



UM-SJTU JOINT INSTITUTE
PHYSICS LABORATORY
(VP241)

LABORATORY REPORT

EXERCISE 5

RC, RL, AND RLC CIRCUITS

Name: Haotian Fu

ID: 520021910012

Group: 1

Date: 26 November 2021

[rev. 4.0]

Contents

1	Introduction	3
1.1	Basic Concepts	3
1.2	Transient Processes in RC , RL , RLC Series Circuits	3
1.2.1	RC Series Circuit	3
1.2.2	RL Series Circuit	4
1.2.3	RLC Series Circuit	4
1.3	RC , RL Steady-State Circuits	5
1.4	RLC Resonant Circuit	5
1.4.1	RLC Series Circuit	5
1.4.2	Resonance	6
1.4.3	Quality Factor in Resonant Circuits	6
2	Apparatus and Measurement Procedure	7
2.1	Apparatus	7
2.2	Measurement Procedure	8
2.2.1	RC , RL Series Circuit	8
2.2.2	RLC Series Circuit	8
2.2.3	RLC Resonant Circuit	8
2.2.4	Cations	8
2.3	Safety Notice	8
3	Experimental Results	9
3.1	$T_{1/2}$ Measurement in RC Series Circuit	9
3.2	$T_{1/2}$ Measurement in RL Series Circuit	10
3.3	$T_{1/2}$ Measurement in RLC Series Circuit	11
3.4	Measurements for RLC Resonance Circuit	14
3.4.1	Relationship of I/I_m vs. f/f_0	14
3.4.2	Phase Shift	16
3.4.3	Calculation of Resonate Frequency f_0 and Quality Factor Q	18
4	Uncertainty Analysis	19
4.1	Uncertainty of Data for RC Circuit	19
4.2	Uncertainty of Data for RL Circuit	20
4.3	Uncertainty of Data for RLC Circuit	21
4.4	Uncertainty of Data for RLC Resonant Circuit	22
4.4.1	Uncertainty of I/I_m vs. f/f_m Relationship Data	22
4.4.2	Uncertainty of Phase Shift Analysis	23
4.4.3	Uncertainty for Resonant Frequency f_0 and Quality Factor Q	24
5	Conclusion and Discussion	26
5.1	Conclusion	26
5.2	Discussion	26
5.2.1	System Errors	26
5.2.2	Random Errors	26
5.2.3	Improvements	26
6	Reference	27

List of Figures

1	<i>RC</i> series circuit	3
2	Charging/discharging curves for a <i>RC</i> series circuit	4
3	Three different regimes of transient processes in a <i>RLC</i> series circuit.	5
4	<i>RLC</i> series circuit	6
5	The impedance, the current and the phase difference as functions of the frequency for a <i>RLC</i> series circuit (generic sketches)	7
6	An example of the connection of <i>RCL</i> circuit	7
7	Experimental Results of <i>RC</i> circuit	10
8	Theoretic Results of <i>RC</i> circuit	10
9	Experimental Results of <i>RL</i> circuit	11
10	Theoretic Results of <i>RL</i> circuit	11
11	Experimental Results of under-damped <i>RCL</i> circuit	12
12	Theoretic Results of under-damped <i>RCL</i> circuit	12
13	Experimental Results of over-damped <i>RCL</i> circuit	12
14	Theoretic Results of over-damped <i>RCL</i> circuit	13
15	Experimental Results of critically damped <i>RCL</i> circuit	13
16	Theoretic Results of critically damped <i>RCL</i> circuit	13
17	Plot of I/I_m vs. f/f_0	15
18	Plot of Theoretic φ_{theo} vs. f/f_0	17
19	Plot of Experimental φ_{exp} vs. f/f_0	18
20	Plot of Experimental φ_{exp} vs. f/f_0 (Adjusted)	18
21	Directly comparing the waves by adjusting the vertical scale	27

List of Tables

1	Precision of the measurement instruments.	7
2	$T_{1/2}$ measurement data for a <i>RC</i> series circuit	9
3	Transformed into SI units	9
4	$T_{1/2}$ measurement data for a <i>RL</i> series circuit	10
5	Transformed into SI units	10
6	$T_{1/2}$ measurement data for a <i>RL</i> series circuit	13
7	Transformed into SI units	14
8	Measurement data for a <i>RLC</i> resonance circuit	14
9	Measurement data for the U_R vs. f dependence for a <i>RLC</i> resonant circuit	15
10	Relationship of I/I_m vs. f/f_0	16
11	Theoretic and experimental phase shift vs. frequency ratio	17
12	Uncertainty of all original data	20
13	Transformed into SI units	20
14	Transformed into SI units	21
15	Uncertainty for the U_R vs. f dependence for a <i>RLC</i> resonant circuit	22
16	Relationship of I/I_m vs. f/f_0	23
17	Uncertainty of Theoretic and experimental phase shift vs. frequency ratio	25
18	Result of time constant in <i>RC</i> , <i>RL</i> and critically damped <i>RLC</i> series circuits	26

1 Introduction

The objective of this exercise is to help students understand the physics of alternating-current circuits, in specific processes of charging/discharging of capacitors, the phenomenon of electromagnetic induction in inductive elements, and other dynamic processes in RC , RL , and RLC series circuits.

In this exercise, students will study methods for measuring the amplitude-frequency and the phase-frequency characteristics of RC , RL , and RLC series circuits. The resonance frequency of an RLC circuit as well as the quality factor of the circuit will be found from the amplitude-frequency curve.

1.1 Basic Concepts

- **Alternating current(AC)^[1]**

Alternating current(AC) is an electric current which periodically reverses direction.

- **Transient Process^[2]**

In a RC circuit, the process of charging or discharging of the capacitor is an example of a transient process.

- **Half-life^[2]**

The half-life period is the time needed for U_C to decrease a half of the initial value (or increase to a half of the terminal value), and may be also used to characterize the dynamics of the transient process.

- **Resonance^[3]**

Electrical resonance occurs in an electric circuit at a particular resonant frequency when the impedance of the circuit is at a minimum in a series circuit or at maximum in a parallel circuit (or when the transfer function is at a maximum).

1.2 Transient Processes in RC , RL , RLC Series Circuits

1.2.1 RC Series Circuit

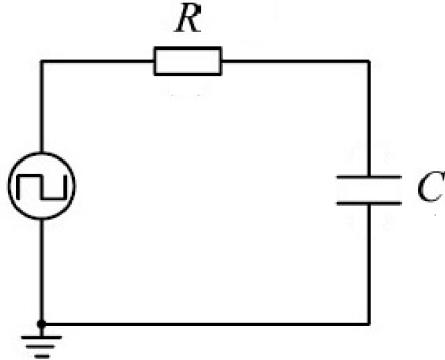


Figure 1: RC series circuit

In a RC circuit, the process of charging or discharging of the capacitor is an example of a transient process. Figure 1 shows a RC series circuit in which a square-wave signal is used as the source signal. In the first half of the cycle, the square-wave voltage is $U(t) = \mathcal{E}$ and it charges the capacitor. In the second half-cycle, the square-wave voltage is zero, and the capacitor discharges through the resistor.

The loop equation (Kirchhoff's loop rule) for the charging process is

$$RC \frac{dU_c}{dt} + U_c = \mathcal{E} \quad (1)$$

With the initial condition $U_C(t=0) = 0$, the solution of Eq.(1) can be found as

$$U_C = \mathcal{E}(1 - e^{-\frac{t}{RC}})$$

$$U_R = iR = \mathcal{E}e^{-\frac{t}{RC}}$$

Hence the voltage across the capacitor U_C increases exponentially with time t , whereas the voltage on the resistor U_R decreases exponentially with time. The curves $U(t)$, $U_C(t)$, and $U_R(t)$ are shown in Fig. 2.

For the discharging process, the loop rule gives

$$RC \frac{dU_C}{dt} + U_C = 0 \quad (2)$$

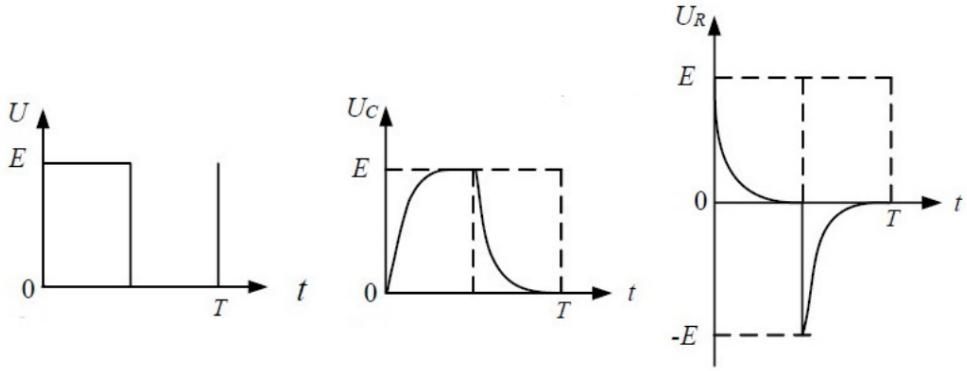


Figure 2: Charging/discharging curves for a *RC* series circuit

The solution of Eq. (2), with the initial condition $U_C(t = 0) = \varepsilon$, is

$$U_C = \mathcal{E} e^{-\frac{t}{RC}}$$

and, consequently,

$$U_R = iR = -\mathcal{E} e^{-\frac{t}{RC}}$$

where the magnitudes of both U_C and U_R decrease exponentially with time.

Since $RC = \tau$ has the units of time, it is called the time constant of the circuit, and characterizes the dynamics of the transient process. There is another characteristic related to the time constant, easier to measure in experiments, which is called the half-life period $T_{1/2}$. The half-life period is the time needed for U_C to decrease a half of the initial value (or increase to a half of the terminal value), and may be also used to characterize the dynamics of the transient process. Both quantities, in the process with exponential dynamics discussed above, are related by the equation

$$T_{1/2} = \tau \ln 2 \approx 0.693\tau.$$

1.2.2 RL Series Circuit

A similar analysis can be carried out for a RL series circuit. In this case

$$\tau = \frac{L}{R} \quad \text{and} \quad T_{1/2} = \frac{L}{R} \ln 2$$

1.2.3 RLC Series Circuit

First, we discuss the situation when a power source is suddenly plugged into a *RLC* circuit. Then the voltage across the capacitor satisfies the differential equation

$$LC \frac{d^2 U_C}{dt^2} + RC \frac{dU_C}{dt} + U_C = \mathcal{E} \quad (3)$$

following again from the loop rule. Dividing both sides of the equation by LC and introducing the symbols

$$\beta = \frac{R}{2L} \quad \text{and} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Eq.(3) can be rewritten as

$$\frac{d^2 U_C}{dt^2} + 2\beta \frac{dU_C}{dt} + \omega_0^2 U_C = \omega_0^2 \mathcal{E} \quad (4)$$

Note that Eq.(4) is an inhomogeneous differential equation and it is mathematically equivalent to the equation of motion of a damped harmonic oscillator with a constant driving force. Therefore, the complementary homogeneous equation is fully analogous to the equation of motion of a damped harmonic oscillator, with β being the damping coefficient, and ω_0 — the natural angular frequency. Moreover, after a specific solution to the inhomogeneous equation is found, a unique solution to the initial value problem consisting of Eq.5 and the initial conditions

$$U_C(t = 0) = 0 \quad \text{and} \quad \left. \frac{dU_C}{dt} \right|_{t=0} = 0 \quad (5)$$

can be found.

Exactly as for mechanical oscillations, depending on the relation between β and ω_0 , there are three regimes, as implied by the solution of the complementary homogeneous equation:

- i) If $\beta^2 - \omega_0^2 < 0$ (weak damping), the system is in the under-damped regime and the solution to the initial value problem is of the form

$$U_C = \mathcal{E} - \mathcal{E}e^{-\beta t} \left(\cos \omega t + \frac{\beta}{\omega} \sin \omega t \right)$$

$$\text{where } \omega = \sqrt{\omega_0^2 - \beta^2}$$

- ii) If $\beta^2 - \omega_0^2 > 0$ (strong damping), the system is in the over-damped regime with the solution of the form

$$U_C = \mathcal{E} - \frac{\mathcal{E}}{2\gamma} e^{-\beta t} [(\beta + \gamma)e^{\gamma t} - (\beta - \gamma)e^{-\gamma t}]$$

$$\text{where } \gamma = \sqrt{\beta^2 - \omega^2}$$

- iii) Finally, if $\beta^2 - \omega_0^2 = 0$, the system is said to be critically damped, and

$$U_C = \mathcal{E} - \mathcal{E}(1 + \beta t)e^{-\beta t} \quad (7)$$

When the circuit reaches a steady state, the power source is suddenly removed ($E = 0$). The differential equation for the discharging process is similar to that of the charging process, and there are also three regimes of the process.

The above discussion is valid for an ideal circuit and a step-signal source with zero internal resistance. In the experiment, the ideal system is replaced by a square-wave source with a small internal resistance. The period of the square-signal must be much greater than the time constant of the circuit. Note that, according to the above equations, the voltage across the capacitor U_C will finally reach E regardless of the regime (Fig. 3).

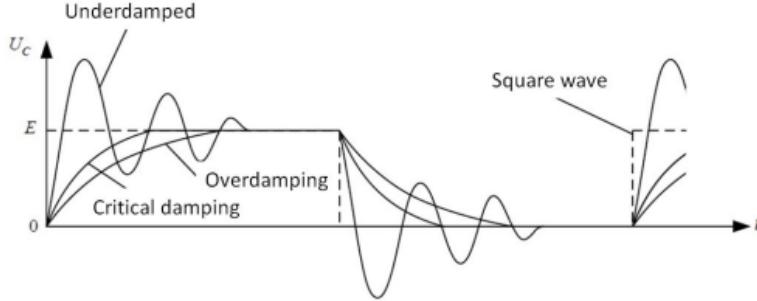


Figure 3: Three different regimes of transient processes in a RLC series circuit.

1.3 RC, RL Steady-State Circuits

When providing a sinusoidal alternating input voltage is provided to a RC (or RL) series circuit, the amplitude and the phase of the voltage across the capacitor and the resistor will change with the frequency of the input voltage. Then the amplitude vs. frequency relation and the phase vs. frequency relation can be obtained by measuring the voltage across the elements in the circuit for different input signal frequencies

$$\begin{aligned} \varphi &= \tan^{-1}\left(\frac{U_L}{U_R}\right) = \tan^{-1}\left(\frac{\omega L}{R}\right) \\ \varphi &= \tan^{-1}\left(-\frac{U_C}{U_R}\right) = \tan^{-1}\left(-\frac{1}{\omega RC}\right) \end{aligned}$$

1.4 RLC Resonant Circuit

1.4.1 RLC Series Circuit

A generic *RLC* series circuit is shown in Fig. 4.

The impedance and the phase difference in the RLC circuit can be calculated, e.g., by using the phasors technique. Representing the current I by a vector along the horizontal axis, the phase differences between the current and the voltages across the resistor, coil, and capacitor are

$$\varphi_R = 0, \quad \varphi_L = \frac{\pi}{2}, \quad \varphi_C = -\frac{\pi}{2},$$

respectively. The corresponding voltage amplitudes across the elements are

$$U_R = IZ = IR, \quad U_L = IZ_L = I\omega L, \quad U_C = IZ_C = \frac{I}{\omega C}.$$

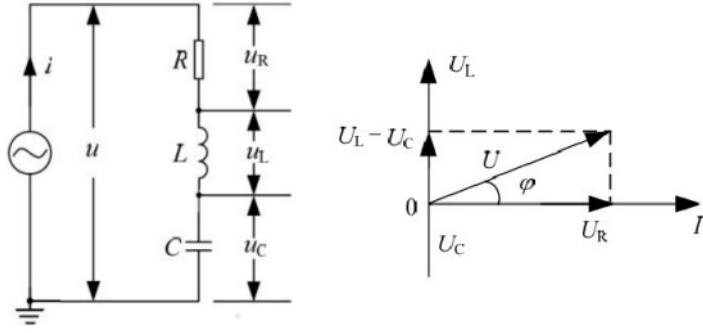


Figure 4: RLC series circuit

Hence, the voltage amplitude

$$U = \sqrt{U_R^2 + (U_L - U_C)^2} \quad \text{or} \quad U = I \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}, \quad (8)$$

and the total impedance

$$Z = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}, \quad (9)$$

with the phase difference between the current and the voltage in the circuit

$$\varphi = \tan^{-1} \frac{U_L - U_C}{U_R} = \tan^{-1} \frac{\omega L - \frac{1}{\omega C}}{R}.$$

1.4.2 Resonance

If the frequency of the input signal provided by the source satisfies the condition

$$\omega_0 L = \frac{\omega_0}{C}, \quad \Leftrightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}},$$

the total impedance will reach a minimum, $Z_0 = R$. Note that the resistance R in a real circuit includes the internal resistance and all kinds of alternating-current power losses, so its actual value will be greater than the theoretical one.

When the current reaches its maximum, $I_m = U/R$, the circuit is said to be at resonance. The frequency

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}},$$

at which the resonance phenomenon occurs, is called the *resonance frequency*.

The total impedance Z , current I , and the phase difference $\varphi = \varphi_u - \varphi_i$ all depend on frequency, with the generic shape of the curves shown in Figure 5. According to Eqs. (8) and (9), when the frequency is low ($f < f_0$, i.e. $1/\omega C > \omega L$), then $\varphi < 0$. In this situation the total voltage lags behind the current and the circuit is said to be capacitive.

When the circuit is resonant ($f = f_0$, i.e. $1/\omega C = \omega L$), then $\varphi < 0$, the voltage across the capacitor and the inductor should be equal, and the circuit is said to be resistive.

Finally, when the frequency is high ($f > f_0$, i.e. $\omega L > 1/\omega C$), and $\varphi > 0$. In this situation the total voltage leads the current, and the circuit is said to be inductive.

1.4.3 Quality Factor in Resonant Circuits

Since $I_m = U/R$, the voltages across the resistor, the inductor, and the capacitor are

$$\begin{aligned} U_R &= I_m R = U, \\ U_L &= I_m Z_L = \frac{U}{R} \omega L, \\ U_C &= I_m Z_C = \frac{U}{R} \frac{1}{\omega_0 C} = U_L, \end{aligned}$$

respectively. For a circuit driven at the resonance frequency, the ratio of U_L (or U_C) to U is called the quality factor Q of a resonant circuit

$$Q = \frac{U_L}{U} = \frac{\omega_0 L}{R} \quad \text{or} \quad Q = \frac{U_C}{U} = \frac{1}{\omega_0 R C}.$$

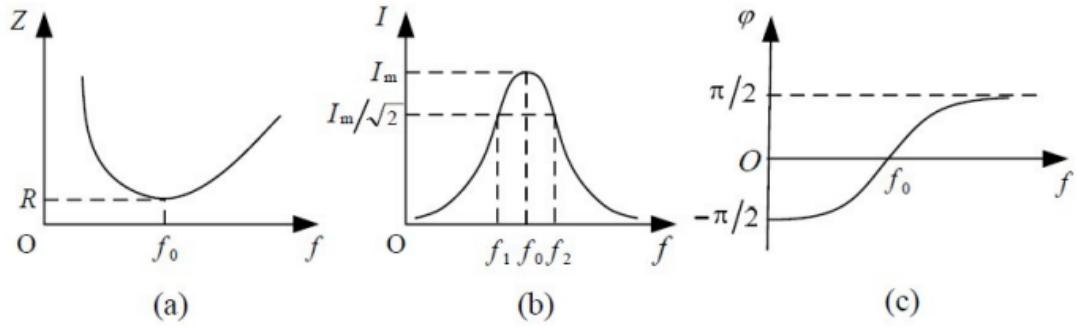


Figure 5: The impedance, the current and the phase difference as functions of the frequency for a *RLC* series circuit (generic sketches)

When the total voltage is fixed, the greater Q is, the greater U_L and U_C are. The value of Q can be used to quantify the efficiency of resonant circuits.

The quality factor can also be found as

$$Q = \frac{f_0}{f_2 - f_1},$$

where f_1 and f_2 are two frequencies such that $I(f_1) = I(f_2) = I_m/\sqrt{2}$ (Fig. 5).

2 Apparatus and Measurement Procedure

2.1 Apparatus

The apparatus involved in this lab include a signal generator, an oscilloscope, a digital multimeter, a wiring board, a fixed resistor $100\ \Omega$ (2 W), a variable resistor $2\ k\Omega$ (2 W), two capacitors $0.47\ \mu F$ and $0.1\ \mu F$, and two inductors ($10\ mH$ and $33\ mH$).

The following shows how to set up the apparatus. Remember to all black wires in one block to grounding.

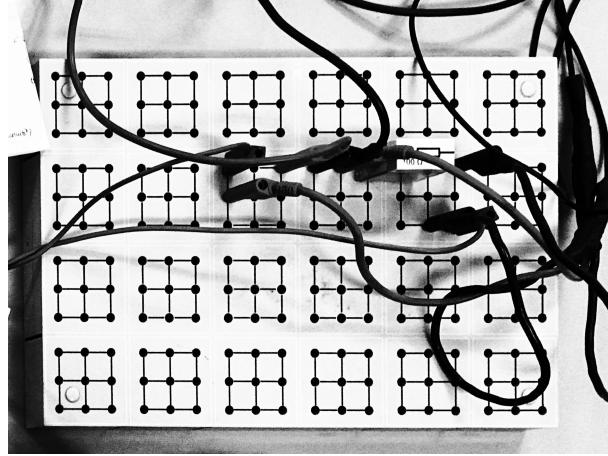


Figure 6: An example of the connection of RCL circuit

The precision of the apparatus is shown in Table 1.

Apparatus	Measurement	Unit	Precision
Signal generator	Frequency	[Hz]	0.001
	Amplitude	[V _{pp}]	0.001
Oscilloscope	Time	[μs]	0.01/0.001
	Voltage	[V _{pp}]	0.01
Digital multimeter	Resistance	[Ω]	0.01
	Capacitance	[nF]	0.01
Inductor	Inductance	[H]	0

Table 1: Precision of the measurement instruments.

2.2 Measurement Procedure

2.2.1 RC, RL Series Circuit

1. We chose a capacitor and an inductor to assemble a circuit with the fixed-resistance 100Ω resistor.
2. We adjusted the output frequency of the square-wave signal provided by the signal generator and observed the change of the waveform (curve shown on the oscilloscope screen) when the time constant is smaller or greater than the half period of the square wave.
3. We used the PRINT function of the oscilloscope to record the waveforms.
4. We adjust display parameters of the oscilloscope and measure $T_{1/2}$ for the studied circuits.
5. We calculated the time constant and compare it with the theoretical value. In order to find the time constant accurately, only one period should be displayed on the oscilloscope screen.

2.2.2 RLC Series Circuit

1. We chose a capacitor and an inductor to assemble a RLC series circuit with the variable resistor.
2. We observed the waveform of the capacitor voltage in the under-damped, critically damped, and over-damped regimes.
3. We used the PRINT function of the oscilloscope to store the waveforms.
4. We adjusted the variable resistor to the critically damped regime. According to the definition of the half-life period $T_{1/2}$, we have $\beta T_{1/2} = 1.68$.
5. We found the value of $T_{1/2}$, and then found the time constant as $\tau = 1/\beta = T_{1/2}/1.68$.
6. we compared your result with the theoretical value.

2.2.3 RLC Resonant Circuit

1. We applied a sinusoidal input voltage U_i to the *RLC* series circuit, and changed the frequency to observe the change of the voltage U_R for a fixed resistor R .
2. We observed the phase difference between U_R and U_i .
3. We measured how U_R changes with U_i and calculate the phase difference.
4. We plotted the graphs f/f_0 vs. I/I_m and f/f_0 vs. φ .
5. We estimated the resonance frequency and calculate the quality factor Q .

2.2.4 Cations

1. The circuit should be grounded to the same point as the instruments used in the measurements. In practice, all black wires should be inserted into the same block.
2. When measuring the corresponding element, it should be put at the side of the whole series connection. Otherwise it's hard to make sure that all black wires can be grounded.
3. When adjusting the waves in phase, we use the feature of Lissajous' figures. The figure should present a straight line of 45° .
4. In order to find the time constant accurately, only one period should be displayed on the oscilloscope screen.
5. We should read manuals carefully before operating the instruments.

2.3 Safety Notice

- a. Do not direct the laser beam into the eye.
- b. Do not touch the surface of the polarizers or the wave plates.
- c. Please leave the equipment in order before leaving.

3 Experimental Results

3.1 $T_{1/2}$ Measurement in RC Series Circuit

For RC circuit, the recorded data is as follows.

$R[\Omega]$	$f[\text{Hz}]$	$\varepsilon[\text{Vpp}]$	$C[\text{nF}]$	$T_{1/2}[\mu\text{s}]$
100.15 ± 0.01	1000.000 ± 0.001	4.000 ± 0.001	100.57 ± 0.01	9.000 ± 0.001

Table 2: $T_{1/2}$ measurement data for a RC series circuit

We first transform the above data into SI units for further calculations.

	Value	Uncertainty
$R [\Omega]$	100.15	0.01
$f [\text{Hz}]$	1000.000	0.001
$\varepsilon [\text{Vpp}]$	4.000	0.001
$C [\text{F}]$	1.0057×10^{-7}	1×10^{-11}
$T_{1/2} [\text{s}]$	9.000×10^{-6}	1×10^{-9}

Table 3: Transformed into SI units

The experimental determined time constant τ can be calculated by

$$\begin{aligned}\tau_{exp} &= \frac{T_{1/2}}{\ln 2} \\ &= \frac{9.000 \times 10^{-6}}{\ln 2} \\ &= 1.29843 \times 10^{-5} \pm 1.4 \times 10^{-9} [\text{s}] \\ &= 12.9843 \pm 0.0014 [\mu\text{s}]\end{aligned}$$

Theoretically, we find that

$$\begin{aligned}\tau_{theo} &= RC \\ &= 100.15 \times 1.0057 \times 10^{-7} \\ &= 1.00721 \times 10^{-5} \pm 1.4 \times 10^{-9} [\text{s}] \\ &= 10.0721 \pm 0.0014 [\mu\text{s}]\end{aligned}$$

The relative error δ can be found as

$$\begin{aligned}\delta_\tau &= \frac{\tau_{exp} - \tau_{theo}}{\tau_{theo}} \times 100\% \\ &= \frac{1.29843 \times 10^{-5} - 1.00721 \times 10^{-5}}{1.00721 \times 10^{-5}} \times 100\% \\ &= 28.91\%\end{aligned}$$

The relatively error is relatively large, thus we conclude that there might be some errors in this experiment. Hence repeated exercise is needed. Uncertainty will be analysis in the next section and the theoretic and experimental wave graphs are plotted as follows. The theoretic graph is plotted by Origin.



Figure 7: Experimental Results of RC circuit

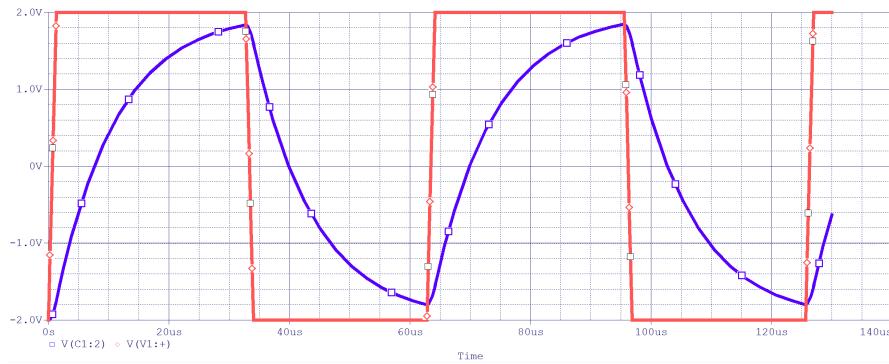


Figure 8: Theoretic Results of RC circuit

3.2 $T_{1/2}$ Measurement in RL Series Circuit

For RL circuit, the recorded data is as follows.

$R[\Omega]$	$f[\text{Hz}]$	$\varepsilon[\text{Vpp}]$	$L[\text{H}]$	$T_{1/2}[\mu\text{s}]$
100.15 ± 0.01	1000.000 ± 0.001	4.000 ± 0.001	0.01 ± 0.01	80.00 ± 0.01

Table 4: $T_{1/2}$ measurement data for a RL series circuit

We first transform the above data into SI units for further calculations.

	Value	Uncertainty
$R [\Omega]$	100.15	0.01
$f [\text{Hz}]$	1000.000	0.001
$\varepsilon [\text{Vpp}]$	4.000	0.001
$L [\text{H}]$	0.01	0
$T_{1/2} [\text{s}]$	8.000×10^{-5}	1×10^{-8}

Table 5: Transformed into SI units

The experimental determined time constant τ can be calculated by

$$\begin{aligned}\tau_{exp} &= \frac{T_{1/2}}{\ln 2} \\ &= \frac{8.000 \times 10^{-5}}{\ln 2} \\ &= 1.154156 \times 10^{-4} \pm 1.4 \times 10^{-9} [\text{s}] \\ &= 115.4156 \pm 0.0014 [\mu\text{s}]\end{aligned}$$

Theoretically, we find that

$$\begin{aligned}
\tau_{theo} &= \frac{L}{R} \\
&= \frac{0.01}{100.15} \\
&= 9.98502 \times 10^{-5} \pm 1.4 \times 10^{-9} [\text{s}] \\
&= 99.8502 \pm 0.0014 [\mu\text{s}]
\end{aligned}$$

The relative error δ can be found as

$$\begin{aligned}
\delta_\tau &= \frac{\tau_{exp} - \tau_{theo}}{\tau_{theo}} \times 100\% \\
&= \frac{11.54156 \times 10^{-5} - 9.98502 \times 10^{-5}}{9.98502 \times 10^{-5}} \times 100\% \\
&= 15.59\%
\end{aligned}$$

The relatively error is relatively large, thus we conclude that potential repeated experiments are needed. Uncertainty will be analysis in the next section and the theoretic and experimental wave graphs are plotted as follows. The theoretic graph is plotted by Origin.

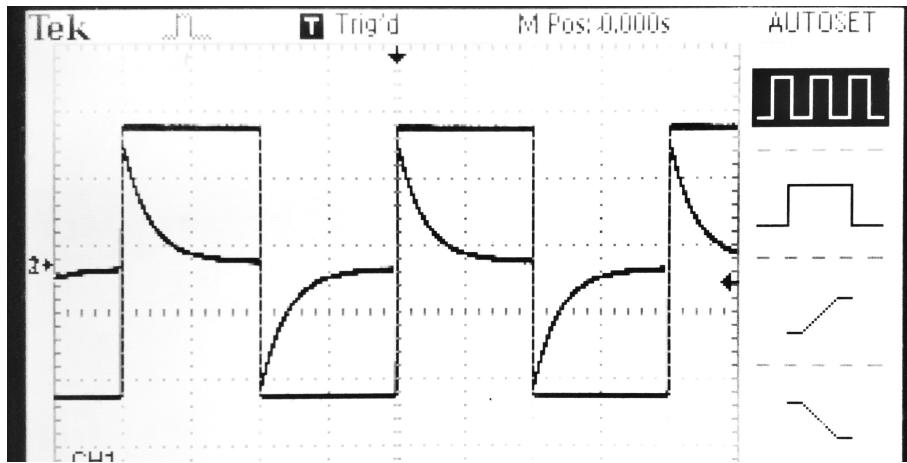


Figure 9: Experimental Results of RL circuit

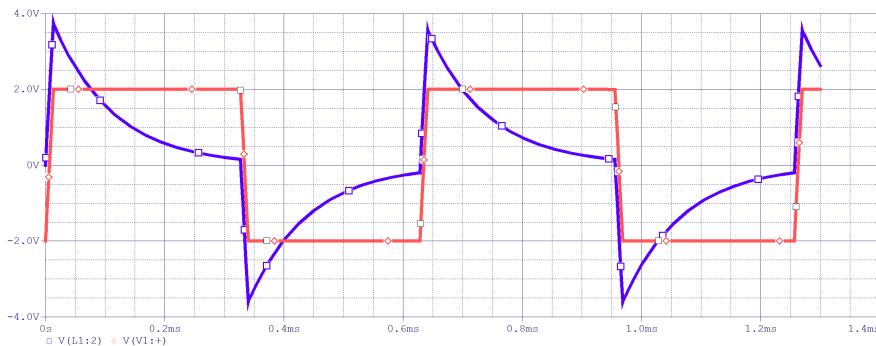


Figure 10: Theoretic Results of RL circuit

3.3 $T_{1/2}$ Measurement in RLC Series Circuit

In this section, we first sow the 3 possible regimes of the *RLC* circuit by adjusting the variable resistance. The 3 kinds of regimes are under-, over- and critically damped. The following is a display of all wave presentation of the regimes.

The following is the under-damped situation.

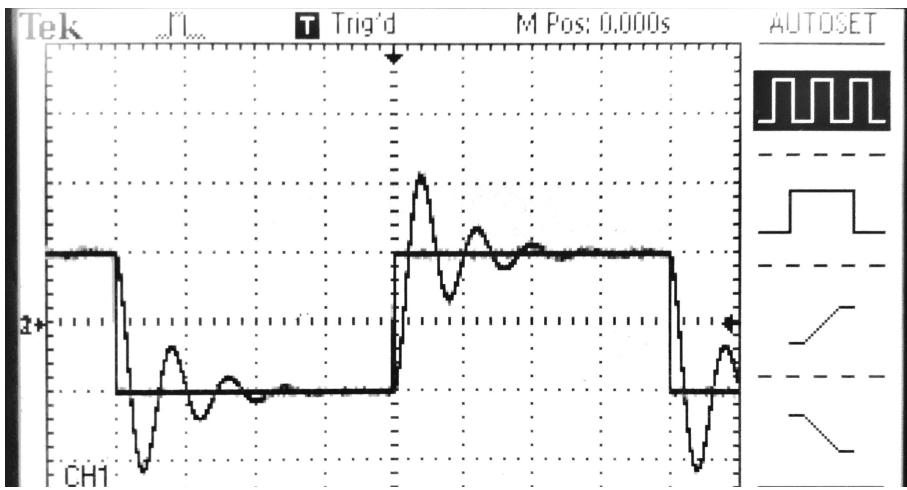


Figure 11: Experimental Results of under-damped RCL circuit



Figure 12: Theoretic Results of under-damped RCL circuit

The following is the over-damped situation.

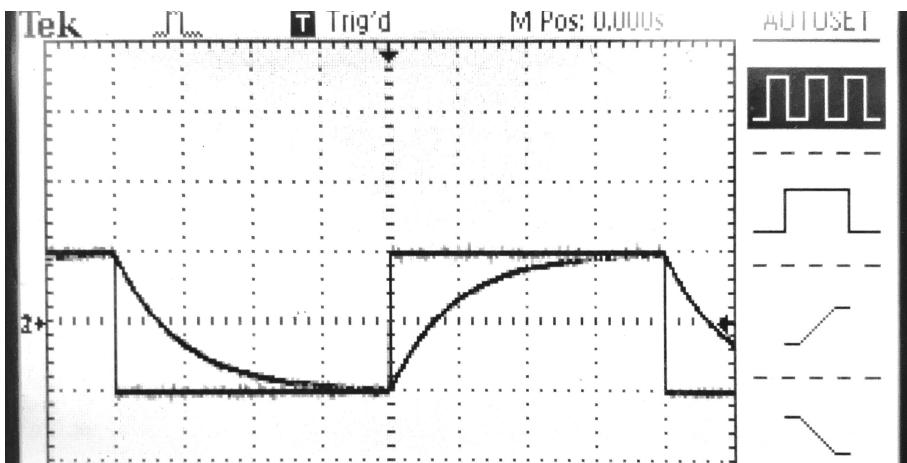


Figure 13: Experimental Results of over-damped RCL circuit

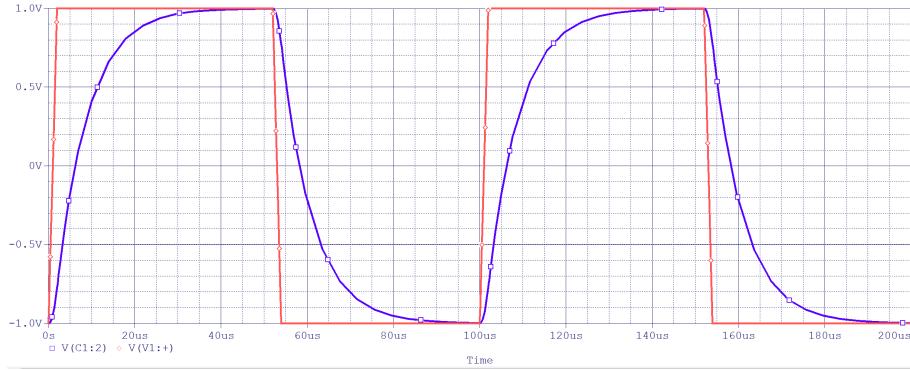


Figure 14: Theoretic Results of over-damped RCL circuit

The following is the critically damped situation.

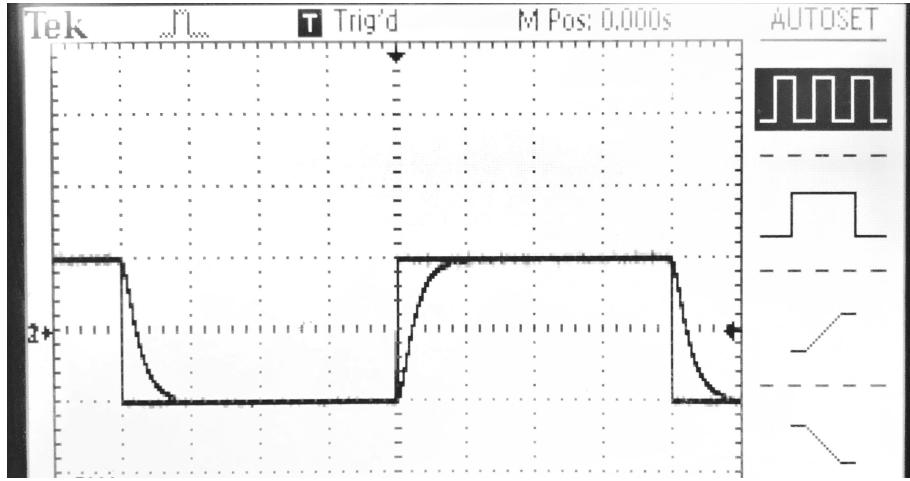


Figure 15: Experimental Results of critically damped RCL circuit

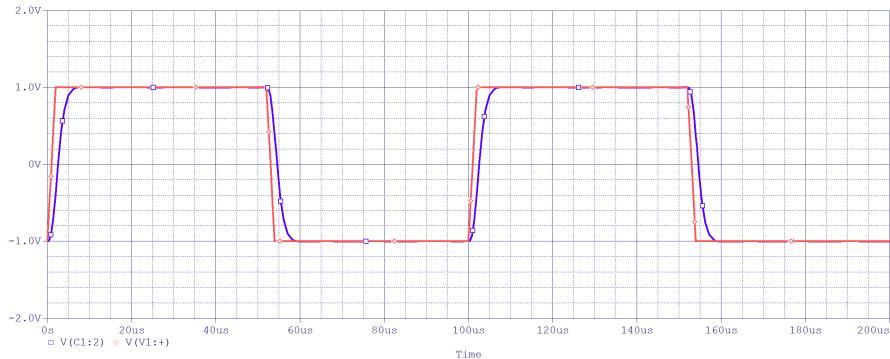


Figure 16: Theoretic Results of critically damped RCL circuit

We conducted $T_{1/2}$ measurement data to critically damped RLC series circuit. the recorded data is as follows.

L[H]	C[nF]	f[Hz]	ε [Vpp]	$\beta t[s]$	$T_{1/2}[\mu s]$
0.01 ± 0	100.57 ± 0.01	1000.000 ± 0.001	4.000 ± 0.001	1.68	30.00 ± 0.01

Table 6: $T_{1/2}$ measurement data for a RL series circuit

where βt is calculated as 1.68 (Measurement Procedure Part).

We first transform the above data into SI units for further calculations.

	Value	Uncertainty
f [Hz]	1000.000	0.001
ε [Vpp]	4.000	0.001
C [F]	1.0057×10^{-7}	1×10^{-11}
L [H]	0.01	0
$T_{1/2}$ [s]	3.000×10^{-5}	1×10^{-8}

Table 7: Transformed into SI units

The experimental determined time constant τ can be calculated by

$$\begin{aligned}\tau_{exp} &= \frac{T_{1/2}}{\beta t} \\ &= \frac{3.000 \times 10^{-5}}{1.68} \\ &= 1.7857 \times 10^{-5} \pm 1.4 \times 10^{-8} [\text{s}] \\ &= 17.857 \pm 0.014 [\mu\text{s}]\end{aligned}$$

Theoretically, we find that

$$\begin{aligned}\tau_{theo} &= \sqrt{LC} \\ &= \sqrt{0.01 \times 1.0057 \times 10^{-7}} \\ &= 3.1713 \times 10^{-5} \pm 1.0 \times 10^{-8} [\text{s}] \\ &= 31.713 \pm 0.010 [\mu\text{s}]\end{aligned}$$

The relative error δ can be found as

$$\begin{aligned}\delta_\tau &= \frac{\tau_{exp} - \tau_{theo}}{\tau_{theo}} \times 100\% \\ &= \frac{1.7857 \times 10^{-5} - 3.1713 \times 10^{-5}}{3.1713 \times 10^{-5}} \times 100\% \\ &= -43.69\%\end{aligned}$$

3.4 Measurements for RLC Resonance Circuit

For *RLC* circuit, the recorded condition data is as follows.

R[Ω]	L[H]	C[nF]	f_0 [Hz]	ε [Vpp]
100.15 ± 0.01	0.01 ± 0	100.57 ± 0.01	5000.000 ± 0.001	4.000 ± 0.001

Table 8: Measurement data for a *RLC* resonance circuit

Measurement data for the U_R vs. f dependence for a RLC resonant circuit are presented in the following table.

3.4.1 Relationship of I/I_m vs. f/f_0

At $f = f_0$, we find $U_m = 3.84\text{V}$. To find the relationship between I/I_m and f/f_0 , we use the formula in the previous section and obtain the formula as follows:

$$\begin{aligned}\frac{I}{I_m} &= \frac{U/R}{U_m/R} \\ &= \frac{U}{U_m} \\ &= \frac{U_R}{U_m}\end{aligned}$$

Thus we obtain:

$$\frac{I}{I_m} = \frac{U_R}{U_m} \tag{6}$$

	U_R [Vpp]	u_u [Vpp]	f [Hz]	u_f [Hz]
1	0.520	0.001	1500.000	0.001
2	0.720	0.001	2000.000	0.001
3	0.960	0.001	2500.000	0.001
4	1.28	0.01	3000.000	0.001
5	1.68	0.01	3500.000	0.001
6	2.40	0.01	4000.000	0.001
7	2.68	0.01	4200.000	0.001
8	3.08	0.01	4400.000	0.001
9	3.44	0.01	4600.000	0.001
10	3.72	0.01	4800.000	0.001
11	3.84	0.01	5000.000	0.001
12	3.76	0.01	5200.000	0.001
13	3.56	0.01	5400.000	0.001
14	3.24	0.01	5600.000	0.001
15	2.96	0.01	5800.000	0.001
16	2.64	0.01	6000.000	0.001
17	2.12	0.01	6500.000	0.001
18	1.72	0.01	7000.000	0.001
19	1.48	0.01	7500.000	0.001
20	1.36	0.01	8000.000	0.001
21	1.16	0.01	8500.000	0.001

Table 9: Measurement data for the U_R vs. f dependence for a RLC resonant circuit

A sample calculation for the first set of data is provided as follows.

$$\begin{aligned} \frac{I}{I_m} &= \frac{U_R}{U_m} \\ &= \frac{0.520}{3.84} \\ &= 0.1354 \pm 0.0004 \end{aligned}$$

$$\begin{aligned} \frac{f}{f_0} &= \frac{200.000}{5000.000} \\ &= 0.0400000 \pm 0.0000002 \end{aligned}$$

Thus we can calculate all the value of I/I_m vs. f/f_0 as follows.

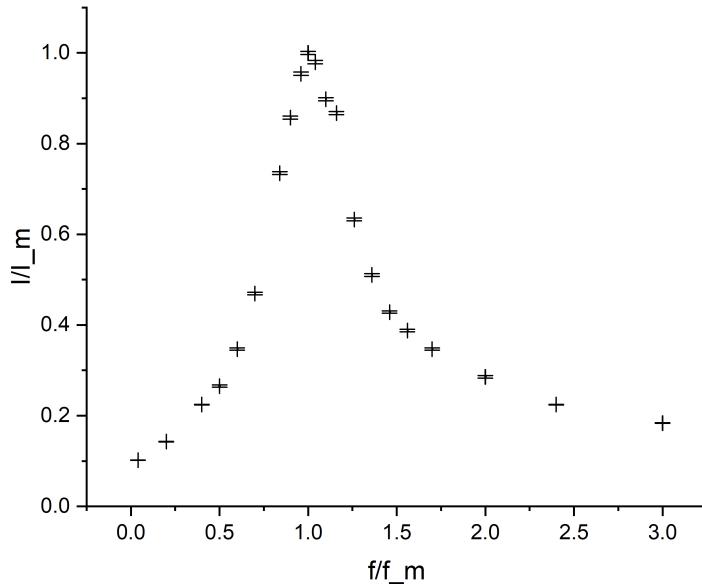


Figure 17: Plot of I/I_m vs. f/f_0

	U_R [Vpp]	u_{U_R} [Vpp]	f [Hz]	u_f [Hz]	I/I_m	u_{I/I_m}	f/f_0	u_{f/f_m}
1	0.520	0.001	1500.000	0.001	0.1354	0.0004	0.3000000	0.0000002
2	0.720	0.001	2000.000	0.001	0.1875	0.0006	0.4000000	0.0000002
3	0.960	0.001	2500.000	0.001	0.2500	0.0007	0.5000000	0.0000002
4	1.28	0.01	3000.000	0.001	0.333	0.003	0.6000000	0.0000002
5	1.68	0.01	3500.000	0.001	0.438	0.003	0.7000000	0.0000002
6	2.40	0.01	4000.000	0.001	0.625	0.003	0.8000000	0.0000003
7	2.68	0.01	4200.000	0.001	0.698	0.003	0.8400000	0.0000003
8	3.08	0.01	4400.000	0.001	0.802	0.003	0.8800000	0.0000003
9	3.44	0.01	4600.000	0.001	0.896	0.003	0.9200000	0.0000003
10	3.72	0.01	4800.000	0.001	0.969	0.004	0.9600000	0.0000003
11	3.84	0.01	5000.000	0.001	1.000	0.004	1.0000000	0.0000003
12	3.76	0.01	5200.000	0.001	0.979	0.004	1.0400000	0.0000003
13	3.56	0.01	5400.000	0.001	0.927	0.004	1.0800000	0.0000003
14	3.24	0.01	5600.000	0.001	0.844	0.003	1.1200000	0.0000003
15	2.96	0.01	5800.000	0.001	0.771	0.003	1.1600000	0.0000003
16	2.64	0.01	6000.000	0.001	0.688	0.003	1.2000000	0.0000003
17	2.12	0.01	6500.000	0.001	0.552	0.003	1.3000000	0.0000003
18	1.72	0.01	7000.000	0.001	0.448	0.003	1.4000000	0.0000003
19	1.48	0.01	7500.000	0.001	0.385	0.003	1.5000000	0.0000004
20	1.36	0.01	8000.000	0.001	0.354	0.003	1.6000000	0.0000004
21	1.16	0.01	8500.000	0.001	0.302	0.003	1.7000000	0.0000004

Table 10: Relationship of I/I_m vs. f/f_0

3.4.2 Phase Shift

The theoretical and experimental time constant can be calculated from frequency and U_R correspondingly. By using the formula mentioned in the above section, we deduce that:

$$\begin{aligned}\varphi_{theo} &= \tan^{-1} \left(\frac{U_L - U_C}{U_R} \right) \\ &= \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \\ &= \tan^{-1} \left(\frac{2\pi f L - \frac{1}{2\pi f C}}{R} \right)\end{aligned}$$

$$\begin{aligned}\varphi_{exp} &= \tan^{-1} \left(\frac{U_L - U_C}{U_R} \right) \\ &= \tan^{-1} \left(\frac{\sqrt{\varepsilon^2 - U_R^2}}{U_R} \right)\end{aligned}$$

Thus we obtain:

$$\varphi_{theo} = \tan^{-1} \left(\frac{2\pi f L - \frac{1}{2\pi f C}}{R} \right) \quad (7)$$

$$\varphi_{exp} = \tan^{-1} \left(\frac{\sqrt{\varepsilon^2 - U_R^2}}{U_R} \right) \quad (8)$$

A sample calculation of the first set of data is provided as follows. Thus we obtain:

$$\begin{aligned}\varphi_{theo} &= \tan^{-1} \left(\frac{2\pi f L - \frac{1}{2\pi f C}}{R} \right) \\ &= \tan^{-1} \left(\frac{2\pi \times 0.01 \times 1500.000 - \frac{1}{2\pi \times 1500.000 \times 1.0057 \times 10^{-7}}}{100.15} \right) \\ &= -1.46693 \pm 0.00011 [\text{rad}]\end{aligned}$$

$$\begin{aligned}
\varphi_{exp} &= \tan^{-1} \left(\frac{\sqrt{\varepsilon^2 - U_R^2}}{U_R} \right) \\
&= \tan^{-1} \left(\frac{\sqrt{4.000^2 - 0.520^2}}{0.520} \right) \\
&= 1.4350 \pm 0.0004[\text{rad}]
\end{aligned}$$

The following shows all the data calculated with respect to previous example.

f/f_0	u_{f/f_0}	φ_{theo}	$u_{\varphi_{\text{theo}}}$	φ_{ex}	$u_{\varphi_{\text{ex}}}$
0.3000000	0.0000002	-1.46693	0.00011	1.4350	0.0004
0.4000000	0.0000002	-1.4215	0.0002	1.3822	0.0006
0.5000000	0.0000002	-1.3634	0.0003	1.3181	0.0007
0.6000000	0.0000002	-1.2836	0.0004	1.231	0.003
0.7000000	0.0000002	-1.1637	0.0007	1.118	0.003
0.8000000	0.0000003	-0.9641	0.0013	0.896	0.004
0.8400000	0.0000003	-0.845	0.002	0.798	0.004
0.8800000	0.0000003	-0.693	0.002	0.640	0.006
0.9200000	0.0000003	-0.502	0.003	0.460	0.008
0.9600000	0.0000003	-0.274	0.003	0.251	0.015
1.0000000	0.0000003	-0.023	0.003	0.18	0.02
1.0400000	0.0000003	0.220	0.003	0.20	0.02
1.0800000	0.0000003	0.432	0.002	0.384	0.009
1.1200000	0.0000003	0.605	0.002	0.567	0.006
1.1600000	0.0000003	0.7407	0.0015	0.691	0.005
1.2000000	0.0000003	0.8466	0.0012	0.813	0.004
1.3000000	0.0000003	1.0251	0.0007	0.986	0.004
1.4000000	0.0000003	1.1326	0.0004	1.106	0.003
1.5000000	0.0000004	1.2034	0.0003	1.175	0.003
1.6000000	0.0000004	1.2534	0.0002	1.209	0.003
1.7000000	0.0000004	1.29050	0.00014	1.264	0.003

Table 11: Theoretic and experimental phase shift vs. frequency ratio

Thus, we can now plot the experimental and theoretic relationship between φ and f/f_0 . The following is the plot of theoretic determined φ_{theo} vs. f/f_0 .

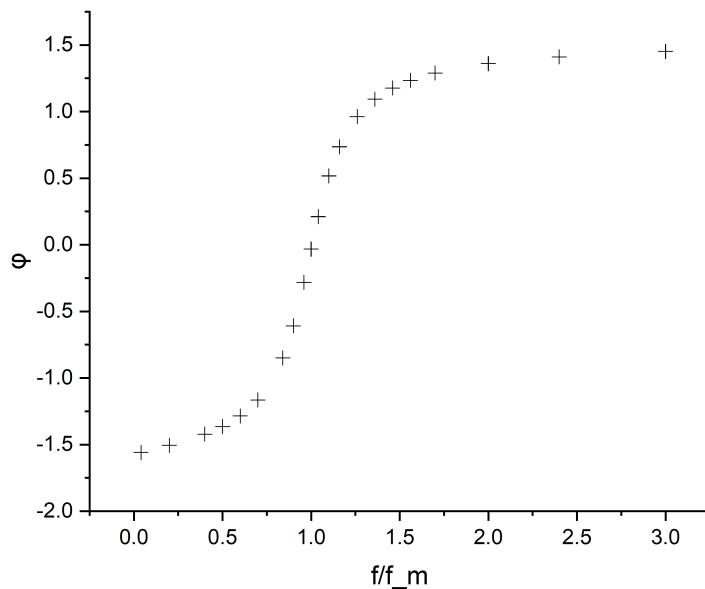


Figure 18: Plot of Theoretic φ_{theo} vs. f/f_0

The following is the experimentally calculated phase shift with respect to frequency ratio.

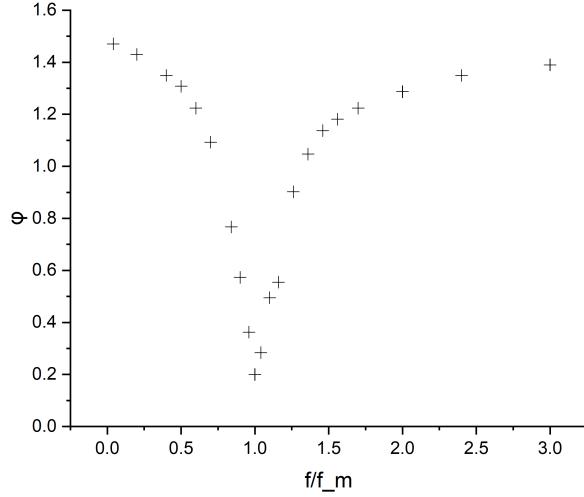


Figure 19: Plot of Experimental φ_{exp} vs. f/f_0

Note that all phase shift data retrieved are positive, because the formula we use

$$\varphi_{exp} = \tan^{-1} \left(\frac{\sqrt{\varepsilon^2 - U_R^2}}{U_R} \right)$$

ensures it's positiveness by square roots. Thus, by applying the correct signs to the data, we can plot the correct graph as:

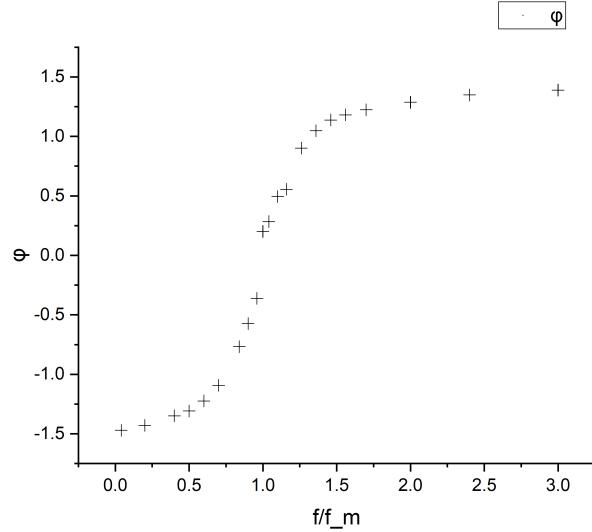


Figure 20: Plot of Experimental φ_{exp} vs. f/f_0 (Adjusted).

From the following figures, we found the two plots match each other. Te experiment is successful.

3.4.3 Calculation of Resonate Frequency f_0 and Quality Factor Q

The theoretic resonate frequency $f_{0,\text{theo}}$ is determined by:

$$f_{0,\text{theo}} = \frac{1}{2\pi\sqrt{LC}} \quad (9)$$

Thus we can calculated it as

$$\begin{aligned} f_{0,\text{theo}} &= \frac{1}{2\pi\sqrt{LC}} \\ &= \frac{1}{2\pi\sqrt{0.01 \times 1.0057 \times 10^{-7}}} \\ &= 5018.6 \pm 0.2[\text{Hz}] \end{aligned}$$

The experimental $f_0 = 5000.000$ [Hz], we can calculated the relative error δ by:

$$\begin{aligned} \delta_{f_0} &= \frac{f_{0,\text{ex}} - f_{0,\text{theo}}}{f_{0,\text{theo}}} \times 100\% \\ &= \frac{5018.6.000 - 5000}{5000} \times 100\% \\ &= 0.372\% \end{aligned}$$

Then we calculate Quality Factor Q. Theoretically,

$$Q_{\text{theo}} = \frac{2\pi f_0 L}{R}$$

and experientially,

$$Q_{\text{exp}} = \frac{f_0}{f_2 - f_1}$$

Thus,

$$\begin{aligned} Q_{\text{theo}} &= \frac{2\pi f_0 L}{R} \\ &= \frac{2\pi \times 5025.6 \times 0.01}{100.15} \\ &= 3.2670 \pm 0.0003[\text{Hz}] \end{aligned}$$

From the U_R calculation, $U_R/\sqrt{2} = 2.715$ [Vpp]. Thus, $f_2 \approx 5800.000$ [Hz] and $f_1 \approx 4200.000$ [Hz].

$$\begin{aligned} Q_{\text{exp}} &= \frac{f_0}{f_2 - f_1} = \frac{5000.0000}{5800.00 - 4200.000} \\ &= 3.125000 \pm 0.000004[\text{Hz}] \end{aligned}$$

we can calculated the relative error δ by:

$$\begin{aligned} \delta_Q &= \frac{Q_{\text{exp}} - Q_{\text{theo}}}{Q_{\text{theo}}} \times 100\% \\ &= \frac{3.125000 - 3.2670}{3.2670} \times 100\% \\ &= -4.35\% \end{aligned}$$

4 Uncertainty Analysis

4.1 Uncertainty of Data for RC Circuit

The parameters, including resistance, frequency, Voltage ε , capacitance and time are all single measurement. They don't have a type-A uncertainty and, their type-B uncertainty can be calculated as their standard deviation. A sample calculation is

$$u_R = \Delta_{\text{dev}} = 1 \times 10^{-2}[\Omega]$$

Thus, the uncertainty of the quantities are given by:

For the experimental time constant,

$$\tau_{\text{exp}} = \frac{T_{1/2}}{\ln 2}$$

	Value	Uncertainty
R [Ω]	100.15	0.01
f [Hz]	1000.000	0.001
E [Vpp]	4.000	0.001
C [F]	1.0057×10^{-7}	1×10^{-11}
T _{1/2} [s]	9.000×10^{-6}	1×10^{-9}

Table 12: Uncertainty of all original data

We can find the uncertainty by:

$$\begin{aligned}
u_{\tau,exp} &= \frac{\partial \tau}{\partial T_{1/2}} u_{T_{1/2}} \\
&= \frac{u_{T_{1/2}}}{\ln 2} \\
&= \frac{1 \times 10^{-9}}{\ln 2} \\
&= 1.4 \times 10^{-9} \text{ [s]}
\end{aligned}$$

For the theoretical time constant,

$$\tau_{\text{theo}} = RC$$

We can find the uncertainty by:

$$\begin{aligned}
u_{\tau,\text{theo}} &= \sqrt{\left(\frac{\partial \tau}{\partial R} u_R\right)^2 + \left(\frac{\partial \tau}{\partial C} u_C\right)^2} \\
&= \sqrt{(Cu_R)^2 + (Ru_C)^2} \\
&= \sqrt{(100.57 \times 10^{-9} \times 0.01)^2 + (100.15 \times 1 \times 10^{-11})^2} \\
&= 1.4 \times 10^{-9} \text{ [s]}.
\end{aligned}$$

4.2 Uncertainty of Data for RL Circuit

The parameters, including resistance, frequency, Voltage ε , inductance and time are all single measurement. They don't have a type-A uncertainty and, their type-B uncertainty can be calculated as their standard deviation. A sample calculation is

$$u_R = \Delta_{dev} = 1 \times 10^{-2} \text{ [Ω]}$$

Thus, the uncertainty of the quantities are given by:

	Value	Uncertainty
R [Ω]	100.15	0.01
f [Hz]	1000.000	0.001
ε [Vpp]	4.000	0.001
L [H]	0.01	0
T _{1/2} [s]	8.000×10^{-5}	1×10^{-8}

Table 13: Transformed into SI units

For the experimental time constant,

$$\tau_{exp} = \frac{T_{1/2}}{\ln 2}$$

We can find the uncertainty by:

$$\begin{aligned}
u_{\tau,exp} &= \frac{\partial \tau}{\partial T_{1/2}} u_{T_{1/2}} \\
&= \frac{u_{T_{1/2}}}{\ln 2} \\
&= \frac{1 \times 10^{-8}}{\ln 2} \\
&= 1.4 \times 10^{-8} \text{ [s]}
\end{aligned}$$

For the experimental time constant,

$$\tau_{\text{theo}} = \frac{L}{R}$$

We can find the uncertainty by:

$$\begin{aligned} u_{\tau,\text{theo}} &= \sqrt{\left(\frac{\partial \tau}{\partial R} u_R\right)^2 + \left(\frac{\partial \tau}{\partial L} u_L\right)^2} \\ &= \sqrt{\left(-\frac{L u_R}{R^2}\right)^2 + \left(\frac{u_L}{R}\right)^2} \\ &= \sqrt{\left(\frac{0.01 \times 0.01}{100.15^2}\right)^2 + \left(\frac{0}{100.15}\right)^2} \\ &= 1.0 \times 10^{-8} [\text{s}]. \end{aligned}$$

4.3 Uncertainty of Data for RLC Circuit

The parameters, including capacitance frequency, Voltage ε , inductance and time are all single measurement. They don't have a type-A uncertainty and, their type-B uncertainty can be calculated as their standard deviation. A sample calculation is

$$u_f = \Delta_{\text{dev}} = 1 \times 10^{-3} [\Omega]$$

Thus, the uncertainty of the quantities are given by:

	Value	Uncertainty
f [Hz]	1000.000	0.001
ε [Vpp]	4.000	0.001
C [F]	1.0057×10^{-7}	1×10^{-11}
L [H]	0.01	0
$T_{1/2}$ [s]	3.000×10^{-5}	1×10^{-8}

Table 14: Transformed into SI units

For the experimental time constant,

$$\tau = \frac{T_{1/2}}{\beta t}$$

We can find the uncertainty by:

$$\begin{aligned} u_{\tau,\text{exp}} &= \frac{\partial \tau}{\partial T_{1/2}} u_{T_{1/2}} \\ &= \frac{u_{T_{1/2}}}{1.68} \\ &= \frac{1 \times 10^{-8}}{1.68} \\ &= 6 \times 10^{-9} [\mu\text{s}] \end{aligned}$$

For the experimental time constant,

$$\tau = \sqrt{LC}$$

We can find the uncertainty by:

$$\begin{aligned} u_{\tau,\text{theo}} &= \sqrt{\left(\frac{\partial \tau_{\text{theo}}}{\partial L} u_L\right)^2 + \left(\frac{\partial \tau_{\text{theo}}}{\partial C} u_C\right)^2} \\ &= \sqrt{\left(\frac{1}{2} \sqrt{\frac{C}{L}} u_L\right)^2 + \left(\frac{1}{2} \sqrt{\frac{L}{C}} u_C\right)^2} \\ &= \sqrt{0^2 + \left(\frac{1}{2} \sqrt{\frac{L}{C}} u_C\right)^2} \\ &= \frac{1}{2} \sqrt{\frac{L}{C}} u_C \\ &= \frac{1}{2} \sqrt{\frac{0.01}{1.0057 \times 10^{-7}}} \times 1 \times 10^{-11} \\ &= 1.6 \times 10^{-9} [\text{s}]. \end{aligned}$$

4.4 Uncertainty of Data for RLC Resonant Circuit

The parameters, including capacitance frequency, Voltage ε , inductance and time are all single measurement. They don't have a type-A uncertainty and, their type-B uncertainty can be calculated as their standard deviation. A sample calculation is

$$u_f = \Delta_{dev} = 1 \times 10^{-3}[\Omega]$$

Thus, the uncertainty of the quantities are given by:

	U_R [Vpp]	u_u [Vpp]	f [Hz]	u_f [Hz]
1	0.520	0.001	1500.000	0.001
2	0.720	0.001	2000.000	0.001
3	0.960	0.001	2500.000	0.001
4	1.28	0.01	3000.000	0.001
5	1.68	0.01	3500.000	0.001
6	2.40	0.01	4000.000	0.001
7	2.68	0.01	4200.000	0.001
8	3.08	0.01	4400.000	0.001
9	3.44	0.01	4600.000	0.001
10	3.72	0.01	4800.000	0.001
11	3.84	0.01	5000.000	0.001
12	3.76	0.01	5200.000	0.001
13	3.56	0.01	5400.000	0.001
14	3.24	0.01	5600.000	0.001
15	2.96	0.01	5800.000	0.001
16	2.64	0.01	6000.000	0.001
17	2.12	0.01	6500.000	0.001
18	1.72	0.01	7000.000	0.001
19	1.48	0.01	7500.000	0.001
20	1.36	0.01	8000.000	0.001
21	1.16	0.01	8500.000	0.001

Table 15: Uncertainty for the U_R vs. f dependence for a RLC resonant circuit

4.4.1 Uncertainty of I/I_m vs. f/f_0 Relationship Data

For $I/I_m = U_R/U_m$, its uncertainty is

$$\begin{aligned} u_{I/I_m} &= \sqrt{\left(\frac{\partial U_R/U_m}{\partial U_R} u_{U_R}\right)^2 + \left(\frac{\partial U_R/U_m}{\partial U_m} u_{U_m}\right)^2} \\ &= \sqrt{\left(\frac{u_{U_R}}{U_m}\right)^2 + \left(-\frac{U_R}{U_m^2} u_{U_m}\right)^2} \end{aligned}$$

For f/f_0 , its uncertainty is calculated as

$$\begin{aligned} u_{f/f_0} &= \sqrt{\left(\frac{\partial f/f_0}{\partial f} u_f\right)^2 + \left(\frac{\partial f/f_0}{\partial f_0} u_{f_0}\right)^2} \\ &= \sqrt{\left(\frac{u_f}{f_0}\right)^2 + \left(-\frac{f}{f_0^2} u_{f_0}\right)^2} \end{aligned}$$

A sample calculation is provided by #1.

$$\begin{aligned} u_{I/I_m} &= \sqrt{\left(\frac{u_{U_R}}{U_m}\right)^2 + \left(-\frac{U_R}{U_m^2} u_{U_m}\right)^2} \\ &= \sqrt{\left(\frac{0.001}{3.84}\right)^2 + \left(\frac{0.520}{3.84^2} \times 0.01\right)^2} \\ &= 0.0004 \end{aligned}$$

$$\begin{aligned}
u_{f/f_0} &= \sqrt{\left(\frac{u_f}{f_0}\right)^2 + \left(-\frac{f}{f_0^2} u_{f_0}\right)^2} \\
&= \sqrt{\left(\frac{0.001}{5000.000}\right)^2 + \left(\frac{1500.000}{5000.000^2} \times 0.001\right)^2} \\
&= 2 \times 10^{-7}
\end{aligned}$$

By following the formula mentioned above we can calculate all the data samples, and they are listed in the following table.

	U_R [Vpp]	u_{U_R} [Vpp]	f [Hz]	u_f [Hz]	I/I_m	u_{I/I_m}	f/f_m	u_{f/f_m}
1	0.520	0.001	1500.000	0.001	0.1354	0.0004	0.3000000	0.0000002
2	0.720	0.001	2000.000	0.001	0.1875	0.0006	0.4000000	0.0000002
3	0.960	0.001	2500.000	0.001	0.2500	0.0007	0.5000000	0.0000002
4	1.28	0.01	3000.000	0.001	0.333	0.003	0.6000000	0.0000002
5	1.68	0.01	3500.000	0.001	0.438	0.003	0.7000000	0.0000002
6	2.40	0.01	4000.000	0.001	0.625	0.003	0.8000000	0.0000003
7	2.68	0.01	4200.000	0.001	0.698	0.003	0.8400000	0.0000003
8	3.08	0.01	4400.000	0.001	0.802	0.003	0.8800000	0.0000003
9	3.44	0.01	4600.000	0.001	0.896	0.003	0.9200000	0.0000003
10	3.72	0.01	4800.000	0.001	0.969	0.004	0.9600000	0.0000003
11	3.84	0.01	5000.000	0.001	1.000	0.004	1.0000000	0.0000003
12	3.76	0.01	5200.000	0.001	0.979	0.004	1.0400000	0.0000003
13	3.56	0.01	5400.000	0.001	0.927	0.004	1.0800000	0.0000003
14	3.24	0.01	5600.000	0.001	0.844	0.003	1.1200000	0.0000003
15	2.96	0.01	5800.000	0.001	0.771	0.003	1.1600000	0.0000003
16	2.64	0.01	6000.000	0.001	0.688	0.003	1.2000000	0.0000003
17	2.12	0.01	6500.000	0.001	0.552	0.003	1.3000000	0.0000003
18	1.72	0.01	7000.000	0.001	0.448	0.003	1.4000000	0.0000003
19	1.48	0.01	7500.000	0.001	0.385	0.003	1.5000000	0.0000004
20	1.36	0.01	8000.000	0.001	0.354	0.003	1.6000000	0.0000004
21	1.16	0.01	8500.000	0.001	0.302	0.003	1.7000000	0.0000004

Table 16: Relationship of I/I_m vs. f/f_0

4.4.2 Uncertainty of Phase Shift Analysis

For $\varphi_{\text{theo}} = \tan^{-1} \left(\frac{2\pi f L - \frac{1}{2\pi f C}}{R} \right)$, the uncertainty

$$\begin{aligned}
u_{\varphi_{\text{theo}}} &= \sqrt{\left(\frac{\partial \varphi_{\text{theo}}}{\partial f} u_f\right)^2 + \left(\frac{\partial \varphi_{\text{theo}}}{\partial C} u_C\right)^2 + \left(\frac{\partial \varphi_{\text{theo}}}{\partial R} u_R\right)^2} \\
&= \sqrt{\left(\frac{R(2\pi L + \frac{1}{2\pi f^2 C})}{R^2 + (2\pi f L - \frac{1}{2\pi f C})^2} u_f\right)^2 + \left(\frac{R}{2\pi f C^2 [R^2 + (2\pi f L - \frac{1}{2\pi f C})^2]} u_C\right)^2 + \left(-\frac{2\pi f L - \frac{1}{2\pi f C}}{R^2 + (2\pi f L - \frac{1}{2\pi f C})^2} u_R\right)^2}.
\end{aligned}$$

Take the first set of data as an example,

$$\begin{aligned}
u_{\varphi_{\text{theo}}} &= \sqrt{\left(\frac{100.15 \times (2\pi \times 0.01 + \frac{1}{2\pi \times 1500^2 \times 100.57})}{100.15^2 + (2\pi \times 1500 \times 0.01 - \frac{1}{2\pi \times 1500 \times 100.57})^2} \times 0.001\right)^2 +} \\
&\quad \left(\frac{100.15}{2\pi \times 1500 \times 100.57^2 \times [100.15^2 + (2\pi \times 1500 \times 0.01 - \frac{1}{2\pi \times 1500 \times 100.57})^2]}\right)^2 + \\
&\quad \left(\frac{2\pi \times 1500 \times 0.01 - \frac{1}{2\pi \times 1500 \times 100.57}}{100.15^2 + (2\pi \times 1500 \times 0.01 - \frac{1}{2\pi \times 1500 \times 100.57})^2}\right)^2} \\
&= 0.00011 \text{ [rad].}
\end{aligned}$$

The uncertainties for other sets of data are calculated in the same way and the results are shown in Table 17.

For $\varphi_{exp} = \tan^{-1} \left(\frac{\sqrt{\varepsilon^2 - U_R^2}}{U_R} \right)$, its uncertainty is

$$u_{\varphi,e} = \left(\frac{\partial \varphi_e}{\partial U_R} u_{U_R} \right)^2$$

Calculate the partial derivative:

$$\frac{\partial \varphi_e}{\partial U_R} = \frac{1}{\sqrt{\varepsilon^2 - U_R^2}}$$

Thus,

$$u_{\varphi,e} = \left(\frac{u_{U_R}}{\sqrt{\varepsilon^2 - U_R^2}} \right)^2$$

Thus, a sample calculation of #1 is provided as follows.

$$\begin{aligned} u_{\varphi,e} &= \left(\frac{u_{U_R}}{\sqrt{\varepsilon^2 - U_R^2}} \right)^2 \\ &= \left(\frac{0.01}{\sqrt{4.000^2 - 0.520^2}} \right)^2 \\ &= 0.000006[\text{Hz}] \end{aligned}$$

$$\begin{aligned} \frac{\partial \varphi_t}{\partial f} &= \frac{R(2\pi L + \frac{1}{2\pi f^2 C})}{R^2 + (2\pi f L - \frac{1}{2\pi f C})^2} \\ &= \frac{100.15 \times (2\pi \times 0.01 + \frac{1}{2\pi \times (1500.000)^2 \times (1.0057 \times 10^{-7})})}{(100.15)^2 + (2\pi \times 1500.000 \times 0.01 - \frac{1}{2\pi \times 1500.000 \times 1.0057 \times 10^{-7}})^2} \\ &= 6.35003 \times 10^{-5} \end{aligned}$$

$$\begin{aligned} \frac{\partial \varphi_t}{\partial C} &= \frac{R}{2\pi f C^2 [R^2 + (2\pi f L - \frac{1}{2\pi f C})^2]} \\ &= \frac{100.15}{2\pi \times 1500.00 \times (1.0057 \times 10^{-7})^2 [(100.15)^2 + (2\pi \times 1500.000 \times 0.01 - \frac{1}{2\pi \times 1500.00 \times (1.0057 \times 10^{-7})})^2]} \\ &= 126433.1015 \end{aligned}$$

$$\begin{aligned} \frac{\partial \varphi_t}{\partial R} &= -\frac{2\pi f L - \frac{1}{2\pi f C}}{R^2 + (2\pi f L - \frac{1}{2\pi f C})^2} \\ &= -\frac{2\pi \times 1500.00 \times 0.01 - \frac{1}{2\pi \times 1500.000 \times 1.0057 \times 10^{-7}}}{(100.15)^2 + (2\pi \times 1500.000 \times 0.01 - \frac{1}{2\pi \times 1500.000 \times 1.0057 \times 10^{-7}})^2} \\ &= 0.000126208 \end{aligned}$$

And thus

$$\begin{aligned} u_{\varphi,t} &= \sqrt{\left(\frac{\partial \varphi_t}{\partial f} u_f \right)^2 + \left(\frac{\partial \varphi_t}{\partial C} u_C \right)^2 + \left(\frac{\partial \varphi_t}{\partial R} u_R \right)^2} \\ &= \sqrt{(6.35003 \times 10^{-5} \times 0.001)^2 + (126433.1015 \times 1 \times 10^{-11})^2 + (0.000126208 \times 0.01)^2} \\ &= 0.0004[\text{Hz}] \end{aligned}$$

The uncertainties for other sets of data are calculated in the same way and the results are shown in Table 17.

By using the above calculation we can get:

4.4.3 Uncertainty for Resonant Frequency f_0 and Quality Factor Q

The theoretic resonant frequency is

$$f_{0,theo} = \frac{1}{2\pi\sqrt{LC}}$$

f/f_0	u_{f/f_0}	φ_{theo}	$u_{\varphi_{\text{theo}}}$	φ_{ex}	$u_{\varphi_{\text{ex}}}$
0.3000000	0.0000002	-1.46693	0.00011	1.4350	0.0004
0.4000000	0.0000002	-1.4215	0.0002	1.3822	0.0006
0.5000000	0.0000002	-1.3634	0.0003	1.3181	0.0007
0.6000000	0.0000002	-1.2836	0.0004	1.231	0.003
0.7000000	0.0000002	-1.1637	0.0007	1.118	0.003
0.8000000	0.0000003	-0.9641	0.0013	0.896	0.004
0.8400000	0.0000003	-0.845	0.002	0.798	0.004
0.8800000	0.0000003	-0.693	0.002	0.640	0.006
0.9200000	0.0000003	-0.502	0.003	0.460	0.008
0.9600000	0.0000003	-0.274	0.003	0.251	0.015
1.0000000	0.0000003	-0.023	0.003	0	/
1.0400000	0.0000003	0.220	0.003	0.20	0.02
1.0800000	0.0000003	0.432	0.002	0.384	0.009
1.1200000	0.0000003	0.605	0.002	0.567	0.006
1.1600000	0.0000003	0.7407	0.0015	0.691	0.005
1.2000000	0.0000003	0.8466	0.0012	0.813	0.004
1.3000000	0.0000003	1.0251	0.0007	0.986	0.004
1.4000000	0.0000003	1.1326	0.0004	1.106	0.003
1.5000000	0.0000004	1.2034	0.0003	1.175	0.003
1.6000000	0.0000004	1.2534	0.0002	1.209	0.003
1.7000000	0.0000004	1.29050	0.00014	1.264	0.003

Table 17: Uncertainty of Theoretic and experimental phase shift vs. frequency ratio

Thus, its uncertainty can be calculated by

$$\begin{aligned}
 u_{f_0,\text{theo}} &= \sqrt{\left(\frac{\partial f}{\partial L} u_L\right)^2 + \left(\frac{\partial f}{\partial C} u_C\right)^2} \\
 &= \sqrt{\left(-\frac{C}{4\pi(LC)^{1.5}} u_L\right)^2 + \left(-\frac{L}{4\pi(LC)^{1.5}} u_C\right)^2} \\
 &= \sqrt{\left(-\frac{1.0057 \times 10^{-7}}{4\pi(0.01 \times 1.0057 \times 10^{-7})^{1.5}} \times 0\right)^2 + \left(-\frac{0.01}{4\pi(0.01 \times 1.0057 \times 10^{-7})^{1.5}} \times 1 \times 10^{-11}\right)^2} \\
 &= 0.3 \text{ [Hz]}
 \end{aligned}$$

Then we calculated the uncertainty of the quality factor. Theoretically, the uncertainty of the quality factor Q is

$$\begin{aligned}
 u_{Q,t} &= \sqrt{\left(\frac{\partial Q_{\text{theo}}}{\partial R} u_R\right)^2 + \left(\frac{\partial Q_{\text{theo}}}{\partial C} u_C\right)^2} \\
 &= \sqrt{\left(-\frac{\sqrt{LC}}{R^2 C} u_R\right)^2 + \left(-\frac{\sqrt{L}}{2RC^{-1.5}} u_C\right)^2} \\
 &= \sqrt{\left(-\frac{\sqrt{0.01 \times 1.0057 \times 10^{-11}}}{100.15^2 \times 1.0057 \times 10^{-11}} \times 0.01\right)^2 + \left(-\frac{\sqrt{0.01}}{2 \times 100.15 \times (1.0057 \times 10^{-11})^{-1.5}} \times 1 \times 10^{-11}\right)^2} \\
 &= 3 \times 10^{-4}
 \end{aligned}$$

To find the quality factor, first find out the frequencies f_1 , f_2 at which I/I_m is about $1/\sqrt{2}$. Consulting Table 16, we obtain that $f_1 = 4200\text{Hz}$ and $f_2 = 5800\text{Hz}$. Then, according to Eq.(??), the experimental quality factor

Experimentally, The uncertainty of the quality factor Q is

$$\begin{aligned}
 u_{Q,e} &= \sqrt{\left(\frac{\partial Q}{\partial f_0} u_{f_0}\right)^2 + \left(\frac{\partial Q}{\partial f_1} u_{f_1}\right)^2 + \left(\frac{\partial Q}{\partial f_2} u_{f_2}\right)^2} \\
 &= \sqrt{\left(\frac{u_{f_0}}{f_2 - f_1}\right)^2 + \left(\frac{f_0}{(f_2 - f_1)^2} u_{f_1}\right)^2 + \left(-\frac{f_0}{(f_2 - f_1)^2} u_{f_2}\right)^2} \\
 &= \sqrt{\left(\frac{0.001}{5800.000 - 4200.000}\right)^2 + \left(\frac{5000.000}{(5800.000 - 4200.000)^2} \times 0.001\right)^2 + \left(-\frac{5000.000}{(5800.000 - 4200.000)^2} \times 0.001\right)^2} \\
 &= 4 \times 10^{-6}
 \end{aligned}$$

5 Conclusion and Discussion

5.1 Conclusion

In exercise 5, we generally studied the properties of alternating-current circuits, especially, the processes of charging and discharging of capacitors, the phenomenon of electromagnetic induction in inductive elements.

In terms of ability to be developed, we obtained acquaintance of the methods for measuring the amplitude-frequency and the phase-frequency characteristics of RC , RL , and RLC series circuits.

The following is a conclusion of the whole experiment.

By applying $T_{1/2}$ measuring, we experimentally determined the time constants of RC , RL and critically damped RLC circuits. The following is the results.

	Experimental τ [μs]	Theoretical τ_{theo} [μs]	Relative error
RC circuit	12.9843 ± 0.0014	10.0721 ± 0.0014	28.91%
RL circuit	115.4156 ± 0.0014	99.8502 ± 0.0014	15.59%
RLC circuit	17.857 ± 0.014	31.713 ± 0.010	-43.69%

Table 18: Result of time constant in RC , RL and critically damped RLC series circuits

The relative errors are relatively huge, and the reasons of errors will be discussed in DISCUSSION part.

5.2 Discussion

Here we will discuss the problems in this exercise, some potential errors which may lead to huge variation, and some suggestions.

5.2.1 System Errors

- In principle, the way f_1 and f_2 are found is not accurate.

We cannot find the corresponding f for $U_R/\sqrt{2}$. The estimation will lead to a difference of around 200[Hz]. This will cause significant errors.

- Inner resistance causes errors.

In ideal condition, the inner resistance of the circuit wire, inductor and the capacitor is ignore. However, this is not the case. For RC and RL circuits, the inner resistance will cause a higher R_{eq} , giving a higher time constant τ . Also, in resonance circuits, the measured U_R will be smaller because voltage division.

5.2.2 Random Errors

- Lissajous figures are always vibrating, and it's not a good way to judge whether the wave is in phase.
- The oscilloscope is not steady when we did the experiment. The readings are always vibrating.
- The oscilloscope's cursor is not precise enough. First, it can't reach exactly $T_{1/2}$ sometimes. Also, the judge is based on human eyes, which is not accurate.
- Due to some signal failure, although voltage is generated from function generators, the frequency is not exactly as the one in the function generator than in the oscilloscope, usually a $\pm 0.5\%$ relative difference.

5.2.3 Improvements

- Lissajous figures are always vibrating, and it's not a good way to judge whether the wave is in phase. Instead, comparing directly the waves by adjusting the vertical scale is more accurate.
- After finding U_R , we should next find the corresponding f for $U_R/\sqrt{2}$, to determine f_1 and f_2 .
- We can measure more data, especially in the first $T_{1/2}$ experiments, to reduce random errors.
- All the measurements in this experiment is single-measurements. Thus, by repeating the procedure for multiple groups of same data, we get rid of random errors (But this will cause type-A uncertainty, which increase the uncertainty).

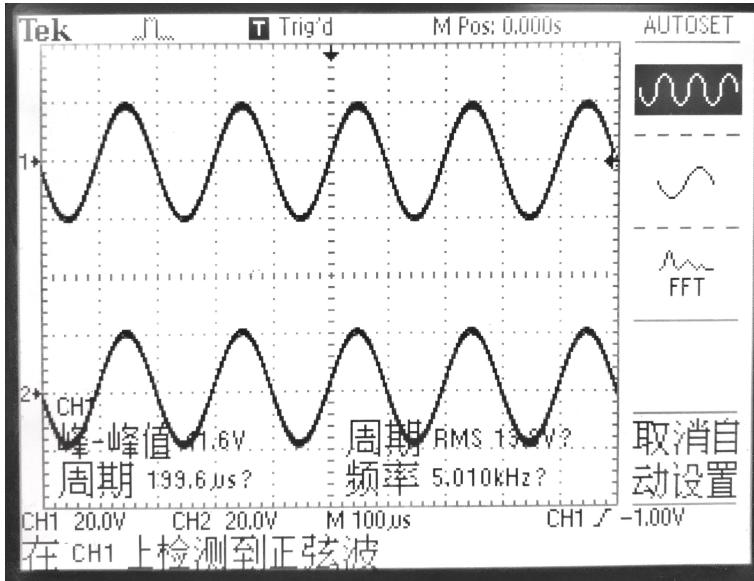


Figure 21: Directly comparing the waves by adjusting the vertical scale

6 Reference

- [1] N. N. Bhargava & D. C. Kulshreshtha (1983). Basic Electronics and Linear Circuits. Tata McGraw-Hill Education. p.90. ISBN 978-0-07-451965-3.
- [2] M. Krzyzosiak (2021). Exercise 5 - lab manual [rev. 2.6].pdf (UMJI-SJTU, Shanghai).
- [3] "The Physics Of Resonance". Intuitor.com (1994-2016).10 July 2017. Web. 3 Dec. 2021. <<http://www.intuitor.com/resonance/circuits.html>>

APPENDIX - DATA SHEET