Problem Set #3

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Exercise 5.1

The condition that characterizes the optimal amount of cake to eat in period 1 is :

$$\max_{c_1 \in [0, W_1]} u\left(c_1\right) \quad s.t. \quad W_2 = W_1 - c_1$$

The condition for the optimal amount of cake to save for the next period W_2 is:

$$\max_{W_2 \in [0, W_1]} u\left(W_1 - W_2\right)$$

In order to maximize utility, we know that If the individual lives for one period, the optimal decision is: $c_1 = W_1$ and $W_2 = 0$.

Exercise 5.2

The condition for the optimal amount of cake to leave for the next period W_3 in period 2 is:

$$\max_{W_3 \in [0, W_2]} u\left(W_2 - W_3\right)$$

In order to maximize utility, we know that If the individual lives for two period, in period 2, the optimal decision is: : $c_2 = W_2$ and $W_3 = 0$.

The condition for the optimal amount of cake leave for the next period W_2 in period 1 is:

$$\max_{W_2 \in [0, W_1]} \left[u\left(W_1 - W_2\right) + \beta \cdot \max_{W_3 \in [0, W_2]} u\left(W_2 - W_3\right)\right]$$

Since we know $W_3 = 0$, the formular above is:

$$\max_{W_2 \in [0, W_1]} \left[u\left(W_1 - W_2\right) + \beta \cdot u\left(W_2\right)\right]$$

Then we get the first order condition of period 1:

$$u'\left(W_1 - W_2\right) = \beta \cdot u'\left(W_2\right)$$

If the utility function and W_1 are known, we can know what W_2 is, which means $W_2 = \psi_1(W_1)$.

Exercise 5.3

The condition for the optimal amount of cake to leave for the next period W_4 in period 3 is:

$$\max_{W_4 \in [0, W_3]} u\left(W_3 - W_4\right)$$

In order to maximize utility, we know that if the individual lives for three period, in period 3, $c_3 = W_3$ and $W_4 = 0$

The condition for the optimal amount of cake leave for the next period W_3 in period 2 is:

$$\max_{W_3 \in [0, W_2]} \left[u\left(W_2 - W_3\right) + \beta \cdot \max_{W_4 \in [0, W_3]} u\left(W_3 - W_4\right) \right]$$

Since we know in last period, $W_4 = 0$, the formular above is:

$$\max_{W_3 \in [0, W_2]} \left[u\left(W_2 - W_3\right) + \beta \cdot u\left(W_3\right) \right]$$

Then we get the first order condition of period 2:

$$u'\left(W_2 - W_3\right) = \beta \cdot u'\left(W_3\right)$$

Then we can get $W_3 = \psi_2(W_2)$

The condition for the optimal amount of cake leave for the next period W_2 in period 1 is:

$$\max_{W_2 \in [0, W_1]} \left[u\left(W_1 - W_2\right) + \beta \cdot \max_{W_3 \in [0, W_2]} u\left(W_2 - W_3\right) + \beta^2 \cdot \max_{W_4 \in [0, W_3]} u\left(W_3 - W_4\right) \right]$$

Since we know $W_4 = 0$ and $W_3 = \psi_2(W_2)$, the formular above is:

$$\max_{W_2 \in [0, W_1]} \left[u\left(W_1 - W_2\right) + \beta \cdot u\left(W_2 - \psi_2(W_2)\right) + \beta^2 \cdot u\left(\psi_2(W_2)\right) \right]$$

Then we get the first order condition for period 1:

$$\begin{aligned} u'\left(W_1 - W_2\right) &= \beta \cdot u'\left(W_2 - \psi_2(W_2)\right) \cdot \left(1 - \psi_2'(W_2)\right) + \beta^2 \cdot u'(\psi_2(W_2)) \cdot \psi_2'(W_2) \\ &= \beta \cdot u'\left(W_2 - W_3\right) \cdot \left(1 - \psi_2'(W_2)\right) + \beta^2 \cdot u'(W_3) \cdot \psi_2'(W_2) \\ &= \beta \cdot u'\left(W_2 - W_3\right) + \beta \cdot \psi_2'(W_2) \cdot \left[-u'(W_2 - W_3) + \beta \cdot u'(W_3)\right] \\ &= \beta \cdot u'\left(W_2 - W_3\right) \end{aligned}$$

If the initial cake size is $W_1 = 1$, the discount factor is $\beta = 0.9$, and the period utility function is $ln(c_t)$, we can solve the equations above and get $W_1 = 1$, $W_2 = 0.631$, $W_3 = 0.299$, and $W_4 = 0$. And $c_1 = 0.369$, $c_2 = 0.332$, $c_3 = 0.299$.

The evolvement of $\{c_t\}_{t=1}^{3}$ and $\{W_t\}_{t=1}^{4}$ over the three periods shows in the Jupyter-Notebook.

Exercise 5.4

The condition that characterizes the optimal choice (the policy function) in period T-1 for $W_T = \psi_{T-1}\left(W_{T-1}\right)$ is :

$$\max_{W_T \in [0, W_{T-1}]} [u(W_{T-1} - W_T) + \beta \cdot u(W_T)]$$

Then we get the first order condition:

$$u'(W_{T-1} - W_T) = \beta \cdot u'(W_T)$$

$$u'(W_{T-1} - \psi_{T-1}(W_{T-1})) = \beta \cdot u'(\psi_{T-1}(W_{T-1}))$$

The value function V_{T-1} is:

$$V_{T-1}(W_{T-1}) = u(W_{T-1} - \psi_{T-1}(W_{T-1})) + \beta \cdot u(\psi_{T-1}(W_{T-1}))$$

Exercise 5.5

Since $u(c) = ln(c)$,
in period T, we know that

$$V_T(\overline{W}) = u(\overline{W}) = ln(\overline{W})$$

And

$$\psi_T(\overline{W}) = 0$$

In period T-1, from the equation in Exercise 5.4, we can get

$$\psi_{T-1}(\overline{W}) = \frac{\beta}{1+\beta} \cdot \overline{W}$$

And

$$V_{T-1}(\overline{W}) = ln(\frac{1}{1+\beta}\overline{W}) + \beta \cdot ln(\frac{\beta}{1+\beta}\overline{W})$$

Thus they are not equal.

Exercise 5.6 In period T-2,

$$\max_{W_{T-1}\in[0,W_{T-2}]} \left[u\left(W_{T-2} - W_{T-1}\right) + \beta \cdot u\left(W_{T-1} - \frac{\beta}{1+\beta} \cdot W_{T-1}\right) + \beta^2 \cdot u(\frac{\beta}{1+\beta} \cdot W_{T-1}) \right]$$

Then using the envelope theorem(like the equation in Exercise 5.3), we can get the first order condition:

$$u'(W_{T-2} - W_{T-1}) = \beta \cdot u'(W_{T-1} - \frac{\beta}{1+\beta} \cdot W_{T-1}) = \beta \cdot u'(\frac{1}{1+\beta} \cdot W_{T-1})$$

Since $u(c) = ln(c)$, from the above equation we can get:

$$W_{T-1} = \psi_{T-2}(W_{T-2}) = \frac{\beta+\beta^2}{1+\beta+\beta^2} \cdot W_{T-2}$$

and

$$V_{T-2} = ln(\frac{1}{1+\beta+\beta^2} \cdot W_{T-2}) + \beta \cdot ln(\frac{\beta}{1+\beta+\beta^2} \cdot W_{T-2}) + \beta^2 \cdot ln(\frac{\beta^2}{1+\beta+\beta^2} \cdot W_{T-2})$$

Exercise 5.7 For the general integer $s \geq 1$ using induction, we can get:

$$\psi_{T-s}(W_{T-s}) = \frac{\sum\limits_{i=1}^{s} \beta^i}{\sum\limits_{j=0}^{s} \beta^j} W_{T-s}$$

$$V_{T-s}(W_{T-s}) = \sum_{i=0}^{s} \beta^i ln\left(\frac{\beta^i \cdot W_{T-s}}{\sum\limits_{j=0}^{s} \beta^j}\right)$$

As s becomes infinite,

$$\psi_{T-s}(W_{T-s}) = \psi(W_{T-s}) = \frac{\frac{\beta}{1-\beta}}{\frac{1}{1-\beta}} \cdot W_{T-s} = \beta \cdot W_{T-s}$$

$$\begin{aligned} V_{T-s}(W_{T-s}) = V(W_{T-s}) &= \sum_{i=0}^{s} \beta^i ln(\beta^i) + \sum_{i=0}^{s} \beta^i ln(\frac{1}{\sum\limits_{j=0}^{s} \beta^j}) + \sum_{i=0}^{s} \beta^i ln(W_{T-s}) \\ &= \frac{\beta}{(1-\beta)^2} ln(\beta) + \frac{1}{1-\beta} ln(1-\beta) + \frac{1}{1-\beta} ln(W_{T-s}) \end{aligned}$$

As the horizon becomes further and further away (infinite), the value function and policy function become independent of time.

Exercise 5.8

$$V(W) \equiv \max_{W' \in [0,W]} u(W - W') + \beta V(W')$$

W' is cake to leave for the next period.