Problem 1 *(3 points)*

Two conducting wires in the shape of cylinders of the same cross-sectional area, at 0 $°C$ have resistivities ρ_{01}, ρ_{02} and temperature coefficients of resistivity α_1 and α_2, respectively. What is the effective temperature coefficient of resistivity if the conductors are connected (a) in series, (b) in parallel.

Solution:

We may as well assume both the two conducting wires have the same length. Then we denote their resistance

$$R_1 = \rho_1 \frac{L}{A}$$
$$R_2 = \rho_2 \frac{L}{A}$$

Then for **series**, the cross-sectional area will not change but the length doubles.

$$R_s = \rho_s \frac{2L}{A} = R_1 + R_2$$

Hence

$$\rho_s = \frac{\rho_1 + \rho_2}{2} \tag{1}$$

For **parallel**, the length will not change but the cross-sectional area doubles.

$$R_p = \rho_p \frac{L}{2A} = \frac{R_1 R_2}{R_1 + R_2}$$

Hence

$$\rho_p = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2} \tag{2}$$

While we know

$$\rho_i = \rho_{0i}(1 + \alpha_i(T - T_0)) \qquad \text{where } i = 1, 2 \tag{3}$$

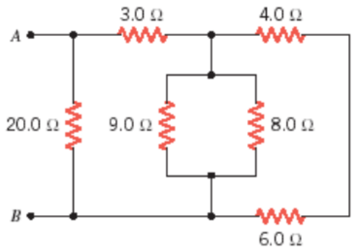
Namely

$$\rho_{0s}(1 + \alpha_s(T - T_0)) = \frac{\rho_{01} + \rho_{02}}{2}(1 + \alpha_s(T - T_0)) = \frac{\rho_{01}(1 + \alpha_1(T - T_0)) + \rho_{02}(1 + \alpha_2(T - T_0))}{2}$$
$$\frac{2\rho_{01}\rho_{02}}{\rho_{01} + \rho_{02}}(1 + \alpha_p(T - T_0)) = \frac{2\rho_{01}(1 + \alpha_1(T - T_0))\rho_{02}(1 + \alpha_2(T - T_0))}{\rho_{01}(1 + \alpha_1(T - T_0)) + \rho_{02}(1 + \alpha_2(T - T_0))}$$

while $T_0 = 0$,hence

$$\alpha_s = \frac{\rho_{01}\alpha_1 + \rho_{02}\alpha_2}{\rho_{01} + \rho_{02}} \tag{4}$$
$$\alpha_p = \frac{\rho_{01}\alpha_2 + \rho_{02}\alpha_1 + (\rho_{01} + \rho_{02})\alpha_1\alpha_2(T - T_0)}{\rho_{01} + \rho_{02} + \rho_{01}\alpha_1(T - T_0) + \rho_{02}\alpha_2(T - T_0)} = \frac{\rho_{01}\alpha_2 + \rho_{02}\alpha_1 + (\rho_{01} + \rho_{02})\alpha_1\alpha_2 T}{\rho_{01} + \rho_{02} + \rho_{01}\alpha_1 T + \rho_{02}\alpha_2 T} \tag{5}$$

Problem 2 *(4 points)*
For the system of resistors shown in the figure, find the equivalent resistance between points A and B.



Solution:
We first analyze the circuit diagram as follows.

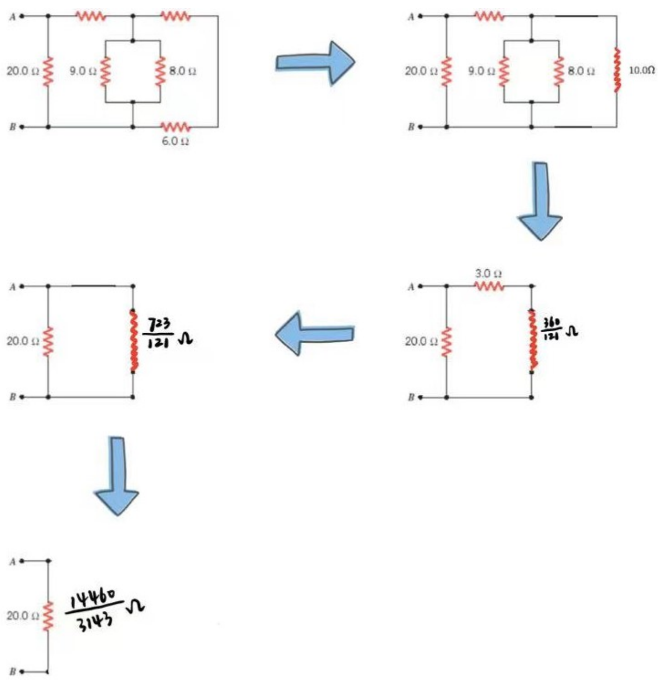


Figure 1: Analysis Process

As the diagram shows, we calculate the parallel resistance with 4Ω and 6Ω being a series first.

$$R_p = \frac{1}{\frac{1}{9} + \frac{1}{8} + \frac{1}{4+6}} = \frac{360}{121}\ [\Omega]$$

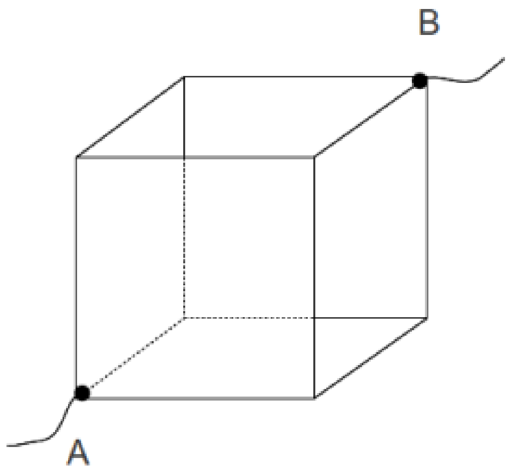
Then calculate the series

$$R_s = 3 + \frac{360}{121} = \frac{723}{121}\ [\Omega]$$

At last calculate the **equivalent resistance**

$$R_{eq} = \frac{1}{\frac{1}{20} + \frac{1}{723/121}} = \frac{14460}{3143}\ [\Omega] \tag{6}$$

Problem 3 *(4 points)*
Twelve identical resistors, each of resistance R, are connected to form a cube-shaped circuit (see the figure). Find the equivalent resistance between points A and B.
Hint: Use symmetry.



Solution:
We may first consider the symmetry to divide the cube into three parts as follows.

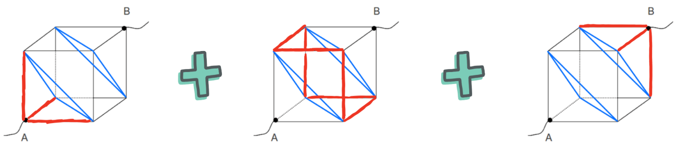


Figure 2: Symmetry Analysis

Apparently, it is equivalent to three resistors in serires which are all composed of parallel resistors, shown as follows.

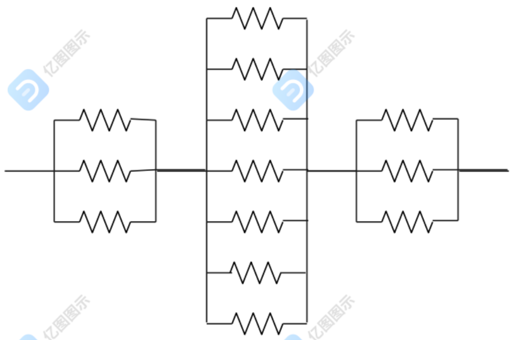


Figure 3: Equivalent Circuit

We then calculate the equivalent resistance separately.
For the left side

$$R_{left} = \frac{R}{3}$$

For the middle part

$$R_{middle} = \frac{R}{6}$$

For the right side

$$R_{right} = \frac{R}{3}$$

Hence the total equivalent resistance is

$$R_{eq} = \frac{5R}{6} \tag{7}$$

Problem 4 *(5 points)*
Consider the circuit shown in the figure below ($E_1 = 12$ V, $E_2 = 8$ V, $r = 1\ \Omega$, $R = 8\ \Omega$).

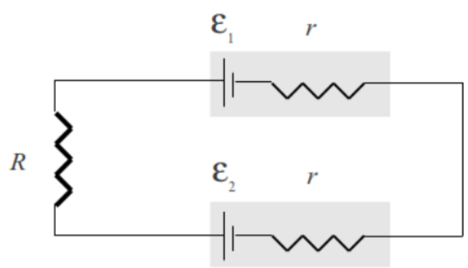
(a) Find the current through the resistor R,

(b) and the total rate of dissipation of electrical energy in the resistor R and in the internal resistance of the batteries.

(c) In one of the batteries, chemical energy is being converted into electrical energy. In which one it is happening, and at what rate?

(d) In one of the batteries, electrical energy is being converted into chemical energy. In which one it is happening, and at what rate?

(e) Show that the overall rate of production of electrical energy is equal to the overall rate of consumption of electrical energy in the circuit.



Solution:

(a) Suppose there is a current flowing clockwise, whose magnitude is i. Then

$$\varepsilon_1 + 2ir + iR - \varepsilon_2 = 0 \tag{8}$$
$$\Rightarrow \quad i = -0.4\ [A] \tag{9}$$

Showing that the current is flowing counter-clockwise with magnitude of 0.4 A.

(b) The rate of dissipation of electrical energy is equal to the power of elements. Hence

$$P_R = i^2 R = 1.28\ [W]$$
$$P_r = 2i^2 r = 0.32\ [W]$$

(c) ε_1, $P_1 = 12 \times 0.4 = 4.8\ [J/s]$.

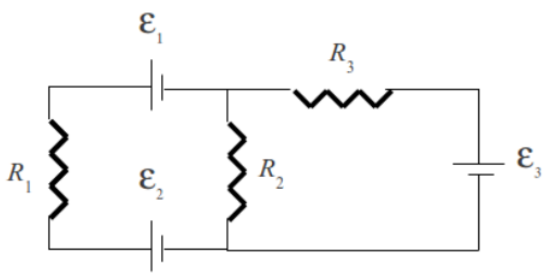
(d) ε_2, $P_2 = 8 \times 0.4 = 3.2\ [J/s]$.

(e) Notice that

$$P_R + P_r + P_2 = P_1 \tag{10}$$

Namely, the rates of consumption and production are identical.

Problem 5 *(4 points)*
For the circuit shown in the figure below, find the current through each of the resistors. For numerical calculations assume: $R_1 = 2\Omega$, $R_2 = 4\Omega$, $R_3 = 5\Omega$, $\varepsilon_1 = 20V$, $\varepsilon_2 = 14V$, $\varepsilon_3 = 36V$. The internal resistance of the emfs is negligible.



Solution:
Suppose the current in the left loop(mesh) is i_1, and the current in the right loop(mesh) is i_2. Then we have

$$-\varepsilon_1 + i_1 R_1 + \varepsilon_2 + (i_1 - i_2) R_2 = 0 \tag{11}$$
$$-\varepsilon_3 + i_2 R_3 + (i_2 - i_1) R_2 = 0 \tag{12}$$

Then we get

$$\begin{cases} i_1 = \frac{99}{19}\ [A] \\ i_2 = \frac{120}{19}\ [A] \end{cases} \tag{13}$$

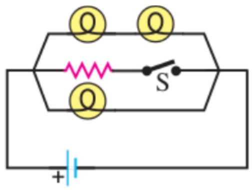
Hence we know the current passing through each resistor.

$$i_{R_1} = \frac{99}{19}\ [A] \qquad i_{R_2} = \frac{21}{19}\ [A] \qquad i_{R_3} = \frac{120}{99}\ [A] \tag{14}$$

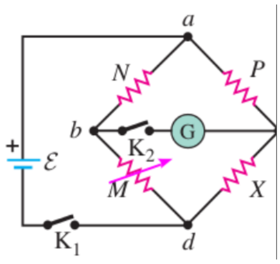
Problem 6 *(3 points)*
For the circuit shown in the figure below what happens to the brightness of the bulbs when the switch S is closed if the battery (a) has no internal resistance and (b) has non-negligible internal resistance? Explain why.

Solution:
For (a), the brightness of the bulb will not change since the voltage applied on the bulb has not changed. The **terminal voltage** is still equal to **emf**.

For (b), the brightness of the bulb will decrease since the whole resistance of the system will decrease when the switch closed, thus the **terminal voltage** will decrease as well. Hence the brightness decreases.



Problem 7 *(4 points)* Four resistors are connected to form a Wheatstone bridge - a circuit that can be used to measure unknown resistance X, provided the resistances of N, M and P are known. The idea of the measurement method is to tune (with the switches K_1 and K_2 closed) the variable resistance M so that the potential difference between points b and c is zero and the galvanometer does not show any current. The bridge is then said to be balanced. Show that in this configuration $X = MP/N$.



Solution: Suppose the current flowing in `a-b-d` is i_1 while the current flowing in `a-c-d` is i_2. Then according to KVL

$$i_1(N+M) - i_2(P+X) = 0 \tag{15}$$

$$\tag{16}$$

According to the balance condition

$$\varepsilon - i_1 N = \varepsilon - i_2 P \tag{17}$$

Hence

$$\frac{N}{P} = \frac{i_2}{i_1} = \frac{N+M}{P+X}$$

Hence

$$\frac{N}{P} = \frac{M}{X}$$

Namely

$$X = \frac{MP}{N} \tag{18}$$

Problem 8 *(3 points)* Strictly speaking, the formula $q(t) = Q_{\max} e^{-t/RC}$ implies that an infinite amount of time is required to discharge a capacitor in a $R-C$ circuit completely. Yet for practical purposes, a capacitor may be considered to be fully discharged after a finite time t_{d}, defined as the time when the charge on the capacitor $q(t_{\mathrm{d}})$ differs from zero by no more than the charge of one electron.

(a) Find t_{d} if $C = 0.92\mu\mathrm{F}$, $R = 670\mathrm{k}\Omega$, and $Q_{\max} = 7\mu\mathrm{C}$.

(b) For a given $Q_{\max}$ is the time required to reach this state always the same number of time constants, independent of R and C. Why or why not?

Solution:

(a) Plug in the data

$$\begin{aligned} q_e &= Q_{max}^{-\frac{t_d}{RC}} \\ \Rightarrow \quad t_d &= 19.36\ [s] \end{aligned}$$

(b) Based on (a)

$$\ln\left(\frac{q_e}{Q_{max}}\right) = -\frac{t_d}{RC} \tag{19}$$

Hence we conclude that, t_d does not depend on R or C separately but depend on RC as a whole. Namely, if RC is constant, t_d is also a constant.