ProblemSet2_Q1

January 21, 2019

0.0.1 Problem Set #2

MACS 30150, Dr. Evans

Due Monday, Jan. 21 at 11:30am

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1. ACME: Numerical Differentiation lab.

```
In [1]: import sympy as sy
        import numpy as np
        import pandas as pd
        from matplotlib import pyplot as plt
        from autograd import grad
        from autograd import numpy as anp
        import math
        import time
        import warnings
```

Problem 1

```
In [2]: #define the function
        x = sy.symbols('x')
        sin = sy.sin
        cos = sy.cos

        def y(x):
            y = (sin(x)+1)**sin(cos(x))
            return y

        print("The function is: f(x) = {}".format(y(x)))
```

```
The function is: f(x) = (sin(x) + 1)**sin(cos(x))
```

```
In [3]: #take the function's symbolic derivative with respect to x
        yprime = sy.diff(y(x),x)
        print("The symbolic derivative of the function with respect to x is: ")
        print(yprime)
```

```
The symbolic derivative of the function with respect to x is:
(-log(sin(x) + 1)*sin(x)*cos(cos(x)) + sin(cos(x))*cos(x)/(sin(x) + 1))*(sin(x) + 1)**sin(cos(
```

```
In [4]: #Lambdify the resulting function
        x_vec = np.linspace(-math.pi, math.pi, 100)
        f = sy.lambdify(x, y(x), "numpy")
        fprime = sy.lambdify(x, yprime, "numpy")
        f(x_vec).round(2)
```

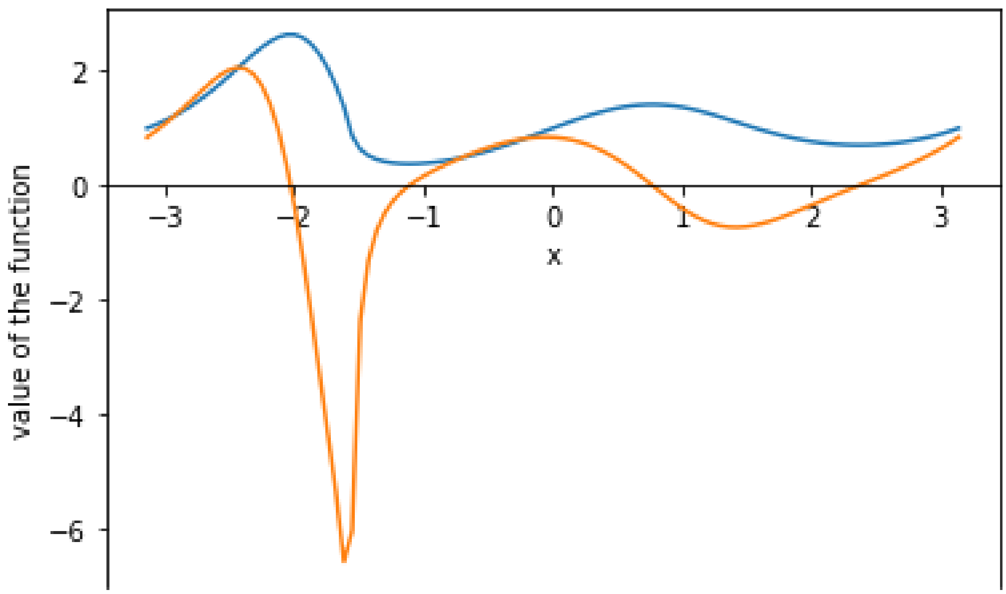
```
Out[4]: array([1.  , 1.06, 1.12, 1.19, 1.27, 1.36, 1.45, 1.55, 1.67, 1.79, 1.91,
               2.04, 2.17, 2.3 , 2.42, 2.52, 2.6 , 2.64, 2.64, 2.59, 2.48, 2.3 ,
               2.06, 1.76, 1.38, 0.87, 0.63, 0.52, 0.46, 0.42, 0.39, 0.38, 0.38,
               0.38, 0.39, 0.4 , 0.42, 0.45, 0.47, 0.51, 0.54, 0.58, 0.62, 0.67,
               0.71, 0.76, 0.81, 0.87, 0.92, 0.97, 1.03, 1.08, 1.13, 1.18, 1.23,
               1.27, 1.31, 1.34, 1.37, 1.39, 1.41, 1.41, 1.42, 1.41, 1.4 , 1.38,
               1.35, 1.32, 1.28, 1.24, 1.19, 1.15, 1.1 , 1.06, 1.01, 0.97, 0.93,
               0.89, 0.85, 0.82, 0.79, 0.77, 0.75, 0.73, 0.72, 0.71, 0.71, 0.71,
               0.71, 0.71, 0.72, 0.74, 0.76, 0.78, 0.8 , 0.83, 0.87, 0.91, 0.95,
               1.  ])
```

```
In [5]: fprime(x_vec).round(2)
```

```
Out[5]: array([ 0.84,  0.94,  1.05,  1.17,  1.3 ,  1.43,  1.57,  1.7 ,  1.83,
                1.94,  2.02,  2.06,  2.04,  1.94,  1.75,  1.43,  0.97,  0.37,
               -0.38, -1.26, -2.24, -3.27, -4.3 , -5.33, -6.59, -6.05, -2.36,
               -1.31, -0.79, -0.48, -0.27, -0.12,  0.  ,  0.1 ,  0.19,  0.26,
                0.33,  0.4 ,  0.46,  0.52,  0.58,  0.63,  0.68,  0.72,  0.76,
                0.79,  0.82,  0.83,  0.84,  0.84,  0.84,  0.82,  0.79,  0.75,
                0.7 ,  0.64,  0.57,  0.49,  0.4 ,  0.3 ,  0.19,  0.07, -0.04,
               -0.16, -0.27, -0.37, -0.47, -0.55, -0.62, -0.67, -0.71, -0.73,
               -0.73, -0.72, -0.7 , -0.67, -0.63, -0.58, -0.52, -0.47, -0.41,
               -0.35, -0.29, -0.23, -0.17, -0.11, -0.05,  0.01,  0.07,  0.12,
                0.18,  0.24,  0.3 ,  0.37,  0.43,  0.5 ,  0.58,  0.66,  0.75,
                0.84])
```

```
In [6]: #plot the function
        ax = plt.gca()
        ax.spines["bottom"].set_position("zero")

        plt.plot(x_vec, f(x_vec))
        plt.plot(x_vec, fprime(x_vec))
        plt.xlabel("x")
        plt.ylabel("value of the function")
```

Out[6]: Text(0, 0.5, 'value of the function')



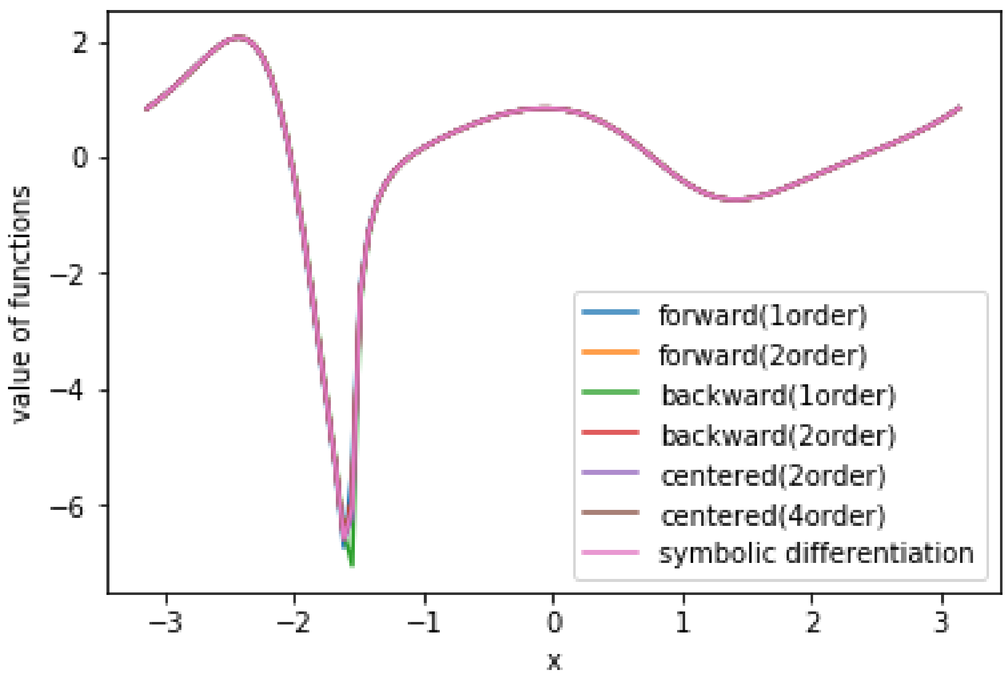
Problem 2

In [7]:
```
#define functions of finite difference quotients

def for_diff1(fn, x, h):
    quotient = (fn(x+h)-fn(x))/h
    return quotient

def for_diff2(fn, x, h):
    quotient = (-3*fn(x)+4*fn(x+h)-f(x+2*h))/(2*h)
    return quotient

def back_diff1(fn, x, h):
    quotient = (fn(x)-fn(x-h))/h
    return quotient

def back_diff2(fn, x, h):
    quotient = (-3*fn(x)+4*fn(x+h)-f(x+2*h))/(2*h)
    return quotient

def cen_diff1(fn, x, h):
    quotient =  (fn(x+h)-fn(x-h))/(2*h)
    return quotient

def cen_diff2(fn, x, h):
```

```
        quotient =  (fn(x-2*h)-8*fn(x-h)+8*fn(x+h)-fn(x+2*h))/(12*h)
        return quotient
```

```
In [8]: #plot the functions
        plt.plot(x_vec, for_diff1(f, x_vec, 0.01), label = "forward(1order)")
        plt.plot(x_vec, for_diff2(f, x_vec, 0.01), label = "forward(2order)")
        plt.plot(x_vec, back_diff1(f, x_vec, 0.01), label = "backward(1order)")
        plt.plot(x_vec, back_diff2(f, x_vec, 0.01), label = "backward(2order)")
        plt.plot(x_vec, cen_diff1(f, x_vec, 0.01), label = "centered(2order)")
        plt.plot(x_vec, cen_diff2(f, x_vec, 0.01), label = "centered(4order)")

        plt.plot(x_vec, fprime(x_vec), label = "symbolic differentiation")

        plt.legend()
        plt.xlabel("x")
        plt.ylabel("value of functions")
```

```
Out[8]: Text(0, 0.5, 'value of functions')
```



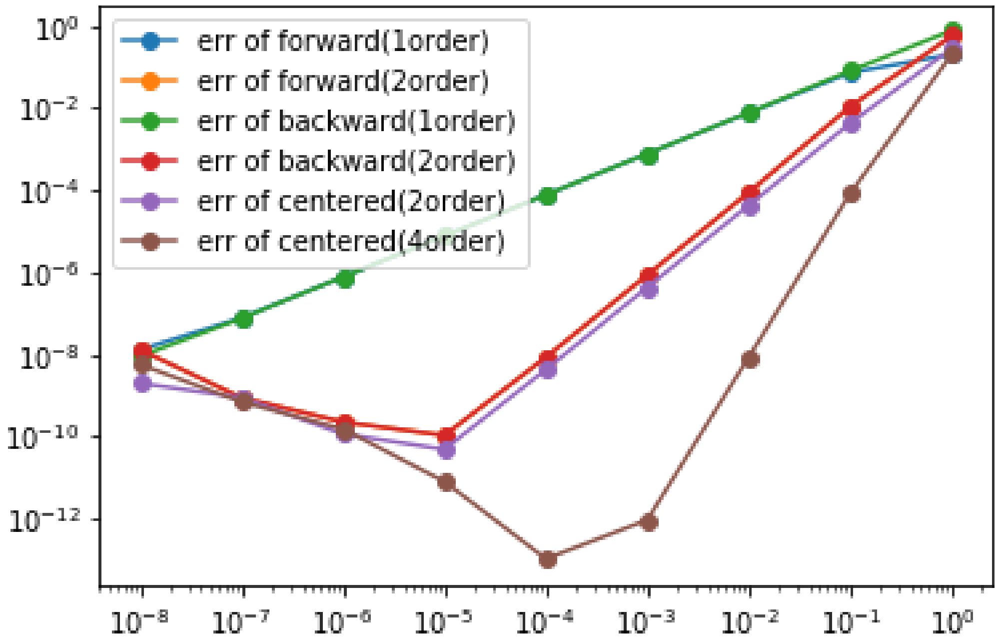
From the above plots, we can see that the result of symbolic differentiation and the result of numerical differentiation are very similar.

Problem 3

```
In [9]: #Write a function that accepts a point x0 at which to compute the derivative--The func
```

```python
#compute the exact value of f_prime at point x0
x0 = 1
f_prime = fprime(x0)

#get the approximate derivatives
h_vec=np.logspace(-8, 0, 9)
app_deri1 = for_diff1(f, x0, h_vec)
app_deri2 = for_diff2(f, x0, h_vec)
app_deri3 = back_diff1(f, x0, h_vec)
app_deri4 = back_diff2(f, x0, h_vec)
app_deri5 = cen_diff1(f, x0, h_vec)
app_deri6 = cen_diff2(f, x0, h_vec)

#track the absolute error
abs_error1 = abs(app_deri1 - f_prime)
abs_error2 = abs(app_deri2 - f_prime)
abs_error3 = abs(app_deri3 - f_prime)
abs_error4 = abs(app_deri4 - f_prime)
abs_error5 = abs(app_deri5 - f_prime)
abs_error6 = abs(app_deri6 - f_prime)

#plot the absolute error against h

plt.plot(h_vec, abs_error1, marker = 'o', label='err of forward(1order)')
plt.plot(h_vec, abs_error2, marker = 'o', label='err of forward(2order)')
plt.plot(h_vec, abs_error3, marker = 'o', label='err of backward(1order)')
plt.plot(h_vec, abs_error4, marker = 'o', label='err of backward(2order)')
plt.plot(h_vec, abs_error5, marker = 'o',  label='err of centered(2order)')
plt.plot(h_vec, abs_error6, marker = 'o', label='err of centered(4order)')
plt.loglog()
plt.legend()
```

Out[9]: <matplotlib.legend.Legend at 0x12019b9b0>



Problem 4

```
In [10]: #Load the data
         radar_data = np.load("plane.npy")
         radar_df = pd.DataFrame(radar_data, columns=['t','alpha','beta'])
```

```
In [11]: #Convert degrees to radians
         radar_df['alpha']=np.deg2rad(radar_df['alpha'])
         radar_df['beta']=np.deg2rad(radar_df['beta'])
```

```
In [12]: # Calculate the Cartesian coordinates of the plane
         a = 500
         radar_df["x"] = (a*np.tan(radar_df['beta'])) / (np.tan(radar_df['beta'])-np.tan(radar_
         radar_df["y"] = (a*np.tan(radar_df['beta']) * np.tan(radar_df['alpha'])) / (np.tan(rad
```

```
In [13]: warnings.filterwarnings("ignore")

         #Approxiamate x_prime and y_prime
         radar_df["x_prime"]= None
         radar_df["y_prime"]= None

         # Use forward difference quotient1 for t = 7
         radar_df["x_prime"][0] = radar_df["x"][1]-radar_df["x"][0]
         radar_df["y_prime"][0] = radar_df["y"][1]-radar_df["y"][0]

         #Use backward difference quotient1 for t = 14
```

```
radar_df["x_prime"][7] = radar_df["x"][7]-radar_df["x"][6]
radar_df["y_prime"][7] = radar_df["y"][7]-radar_df["y"][6]

# Use centered difference quotient1 for t = 8, 9, . . . , 13
for i in range(1, 7):
    radar_df["x_prime"][i] = (radar_df["x"][i + 1]-radar_df["x"][i - 1])/2
    radar_df["y_prime"][i] = (radar_df["y"][i + 1]-radar_df["y"][i - 1])/2

#Calculate the values of the speed
radar_df['speed']=(radar_df["x_prime"]**2+radar_df["y_prime"]**2)**0.5

radar_df
```

Out[13]:

```
      t     alpha      beta            x            y  x_prime  y_prime  \
0   7.0  0.981748  1.178795  1311.271337  1962.456239  44.6651  12.6583
1   8.0  0.969181  1.161866  1355.936476  1975.114505  45.3235  12.4449
2   9.0  0.956440  1.144761  1401.918398  1987.346016  47.2803  12.8631
3  10.0  0.943525  1.127308  1450.497006  2000.840713   48.361  13.0832
4  11.0  0.930959  1.110378  1498.640350  2013.512411   46.651  12.4758
5  12.0  0.919614  1.095020  1543.798955  2025.792234  49.7005  13.7391
6  13.0  0.906524  1.077217  1598.041382  2040.990583  51.8986  14.6367
7  14.0  0.895005  1.061509  1647.596093  2055.065571  49.5547   14.075

     speed
0  46.4242
1   47.001
2  48.9988
3  50.0994
4  48.2904
5  51.5646
6   53.923
7  51.5148
```

In [14]:

```
for i in range(8):
    print("when t is {}, speed is {}".format(int(radar_df["t"][i]),  radar_df["speed"]
```

```
when t is 7, speed is 46.42420062213374
when t is 8, speed is 47.001039380953344
when t is 9, speed is 48.99880514036797
when t is 10, speed is 50.09944162965227
when t is 11, speed is 48.290350838204944
when t is 12, speed is 51.564559049272255
when t is 13, speed is 53.923033545053535
when t is 14, speed is 51.51480056963696
```

Problem 5

In [15]:
```
#Define a function that approximate the Jacobian matrix of f at x using the second or
def Jacobian(fn_vec, x_vec, h):
    m = len(fn_vec)
    n = len(x_vec)
    I = np.identity(n)
    Jac = np.zeros((m, n))
    if n == 1:
        for i, fn in enumerate(fn_vec):
            for j in range(n):
                f= sy.lambdify([x1], fn, 'numpy')
                new_x_vec1 = x_vec+h*I[:,j]
                new_x_vec2 = x_vec-h*I[:,j]
                Jac[i,j] = (f(new_x_vec1[0]) - f(new_x_vec2[0]))/(2*h)
    if n == 2:
        for i, fn in enumerate(fn_vec):
            for j in range(n):
                f= sy.lambdify([x1,x2], fn, 'numpy')
                new_x_vec1 = x_vec+h*I[:,j]
                new_x_vec2 = x_vec-h*I[:,j]
                Jac[i,j] = (f(new_x_vec1[0], new_x_vec1[1]) - f(new_x_vec2[0], new_x_v
    if n == 3:
        for i, fn in enumerate(fn_vec):
            for j in range(n):
                f= sy.lambdify([x1,x2,x3], fn, 'numpy')
                new_x_vec1 = x_vec+h*I[:,j]
                new_x_vec2 = x_vec-h*I[:,j]
                Jac[i,j] = (f(new_x_vec1[0], new_x_vec1[1], new_x_vec1[2]) - f(new_x_v

    return Jac
```

In [16]:
```
#Test the function
# f(x1,x2) = [x1**2, x1**3 -x2]
x1 = sy.symbols("x1")
x2 = sy.symbols("x2")
fn1=x1**2
fn2=x1**3 - x2
fn_vec = [fn1, fn2]
x_vec = [1, 2]
h = 0.01
Jacobian(fn_vec, x_vec, h)
```

Out[16]:
```
array([[ 2.    ,  0.    ],
       [ 3.0001, -1.    ]])
```

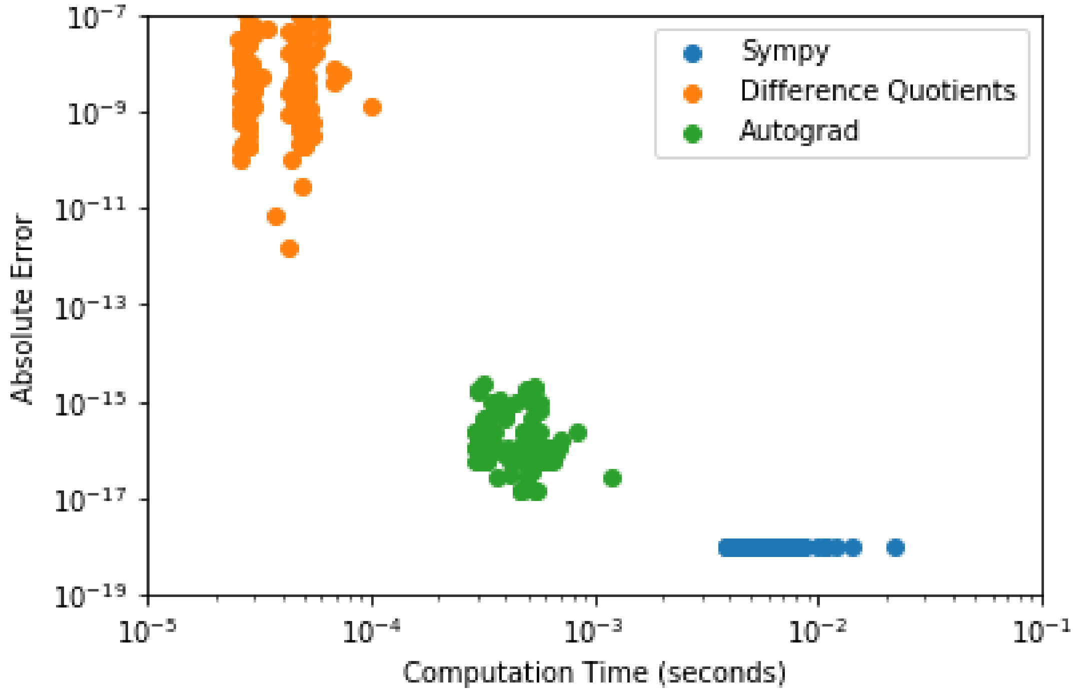
Problem7

In [17]:
```
def Time(N):
        t1 = np.zeros(N,dtype='float')
```

```
        t2 = np.zeros(N,dtype='float')
        t3 = np.zeros(N,dtype='float')
        abs_e1 = np.array([1e-18] * N)
        abs_e2 = np.zeros(N,dtype='float')
        abs_e3 = np.zeros(N,dtype='float')



        y = lambda x: (anp.sin(x)+1)**(anp.sin(anp.cos(x)))
        auto_yprime = grad(y)

        for i in range(N):
            x = np.random.uniform(-math.pi, math.pi)

            time1 = time.clock()
            z =sy.symbols('z')

            yprime = sy.diff((sy.sin(z)+1)**sy.sin(sy.cos(z)), z)
            fprime = sy.lambdify(z, yprime, "numpy")

            prime = fprime(x)
            time2 = time.clock()
            t1[i] = time2 - time1

            time3 = time.clock()
            appr_prime = cen_diff2(f, x, h = 0.01)
            time4 = time.clock()
            t2[i] = time4 - time3
            abs_e2[i] = abs(appr_prime - prime)

            time5 = time.clock()
            auto_appr_prime = auto_yprime(x)
            time6 = time.clock()
            t3[i] = time6 - time5
            abs_e3[i] = abs(auto_appr_prime- prime)

        return t1, t2, t3, abs_e1, abs_e2, abs_e3

t1, t2, t3, abs_e1, abs_e2, abs_e3 = Time(200)

plt.scatter(t1, abs_e1, label='Sympy')
plt.scatter(t2 ,abs_e2, label='Difference Quotients')
plt.scatter(t3, abs_e3, label='Autograd')
plt.loglog()
plt.xlim(10**-5,10**-1)
plt.ylim(10**-19,10**-7)
plt.xlabel("Computation Time (seconds)")
plt.ylabel("Absolute Error")
```

```
plt.legend()
```

Out[17]: <matplotlib.legend.Legend at 0x12042a908>



In [ ]: