ECE, Signal Processing

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Contents

1 Sinusoidal Signals

1. Sinusoidal signals are the basic building blocks in the theory of signals & systems
2. Arise as soutions to differential equations that through the laws of physics, describe common physical phenomena

1.1 Mathematical Formula

let a signal be $s(t)$, where t represents the time t of a given time, we have:

$$s(t) = Acos(2\pi f_0 + \phi) \tag{1}$$

A linear function of time t, in which **A is the amplitude** that scales the cosine signal in y dimention, since

$$-1 \leq cos(x) \leq 1 \text{for all } x, s(t) \in [-A, +A] \tag{2}$$

ϕ **is the phase in (rad)** that determines the time locations of the maxima and minima of $s(t)$

eg: $\phi = 0 \implies s(0) = Acos(0) = A$, so $s(t)$ has a local maxima at 0
if $\phi! = 0 \implies s(0) = Acos(\phi)$, which is not necessary A

Finding Maximum of the signal after t = 0

$$Acos(2\pi f_0 t + \phi) = A \equiv cos(2\pi f_0 t + \phi) = 1 \tag{3}$$
$$2\pi f_0 t + \phi = 2kpi \tag{4}$$
$$t = \frac{2kpi - \phi}{2\pi f_0},\ k \in \mathbb{Z} \tag{5}$$

f_0 **is the frequency in (Hz)**, determines the rate the signal oscillates

$$\text{Period } T_0 = \frac{1}{f_0} \tag{6}$$

1.2 Periodic Signals

A periodic signal persists for an infinity amount of time, and will always output the same waveform in any integer multiples of periods

$$s(t + T_0) = s(t + T_0), \text{ for all t} \tag{7}$$

We can express ϕ as

$$\phi = 2k\pi + \phi_0, k \in F, \phi_0 \in [-\pi, \pi] \tag{8}$$

if $f_0 = 0$, for all values of $Acos(\phi)$ is a constant, hence a DC signal

We can ecpress any arbitrary sinusoidal signal as a linear combination of the two pure sinusoidal signals:

$$s(t) = R \times cos(2\pi f_0 t + \phi) \tag{9}$$
$$s(t) = A \times cos(s\pi f_0 t) + B \times sin(s\pi f_0 t) \tag{10}$$

where $A = Rcos(\phi)$ and $B = Rsin(\phi)$
Using the formulae:

$$sin(A \pm B) \equiv sin(A)cos(B) \pm cos(A)sin(B) \tag{11}$$
$$cos(A \pm B) \equiv cos(A)cos(B) \mp sin(A)sin(B) \tag{12}$$
$$tan(A \pm B) \equiv \frac{tan(A) \pm tan(B)}{1 \mp Atan(B)} \tag{13}$$
$$sin(3A) \equiv 3sin(A) - 4sin^3(A) \tag{14}$$
$$cos(3A) \equiv 4cos^3(A) - 3cos(A) \tag{15}$$
$$sin(P) + sin(Q) \equiv 2sin(\frac{1}{2}(P+Q))cos(\frac{1}{2}(P-Q)) \tag{16}$$
$$sin(P) - sin(Q) \equiv 2cos(\frac{1}{2}(P+Q))sin(\frac{1}{2}(P-Q)) \tag{17}$$
$$cos(P) + cos(Q) \equiv 2cos(\frac{1}{2}(P+Q))cos(\frac{1}{2}(P-Q)) \tag{18}$$
$$cos(P) - cos(Q) \equiv -2sin(\frac{1}{2}(P+Q))sin(\frac{1}{2}(P-Q)) \tag{19}$$
$$\tag{20}$$

1.3 Harmonics

Lets say we have Harmonic Frequencies $f_k, k \in \mathbb{Z}$

$$x_1(t) = cos(2\pi f_1 + \phi) \tag{21}$$
$$x_2(t) = cos(2\pi f_2 + \phi) \tag{22}$$
$$f_k = k \times f_0, k \in \mathbb{Z} \tag{23}$$

Proof Any linear combination of sinusoidal signals of harmonics of some frequency f_0 is a periodic signal

$$\text{For arbitrary } \{A_k\}\{\phi\} \tag{24}$$

$$s(t) = \sum_{k=1}^{N} A_k cos(2\pi f_0 t + \phi_k) \tag{25}$$

$$s(t) = \sum_{k=1}^{N} A_k cos(2\pi f_0 t + \phi_k) \tag{26}$$

$$s(t + T_0) = \sum_{k=1}^{N} A_k cos(2\pi f_0 t + 2\pi k + \phi_k) \tag{27}$$

$$\text{hence: } s(t + T_0) = s(t) \tag{28}$$

$$\tag{29}$$

1.4 Fourier Series

We can express any arbitrary periodic signal as a sum of harmonic sinusoids, let $x(t) = x(t + T_0)$ be a periodic signal with period T_0
Then there exist coefficients:

$$\{A_k\}_{k=0}^{\infty}, \{B_k\}_{k=0}^{\infty} \tag{30}$$

$$x(t) = \frac{A_0}{2} + \sum_{k=0}^{\infty} \left(A_k cos(2\pi f_0 kt) + B_k sin(2\pi f_0 kt)\right), \forall t \tag{31}$$

Calculating $\{A_k\}, \{B_k\}$ starting from $x(t)$ It can be shown that:

$$A_k = \frac{2}{T_0} \int_{-0.5T_0}^{0.5T_0} x(t) cos(2\pi kt) dt \tag{32}$$

$$B_k = \frac{2}{T_0} \int_{-0.5T_0}^{0.5T_0} x(t) sin(2\pi kt) dt \tag{33}$$

Fourier synthesis is the reverse process of starting from $\{A_k\}_{k=0}^{\infty}, \{B_k\}_{k=0}^{\infty}$ and generates a periodic signal

Equivalent Form:

$$s(t) = \frac{A_0}{2} \sum_{k=1}^{\infty} C_k cos(2\pi f_0 t + \phi_k) \tag{34}$$

where

$$C_k = \sqrt{A_k^2 + B_k^2} and \phi_k = tan^{-1}(\frac{B}{A}) \tag{35}$$

1.5 Spectrum

Fourier thyrm showed that all periodic signals can be written as an addictive linear combination of sinusoid signals

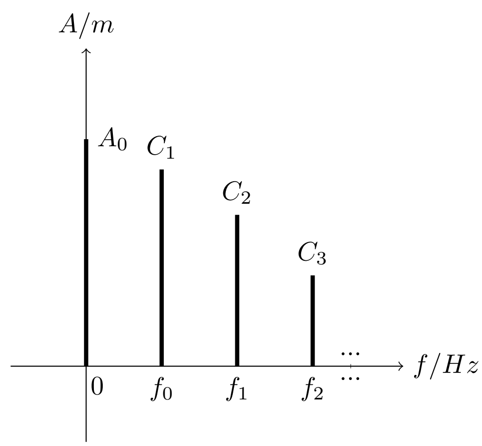
The spectrum of a periodic signal is the collection of amplitude, frequency and phase information that allow us to represent the signal as a linear combination

1. Time-domain: need knowledge of $f_0, A_0,$ and $\{x(t), t \in [-\frac{T_0}{2}, \frac{T_0}{2}]\}$
2. Frequency-domain: Would require $f_0, A_0, \{C_k\}_{k=1}^{\infty}, \{\phi_k\}_{k=1}^{\infty}$

The spectrum is given by the following collection:

Frequency	Amplitude	Phase
0	A_0	0
f_0	C_1	ϕ_1
$2f_0$	C_2	ϕ_2
$\vdots$	$\vdots$	$\vdots$
f_0	A_0	ϕ_1

1.5.1 Graphical plot of the spectrum



In which the vertical plots are the spectral lines

$$\begin{array}{c} \rightarrow \text{analysis} \rightarrow \\ \text{Time Domain} \Longleftrightarrow \text{Frequency Domain} \\ \leftarrow \text{syntheses} \leftarrow \end{array}$$

1.5.2 Benefits of spectrum

1. Often time-waveforms are very complicated while spectrun is more straight forward

2. We can compress data by only storing the important frequencies, shrinking file size (mp3)

3. Understanding the properties of the signal is often insightful on how to process item

    - Think of audio processing, mp3 is a format which removes all frequencies sampled beyond human's limits, also shrinking the file size
    - We can remove noise

4. Often easy to see how system affect a signal by determining what-happens to the signal spectrum as it is transmitted through the system

    - A radio receiver uses a susem called filler that filters out all frequencies in the received radio wave other than the frequeny of the channel we choosed

1.5.3 Fourier integral

For any periodic signal we might need to go beyond just harmonic signals:

$$x(t) = \int_0^\infty A(f)cos(2\pi ft) + B(f)sin(2\pi ft)df \tag{36}$$

where:

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} x(t)cos(2\pi ft)dt \tag{37}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} x(t)cos(2\pi ft)dt \tag{38}$$

1.5.4 Limitations

1. Arbitrary continuous functions cannot be represented in practice and cannot be stored in computer

2. The involved integrals cannot be computed in general

1.6 Fourier Transform summary

Let $x(t) = x(t + T_0)$ be a periodic signal with period T_0, then there exist coefficients $\{A_k\}_{k=0}^{\infty}$, $\{B_k\}_{k=0}^{\infty}$ such that

$$x(t) = \frac{A_0}{2} + \sum_{k=0}^{\infty} \left(A_k cos(2\pi f_0 kt) + B_k sin(2\pi f_0 kt)\right), \forall t \tag{39}$$

$$A_0 = \frac{2}{T_0} \int_{-0.5T_0}^{0.5T_0} x(t)dt \tag{40}$$

$$A_k = \frac{2}{T_0} \int_{-0.5T_0}^{0.5T_0} x(t)cos(2\pi kt)dt \tag{41}$$

$$B_k = \frac{2}{T_0} \int_{-0.5T_0}^{0.5T_0} x(t)sin(2\pi kt)dt \tag{42}$$

$$or$$

$$s(t) = \frac{A_0}{2} \sum_{k=1}^{\infty} C_k cos(2\pi f_0 t + \phi_k) \tag{43}$$

$$where$$

$$C_k = \sqrt{A_k^2 + B_k^2 and \phi_k} = tan^{-1}(\frac{B}{A}) \tag{44}$$

A spectrum of a periodic signal is the collection of amplitude, frequency and phase information that allows us to represent the signal as a linear combination A_0 is the DC part of the signal Think of the integral as a Riemannian sum, that is, the average of signal over a period

1.7 Complex number involved in fourier transform

Complex number are an elegant way of expressing rotations, something periodic.

$$Euler's\ Formula$$

$$R(cos(\theta) + isin(\theta))\ (orRcis(\theta)) \equiv Re^{i\theta} \tag{45}$$

Adding two complex number

$$z = a + bi \tag{46}$$

$$z_1 = a_1 + b_1 i \tag{47}$$

$$z + z_1 = a + a_1 + b + b_1 i \tag{48}$$

We can easily understand the concept of multiplication and division using Euler's formula

$$z = Re^{i\theta} \tag{49}$$
$$z_1 = R_1 e^{i\theta_1} \tag{50}$$
$$z \times z_1 = R \times R_1 \times e^{i(\theta+\theta_1)} \tag{51}$$

A conjugate of an imaginary number is an symmetric image of itself by the x-axis

$$z = a + bi \tag{52}$$
$$z* = a - bi \tag{53}$$
$$z \times z* = a^2 + b^2 \tag{54}$$
$$\Re(z) = \frac{z + z*}{2} \tag{55}$$
$$\Im(z) = \frac{z - z*}{2i} \tag{56}$$

hence

$$cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{57}$$
$$sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{58}$$

1.7.1 Complex Exponential Signal

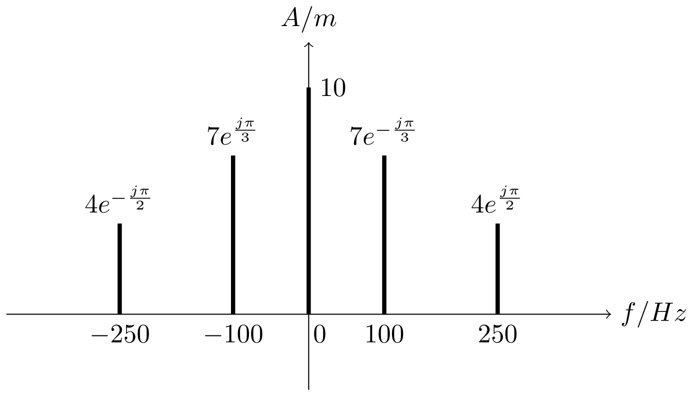
We can use the conclusion above to express a signal with an complex exponential signal

$$z(t) = Ae^{2\pi f_0 t + \phi} \tag{59}$$

Where A is the magnitude and $\omega_0 + \phi$ is the angle linearly variating in time

We can plot an complex exponential signal spectrum for any given signal by seperating them into real part and imaginary part

$$let\ x(t) = 10 + 14cos(200\pi t - \frac{\pi}{3}) + 8cos(500\pi t + \pi/2) \tag{60}$$
$$\begin{aligned} x(t) = 10 + 7e^{-\frac{j\pi}{3}} e^{2j\pi(100)t} + 7e^{\frac{j\pi}{3}} e^{-2j\pi(100)t} \\ + 4e^{\frac{j\pi}{2}} e^{2j\pi(250)t} + 4e^{-\frac{j\pi}{2}} e^{2j\pi(250)t} \end{aligned} \tag{61}$$



Multiplying something by i jas an effect of rotating that something by 90 degrees. Using euler's formula, we can express any unit-length complex number as $e^{(2\pi i t)}$ Where t is a real number, if t represents time, and plot the location of the complexnumber at the given time t. What we will see is a circle being drawn with a frequency of one cycle per second(Hz). We could adjust the speed of the rotation by multiplying a frequency f to the power, which gives us $e^{(2\pi i f t)}$

Can it have negative frequencies? negative frequencies seems pretty abstract at first, but once you start to understand the idea of drawing a circle in the anti-clockwise direction, you will soon realise that having negative frequencies simply implies rotating in the clockwise direction.

Multipying the circle by a function $f(t)$ would then suggest wrapping the function around the circle, By taking the mean, finding the center of mass of the wrapped signal, at different time t with an integral, we can plot a diagram of the spectrum. When it all lined up, there will be an obvious peak. where the center of mass is shiped the farthest away from the signal.

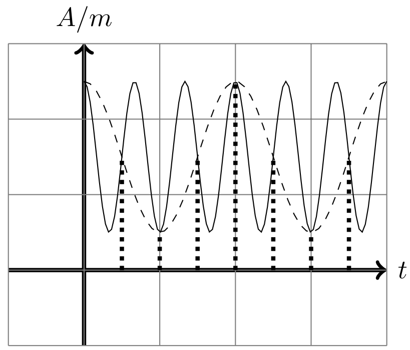
So the complex version of fourier integrals is:

$$x_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi f_0 k t} \tag{62}$$

2 Discrete Signals

2.1 Aliasing

Aliasing in the context of signal processing is simply the same name of two signals. When sampling a signal at a certain frequency, multiple signals at different frequency can produce the same signal For, example: $\hat{\omega} + 2\pi l$ and $2\pi l - \hat{\omega}$ are aliases of $\hat{\omega}$ as they are different discrete times signals defining the same signal values



As you can see from the diagram above, the two signal, both sampled at 2 Hz, produced the same value each sample, but are fundamentally, two different frequencies

The principle alias is the frequency in $[0, \pi]$

$$\hat{\omega} \hat{=} \omega_0 T_s = \frac{\omega_0}{f_s} \tag{63}$$

Fix a discrete sinusioid of discrete-time frequency $\hat{\omega}$, then the frequency f_0 of the original continuous time sinusoid could have been any of the following:

$$f_0 = (\frac{\hat{\omega}}{2\pi})f_S \text{ or } f_0 = (\frac{\hat{\omega}}{2\pi})f_S + lf_s \text{ or } f_0 = lf_s - (\frac{\hat{\omega}}{2\pi})f_s \ wherel \in \mathbb{Z} \tag{64}$$

We cannot know which one of them is the true signal, unless we can fix more conditions

2.1.1 Sampling sinusoids

Take a sample of $x[n] = cos(0.4\pi n)$, and is equivalent to $x_2[n] = cos(2.4\pi n)$. An alias corresponding to the principal alias , by convention, is used. Once the name is fixed and f_s is known, we canguarantee that the samples came from

$$x[n] = \ cos(0.4\pi n) \text{ since } \hat{\omega} \in [0, \pi] \tag{65}$$

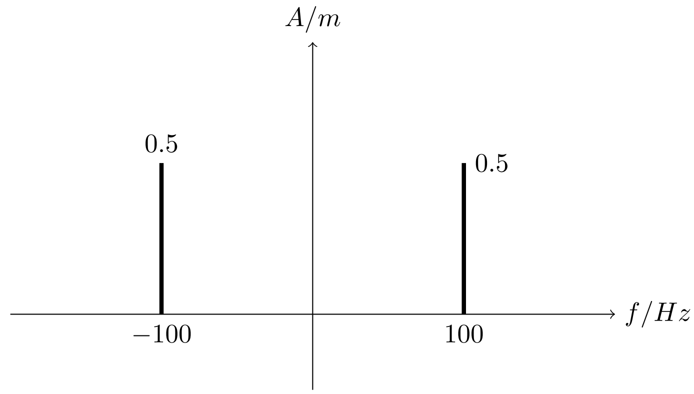
$$\implies \omega_0 = \hat{\omega} f_s \in [0, \pi \times f_s] \tag{66}$$

$$\implies f_0 = \frac{f_0}{2\pi} \in [0, \frac{f_s}{2}] \tag{67}$$

If the frequency of the original continuous time signal does not satisfy the condition mentioned above, then the resulting samples are identical to those obtained by sampling a lower frequency signal. So When $f_0 \notin [0, \frac{f_s}{2}]$, aliasing occurs.

Example:

Continuous-time signal $x(t) = cos(2\pi(100)t)$ with sampling frequency $f_s = 500Hz$.



The signal is baud-limited where $f_{max} = 100Hz$. The sampling frequency is larger than the Nyquist rate($2f_{max} = 200Hz$), so this is an over-sampling

$$\hat{\omega} = \frac{2\pi \times 100}{f_s} = 0.4\pi \tag{68}$$

$$\hat{\omega} = 0.4\pi + 2\pi l \tag{69}$$

$$\hat{\omega} = 2\pi l - 0.4\pi \tag{70}$$

$$\text{where} l \in \mathbb{Z}$$

Hence the discrete-time spectrum plot looks like: