Math 496 the solution of $x! = x^t$ and its variants

Haotian Ju, Saibo Xu

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1 Introduction

The main topic for this paper is to discuss the solutions of the equation $x! = x^t$ and its variants. The methods to approximate the solutions can be various. For example, some algorithms can be used on this equation to approximate the solution. Moreover, we can use some theorems and lemmas in number theory to find an accurate solution. In this paper, the main function that will be used is some number theory theorems together with some knowledge of gamma functions to deal with situations where x is a decimal number. Also, we will use Stirling Approximation to simplify the equation.

2 Abstract

We can divide this whole problem into several categories. The first one, also the easiest one to work with is the situation when x is a positive integer number. The second situation is when x is a negative integer number. The third situation is when x is a positive decimal number. The last one, also the most complicated one to work on, is when x is a negative decimal number. This paper will show all the solutions to those 4 different situations in the order above. This paper will also show some variants of this problem so that it will have a wider range of applications to some future problems in number theory and other areas.

3 solution of the equation when x is a positive integer number

Theorem 1.1: For positive integers x and t, the equation $x! - x^t = 0$ has a unique solution for $x = 1$ for all $t \geq 1$, and for $x = 2$, $t = 1$. For $x > 2$, there are no integer solutions for t.

Proof:

Case $x = 1$:

For $x = 1$, the equation simplifies to $1! - 1^t = 0$ for all $t \geq 1$. This is because $1! = 1$ and $1^t = 1$ for any t, proving that $x = 1$ is always a solution.

Case $x = 2$:

For $x = 2$, the equation becomes $2! - 2^t = 0$. Simplifying, we get $2 - 2^t = 0$. This holds true only when $t = 1$, since $2^1 = 2$. Thus, for $x = 2$, the only solution is $t = 1$.

Case $x > 2$:

For $x > 2$, we argue that there are no integer solutions for t. This is because for $x!$ to equal x^t, x^t must include all prime factors of $x!$. However, $x! = x \cdot (x-1) \cdot \ldots \cdot 2 \cdot 1$, and for any $x > 2$, $x!$ will contain factors that are greater than x (e.g., $x-1$ when $x>2$) which are not factors of x^t (since x^t contains only x as its prime factor repeated t times).

Furthermore, for any $x>2$, $x-1>1$ and is a divisor of $x!$. Given that $gcd(x-1,x)=1$ (since x and $x-1$ are consecutive integers and hence coprime), it implies that $x!$ and x^t cannot be equal for any t when $x>2$, as $x!$ will include prime factors not present in x^t.

Conclusion:

The analysis concludes that the equation $x!-x^t=0$ has a very limited set of solutions in the integers:

For $x=1$, any $t\geq 1$ is a solution. For $x=2$, the only solution is $t=1$. For $x>2$, there are no integer solutions for t. This theorem and its proof demonstrate the uniqueness of solutions for the given equation in the positive integers, aligning with the initial problem statement's exploration.

4 Approximate solution of the equation when x is a positive number

Theorem: For the equation $x!=x^t$ where x is a positive real number and t is a real number, the value of t as a function of x can be approximated by

$$t\approx x-\frac{x}{\ln(x)}+\frac{1}{\ln(x)}.$$

Furthermore, as x increases, t also increases for $x>1$.

Proof: Given the Stirling approximation $\ln(x!)\approx x\ln(x)-x+1$, and substituting $\Gamma(x+1)=x!$ for positive real x, we obtain the equation

$$\ln(\Gamma(x+1))=t\ln(x).$$

Simplifying and rearranging gives us an expression for t in terms of x:

$$t\approx\frac{x\ln(x)-x+1}{\ln(x)}=x-\frac{x}{\ln(x)}+\frac{1}{\ln(x)}.$$

Derivative Analysis: To analyze the behavior of t as x varies, we examine the derivative of t with respect to x:

$$\frac{dt}{dx}=\frac{d}{dx}\left(x-\frac{x}{\ln(x)}+\frac{1}{\ln(x)}\right).$$

This derivative helps to understand how t changes as x increases, demonstrating that t increases as x increases for $x>1$.

4.1 Another approach for the problem

Another approach would be to make a table of $\ln(x!)/(\ln x)$ for various values of x and make an estimate for this as a function of x. Here we will use the Sterling formula for this problem

We use the $log_e(x)$ for all the log() functions.

$x = 1$	$\log(x!)/(\log\ x) = 1$
$x = 5$	$\log(x!)/(\log\ x) = 2.9746358687061645$
$x = 10$	$\log(x!)/(\log\ x) = 6.559763032876793$
$x = 20$	$\log(x!)/(\log\ x) = 14.131975956091887$
$x = 30$	$\log(x!)/(\log\ x) = 21.950574451031976$
$x = 40$	$\log(x!)/(\log\ x) = 29.906274001912664$
$x = 50$	$\log(x!)/(\log\ x) = 37.95421620623191$
$x = 100$	$\log(x!)/(\log\ x) = 78.98500182735788$
$x = 200$	$\log(x!)/(\log\ x) = 162.92568516989743$
$x = 500$	$\log(x!)/(\log\ x) = 420.19229806673394$

So, we can see an increasing trend for the t as we would like to have an intersection point for $(x,\ y = 0)$.

Therefore, we can make the assumption that as the intersection point for x increases, t will also increase. To make a good estimation of t for any given x, we use Sterling Theorem to expand the $\log(x!)$.

For the equation $\frac{\ln(x!)}{x\ln(x)}$, we can see that its value approaches 1 as we increase the values of x. We can see this from the graph:

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Plot[Log[Factorial[x]] / (Log[x] * x), {x, 1, 10000000000000}]
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Figure 1: The code for the function $\frac{\ln(x!)}{x\ln(x)}$ with x ranges from 1 to 10000000000000

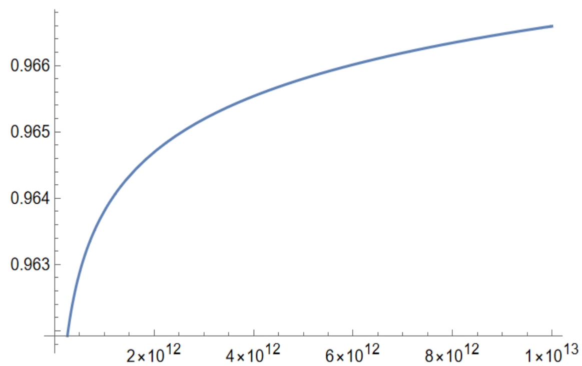

Figure 2: The graph of the function $\frac{\ln(x!)}{x\ln(x)}$

To mathematically prove it, we need to use the Stirling again.

$$\frac{\ln(x!)}{x\ln(x)} \approx \frac{x\ln(x) - x + 1}{x\ln(x)}$$

which can be written into:

$$\frac{\ln(x!)}{x\ln(x)} \approx 1 - \frac{1}{\ln(x)} + \frac{1}{x\ln(x)}$$

So as the x increases, the latter 2 terms approach 0. Therefore the whole equation approaches 1 as x is big enough.

5 Lower bound and Upper bound of the solutions for $x! = x^t$

Theorem: Given the equation $x! = x^t$ and the approximation for $n!$ in the range of

$$\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}}, \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}\right),$$

to determine the lower and upper bounds of x where the solution to $x! = x^t$ is sure to exist.

Proof: The bounds for the solution of $x! = x^t$ are derived from the given approximation for the factorial function. By substituting the range of values for $\ln(x!)$ into the formula for $t = \frac{\ln(x!)}{\ln(x)}$, we get two equations that represent the lower and upper bounds of t as functions of x:

1. For the lower bound:

$$\frac{\ln\left(\sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{\frac{1}{12x+1}}\right)}{\ln(x)} - t = 0,$$

2. For the upper bound:

$$\frac{\ln\left(\sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{\frac{1}{12x}}\right)}{\ln(x)} - t = 0.$$

the logarithm of $x!$ falls in the range

$$\left(\ln\left(\sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{\frac{1}{12x+1}}\right), \ln\left(\sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{\frac{1}{12x}}\right)\right).$$

Therefore, for a fixed value of t, the bounds for x can be found by solving the equations

$$\begin{cases} \frac{\ln\left(\sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{\frac{1}{12x+1}}\right)}{\ln(x)} = t, \\ \frac{\ln\left(\sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{\frac{1}{12x}}\right)}{\ln(x)} = t, \end{cases}$$

Solving these equations provides the interval for x that satisfies the original equation $x! = x^t$ for a given t.

Next, for some given t value, we can see the intervals where the solutions are sure to exist:

6 Some changes for the form of the equation $x! = x^t$ but still the same problem

6.1 The approximate solution for $(\alpha x)! = (\alpha x)^t$

Suppose that we substitute x to αx, then we try to find the approximate solution for this new equation. In this problem, we can still use the methods above to find the approximate solution.

For the first step of the solution, we take log_e for both sides, then we have:

$$\ln((\alpha x)!) = t * \ln(\alpha x)$$

Then, we rewrite the formula as:

$$t = \frac{\ln((\alpha x)!)}{\ln(\alpha x)}$$

At this time, to simplify the computation, we define $y = \alpha x$, so that:

$$t = \frac{ln(y!)}{ln(y)}$$

Therefore, we can expand $\ln(y!)$ to:

$$\ln(y!) \approx y * \ln(y) - y + 1$$

So that the t can be written as an equation of y:

$$t \approx \frac{y * \ln(y) - y + 1}{\ln(y)} = y - \frac{y}{\ln(y)} + \frac{1}{\ln(y)}$$

Next, we bring $y = \alpha x$ back to the equation above and get:

$$t \approx \alpha x - \frac{\alpha x}{\ln(\alpha x)} + \frac{1}{\ln(\alpha x)}$$

In this problem, we can still find the lower bound and upper bound for the solution x corresponding to given t:

6.2 The approximate solution for $(\alpha x)! = (\alpha x)^{\beta t}$

Next, we more generalize the original equation by substituting the t to βt, where $\beta \in \mathbb{R}$.

We still take ln to both sides, and we get:

$$\ln((\alpha x)!) = \beta t \ln(\alpha x)$$

Therefore, t is equal to:

$$t = \frac{\ln((\alpha x)!)}{\beta \ln(\alpha x)}$$

We use the Sterling formula to expand $\ln((\alpha x))!$:

$$\ln(\alpha x!) \approx \alpha x * \ln(\alpha x) - \alpha x + 1$$

Therefore, we got the solution for t:

$$t \approx \frac{\alpha x}{\beta} - \frac{\alpha x}{\beta \ln(\alpha x)} + \frac{1}{\beta \ln(\alpha x)}$$

This is only an approximate solution for t given the value of x. We can find a region where an accurate solution is sure to exist.

We still use this region for n! to solve the problem:

$$(\sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n+1}}, \sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n}})$$

This can be used to find the interval of t:

$$t \in (\frac{ln(\sqrt{2\pi\alpha x}(\frac{\alpha x}{e})^{\alpha x} e^{\frac{1}{12(\alpha x)+1}})}{ln(\alpha x)}, \frac{ln(\sqrt{2\pi\alpha x}(\frac{\alpha x}{e})^{\alpha x} e^{\frac{1}{12(\alpha x)}})}{ln(\alpha x)})$$

Therefore, if we want a solution at x, we can bring this value back to the interval above, and there exists a t such that the equation holds for given α and β.

7 Some variants of the equation $x! = x^t$

7.1 The approximate solution of $x! = (\alpha x)^t$

we still take ln to both sides, then we got:

$$\ln(x!) = t * \ln(\alpha x)$$

Then, we use Stirling approximation to rewrite the equation:

$$x * \ln(x) - x + 1 \approx t * \ln(\alpha x)$$

Therefore, we got t as an equation of x:

$$t \approx \frac{x * \ln(x) - x + 1}{\ln(\alpha) + \ln(x)}$$

From this equation, if we assume that there is a solution at the point $x = \beta$, we can use the equation above to find the approximated t corresponding to the $x = \beta$:

$$t \approx \frac{\beta * \ln(\beta) - \beta + 1}{\ln(\alpha) + \ln(\beta)}$$

We can compare $x! = (\alpha x)^t$ with $x! = x^t$, if we would like to find the "t"s corresponding to $x = \beta$ for both of the equations. There should be a difference between those 2 equations:

$$\begin{cases} \beta! = \beta^t (1) \\ \beta! = (\alpha * \beta)^t (2) \end{cases}$$

We solve for t for both equations and get:

$$\begin{cases} t_1 = \frac{\ln(\beta!)}{\ln(\beta)} (1) \\ t_2 = \frac{\ln(\beta!)}{\ln(\beta)+\ln(\alpha)} (2) \end{cases}$$

Then, we rewrite t_2 to a function contains t_1:

$$t_2 = \frac{\ln(\beta!)}{\ln(\beta)} * \frac{\ln(\beta)}{\ln(\beta) + \ln(\alpha)}$$

Where the first term is t_1, so:

$$t_2 = t_1 * \frac{\ln(\beta)}{\ln(\beta) + \ln(\alpha)}$$

For the second term, we can rewrite it as:

$$\frac{\ln(\beta)}{\ln(\beta) + \ln(\alpha)} = 1 - \frac{\ln(\alpha)}{\ln(\beta) + \ln(\alpha)}$$

Therefore, t_2 is equal to:

$$t_2 = t_1 * (1 - \frac{\ln(\alpha)}{\ln(\beta) + \ln(\alpha)})$$

The term $1 - \frac{\ln(\alpha)}{\ln(\beta)+\ln(\alpha)}$ will reach to 0 if we take β big enough. Therefore, t_2 will be closer and closer to t_1 if we increase the value of β

7.2 The approximate solution of $(\alpha x)! = x^t$

We still try to use the Stirling function to solve this problem

We take ln for both sides and got:

$$\ln((\alpha x)!) = t * \ln(x)$$

Then we can rewrite t as a function of x:

$$t = \frac{\ln((\alpha x)!)}{\ln(x)}$$

Since we have $\ln(x!) \approx x * \ln(x) - x + 1$, we can write t as:

$$t \approx \frac{\alpha * x * \ln(\alpha * x) - \alpha * x + 1}{\ln(x)}$$

Next, we rewrite the equation as:

$$t \approx \frac{\alpha * x * (\ln(x) + \ln(\alpha)) - \alpha * x + 1}{\ln(x)}$$

Then, t is equal to:

$$t \approx \alpha * x * \frac{\ln(x) - \ln(\alpha) + 1}{\ln(x)} + 1$$

We can also consider the upper and lower bound for t as we assume x to be a fixed point:

From the formulas above, we have:

$$x! \in (\sqrt{2\pi x}(\frac{x}{e})^x e^{\frac{1}{12x+1}}, \sqrt{2\pi x}(\frac{x}{e})^x e^{\frac{1}{12x}})$$

And,

$$\ln(x!) \in (ln(\sqrt{2\pi x}(\frac{x}{e})^x e^{\frac{1}{12x+1}}), ln(\sqrt{2\pi x}(\frac{x}{e})^x e^{\frac{1}{12x}}))$$

Which is equal to:

$$\ln(x!) \in ((\frac{1}{12x+1} - x)\ln(\sqrt{2\pi x} * x^x), (\frac{1}{12x} - x)\ln(\sqrt{2\pi x} * x^x))$$

So the $\ln(\alpha x)$ is in the range of:

$$\ln(\alpha x!) \in ((\frac{1}{12(\alpha x)+1} - \alpha x)\ln(\sqrt{2\pi\alpha x} * (\alpha x)^{\alpha x}), (\frac{1}{12(\alpha x)} - \alpha x)\ln(\sqrt{2\pi\alpha x} * (\alpha x)^{\alpha x}))$$

Finally, we bring the interval back to the equation, and we have

$$t \in (\frac{(\frac{1}{12(\alpha x)+1} - \alpha x)\ln(\sqrt{2\pi\alpha x} * (\alpha x)^{\alpha x})}{\ln(x)}, \frac{(\frac{1}{12(\alpha x)} - \alpha x)\ln(\sqrt{2\pi\alpha x} * (\alpha x)^{\alpha x})}{\ln(x)})$$

So, from this equation, if we would like to find $x = \beta$ to satisfy the equation, we substitute x by β into the interval, and there exists a $t = t^{'}$ such that the equation

$$(\alpha\beta)! = \beta^{t'}$$

holds.

8 Solutions for $n^n = n^t$

In this part, we want to find a certain form of $n! = n^t$ for an integer n. It is natural to do some research on this form, since if n is a prime, we can factorize it exactly into n items, which both sides equal.

First, we want to state that

$$n^n \equiv kn \pmod{n!}$$

for some $0 \leq k < n^{n-1}$.

Proof:

Given an integer n, $n^n - kn = n(n^{n-1} - k)$. Then if $n! \mid n^n - kn = n(n^{n-1} - k)$, we can let $k = n^{n-1} - (n-1)! < n^{n-1}$, then $n! = n(n^{n-1} - k)$. Therefore, $n^n \equiv kn \pmod{n}!$.

With this statement, we can write n^n into $pn! + kn$ for some p and k. Then the equation becomes $n! = pn! + kn$, that is, $(1-p)(n-1)! = k$. Therefore, $p = 0$. Then, $n^n = kn$, that is, $n^{n-1} = k = n^{n-1} - (n-1)!$, then $n = 1$ is the only solution.

9 Conclusions

In this paper, we discussed the accurate solution of the equation $x! = x^t$ when x is an integer and the approximate solution for t when x is any given value that is greater than 0. We also discussed the upper and lower bound of the equation $x! = x^t$, and provided a pair of equations to find the lower bound of t and upper bound of the given value of x.

Moreover, we generalized the equation $x! = x^t$ to $(\alpha x)! = (\alpha x)^t$ and $(\alpha x)! = (\alpha x)^{\beta t}$. Then, we find the approximate solution of t given x and the lower bound and upper bound of t given x.

10 Future work

In this paper, we find the approximate solution for the equation $(x)! = x^t$ and its variants. However, we fixed x to find the approximate t and its lower and upper bounds to make the solution sure to exist.

In the future, we try to find the approximate solution by writing x as a function of t.

What is more, we can try to complicate the equation by adding constant terms on the left and right-hand sides.

11 References

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