\overline{r}(t)$ — displacement over the time interval (t_1, t_2)

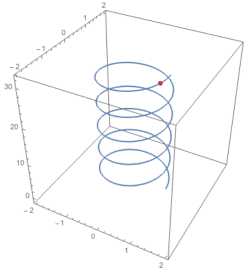
The curve traced out by the tip of the position vector of a moving particle is called the particle's **trajectory**.

The vector-valued function $\overline{r} = \overline{r}(t)$ defines the trajectory in the **parametric form**. In terms of individual components $\overline{r}(t) = x(t)\hat{n}_x + y(t)\hat{n}_y + z(t)\hat{n}_z$, that is

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \qquad t_0 \leq t \leq t_1$$

Examples

Example 1. What is the shape of trajectory if $x(t) = R\sin\omega t$, $y(t) = R\cos\omega t$, and $z(t) = vt$, where R, ω, v are positive constants?



Example 2. For motion in the x-y plane ($z \equiv 0$), we have $x(t) = R\sin\omega t$, $y(t) = R\cos\omega t$. The parameter (time) can be eliminated and

$$x^2 + y^2 = R^2.$$

This circle of radius R, centered at the origin, is an example of a trajectory defined in an **implicit form** as $F(x, y) = 0$.

Velocity

Recall that

$$\bar{r}(t) = x(t)\hat{n}_x + y(t)\hat{n}_y + z(t)\hat{n}_z.$$

and $\hat{n}_x, \hat{n}_y, \hat{n}_z$ are <u>fixed</u> (that is <u>time-independent</u>) unit vectors.
That is, $\dot{\hat{n}}_x = \dot{\hat{n}}_y = \dot{\hat{n}}_z = 0$.

Average velocity

$$\bar{v}_{\text{av}} = \frac{\Delta \bar{r}}{\Delta t} \quad = \quad \frac{x(t+\Delta t) - x(t)}{\Delta t}\hat{n}_x + \frac{y(t+\Delta t) - y(t)}{\Delta t}\hat{n}_y + \\ + \frac{z(t+\Delta t) - z(t)}{\Delta t}\hat{n}_z$$

Instantaneous velocity

$$\bar{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \bar{r}}{\Delta t} \quad = \quad \lim_{\Delta t \to 0} \left[\frac{x(t+\Delta t) - x(t)}{\Delta t}\hat{n}_x + \frac{y(t+\Delta t) - y(t)}{\Delta t}\hat{n}_y \right. \\ \left. + \frac{z(t+\Delta t) - z(t)}{\Delta t}\hat{n}_z \right]$$

Velocity

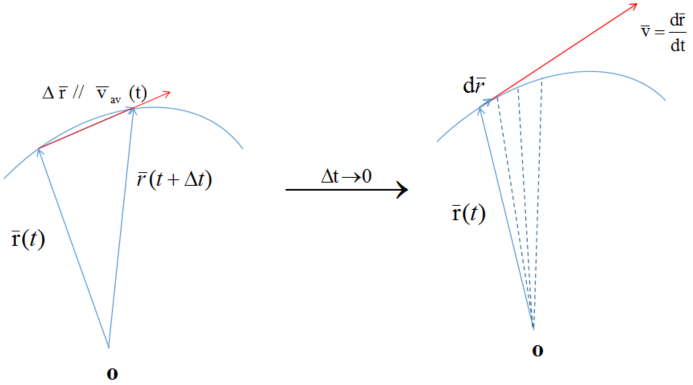
Eventually,

$$\begin{aligned}\bar{v}(t) &= \underbrace{\dot{x}(t)}_{v_x(t)}\hat{n}_x + \underbrace{\dot{y}(t)}_{v_y(t)}\hat{n}_y + \underbrace{\dot{z}(t)}_{v_z(t)}\hat{n}_z = \\ &= (\dot{x}(t), \dot{y}(t), \dot{z}(t))\end{aligned}$$

Instantaneous speed

$$v(t) = |\bar{v}(t)| = \sqrt{[\dot{x}(t)]^2 + [\dot{y}(t)]^2 + [\dot{z}(t)]^2}$$

Instantaneous Velocity Vector



Observation: The instantaneous velocity vector is always tangential to the trajectory.

Similarly, we can define acceleration.

Average acceleration

$$\overline{a}_{\mathrm{av}} = \frac{\Delta\overline{v}}{\Delta t}, \qquad \Delta\overline{v} = \overline{v}(t+\Delta t) - \overline{v}(t)$$

Instantaneous acceleration

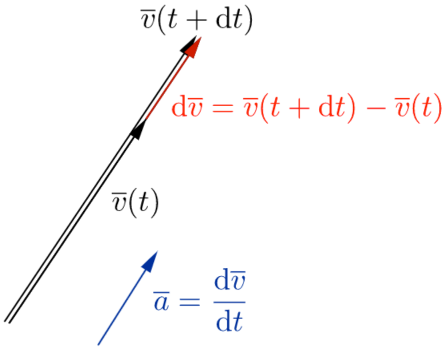
$$\begin{aligned}\overline{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta\overline{v}}{\Delta t} &= \dot{v}_x(t)\hat{n}_x + \dot{v}_y(t)\hat{n}_y + \dot{v}_z(t)\hat{n}_z = \\ &= \underbrace{\ddot{x}(t)}_{a_x(t)}\hat{n}_x + \underbrace{\ddot{y}(t)}_{a_y(t)}\hat{n}_y + \underbrace{\ddot{z}(t)}_{a_z(t)}\hat{n}_z\end{aligned}$$

Magnitude

$$a(t) = |\overline{a}(t)| = \sqrt{[\ddot{x}(t)]^2 + [\ddot{y}(t)]^2 + [\ddot{z}(t)]^2}$$

Observation 1. The tangential component of instantaneous acceleration changes the magnitude of instantaneous velocity (that is the speed) only.

Suppose that only the magnitude of the instantaneous velocity $\overline{v}$ changes. Then the acceleration vector $\overline{a}$ must be parallel to $\overline{v}$ (which is always tangential to trajectory).



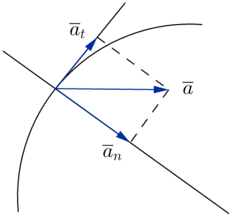
Observation 2. The normal component of instantaneous acceleration changes the direction of the instantaneous velocity, but not its magnitude.

Suppose now that the instantaneous speed is constant, that is $v = \text{const}$. Then, of course, $v^2 = \text{const}$, and

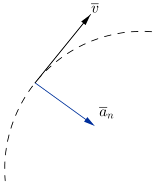
$$\frac{dv^2}{dt} = \frac{d}{dt}\left(\overline{v} \circ \overline{v}\right) = \dot{\overline{v}} \circ \overline{v} + \overline{v} \circ \dot{\overline{v}} = 2\,\dot{\overline{v}} \circ \overline{v} = 0$$

Hence $\overline{a} \perp \overline{v}$, that is the instantaneous acceleration vector points along the normal to the trajectory (recall that the normal direction is perpendicular to the tangential direction).

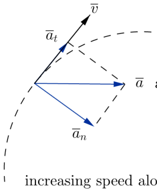
What if both the magnitude and the direction of the instantaneous velocity change?



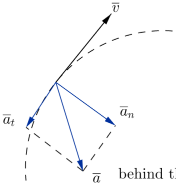
Special cases



constant speed along curved path

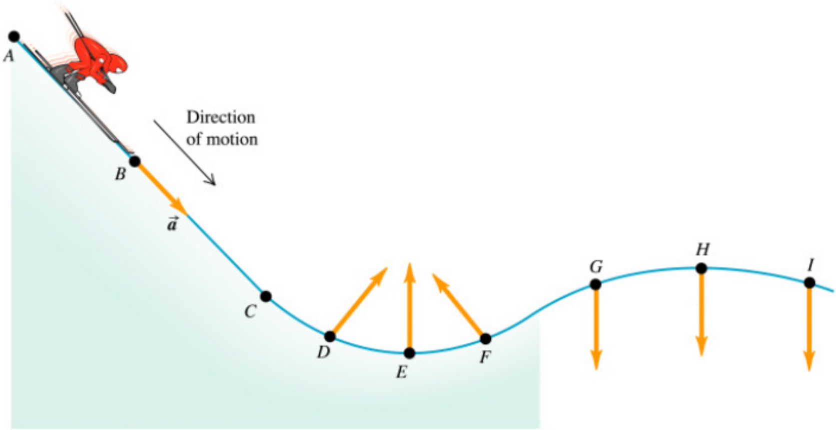


increasing speed along curved path



decreasing speed along curved path

Example



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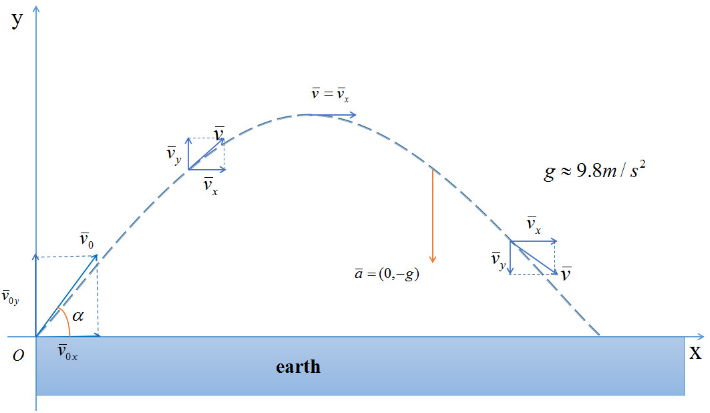
Example: Projectile Motion

Statement of the Problem



Initial conditions $(t = 0)$

$$\begin{cases} x(0) = 0 \\ y(0) = 0 \end{cases} \qquad \begin{cases} v_x(0) = v_{0x} = v_0 \cos\alpha \\ v_y(0) = v_{0y} = v_0 \sin\alpha \end{cases}$$

Observation: Constant non-zero acceleration only in the vertical direction (downwards) with the magnitude of $\approx 9.8\ m/s^2$.

$$\begin{cases} a_x(t) \equiv 0 \\ a_y(t) = -g \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \frac{\mathrm{d}v_x}{\mathrm{d}t} \equiv 0 \\ \frac{\mathrm{d}v_y}{\mathrm{d}t} = -g \end{cases}$$

Velocity

$$\boxed{v_x(t) = v_0 \cos\alpha = \text{const}}$$

because $\int_{v_{0x}}^{v_x(t)} \mathrm{d}v_x = \int_0^t 0\, \mathrm{d}t = 0$. Similarly,

$$\int_{v_0 \sin\alpha}^{v_y(t)} \mathrm{d}v_y = -\int_0^t g \mathrm{d}t \Rightarrow \boxed{v_y(t) = v_0 \sin\alpha - gt}$$

Velocity

Position

$$v_x(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = v_0 \cos\alpha \qquad \Rightarrow \qquad \int_0^{x(t)} \mathrm{d}x = \int_0^t v_0 \cos\alpha \, \mathrm{d}t$$

$$\boxed{x(t) = v_0 t \cos\alpha}$$

$$v_y(t) = \frac{\mathrm{d}y}{\mathrm{d}t} = v_0 \sin\alpha - gt \qquad \Rightarrow \qquad \int_0^{y(t)} \mathrm{d}y = \int_0^t [v_0 \sin\alpha - gt] \, \mathrm{d}t$$

$$\boxed{y(t) = v_0 t \sin\alpha - \frac{1}{2} g t^2}$$

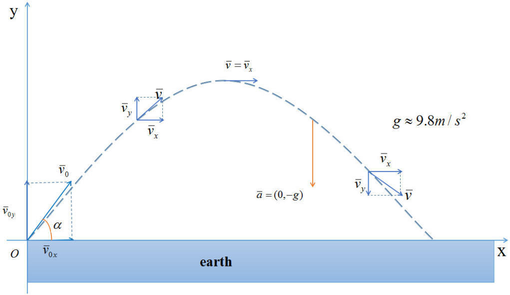
Trajectory

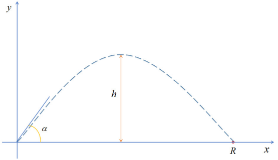
Hence, the parametric equations of the trajectory

$$\begin{cases} x(t) = v_0 t \cos\alpha \;\Rightarrow\; t = \dfrac{x}{v_0\cos\alpha} \\ y(t) = v_0 t \sin\alpha - \frac{1}{2} g t^2 \end{cases}$$

Eliminating time, we find $y = y(x)$ as

$$\boxed{y(x) = x\tan\alpha - \frac{1}{2}\frac{g}{v_0^2\cos^2\alpha}x^2}$$





Maximum Height. Range

Maximum height — at the highest point of the trajectory $v_y(t_h) = 0$. It is reached at the instant

$$t_h = \frac{v_0 \sin\alpha}{g}.$$

Using the parametric equations of the trajectory, we find

$$\boxed{y(t_h) = \frac{v_0^2 \sin^2\alpha}{2g} = h_{\mathrm{max}}}$$

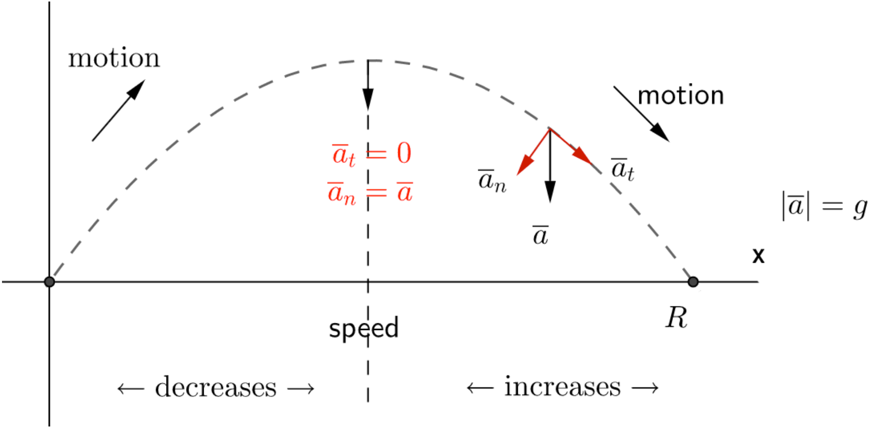
Range

$$y(x_R) = 0 \;\Rightarrow\; x_R \tan\alpha - \frac{1}{2}\frac{g}{v_0^2\cos^2\alpha}x_R^2 = 0$$

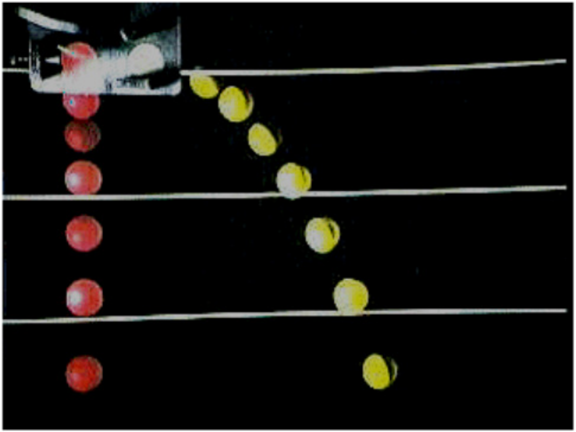
$$x_R = 0 \text{ (starting point) or } \boxed{x_R = \frac{v_0^2 \sin 2\alpha}{g}}$$

Observation. Maximum range for $\alpha = \pi/4$.

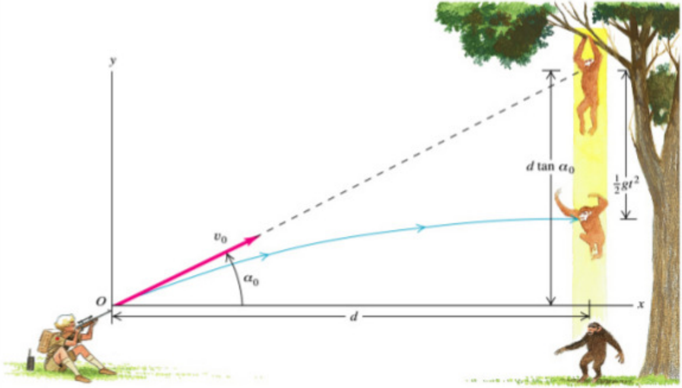
Tangential and Normal Components of Acceleration in Projectile Motion



Example. Free Fall and Projectile Motion Combined



Example. Vet and Monkey



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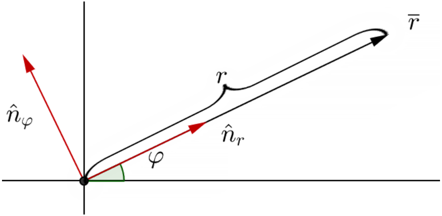
Kinematics in Polar Coordinates (2D)

Position and Trajectory



Position vector

$$\overline{r} = r\hat{n}_r$$

Trajectory (parametric form)

$$\begin{cases} r = r(t) \\ \varphi = \varphi(t) \end{cases}$$

or in the implicit form

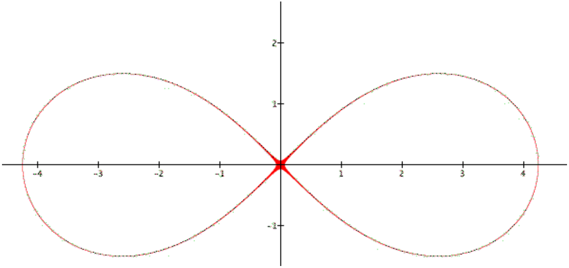
$$r = r(\varphi) \qquad \text{or} \qquad \varphi = \varphi(r) \qquad \text{or} \qquad F(r, \varphi) = 0$$

Trajectory. Examples

Example 1. $r(t) \equiv R = \text{const}$ and $\varphi(t) = \omega t$, where $\omega > 0$.

Circle with radius R, centered at the origin.

Example 2. Lemniscate: $r^2 = 2A^2 \cos 2\varphi$.

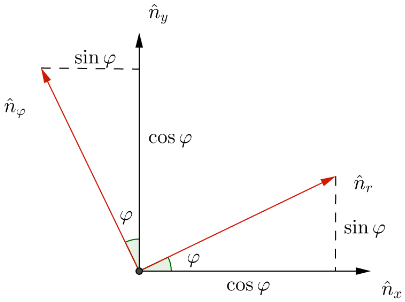


Velocity. How to Handle Time-dependent Unit Vectors?

Velocity $\overline{v} = \dot{\overline{r}} = \frac{d}{dt}\left(r\hat{n}_r\right) = \dot{r}\hat{n}_r + r\dot{\hat{n}}_r$

Note that the derivative of the unit vector $\dot{\hat{n}}_r$ is not zero, unlike in Cartesian coordinates. Here $\hat{n}_r, \hat{n}_\varphi$ are not fixed.

How to find the derivative $\hat{n}_r$ (and $\hat{n}_\varphi$) ?



$$
\begin{aligned}
\hat{n}_r &= \cos\varphi\,\hat{n}_x + \sin\varphi\,\hat{n}_y \\
\hat{n}_\varphi &= -\sin\varphi\,\hat{n}_x + \cos\varphi\,\hat{n}_y
\end{aligned}
$$

$$
\begin{aligned}
\boxed{\dot{\hat{n}}_r} &= -\dot{\varphi}\sin\varphi\,\hat{n}_x + \dot{\varphi}\cos\varphi\,\hat{n}_y \\
&= \dot{\varphi}(-\sin\varphi\,\hat{n}_x + \cos\varphi\,\hat{n}_y) = \\
&= \boxed{\dot{\varphi}\hat{n}_\varphi}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\boxed{\dot{\hat{n}}_\varphi} &= -\dot{\varphi}\cos\varphi\,\hat{n}_x - \dot{\varphi}\sin\varphi\,\hat{n}_y \\
&= -\dot{\varphi}(\cos\varphi\,\hat{n}_x + \sin\varphi\,\hat{n}_y) = \\
&= \boxed{-\dot{\varphi}\hat{n}_r}
\end{aligned}
$$

Velocity and Acceleration

Use the result to find the **velocity** vector in polar coordinates

$$\boxed{\overline{v} = \dot{r}\hat{n}_r + r\dot{\hat{n}}_r = \underbrace{\dot{r}\hat{n}_r}_{\text{radial component}} + \underbrace{r\dot{\varphi}\hat{n}_\varphi}_{\text{transverse component}}}$$

Interpretation?
Speed: $v = |\overline{v}| = \sqrt{(\dot{r})^2 + (r\dot{\varphi})^2}$

Similarly, the **acceleration** vector

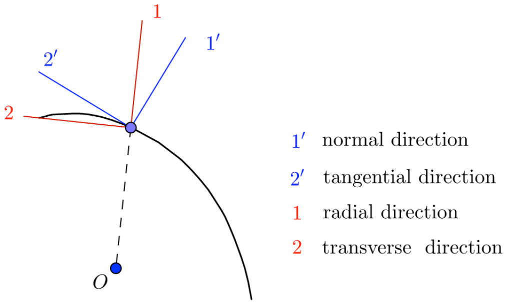
$$\begin{aligned}
\overline{a} = \dot{\overline{v}} \quad &= \quad \ddot{r}\hat{n}_r + \dot{r}\dot{\hat{n}}_r + \dot{r}\dot{\varphi}\hat{n}_\varphi + r\ddot{\varphi}\hat{n}_\varphi + r\dot{\varphi}\dot{\hat{n}}_\varphi \\
&= \quad \ddot{r}\hat{n}_r + \dot{r}\dot{\varphi}\hat{n}_\varphi + \dot{r}\dot{\varphi}\hat{n}_\varphi + r\ddot{\varphi}\hat{n}_\varphi - r(\dot{\varphi})^2\hat{n}_r \\
&= \quad (\ddot{r} - r\dot{\varphi}^2)\hat{n}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{n}_\varphi
\end{aligned}$$

Interpretation?

Radial $\neq$ Normal. Transverse $\neq$ Tangential

CAUTION!

In general, radial $\neq$ normal, nor transverse $\neq$ tangential!



Positive (i.e., radial = normal, transverse = tangential) example?
Circular motion.

Example. Circular Motion

For circular motion

- $r = R =$ const. Hence $\dot{r} = \ddot{r} = 0$.
- $\varphi = \varphi(t)$ — any function of time in general

Two types of circular motion

- uniform
- non-uniform

Example. Circular Motion: (A) Uniform

(A) **uniform circular motion** (*uniform* — particle travels at constant speed; assume counter-clockwise rotation)

Velocity

$$\overline{v} = \underbrace{\dot{r}}_{=0}\hat{n}_r + r\dot{\varphi}\hat{n}_\varphi = R\dot{\varphi}\hat{n}_\varphi$$

Uniform motion, so $|\overline{v}| = v = \text{const}$, and

$$R\dot{\varphi} = v \qquad \Longrightarrow \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{v}{R} \qquad \xRightarrow{\varphi(0)=0} \qquad \int\limits_0^{\varphi(t)} \mathrm{d}\varphi = \int\limits_0^{t} \frac{v}{R}\,\mathrm{d}t$$

Hence $\varphi(t) = vt/R = \omega t$, where $\omega = v/R$ is the angular velocity (constant here; in general a vector).

Example. Circular Motion: (A) Uniform

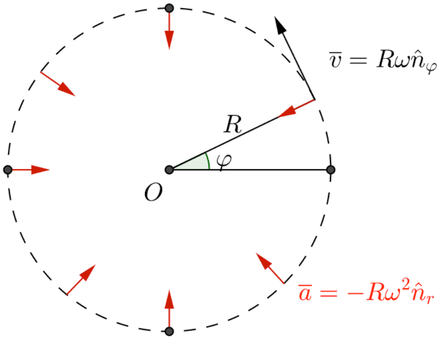
Acceleration

$$\bar{a} = (\underbrace{\ddot{r}}_{=0} - r\dot{\varphi}^2)\hat{n}_r + (r\underbrace{\ddot{\varphi}}_{=0} + 2\underbrace{\dot{r}}_{=0}\dot{\varphi})\hat{n}_\varphi = -R\omega^2\hat{n}_r$$

Summary

$$\boxed{\bar{v} = R\omega\hat{n}_\varphi, \qquad \bar{a} = -R\omega^2\hat{n}_r}$$

- $\hat{n}_\varphi$ corresponds to the tangential direction; $\hat{n}_r$ corresponds to the normal direction
- both $|\bar{v}|$ and $|\bar{a}|$ are constant in time



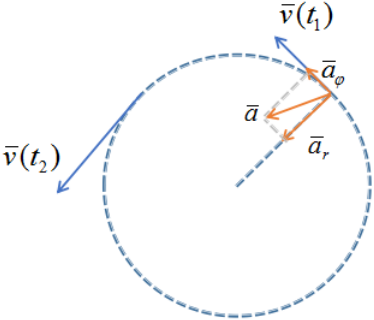
(B) **Non-Uniform Circular Motion**

Still $r = R = \mathrm{const}$, but now $\varphi = \varphi(t)$ is an arbitrary function of time.

$$
\begin{aligned}
\dot{\varphi} &= \dot{\varphi}(t) = \omega(t) && \text{angular velocity} \\
\ddot{\varphi} &= \ddot{\varphi}(t) = \dot{\omega}(t) = \varepsilon(t) && \text{angular acceleration}
\end{aligned}
$$

Note. Angular acceleration is in general defined as a vector quantity.

$$
\begin{aligned}
\bar{v} &= R\omega(t)\hat{n}_\varphi \\
\bar{a} &= \underbrace{-R\omega^2(t)\hat{n}_r}_{\text{curves the trajectory}} + \underbrace{R\varepsilon(t)\hat{n}_\varphi}_{\text{changes the speed}}
\end{aligned}
$$



Another Example: Beetle on a Vinyl

A beetle starts out from the center of a vinyl put on a gramophone, moving along its radius with constant speed v_0 with respect to the vinyl. The plate of the gramophone is set to rotate counter-clockwise (when looking from above) with constant angular speed Ω. At $t = 0\ s$, we have $\varphi(0) = 0$.

In the polar coordinates, with the origin set at the center of the vinyl, find: the trajectory of the beetle, its velocity and acceleration vectors and their magnitudes, as well as the tangential and the normal components of acceleration.

From the information provided

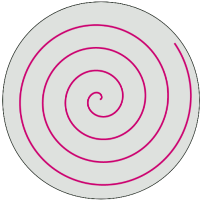
$$\begin{aligned} \dot{r} &= v_0 \\ \dot{\varphi} &= \Omega \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} r(t) &= \int_0^t v_0 \, \mathrm{d}t = v_0 t \\ \varphi(t) &= \int_0^t \Omega \, \mathrm{d}t = \Omega t \end{aligned}$$

Another Example: Beetle on a Vinyl

Eliminating time

$$\boxed{r = \frac{v_0}{\Omega}\varphi}$$

(**trajectory**: Archimedes' spiral)



Velocity

$$v_r = \dot{r} = v_0$$

$$v_\varphi = r\dot{\varphi} = v_0 \Omega t$$

$$v = \sqrt{v_r^2 + v_\varphi^2} = v_0\sqrt{1 + \Omega^2 t^2}$$

Another Example: Beetle on a Vinyl

Acceleration

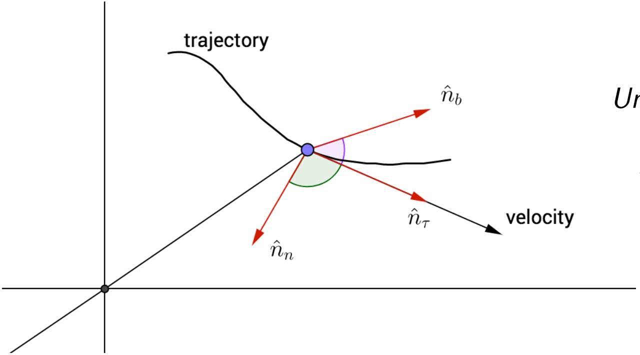
$$a_r = \ddot{r} - r\dot{\varphi}^2 = -v_0 t\Omega^2$$
$$a_\varphi = r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = 2v_0\Omega$$
$$a = \sqrt{a_r^2 + a_\varphi^2} = v_0\Omega\sqrt{4 + \Omega^2 t^2}$$

Tangential and normal components of acceleration

$$a_\tau = \dot{v} = v_0 \frac{\Omega^2 t}{\sqrt{1 + \Omega^2 t^2}}$$
$$a_n = \sqrt{a^2 - a_\tau^2} = \frac{v_0\Omega(2 + \Omega^2 t^2)}{\sqrt{1 + \Omega^2 t^2}}$$

Exercise: Analyze motion of the beetle using Cartesian coordinates.

Natural (or Kinematic) Coordinate System



Unit vectors

$\hat{n}_\tau$ — tagential ($\hat{n}_\tau || \overline{v}$)

$\hat{n}_n$ — normal

$\hat{n}_b$ — binormal

Velocity

$$\overline{v}(t) = v\hat{n}_\tau$$

Hence

$$\boxed{\hat{n}_\tau = \frac{\overset{\text{velocity (vector)}}{\overline{v}}}{\underset{\text{speed (scalar)}}{v}} = \frac{\dot{\overline{r}}}{|\dot{\overline{r}}|}}$$

Normal Unit Vector

Normal unit vector is perpendicular (orthogonal) to $\hat{n}_\tau$. *Problem*: many choices possible in 3D!

Unique choice:

$$\boxed{\hat{n}_n \stackrel{\text{def}}{=} \frac{\dfrac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t}}{\left|\dfrac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t}\right|}}$$

Note. We need to normalize the vector, because $\mathrm{d}\hat{n}_\tau/\mathrm{d}t$ is not of a unit length in general.

Is $\hat{n}_n \perp \hat{n}_\tau$? YES!

$$\hat{n}_\tau \circ \hat{n}_\tau = 1 \quad \xrightarrow[\text{with respect to } t]{\text{differentiate}} \quad \frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t} \circ \hat{n}_\tau + \hat{n}_\tau \circ \frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t} \circ \hat{n}_\tau = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t} \perp \hat{n}_\tau \quad \text{and} \quad \hat{n}_n \perp \hat{n}_\tau$$

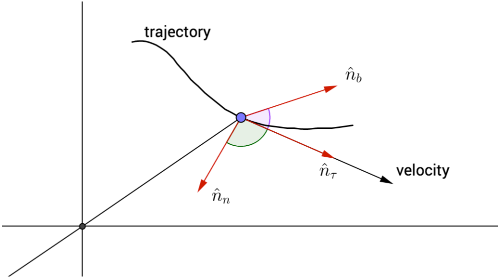
Binormal Unit Vector

Note. The normal unit vector $\hat{n}_n$ points along the radius of curvature.

The binormal unit vector is defined as

$$\boxed{\hat{n}_b = \hat{n}_\tau \times \hat{n}_n} \qquad \text{(right-handed system)}$$

The three vectors $\hat{n}_\tau$, $\hat{n}_n$ and $\hat{n}_b$, "sliding" along the particle's trajectory, are the unit vectors of the **natural coordinate system**.



Acceleration

$$\bar{a} = \dot{\bar{v}} = \dot{v}\hat{n}_\tau + v\dot{\hat{n}}_\tau = \dot{v}\hat{n}_\tau + v\left|\frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t}\right|\hat{n}_n$$

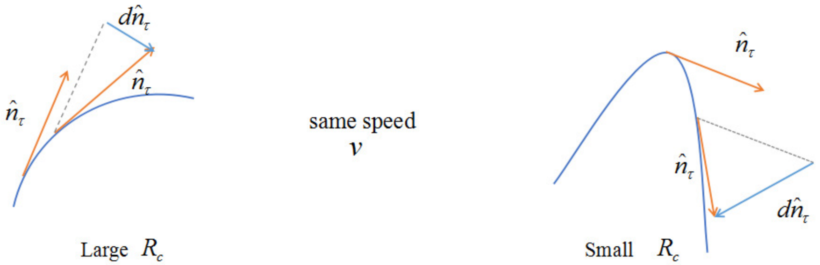
Define the (instantaneous) radius of trajectory's curvature as $R_c \stackrel{\text{def}}{=} \dfrac{v}{\left|\dfrac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t}\right|}$. Then

$$\bar{a} = \underbrace{\dot{v}\hat{n}_\tau}_{\text{tangential component } \bar{a}_\tau} + \underbrace{\frac{v^2}{R_c}\hat{n}_n}_{\text{normal component } \bar{a}_n}$$

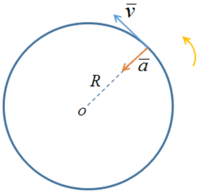
Both components are mutually perpendicular and $|\bar{a}| = \sqrt{a_\tau^2 + a_n^2}$

Instantaneous Radius of Curvature

Interpretation of $R_c \stackrel{\text{def}}{=} \dfrac{v}{\left|\dfrac{d\hat{n}_\tau}{dt}\right|}$ (assume same speed v)



Example 1. Uniform circular motion



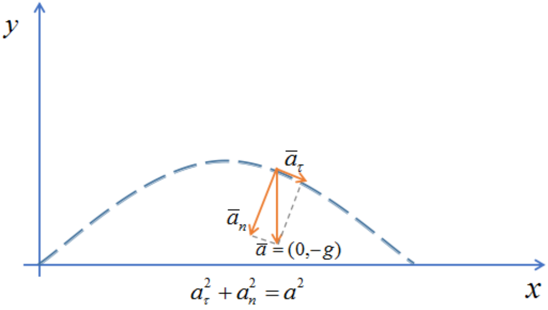
$$\begin{aligned} R_c &= R \qquad \text{(exercise)} \\ a_\tau &= 0 \\ a_n &= \frac{v^2}{R} \end{aligned}$$

Again!

$$a_\tau = \dot{v} \qquad \text{changes the } \underline{\text{magnitude}} \text{ of } \bar{v}$$

$$a_n = \frac{v^2}{R_c} \qquad \text{changes the } \underline{\text{direction}} \text{ of } \bar{v}$$

Example 2. Projectile motion (also, see Problem Set 2)

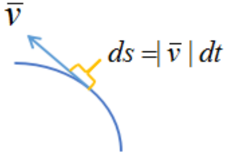


$$\begin{aligned}
\bar{v} &= (v_0 \cos\alpha)\hat{n}_x + (v_0 \sin\alpha - gt)\hat{n}_y \\
|\bar{v}| &= \sqrt{[v_x(t)]^2 + [v_y(t)]^2} \\
a_\tau &= \dot{v} \\
a_n &= \sqrt{g^2 - (\dot{v})^2}
\end{aligned}$$

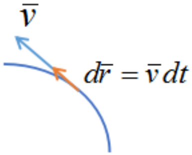
Final Remarks

Average Speed vs Average Velocity

$$\underset{\text{(scalar)}}{\text{average speed}} = \frac{\overbrace{\boxed{\int_{t_1}^{t_2} |\bar{v}(t)|\,\mathrm{d}t}}^{\substack{\text{distance traveled over} \\ \text{time interval } (t_1,t_2)}}}{t_2 - t_1}$$

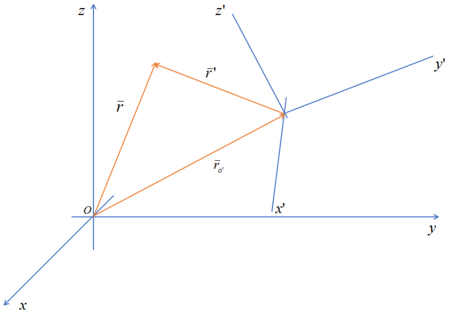


$$\underset{\text{(vector)}}{\text{average velocity}} = \frac{\overbrace{\boxed{\int_{t_1}^{t_2} \bar{v}(t)\,\mathrm{d}t}}^{\substack{\text{net displacement over} \\ \text{time interval } (t_1,t_2)}}}{t_2 - t_1}$$



Relative Motion and Galilean Transformation

Consider two frames of reference A and A'



Position vectors in both FoR are related

$$\bar{r} = \bar{r}_{o'} + \bar{r}'$$

Relative Motion and Galilean Transformation

Assume $\dot{\bar{r}}_{o'} = \bar{v}_{o'} = \mathrm{const}$, that is A' moves in a straight line (no rotations, either) with respect to A.
Then

$$\bar{v} = \bar{v}_{o'} + \bar{v}' \qquad \text{(velocity addition rule)}$$

Note that

$$\bar{r}_{o'} = \bar{v}_{o'} t + \bar{r}_{o'_{init}}.$$

Assuming that initially ($t = 0$) the origins of A and A' coincide, that is $\bar{r}_{o'_{init}} = 0$, we have

$$\boxed{\bar{r} = \bar{v}_{o'} t + \bar{r}'} \qquad \textbf{Galilean Transformation}$$