Math for Mass Spring System

Problem Setup

- Two nodes in 3D space
  - vector p1 = [x1, y1, z1]
  - vector p2 = [x2, y2, z2]
- One spring
  - Stiffness k_s
  - Rest length L
- Notation
  - $||p1 - p2||$ is the **Euclidean** norm
  - V is potential energy scalar
  - F is the force vector
  - K is the stiffness matrix

Spring Potential Energy

$V = \frac{1}{2}k_s(\Delta x)^2$

where $\Delta x = ||p1 - p2|| - L$

$V = \frac{1}{2}k_s(||p1 - p2|| - L)^2$

Differentiate for forces:

$\nabla V = 2\frac{1}{2}k_s(||p1 - p2|| - L) * \nabla(||p1 - p2|| - L)$ by the chain rule

$$= \nabla V = 2\frac{1}{2}k_s(||p1 - p2|| - L) * (\nabla||p1 - p2|| - 0)$$

$$= k_s(||p1 - p2|| - L)\frac{(p1-p2)}{||p1-p2||} \text{ since } \nabla||v|| = \frac{v}{||v||}$$

Spring Force

$F = -\nabla V = k_s(||p1 - p2|| - L)\frac{(p1-p2)}{||p1-p2||}$

where

$k_s(||p1 - p2|| - L)$ is the magnitude of F
$\frac{(p1-p2)}{||p1-p2||}$ is the direction of F

so

$F_{12} = -F$ force on $p1$ by $p2$
$F_{21} = F$ force on $p2$ by $p1$

Differentiate for Stiffness

$$\nabla F = \nabla k_s(||p1 - p2|| - L)\frac{(p1-p2)}{||p1-p2||}$$

$$= k_s(\nabla(p1 - p2) - L\nabla\frac{(p1-p2)}{||p1-p2||})$$ multiply and simplify

$$= k_s(I - L * (\frac{||p1-p2||\nabla(p1-p2) - (p1-p2)\nabla||p1-p2||}{||p1-p2||^2}))$$ division rule

$$= k_s(I - L * (\frac{I}{||p1-p2||} - \frac{(p1-p2)(p1-p2)^T}{||p1-p2||^3}))$$ simplify

Stiffness: Jacobian of Forces

$$K = \nabla F = k_s(I - L * (\frac{I}{||p1-p2||} - \frac{(p1-p2)(p1-p2)^T}{||p1-p2||^3}))$$

so

$K_{11} = -K$
$K_{12} = K$
$K_{21} = K$
$K_{22} = -K$

Implicit Euler Equations

$x_{t+1} = x_t + h * v_{t+1}$
$v_{t+1} = v_t + ha_{t+1}$

where

- x_t is the vector of positions at time t
- v_t is the vector of velocities at time t
- a_t is the acceleration

- h is the time step

From Newton's Second Law
$f(x_t) = Ma_t = F$ from spring forces

- $f(x)$ is the force vector function
- M is the diagonal mass matrix

$x_{t+1} = x_t + h * v_{t+1}$
$v_{t+1} = v_t + hM^{-1}f(x_{t+1})$

Then use Newton's Method or LBFGS or whatever to find x_{t+1} and v_{t+1}

Newtons Method

$g(x_{t+1}) = x_{t+1} - (x_t + h(v_t + hM^{-1}f(x_{t+1})))$
$\nabla g(x_t) = I - h^2M^{-1}\nabla f(x_{t+1})$

We want to minimize the error g, so use the gradient of g:
$0 = g(x) + \nabla g(x)dx$
$dx = -\nabla g(x)^{-1}g(x)$
Keep updating x while $g(x) < \epsilon$