PHYSICS II Problem Set 1

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Problem 1

Solution

Free-body Diagrams are show as follows.

For Point A

$$\begin{cases} E_{ya} = \frac{1}{4\pi\varepsilon_0} \cdot \left(-\frac{q}{(a/2)^2} + \frac{2q}{(a/2)^2} - \frac{q}{(a/2)^2+a^2} \times \frac{1}{\sqrt{5}} \right) \\ E_{xa} = -\frac{1}{4\pi\varepsilon_0} \cdot \left(\frac{-q}{(a/2)^2+a^2} \times \frac{2}{\sqrt{5}} \right) \end{cases} \tag{1}$$

For Point B

$$\begin{cases} E_{yb} = \frac{1}{4\pi\varepsilon_0} \cdot \left(-\frac{-q}{a^2} - \frac{2q}{a^2+a^2} \times \frac{1}{\sqrt{2}} \right) \\ E_{xb} = \frac{1}{4\pi\varepsilon_0} \cdot \left(\frac{q}{a^2} + \frac{2q}{a^2+a^2} \times \frac{1}{\sqrt{2}} \right) \end{cases} \tag{2}$$

For Point C

$$\begin{cases} E_{yc} = \frac{1}{4\pi\varepsilon} \cdot \left(\frac{q}{(a/2)^2+(a/2)^2} \times \frac{1}{\sqrt{2}} - \frac{-q}{(a/2)^2+(a/2)^2} \times \frac{1}{\sqrt{2}} - \frac{2q}{(a/2)^2+(a/2)^2} \times \frac{1}{\sqrt{2}} \right) \\ E_{xc} = \frac{1}{4\pi\varepsilon} \cdot \left(\frac{q}{(a/2)^2+(a/2)^2} \times \frac{1}{\sqrt{2}} - \frac{-q}{(a/2)^2+(a/2)^2} \times \frac{1}{\sqrt{2}} + \frac{2q}{(a/2)^2+(a/2)^2} \times \frac{1}{\sqrt{2}} \right) \end{cases} \tag{3}$$

Solving Eq.(1)(2)(3) we get

$$\begin{cases} E_{ya} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{100-4\sqrt{5}q}{25a^2} \\ E_{xa} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{8\sqrt{5}q}{25a^2} \end{cases} \quad \begin{cases} E_{yb} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{(2-\sqrt{2})q}{2a^2} \\ E_{xb} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{(2+\sqrt{2})q}{2a^2} \end{cases} \quad \begin{cases} E_{yc} = 0 \\ E_{xc} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{4\sqrt{2}q}{a^2} \end{cases}$$

Hence

$$E_A = \frac{1}{4\pi\varepsilon_0} \cdot \frac{8\sqrt{5}q}{25a^2}\hat{n_x} + \frac{1}{4\pi\varepsilon_0} \cdot \frac{100-4\sqrt{5}q}{25a^2}\hat{n_y} \tag{4}$$

$$E_B = \frac{1}{4\pi\varepsilon_0} \cdot \frac{(2+\sqrt{2})q}{2a^2}\hat{n_x} + \frac{1}{4\pi\varepsilon_0} \cdot \frac{(2-\sqrt{2})q}{2a^2}\hat{n_y} \tag{5}$$

$$E_C = \frac{1}{4\pi\varepsilon_0} \cdot \frac{(2-\sqrt{2})q}{2a^2}\frac{4\sqrt{2}q}{a^2}\hat{n_x} \tag{6}$$

where units are all [V/m].

Plug in all the data, we get

$$\mathbf{E}_A = 6.43 \times 10^{20}\hat{n_x} + 3.27 \times 10^{21}\hat{n_y} \tag{7}$$

$$\mathbf{E}_B = 1.53 \times 10^{21}\hat{n_x} - 2.63 \times 10^{20}\hat{n_y} \tag{8}$$

$$\mathbf{E}_C = 5.08 \times 10^{21}\hat{n_x} \tag{9}$$

where units are all [V/m].

Problem 2

Solution

According to symmetry, we deduce that electric fields formed by any rods symmetric about origin will be cancelled out.

Hence, only considering the magnitude of charges, A has $\frac{1}{3}Q$, B has $\frac{\sqrt{2}}{2}Q$, C has Q and D has 0.

Thus

$$C > B > A > D$$

Problem 3

Solution

(a) For every tiny slice of the rod, suppose its distance from the origin is x, then we can denote the electric field as

$$E = \int_0^l \frac{1}{4\pi\varepsilon_0} \cdot \frac{\lambda dx}{(l+a-x)^2} = \frac{\lambda}{4\pi\varepsilon_0} \int_0^l \frac{1}{(l+a-x)^2} dx \tag{10}$$

Hence the magnitude of electric filed is

$$E = \frac{\lambda}{4\pi\varepsilon_0} \left(\frac{1}{a} - \frac{1}{a+l} \right) \tag{11}$$

(b) Analogously, we denote the electric field as

$$E = \int_0^l \frac{1}{4\pi\varepsilon_0} \cdot \frac{\lambda dx}{(l+a-x)^2} = \int_0^l \frac{1}{4\pi\varepsilon_0} \cdot \frac{Axdx}{(l+a-x)^2} \tag{12}$$

Hence we get

$$E = \frac{A}{4\pi\varepsilon_0} \left(\ln\left(\frac{a}{a+l} \right) + \frac{l}{a} \right) \tag{13}$$

Problem 4

Solution

(a) For tiny piece dx of the right rod, the force exerted on it is

$$dF = \int_{-a/2}^{-a/2-l} \frac{1}{4\pi\varepsilon_0} \cdot \frac{\eta_e d(-s) \cdot \eta_e dx}{(x-s)^2} \tag{14}$$

where η_e denotes the linear charge density of the rod and s denotes the coordinate of the tiny piece $d(-s)$ of the left rod.

Hence we can denote the force exerted on the whole right rod

$$F = \int_{a/2}^{a/2+l} \int_{-a/2}^{-a/2-l} \frac{1}{4\pi\varepsilon_0} \cdot \frac{\eta_e d(-s) \cdot \eta_e dx}{(x-s)^2} \tag{15}$$

Then through calculation

$$
\begin{aligned}
F &= \int_{a/2}^{a/2+l} \int_{-a/2}^{-a/2-l} \frac{1}{4\pi\varepsilon_0} \cdot \frac{\eta_e d(-s) \cdot \eta_e dx}{(x-s)^2} \\
&= \frac{\eta_e^2}{4\pi\varepsilon_0} \int_{a/2}^{a/2+l} \int_{-a/2}^{-a/2-l} \frac{1}{(x-s)^2} d(-s)\ dx \\
&= \frac{\eta_e^2}{4\pi\varepsilon_0} \int_{a/2}^{a/2+l} \int_{a/2}^{a/2+l} \frac{1}{(x+s)^2} ds\ dx \\
&= \frac{\eta_e^2}{4\pi\varepsilon_0} \int_{x+a/2}^{x+a/2+l} \frac{1}{s^2} ds\ dx \\
&= \frac{\eta_e^2}{4\pi\varepsilon_0} \int_{a/2}^{a/2+l} \left(\frac{1}{x+\frac{a}{2}} - \frac{1}{x+\frac{a}{2}+l} \right) dx \\
&= \frac{\eta_e^2}{4\pi\varepsilon_0} \int_{a}^{a+l} \left(\frac{1}{x} - \frac{1}{x+l} \right) dx \\
&= \frac{\eta_e^2}{4\pi\varepsilon_0} \ln \left(\frac{(a+l)^2}{a(a+2l)} \right)
\end{aligned} \tag{16}
$$

Since η_e is the linear charge density, we know

$$\eta_e l = Q \tag{17}$$

Thus

$$F = \frac{Q^2}{4\pi\varepsilon_0 l^2} \ln \left(\frac{(a+l)^2}{a(a+2l)} \right) \tag{18}$$

(b) since

$$\ln(1+u) = u - u^2/2 + u^3/3 - \cdots \tag{19}$$

when $u \ll 1$.

For Eq.(18),

$$\frac{(a+l)^2}{a(a+2l)} = 1 + \frac{l^2}{a^2+2al}$$

while the latter part $\frac{l^2}{a^2+2al} \ll 1$ when $a \gg l$.

Thus we denote Eq.(18) as

$$\frac{Q^2}{4\pi\varepsilon_0 l^2} \left(u - u^2/2 + u^3/3 - \cdots \right) \tag{20}$$

where $u = l^2/(a^2+2al)$.

Since $u \ll 1$, $u^2 \ll u$, thus we can ignore $u^2/2$, $u^3/3$, ... in Eq.(20)

Therefore, Eq.(18) can be deduced as

$$F = \frac{Q^2}{4\pi\varepsilon_0 l^2} \cdot \frac{l^2}{a^2+2al} = \frac{Q^2}{4\pi\varepsilon_0 (a^2+2al)} \tag{21}$$

Since $a \gg l$, $a^2 \gg 2al$, then the most simplified force is

$$F = \frac{Q^2}{4\pi\varepsilon_0 a^2} \tag{22}$$

which interprets that the rod that exerts force on the other rod can be regard as a point charge as long as the distance between these two rods are far enough.

Problem 5

Solution

(a) According to what we learned in the lecture, the electric field on the axis of symmetry perpendicular to a single ring with radius r is

$$dE = \frac{1}{4\pi\varepsilon_0} \cdot \frac{x(\sigma \cdot 2\pi r dr)}{(x^2 + r^2)^{3/2}} \tag{23}$$

where x denotes the distance between the any field point and the geometric center of the disk.

Then we can calculate the total electric field

$$\begin{aligned} E &= \int_{R_1}^{R_2} \frac{1}{4\pi\varepsilon_0} \cdot \frac{x(\sigma \cdot 2\pi r dr)}{(x^2 + r^2)^{3/2}} \\ &= \frac{\sigma x}{4\varepsilon_0} \int_{x^2+R_1^2}^{x^2+R_2^2} \frac{1}{u^{3/2}} du \\ &= \frac{\sigma x}{4\varepsilon_0} \left(-2u^{-1/2} \Big|_{x^2+R_1^2}^{x^2+R_2^2} \right) \end{aligned}$$

where $u = x^2 + r^2$ discussed in the lecture.

Hence

$$E = \frac{\sigma x}{2\varepsilon_0} \left(\frac{1}{\sqrt{x^2 + R_1^2}} - \frac{1}{\sqrt{x^2 + R_2^2}} \right) \tag{24}$$

(b) When the point is sufficiently close to the geometric center of the disk, $x \ll R_1$ and $x \ll R_2$. Hence

$$E \approx \frac{\sigma x}{2\varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = \frac{\sigma(1/R_1 - 1/R_2)}{2\varepsilon_0} x \tag{25}$$

which shows that the magnitude of the electric field is approximately proportional to the distance from the center.

Consequence of this fact: We may use two parallel disks with holes at the center to form uniform electric field.