ECE, Signal Processing

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1 Sinusoidal Signals

1. Sinusoidal signals are the basic building blocks in the theory of signals & systems
2. Arise as soutions to differential equations that through the laws of physics, describe common physical phenomena

1.1 Mathematical Formula

let a signal be $s(t)$, where t represents the time t of a given time, we have:

$$s(t) = Acos(2\pi f_0 + \phi) \tag{1}$$

A linear function of time t, in which **A is the amplitude** that scales the cosine signal in y dimention, since

$$-1 \leq cos(x) \leq 1 \text{for all } x, s(t) \in [-A, +A] \tag{2}$$

ϕ **is the phase in (rad)** that determines the time locations of the maxima and minima of $s(t)$

eg: $\phi = 0 \implies s(0) = Acos(0) = A$, so $s(t)$ has a local maxima at 0
if $\phi! = 0 \implies s(0) = Acos(\phi)$, which is not necessary A

Finding Maximum of the signal after t = 0

$$Acos(2\pi f_0 t + \phi) = A \equiv cos(2\pi f_0 t + \phi) = 1 \tag{3}$$

$$2\pi f_0 t + \phi = 2kpi \tag{4}$$

$$t = \frac{2kpi - \phi}{2\pi f_0},\ k \in \mathbb{Z} \tag{5}$$

f_0 **is the frequency in (Hz)**, determines the rate the signal oscillates

$$\text{Period } T_0 = \frac{1}{f_0} \tag{6}$$

1.2 Periodic Signals

A periodic signal persists for an infinity amount of time, and will always output the same waveform in any integer multiples of periods

$$s(t+T_0) = s(t+T_0),\ \text{for all t} \tag{7}$$

We can express ϕ as

$$\phi = 2k\pi + \phi_0, k \in F, \phi_0 \in [-\pi, \pi] \tag{8}$$

if $f_0 = 0$, for all values of $Acos(\phi)$ is a constant, hence a DC signal

We can ecpress any arbitrary sinusoidal signal as a linear combination of the two pure sinusoidal signals:

$$s(t) = R \times cos(2\pi f_0 t + \phi) \tag{9}$$

$$s(t) = A \times cos(s\pi f_0 t) + B \times sin(s\pi f_0 t) \tag{10}$$

where $A = Rcos(\phi)$ and $B = Rsin(\phi)$
Using the formulae:

$$sin(A \pm B) \equiv sin(A)cos(B) \pm cos(A)sin(B) \tag{11}$$

$$cos(A \pm B) \equiv cos(A)cos(B) \mp sin(A)sin(B) \tag{12}$$

$$tan(A \pm B) \equiv \frac{tan(A) \pm tan(B)}{1 \mp Atan(B)} \tag{13}$$

$$sin(3A) \equiv 3sin(A) - 4sin^3(A) \tag{14}$$

$$cos(3A) \equiv 4cos^3(A) - 3cos(A) \tag{15}$$

$$sin(P) + sin(Q) \equiv 2sin(\frac{1}{2}(P+Q))cos(\frac{1}{2}(P-Q)) \tag{16}$$

$$sin(P) - sin(Q) \equiv 2cos(\frac{1}{2}(P+Q))sin(\frac{1}{2}(P-Q)) \tag{17}$$

$$cos(P) + cos(Q) \equiv 2cos(\frac{1}{2}(P+Q))cos(\frac{1}{2}(P-Q)) \tag{18}$$

$$cos(P) - cos(Q) \equiv -2sin(\frac{1}{2}(P+Q))sin(\frac{1}{2}(P-Q)) \tag{19}$$

$$\tag{20}$$

1.3 Harmonics

Lets say we have Harmonic Frequencies $f_k, k \in \mathbb{Z}$

$$x_1(t) = cos(2\pi f_1 + \phi) \tag{21}$$

$$x_2(t) = cos(2\pi f_2 + \phi) \tag{22}$$

$$f_k = k \times f_0, k \in \mathbb{Z} \tag{23}$$

Proof Any linear combination of sinusoidal signals of harmonics of some frequency f_0 is a periodic signal

$$\text{For arbitrary } \{A_k\}\{\phi\} \tag{24}$$

$$s(t) = \sum_{k=1}^{N} A_k cos(2\pi f_0 t + \phi_k) \tag{25}$$

$$s(t) = \sum_{k=1}^{N} A_k cos(2\pi f_0 t + \phi_k) \tag{26}$$

$$s(t + T_0) = \sum_{k=1}^{N} A_k cos(2\pi f_0 t + 2\pi k + \phi_k) \tag{27}$$

$$\text{hence: } s(t + T_0) = s(t) \tag{28}$$

$$\tag{29}$$

1.4 Fourier Series

We can express any arbitrary periodic signal as a sum of harmonic sinusoids, let $x(t) = x(t + T_0)$ be a periodic signal with period T_0
Then there exist coefficients:

$$\{A_k\}_{k=0}^{\infty}, \{B_k\}_{k=0}^{\infty} \tag{30}$$

$$x(t) = \frac{A_0}{2} + \sum_{k=0}^{\infty} \left(A_k cos(2\pi f_0 kt) + A_k sin(2\pi f_0 kt)\right), \forall t \tag{31}$$

Calculating $\{A_k\}, \{B_k\}$ starting from $x(t)$ It can be shown that:

$$A_k = \frac{2}{T_0} \int_{-0.5T_0}^{0.5T_0} x(t) cos(2\pi kt) dt \tag{32}$$

$$B_k = \frac{2}{T_0} \int_{-0.5T_0}^{0.5T_0} x(t) sin(2\pi kt) dt \tag{33}$$

Fourier synthesis is the reverse process of starting from $\{A_k\}_{k=0}^{\infty}, \{B_k\}_{k=0}^{\infty}$ and generates a periodic signal

Equivalent Form:

$$s(t) = \frac{A_0}{2} \sum_{k=1}^{\infty} C_k cos(2\pi f_0 t + \phi_k) \tag{34}$$

where

$$C_k = \sqrt{A_k^2 + B_k^2} and \phi_k = tan^{-1}(\frac{B}{A}) \tag{35}$$

1.5 Spectrum

Fourier thyrm showed that all periodic signals can be written as an addictive linear combination of sinusoid signals

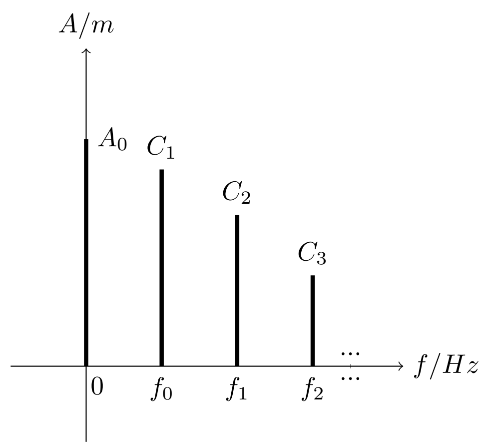
The spectrum of a periodic signal is the collection of amplitude, frequency and phase information that allow us to represent the signal as a linear combination

1. Time-domain: need knowledge of $f_0, A_0,$ and $\{x(t), t \in [-\frac{T_0}{2}, \frac{T_0}{2}]\}$
2. Frequency-domain: Would require $f_0, A_0, \{C_k\}_{k=1}^{\infty}, \{\phi_k\}_{k=1}^{\infty}$

The spectrum is given by the following collection:

Frequency	Amplitude	Phase
0	A_0	0
f_0	C_1	ϕ_1
$2f_0$	C_2	ϕ_2
$\vdots$	$\vdots$	$\vdots$
f_0	A_0	ϕ_1

1.5.1 Graphical plot of the spectrum



In which the vertical plots are the spectral lines

$$\begin{gathered} \rightarrow \text{analysis} \rightarrow \\ \text{Time Domain} \Longleftrightarrow \text{Frequency Domain} \\ \leftarrow \text{syntheses} \leftarrow \end{gathered}$$

1.5.2 Benefits of spectrum

1. Often time-waveforms are very complicated while spectrun is more straight forward

2. We can compress data by only storing the important frequencies, shrinking file size (mp3)

3. Understanding the properties of the signal is often insightful on how to process item

   - Think of audio processing, mp3 is a format which removes all frequencies sampled beyond human's limits, also shrinking the file size
   - We can remove noise

4. Often easy to see how system affect a signal by determining what-happens to the signal spectrum as it is transmitted through the system

   - A radio receiver uses a susem called filler that filters out all frequencies in the received radio wave other than the frequeny of the channel we choosed

1.5.3 Fourier integral

For any periodic signal we might need to go beyond just harmonic signals:

$$x(t) = \int_0^{\infty} A(f)cos(2\pi ft) + B(f)sin(2\pi ft)df \tag{36}$$

where:

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} x(t)cos(2\pi ft)dt \tag{37}$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} x(t)cos(2\pi ft)dt \tag{38}$$

1.5.4 Limitations

1. Arbitrary continuous functions cannot be represented in practice and cannot be stored in computer

2. The involved integrals cannot be computed in general