alecjacobson / **geometry-processing-registration**

Registration assignment for Geometry Processing course

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README.md

Geometry Processing — Registration

> **To get started:** Clone this repository then issue
>
> ```
> git clone --recursive http://github.com/[username]/geometry-
> processing-registration.git
> ```

Installation, Layout, and Compilation

See introduction.

Execution

Once built, you can execute the assignment from inside the `build/` using

```
./registration [path to mesh1.obj] [path to mesh2.obj]
```

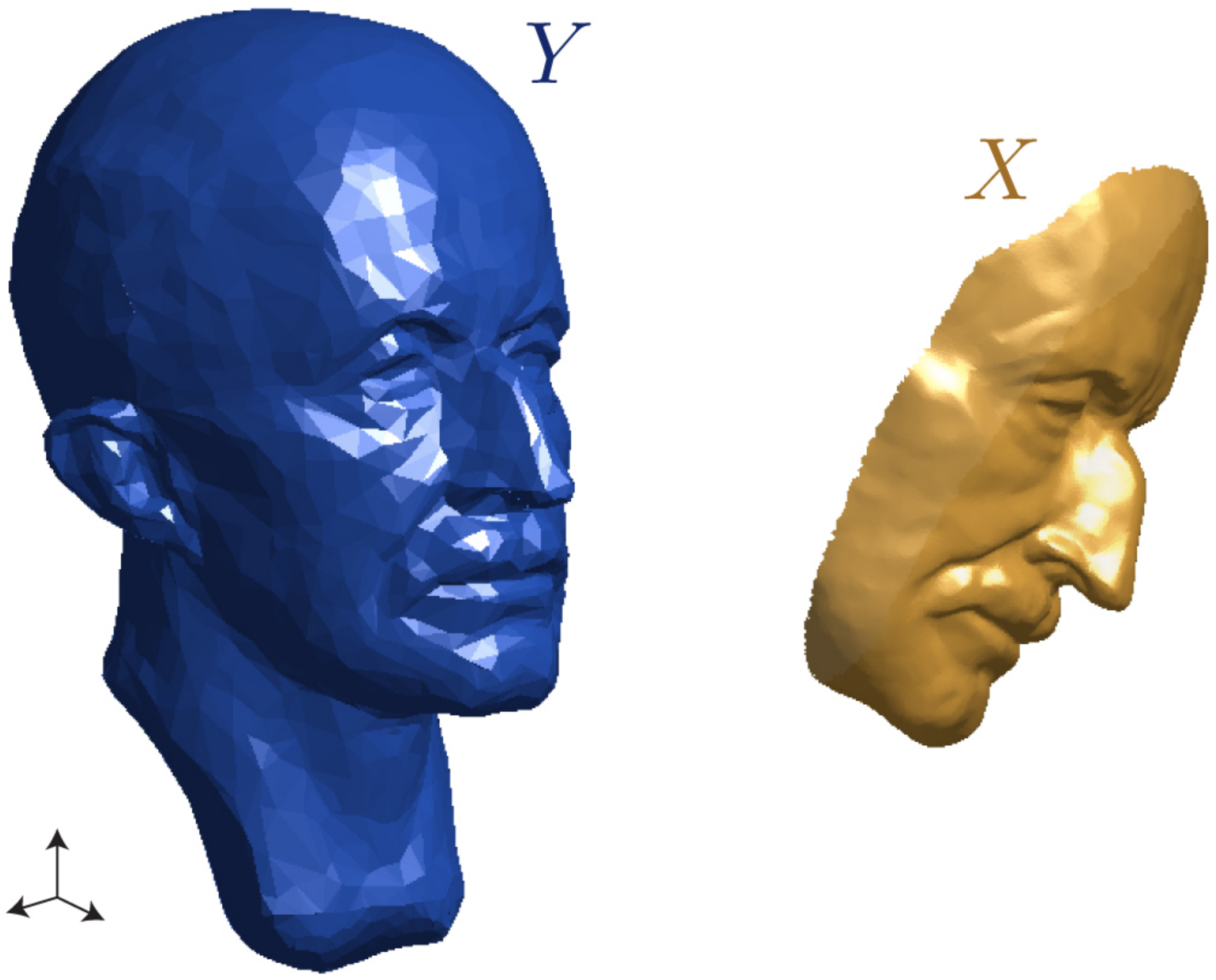
Background

In this assignment, we will be implementing a version of the iterative closest point (ICP), not to be confused with Insane Clown Posse.

Rather than registering multiple point clouds, we will register multiple triangle mesh surfaces.

This *algorithm* and its many variants has been used for quite some time to align discrete shapes. One of the first descriptions is given in "A Method for Registration of 3-D Shapes" by Besl & McKay 1992. However, the award-winning PhD thesis of Sofien Bouaziz ("Realtime Face Tracking and Animation" 2015, section 3.2-3.3) contains a more modern view that unifies many of the variants with respect to how they impact the same core optimization problem.

For our assignment, we will assume that we have a triangle mesh representing a complete scan of the surface Y of some rigid object and a new partial scan of that surface X.



These meshes will not have the same number of vertices or the even the same topology. We will first explore different ways to *measure* how well aligned two surfaces are and then how to optimize the *rigid* alignment of the partial surface X to the complete surface Y.

Hausdorff distance

We would like to compute a single scalar number that measures how poorly two surfaces are matched. In other words, we would like to measure the *distance* between two surfaces. Let's start by reviewing more familiar distances:

Point-to-point distance

The usually Euclidean distance between *two points* $\mathbf{x}$ and $\mathbf{y}$ is the L^2 norm of their difference :

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|.$$

Point-to-projection distance

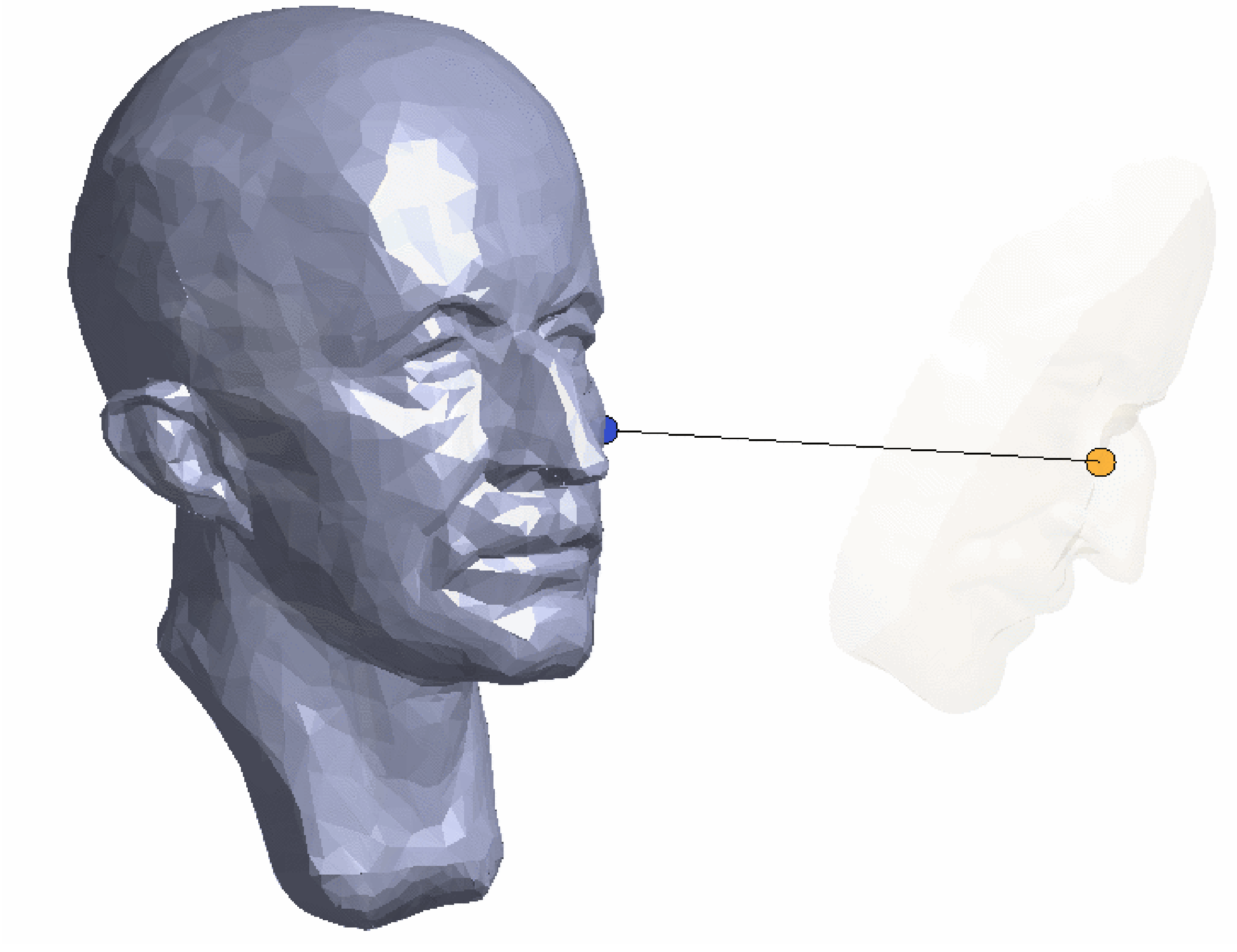
When we consider the distance between a point $\mathbf{x}$ and some *larger* object Y (a line, a circle, a surface), the natural extension is to take the distance to the closest point $\mathbf{y}$ on Y:

$$d(\mathbf{x},Y) = \inf_{\mathbf{y}\in Y} d(\mathbf{x},\mathbf{y}).$$

written in this way the infimum considers all possible points $\mathbf{y}$ and keeps the minimum distance. We may equivalently write this distance instead as simply the point-to-point distance between $\mathbf{x}$ and the *closest-point projection* $P_Y(\mathbf{x})$:

$$d(\mathbf{x},Y) = d((\mathbf{x},P_Y(\mathbf{x})) = \|\mathbf{x}-P_Y(\mathbf{x})\|.$$

If Y is a smooth surface, this projection will also be an orthogonal projection.



Directed Hausdorff distance

We might be tempted to define the distance from surface X to Y as the *infimum* of *point-to-projection* distances over all points $\mathbf{x}$ on X:

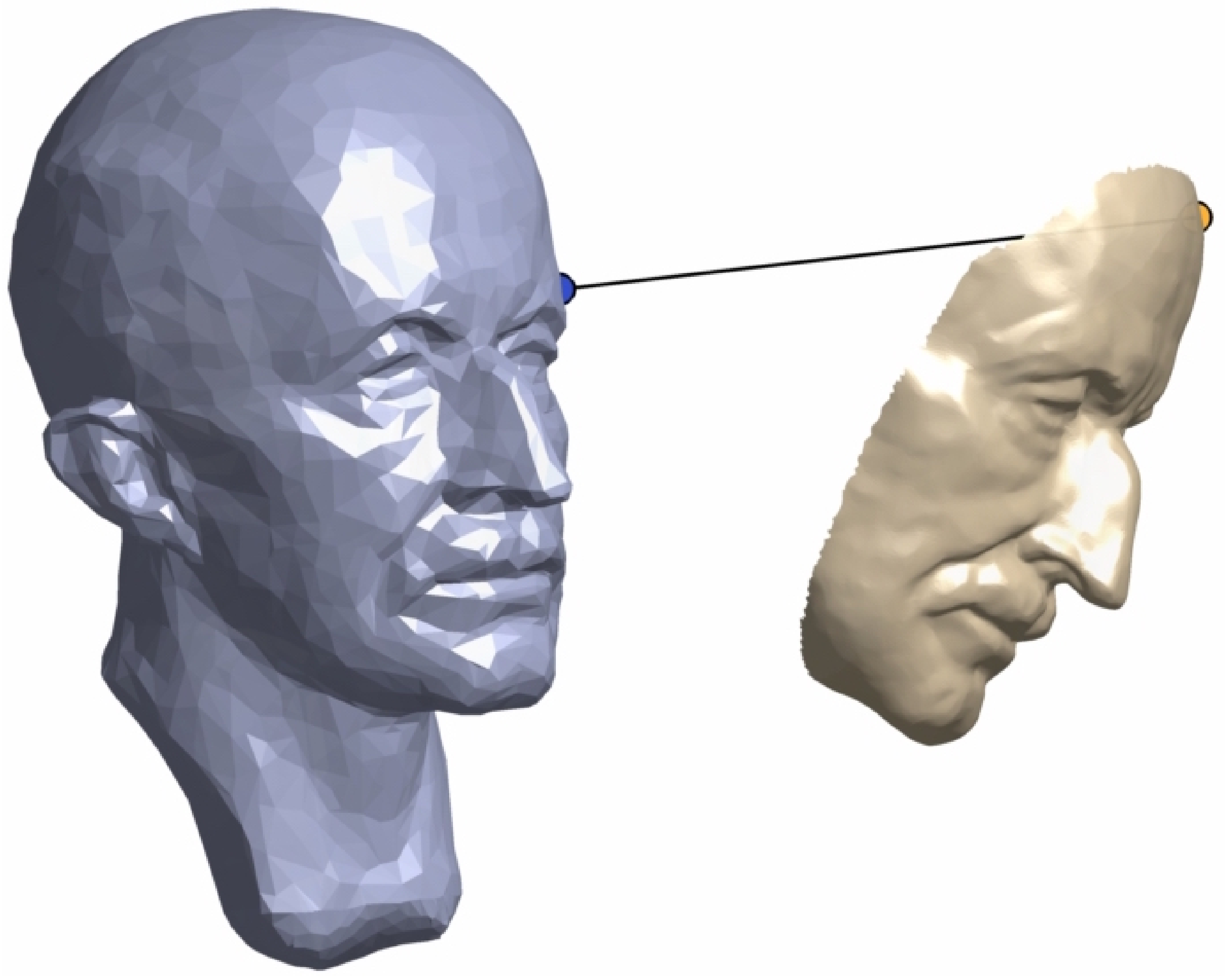
$$D_{\text{inf}}(X,Y) = \inf_{\mathbf{x} \in X} \|\mathbf{x} - P_Y(\mathbf{x})\|,$$

but this will not be useful for registering two surfaces: it will measure zero if even just a single point of $\mathbf{x}$ happens to lie on Y. Imagine the noses of two faces touching at their tips.

Instead, we should take the *supremum* of *point-to-projection* distances over all points $\mathbf{x}$ on X:

$$D_{\overrightarrow{H}}(X,Y) = \sup_{\mathbf{x} \in X} \|\mathbf{x} - P_Y(\mathbf{x})\|.$$

This surface-to-surface distance measure is called the *directed* Hausdorff distance. We may interpret this as taking the worst of the best: we let each point $\mathbf{x}$ on X declare its shortest distance to Y and then keep the longest of those.



It is easy to verify that $D_{\overrightarrow{H}}$ will only equal zero if all points on X also lie exactly on Y.

The converse is not true: if $D_{\overrightarrow{H}} = 0$ there may still be points on Y that do not lie on X. In other words, *in general* the directed Hausdorff distance from surface X to surface Y will not equal the Hausdorff distance from surface Y to surface X:

$$D_{\overrightarrow{H}}(X,Y) \neq D_{\overrightarrow{H}}(Y,X).$$

directed Hausdorff distance between triangle meshes

We can approximate a *lower bound* on the Hausdorff distance between two meshes by densely sampling surfaces X and Y. We will discuss sampling methods, later. For now consider that we have chosen a set $\mathbf{P}_X$ of k points on X (each point might lie at a vertex, along an edge, or inside a triangle). The directed Hausdorff distance from X to another triangle mesh Y must be *greater* than the directed Hausdorff distance from this point cloud $\mathbf{P}_X$ to Y:

$$D_{\overrightarrow{H}}(X,Y) \geq D_{\overrightarrow{H}}(\mathbf{P}_X,Y) = \max_{i=1}^{k} \|\mathbf{p}_i - P_Y(\mathbf{p}_i)\|,$$

where we should be careful to ensure that the projection $P_Y(\mathbf{p}_i)$ of the point $\mathbf{p}_i$ onto the triangle mesh Y might lie at a vertex, along an edge or inside a triangle.

As our sampling $\mathbf{P}_X$ becomes denser and denser on X this lower bound will approach the true directed Hausdorff distance. Unfortunately, an efficient *upper bound* is significantly more difficult to design.

Hausdorff distance for alignment optimization

Even if it *were* cheap to compute, Hausdorff distance is difficult to *optimize* when aligning two surfaces. If we treat the Hausdorff distance between surfaces X and Y as an energy to be minimized, then only change to the surfaces that will decrease the energy will be moving the (in general) isolated point on X and isolated point on Y generating the maximum-minimum distance. In effect, the rest of the surface does not even matter or effect the Hausdorff distance. This, or any type of L^∞ norm, will be much more difficult to optimize.

Hausdorff distance can serve as a validation measure, while we turn to L^2 norms for optimization.

Integrated closest-point distance

We would like a distance measure between two surfaces that — like Hausdorff distance — does not require a shared parameterization. Unlike Hausdorff distance, we would like this distance to *diffuse* the measurement over the entire surfaces rather than generate it from the sole *worst offender*. We can accomplish this by replacing the *supremum* in the Hausdorff distance (L^∞) with a integral of squared distances (L^2). Let us first define a directed *closest-point distance* from a surface X to another surface Y, as the integral of the squared distance from every point $\mathbf{x}$ on X to its closest-point projection $P_Y(\mathbf{x})$ on the surfaces Y:

$$D_{\overrightarrow{C}}(X,Y) = \sqrt{\int\limits_{\mathbf{x}\in X} \|\mathbf{x} - P_Y(\mathbf{x})\|^2 \; dA}.$$

This distance will only be zero if all points on X also lie on Y, but when it is non-zero it is summing/averaging/diffusing the distance measures of all of the points.

This distance is suitable to define a matching energy, but is not necessarily welcoming for optimization: the function inside the square is non-linear. Let's dig into it a bit. We'll define a directed *matching energy* $E_{\overrightarrow{C}}(Z,Y)$ from Z to Y to be the squared directed closest point distance from X to Y:

$$E_{\overrightarrow{C}}(Z,Y) = \int\limits_{\mathbf{z}\in Z} \|\mathbf{z} - P_Y(\mathbf{z})\|^2 \, dA = \int\limits_{\mathbf{z}\in Z} \|f_Y(\mathbf{z})\|^2 \, dA$$

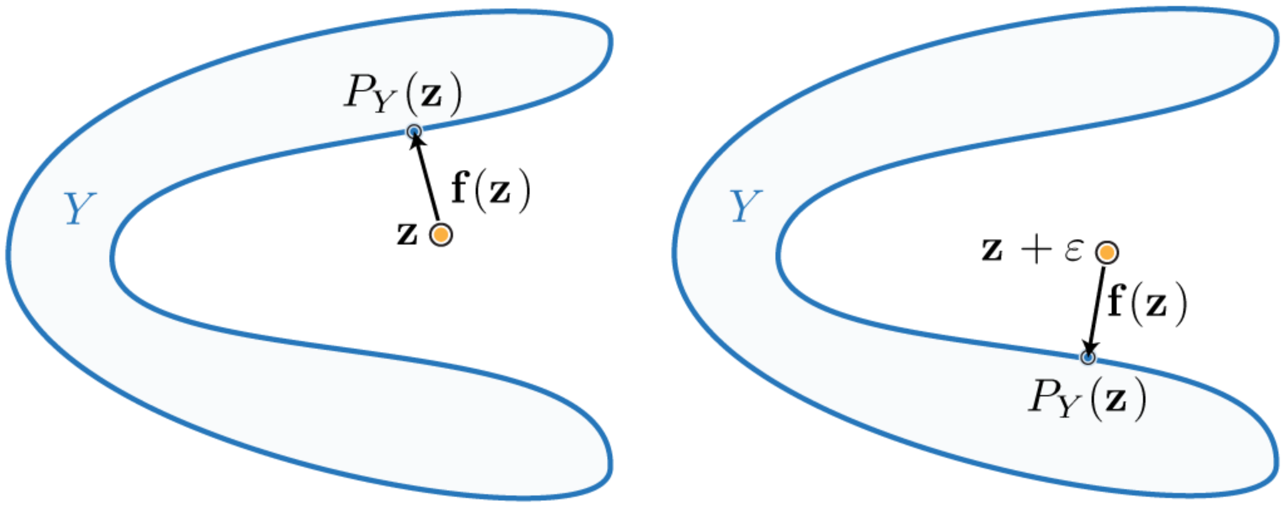
where we introduce the proximity function $\mathbf{f}_Y : \mathbb{R}^3 \Rightarrow \mathbb{R}^3$ defined simply as the vector from a point $\mathbf{z}$ to its closest-point projection onto Y:

$$\mathbf{f}(\mathbf{z}) = \mathbf{z} - P_Y(\mathbf{z}).$$

Suppose Y was not a surface, but just a single point $Y = \{\mathbf{y}\}$. In this case, $\mathbf{f}(\mathbf{z}) = \mathbf{z} - \mathbf{y}$ is clearly linear in $\mathbf{z}$.

Similarly, suppose Y was an infinite plane $Y = \{\mathbf{y} | (\mathbf{y} - \mathbf{p}) \cdot \mathbf{n} = 0\}$ defined by some point $\mathbf{p}$ on the plane and the plane's unit normal vector $\mathbf{n}$. Then $\mathbf{f}(\mathbf{z}) = ((\mathbf{z} - \mathbf{p}) \cdot \mathbf{n})\mathbf{n})$ is also linear in $\mathbf{z}$.

But in general, if Y is an interesting surface $\mathbf{f}(\mathbf{z})$ will be non-linear; it might not even be a continuous function.



In optimization, a common successful strategy to minimize energies composed of squaring a non-linear functions $\mathbf{f}$ is to linearize the function about a current input value (i.e., a current guess $\mathbf{z}_0$), minimize the energy built from this linearization, then re-linearize around that solution, and then repeat.

This is the core idea behind gradient descent and the Gauss-Newton methods:

```
minimize f(z)^{2}
  z_{0} ← initial guess
  repeat until convergence
    f_{0} ← linearize f(z) around z_{0}
    z_{0} ← minimize f_{0}(z)^{2}
```

Since our $\mathbf{f}$ is a geometric function, we can derive its linearizations *geometrically*.

Constant function approximation

If we make the convenient—however unrealistic—assumption that in the neighborhood of the closest-point projection $P_Y(\mathbf{z}_0)$ of the current guess $\mathbf{z}_0$ the surface Y is simply the point $P_Y(\mathbf{z}_0)$ (perhaps imagine that Y is makes a sharp needle-like point at $P_Y(\mathbf{z}_0)$ or that Y is very far away from $\mathbf{x}$), then we can approximate $\mathbf{f}(\mathbf{z})$ in the proximity of our current guess $\mathbf{z}_0$ as the vector between the input point $\mathbf{z}$ and $P_Y(\mathbf{z}_0)$:

$$\mathbf{f}(\mathbf{z}) \approx \mathbf{f}_{\text{point}}(\mathbf{z}) = \mathbf{z} - P_Y(\mathbf{z}_0)$$

In effect, we are assuming that the surface Y is *constant* function of its parameterization: $\mathbf{y}(u, v) = P_Y(\mathbf{z}_0)$.

Minimizing $E_{\overrightarrow{C}}$ iteratively using this linearization of $\mathbf{f}$ is equivalent to gradient descent. We have simply derived our gradients geometrically.

Linear function approximation

If we make make a slightly more appropriate assumption that in the neighborhood of the $P_Y(\mathbf{z}_0)$ the surface Y is a plane, then we can improve this approximation while keeping $\mathbf{f}$ linear in $\mathbf{z}$:

$$\mathbf{f}(\mathbf{z}) \approx \mathbf{f}_{\text{plane}}(\mathbf{z}) = ((\mathbf{z} - P_Y(\mathbf{z}_0)) \cdot \mathbf{n})\mathbf{n}.$$

where the plane that *best* approximates Y locally near $P_Y(\mathbf{z}_0)$ is the tangent plane defined by the normal vector $\mathbf{n}$ at $P_Y(\mathbf{z}_0)$.

Minimizing $E_{\overrightarrow{C}}$ iteratively using this linearization of $\mathbf{f}$ is equivalent to the Gauss-Newton method. We have simply derived our linear approximation geometrically.

Equipped with these linearizations, we may now describe an optimization algorithm for minimizing the matching energy between a surface Z and another surface Y.

Iterative closest point algorithm

So far we have derived distances between a surface Z and another surface Y. In our *rigid* alignment and registration problem, we would like to transform one surface X into a new surface $T(X) = Z$ so that it best aligns with/matches the other surface Y. Further we require that T is a rigid transformation: $T(\mathbf{x}) = \mathbf{R}\mathbf{x} + \mathbf{t}$ for some rotation matrix $\mathbf{R} \in SO(3) \subset \mathbb{R}^{3\times 3}$ (i.e., an orthogonal matrix with determinant 1) and translation vector $\mathbf{t} \in \mathbb{R}^3$.

Our matching problem can be written as an optimization problem to find the best possible rotation $\mathbf{R}$ and translation $\mathbf{t}$ that match surface X to surface Y:

$$\mathop{\text{minimize}}_{\mathbf{t}\in\mathbb{R}^3,\ \mathbf{R}\in SO(3)} \int\limits_{\mathbf{x}\in X} \left\|\mathbf{R}\mathbf{x} + \mathbf{t} - P_Y(T(\mathbf{x}))\right\|^2 \, dA$$

Even if X is a triangle mesh, it is difficult to *integrate* over *all* points on the surface of X. *At any point*, we can approximate this energy by *summing* over a point-sampling of X:

$$\mathop{\text{minimize}}_{\mathbf{t}\in\mathbb{R}^3,\ \mathbf{R}\in SO(3)} \sum_{i=1}^{k} \left\|\mathbf{R}\mathbf{x}_i + \mathbf{t} - P_Y(T(\mathbf{x}_i))\right\|^2,$$

where $\mathbf{X} \in \mathbb{R}^{k\times 3}$ is a set of k points on X so that each point $\mathbf{x}_i$ might lie at a vertex, along an edge, or inside a triangle. We defer discussion of *how* to sample a triangle mesh surface.

Pseudocode

As the name implies, the method proceeds by iteratively finding the closest point on Y to the current rigid transformation $\mathbf{R}\mathbf{x} + \mathbf{t}$ of each sample point $\mathbf{x}$ in $\mathbf{X}$ and then minimizing the *linearized* energy to update the rotation $\mathbf{R}$ and translation $\mathbf{t}$.

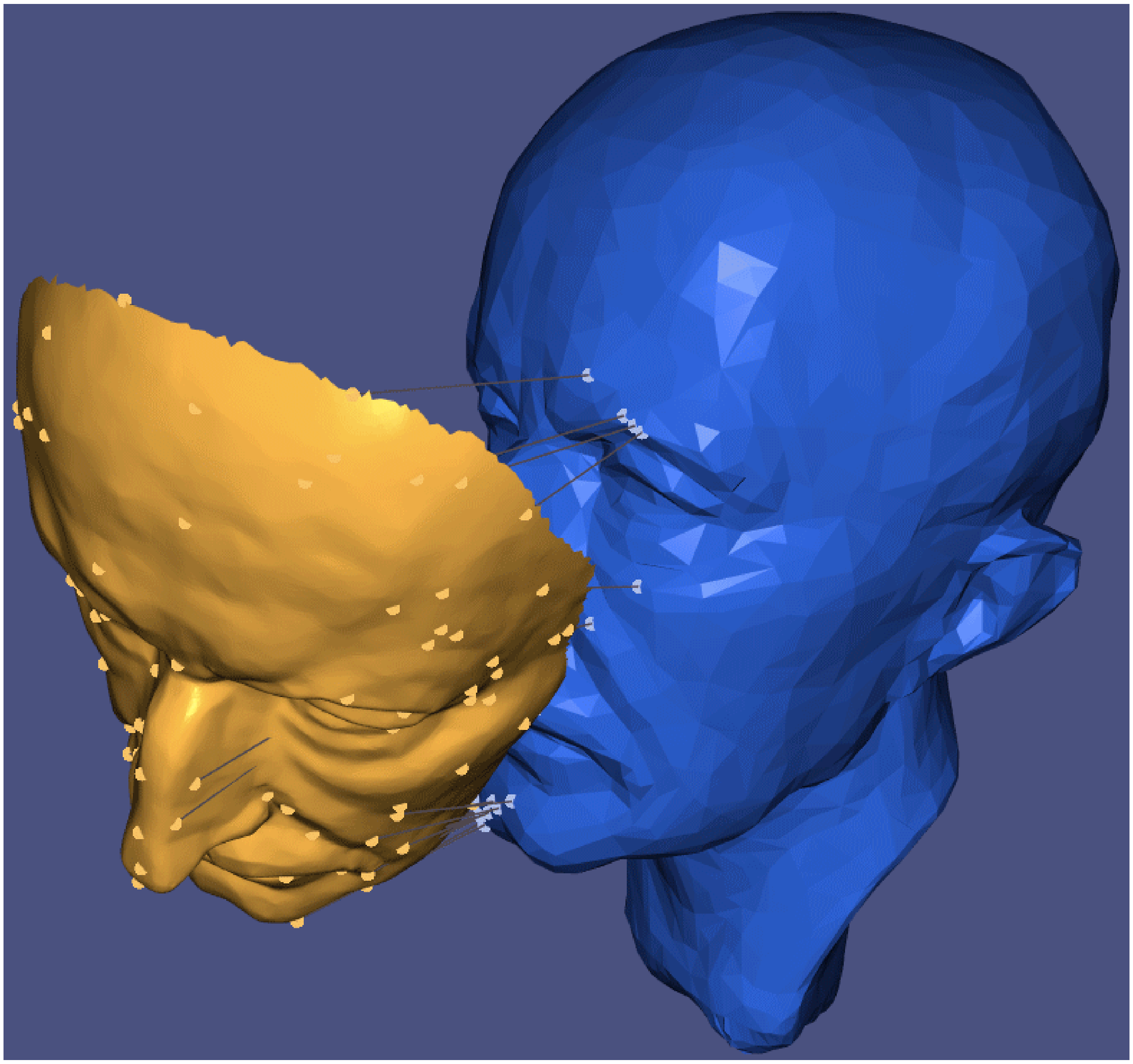
If `V_X` and `F_X` are the vertices and faces of a triangle mesh surface X (and correspondingly for Y), then we can summarize a generic ICP algorithm in pseudocode:

```
icp V_X, F_X, V_Y, F_Y
  R,t \Leftarrow  initialize (e.g., set to identity transformation)
  repeat until convergence
    X \Leftarrow  sample source mesh (V_X,F_X)
    P0 \Leftarrow  project all X onto target mesh (V_Y,F_Y)
    R,t \Leftarrow  update rigid transform to best match X and P0
    V_X \Leftarrow  rigidly transform original source mesh by R and t
```

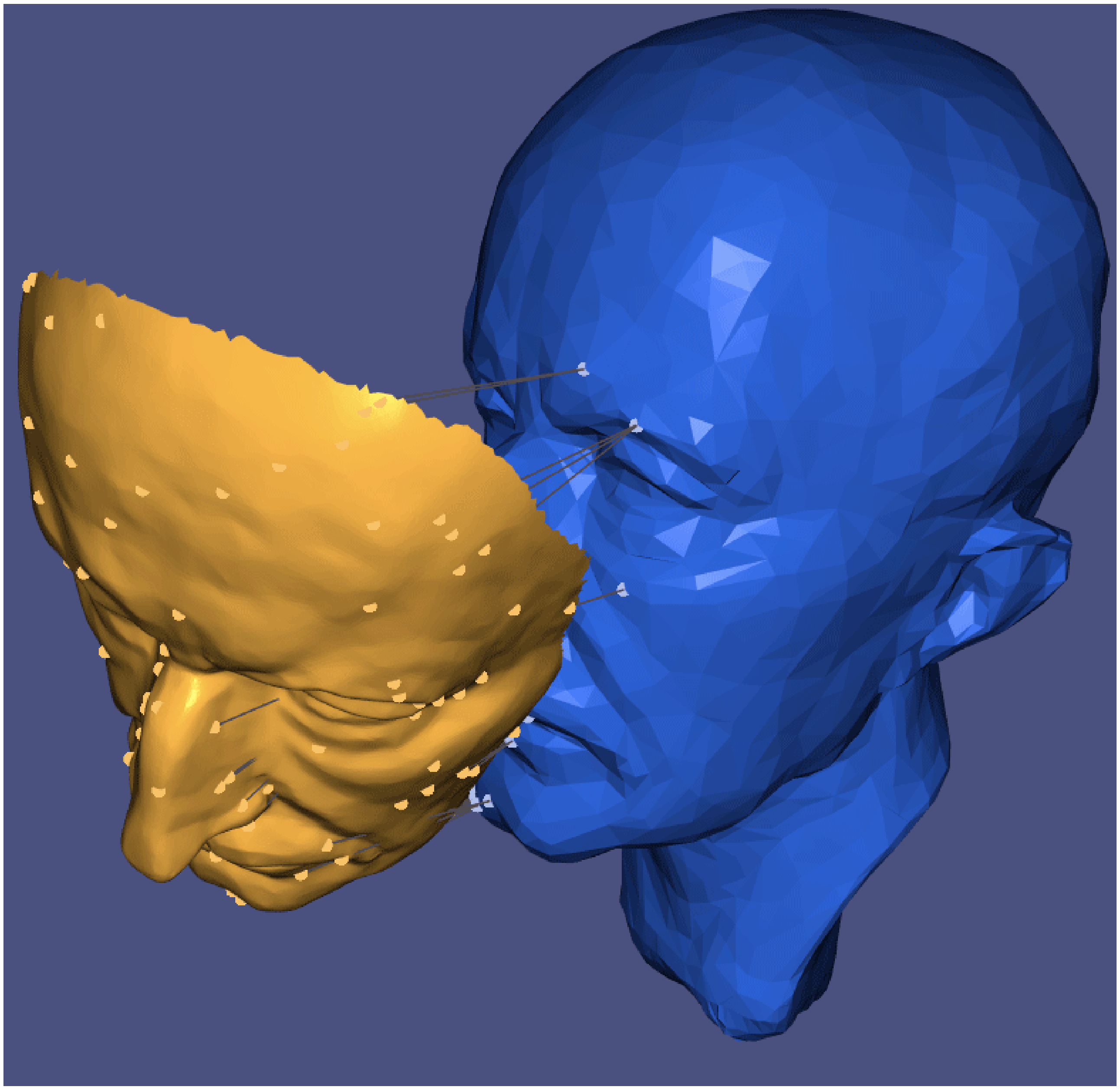
Updating the rigid transformation

We would like to find the rotation matrix $\mathbf{R} \in SO(3) \subset \mathbb{R}^{3\times3}$ and translation vector $\mathbf{t} \in \mathbb{R}^3$ that *best* aligns a given a set of points $\mathbf{X} \in \mathbb{R}^{k\times3}$ on the source mesh and their current closest points $\mathbf{P} \in \mathbb{R}^{k\times3}$ on the target mesh. We have two choices for *linearizing* our matching energy: point-to-point (gradient descent) and point-to-plane (Gauss-Newton).

ICP using the point-to-point matching energy linearization is slow to converge.



ICP using the point-to-plane matching energy linearization is faster.



In either case, this is still a non-linear optimization problem. This time due to the constraints rather than the energy term.

Closed-form solution for point-to-point rigid matching

> In an effort to provide an alternative from "Least-Squares Rigid Motion Using SVD" [Sorkine 2009], this derivation purposefully *avoids* the trace operator and its various nice properties.

The point-to-point (gradient descent) rigid matching problem solves:

$$\mathop{\text{minimize}}_{\mathbf{R}\in SO(3),\mathbf{t}\in\mathbb{R}^3} \sum_{i=1}^{k} \|\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{p}_i\|^2$$

This is a variant of what's known as a Procrustes problem, named after a mythical psychopath who would kidnap people and force them to fit in his bed by stretching them or cutting off their legs. In our case, we are forcing $\mathbf{R}$ to be perfectly orthogonal (no "longer", no "shorter").

Substituting out the translation terms

This energy is *quadratic* in $\mathbf{t}$ and there are no other constraints on $\mathbf{t}$. We can immediately solve for the optimal $\mathbf{t}^*$ — leaving $\mathbf{R}$ as an unknown — by setting all derivatives with respect to unknowns in $\mathbf{t}$ to zero:

$$\begin{aligned}\mathbf{t}^* &= \operatorname*{argmin}_{\mathbf{t}} \sum_{i=1}^{k} \|\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{p}_i\|^2 \\ &= \operatorname*{argmin}_{\mathbf{t}} \left\|\mathbf{R}\mathbf{X}^\top + \mathbf{t}\mathbf{1}^\top - \mathbf{P}^\top\right\|_F^2,\end{aligned}$$

where $\mathbf{1} \in \mathbb{R}^k$ is a vector ones and $\|\mathbf{X}\|_F^2$ computes the squared Frobenius norm of the matrix $\mathbf{X}$ (i.e., the sum of all squared element values. In MATLAB syntax: `sum(sum(A.^2))`). Setting the partial derivative with respect to $\mathbf{t}$ of this quadratic energy to zero finds the minimum:

$$\begin{aligned}0 &= \frac{\partial}{\partial \mathbf{t}} \left\|\mathbf{R}\mathbf{X}^\top + \mathbf{t}\mathbf{1}^\top - \mathbf{P}^\top\right\|_F^2 \\ &= \mathbf{1}^\top\mathbf{1}\mathbf{t} + \mathbf{R}\mathbf{X}^\top\mathbf{1} - \mathbf{P}^\top\mathbf{1},\end{aligned}$$

Rearranging terms above reveals that the optimal $\mathbf{t}$ is the vector aligning the centroids of the points in $\mathbf{P}$ and the points in $\mathbf{X}$ rotated by the — yet-unknown — $\mathbf{R}$. Introducing variables for the respective centroids $\overline{\mathbf{p}} = \frac{1}{k}\sum_{i=1}^{k} \mathbf{p}_i$ and $\overline{\mathbf{x}} = \frac{1}{k}\sum_{i=1}^{k} \mathbf{x}_i$, we can write the formula for the optimal $\mathbf{t}$:

$$\begin{aligned}\mathbf{t} &= \frac{\mathbf{P}^\top\mathbf{1} - \mathbf{R}\mathbf{X}^\top\mathbf{1}}{\mathbf{1}^\top\mathbf{1}} \\ &= \overline{\mathbf{p}} - \mathbf{R}\overline{\mathbf{x}}.\end{aligned}$$

Now we have a formula for the optimal translation vector $\mathbf{t}$ in terms of the unknown rotation $\mathbf{R}$. Let us substitute this formula for all occurrences of $\mathbf{t}$ in our energy written in its original summation form:

$$\underset{\mathbf{R}\in SO(3)}{\text{minimize}} \sum_{i=1}^{k} \left\|\mathbf{R}\mathbf{x}_i + (\overline{\mathbf{p}} - \mathbf{R}\overline{\mathbf{x}}) - \mathbf{p}_i\right\|^2$$

$$\underset{\mathbf{R}\in SO(3)}{\text{minimize}} \sum_{i=1}^{k} \left\|\mathbf{R}(\mathbf{x}_i - \overline{\mathbf{x}}) - (\mathbf{p}_i - \overline{\mathbf{p}})\right\|^2$$

$$\underset{\mathbf{R}\in SO(3)}{\text{minimize}} \sum_{i=1}^{k} \left\|\mathbf{R}\overline{\mathbf{x}}_i - \overline{\mathbf{p}}_i\right\|^2$$

$$\underset{\mathbf{R}\in SO(3)}{\text{minimize}} \left\|\mathbf{R}\overline{\mathbf{X}}^\top - \overline{\mathbf{P}}^\top\right\|_F^2,$$

where we introduce $\overline{\mathbf{X}} \in \mathbb{R}^{k\times 3}$ where the ith row contains the *relative position* of the ith point to the centroid $\overline{\mathbf{x}}$: i.e., $\overline{\mathbf{x}}_i = (\mathbf{x}_i - \overline{\mathbf{x}})$ (and analagously for $\overline{\mathbf{P}}$).

Now we have the canonical form of the orthogonal procrustes problem. To find the optimal rotation matrix $\mathbf{R}^*$, using the associativity property of the Frobenius norm, we will massage the terms in the *minimization* until we have a *maximization* problem involving the Frobenius inner-product of the unknown rotation $\mathbf{R}$ and covariance matrix of $\mathbf{X}$ and $\mathbf{P}$:

$$\begin{aligned}\mathbf{R}^* &= \underset{\mathbf{R}\in SO(3)}{\text{argmin}} \left\|\mathbf{R}\overline{\mathbf{X}}^\top - \overline{\mathbf{P}}^\top\right\|_F^2 \\ &= \underset{\mathbf{R}\in SO(3)}{\text{argmin}} \left\langle \mathbf{R}\overline{\mathbf{X}}^\top - \overline{\mathbf{P}}^\top, \mathbf{R}\overline{\mathbf{X}}^\top - \overline{\mathbf{P}}^\top \right\rangle_F \\ &= \underset{\mathbf{R}\in SO(3)}{\text{argmin}} \left\langle \mathbf{R}\overline{\mathbf{X}}^\top, \mathbf{R}\overline{\mathbf{X}}^\top \right\rangle_F - 2\left\langle \mathbf{R}\overline{\mathbf{X}}^\top, \overline{\mathbf{P}}^\top \right\rangle_F + \left\langle \overline{\mathbf{P}}^\top, \overline{\mathbf{P}}^\top \right\rangle_F\end{aligned}$$

where $\langle \mathbf{A}, \mathbf{B}\rangle_F$ is the Frobenius inner product of $\mathbf{A}$ and $\mathbf{B}$ (i.e., the sum of all per-element products. In MATLAB syntax: `sum(sum(A.*B))`). This can be further reduced:

$$\mathbf{R}^* = \underset{\mathbf{R}\in SO(3)}{\text{argmin}} \left\|\overline{\mathbf{X}}\right\|_F^2 + \left\|\overline{\mathbf{P}}\right\|_F^2 - 2\left\langle \mathbf{R}\overline{\mathbf{X}}^\top, \overline{\mathbf{P}}^\top \right\rangle_F$$

> **Question:** what is $\mathbf{R}^\top\mathbf{R}$?
>
> **Hint:** 👁

$$\begin{aligned}\mathbf{R}^* &= \underset{\mathbf{R}\in SO(3)}{\text{argmax}} \left\langle \mathbf{R}, \overline{\mathbf{P}}^\top \overline{\mathbf{X}} \right\rangle_F \\ &= \underset{\mathbf{R}\in SO(3)}{\text{argmax}} \left\langle \mathbf{R}, \mathbf{M} \right\rangle_F\end{aligned}$$

Letting $\mathbf{M} = \overline{\mathbf{P}}^\top \overline{\mathbf{X}}$. We can understand this problem as *projecting* the covariance matrix $\mathbf{M}$ to the nearest rotation matrix $\mathbf{R}$.

Question: How can we prove that $\left\langle \mathbf{R}\overline{\mathbf{X}}^{\top}, \overline{\mathbf{P}}^{\top} \right\rangle \left\langle \mathbf{R}, \overline{\mathbf{P}}^{\top}\overline{\mathbf{X}} \right\rangle$?

Hint: Recall some linear algebra properties:

1. Matrix multiplication (on the left) can be understood as *acting* on each column: $\mathbf{A}\mathbf{B} = \mathbf{A}[\mathbf{B}_1\ \mathbf{B}_2\ \dots\ \mathbf{B}_n] = [\mathbf{A}\mathbf{B}_1\ \mathbf{A}\mathbf{B}_2\ \dots\ \mathbf{A}\mathbf{B}_n]$,
2. The Kronecker product $\mathbf{I} \otimes \mathbf{A}$ of the identity matrix $\mathbf{I}$ of size k and a matrix $\mathbf{A}$ simply repeats $\mathbf{A}$ along the diagonal k times. In MATLAB, `repdiag(A,k)` ,
3. Properties 1. and 2. imply that the vectorization of a matrix product $\mathbf{B}\mathbf{C}$ can be written as the Kronecker product of the #-columns-in-$\mathbf{C}$ identity matrix and $\mathbf{B}$ times the vectorization of $\mathbf{C}$: $\text{vec}(\mathbf{B}\mathbf{C}) = (\mathbf{I} \otimes \mathbf{B})\text{vec}(\mathbf{C})$,
4. The transpose of a Kronecker product is the Kronecker product of transposes: $(\mathbf{A} \otimes \mathbf{B})^{\top} = \mathbf{A}^{\top} \otimes \mathbf{B}^{\top}$,
5. The Frobenius inner product can be written as a dot product of vectorized matrices: $< \mathbf{A}, \mathbf{B} >_F = \text{vec}(\mathbf{A}) \cdot \text{vec}(\mathbf{B}) = \text{vec}(\mathbf{A})^{\top}\text{vec}(\mathbf{B})$,
6. Properties 3., 4., and 5. imply that Frobenius inner product of a matrix $\mathbf{A}$ and the matrix product of matrix $\mathbf{B}$ and $\mathbf{C}$ is equal to the Frobenius inner product of the matrix product of the transpose of $\mathbf{B}$ and $\mathbf{A}$ and the matrix $\mathbf{C}$:

$$\begin{aligned}\langle \mathbf{A}, \mathbf{B}\mathbf{C} \rangle_F &= \text{vec}(\mathbf{A})^{\top}\text{vec}(\mathbf{B}\mathbf{C}) \\ &= \text{vec}(\mathbf{A})^{\top}(\mathbf{I} \otimes \mathbf{B})\text{vec}(\mathbf{C}) \\ &= \text{vec}(\mathbf{A})^{\top}(\mathbf{I} \otimes \mathbf{B}^{\top})^{\top}\text{vec}(\mathbf{C}) \\ &= \text{vec}(\mathbf{B}^{\top}\mathbf{A})^{\top}\text{vec}(\mathbf{C}) \\ &= \left\langle \mathbf{B}^{\top}\mathbf{A}, \mathbf{C} \right\rangle_F\end{aligned}$$

.

Any matrix can be written in terms of its singular value decomposition. Let's do this for our covariance matrix: $\mathbf{M} = \mathbf{U}\sigma\mathbf{V}^{\top}$, where $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{3\times3}$ are orthonormal matrices and $\sigma \in \mathbb{R}^{3\times3}$ is a non-negative diagonal matrix:

$$\mathbf{R}^* = \underset{\mathbf{R}\in SO(3)}{\text{argmax}} \left\langle \mathbf{R}, \mathbf{U}\sigma\mathbf{V}^{\top} \right\rangle_F .$$

We can use the permutation property of Frobenius inner product again to move the products by $\mathbf{V}$ and $\mathbf{U}$ from the right argument to the left argument:

$$\mathbf{R}^* = \underset{\mathbf{R}\in SO(3)}{\text{argmax}} \left\langle \mathbf{U}^{\top}\mathbf{R}\mathbf{V}, \sigma \right\rangle_F .$$

Now, $\mathbf{U}$ and $\mathbf{V}$ are both orthonormal, so multiplying them against a rotation matrix $\mathbf{R}$ does not change its orthonormality. We can pull them out of the maximization if we account for the reflection they *might* incur: introduce $\Omega = \mathbf{U}^T\mathbf{R}\mathbf{V} \in O(3)$ with $\det\Omega = \det\mathbf{U}\mathbf{V}^\top$. This implies that the optimal rotation for the original problem is recovered via $\mathbf{R}^* = \mathbf{U}\Omega^*\mathbf{V}^\top$. When we move the argmax inside, we now look for an orthonormal matrix $\Omega \in O(3)$ that is a reflection (if $\det\mathbf{U}\mathbf{V}^\top = -1$) or a rotation (if $\det\mathbf{U}\mathbf{V}^\top = 1$):

$$\mathbf{R}^* = \mathbf{U}\left(\underset{\Omega\in O(3),\ \det\Omega=\det\mathbf{U}\mathbf{V}^\top}{\mathrm{argmax}} \left\langle \Omega, \sigma\right\rangle_F\right)\mathbf{V}^\top.$$

This ensures that as a result $\mathbf{R}^*$ will be a rotation: $\det\mathbf{R}^* = 1$.

> Recall that $\sigma \in \mathbb{R}^{3\times3}$ is a non-negative diagonal matrix of singular values sorted so that the smallest value is in the bottom right corner.

Because Ω is orthonormal, each column (or row) of Ω must have unit norm. Placing a non-zero on the off-diagonal will get "killed" when multiplied by the corresponding zero in σ. So the optimal choice of Ω is to set all values to zero except on the diagonal. If $\det\mathbf{U}\mathbf{V}^\top = -1$, then we should set one (and only one) of these values to -1. The best choice is the bottom right corner since that will multiply against the smallest singular value in $\sum$ (add negatively affect the maximization the least):

$$\Omega^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det\mathbf{U}\mathbf{V}^\top \end{bmatrix}$$

Finally, we have a formula for our optimal rotation:

$$\mathbf{R} = \mathbf{U}\Omega^*\mathbf{V}^\top.$$

Iterative linearization for point-to-plane rigid matching

The point-to-plane (Gauss-Newton) rigid matching problem solves:

$$\underset{\mathbf{R}\in SO(3),\mathbf{t}\in\mathbb{R}^3}{\text{minimize}} \sum_{i=1}^{k} \|\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{p}_i) \cdot \hat{\mathbf{n}}_i\hat{\mathbf{n}}_i\|^2,$$

where $\hat{\mathbf{n}}_i \in \mathbb{R}^3$ is the unit normal at the located closest point $\mathbf{p}_i$. Since $\hat{\mathbf{n}}$ is a unit vector the norm is only measuring the proceeding term $\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{p}_i) \cdot \hat{\mathbf{n}}_i$, so we can reduce this problem to:

$$\mathop{\text{minimize}}_{\mathbf{R}\in SO(3),\mathbf{t}\in\mathbb{R}^3} \sum_{i=1}^{k} \left((\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{p}_i)\cdot\hat{\mathbf{n}}_i\right)^2,$$

Unlike the point-to-point problem above, there is closed-form solution to this problem. Instead we will ensure that that $\mathbf{R}$ is not just any 3×3 matrix, but a rotation matrix by iteartive linearization.

If we simply optimize the 9 matrix entries of $\mathbf{R}$ directly, the result will be far from a rotation matrix: for example, if $\mathbf{X}$ is a twice scaled version of $\mathbf{P}$, then this unconstrained optimization would happily declare the entries of $\mathbf{R}$ to describe a (non-orthonormal) scaling matrix.

Instead, we *linearize* the constraint that $\mathbf{R}$ stays a rotation matrix and work with a reduced set of variables.

Any rotation $\mathbf{R}$ in 3D can be written as scalar rotation angle θ around a rotation axis defined by a unit vector $\hat{\mathbf{w}} \in \mathbb{R}^3$.

If $\hat{\mathbf{w}} = \hat{\mathbf{z}} = [0, 0, 1]$, we know that a rotation by θ can be written as:

$$\mathbf{R}_{\hat{\mathbf{z}}}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For a general, rotation axis $\hat{\mathbf{w}}$, we can write a generalized ***axis-angle to matrix formula***:

$$\mathbf{R}_{\hat{\mathbf{w}}}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \underbrace{\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}}_{\mathbf{W}} + (1-\cos\theta) \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}^2$$

where $\mathbf{W} \in \mathbb{R}^{3\times 3}$ is the skew-symmetric cross product matrix of $\hat{\mathbf{w}}$ so that $\mathbf{W}\mathbf{x} = \hat{\mathbf{w}} \times \mathbf{x}$

In this form, we can linearize by considering a small change in θ and $\hat{\mathbf{w}}$:

$$\mathbf{R} \approx \mathbf{I} + \begin{bmatrix} 0 & -\theta w_3 & \theta w_2 \\ \theta w_3 & 0 & -\theta w_1 \\ -\theta w_2 & \theta w_1 & 0 \end{bmatrix}.$$

By defining $\mathbf{a} = \theta\hat{\mathbf{w}}$, we can write this in terms of only three simple scalar variables:

$$\mathbf{R} \approx \mathbf{I} + \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

or written in terms of its action on a vector $\mathbf{x}$, we can simply write in terms of the cross product:

$$\mathbf{R}\mathbf{x} \approx \mathbf{x} + \mathbf{a} \times \mathbf{x}.$$

If we apply our linearization of $\mathbf{R}$ to the **point-to-plane** distance linearization of the matching energy, our minimization is:

$$\mathop{\text{minimize}}_{\mathbf{t}\in\mathbb{R}^3, a_1, a_2, a_2} \sum_{i=1}^{k} \left((\mathbf{x}_i + \mathbf{a} \times \mathbf{x}_i + \mathbf{t} - \mathbf{p}_i) \cdot \mathbf{n}_i \right)^2.$$

Let's gather a vector of unknowns: $\mathbf{u}^\top = [\mathbf{a}^\top \ \mathbf{t}^\top] \in \mathbb{R}^6$. Then we can use properties of the triple product to rewrite our problem as:

$$\mathop{\text{minimize}}_{\mathbf{a},\mathbf{t}\in\mathbb{R}^3} \sum_{i=1}^{k} \left((\mathbf{x}_i \times \hat{\mathbf{n}}_i)^\top \mathbf{a} + \hat{\mathbf{n}}_i^\top \mathbf{t} - \hat{\mathbf{n}}_i^\top (\mathbf{p}_i - \mathbf{x}_i) \right)^2, \tag{1}$$

$$\mathop{\text{minimize}}_{\mathbf{u}\in\mathbb{R}^6} \sum_{i=1}^{k} \left(\left[(\mathbf{x}_i \times \hat{\mathbf{n}}_i)^\top \ \hat{\mathbf{n}}_i^\top \right] \mathbf{u} - \hat{\mathbf{n}}_i^\top (\mathbf{p}_i - \mathbf{x}_i) \right)^2. \tag{2}$$

Expanding all terms, moving the summations inside like terms, we can expose this in familiar quadratic energy minimization form:

$$\mathop{\text{minimize}}_{\mathbf{u}\in\mathbb{R}^6} \mathbf{u}^\top \underbrace{\left(\sum_{i=1}^{k} \begin{bmatrix} (\mathbf{x}_i \times \hat{\mathbf{n}}_i) \\ \hat{\mathbf{n}}_i \end{bmatrix} \left[(\mathbf{x}_i \times \hat{\mathbf{n}}_i)^\top \ \hat{\mathbf{n}}_i^\top \right] \right)}_{\mathbf{A}} \mathbf{u} - 2\mathbf{u}^\top \underbrace{\left(\sum_{i=1}^{k} \begin{bmatrix} (\mathbf{x}_i \times \hat{\mathbf{n}}_i) \\ \hat{\mathbf{n}}_i \end{bmatrix} \hat{\mathbf{n}}_i^\top (\mathbf{p}_i - \mathbf{x}_i) \right)}_{\mathbf{b}} + \underbrace{\left(\sum_{i=1}^{k} (\mathbf{p}_i - \mathbf{x}_i)^\top \hat{\mathbf{n}}_i \hat{\mathbf{n}}_i^\top (\mathbf{p}_i - \mathbf{x}_i) \right)}_{\text{constant}}$$

Gather coefficients into $\mathbf{A} \in \mathbb{R}^{6\times 6}$ and $\mathbf{b} \in \mathbb{R}^6$, we have a compact quadratic minimization problem in $\mathbf{u}$:

$$\mathop{\text{minimize}}_{\mathbf{u}\in\mathbb{R}^6} \mathbf{u}^\top \mathbf{A} \mathbf{u} - 2\mathbf{u}^\top \mathbf{b}$$

whose solution is revealed as $\mathbf{u}^* = \mathbf{A}^{-1}\mathbf{b}$.

> **Question:** How do we know that $\mathbf{u}^*$ is a minimizer and not a maximizer of the quadratic expression above?
>
> **Hint:** 🥣

> **Question:** For our problem can we reasonably assume that $\mathbf{A}$ will be invertible?
>
> **Hint:** 🎰

Solving this small 6×6 system gives us our translation vector $\mathbf{t}$ and the linearized rotation $\mathbf{a}$. If we simply assign

$$\mathbf{R} \leftarrow \mathbf{I} + \begin{pmatrix} 0 & -a_2 & a_2 \\ a_2 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

then our transformation will *not* be rigid. Instead, we should *recover* the axis and angle of rotation from $\mathbf{a}$ via $\theta = \|\mathbf{a}\|$ and $\hat{\mathbf{w}} = \mathbf{a}/\theta$ and then update our rotation via the ***axis-angle to matrix formula*** above. Because we used a linearization of the rotation constraint, we cannot assume that we have *successfully* found the best rigid transformation. To converge on an optimal value, we must set $\mathbf{x}_i \leftarrow \mathbf{R}\mathbf{x}_i + \mathbf{t}$ and repeat this process (usually 5 times or so is sufficient).

Uniform random sampling of a triangle mesh

Our last missing piece is to sample the surface of a triangle mesh X with m faces uniformly randomly. This allows us to approximate *continuous* integrals over the surface X with a summation of the integrand evaluated at a finite number of randomly selected points. This type of numerical integration is called the Monte Carlo method.

We would like our random variable $\mathbf{x} \in X$ to have a uniform probability density function $f(\mathbf{x}) = 1/A_X$, where A_X is the surface area of the triangle mesh X. We can achieve this by breaking the problem into two steps: uniformly sampling in a single triangle and sampling triangles non-uniformly according to their area.

Suppose we have a way to evaluate a continuous random point $\mathbf{x}$ in a triangle T with uniform probability density function $g_T(\mathbf{x}) = 1/A_T$ *and* we have a away to evaluate a discrete random triangle index $T \in \{1, 2, \dots, m\}$ with discrete probability distribution $h(T) = A_T/A_X$, then the joint probability of evaluating a certain triangle index T and then uniformly random point in that triangle $\mathbf{x}$ is indeed uniform over the surface:

$$h(T)g_T(\mathbf{x}) = \frac{A_T}{A_X}\frac{1}{A_T} = \frac{1}{A_X} = f(\mathbf{x}).$$

Uniform random sampling of a single triangle

In order to pick a point uniformly randomly in a triangle with corners $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ we will *first* pick a point uniformly randomly in the parallelogram formed by reflecting $\mathbf{v}_1$ across the line $\overline{\mathbf{v}_2\mathbf{v}_3}$:

$$\mathbf{x} = \mathbf{v}_1 + a_1(\mathbf{v}_2 - \mathbf{v}_1) + a_2(\mathbf{v}_3 - \mathbf{v}_1)$$

where a_1, a_2 are uniformly sampled from the unit interval $[0, 1]$. If $a_1 + a_2 > 1$ then the point $\mathbf{x}$ above will lie in the reflected triangle rather than the original one. In this case, preprocess a_1 and a_2 by setting $a_1 \Leftarrow 1 - a_1$ and $a_2 \Leftarrow 1 - a_2$ to reflect the point $\mathbf{x}$ back into the original triangle.

Area-weighted random sampling of triangles

Assuming we know how to draw a *continuous* uniform random variable a_2 from the unit interval $[0, 1]$, we would now like to draw a *discrete* random triangle index T from the sequence $1, \dots, m$ with likelihood proportional to the relative area of each triangle in the mesh.

We can achieve this by first computing the cumulative sum $\mathbf{C} \in \mathbb{R}^m$ of the relative areas:

$$C_i = \sum_{j=1}^{i} \frac{A_j}{A_X},$$

Then our random index is found by identifying the first entry in $\mathbf{C}$ whose value is greater than a uniform random variable a_2. Since $\mathbf{C}$ is sorted, locating this entry can be done in $O(\log m)$ time.

Why is my code so slow?

Try profiling your code. Where is most of the computation time spent?

If you have done things right, the majority of time is spent computing point-to-mesh distances. For each query point, the computational complexity of computing its distance to a mesh with m faces is $O(m)$.

This can be *dramatically* improved (e.g., to $O(\log m)$ on average) using an space partitioning data structure such as a kd tree, a bounding volume hierarchy, or spatial hash.

You could follow this assignment from our graphics course to learn how to implement an AABB tree.

Tasks

Read [Bouaziz 2015]

This reading task is not directly graded, but it's expected that you read and understand sections 3.2-3.3 of Sofien Bouaziz's PhD thesis "Realtime Face Tracking and Animation" 2015. *Understanding* this may require digging into wikipedia, other online resources or other papers.

Blacklist

You may not use the following libigl functions:

- `igl::AABB`
- `igl::fit_rotations`
- `igl::hausdorff`
- `igl::iterative_closest_point`
- `igl::point_mesh_squared_distance`
- `igl::point_simplex_squared_distance`
- `igl::polar_dec`
- `igl::polar_svd3x3`
- `igl::polar_svd`
- `igl::random_points_on_mesh`
- `igl::rigid_alignment`
- `Eigen::umeyama`

Whitelist

You are encouraged to use the following libigl functions:

- `igl::cumsum` computes cumulative sum
- `igl::doublearea` computes triangle areas
- `igl::per_face_normals` computes normal vectors for each triangle face

`src/random_points_on_mesh.cpp`

Generate `n` random points uniformly sampled *on* a given triangle mesh with vertex positions `VX` and face indices `FX` .

`src/point_triangle_distance.cpp`

Compute the distance `d` between a given point `x` and the closest point `p` on a given triangle with corners `a` , `b` , and `c` .

`src/point_mesh_distance.cpp`

Compute the distances `D` between a set of given points `X` and their closest points `P` on a given mesh with vertex positions `VY` and face indices `FY`. For each point in `P` also output a corresponding normal in `N`.

> It is OK to assume that all points in `P` lie inside (rather than exactly at vertices or exactly along edges) for the purposes of normal computation in `N`.

`src/hausdorff_lower_bound.cpp`

Compute a lower bound on the *directed* Hausdorff distance from a given mesh (`VX`, `FX`) to another mesh (`VY`, `FY`). This function should be implemented by randomly sampling the X mesh.

`src/closest_rotation.cpp`

Given a 3×3 matrix `M`, find the closest rotation matrix `R`.

`src/point_to_point_rigid_matching.cpp`

Given a set of source points X and corresponding target points P, find the optimal rigid transformation (R,t) that aligns X to P, minimizing the point-to-point matching energy.

`src/point_to_plane_rigid_matching.cpp`

Given a set of source points `X` and corresponding target points `P` and their normals `N`, find the optimal rigid transformation (`R`, `t`) that aligns `X` to planes passing through `P` orthogonal to `N`, minimizing the point-to-point matching energy.

`src/icp_single_iteration.cpp`

Conduct a *single iteration* of the iterative closest point method align (`VX`, `FX`) to (`VY`, `FY`) by finding the rigid transformation (`R`, `t`) minimizing the matching energy.

The caller can specify the number of samples `num_samples` used to approximate the integral over X and specify the `method` (point-to-point or point-to-plane).

Releases

No releases published

Packages

No packages published

Contributors 3

alecjacobson Alec Jacobson

ErisZhang Jiayi Eris Zhang

psarahdactyl Sarah Kushner

Languages

● **C++** 62.7% ● **C** 24.3% ● **CMake** 13.0%