ProblemSet3-HaowenShang

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0.0.1 Problem Set 3

MACS 30150, Dr. Evans

Due Wednesday, Jan. 30 at 11:30am

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```
In [1]: import matplotlib.pyplot as plt
        import numpy as np
        from scipy.stats import norm
        from mpl_toolkits import mplot3d
```

Exercise 5.1 The condition that characterizes the optimal amount of cake to eat in period 1 is :

$$\max_{c_1 \in [0, W_1]} u(c_1) \quad s.t. \quad W_2 = W_1 - c_1$$

The condition for the optimal amount of cake to save for the next period W_2 is:

$$\max_{W_2 \in [0, W_1]} u(W_1 - W_2)$$

In order to maximize utility, we know that If the individual lives for one period, the optimal decision is: $c_1 = W_1$ and $W_2 = 0$.

Exercise 5.2 The condition for the optimal amount of cake to leave for the next period W_3 in period 2 is:

$$\max_{W_3 \in [0, W_2]} u(W_2 - W_3)$$

In order to maximize utility, we know that If the individual lives for two period, in period 2, the optimal decision is: : $c_2 = W_2$ and $W_3 = 0$.

The condition for the optimal amount of cake leave for the next period W_2 in period 1 is:

$$\max_{W_2 \in [0, W_1]} \left[u(W_1 - W_2) + \beta \cdot \max_{W_3 \in [0, W_2]} u(W_2 - W_3) \right]$$

Since we know $W_3 = 0$, the formular above is:

$$\max_{W_2 \in [0, W_1]} \left[u\left(W_1 - W_2\right) + \beta \cdot u\left(W_2\right) \right]$$

Then we get the first order condition of period 1:

$$u'\left(W_1 - W_2\right) = \beta \cdot u'\left(W_2\right)$$

If the utility function and W_1 are known, we can get what W_2 is, which means $W_2 = \psi_1(W_1)$.

Exercise 5.3 The condition for the optimal amount of cake to leave for the next period W_4 in period 3 is:

$$\max_{W_4 \in [0, W_3]} u\left(W_3 - W_4\right)$$

In order to maximize utility, we know that If the individual lives for three period, in period 3, $c_3 = W_3$ and $W_4 = 0$

The condition for the optimal amount of cake leave for the next period W_3 in period 2 is:

$$\max_{W_3 \in [0, W_2]} \left[u\left(W_2 - W_3\right) + \beta \cdot \max_{W_4 \in [0, W_3]} u\left(W_3 - W_4\right) \right]$$

Since we know in last period, $W_4 = 0$, the formular above is:

$$\max_{W_3 \in [0, W_2]} \left[u\left(W_2 - W_3\right) + \beta \cdot u\left(W_3\right) \right]$$

Then we get the first order condition of period 2:

$$u'\left(W_2 - W_3\right) = \beta \cdot u'\left(W_3\right)$$

Then we can get $W_3 = \psi_2(W_2)$

The condition for the optimal amount of cake leave for the next period W_2 in period 1 is:

$$\max_{W_2 \in [0, W_1]} \left[u\left(W_1 - W_2\right) + \beta \cdot \max_{W_3 \in [0, W_2]} u\left(W_2 - W_3\right) + \beta^2 \cdot \max_{W_4 \in [0, W_3]} u\left(W_3 - W_4\right) \right]$$

Since we know $W_4 = 0$ and $W_3 = \psi_2(W_2)$, the formular above is:

$$\max_{W_2 \in [0, W_1]} \left[u\left(W_1 - W_2\right) + \beta \cdot u\left(W_2 - \psi_2(W_2)\right) + \beta^2 \cdot u\left(\psi_2(W_2)\right) \right]$$

Then we get the first order condition for period 1:

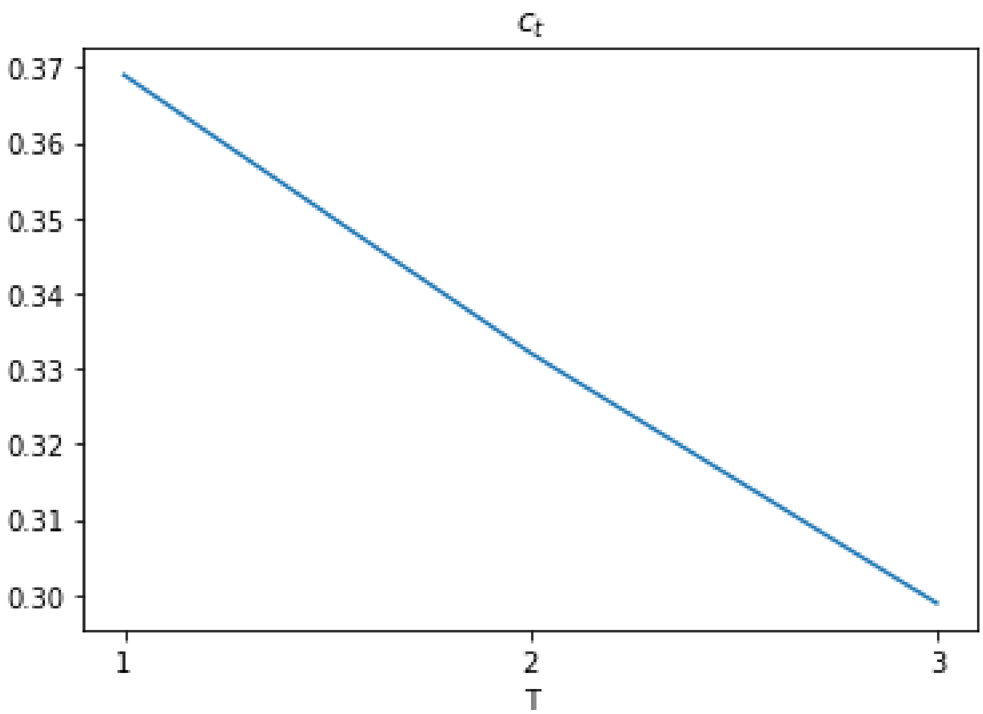
$$u'\left(W_1 - W_2\right) = \beta \cdot u'\left(W_2 - \psi_2(W_2)\right) \cdot \left(1 - \psi_2'(W_2)\right) + \beta^2 \cdot u'(\psi_2(W_2)) \cdot \psi_2'(W_2) = \beta \cdot u'\left(W_2 - W_3\right) \cdot \left(1 - \psi_2'(W_2)\right) +$$

If the initial cake size is $W_1 = 1$, the discount factor is $\beta = 0.9$, and the period utility function is $ln(c_t)$, we can solve the equations above and get $W_1 = 1$, $W_{2} =0.631 $, $W_3 = 0.299$, and $W_4 = 0$. And $c_1 = 0.369$, $c_{2} =0.332 $, $c_3 = 0.299$.

The evolvement of $\{c_t\}_{t=1}^{3}$ and $\{W_t\}_{t=1}^{4}$ over the three periods shows like following:

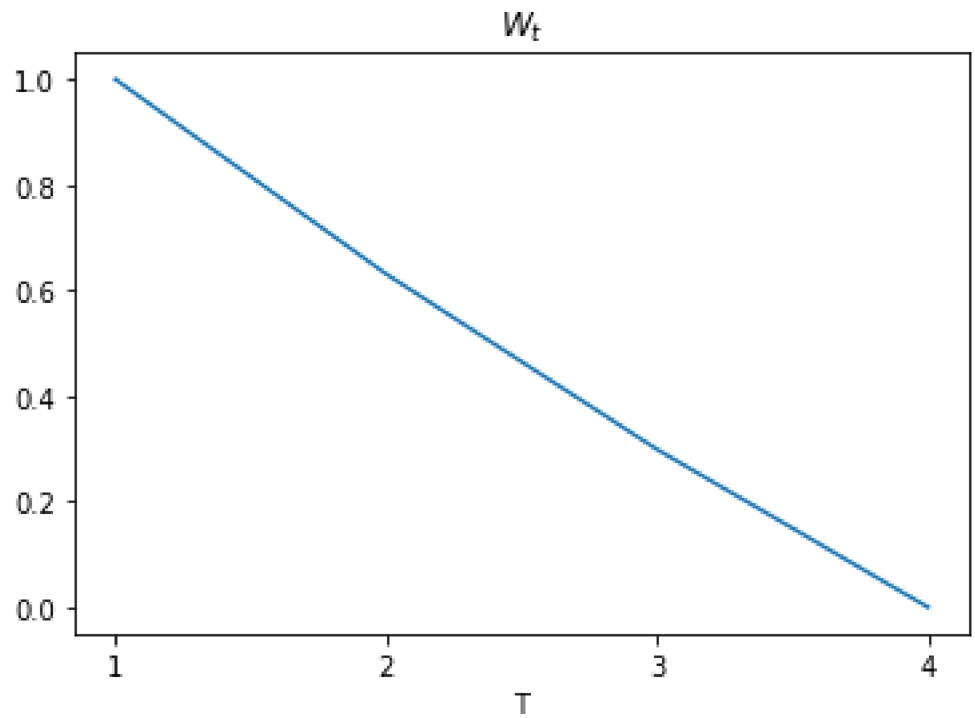
In [2]:
```
c = [1-171/271, 171/271- 9/19*171/271 , 9/19*171/271]
W = [1, 171/271, 9/19*171/271, 0]
plt.plot([1,2,3], c)
plt.title(r"$c_{t}$")
plt.xlabel('T')
plt.xticks([1,2,3])
```

Out[2]:
```
([<matplotlib.axis.XTick at 0x1175b3668>,
  <matplotlib.axis.XTick at 0x11759df28>,
  <matplotlib.axis.XTick at 0x11759dc50>],
 <a list of 3 Text xticklabel objects>)
```



In [3]:
```
plt.plot([1,2,3,4], W)
plt.title(r"$W_{t}$")
plt.xlabel('T')
plt.xticks([1,2,3,4])
```

Out[3]:
```
([<matplotlib.axis.XTick at 0x124162550>,
  <matplotlib.axis.XTick at 0x124197be0>,
  <matplotlib.axis.XTick at 0x12419bf28>,
  <matplotlib.axis.XTick at 0x1241c4748>],
 <a list of 4 Text xticklabel objects>)
```



Exercise 5.4 The condition that characterizes the optimal choice (the policy function) in period T 1 for $W_T = \psi_{T1}(W_{T1})$ is :

$$\max_{W_T \in [0, W_{T-1}]} [u(W_{T-1} - W_T) + \beta \cdot u(W_T)]$$

Then we get the first order condition:

$$u'(W_{T-1} - W_T) = \beta \cdot u'(W_T)$$
$$u'(W_{T-1} - \psi_{T1}(W_{T1})) = \beta \cdot u'(\psi_{T1}(W_{T1}))$$

The value function V_{T-1} is:

$$V_{T-1}(W_{T-1}) = u(W_{T-1} - \psi_{T1}(W_{T1})) + \beta \cdot u(\psi_{T1}(W_{T1}))$$

Exercise 5.5 Since $u(c) = ln(c)$,
in period T, we know that

$$V_T(\overline{W}) = u(\overline{W}) = ln(\overline{W})$$

And

$$\psi_T(\overline{W}) = 0$$

In period T-1, from the equation in Exercise 5.4, we can get

$$\psi_{T-1}(\overline{W}) = \frac{\beta}{1+\beta} \cdot \overline{W}$$

And

$$V_{T-1}(\overline{W}) = ln(\frac{1}{1+\beta}\overline{W}) + \beta \cdot ln(\frac{\beta}{1+\beta}\overline{W})$$

Thus $V_{T1}(\overline{W})$ does not equal $V_T(\overline{W})$ and that $\psi_{T1}(\overline{W})$ does not equal $\psi_T(\overline{W})$, because they depend on time T .

Exercise 5.6 In period T-2,

$$\max_{W_{T-1}\in[0,W_{T-2}]} \left[u\left(W_{T-2} - W_{T-1}\right) + \beta \cdot u\left(W_{T-1} - \frac{\beta}{1+\beta} \cdot W_{T-1}\right) + \beta^2 \cdot u(\frac{\beta}{1+\beta} \cdot W_{T-1}) \right]$$

Then using the envelope theorem(like the equation in Exercise 5.3), we can get the first order condition:

$$u'(W_{T-2} - W_{T-1}) = \beta \cdot u'(W_{T-1} - \frac{\beta}{1+\beta} \cdot W_{T-1}) = \beta \cdot u'(\frac{1}{1+\beta} \cdot W_{T-1})$$

Since $u(c) = ln(c)$, from the above equation we can get:

$$W_{T1} = \psi_{T2}(W_{T2}) = \frac{\beta+\beta^2}{1+\beta+\beta^2} \cdot W_{T2}$$

and

$$V_{T-2} = ln(\frac{1}{1+\beta+\beta^2} \cdot W_{T-2}) + \beta \cdot ln(\frac{\beta}{1+\beta+\beta^2} \cdot W_{T2}) + \beta^2 \cdot ln(\frac{\beta^2}{1+\beta+\beta^2} \cdot W_{T2})$$

Exercise 5.7 For the general integer s 1 using induction, we can get:

$$\psi_{T-s}(W_{T-s}) = \frac{\sum\limits_{i=1}^{s} \beta^i}{\sum\limits_{j=0}^{s} \beta^j} W_{T-s}$$

$$V_{T-s}(W_{T-s}) = \sum_{i=0}^{s} \beta^i ln\left(\frac{\beta^i \cdot W_{T-s}}{\sum\limits_{j=0}^{s} \beta^j}\right)$$

As s becomes infinite,

$$\psi_{T-s}(W_{T-s}) = \psi(W_{T-s}) = \frac{\frac{\beta}{1-\beta}}{\frac{1}{1-\beta}} \cdot W_{T-s} = \beta \cdot W_{T-s}$$

$$V_{T-s}(W_{T-s}) = V(W_{T-s}) = \sum_{i=0}^{s} \beta^i ln(\beta^i) + \sum_{i=0}^{s} \beta^i ln(\frac{1}{\sum\limits_{j=0}^{s} \beta^j}) + \sum_{i=0}^{s} \beta^i ln(W_{T-s}) = \frac{\beta}{(1-\beta)^2} ln(\beta) + \frac{1}{1-\beta} ln(1-\beta) + \frac{1}{1-}$$

As the horizon becomes further and further away (infinite), the value function and policy function become independent of time.

Exercise 5.8

$$V(W) \equiv \max_{W' \in [0,W]} u(W - W') + \beta V(W')$$

W' is cake to leave for the next period.

Exercise 5.9

```
In [4]: # Set the vector of discretized cake size:
        W_lb = 1e-2
        W_ub = 1.0
        N = 100
        W_vec = np.linspace(W_lb, W_ub, N)
        print("The possible cake sizes are:",W_vec)
```

```
The possible cake sizes are: [0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.1  0.11 0.12 0.13
 0.15 0.16 0.17 0.18 0.19 0.2  0.21 0.22 0.23 0.24 0.25 0.26 0.27 0.28
 0.29 0.3  0.31 0.32 0.33 0.34 0.35 0.36 0.37 0.38 0.39 0.4  0.41 0.42
 0.43 0.44 0.45 0.46 0.47 0.48 0.49 0.5  0.51 0.52 0.53 0.54 0.55 0.56
 0.57 0.58 0.59 0.6  0.61 0.62 0.63 0.64 0.65 0.66 0.67 0.68 0.69 0.7
 0.71 0.72 0.73 0.74 0.75 0.76 0.77 0.78 0.79 0.8  0.81 0.82 0.83 0.84
 0.85 0.86 0.87 0.88 0.89 0.9  0.91 0.92 0.93 0.94 0.95 0.96 0.97 0.98
 0.99 1.  ]
```

Exercise 5.10

```
In [5]: #set parameters
        beta = 0.9
```

```
In [6]: #choose a utility function u(c) = log(c)
        def utility(c):
            util = np.log(c)
            return util
```

```
In [7]: #initial guess for value function
        V_init = np.zeros_like(W_vec)
        #Set the vector of discretized cake size for tomorrow
        W_prime = np.zeros_like(W_vec)
        u_vec = utility(W_vec-W_prime)
        #Contraction mapping
        V_T = (u_vec + beta * V_init)
        print("The policy function is:", W_prime)
        print("The value function is:", V_T)
```

```
The policy function is: [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 0. 0. 0. 0.]
```

```
The value function is: [-4.60517019 -3.91202301 -3.5065579  -3.21887582 -2.99573227 -2.8134107
 -2.65926004 -2.52572864 -2.40794561 -2.30258509 -2.20727491 -2.12026354
 -2.04022083 -1.96611286 -1.89711998 -1.83258146 -1.77195684 -1.71479843
 -1.66073121 -1.60943791 -1.56064775 -1.51412773 -1.46967597 -1.42711636
 -1.38629436 -1.34707365 -1.30933332 -1.27296568 -1.23787436 -1.2039728
 -1.17118298 -1.13943428 -1.10866262 -1.07880966 -1.04982212 -1.02165125
 -0.99425227 -0.96758403 -0.94160854 -0.91629073 -0.89159812 -0.86750057
 -0.84397007 -0.82098055 -0.7985077  -0.77652879 -0.75502258 -0.73396918
 -0.71334989 -0.69314718 -0.67334455 -0.65392647 -0.63487827 -0.61618614
 -0.597837   -0.5798185  -0.56211892 -0.54472718 -0.52763274 -0.51082562
 -0.49429632 -0.4780358  -0.46203546 -0.4462871  -0.43078292 -0.41551544
 -0.40047757 -0.38566248 -0.37106368 -0.35667494 -0.34249031 -0.32850407
 -0.31471074 -0.30110509 -0.28768207 -0.27443685 -0.26136476 -0.24846136
 -0.23572233 -0.22314355 -0.21072103 -0.19845094 -0.18632958 -0.17435339
 -0.16251893 -0.15082289 -0.13926207 -0.12783337 -0.11653382 -0.10536052
 -0.09431068 -0.08338161 -0.07257069 -0.0618754  -0.05129329 -0.04082199
 -0.03045921 -0.02020271 -0.01005034  0.        ]
```

Exercise 5.11

```
In [8]: def distance(V_new, V_init):
            dist = ((V_new-V_init)**2).sum()
            return dist
```

```
In [10]: delta1 = distance(V_T, V_init)
         print("The distance metric is:", delta1)
```

```
The distance metric is: 178.92611065972804
```

Exercise 5.12

```
In [11]: #Create utility matrix
         c_mat =(np.tile(W_vec.reshape((N, 1)), (1, N))-
                      np.tile(W_vec.reshape((1,N)), (N,1)))
         c_pos = c_mat >0
         c_mat [~c_pos] = 1e-7
         u_mat = utility(c_mat)

         #Contraction mapping
         V_prime = np.tile(V_T.reshape((1,N)), (N,1))
         V_prime[~c_pos] = -9e+4
         V_Tminus1 = (u_mat + beta * V_prime).max(axis = 1)
         W_index = np.argmax(u_mat + beta*V_prime, axis=1)
         W_T = W_vec[W_index]
         print("The policy function is:", W_T)
         print("The value function is:", V_Tminus1)
```

```
The policy function is: [0.01 0.01 0.01 0.02 0.02 0.03 0.03 0.04 0.04 0.05 0.05 0.06 0.06 0.07
 0.07 0.08 0.08 0.09 0.09 0.09 0.1  0.1  0.11 0.11 0.12 0.12 0.13 0.13
 0.14 0.14 0.15 0.15 0.16 0.16 0.17 0.17 0.18 0.18 0.18 0.19 0.19 0.2
 0.2  0.21 0.21 0.22 0.22 0.23 0.23 0.24 0.24 0.25 0.25 0.26 0.26 0.27
 0.27 0.27 0.28 0.28 0.29 0.29 0.3  0.3  0.31 0.31 0.32 0.32 0.33 0.33
 0.34 0.34 0.35 0.35 0.36 0.36 0.36 0.37 0.37 0.38 0.38 0.39 0.39 0.4
 0.4  0.41 0.41 0.42 0.42 0.43 0.43 0.44 0.44 0.45 0.45 0.45 0.46 0.46
 0.47 0.47]
The value function is: [-8.10161181e+04 -8.74982335e+00 -8.05667617e+00 -7.43284371e+00
 -7.02737860e+00 -6.66246000e+00 -6.37477793e+00 -6.11586407e+00
 -5.89272052e+00 -5.69189132e+00 -5.50956976e+00 -5.34548036e+00
 -5.19132968e+00 -5.05259407e+00 -4.91906268e+00 -4.79888442e+00
 -4.68110139e+00 -4.57509666e+00 -4.46973614e+00 -4.37442596e+00
 -4.27960150e+00 -4.19259012e+00 -4.10681096e+00 -4.02676825e+00
 -3.94845801e+00 -3.87435004e+00 -3.80231160e+00 -3.73331873e+00
 -3.66662156e+00 -3.60208303e+00 -3.53998945e+00 -3.47936483e+00
 -3.42128016e+00 -3.36412175e+00 -3.30955959e+00 -3.25549236e+00
 -3.20404979e+00 -3.15275650e+00 -3.10396633e+00 -3.05530583e+00
 -3.00878582e+00 -2.96262185e+00 -2.91817009e+00 -2.87425894e+00
 -2.83169933e+00 -2.78983132e+00 -2.74900932e+00 -2.70900273e+00
 -2.66978202e+00 -2.63147837e+00 -2.59373804e+00 -2.55699824e+00
 -2.52063060e+00 -2.48533196e+00 -2.45024064e+00 -2.41627434e+00
 -2.38237279e+00 -2.34958297e+00 -2.31685209e+00 -2.28510339e+00
 -2.25352120e+00 -2.22274954e+00 -2.19223815e+00 -2.16238519e+00
 -2.13287434e+00 -2.10388681e+00 -2.07531298e+00 -2.04714210e+00
 -2.01944761e+00 -1.99204864e+00 -1.96518097e+00 -1.93851272e+00
 -1.91242394e+00 -1.88644845e+00 -1.86109466e+00 -1.83577685e+00
 -1.81108424e+00 -1.78642517e+00 -1.76232761e+00 -1.73832619e+00
 -1.71479569e+00 -1.69141776e+00 -1.66842824e+00 -1.64564221e+00
 -1.62316935e+00 -1.60094600e+00 -1.57896710e+00 -1.55727930e+00
 -1.53577310e+00 -1.51459565e+00 -1.49354224e+00 -1.47285167e+00
 -1.45223238e+00 -1.43200681e+00 -1.41180411e+00 -1.39200148e+00
 -1.37222046e+00 -1.35280238e+00 -1.33344679e+00 -1.31439860e+00]
```

In [12]:
```
delta2 = distance(V_Tminus1, V_T)
print("The distance metric is:", delta2)
```

```
The distance metric is: 6562865744.5285635
```

In [13]:
```
diff1 = delta2-delta1
print("The difference between delta_{T-1} and delta_T is: ", diff1)
```

```
The difference between delta_{T-1} and delta_T is:  6562865565.602453
```

delta_{T-1} is bigger than delta_T, which means distance gets lager from T to T-1.

Exercise 5.13

```
In [14]: # Perform the contraction on V_T 1
         V_prime = np.tile(V_Tminus1.reshape((1,N)), (N,1))
         V_prime[~c_pos] = -9e+4
         V_Tminus2 = (u_mat + beta * V_prime).max(axis = 1)
         W_index = np.argmax(u_mat + beta*V_prime, axis=1)
         W_Tminus1 = W_vec[W_index]
         print("The policy function is:", W_Tminus1)
         print("The value function is:", V_Tminus2)
```

```
The policy function is: [0.01 0.01 0.02 0.02 0.03 0.04 0.04 0.05 0.06 0.06 0.07 0.08 0.08 0.09
 0.09 0.1  0.11 0.11 0.12 0.13 0.13 0.14 0.15 0.15 0.16 0.17 0.17 0.18
 0.18 0.19 0.19 0.2  0.21 0.21 0.22 0.23 0.23 0.24 0.25 0.25 0.26 0.27
 0.27 0.28 0.28 0.29 0.3  0.3  0.31 0.32 0.32 0.33 0.34 0.34 0.35 0.35
 0.36 0.36 0.37 0.38 0.38 0.39 0.4  0.4  0.41 0.42 0.42 0.43 0.44 0.44
 0.45 0.45 0.46 0.47 0.47 0.48 0.49 0.49 0.5  0.51 0.51 0.52 0.52 0.53
 0.54 0.54 0.55 0.55 0.56 0.57 0.57 0.58 0.59 0.59 0.6  0.61 0.61 0.62
 0.63 0.63]
The value function is: [-8.10161181e+04 -7.29191115e+04 -1.24800112e+01 -1.17868640e+01
 -1.11630316e+01 -1.06015823e+01 -1.01961172e+01 -9.83119864e+00
 -9.50277190e+00 -9.21508983e+00 -8.95617596e+00 -8.72315349e+00
 -8.50000993e+00 -8.29918074e+00 -8.11685918e+00 -7.93611290e+00
 -7.77202350e+00 -7.61787282e+00 -7.47019236e+00 -7.33145675e+00
 -7.19792536e+00 -7.07306331e+00 -6.95288505e+00 -6.83510202e+00
 -6.72694159e+00 -6.62093686e+00 -6.51557634e+00 -6.42017208e+00
 -6.32486190e+00 -6.23003744e+00 -6.14302606e+00 -6.05724690e+00
 -5.97190488e+00 -5.89186218e+00 -5.81355194e+00 -5.73635069e+00
 -5.66224272e+00 -5.59020428e+00 -5.51972507e+00 -5.45073219e+00
 -5.38403502e+00 -5.31920043e+00 -5.25466191e+00 -5.19256832e+00
 -5.13194370e+00 -5.07191624e+00 -5.01383157e+00 -4.95667316e+00
 -4.90078893e+00 -4.84622677e+00 -4.79215955e+00 -4.73988335e+00
 -4.68844078e+00 -4.63714748e+00 -4.58804154e+00 -4.53925138e+00
 -4.49059088e+00 -4.44407086e+00 -4.39777255e+00 -4.35160858e+00
 -4.30715682e+00 -4.26324567e+00 -4.21945122e+00 -4.17689161e+00
 -4.13502359e+00 -4.09347602e+00 -4.05265403e+00 -4.01264744e+00
 -3.97312741e+00 -3.93390670e+00 -3.89560304e+00 -3.85786272e+00
 -3.82018150e+00 -3.78344171e+00 -3.74707406e+00 -3.71106814e+00
 -3.67576949e+00 -3.64067817e+00 -3.60620489e+00 -3.57223859e+00
 -3.53833704e+00 -3.50527122e+00 -3.47248140e+00 -3.43975052e+00
 -3.40798174e+00 -3.37623305e+00 -3.34465086e+00 -3.31387920e+00
 -3.28330953e+00 -3.25279814e+00 -3.22294517e+00 -3.19343433e+00
 -3.16397654e+00 -3.13498901e+00 -3.10641518e+00 -3.07799121e+00
 -3.04982033e+00 -3.02212584e+00 -2.99466558e+00 -2.96726661e+00]
```

```
In [15]: delta3 = distance(V_Tminus2, V_Tminus1)
         print("The distance metric is:", delta3)
```

```
The distance metric is: 5315921432.356884
```

```
In [16]: diff2 = delta3-delta2
         print("The difference between delta_{T-2} and delta_{T-1} is: ", diff2)
```

```
The difference between delta_{T-2} and delta_{T-1} is:  -1246944312.1716795
```

```
In [18]: diff3 = abs(delta3-delta1)
         print("The difference between delta_{T-2} and delta_{T} is: ", diff3)
```

```
The difference between delta_{T-2} and delta_{T} is:  5315921253.430774
```

delta_{T-2} is smaller than delta_{T-1} and delta_{T-2} is bigger than delta_T, which means distance gets smaller from T-1 to T-2.

Exercise 5.14

```
In [19]: V_init = np.zeros_like(W_vec)

         maxiters = 500
         toler = 1e-9
         dist = 10.0
         VF_iter = 0
         W_prime = W_vec
         while dist >= toler and VF_iter < maxiters:
             VF_iter += 1
             W = W_prime
             #Contraction mapping
             V_prime = np.tile(V_init.reshape((1,N)), (N,1))
             V_prime[~c_pos] = -9e+4
             V_new = (u_mat + beta * V_prime).max(axis = 1)
             W_index = np.argmax(u_mat + beta*V_prime, axis=1)
             W_prime = W_vec[W_index]
             dist =((V_new - V_init)**2).sum()
             V_init = V_new
             print("Iter = ",VF_iter,",distance = ", dist)

         print("Yay! It converged.")
         print("psi(W) is", W_prime)
         print("V(W) is", V_init)
         print("After {} times of iterations, V(W) converged.".format(VF_iter))
```

```
Iter =  1 ,distance =  6563611570.214573
Iter =  2 ,distance =  5316525743.271798
Iter =  3 ,distance =  4306386030.006323
Iter =  4 ,distance =  3488172794.5714226
```

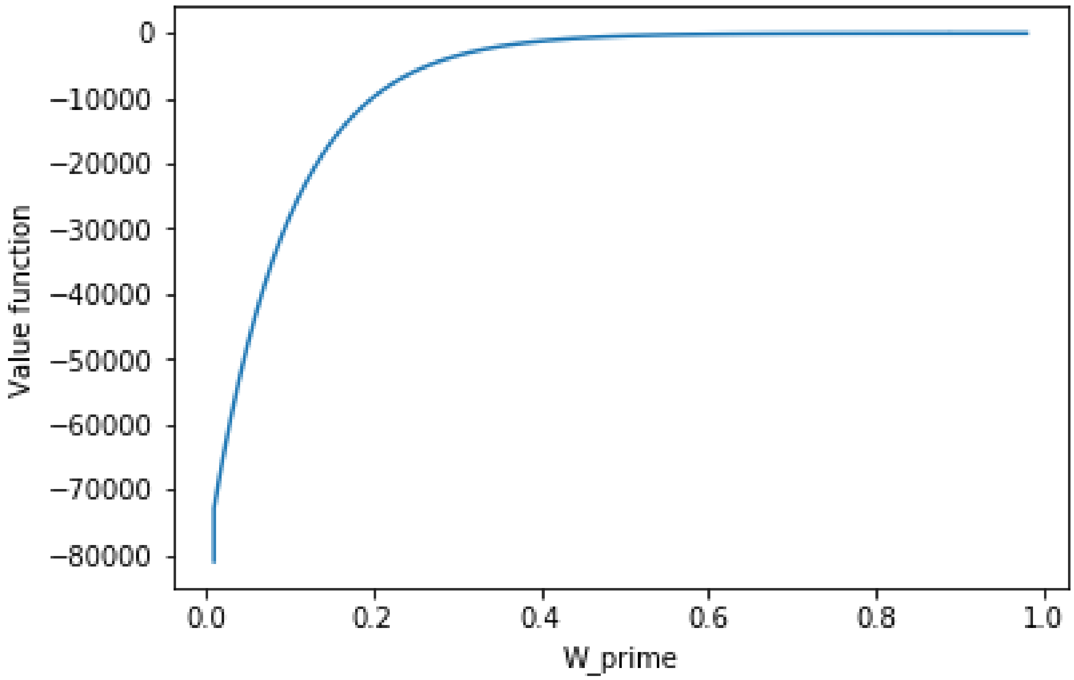
```
Iter =  5 ,distance =  2825420037.630621
Iter =  6 ,distance =  2288590282.5590844
Iter =  7 ,distance =  1853758166.5375955
Iter =  8 ,distance =  1501544142.7426846
Iter =  9 ,distance =  1216250776.642259
Iter =  10 ,distance =  985163145.1419044
Iter =  11 ,distance =  797982160.0373642
Iter =  12 ,distance =  646365559.4266962
Iter =  13 ,distance =  523556110.9935629
Iter =  14 ,distance =  424080456.20615387
Iter =  15 ,distance =  343505174.74340343
Iter =  16 ,distance =  278239195.8852569
Iter =  17 ,distance =  225373752.31221965
Iter =  18 ,distance =  182552742.55996224
Iter =  19 ,distance =  147867724.27005062
Iter =  20 ,distance =  119772859.10622115
Iter =  21 ,distance =  97016018.0385953
Iter =  22 ,distance =  78582976.63315375
Iter =  23 ,distance =  63652212.98545916
Iter =  24 ,distance =  51558294.32776982
Iter =  25 ,distance =  41762220.1200653
Iter =  26 ,distance =  33827399.92370578
Iter =  27 ,distance =  27400195.48398699
Iter =  28 ,distance =  22194159.813998673
Iter =  29 ,distance =  17977270.853317253
Iter =  30 ,distance =  14561590.732883928
Iter =  31 ,distance =  11794889.778709196
Iter =  32 ,distance =  9553861.95285677
Iter =  33 ,distance =  7738629.36668709
Iter =  34 ,distance =  6268290.928397754
Iter =  35 ,distance =  5077316.754233369
Iter =  36 ,distance =  4112627.637429302
Iter =  37 ,distance =  3331229.420618224
Iter =  38 ,distance =  2698296.8357309587
Iter =  39 ,distance =  2185621.415306089
Iter =  40 ,distance =  1770354.30067605
Iter =  41 ,distance =  1433987.914944458
Iter =  42 ,distance =  1161531.1223381404
Iter =  43 ,distance =  940841.1015289264
Iter =  44 ,distance =  762082.1677779292
Iter =  45 ,distance =  617287.4156153646
Iter =  46 ,distance =  500003.6522140527
Iter =  47 ,distance =  405003.7907942261
Iter =  48 ,distance =  328053.89114983805
Iter =  49 ,distance =  265724.4605370663
Iter =  50 ,distance =  215237.6113282474
Iter =  51 ,distance =  174343.25337503717
Iter =  52 ,distance =  141218.81426884216
```

```
Iter =  53 ,distance =  114388.00935499243
Iter =  54 ,distance =  92655.04917523317
Iter =  55 ,distance =  75051.34382720977
Iter =  56 ,distance =  60792.33427200552
Iter =  57 ,distance =  49242.52936902187
Iter =  58 ,distance =  39887.180071882685
Iter =  59 ,distance =  32309.340384184605
Iter =  60 ,distance =  26171.282961856
Iter =  61 ,distance =  21199.44996270106
Iter =  62 ,distance =  17172.25765403326
Iter =  63 ,distance =  13910.225445246653
Iter =  64 ,distance =  11267.97247896265
Iter =  65 ,distance =  9127.741149256373
Iter =  66 ,distance =  7394.146525049693
Iter =  67 ,distance =  5989.928390551506
Iter =  68 ,distance =  4852.503735570291
Iter =  69 ,distance =  3931.1818695174757
Iter =  70 ,distance =  3184.904097831992
Iter =  71 ,distance =  2580.410948027704
Iter =  72 ,distance =  2090.7641764207274
Iter =  73 ,distance =  1694.1410904514291
Iter =  74 ,distance =  1372.8672014517142
Iter =  75 ,distance =  1112.627119310548
Iter =  76 ,distance =  901.8230243640993
Iter =  77 ,distance =  731.0600844858621
Iter =  78 ,distance =  592.7310334744678
Iter =  79 ,distance =  480.6745763286175
Iter =  80 ,distance =  389.89717447236984
Iter =  81 ,distance =  316.3533259689117
Iter =  82 ,distance =  256.76931150496404
Iter =  83 ,distance =  208.4941502491576
Iter =  84 ,distance =  169.37141215479707
Iter =  85 ,distance =  137.665420034222
Iter =  86 ,distance =  111.96869169941148
Iter =  87 ,distance =  91.12990180160644
Iter =  88 ,distance =  74.23008056632061
Iter =  89 ,distance =  60.52291228596271
Iter =  90 ,distance =  49.38428817377354
Iter =  91 ,distance =  40.24315102906038
Iter =  92 ,distance =  32.126682064026454
Iter =  93 ,distance =  25.541799163125216
Iter =  94 ,distance =  19.767115537050348
Iter =  95 ,distance =  15.155025726039097
Iter =  96 ,distance =  11.174263919022518
Iter =  97 ,distance =  8.02000552946439
Iter =  98 ,distance =  5.341082016591021
Iter =  99 ,distance =  3.2347254868217576
Iter =  100 ,distance =  1.4634529907462
```

```
Iter =  101 ,distance =  0.0
Yay! It converged.
psi(W) is [0.01 0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.1  0.11 0.12 0.13
 0.14 0.15 0.16 0.17 0.18 0.19 0.2  0.21 0.22 0.23 0.24 0.25 0.26 0.27
 0.28 0.29 0.3  0.31 0.32 0.33 0.34 0.35 0.36 0.37 0.38 0.39 0.4  0.41
 0.42 0.43 0.44 0.45 0.46 0.47 0.48 0.49 0.5  0.51 0.52 0.53 0.54 0.55
 0.56 0.57 0.58 0.59 0.6  0.61 0.62 0.63 0.64 0.65 0.66 0.67 0.68 0.69
 0.7  0.71 0.72 0.73 0.74 0.75 0.76 0.77 0.78 0.79 0.8  0.81 0.82 0.83
 0.84 0.85 0.86 0.87 0.88 0.89 0.89 0.9  0.91 0.92 0.93 0.94 0.95 0.96
 0.97 0.98]
V(W) is [-8.10161181e+04 -7.29191115e+04 -6.56318055e+04 -5.90732301e+04
 -5.31705123e+04 -4.78580662e+04 -4.30768648e+04 -3.87737835e+04
 -3.49010103e+04 -3.14155144e+04 -2.82785681e+04 -2.54553165e+04
 -2.29143900e+04 -2.06275562e+04 -1.85694057e+04 -1.67170703e+04
 -1.50499685e+04 -1.35495768e+04 -1.21992243e+04 -1.09839070e+04
 -9.89012150e+03 -8.90571452e+03 -8.01974824e+03 -7.22237858e+03
 -6.50474589e+03 -5.85887647e+03 -5.27759400e+03 -4.75443977e+03
 -4.28360096e+03 -3.85984604e+03 -3.47846660e+03 -3.13522511e+03
 -2.82630777e+03 -2.54828216e+03 -2.29805912e+03 -2.07285838e+03
 -1.87017771e+03 -1.68776511e+03 -1.52359377e+03 -1.37583956e+03
 -1.24286077e+03 -1.12317987e+03 -1.01546705e+03 -9.18525516e+02
 -8.31278135e+02 -7.52755491e+02 -6.82085112e+02 -6.18481771e+02
 -5.61238764e+02 -5.09720058e+02 -4.63353223e+02 -4.21623071e+02
 -3.84065934e+02 -3.50264510e+02 -3.19843230e+02 -2.92464077e+02
 -2.67822839e+02 -2.45645726e+02 -2.25686323e+02 -2.07722861e+02
 -1.91555745e+02 -1.77005341e+02 -1.63909977e+02 -1.52124149e+02
 -1.41516905e+02 -1.31970384e+02 -1.23378516e+02 -1.15645835e+02
 -1.08686421e+02 -1.02422949e+02 -9.67858247e+01 -9.17124124e+01
 -8.71463414e+01 -8.30368774e+01 -7.93383599e+01 -7.60096941e+01
 -7.30138948e+01 -7.03176755e+01 -6.78910782e+01 -6.57071405e+01
 -6.37415967e+01 -6.19726072e+01 -6.03805167e+01 -5.89476352e+01
 -5.76580418e+01 -5.64974078e+01 -5.54528372e+01 -5.45127237e+01
 -5.36666215e+01 -5.29051296e+01 -5.22119824e+01 -5.15266396e+01
 -5.09028071e+01 -5.02859987e+01 -4.97245494e+01 -4.91694218e+01
 -4.86641175e+01 -4.81645026e+01 -4.77097288e+01 -4.72600754e+01]
After 101 times of iterations, V(W) converged.
```

In [20]:
```python
plt.plot(W_prime, V_init)
plt.xlabel("W_prime")
plt.ylabel("Value function")
```

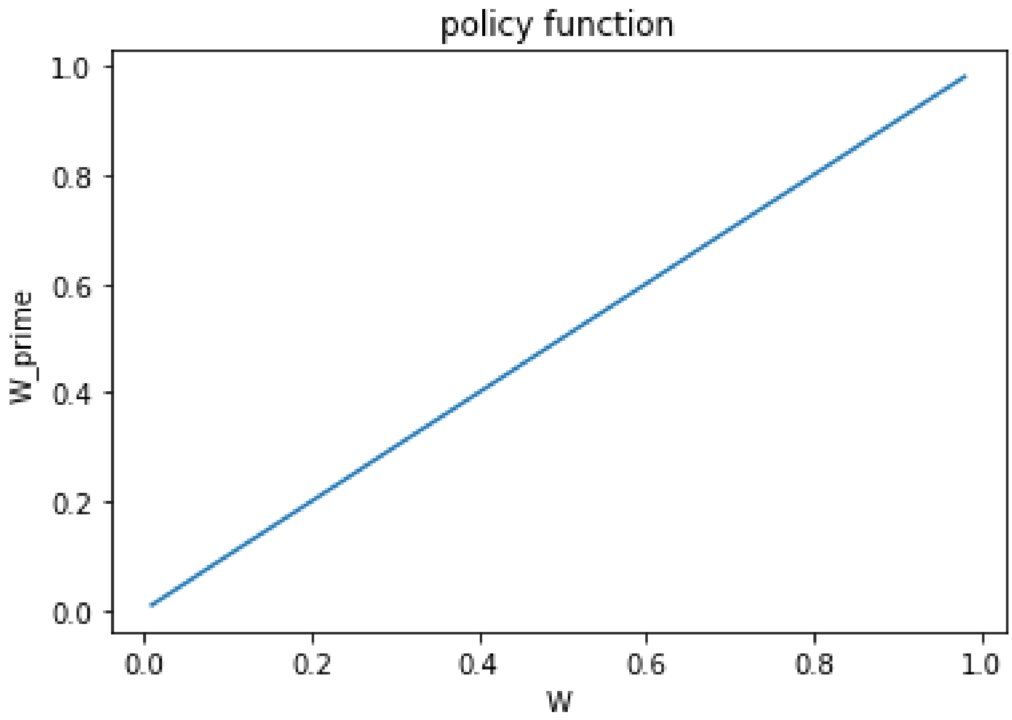
Out[20]:
```
Text(0, 0.5, 'Value function')
```



Exercise 5.15

```
In [21]: plt.plot(W, W_prime)
         plt.title("policy function")
         plt.xlabel('W')
         plt.ylabel("W_prime")

Out[21]: Text(0, 0.5, 'W_prime')
```



Exercise 5.16

```
In [22]: sigma = 0.5
         mu = 4*sigma
         epsilon_lb = mu - 3*sigma
         epsilon_ub = mu + 3*sigma
         M = 7
         epsilon = np.linspace(epsilon_lb, epsilon_ub, M)

         def gamma(x):
             gamma = norm(loc=mu, scale=sigma).pdf(x)
             return gamma

         gamma = gamma(epsilon)
         print("The epsilon is:", epsilon)
         print("The probability distribution is:", gamma)
```

```
The epsilon is: [0.5 1.  1.5 2.  2.5 3.  3.5]
The probability distribution is: [0.0088637  0.10798193 0.48394145 0.79788456 0.48394145 0.1079
 0.0088637 ]
```

Exercise 5.17

```
In [23]: #Set a vetor of discretized cake sizes
         W_lb = 1e-2
         W_ub = 1.0
         N = 100
         W_vec = np.linspace(W_lb, W_ub, N)
         print("The possible cake sizes are:",W_vec)
```

```
The possible cake sizes are: [0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.1  0.11 0.12 0.13
 0.15 0.16 0.17 0.18 0.19 0.2  0.21 0.22 0.23 0.24 0.25 0.26 0.27 0.28
 0.29 0.3  0.31 0.32 0.33 0.34 0.35 0.36 0.37 0.38 0.39 0.4  0.41 0.42
 0.43 0.44 0.45 0.46 0.47 0.48 0.49 0.5  0.51 0.52 0.53 0.54 0.55 0.56
 0.57 0.58 0.59 0.6  0.61 0.62 0.63 0.64 0.65 0.66 0.67 0.68 0.69 0.7
 0.71 0.72 0.73 0.74 0.75 0.76 0.77 0.78 0.79 0.8  0.81 0.82 0.83 0.84
 0.85 0.86 0.87 0.88 0.89 0.9  0.91 0.92 0.93 0.94 0.95 0.96 0.97 0.98
 0.99 1.  ]
```

```
In [24]: # Create utility matrix
         c_mat = (np.tile(W_vec.reshape((N,1)), (1,N))
                  - np.tile(W_vec.reshape((1,N)), (N,1)))
         c_pos = c_mat>0
         c_mat[~c_pos] = 1e-7
         u_mat = utility(c_mat)
         # Create 3-dimensional array
         Three_D_array = np.array([u_mat*e for e in epsilon])

         V_init = np.zeros((N,M))
         EV = (V_init @ gamma).reshape((N,1))
         EV_mat = np.tile(EV.reshape((1,N)), (N,1))
         EV_mat[~c_pos] = -9e+4
         EV_TDarray = np.array([EV_mat for i in range(M)])

         V_new_TDarray = Three_D_array + beta * EV_TDarray
         V_new = np.zeros((N,M))
         W_prime = np.zeros((N,M))
         for i in range(N):
             arr = V_new_TDarray[:, i, :]
             V_new[i] = arr.max(axis=1)
             W_index = np.argmax(arr, axis=1)
             W_prime[i] = W_vec[W_index]

         print("The policy function is:", W_prime)
         print("The value function is:", V_new)
```

```
The policy function is: [[0.01 0.01 0.01 0.01 0.01 0.01 0.01]
 [0.01 0.01 0.01 0.01 0.01 0.01 0.01]
 [0.01 0.01 0.01 0.01 0.01 0.01 0.01]
 [0.01 0.01 0.01 0.01 0.01 0.01 0.01]
```

```
[0.01 0.01 0.01 0.01 0.01 0.01 0.01]
[0.01 0.01 0.01 0.01 0.01 0.01 0.01]
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```

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[0.01 0.01 0.01 0.01 0.01 0.01 0.01]
[0.01 0.01 0.01 0.01 0.01 0.01 0.01]]
```

```
The value function is: [[-8.10080590e+04 -8.10161181e+04 -8.10241771e+04 -8.10322362e+04
  -8.10402952e+04 -8.10483543e+04 -8.10564133e+04]
 [-2.30258509e+00 -4.60517019e+00 -6.90775528e+00 -9.21034037e+00
  -1.15129255e+01 -1.38155106e+01 -1.61180957e+01]
 [-1.95601150e+00 -3.91202301e+00 -5.86803451e+00 -7.82404601e+00
  -9.78005751e+00 -1.17360690e+01 -1.36920805e+01]
 [-1.75327895e+00 -3.50655790e+00 -5.25983685e+00 -7.01311579e+00
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 [-1.40670536e+00 -2.81341072e+00 -4.22011608e+00 -5.62682143e+00
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 [-1.20397280e+00 -2.40794561e+00 -3.61191841e+00 -4.81589122e+00
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  -5.10055207e+00 -6.12066249e+00 -7.14077290e+00]
 [-9.83056428e-01 -1.96611286e+00 -2.94916928e+00 -3.93222571e+00
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  -4.58145366e+00 -5.49774439e+00 -6.41403512e+00]
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 [-7.34837985e-01 -1.46967597e+00 -2.20451396e+00 -2.93935194e+00
  -3.67418993e+00 -4.40902791e+00 -5.14386590e+00]
```

```
[-7.13558178e-01 -1.42711636e+00 -2.14067453e+00 -2.85423271e+00
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[-6.93147181e-01 -1.38629436e+00 -2.07944154e+00 -2.77258872e+00
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[-6.54666660e-01 -1.30933332e+00 -1.96399998e+00 -2.61866664e+00
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 -3.18241419e+00 -3.81889703e+00 -4.45537987e+00]
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[-4.97126137e-01 -9.94252273e-01 -1.49137841e+00 -1.98850455e+00
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 -2.35402135e+00 -2.82482562e+00 -3.29562989e+00]
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 -2.29072683e+00 -2.74887220e+00 -3.20701756e+00]
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```

```
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```

```
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 -2.08454022e-01 -2.50144827e-01 -2.91835631e-01]
[-3.62853464e-02 -7.25706928e-02 -1.08856039e-01 -1.45141386e-01
 -1.81426732e-01 -2.17712079e-01 -2.53997425e-01]
[-3.09377019e-02 -6.18754037e-02 -9.28131056e-02 -1.23750807e-01
 -1.54688509e-01 -1.85626211e-01 -2.16563913e-01]
[-2.56466472e-02 -5.12932944e-02 -7.69399416e-02 -1.02586589e-01
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```

```
[-2.04109973e-02 -4.08219945e-02 -6.12329918e-02 -8.16439890e-02
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[-1.52296037e-02 -3.04592075e-02 -4.56888112e-02 -6.09184150e-02
 -7.61480187e-02 -9.13776225e-02 -1.06607226e-01]
[-1.01013537e-02 -2.02027073e-02 -3.03040610e-02 -4.04054146e-02
 -5.05067683e-02 -6.06081220e-02 -7.07094756e-02]
[-5.02516793e-03 -1.00503359e-02 -1.50755038e-02 -2.01006717e-02
 -2.51258396e-02 -3.01510076e-02 -3.51761755e-02]]
```

Exercise 5.18

```
In [25]: def distance(V_new, V_init):
             dist = ((V_new-V_init)**2).sum()
             return dist

In [26]: delta1 = distance(V_new, V_init)
         print("The distance metric is:", delta1)

The distance metric is: 45963571196.10551
```

Exercise 5.19

```
In [27]: V_init = V_new
         EV = (V_init @ gamma).reshape((N,1))
         EV_mat = np.tile(EV.reshape((1,N)), (N,1))
         EV_mat[~c_pos] = -9e+4
         EV_TDarray = np.array([EV_mat for e in range(M)])
         V_new_TDarray = Three_D_array + beta*EV_TDarray
         V_new = np.zeros((N,M))
         W_prime = np.zeros((N,M))
         for i in range(N):
             arr = V_new_TDarray[:, i, :]
             V_new[i] = arr.max(axis=1)
             W_index = np.argmax(arr, axis=1)
             W_prime[i] = W_vec[W_index]

         print("The policy function is:", W_prime)
         print("The value function is:", V_new)

The policy function is: [[0.01 0.01 0.01 0.01 0.01 0.01 0.01]
 [0.02 0.02 0.02 0.02 0.02 0.02 0.02]
 [0.02 0.02 0.02 0.02 0.02 0.02 0.02]
 [0.03 0.03 0.03 0.03 0.03 0.03 0.03]
 [0.04 0.04 0.04 0.04 0.03 0.03 0.03]
 [0.05 0.05 0.04 0.04 0.04 0.04 0.04]
 [0.06 0.06 0.05 0.05 0.05 0.04 0.04]
 [0.07 0.06 0.06 0.05 0.05 0.05 0.05]
```

```
[0.08 0.07 0.07 0.06 0.06 0.05 0.05]
[0.09 0.08 0.07 0.07 0.06 0.06 0.06]
[0.1  0.09 0.08 0.07 0.07 0.06 0.06]
[0.11 0.1  0.09 0.08 0.07 0.07 0.07]
[0.11 0.1  0.09 0.09 0.08 0.08 0.07]
[0.12 0.11 0.1  0.09 0.09 0.08 0.08]
[0.13 0.12 0.11 0.1  0.09 0.09 0.08]
[0.14 0.13 0.12 0.11 0.1  0.09 0.09]
[0.15 0.14 0.12 0.11 0.1  0.1  0.09]
[0.16 0.14 0.13 0.12 0.11 0.1  0.1 ]
[0.17 0.15 0.14 0.13 0.12 0.11 0.1 ]
[0.18 0.16 0.14 0.13 0.12 0.11 0.11]
[0.19 0.17 0.15 0.14 0.13 0.12 0.11]
[0.19 0.17 0.16 0.14 0.13 0.12 0.12]
[0.2  0.18 0.17 0.15 0.14 0.13 0.12]
[0.21 0.19 0.17 0.16 0.15 0.14 0.13]
[0.22 0.2  0.18 0.16 0.15 0.14 0.13]
[0.23 0.21 0.19 0.17 0.16 0.15 0.14]
[0.24 0.21 0.19 0.18 0.16 0.15 0.14]
[0.25 0.22 0.2  0.18 0.17 0.16 0.15]
[0.26 0.23 0.21 0.19 0.18 0.16 0.15]
[0.26 0.24 0.21 0.2  0.18 0.17 0.16]
[0.27 0.24 0.22 0.2  0.19 0.17 0.16]
[0.28 0.25 0.23 0.21 0.19 0.18 0.17]
[0.29 0.26 0.24 0.22 0.2  0.18 0.17]
[0.3  0.27 0.24 0.22 0.2  0.19 0.18]
[0.31 0.28 0.25 0.23 0.21 0.2  0.18]
[0.32 0.28 0.26 0.23 0.22 0.2  0.19]
[0.33 0.29 0.26 0.24 0.22 0.21 0.19]
[0.33 0.3  0.27 0.25 0.23 0.21 0.2 ]
[0.34 0.31 0.28 0.25 0.23 0.22 0.2 ]
[0.35 0.32 0.29 0.26 0.24 0.22 0.21]
[0.36 0.32 0.29 0.27 0.25 0.23 0.21]
[0.37 0.33 0.3  0.27 0.25 0.23 0.22]
[0.38 0.34 0.31 0.28 0.26 0.24 0.22]
[0.39 0.35 0.31 0.29 0.26 0.24 0.23]
[0.4  0.35 0.32 0.29 0.27 0.25 0.23]
[0.4  0.36 0.33 0.3  0.28 0.26 0.24]
[0.41 0.37 0.33 0.31 0.28 0.26 0.24]
[0.42 0.38 0.34 0.31 0.29 0.27 0.25]
[0.43 0.39 0.35 0.32 0.29 0.27 0.25]
[0.44 0.39 0.36 0.32 0.3  0.28 0.26]
[0.45 0.4  0.36 0.33 0.31 0.28 0.26]
[0.46 0.41 0.37 0.34 0.31 0.29 0.27]
[0.47 0.42 0.38 0.34 0.32 0.29 0.27]
[0.48 0.42 0.38 0.35 0.32 0.3  0.28]
[0.48 0.43 0.39 0.36 0.33 0.3  0.28]
[0.49 0.44 0.4  0.36 0.33 0.31 0.29]
```

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 [0.5  0.45 0.41 0.37 0.34 0.32 0.29]
 [0.51 0.46 0.41 0.38 0.35 0.32 0.3 ]
 [0.52 0.46 0.42 0.38 0.35 0.33 0.3 ]
 [0.53 0.47 0.43 0.39 0.36 0.33 0.31]
 [0.54 0.48 0.43 0.4  0.36 0.34 0.31]
 [0.55 0.49 0.44 0.4  0.37 0.34 0.32]
 [0.55 0.5  0.45 0.41 0.38 0.35 0.32]
 [0.56 0.5  0.45 0.41 0.38 0.35 0.33]
 [0.57 0.51 0.46 0.42 0.39 0.36 0.33]
 [0.58 0.52 0.47 0.43 0.39 0.36 0.34]
 [0.59 0.53 0.48 0.43 0.4  0.37 0.34]
 [0.6  0.53 0.48 0.44 0.41 0.38 0.35]
 [0.61 0.54 0.49 0.45 0.41 0.38 0.35]
 [0.62 0.55 0.5  0.45 0.42 0.39 0.36]
 [0.62 0.56 0.5  0.46 0.42 0.39 0.36]
 [0.63 0.57 0.51 0.47 0.43 0.4  0.37]
 [0.64 0.57 0.52 0.47 0.43 0.4  0.38]
 [0.65 0.58 0.53 0.48 0.44 0.41 0.38]
 [0.66 0.59 0.53 0.49 0.45 0.41 0.39]
 [0.67 0.6  0.54 0.49 0.45 0.42 0.39]
 [0.68 0.6  0.55 0.5  0.46 0.42 0.4 ]
 [0.69 0.61 0.55 0.5  0.46 0.43 0.4 ]
 [0.69 0.62 0.56 0.51 0.47 0.44 0.41]
 [0.7  0.63 0.57 0.52 0.48 0.44 0.41]
 [0.71 0.64 0.57 0.52 0.48 0.45 0.42]
 [0.72 0.64 0.58 0.53 0.49 0.45 0.42]
 [0.73 0.65 0.59 0.54 0.49 0.46 0.43]
 [0.74 0.66 0.6  0.54 0.5  0.46 0.43]
 [0.75 0.67 0.6  0.55 0.51 0.47 0.44]
 [0.76 0.68 0.61 0.56 0.51 0.47 0.44]
 [0.77 0.68 0.62 0.56 0.52 0.48 0.45]
 [0.77 0.69 0.62 0.57 0.52 0.48 0.45]
 [0.78 0.7  0.63 0.58 0.53 0.49 0.46]
 [0.79 0.71 0.64 0.58 0.54 0.5  0.46]
 [0.8  0.71 0.65 0.59 0.54 0.5  0.47]
 [0.81 0.72 0.65 0.59 0.55 0.51 0.47]
 [0.82 0.73 0.66 0.6  0.55 0.51 0.48]
 [0.83 0.74 0.67 0.61 0.56 0.52 0.48]
 [0.84 0.75 0.67 0.61 0.56 0.52 0.49]
 [0.84 0.75 0.68 0.62 0.57 0.53 0.49]
 [0.85 0.76 0.69 0.63 0.58 0.53 0.5 ]
 [0.86 0.77 0.69 0.63 0.58 0.54 0.5 ]
 [0.87 0.78 0.7  0.64 0.59 0.54 0.51]
 [0.88 0.78 0.71 0.65 0.59 0.55 0.51]]
The value function is: [[-8.10080590e+04 -8.10161181e+04 -8.10241771e+04 -8.10322362e+04
  -8.10402952e+04 -8.10483543e+04 -8.10564133e+04]
 [-8.10080590e+04 -8.10161181e+04 -8.10241771e+04 -8.10322362e+04
  -8.10402952e+04 -8.10483543e+04 -8.10564133e+04]
```

```
[-1.88767109e+01 -2.11792960e+01 -2.34818811e+01 -2.57844662e+01
 -2.80870513e+01 -3.03896364e+01 -3.26922215e+01]
[-1.63820564e+01 -1.86846415e+01 -2.09872266e+01 -2.32898117e+01
 -2.55923968e+01 -2.78949819e+01 -3.01975670e+01]
[-1.49227771e+01 -1.72253622e+01 -1.95279472e+01 -2.18305323e+01
 -2.38595288e+01 -2.58155403e+01 -2.77715518e+01]
[-1.38874019e+01 -1.61899870e+01 -1.84882265e+01 -2.04442380e+01
 -2.24002495e+01 -2.43562610e+01 -2.63122725e+01]
[-1.30843025e+01 -1.53868876e+01 -1.74528513e+01 -1.94088628e+01
 -2.13648743e+01 -2.31398657e+01 -2.48931446e+01]
[-1.24281225e+01 -1.46937404e+01 -1.66497519e+01 -1.85979326e+01
 -2.03512115e+01 -2.21044905e+01 -2.38577694e+01]
[-1.18733303e+01 -1.40375605e+01 -1.59935720e+01 -1.77948332e+01
 -1.95481122e+01 -2.12414443e+01 -2.28508822e+01]
[-1.13927474e+01 -1.34827682e+01 -1.53853743e+01 -1.71386533e+01
 -1.88289070e+01 -2.04383449e+01 -2.20477828e+01]
[-1.09688432e+01 -1.30021853e+01 -1.48305820e+01 -1.65632891e+01
 -1.81727270e+01 -1.97689142e+01 -2.12667804e+01]
[-1.05896480e+01 -1.25782811e+01 -1.43499991e+01 -1.60084968e+01
 -1.76148681e+01 -1.91127343e+01 -2.06106004e+01]
[-1.02430744e+01 -1.21728160e+01 -1.39184760e+01 -1.55279139e+01
 -1.70600759e+01 -1.85579420e+01 -1.99724750e+01]
[-9.90005063e+00 -1.17936208e+01 -1.34945719e+01 -1.50816268e+01
 -1.65794930e+01 -1.80109773e+01 -1.94176827e+01]
[-9.58689445e+00 -1.14505970e+01 -1.31153766e+01 -1.46577227e+01
 -1.61236891e+01 -1.75303944e+01 -1.88781553e+01]
[-9.29881868e+00 -1.11374408e+01 -1.27723529e+01 -1.42785275e+01
 -1.56997849e+01 -1.70679424e+01 -1.83975724e+01]
[-9.03210219e+00 -1.08493651e+01 -1.24376375e+01 -1.39138843e+01
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[-8.78379507e+00 -1.05616830e+01 -1.21244814e+01 -1.35708606e+01
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[-8.55151928e+00 -1.02949665e+01 -1.18364056e+01 -1.32577044e+01
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[-8.33332970e+00 -1.00466594e+01 -1.15629233e+01 -1.29494030e+01
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[-7.92488255e+00 -9.59124005e+00 -1.10478996e+01 -1.23942645e+01
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 -1.33626712e+01 -1.45386382e+01 -1.56695013e+01]
[-7.54568735e+00 -9.16733588e+00 -1.05843978e+01 -1.18792409e+01
 -1.30959547e+01 -1.42505625e+01 -1.53563451e+01]
[-7.37009029e+00 -8.97274656e+00 -1.03662083e+01 -1.16436748e+01
 -1.28325534e+01 -1.39646319e+01 -1.50518053e+01]
[-7.20266356e+00 -8.78814067e+00 -1.01604937e+01 -1.14113990e+01
 -1.25842463e+01 -1.36979154e+01 -1.47637296e+01]
```

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[-7.04268053e+00 -8.60581912e+00 -9.96019657e+00 -1.11932094e+01
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[-6.88950738e+00 -8.43022206e+00 -9.76560725e+00 -1.09824884e+01
 -1.21136951e+01 -1.31885742e+01 -1.42168636e+01]
[-6.74258797e+00 -8.26279533e+00 -9.58100137e+00 -1.07767738e+01
 -1.18955055e+01 -1.29484461e+01 -1.39574857e+01]
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 -1.16779770e+01 -1.27161703e+01 -1.37091786e+01]
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 -4.90762628e+00 -5.39044611e+00 -5.82774496e+00]
[-1.99785627e+00 -2.94441349e+00 -3.68348796e+00 -4.30194900e+00
 -4.83907133e+00 -5.31623677e+00 -5.74728165e+00]
[-1.95200819e+00 -2.89312020e+00 -3.62680922e+00 -4.23935567e+00
 -4.77057390e+00 -5.24215893e+00 -5.66817900e+00]
[-1.90673683e+00 -2.84206933e+00 -3.57019873e+00 -4.17781235e+00
 -4.70330043e+00 -5.16944887e+00 -5.58952400e+00]
[-1.86202786e+00 -2.79173249e+00 -3.51439876e+00 -4.11628904e+00
 -4.63662981e+00 -5.09715622e+00 -5.51212262e+00]
[-1.81786748e+00 -2.74208997e+00 -3.45945074e+00 -4.05579979e+00
 -4.57059079e+00 -5.02588605e+00 -5.43519644e+00]
[-1.77424240e+00 -2.69312288e+00 -3.40489927e+00 -3.99609387e+00
 -4.50565208e+00 -4.95529456e+00 -5.35942468e+00]
[-1.73073671e+00 -2.64433271e+00 -3.35077757e+00 -3.93660449e+00
 -4.44080301e+00 -4.88540837e+00 -5.28415296e+00]
[-1.68763409e+00 -2.59602292e+00 -3.29745769e+00 -3.87808245e+00
 -4.37710178e+00 -4.81643981e+00 -5.20994362e+00]
[-1.64504157e+00 -2.54835301e+00 -3.24482071e+00 -3.82010738e+00
 -4.31380726e+00 -4.74788487e+00 -5.13625669e+00]
```

```
 [-1.60294722e+00 -2.50130625e+00 -3.19227926e+00 -3.76252174e+00
  -4.25121393e+00 -4.68046630e+00 -5.06354662e+00]
 [-1.56133953e+00 -2.45478623e+00 -3.14049382e+00 -3.70584300e+00
  -4.18948240e+00 -4.61319284e+00 -4.99137912e+00]]
```

```
In [28]: delta2 = distance(V_init, V_new)
         print("The distance metric is:", delta2)
```

```
The distance metric is: 45953155651.859604
```

```
In [29]: diff2= delta2-delta1
         print("The difference between delta_{T-1} and delta_T is: ", diff2)
```

```
The difference between delta_{T-1} and delta_T is:  -10415544.245903015
```

delta_{T-1} is smaller than delta_T, which means distance gets smaller from T to T-1.

Exercise 5.20

```
In [30]: V_init = V_new
         EV = (V_init @ gamma).reshape((N,1))
         EV_mat = np.tile(EV.reshape((1,N)), (N,1))
         EV_mat[~c_pos] = -9e+4
         EV_TDarray = np.array([EV_mat for i in range(M)])
         V_new_TDarray = Three_D_array + beta*EV_TDarray
         V_new = np.zeros((N,M))
         W_prime = np.zeros((N,M))
         for i in range(N):
             arr = V_new_TDarray[:, i, :]
             V_new[i] = arr.max(axis=1)
             W_index = np.argmax(arr, axis=1)
             W_prime[i] = W_vec[W_index]

         print("The policy function is:", W_prime)
         print("The value function is:", V_new)
```

```
The policy function is: [[0.01 0.01 0.01 0.01 0.01 0.01 0.01]
 [0.02 0.02 0.02 0.02 0.02 0.02 0.02]
 [0.03 0.03 0.03 0.03 0.03 0.03 0.03]
 [0.03 0.03 0.03 0.03 0.03 0.03 0.03]
 [0.04 0.04 0.04 0.04 0.04 0.04 0.04]
 [0.05 0.05 0.05 0.05 0.05 0.05 0.05]
 [0.06 0.06 0.06 0.06 0.06 0.06 0.05]
 [0.07 0.07 0.07 0.07 0.07 0.06 0.06]
 [0.08 0.08 0.08 0.08 0.07 0.07 0.07]
 [0.09 0.09 0.09 0.08 0.08 0.08 0.08]
```

```
[0.1  0.1  0.1  0.09 0.09 0.09 0.08]
[0.11 0.11 0.11 0.1  0.1  0.09 0.09]
[0.12 0.12 0.11 0.11 0.11 0.1  0.1 ]
[0.13 0.13 0.12 0.12 0.11 0.11 0.11]
[0.14 0.14 0.13 0.13 0.12 0.12 0.11]
[0.15 0.15 0.14 0.14 0.13 0.13 0.12]
[0.16 0.15 0.15 0.14 0.14 0.13 0.13]
[0.17 0.16 0.16 0.15 0.15 0.14 0.14]
[0.18 0.17 0.17 0.16 0.15 0.15 0.14]
[0.19 0.18 0.18 0.17 0.16 0.16 0.15]
[0.2  0.19 0.18 0.18 0.17 0.16 0.16]
[0.21 0.2  0.19 0.18 0.18 0.17 0.17]
[0.22 0.21 0.2  0.19 0.19 0.18 0.17]
[0.23 0.22 0.21 0.2  0.19 0.19 0.18]
[0.24 0.23 0.22 0.21 0.2  0.2  0.19]
[0.25 0.24 0.23 0.22 0.21 0.2  0.2 ]
[0.26 0.25 0.24 0.23 0.22 0.21 0.2 ]
[0.27 0.26 0.24 0.24 0.23 0.22 0.21]
[0.28 0.26 0.25 0.24 0.23 0.23 0.22]
[0.29 0.27 0.26 0.25 0.24 0.23 0.23]
[0.3  0.28 0.27 0.26 0.25 0.24 0.23]
[0.3  0.29 0.28 0.27 0.26 0.25 0.24]
[0.31 0.3  0.29 0.28 0.27 0.26 0.25]
[0.32 0.31 0.3  0.29 0.27 0.26 0.25]
[0.33 0.32 0.31 0.29 0.28 0.27 0.26]
[0.34 0.33 0.31 0.3  0.29 0.28 0.27]
[0.35 0.34 0.32 0.31 0.3  0.29 0.28]
[0.36 0.35 0.33 0.32 0.31 0.29 0.28]
[0.37 0.36 0.34 0.33 0.31 0.3  0.29]
[0.38 0.36 0.35 0.34 0.32 0.31 0.3 ]
[0.39 0.37 0.36 0.34 0.33 0.32 0.31]
[0.4  0.38 0.37 0.35 0.34 0.33 0.31]
[0.41 0.39 0.38 0.36 0.35 0.33 0.32]
[0.42 0.4  0.38 0.37 0.35 0.34 0.33]
[0.43 0.41 0.39 0.38 0.36 0.35 0.34]
[0.44 0.42 0.4  0.39 0.37 0.36 0.34]
[0.45 0.43 0.41 0.39 0.38 0.36 0.35]
[0.46 0.44 0.42 0.4  0.39 0.37 0.36]
[0.47 0.45 0.43 0.41 0.39 0.38 0.37]
[0.48 0.46 0.44 0.42 0.4  0.39 0.37]
[0.49 0.46 0.45 0.43 0.41 0.4  0.38]
[0.5  0.47 0.45 0.44 0.42 0.4  0.39]
[0.51 0.48 0.46 0.44 0.43 0.41 0.4 ]
[0.51 0.49 0.47 0.45 0.43 0.42 0.4 ]
[0.52 0.5  0.48 0.46 0.44 0.43 0.41]
[0.53 0.51 0.49 0.47 0.45 0.43 0.42]
[0.54 0.52 0.5  0.48 0.46 0.44 0.43]
[0.55 0.53 0.51 0.49 0.47 0.45 0.43]
```

```
 [0.56 0.54 0.51 0.49 0.48 0.46 0.44]
 [0.57 0.55 0.52 0.5  0.48 0.46 0.45]
 [0.58 0.56 0.53 0.51 0.49 0.47 0.46]
 [0.59 0.56 0.54 0.52 0.5  0.48 0.46]
 [0.6  0.57 0.55 0.53 0.51 0.49 0.47]
 [0.61 0.58 0.56 0.54 0.52 0.49 0.48]
 [0.62 0.59 0.57 0.54 0.52 0.5  0.48]
 [0.63 0.6  0.58 0.55 0.53 0.51 0.49]
 [0.64 0.61 0.58 0.56 0.54 0.52 0.5 ]
 [0.65 0.62 0.59 0.57 0.55 0.53 0.51]
 [0.66 0.63 0.6  0.58 0.55 0.53 0.52]
 [0.67 0.64 0.61 0.59 0.56 0.54 0.52]
 [0.68 0.65 0.62 0.59 0.57 0.55 0.53]
 [0.69 0.66 0.63 0.6  0.58 0.56 0.54]
 [0.7  0.66 0.64 0.61 0.59 0.56 0.54]
 [0.71 0.67 0.65 0.62 0.59 0.57 0.55]
 [0.71 0.68 0.65 0.63 0.6  0.58 0.56]
 [0.72 0.69 0.66 0.64 0.61 0.59 0.57]
 [0.73 0.7  0.67 0.64 0.62 0.6  0.57]
 [0.74 0.71 0.68 0.65 0.63 0.6  0.58]
 [0.75 0.72 0.69 0.66 0.63 0.61 0.59]
 [0.76 0.73 0.7  0.67 0.64 0.62 0.6 ]
 [0.77 0.74 0.71 0.68 0.65 0.63 0.6 ]
 [0.78 0.75 0.71 0.69 0.66 0.63 0.61]
 [0.79 0.76 0.72 0.69 0.67 0.64 0.62]
 [0.8  0.76 0.73 0.7  0.68 0.65 0.63]
 [0.81 0.77 0.74 0.71 0.68 0.66 0.63]
 [0.82 0.78 0.75 0.72 0.69 0.66 0.64]
 [0.83 0.79 0.76 0.73 0.7  0.67 0.65]
 [0.84 0.8  0.77 0.74 0.71 0.68 0.66]
 [0.85 0.81 0.78 0.74 0.72 0.69 0.66]
 [0.86 0.82 0.79 0.75 0.72 0.7  0.67]
 [0.87 0.83 0.79 0.76 0.73 0.7  0.68]
 [0.88 0.84 0.8  0.77 0.74 0.71 0.69]
 [0.89 0.85 0.81 0.78 0.75 0.72 0.69]
 [0.9  0.86 0.82 0.79 0.76 0.73 0.7 ]
 [0.91 0.87 0.83 0.79 0.76 0.73 0.71]
 [0.91 0.87 0.84 0.8  0.77 0.74 0.72]
 [0.92 0.88 0.85 0.81 0.78 0.75 0.72]
 [0.93 0.89 0.85 0.82 0.79 0.76 0.73]
 [0.94 0.9  0.86 0.83 0.79 0.76 0.74]
 [0.95 0.91 0.87 0.84 0.8  0.77 0.74]]
The value function is: [[-8.10080590e+04 -8.10161181e+04 -8.10241771e+04 -8.10322362e+04
  -8.10402952e+04 -8.10483543e+04 -8.10564133e+04]
 [-8.10080590e+04 -8.10161181e+04 -8.10241771e+04 -8.10322362e+04
  -8.10402952e+04 -8.10483543e+04 -8.10564133e+04]
 [-8.10080590e+04 -8.10161181e+04 -8.10241771e+04 -8.10322362e+04
  -8.10402952e+04 -8.10483543e+04 -8.10564133e+04]
```

```
[-4.87020633e+01 -5.10046483e+01 -5.33072334e+01 -5.56098185e+01
 -5.79124036e+01 -6.02149887e+01 -6.25175738e+01]
[-4.42129004e+01 -4.65154855e+01 -4.88180706e+01 -5.11206557e+01
 -5.34232408e+01 -5.57258259e+01 -5.80284110e+01]
[-4.13997661e+01 -4.37023512e+01 -4.60049362e+01 -4.83075213e+01
 -5.06101064e+01 -5.29126915e+01 -5.52152766e+01]
[-3.90535002e+01 -4.13560853e+01 -4.36586704e+01 -4.59612555e+01
 -4.82638406e+01 -5.05664257e+01 -5.27892615e+01]
[-3.71941021e+01 -3.94966872e+01 -4.17992723e+01 -4.41018574e+01
 -4.64044425e+01 -4.84869841e+01 -5.04429956e+01]
[-3.56390121e+01 -3.79415972e+01 -4.02441822e+01 -4.25467673e+01
 -4.46715745e+01 -4.66275860e+01 -4.85835975e+01]
[-3.42666222e+01 -3.65692073e+01 -3.88717924e+01 -4.11604730e+01
 -4.31164845e+01 -4.50724960e+01 -4.70285075e+01]
[-3.30750699e+01 -3.53776550e+01 -3.76802401e+01 -3.97880832e+01
 -4.17440947e+01 -4.37001062e+01 -4.56093796e+01]
[-3.20130929e+01 -3.43156780e+01 -3.66182631e+01 -3.85965308e+01
 -4.05525423e+01 -4.24837108e+01 -4.42369898e+01]
[-3.10491821e+01 -3.33517672e+01 -3.55785423e+01 -3.75345538e+01
 -3.94905653e+01 -4.12921585e+01 -4.30454374e+01]
[-3.01733137e+01 -3.24758988e+01 -3.46146315e+01 -3.65706430e+01
 -3.84769025e+01 -4.02301815e+01 -4.19834604e+01]
[-2.93617199e+01 -3.16643050e+01 -3.37387631e+01 -3.56947746e+01
 -3.75129917e+01 -3.92662707e+01 -4.09765732e+01]
[-2.86067910e+01 -3.09093761e+01 -3.29271693e+01 -3.48831808e+01
 -3.66371233e+01 -3.83904023e+01 -4.00126624e+01]
[-2.79189507e+01 -3.02162289e+01 -3.21722405e+01 -3.40722506e+01
 -3.58255295e+01 -3.75273561e+01 -3.91367940e+01]
[-2.72684189e+01 -2.95283886e+01 -3.14844001e+01 -3.33173217e+01
 -3.50706007e+01 -3.67157623e+01 -3.83252002e+01]
[-2.66482914e+01 -2.88778568e+01 -3.08338683e+01 -3.26294814e+01
 -3.43513955e+01 -3.59608334e+01 -3.75441978e+01]
[-2.60806278e+01 -2.82577293e+01 -3.02137408e+01 -3.19789496e+01
 -3.36635552e+01 -3.52729931e+01 -3.67892689e+01]
[-2.55314885e+01 -2.76900657e+01 -2.96055431e+01 -3.13588221e+01
 -3.30130234e+01 -3.46035624e+01 -3.61014285e+01]
[-2.50115679e+01 -2.71409264e+01 -2.90378796e+01 -3.07834579e+01
 -3.23928958e+01 -3.39530306e+01 -3.54508968e+01]
[-2.45266338e+01 -2.66210058e+01 -2.84887402e+01 -3.02157944e+01
 -3.18252323e+01 -3.33329031e+01 -3.48127713e+01]
[-2.40526164e+01 -2.61360717e+01 -2.79688196e+01 -2.96666550e+01
 -3.12673734e+01 -3.27652395e+01 -3.41926438e+01]
[-2.36054713e+01 -2.56620543e+01 -2.74838856e+01 -2.91467344e+01
 -3.07182341e+01 -3.22161002e+01 -3.36249802e+01]
[-2.31760277e+01 -2.52149092e+01 -2.70098682e+01 -2.86618004e+01
 -3.01983134e+01 -3.16691355e+01 -3.30758409e+01]
[-2.27639886e+01 -2.47854656e+01 -2.65627231e+01 -2.81877830e+01
 -2.97133794e+01 -3.11492149e+01 -3.25363135e+01]
```

```
[-2.23696903e+01 -2.43734265e+01 -2.61312000e+01 -2.77406379e+01
 -2.92393620e+01 -3.06642809e+01 -3.20163929e+01]
[-2.19879063e+01 -2.39679614e+01 -2.57017564e+01 -2.72943508e+01
 -2.87835581e+01 -3.01902635e+01 -3.15314589e+01]
[-2.16218981e+01 -2.35736631e+01 -2.52897172e+01 -2.68649072e+01
 -2.83364130e+01 -2.97278114e+01 -3.10574415e+01]
[-2.12692207e+01 -2.31918791e+01 -2.48954190e+01 -2.64528680e+01
 -2.79069694e+01 -2.92806663e+01 -3.05900816e+01]
[-2.09226471e+01 -2.28258709e+01 -2.45136349e+01 -2.60585698e+01
 -2.74949303e+01 -2.88512227e+01 -3.01429365e+01]
[-2.05800445e+01 -2.24731935e+01 -2.41476267e+01 -2.56767857e+01
 -2.71006320e+01 -2.84391836e+01 -2.97134929e+01]
[-2.02483746e+01 -2.21305909e+01 -2.37949493e+01 -2.53107775e+01
 -2.67152553e+01 -2.80385894e+01 -2.93012523e+01]
[-1.99306068e+01 -2.17989210e+01 -2.34523468e+01 -2.49461344e+01
 -2.63334713e+01 -2.76442912e+01 -2.88892131e+01]
[-1.96196554e+01 -2.14811532e+01 -2.31176314e+01 -2.45934570e+01
 -2.59674631e+01 -2.72625071e+01 -2.84949149e+01]
[-1.93175309e+01 -2.11702018e+01 -2.27859615e+01 -2.42508545e+01
 -2.56147857e+01 -2.68964989e+01 -2.81131308e+01]
[-1.90277272e+01 -2.08680773e+01 -2.24681937e+01 -2.39191846e+01
 -2.52721831e+01 -2.65431498e+01 -2.77443690e+01]
[-1.87422834e+01 -2.05782736e+01 -2.21572423e+01 -2.36014167e+01
 -2.49383546e+01 -2.61904724e+01 -2.73783608e+01]
[-1.84654965e+01 -2.02905915e+01 -2.18551178e+01 -2.32904653e+01
 -2.46066847e+01 -2.58478698e+01 -2.70256834e+01]
[-1.81983137e+01 -2.00051477e+01 -2.15653141e+01 -2.29821640e+01
 -2.42889169e+01 -2.55161999e+01 -2.66830808e+01]
[-1.79356602e+01 -1.97283609e+01 -2.12798703e+01 -2.26800395e+01
 -2.39779655e+01 -2.51984321e+01 -2.63494952e+01]
[-1.76807537e+01 -1.94611781e+01 -2.10030835e+01 -2.23902358e+01
 -2.36758410e+01 -2.48823506e+01 -2.60178253e+01]
[-1.74315696e+01 -1.91985245e+01 -2.07296011e+01 -2.21047920e+01
 -2.33813835e+01 -2.45713992e+01 -2.57000575e+01]
[-1.71884396e+01 -1.89436180e+01 -2.04624183e+01 -2.18280051e+01
 -2.30915797e+01 -2.42692747e+01 -2.53891061e+01]
[-1.69524203e+01 -1.86944339e+01 -2.01997648e+01 -2.15608223e+01
 -2.28061359e+01 -2.39794710e+01 -2.50845663e+01]
[-1.67196809e+01 -1.84513039e+01 -1.99448583e+01 -2.12937595e+01
 -2.25293491e+01 -2.36935405e+01 -2.47824418e+01]
[-1.64936415e+01 -1.82152847e+01 -1.96956742e+01 -2.10311060e+01
 -2.22621662e+01 -2.34080967e+01 -2.44926381e+01]
[-1.62726903e+01 -1.79825452e+01 -1.94525442e+01 -2.07761995e+01
 -2.19987650e+01 -2.31313098e+01 -2.42071943e+01]
[-1.60551456e+01 -1.77565058e+01 -1.92165249e+01 -2.05270154e+01
 -2.17361114e+01 -2.28641270e+01 -2.39270448e+01]
[-1.58429837e+01 -1.75333623e+01 -1.89837855e+01 -2.02838854e+01
 -2.14812049e+01 -2.26014734e+01 -2.36502579e+01]
```

```
[-1.56361362e+01 -1.73124111e+01 -1.87525594e+01 -2.00478661e+01
 -2.12320208e+01 -2.23404393e+01 -2.33830751e+01]
[-1.54320215e+01 -1.70948663e+01 -1.85265201e+01 -1.98123001e+01
 -2.09888908e+01 -2.20855328e+01 -2.31204216e+01]
[-1.52292889e+01 -1.68827045e+01 -1.83055689e+01 -1.95795606e+01
 -2.07506154e+01 -2.18363487e+01 -2.28610437e+01]
[-1.50291293e+01 -1.66758570e+01 -1.80880241e+01 -1.93535212e+01
 -2.05145961e+01 -2.15932187e+01 -2.26061372e+01]
[-1.48343586e+01 -1.64717423e+01 -1.78758623e+01 -1.91325700e+01
 -2.02818567e+01 -2.13530906e+01 -2.23569531e+01]
[-1.46419558e+01 -1.62715826e+01 -1.76690147e+01 -1.89150253e+01
 -2.00558173e+01 -2.11170713e+01 -2.21138231e+01]
[-1.44534690e+01 -1.60768119e+01 -1.74649001e+01 -1.87028634e+01
 -1.98348661e+01 -2.08843319e+01 -2.18723481e+01]
[-1.42689457e+01 -1.58844091e+01 -1.72646030e+01 -1.84921424e+01
 -1.96173213e+01 -2.06582925e+01 -2.16363288e+01]
[-1.40870996e+01 -1.56959223e+01 -1.70644433e+01 -1.82852949e+01
 -1.93997929e+01 -2.04359686e+01 -2.14035893e+01]
[-1.39090025e+01 -1.55113990e+01 -1.68696726e+01 -1.80811802e+01
 -1.91876310e+01 -2.02150174e+01 -2.11775500e+01]
[-1.37334501e+01 -1.53290774e+01 -1.66772698e+01 -1.78810205e+01
 -1.89807835e+01 -1.99974726e+01 -2.09516651e+01]
[-1.35610925e+01 -1.51472313e+01 -1.64887830e+01 -1.76862498e+01
 -1.87766688e+01 -1.97853108e+01 -2.07307139e+01]
[-1.33923892e+01 -1.49691343e+01 -1.63042597e+01 -1.74938470e+01
 -1.85765091e+01 -1.95783322e+01 -2.05131692e+01]
[-1.32250883e+01 -1.47935818e+01 -1.61224136e+01 -1.73032267e+01
 -1.83764024e+01 -1.93714846e+01 -2.03009830e+01]
[-1.30615679e+01 -1.46212243e+01 -1.59443165e+01 -1.71147399e+01
 -1.81816317e+01 -1.91673699e+01 -2.00888212e+01]
[-1.29008259e+01 -1.44525210e+01 -1.57676420e+01 -1.69302165e+01
 -1.79892289e+01 -1.89672103e+01 -1.98819736e+01]
[-1.27413693e+01 -1.42852201e+01 -1.55920895e+01 -1.67483704e+01
 -1.78007421e+01 -1.87724396e+01 -1.96778589e+01]
[-1.25853413e+01 -1.41216997e+01 -1.54197320e+01 -1.65702734e+01
 -1.76154722e+01 -1.85788240e+01 -1.94776993e+01]
[-1.24317843e+01 -1.39609576e+01 -1.52510287e+01 -1.63947209e+01
 -1.74309489e+01 -1.83864212e+01 -1.92776448e+01]
[-1.22801370e+01 -1.38015011e+01 -1.50837278e+01 -1.62206982e+01
 -1.72491028e+01 -1.81979344e+01 -1.90828741e+01]
[-1.21305817e+01 -1.36454731e+01 -1.49202074e+01 -1.60483406e+01
 -1.70710057e+01 -1.80134111e+01 -1.88904713e+01]
[-1.19837526e+01 -1.34913224e+01 -1.47594653e+01 -1.58796373e+01
 -1.68954532e+01 -1.78315372e+01 -1.87012361e+01]
[-1.18386338e+01 -1.33377653e+01 -1.46000088e+01 -1.57123365e+01
 -1.67229711e+01 -1.76496911e+01 -1.85127493e+01]
[-1.16947928e+01 -1.31861181e+01 -1.44419680e+01 -1.55488160e+01
 -1.65506135e+01 -1.74715941e+01 -1.83282259e+01]
```

```
[-1.15518835e+01 -1.30365628e+01 -1.42859400e+01 -1.53880740e+01
 -1.63819102e+01 -1.72960416e+01 -1.81463798e+01]
[-1.14111095e+01 -1.28897337e+01 -1.41323829e+01 -1.52279886e+01
 -1.62146094e+01 -1.71236841e+01 -1.79668533e+01]
[-1.12719193e+01 -1.27446149e+01 -1.39807357e+01 -1.50685320e+01
 -1.60510889e+01 -1.69522088e+01 -1.77887563e+01]
[-1.11350807e+01 -1.26017056e+01 -1.38311804e+01 -1.49125040e+01
 -1.58897426e+01 -1.67835055e+01 -1.76132038e+01]
[-1.09994966e+01 -1.24609316e+01 -1.36843513e+01 -1.47589470e+01
 -1.57290006e+01 -1.66162047e+01 -1.74408463e+01]
[-1.08661585e+01 -1.23217414e+01 -1.35392325e+01 -1.46072997e+01
 -1.55695440e+01 -1.64526842e+01 -1.72700807e+01]
[-1.07348391e+01 -1.21849028e+01 -1.33962672e+01 -1.44577444e+01
 -1.54135160e+01 -1.62904826e+01 -1.71013774e+01]
[-1.06042515e+01 -1.20493187e+01 -1.32533579e+01 -1.43095284e+01
 -1.52599590e+01 -1.61297405e+01 -1.69340765e+01]
[-1.04760709e+01 -1.19157873e+01 -1.31125840e+01 -1.41626994e+01
 -1.51083117e+01 -1.59702840e+01 -1.67705561e+01]
[-1.03496022e+01 -1.17824493e+01 -1.29733937e+01 -1.40175806e+01
 -1.49567502e+01 -1.58142560e+01 -1.66077360e+01]
[-1.02240129e+01 -1.16511299e+01 -1.28365552e+01 -1.38746713e+01
 -1.48071948e+01 -1.56603761e+01 -1.64469940e+01]
[-1.01005537e+01 -1.15205423e+01 -1.27009710e+01 -1.37338973e+01
 -1.46603658e+01 -1.55068190e+01 -1.62875374e+01]
[-9.97846307e+00 -1.13923616e+01 -1.25676330e+01 -1.35947071e+01
 -1.45152470e+01 -1.53551718e+01 -1.61315094e+01]
[-9.85772198e+00 -1.12658930e+01 -1.24363136e+01 -1.34567213e+01
 -1.43723377e+01 -1.52056165e+01 -1.59759283e+01]
[-9.73849894e+00 -1.11403037e+01 -1.23057260e+01 -1.33198828e+01
 -1.42294416e+01 -1.50587874e+01 -1.58223712e+01]
[-9.62063755e+00 -1.10168444e+01 -1.21752089e+01 -1.31842986e+01
 -1.40886677e+01 -1.49124169e+01 -1.56707240e+01]
[-9.50417119e+00 -1.08947538e+01 -1.20470283e+01 -1.30509606e+01
 -1.39494774e+01 -1.47672981e+01 -1.55211686e+01]
[-9.38900925e+00 -1.07740127e+01 -1.19205596e+01 -1.29196412e+01
 -1.38126389e+01 -1.46243888e+01 -1.53722100e+01]
[-9.27521484e+00 -1.06547897e+01 -1.17949703e+01 -1.27890536e+01
 -1.36770547e+01 -1.44836149e+01 -1.52253809e+01]
[-9.16256586e+00 -1.05369283e+01 -1.16715111e+01 -1.26599766e+01
 -1.35418867e+01 -1.43440548e+01 -1.50802621e+01]
[-9.05099408e+00 -1.04191452e+01 -1.15494205e+01 -1.25317959e+01
 -1.34085486e+01 -1.42048646e+01 -1.49373528e+01]
[-8.93982659e+00 -1.03026789e+01 -1.14286794e+01 -1.24053272e+01
 -1.32772292e+01 -1.40680260e+01 -1.47944758e+01]
[-8.82941351e+00 -1.01875169e+01 -1.13086153e+01 -1.22797379e+01
 -1.31466416e+01 -1.39324419e+01 -1.46537019e+01]
[-8.72056413e+00 -1.00737225e+01 -1.11893923e+01 -1.21562787e+01
 -1.30184084e+01 -1.37990866e+01 -1.45145116e+01]
```

```
 [-8.61307430e+00 -9.96107355e+00 -1.10715309e+01 -1.20341881e+01
  -1.28902277e+01 -1.36657485e+01 -1.43772391e+01]]
```

```
In [31]: delta3 = distance(V_init, V_new)
         print("The distance metric is:", delta3)
```

```
The distance metric is: 45934472596.166245
```

```
In [32]: diff3 = delta3-delta2
         print("The difference between delta_{T-2} and delta_{T-1} is: ", diff3)
```

```
The difference between delta_{T-2} and delta_{T-1} is:  -18683055.693359375
```

```
In [33]: diff4 = delta3-delta1
         print("The difference between delta_{T-2} and delta_{T-1} is: ", diff4)
```

```
The difference between delta_{T-2} and delta_{T-1} is:  -29098599.93926239
```

delta_{T-2} is smaller than delta_{T-1} and delta_{T-2} is much smaller than delta_T, Which means distance gets smaller from T-1 to T-2.

Exercise 5.21

```
In [34]: maxiters = 500
         toler = 1e-9
         delta = 10.0
         VF_iter = 0
         V_init = np.zeros((N,M))

         while delta>toler and VF_iter<maxiters:
             VF_iter += 1
             EV = (V_init @ gamma).reshape((N,1))
             EV_mat = np.tile(EV.reshape((1,N)), (N,1))
             EV_mat[~c_pos] = -9e+4
             EV_TDarray = np.array([EV_mat for e in range(M)])
             V_new_TDarray = Three_D_array + beta*EV_TDarray
             V_new = np.zeros((N,M))
             W_prime = np.zeros((N,M))
             for i in range(N):
                 arr = V_new_TDarray[:, i, :]
                 V_new[i] = arr.max(axis=1)
                 W_index = np.argmax(arr, axis=1)
                 W_prime[i] = W_vec[W_index]
             delta = distance(V_init, V_new)
             V_init = V_new
```

```
            print('Iter=', VF_iter, ', distance= ', delta)

        print("Yay! It converged.")
        print("psi(W) is", W_prime)
        print("V(W) is", V_init)
        print("After {} times of iterations, V(W) converged.".format(VF_iter))
```

```
Iter= 1 , distance=  45963571196.10551
Iter= 2 , distance=  45953155651.859604
Iter= 3 , distance=  45934472596.166245
Iter= 4 , distance=  45901033774.70888
Iter= 5 , distance=  45841440579.44991
Iter= 6 , distance=  45736060684.31199
Iter= 7 , distance=  45552402370.20611
Iter= 8 , distance=  45241165831.95568
Iter= 9 , distance=  44743295702.61597
Iter= 10 , distance=  44048017695.89585
Iter= 11 , distance=  43443854297.730034
Iter= 12 , distance=  44442974185.644424
Iter= 13 , distance=  52967967667.04762
Iter= 14 , distance=  89997648384.71588
Iter= 15 , distance=  226554825804.06644
Iter= 16 , distance=  457375601904.8917
Iter= 17 , distance=  192167632805.63055
Iter= 18 , distance=  0.0
Yay! It converged.
psi(W) is [[0.01 0.01 0.01 0.01 0.01 0.01 0.01]
 [0.02 0.02 0.02 0.02 0.02 0.02 0.02]
 [0.03 0.03 0.03 0.03 0.03 0.03 0.03]
 [0.04 0.04 0.04 0.04 0.04 0.04 0.04]
 [0.05 0.05 0.05 0.05 0.05 0.05 0.05]
 [0.06 0.06 0.06 0.06 0.06 0.06 0.06]
 [0.07 0.07 0.07 0.07 0.07 0.07 0.07]
 [0.08 0.08 0.08 0.08 0.08 0.08 0.08]
 [0.09 0.09 0.09 0.09 0.09 0.09 0.09]
 [0.1  0.1  0.1  0.1  0.1  0.1  0.1 ]
 [0.11 0.11 0.11 0.11 0.11 0.11 0.11]
 [0.12 0.12 0.12 0.12 0.12 0.12 0.12]
 [0.13 0.13 0.13 0.13 0.13 0.13 0.13]
 [0.14 0.14 0.14 0.14 0.14 0.14 0.14]
 [0.15 0.15 0.15 0.15 0.15 0.15 0.15]
 [0.16 0.16 0.16 0.16 0.16 0.16 0.16]
 [0.17 0.17 0.17 0.17 0.17 0.17 0.17]
 [0.18 0.18 0.18 0.18 0.18 0.18 0.18]
 [0.19 0.19 0.19 0.19 0.19 0.19 0.19]
 [0.2  0.2  0.2  0.2  0.2  0.2  0.2 ]
 [0.21 0.21 0.21 0.21 0.21 0.21 0.21]
 [0.22 0.22 0.22 0.22 0.22 0.22 0.22]
```

```
[0.23 0.23 0.23 0.23 0.23 0.23 0.23]
[0.24 0.24 0.24 0.24 0.24 0.24 0.24]
[0.25 0.25 0.25 0.25 0.25 0.25 0.25]
[0.26 0.26 0.26 0.26 0.26 0.26 0.26]
[0.27 0.27 0.27 0.27 0.27 0.27 0.27]
[0.28 0.28 0.28 0.28 0.28 0.28 0.28]
[0.29 0.29 0.29 0.29 0.29 0.29 0.29]
[0.3  0.3  0.3  0.3  0.3  0.3  0.3 ]
[0.31 0.31 0.31 0.31 0.31 0.31 0.31]
[0.32 0.32 0.32 0.32 0.32 0.32 0.32]
[0.33 0.33 0.33 0.33 0.33 0.33 0.33]
[0.34 0.34 0.34 0.34 0.34 0.34 0.34]
[0.35 0.35 0.35 0.35 0.35 0.35 0.35]
[0.36 0.36 0.36 0.36 0.36 0.36 0.36]
[0.37 0.37 0.37 0.37 0.37 0.37 0.37]
[0.38 0.38 0.38 0.38 0.38 0.38 0.38]
[0.39 0.39 0.39 0.39 0.39 0.39 0.39]
[0.4  0.4  0.4  0.4  0.4  0.4  0.4 ]
[0.41 0.41 0.41 0.41 0.41 0.41 0.41]
[0.42 0.42 0.42 0.42 0.42 0.42 0.42]
[0.43 0.43 0.43 0.43 0.43 0.43 0.43]
[0.44 0.44 0.44 0.44 0.44 0.44 0.44]
[0.45 0.45 0.45 0.45 0.45 0.45 0.45]
[0.46 0.46 0.46 0.46 0.46 0.46 0.46]
[0.47 0.47 0.47 0.47 0.47 0.47 0.47]
[0.48 0.48 0.48 0.48 0.48 0.48 0.48]
[0.49 0.49 0.49 0.49 0.49 0.49 0.49]
[0.5  0.5  0.5  0.5  0.5  0.5  0.5 ]
[0.51 0.51 0.51 0.51 0.51 0.51 0.51]
[0.52 0.52 0.52 0.52 0.52 0.52 0.52]
[0.53 0.53 0.53 0.53 0.53 0.53 0.53]
[0.54 0.54 0.54 0.54 0.54 0.54 0.54]
[0.55 0.55 0.55 0.55 0.55 0.55 0.55]
[0.56 0.56 0.56 0.56 0.56 0.56 0.56]
[0.57 0.57 0.57 0.57 0.57 0.57 0.57]
[0.58 0.58 0.58 0.58 0.58 0.58 0.58]
[0.59 0.59 0.59 0.59 0.59 0.59 0.59]
[0.6  0.6  0.6  0.6  0.6  0.6  0.6 ]
[0.61 0.61 0.61 0.61 0.61 0.61 0.61]
[0.62 0.62 0.62 0.62 0.62 0.62 0.62]
[0.63 0.63 0.63 0.63 0.63 0.63 0.63]
[0.64 0.64 0.64 0.64 0.64 0.64 0.64]
[0.65 0.65 0.65 0.65 0.65 0.65 0.65]
[0.66 0.66 0.66 0.66 0.66 0.66 0.66]
[0.67 0.67 0.67 0.67 0.67 0.67 0.67]
[0.68 0.68 0.68 0.68 0.68 0.68 0.68]
[0.69 0.69 0.69 0.69 0.69 0.69 0.69]
[0.7  0.7  0.7  0.7  0.7  0.7  0.7 ]
```

```
 [0.71 0.71 0.71 0.71 0.71 0.71 0.71]
 [0.72 0.72 0.72 0.72 0.72 0.72 0.72]
 [0.73 0.73 0.73 0.73 0.73 0.73 0.73]
 [0.74 0.74 0.74 0.74 0.74 0.74 0.74]
 [0.75 0.75 0.75 0.75 0.75 0.75 0.75]
 [0.76 0.76 0.76 0.76 0.76 0.76 0.76]
 [0.77 0.77 0.77 0.77 0.77 0.77 0.77]
 [0.78 0.78 0.78 0.78 0.78 0.78 0.78]
 [0.79 0.79 0.79 0.79 0.79 0.79 0.79]
 [0.8  0.8  0.8  0.8  0.8  0.8  0.8 ]
 [0.81 0.81 0.81 0.81 0.81 0.81 0.81]
 [0.82 0.82 0.82 0.82 0.82 0.82 0.82]
 [0.83 0.83 0.83 0.83 0.83 0.83 0.83]
 [0.84 0.84 0.84 0.84 0.84 0.84 0.84]
 [0.85 0.85 0.85 0.85 0.85 0.85 0.85]
 [0.86 0.86 0.86 0.86 0.86 0.86 0.86]
 [0.87 0.87 0.87 0.87 0.87 0.87 0.87]
 [0.88 0.88 0.88 0.88 0.88 0.88 0.88]
 [0.89 0.89 0.89 0.89 0.89 0.89 0.89]
 [0.9  0.9  0.9  0.9  0.9  0.9  0.9 ]
 [0.91 0.91 0.91 0.91 0.91 0.91 0.91]
 [0.92 0.92 0.92 0.92 0.92 0.92 0.92]
 [0.93 0.93 0.93 0.93 0.93 0.93 0.93]
 [0.94 0.94 0.94 0.94 0.94 0.94 0.94]
 [0.95 0.95 0.95 0.95 0.95 0.95 0.95]
 [0.96 0.96 0.96 0.96 0.96 0.96 0.96]
 [0.97 0.97 0.97 0.97 0.97 0.97 0.97]
 [0.98 0.98 0.98 0.98 0.98 0.98 0.98]
 [0.99 0.99 0.99 0.99 0.99 0.99 0.99]
 [1.   1.   1.   1.   1.   1.   1.  ]]
V(W) is [[-81008.05904783 -81016.11809565 -81024.17714348 -81032.2361913
  -81040.29523913 -81048.35428695 -81056.41333478]
 [-81008.05904783 -81016.11809565 -81024.17714348 -81032.2361913
  -81040.29523913 -81048.35428695 -81056.41333478]
 [-81008.05904783 -81016.11809565 -81024.17714348 -81032.2361913
  -81040.29523913 -81048.35428695 -81056.41333478]
 [-81008.05904783 -81016.11809565 -81024.17714348 -81032.2361913
  -81040.29523913 -81048.35428695 -81056.41333478]
 [-81008.05904783 -81016.11809565 -81024.17714348 -81032.2361913
  -81040.29523913 -81048.35428695 -81056.41333478]
 [-81008.05904783 -81016.11809565 -81024.17714348 -81032.2361913
  -81040.29523913 -81048.35428695 -81056.41333478]
 [-81008.05904783 -81016.11809565 -81024.17714348 -81032.2361913
  -81040.29523913 -81048.35428695 -81056.41333478]
 [-81008.05904783 -81016.11809565 -81024.17714348 -81032.2361913
  -81040.29523913 -81048.35428695 -81056.41333478]
 [-81008.05904783 -81016.11809565 -81024.17714348 -81032.2361913
  -81040.29523913 -81048.35428695 -81056.41333478]
```

```
[-81008.05904783 -81016.11809565 -81024.17714348 -81032.2361913
 -81040.29523913 -81048.35428695 -81056.41333478]
[-81008.05904783 -81016.11809565 -81024.17714348 -81032.2361913
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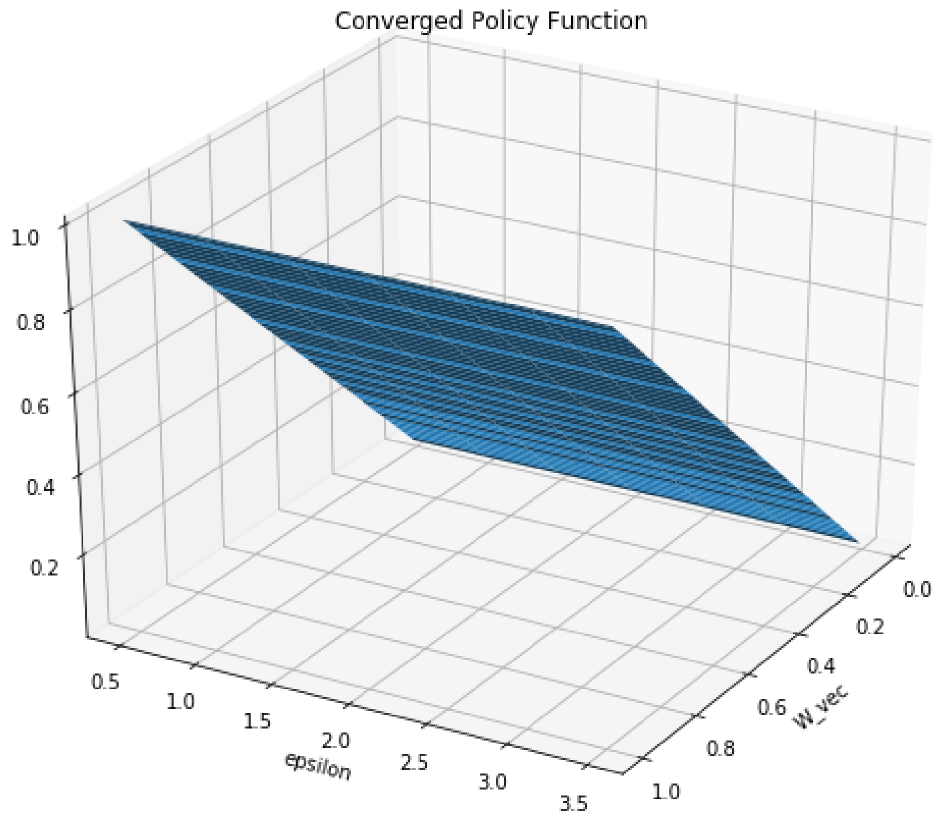
After 18 times of iterations, V(W) converged.
```

Exercise 5.22

```
In [35]: X, Y = np.meshgrid(W_vec, epsilon)
         fig = plt.figure(figsize=(10, 8))
         ax = fig.add_subplot(111, projection='3d')
         ax.plot_surface(X.T, Y.T, W_prime)
         ax.set_xlabel('W_vec')
```

```
ax.set_ylabel('epsilon')
ax.set_title('Converged Policy Function')
ax.view_init(elev=30,azim=30)
plt.show()
```




```
In [ ]:
```